Triangle Inequality - A Curious Counting Result

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Counting and the study of mathematics go a long way back. In this article, geometry and counting come together to provide an interesting window to mathematical thinking and reasoning. Geometrical constructions provide a hands-on aspect and teachers of classes 6-10 can use this article to design GeoGebra investigations, mathematical discussions with trigger questions or even an unusual revision worksheet.

Geometric constructions using ruler-and-compass can be a rich learning experience in middle school and high school. Properties of parallel lines, triangles, various types of quadrilaterals and circles can be reinforced using constructions of geometrical figures, giving different combinations of parameters such as angles, lengths of sides, medians, altitudes, sums of sides, etc. Ideas from the locus concept can also be used to create interesting problems (e.g., [1]).

In the middle school, the triangle inequality can be appreciated by following a graded sequence of construction problems. Once they feel comfortable with the task of constructing triangles given the lengths of the sides or the measures of the angles (in different combinations), students can be asked whether it is always possible to construct a triangle given the lengths of the sides; e.g., "Is it possible to construct a triangle with sides of lengths 5 cm, 6 cm, 12 cm?" The students soon find that after drawing one side, say the one with length 5 cm, while trying to construct the remaining two sides, the relevant arcs do not intersect, so it is not possible to construct the triangle.

Keywords: Triangle, triangle inequality, side, length

From this, they can be guided to explore the triangle inequality. Let lengths a, b be given, with $b < a$; we wish to draw a triangle ABC in which BC has length a and AC has length b. Draw a segment BC with length a (Figure 1). With C as centre, draw a circle Γ with b as radius. Let U, V be the points where the circle intersects line BC, with U between B and C. Then $BU = a - b$ and $BV = a + b$. Vertex A clearly must lie somewhere on Γ (except points U and V). It should be clear that the third side of the triangle cannot be shorter than BU, nor can it be longer than BV. That is, the length c of the third side of the triangle must lie between $a - b$ and $a + b$, i.e., $c > a - b$ and $c < a + b$. Since there are infinitely many numbers between $a - b$ and $a + b$, the number of distinct triangles with two sides as a and b is infinite.

A remarkable fact emerges if we restrict the lengths of the sides of the triangle to integer values. For example, say we want to find the number of integer-sided triangles in which two sides have lengths 11 cm and 7 cm. The length of the third side must lie between 11 − 7 = 4 cm and 11 + 7 = 18 cm, and the lengths 4 and 18 are not possible, so the possibilities for the sides are (11, 7, 5), (11, 7, 6), (11, 7, 7), . . . , (11, 7, 16), (11, 7, 17). The total number of possibilities is 13 (= 17 − 5 + 1).

If the two sides have lengths 14 and 6, then the possibilities are (14, 6, 9), (14, 6, 10), (14, 6, 11), (14, 6, 12), . . . , (14, 6, 17), (14, 6, 18), (14, 6, 19); the number of possibilities is 11 (= 19 − 9 + 1).

Now consider the general case of an integer-sided triangle. Let two of the sides have specified integer lengths a, b, where $a > b$. Let c be the length of the third side (c too is an integer). Then the least possible value of c is $a - b + 1$, and the largest possible value of c is $a + b - 1$. Thus, c can take all integer values from $a - b + 1$ to $a + b - 1$. Now the number of integers from $a - b + 1$ to $a + b - 1$ is

$$(a + b - 1) - (a - b + 1) + 1 = 2b - 1.$$

So it is possible to construct precisely $2b - 1$ different integer-sided triangles with a and b as the lengths of two of its sides ($a > b$).

The striking fact here is that ***the number of different triangles depends only on the length of the smaller side***. So the number of integer-sided triangles with 11, 7 as the lengths of two sides is exactly the same as the number of integer-sided triangles with 1234,7 as the lengths of two sides, or with 157869,7 as the lengths of two sides; the number in each case is 13. All the possible integer-sided triangles with two sides having lengths 11 cm and 7 cm are drawn in Figure 2 (all with base BC). This yields a nice construction pattern!

