Problem Corner

Problems for the Senior School

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Problems for Solution

Problem VII-2-S.1 Let AB be a fixed line segment in the plane. Let O and P be two points in the plane and on the same side of AB. If $\measuredangle AOB = 2\measuredangle APB$, does it necessarily follow that P lies on the circle with centre O and passing through A and B?

Problem VII-2-S.2 Let ABC be an equilateral triangle with centre O. A line through C meets the circumcircle of triangle AOB at points D and E. Prove that the points A, O and the midpoints of segments BD, BE are concyclic. [Tournament of Towns]

Problem VII-2-S.3 Three non-zero real numbers are given. If they are written in any order as coefficients of a quadratic trinomial, then each of these trinomials has a real root. Does it follow that each of these trinomials has a positive root? [Tournament of Towns]

Problem VII-2-S.4 D is the midpoint of the side BC of triangle ABC. E and F are points on CA and AB respectively, such that BE is perpendicular to CA and CF is perpendicular to AB. If DEF is an equilateral triangle, does it follow that ABC is equilateral? [Tournament of Towns]

Problem VII-2-S.5 A boy computed the product of the first n positive integers and his sister computed the product of the first m even positive integers where $m \geq 2$. Is it possible for them to get the same result?

Solution to problem VII-1-S.1 Two hundred students are positioned in 10 rows, each containing 20 students. From each of the 20 columns thus formed, the shortest student is selected, and the tallest of these 20 (short) students is labelled A. These students return to their initial places. Next, the tallest student in each row is selected, and from these 10 (tall) students, the shortest is labelled B. Who is taller, A or B?

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If A and B stand in the same row, then B is taller than A, since B is the tallest student in that row. If A and B stand in the same column, then again B is taller than A, since A is the shortest student in that column. Finally, if A and B do not stand in either the same column or the same row, let C be that student standing in the same column as A and in the same row as B. Then B is taller than C (since B is the tallest in that row), and C is taller than A (since A is the shortest in that column). Hence, in all cases, B is taller than A.

Solution to problem VII-1-S.2 Given 13 coins, each weighing an integral number of grams. It is known that if any coin is removed, then the remaining 12 coins can be divided into two groups of 6 with equal total weight. Prove that all the coins are of the same weight.

First, it follows from the conditions of the problem that each coin weighs either an even number of grams or an odd number of grams. Since any set of twelve coins can be divided into two groups of equal weight, a set of twelve coins weighs an even number of grams. This total weight remains an even number if one of the twelve coins is exchanged with the thirteenth coin. But this is possible only if the weights of the coins interchanged are of the same parity, and this holds for any of the twelve coins initially weighed. Hence, either each coin weighs an even number of grams or each coin weighs an odd number of grams.

Now subtract from the weight of each coin, the weight of the lightest coin (possibly two or more coins may have the same weight, but this is unimportant). This may be thought of as producing a *new* set of coins, and this new set clearly satisfies the conditions of the problem. (One or more coins may be thought of as having *zero weight*.) It is clear that each coin in the *new* set weighs an even number of grams. If now we divide each weight by 2 and think of this as providing a *new* set of weights, this new set again satisfies the conditions of the problem.

Assume now that **not** all the coins are of the same weight. In this case, not all the weights of the second set (obtained by subtracting the weight of the lightest coin from the original weights of each of the coins) will be zero. If we continue to divide by 2, thus obtaining *new* sets satisfying the conditions of the problem, we finally arrive at a set of coins of which some are of even weight (at least one is of zero weight) and some are of odd weight (continued division of an even number by 2 finally produces an odd number). But such a set satisfying the conditions of the problem has been shown to be impossible. This contradiction proves the assertion of the problem statement.

Solution to problem VII-1-S.3 Show that there are infinitely many positive integers A such that $2A$ is a square, $3A$ is a cube and $5A$ is a fifth power.

First, observe that 2,3,5 divide A. So we may take $A = 2^{\alpha}3^{\beta}5^{\gamma}$. Considering $2A$, $3A$ and $5A$, we observe that $\alpha + 1, \beta, \gamma$ are divisible by 2; $\alpha, \beta + 1, \gamma$ are divisible by 3, and $\alpha, \beta, \gamma + 1$ are divisible by 5. We can choose $\alpha = 15 + 30n$; $\beta = 20 + 30n$; $\gamma = 24 + 30n$. As n varies over the set of natural numbers, we get an infinite set of numbers of required type.

Solution to problem VII-1-S.4 An infinite sequence of positive integers $a_1, a_2, \ldots, a_n, \ldots$ satisfies the condition $\sum_{k=1}^{m} a_k^3 = \left(\sum_{k=1}^{m} a_k\right)^2$, *i.e.*,

$$a_1^3 + a_2^3 + a_3^3 + \cdots a_m^3 = \left(a_1 + a_2 + a_3 + \cdots + a_m\right)^2$$ for each positive integer m. Determine the sequence.

Putting $m = 1$ we get $a_1 = 1$. If $m = 2$, then $1 + a_2^3 = (1 + a_2)^2$ and this implies $a_2 = 2$. If we assume that we have shown $a_j = j$ for $1 \le j \le k$ for some $k \ge 3$ then

$$1^3 + 2^3 + \cdots + k^3 + a_{k+1}^3 = (1 + 2 + \cdots + k + a_{k+1})^2,$$

and noting that $1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^2 = \left(\tfrac{1}{2}k(k+1)\right)^2$, we get after simplification,

$$(a_{k+1} - k - 1)(a_{k+1} + k) = 0,$$

whence $a_{k+1} = k + 1$. Thus by induction it follows that $a_n = n$ for every positive integer n.

Solution to problem VII-1-S.5 The function $f(n) = an+b$, where a and b are integers, is such that for every integer n, the numbers $f(3n+1)$, $f(3n)+1$ and $3f(n)+1$ are three consecutive integers in some order. Determine all such functions $f(n)$.

We have

$$f(3n+1) = 3an + a + b,$$

$$f(3n) + 1 = 3an + b + 1,$$

$$3f(n) + 1 = 3an + 3b + 1.$$

Observe that $b \neq 0$, for if $b = 0$, then $f(3n)+1 = 3f(n) + 1$, which contradicts the fact that they are consecutive integers. Also, $3f(n) + 1 - f(3n) - 1 = 2b$, an even non-zero integer. Thus these two numbers cannot be consecutive. Therefore $f(3n+1)$ is the middle number and hence

$$2f(3n+1) = f(3n) + 1 + 3f(n) + 1,$$

whence $a = b + 1$ and $f(n) = (b + 1)\,n + b$. Also,

$$|f(3n+1) - f(3n) - 1| = |b| = 1.$$

Thus $b = \pm 1$ and hence $f(n) = 2n + 1$, and $f(n) = -1$ are the only solutions. It is easy to check that both satisfy the condition of the problem.