Fun Problems

Problems for the Middle School

Problems for the Senior School

problem corner

Fun Problems

1. A Problem from the 'Kangaroo' Math Competition

Here is a charming and memorable problem I came across the other day, adapted from a similar problem posed in the 'Kangaroo' Math Competition of the USA.

In a particular month of some year, there are three Mondays which have even dates. On which day of the week does the 15th of that month fall?

At first sight this look baffling. But as one looks more closely, a solution emerges. Please try it out before reading any further!

Let x denote the date of the first Monday of that month. Clearly, x is one of the numbers 1, 2, 3, 4, 5, 6, 7. The Mondays of that month have the following dates:

$$x,\ x+7,\ x+14,\ x+21,\ x+28\,(?),$$

with a possible question mark against $x+28$; for that particular day may fall in the next month (this will depend on which month it is, and on the size of x). For example, if $x = 4$, then $x+28$ is not a valid date, whichever month it is; and if $x = 2$ and the month is February, then too $x+28$ is not a valid date.

Now the numbers $x,\ x+14,\ x+28$ are either all odd or all even; and $x+7,\ x+21$ are either both odd or both even.

Since we are told that the month has *three* Mondays on even dates, it is the first possibility which must apply. The Mondays of that month thus come on dates $x,\ x+14,\ x+28$.

From this we deduce two things: (i) x is even; (ii) $x+28$ is a valid date for that month.

From (ii) we deduce that x is either 1, 2 or 3. Combining this with (i), we get $x = 2$.

So the 2nd of the month falls on Monday, as does 16th. Hence the 15th falls on Sunday.

2. Problems For Solution - What is a cryptarithm?

Two of the fun problems in this issue deal with *cryptarithms*, so we first explain what they are and how they are to be approached.

A cryptarithm is a disguised arithmetic problem, in which digits have been replaced by letters — each digit being mapped to a different letter. (This implies that two different letters cannot represent the same digit.) The problem, of course, is to 'decode' the mapping — i.e., find which digit stands for which letter.

For example, consider the following 'long multiplication' cryptarithm:

$$\begin{array}{cc} A & B \\ & 4 \\ \hline C & A \\ \hline \end{array}$$

To solve this we argue as follows. Since 4 times the two-digit number AB yields another two-digit number, the tens digit of AB cannot exceed 2. So $A = 0$, 1 or 2. But since A is the tens digit of AB, we cannot have $A = 0$ (else, AB would be a one-digit number); hence $A = 1$ or 2.

Again, since 4 times any number is an even number, the units digit of CA is even; i.e., A is even.

Combining this what we got earlier, we find that $A = 2$.

Now we ask: What can B be, so that the units digit of 4 B has units digit 2? Clearly it must be 3 or 8 (because 4   3 = 12 and 4   8 = 32).

But if $B = 8$ then $AB = 28$, and 4   28 = 112 is a three-digit number; too large.

So $B = 3$, and the answer is: 23   4 = 92. Thus: $A = 2$, $B = 3$, $C = 9$.

Cryptarithms do not always come this easy! But they generally yield to persistence and a long, careful examination of the underlying arithmetic. And, of course, there is no harm in doing a bit of 'trial and error'! Once one solves a cryptarithm there is a great feeling of satisfaction, and one finds that one has learnt some useful mathematics in the process.

Sometimes we come across a cryptarithm with more than one solution. People who design cryptarithms consider this to be a 'design flaw'. *They maintain that a really well designed cryptarithm has a unique solution, and it should be possible to find it using 'pure' arithmetical reasoning, possibly with a small component of trial.*

Problem I-1-F.1
Solve this cryptarithm:

$$ABCD \quad 4 = DCBA.$$

Thus, $ABCD$ is a four digit number whose digits come in reverse order when the number is multiplied by 4.

Problem I-1-F.2
Solve this cryptarithm: $(TWO)^2 = THREE$.

Problem I-1-F.3
The numbers 1, 2, 3, . . . , 99, 100, 101, . . . , 999998, 999999 are written in a line. In this enormously long string of numbers, what is the total number of 1s?

