Features

Addendum to Power Triangle

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In Part II of this article [1], we had presented a unified approach by which, for any given positive integer k, the formula for the sum of the k-th powers of the first n natural numbers can be obtained. The method made use of a triangular arrangement of numbers called the *Power Triangle*. Its first few rows are given in Figure 1.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	3	2			
$n=3$	1	7	12	6		
$n=4$	1	15	50	60	24	
$n=5$	1	31	180	390	360	120

Figure 1. The first few rows of the Power Triangle

Here are the rules governing the formation of the Power Triangle. Denote the number in row n and column r by $T(n, r)$; here $n = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots, n+1$. Then:

Rule 1: Row n has $n + 1$ numbers, $T(n, 1), T(n, 2), T(n, 3), \ldots, T(n, n+1)$. We adopt the convention that $T(n, r) = 0$ if $r < 1$ or if $r > n + 1$. (In words: if the element at any position is absent, it is taken to be 0.)

Rule 2: The first number of every row is 1; so $T(n, 1) = 1$ for $n = 0,1,2, \ldots$.

Rule 3: The numbers in the successive rows of the power triangle are determined recursively as follows: for $n = 1,2,3\ldots$ and $r = 1,2,3, \ldots, n+1$,

$$T(n, r) = (r-1) \cdot T(n-1, r-1) + r \cdot T(n-1, r). \qquad \text{(I)}$$

Keywords: Pascal triangle, power triangle, sums of powers

The Power Triangle with subscripts. In its original form, the Power Triangle had been presented using **subscripts**. See Figure 2; observe that each entry has a subscript which is identical to the *r*-value of its column. So this brief addendum is being offered for the sake of historical correctness.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=0$	1_1					
$n=1$	1_1	1_2				
$n=2$	1_1	3_2	2_3			
$n=3$	1_1	7_2	12_3	6_4		
$n=4$	1_1	15_2	50_3	60_4	24_5	
$n=5$	1_1	31_2	180_3	390_4	360_5	120_6

Figure 2. The first few rows of the Power Triangle, using subscripts

The law of formation of the Power Triangle when presented in this form is the following. The zeroth row has a single entry: 1, with subscript 1. For subsequent rows, the first entry (corresponding to $r = 1$) is 1, with subscript 1. Each subsequent entry (i.e., corresponding to $r > 1$) is given by the following sum, with the understanding that an empty space means that the corresponding entry is 0: *number in row n, column r is equal to the number in row* $n-1$*, column r times its subscript plus the number in row* $n-1$*, column* $r-1$ *times its subscript*. Note that this is simply the following rule expressed in words:

$$T(n, r) = (r - 1) \cdot T(n - 1, r - 1) + r \cdot T(n - 1, r).$$

We have already explained how the Power Triangle is used to get the desired formulas; so we do not elaborate on that now. All we need is this formula:

$$1^k + 2^k + \cdots + n^k = \sum_{r=1}^{k+1} \binom{n}{r} \cdot T\left(k, r\right).$$

References

1. V G Tikekar, “On the sums of powers of natural numbers, Part II”, *At Right Angles*, March 2018, http://azimpremjiuniversity.edu.in/SitePages/ resources-ara-march-2018-sums-of-powers.aspx

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