Extending the Definitions of GCD and LCM to Fractions

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With the LCM and GCD of natural numbers well-defined and an integral part of the middle school curriculum, one may wonder why this article embarks on a rather theoretical study of the LCM and GCD of rational numbers. But this article depicts exactly what a mathematician does – take a well-known concept and extend it to larger sets, testing the extended definition with backward compatibility with the original set. For the more able middle-schooler, this is an excellent opportunity to flex the muscles of conceptual understanding and constructive reasoning. Dive in!

It happens quite frequently in mathematics that we need to extend the definition of a mathematical concept to cover a larger domain than the one on which the concept was originally defined. Historically, such a progression is part of the very evolution of mathematics.

To give a simple example, consider the notion of sine and cosine of an angle. These notions arise from the consideration of right-angled triangles. If we stick strictly to the original definition, then it becomes absurd to talk of the sine and cosine of an obtuse angle. But one can easily extend the domains of definition of these functions to cover angles of arbitrary measure by considering, instead of a right-angled triangle, a circle of unit radius centred at the origin of a rectangular coordinate plane.

In such cases, it becomes imperative to check for *backward compatibility* (to borrow a term used more often in relation to software packages as they evolve and grow over time). That is, we must verify that the extended definition reduces to the original definition when considered over the original (reduced) domain. It is easy to verify that backward compatibility does hold in the case of the trigonometric functions.

And just as we have the notion of 'vibration testing' in engineering, in which we test the response of a newly engineered device in a stressful vibration environment to ascertain its points of weakness, so must we subject an extended definition to tests to ascertain *its* points of weakness.

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We consider one such case here. We ask:

Is it possible to extend the definitions of GCD *and* LCM *to the rational numbers?*

In their current form, these functions are defined only for pairs of integers (not both zero). We recall their definitions here.

- Let a, b be integers, not both equal to 0 (note that $\text{GCD}(0,0)$ is not defined). Then:
  - → We define $\text{GCD}(a,0)$ to be $|a|$, provided that $a \neq 0$.
  - → If $a \neq 0$ and $b \neq 0$, then $\text{GCD}(a,b)$ is defined to be the largest positive integer c such that a/c and b/c are integers. (Note that the two conditions "$a \neq 0$ and $b \neq 0$" can be captured as a single condition by writing: "if $ab \neq 0$".)

  The definition will always yield a number satisfying the stated conditions since the set of positive integers c such that a/c and b/c are integers is nonempty (1 belongs to this set) and finite (since $c \leq |a|$ and $c \leq |b|$).
- Let $a \neq 0,\ b \neq 0$ be integers. We define $\text{LCM}(a,b)$ to be the smallest positive integer c such that c/a and c/b are integers. The definition will always yield a number satisfying the stated conditions since the set of positive integers c such that c/a and c/b are integers is nonempty (since ab belongs to this set).

The GCD can be efficiently computed using Euclid's division algorithm, and the LCM can then be computed using the relation

$$\text{GCD}(a,b) \times \text{LCM}(a,b) = ab, \tag{1}$$

which is true for all $a \neq 0,\ b \neq 0$.

Can we extend these definitions to cover rational numbers as well, keeping in mind the comments made earlier? Let us apply commonsense logic and see where it leads us.

Greatest common divisor of two positive rational numbers

We do not ordinarily use the term 'divisor' and 'multiple' in connection with non-integral rational numbers; so it is best to be clear at the start as to what these notions mean. We adopt the simplest approach here; the two terms are assumed to mean exactly the same thing as what they mean when used with reference to integers. So, if r and s are non-zero rational numbers, we say that r is a divisor of s if s/r is an integer; and in this situation we also say that s is a multiple of r. For example, $1/6$ is a divisor of $2/3$ (for $2/3 \div 1/6 = 4$, an integer) and $2/3$ is a multiple of $1/6$.

Study of a particular case. To start with, let us try to find the GCD of a specific pair of fractions, say $15/4$ and $9/14$. Note that both the fractions have been given in their lowest terms. Suppose that the required GCD is $1/m$ times the first fraction and also $1/n$ times the second fraction, where m and n are positive integers. By the definition of greatest common divisor, m and n cannot have any divisors in common other than 1; i.e., $\text{GCD}(m,n) = 1$. So we have:

$$\frac{15}{4m} = \frac{9}{14n} \qquad \text{GCD}(m,n) = 1.$$

Cancelling common factors and simplifying, we get

$$\frac{m}{n} = \frac{35}{6}, \qquad \text{GCD}(m,n) = 1.$$

Since 35 and 6 have no factors in common other than 1, the relation $m/n = 35/6$ tells us that m is a multiple of 35 and n is a multiple of 6. And since $\text{GCD}\,(m, n) = 1$, it must be that $m = 35$ and $n = 6$. Hence the GCD of the two fractions is equal to

$$\frac{15}{4 \times 35} = \frac{3}{28} = \frac{9}{14 \times 6},$$

i.e., the GCD is equal to $3/28$. Now the numerator of $3/28$ is 3, which is equal to $\text{GCD}\,(15, 9)$, and the denominator of $3/28$ is 28, which is equal to $\text{LCM}\,(4, 14)$. So our reasoning has led us to the following:

$$\text{GCD}\left(\frac{15}{4}, \frac{9}{14}\right) = \frac{\text{GCD}\,(15, 9)}{\text{LCM}\,(4, 14)}.$$