Middle School

Problem Editor : R ATHMARAMAN

1. An Unusual Multiple of 15

The problem we take up for discussion is the following; it was asked in the American Invitational Mathematics Examination (AIME), which is one of the examinations taken by students aspiring to do the national level olympiad of USA:

Find the least positive integer n such that every digit of $15n$ *is either* 0 *or* 8.

Such questions look a bit baffling at first sight, but if one looks carefully, then some facts emerge. Let us try to solve to this problem.

A number whose digits are all 0s and 8s is clearly divisible by 8; hence $15n$ is divisible by 8. But $15 = 3 \quad 5$ and thus has no factors in common with 8 (we say that 8 and 15 are 'coprime'); hence n itself is divisible by 8. Let $m = n/8$. Then m has the property that every digit of $15m$ is 0 or 1. So the problem to solve is:

> *Find the least positive integer m such that every digit of* $15m$ *is* 0 *or* 1.

Once we get this least m, we multiply by 8 to get the required n. So let us look for the least such m.

Since $15m$ is a multiple of 3, the test for divisibility by 3 must apply. Thus, the sum of the digits of $15m$ must be a multiple of 3. Since 0 does not contribute to the sum of the digits it follows that *the number of* 1*s must be a mutiple of* 3. That is, there must be three 1s, or six 1s, or nine 1s, and so on. The least positive multiple of 3 which has no digit outside the set. (0, 1) is clearly the number 111. Since we want the number to be divisible by 5 as well, we simply append a 0 at the end; we get 1110. Hence: 1110 *is the least multiple of* 15 *which has the stated property*.

Since $1110 = 15 \quad 74$, it follows that $m = 74$. Hence the required value of n is $8 \quad 74 = 592$. (Please check for yourself that $592 \quad 15 = 8880$.)

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2. Problems for Solution

Problem I-1-M.1

Find: (a) Four examples of right triangles in which the lengths of the longer leg and the hypotenuse are consecutive natural numbers.

(b) Two examples of right triangles in which the lengths of the legs are consecutive natural numbers.

(c) Two differently shaped rectangles having integer sides and a diagonal of length 25.

(d) Two PPTs free from prime numbers.

Problem I-1-M.2

Let a right triangle have legs a and b and hypotenuse c, where a, b, c are integers. Is it possible that among the numbers a, b, c:

1. All three are even?
2. Exactly two of them are even?
3. Exactly one of them is even?
4. None of them is even?

Either give an example for each or prove why the statement is false.

Problem I-1-M.3

Let a right triangle have legs a and b and hypotenuse c, where a, b, c are integers. Is it possible that among the numbers a, b, c:

1. All three are multiples of 3?
2. Exactly two of them are multiples of 3?
3. Exactly one of them is a multiple of 3?
4. None of them is a multiple of 3?

Either give an example for each or prove why the statement is false.

Problem I-1-M.4

How many PPTs are there in which one of the numbers in the PPT is 60?

Problem I-1-M.5

Take any two fractions whose product is 2. Add 2 to each fraction.

Multiply each of them by the LCM of the denominators of the fractions. You now get two natural numbers. Show that they are the legs of an integer sided right triangle. (Example: Take the fractions $\frac{5}{2}$ and $\frac{4}{5}$; their product is 2. Adding 2 to each we get $\frac{9}{2}$ and $\frac{14}{5}$. Multiplying by 10 which is the LCM of 2 and 5, we get 45 and 28. Now observe that $45^2 + 28^2 = 53^2$.)

Problem I-1-M.6

The medians of a right triangle drawn from the vertices of the acute angles have lengths 5 and $\sqrt{14}$. What is the length of the hypotenuse?

Problem I-1-M.7

Let $ABCD$ be a square of side 1. Let P and Q be the midpoints of sides AB and BC respectively. Join PC, PD and DQ. Let PC and DQ meet at R. What type of triangle is $\triangle PRD$? What are the lengths of the sides of this triangle?

Problem I-1-M.8

Find all right triangles with integer sides such that their perimeter and area are *numerically* equal.