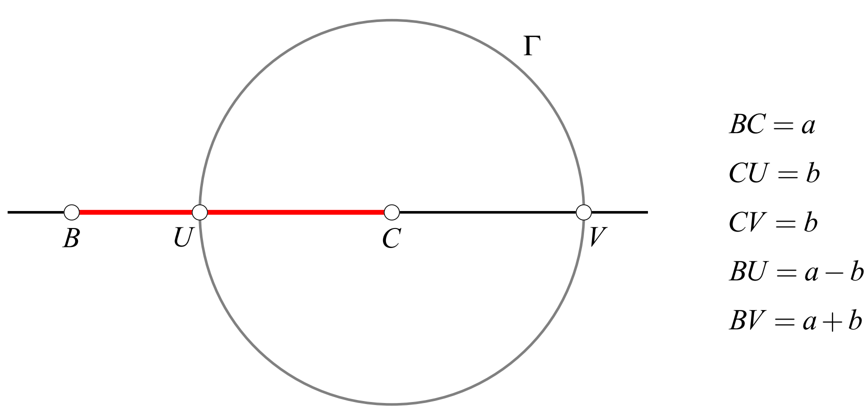

Figure 1. Constructing a triangle given the lengths of two of its sides

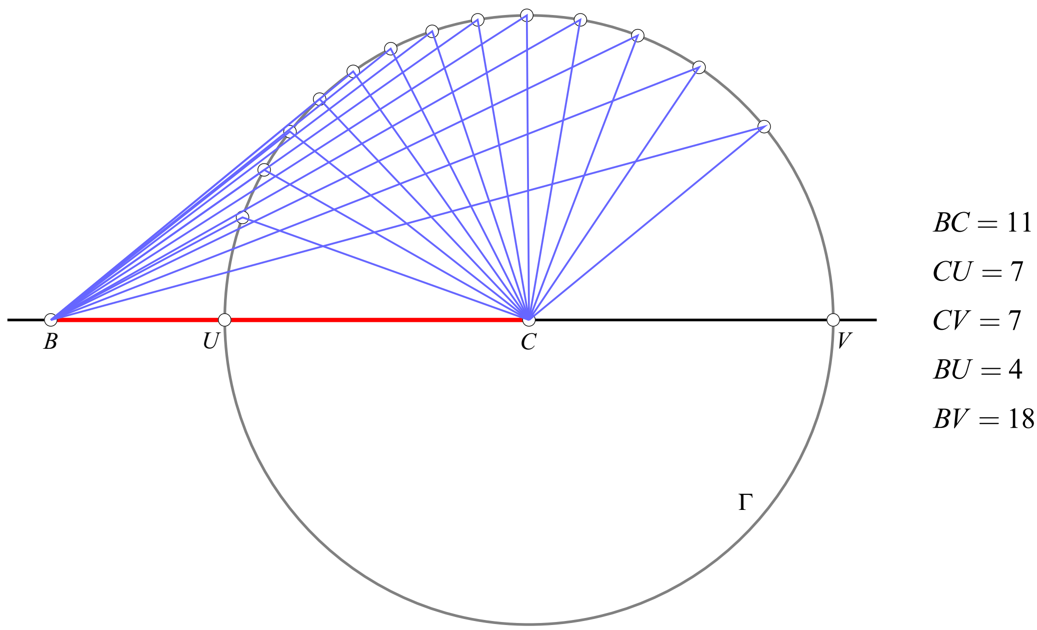

Figure 2. 13 different integer-sided triangles with 11 and 7 as the lengths of two of its sides

References

1. Sneha Titus & R Athmaraman, "Problems of the Middle School", http://azimpremjiuniversity.edu.in/SitePages/ resources-ara-march-2018-problems-of-middle-school.aspx.

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