The general case. Will this reasoning work in general? Let us apply the same reasoning to the pair of fractions a/b and c/d; here a, b, c, d are positive integers with $\text{GCD}\,(a, b) = 1$ and $\text{GCD}\,(c, d) = 1$. Suppose that the required GCD is $1/m$ times the first fraction and also $1/n$ times the second fraction, where m and n are positive integers with no divisors in common other than 1; i.e., $\text{GCD}\,(m, n) = 1$. So we have:

$$\frac{a}{bm} = \frac{c}{dn}, \qquad \text{GCD}\,(m, n) = 1,$$

$$\therefore \quad \frac{m}{n} = \frac{ad}{bc}, \qquad \text{GCD}\,(m, n) = 1.$$

By assumption, $\text{GCD}\,(a, b) = 1 = \text{GCD}\,(c, d)$. However, a and c may have common divisors other than 1; likewise for b and d. Let $\text{GCD}\,(a, c) = u$ and $\text{GCD}\,(b, d) = v$. We have, then:

$$\frac{ad}{bc} = \frac{a}{c} \times \frac{d}{b} = \frac{a/u}{c/u} \times \frac{d/v}{b/v} = \frac{(ad)/(uv)}{(bc)/(uv)}.$$

Hence:

$$\frac{m}{n} = \frac{(ad)/(uv)}{(bc)/(uv)},$$

and since $\dfrac{ad}{uv}$ and $\dfrac{bc}{uv}$ can have no factors in common, it must be that

$$m = \frac{ad}{uv}, \qquad n = \frac{bc}{uv}.$$

Hence the required GCD is

$$\frac{a}{b \times m} = \frac{uv}{bd} = \frac{c}{d \times n}.$$

Let us look more closely at the fraction in the middle; it can be written as:

$$\frac{uv}{bd} = \frac{u}{bd/v}.$$

The numerator of the fraction on the right side is $\text{GCD}\,(a, c)$. The denominator of the fraction is

$$\frac{bd}{v} = \frac{b \times d}{\text{GCD}\,(b, d)} = \text{LCM}\,(b, d).$$

We see, therefore, that

$$\text{GCD}\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{GCD}\,(a, c)}{\text{LCM}\,(b, d)}.$$

Least common multiple of two positive rational numbers

Study of a particular case. To start with, let us try to find the LCM of the same pair of fractions we studied earlier, $15/4$ and $9/14$. Suppose that the required GCD is m times the first fraction and also n times the second fraction, where m and n are positive integers. By the definition of least common multiple, m and n cannot have any divisors in common other than 1; i.e., $\gcd(m, n) = 1$. So we have:

$$\frac{15m}{4} = \frac{9n}{14}, \quad \gcd(m, n) = 1.$$

Cancelling common factors and simplifying, we get

$$\frac{m}{n} = \frac{6}{35}, \quad \gcd(m, n) = 1.$$

Since 6 and 35 have no factors in common other than 1, the relation $m/n = 6/35$ tells us that m is a multiple of 6 and n is a multiple of 35. And since $\gcd(m, n) = 1$, it must be that $m = 6$ and $n = 35$. Hence the LCM of the two fractions is equal to

$$\frac{15 \times 6}{4} = \frac{45}{2} = \frac{9 \times 35}{14},$$

i.e., the GCD is equal to $45/2$. Now the numerator of $45/2$ is 45, which is equal to $\operatorname{lcm}(15, 9)$, and the denominator of $45/2$ is 2, which is equal to $\gcd(4, 14)$. So our reasoning has led us to the following:

$$\operatorname{lcm}\left(\frac{15}{4}, \frac{9}{14}\right) = \frac{\operatorname{lcm}(15, 9)}{\gcd(4, 14)}.$$

The general case. Just as we did earlier, let us apply the same reasoning to the pair of fractions a/b and c/d; here a, b, c, d are positive integers with $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$. Suppose that the required GCD is m times the first fraction and also n times the second fraction, where m and n are positive integers with no divisors in common other than 1; i.e., $\gcd(m, n) = 1$. So we have:

$$\frac{am}{b} = \frac{cn}{d}, \quad \gcd(m, n) = 1,$$

$$\therefore \quad \frac{m}{n} = \frac{bc}{ad}, \quad \gcd(m, n) = 1.$$

Let $\gcd(a, c) = u$ and $\gcd(b, d) = v$. We have, then:

$$\frac{bc}{ad} = \frac{b}{d} \times \frac{c}{a} = \frac{b/v}{d/v} \times \frac{c/u}{a/u} = \frac{(bc)/(uv)}{(ad)/(uv)}.$$

