in the classroom

A math connect across the centuries

Ramanujan and Pythagoras!

An interesting extract from Ramanujan's notebooks which makes for a great classroom exercise in geometry, with a dash of algebra thrown in. An enterprising teacher could do this proof in stages — starting from showing students the figure and asking them to prove the theorem; if they can't, providing them with enough scaffolding to help them complete the proof.

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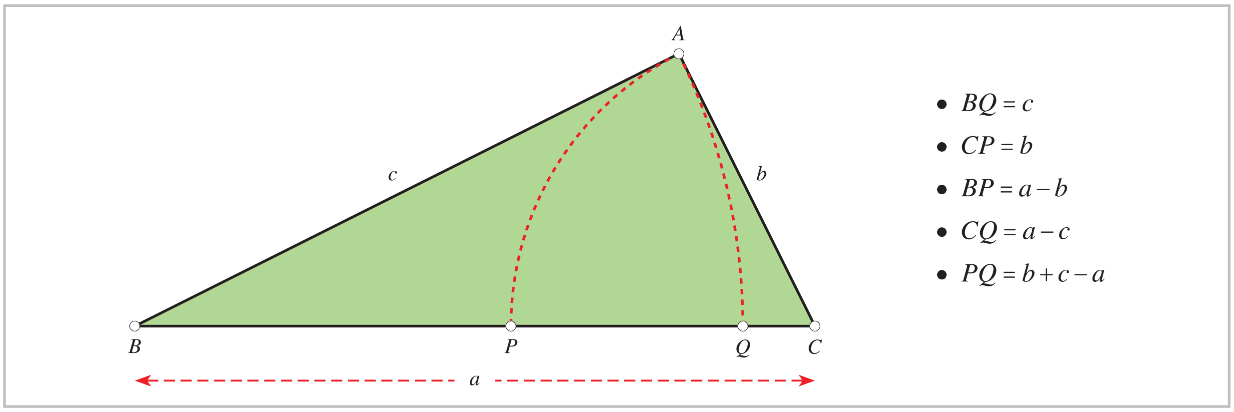
What connection could there possibly be between Ramanujan and Pythagoras, when they lived twenty five centuries apart? Here is one such: an entry in one of Ramanujan's famous NOTEBOOKS, about a right angled triangle, which turns to be a consequence of Pythagoras's theorem. (See Remark 1, below, for some information about these notebooks.)

In the figure we see a right $\triangle ABC$, with $\angle A = 90^\circ$. An arc is drawn with C as centre and radius CA, cutting BC at P, and an arc is drawn with B as centre and radius BA, cutting BC at Q.



Here is Ramanujan's claim about this diagram: $PQ^2 = 2BP \times CQ$. See if you can prove it for yourself, before reading on.

Proof. We have $BP = a - b$, $CQ = a - c$, $PQ = b + c - a$. So Ramanujan's claim is:

$$(b + c - a)^2 = 2(a - b)(a - c).$$

We must verify this equality. Expanding the terms and subtracting the quantity on the right side from the quantity on the left, we get the following:

$$\begin{aligned}
&(b + c - a)^2 - 2(a - b)(a - c) \\
&\quad = (a^2 + b^2 + c^2 + 2bc - 2ab - 2ac) \\
&\qquad - (2a^2 + 2bc - 2ab - 2ac) \\
&\quad = b^2 + c^2 - a^2.
\end{aligned}$$

Hence the claim that $PQ^2 = 2BP \times CQ$ is identical to the claim that $a^2 = b^2 + c^2$, which is nothing but the PT. So Ramanujan's claim follows from the PT.

Remark 1. The entry we have described here is one of the few entries in Ramanujan's NOTEBOOKS that deal with geometry. Most of the entries deal with topics in algebra and trigonometry (identities and systems of equations, continued fractions), number theory (properties of various functions, solutions of some equations) and analysis (summations of series). Some entries also deal with magic squares. Probably these were written when he was very much younger. You will find more information on the NOTEBOOKS on this page: http://en.wikipedia.org/wiki/Srinivasa_Ramanujan.

It is not easy to access these notebooks. You can view individual pages at: http://www.imsc.res.in/~rao/ramanujan/NotebookFirst.htm. And here is the page where you find the '$PQ^2 = 2BP \times CQ$' entry: http://www.imsc.res.in/~rao/ramanujan/NoteBooks/NoteBook2/chapterXXI/page11.htm

Remark 2. On studying the entry closely, one gets the clear impression that Ramanujan *first* discovered the underlying algebraic identity, and then 'cooked up' a theorem based on the identity! For, just below the figure one finds the following statement:

$$\begin{aligned}
&(a + b - \sqrt{a^2 + b^2})^2 \\
&\quad = 2(\sqrt{a^2 + b^2} - a)(\sqrt{a^2 + b^2} - b).
\end{aligned}$$

Now this is an algebraic identity — a 'stand-alone' relation which does not need to rest on any geometric result either for its meaning or for it proof. It can be verified independently. (Please do try it.) But what is *most* interesting is the statement that appears immediately below this one in the notebook:

$$\begin{aligned}
&(\sqrt[3]{(a + b)^2} - \sqrt[3]{a^2 - a\,b + b^2})^3 \\
&\quad = 3(\sqrt[3]{a^3 + b^3} - a)(\sqrt[3]{a^3 + b^3} - b).
\end{aligned}$$

Examining the two relations we see an amazing resemblance between them. But it is clearly not a *logical* connection, for neither relation implies the other one. What seems plausible is that Ramanujan found the first relation the 'ordinary' way, which (perhaps) others could have done, and then in a leap of intuition he 'saw' the second relation too. (The relation is far from obvious! Try proving it.)