Problem I-1-M.9

If a and b are the legs of a right triangle, show that

$$\sqrt{a^2 + b^2} < a + b \leq \sqrt{2(a^2 + b^2)}$$

Hint. A suitable diagram with a right triangle inscribed in a square may reveal the answer.

Acknowledgement

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Senior School

Problem Editors : PRITHWIJIT DE & SHAILESH SHIRALI

1. A problem in number theory

We start this column with a discussion of the following problem which is adapted from one asked in the first Canadian Mathematical Olympiad (1969):

Find all integer solutions of the equation $a^2 + b^2 = 8c + 6$.

At first sight it looks rather daunting, doesn't it? — a single equation with *three* unknowns, and we are asked to find *all* its integer solutions! But as we shall see, it isn't as bad as it looks.

Note the expression on the right side: 8c + 6, eight times some integer plus six. That means it leaves remainder 6 when divided by 8. Hmmm ...; so we want pairs of integers such that their sum of squares leaves remainder 6 when divided by 8. Put this way, it invites us to first examine what kinds of remainders are left when squared numbers are divided by 8. We build the following table. We have used a shortform in the table: 'Rem' means 'remainder', so 'Rem $(n^2 \div 8)$' means 'the remainder when n^2 is divided by 8'.

n	1	2	3	4	5	6	7	8	9	10	...
n^2	1	4	9	...	...	...	...	...	...	...	...
Rem $(n^2 \div 8)$	1	4	1	...	...	...	...	...	...	...	...

Please complete the table on your own and study the data. What do you see?

Here are some striking patterns we see (and there may be more such patterns):

1. *Every odd square leaves remainder* 1 *when divided by* 8.

2. *The even squares leave remainders* 0 *and* 4 *when divided by* 8*, in alternation:* 4, 0, 4, 0,.... (To say that 'the remainder is 0' means that there is no remainder, i.e., the square is divisible by 8.)

Some thought will convince us that these patterns are 'real'; *they stay all through the sequence of squares and so are genuine properties of the squares*.

For example, consider pattern (1). Every even number can be represented as $2n$ and every odd number as $2n + 1$, for some integer n. The quoted property concerns odd squares.

Hence we have:

$$\begin{aligned}(2n+1)^2 &= 4n^2+4n+1\\ &= 4n(n+1)+1\\ &= (4 \quad \text{an even number})+1\\ &= (\text{a multiple of } 8)+1.\end{aligned}$$

We see that every odd square leaves remainder 1 under division by 8.

In the same way, we find that the even squares leave remainders 0 and 4 under division by 8. Please prove this on your own. (You may want to figure out which squares leave remainder 0, and which squares leave remainder 4.)

With these findings let us look again at the expression a^2+b^2, which is a sum of two squares. We have just seen that under division by 8, the only remainders possible are 0, 1 or 4. So the possible remainders when a^2+b^2 is divided by 8 are the following:

$$0+0,\quad 0+1,\quad 1+1,\quad 4+0,\quad 4+1,\quad 4+4.$$

Hence, the possible remainders are 0, 1, 2, 4 *and* 5*.*

Some numbers are missing in this list. We see that *a sum of two squared numbers cannot leave remainder* 3 *under division by* 8*; nor can it leave remainder* 6*; nor remainder* 7. Note in particular that 'remainder 6' is not possible.

So we have found our answer: *The equation $a^2+b^2=8c+6$ has* ***no*** *integer solutions!*

In fact our analysis has shown us rather more: There are no integer solutions to *any* of the following three equations:

$$a^2+b^2=8c+3,\quad a^2+b^2=8c+6,\quad a^2+b^2=8c+7.$$

The reasoning we have used in solving this problem is typical of such solutions. We call it 'number theoretic reasoning'.

Another example of number theoretic reasoning

Here is another problem from number theory, of a kind often encountered. It has clearly been composed keeping in mind the year when India became independent.

Find all possible square values taken by the expression $n^2+19n+47$ as n takes on all integer values.

Let $n^2+19n+47=m^2$. We need to find the possible values of m.