Hence:

$$\frac{m}{n} = \frac{(bc)/(uv)}{(ad)/(uv)},$$

and since $\dfrac{bc}{uv}$ and $\dfrac{ad}{uv}$ have no factors in common, it must be that

$$m = \frac{bc}{uv}, \quad n = \frac{ad}{uv}.$$

Hence the required GCD is

$$\frac{a \times m}{b} = \frac{ac}{uv} = \frac{c \times n}{d}.$$

Let us look more closely at the fraction in the middle; it can be written as:

$$\frac{ac}{uv} = \frac{ac/u}{v}.$$

The numerator of the fraction on the right side is $ac/\text{GCD}\,(a,c) = \text{LCM}\,(a,c)$. The denominator of the fraction is $v = \text{GCD}\,(b,c)$. We see, therefore, that

$$\text{LCM}\,\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{LCM}\,(a,c)}{\text{GCD}\,(b,d)}.$$

Subjecting the extended definition to stress tests…

First test. Let us subject our formula to the simplest possible stress test, that of backward compatibility. That is, let us see if the formula reduces to the known formula for GCD if the rational numbers under consideration happen to be positive integers, i.e., with unit denominator.

In the statement,

$$\text{GCD}\,\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{GCD}\,(a,c)}{\text{LCM}\,(b,d)},$$

let $b = 1$, $d = 1$. The result then assumes the following form:

$$\text{GCD}\,(a,c) = \frac{\text{GCD}\,(a,c)}{\text{LCM}\,(1,1)}.$$

But $\text{LCM}\,(1,1) = 1$, so the statement assumes the form $\text{GCD}\,(a,b) = \text{GCD}\,(a,b)$, which is vacuously true. So the test of backward compatibility has been passed (though in a rather trivial manner).

Second test. We know that if a, b are positive integers, then

$$\text{GCD}\,(a,b) \times \text{LCM}\,(a,b) = ab.$$

Let us check whether such a relation holds for the GCD and the LCM of two positive rational numbers. Let the two positive rational numbers be a/b and c/d; here a, b, c, d are positive integers with $\text{GCD}\,(a,b) = 1$ and $\text{GCD}\,(c,d) = 1$. Then we have:

$$\begin{aligned} \text{GCD}\,\left(\frac{a}{b}, \frac{c}{d}\right) \times \text{LCM}\,\left(\frac{a}{b}, \frac{c}{d}\right) &= \frac{\text{GCD}\,(a,c)}{\text{LCM}\,(b,d)} \times \frac{\text{LCM}\,(a,c)}{\text{GCD}\,(b,d)} \\ &= \frac{\text{GCD}\,(a,c) \times \text{LCM}\,(a,c)}{\text{GCD}\,(b,d) \times \text{LCM}\,(b,d)} \\ &= \frac{ac}{bd} = \frac{a}{b} \times \frac{c}{d}. \end{aligned}$$

We see that the relation does hold in the domain of rational numbers. Our extended definition has just passed an important test!

Third test. We know that if a, b are positive integers, then there exist integers x and y such that

$$ax + by = \text{GCD}\,(a,b).$$

Let us check whether such a relation holds for the GCD of two positive rational numbers. Let the two positive rational numbers be a/b and c/d; here a, b, c, d are positive integers with $\text{GCD}\,(a,b) = 1$ and $\text{GCD}\,(c,d) = 1$. We must examine whether there exist integers x and y such that

$$\frac{ax}{b} + \frac{cy}{d} = \frac{\text{GCD}\,(a,c)}{\text{LCM}\,(b,d)}.$$

Multiplying both sides of the above relation by bd, we see that the question reduces to asking whether there exist integers x and y such that

$$ad \cdot x + bc \cdot y = \textsc{gcd}\,(a, c) \cdot \textsc{gcd}\,(b, d).$$

Since the following relation is clearly true,

$$\textsc{gcd}\,(ad, bc) = \textsc{gcd}\,(a, c) \cdot \textsc{gcd}\,(b, d),$$

it follows that there must indeed exist integers x and y such that

$$ad \cdot x + bc \cdot y = \textsc{gcd}\,(a, c) \cdot \textsc{gcd}\,(b, d),$$

and, therefore, that there must indeed exist integers x and y such that

$$\frac{ax}{b} + \frac{cy}{d} = \frac{\textsc{gcd}\,(a, c)}{\textsc{lcm}\,(b, d)}.$$

So the same kind of relation holds in the domain of rational numbers. Our extended definition has now passed a third important test!

It is impressive to see how far commonsense logic has led us. We were true to it and it has not let us down!

References

1. Wikipedia, "Greatest common divisor", https://en.wikipedia.org/wiki/Greatest_common_divisor.
2. Wikipedia, "Least common multiple", https://en.wikipedia.org/wiki/Least_common_multiple.

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