We shall now use the humble and time honoured technique of 'completing the square'. However to avoid fractions we first multiply by 4; this is acceptable: if $n^2+19n+47$ is a square number, then so is $4\,(n^2+19n+47)$. Here is what we get:

$$4(n^2+19n+47) = 4n^2+76n+188$$
$$= (4n^2+76n+19^2)+(188-19^2)$$
$$=(2n+19)^2-173.$$

It so happens that 173 is a prime number. This will play a part in the subsequent analysis!

Now we transpose terms and factorize:

$$(2n+19)^2-173 = (2m)^2,$$
$$\therefore (2n+19-2m)\cdot(2n+19+2m) = 173.$$

As 173 is prime, it can be written as a product of two integers in the following four ways (where we have permitted the use of negative integers):

$$-1 \quad -173 = 1 \quad 173 = -173 \quad -1 = 173 \quad 1.$$

Hence the pair $(2n+19-2m,\ 2n+19+2m)$ must be one of the following:

$$(-1,-173),\ (1,173),\ (-173,-1),\ (173,1).$$

By addition we get $4n+38 = \pm 174$, i.e., $n = 39$ or $n = -53$.

So there are precisely two values of n for which $n^2 + 19n + 47$ *is a perfect square, namely:* $n = 39$ *and* $n = -53$.

Next, by subtraction we get $4m = \pm 172$, i.e., $m = \pm 43$.

Hence, $n^2+19n+47$ *takes precisely one square value,* namely: 43^2 or 1849.

(Query: Was there something noteworthy happening in India in 1849?)

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2. Problems for solution

Problem I-1-S.1

Let (a, b, c) be a PPT.

1. Show that of the numbers a and b, one is odd and the other is even.
2. Show that the even number in $\{a, b\}$ is a multiple of 4.

Problem I-1-S.2

Let (a, b, c) be a PPT. Show that abc is a multiple of 60.

Problem I-1-S.3

Show that any right-angled triangle with integer sides is similar to one in the Cartesian plane whose hypotenuse is on the x-axis and whose three vertices have integer coordinates. (Source: Problem of the Week column, Purdue University.)

Problem I-1-S.4

Let a, b and c be the sides of a right-angled triangle. Let θ be the smallest angle of this triangle. Show that if $1/a$, $1/b$ and $1/c$ too are the sides of a right-angled triangle, then

$$\sin\theta = \tfrac{1}{2}\left(\sqrt{5}-1\right)$$

(Source: B Math entrance examination of the Indian Statistical Institute.)

Problem I-1-S.5

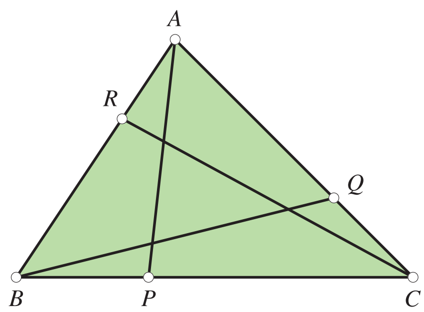
Find all Pythagorean triples (a, b, c) in which:
(i) one of a, b, c equals 2011; (ii) one of a, b, c equals 2012.

Problem I-1-S.6

Find all PPTs (a, b, c) in which a, b, c are in *geometric progression*; or show that no such PPT exists.

Problem I-1-S.7

In any triangle, show that the sum of the squares of the medians equals $\frac{3}{4}$ of the sum of the squares of the sides.



- $BP = \frac{1}{3}BC$
- $CQ = \frac{1}{3}CA$
- $AR = \frac{1}{3}AB$

Problem I-1-S.8

The figure shows a $\triangle ABC$ in which P, Q, R are points of trisection of the sides, with $BP = \frac{1}{3}BC$, $CQ = \frac{1}{3}CA$, $AR = \frac{1}{3}AB$. Show that the fraction

$$\frac{AP^2 + BQ^2 + CR^2}{BC^2 + CA^2 + AB^2}$$

has the same value for every triangle. What is the value of the constant?

The Closing Bracket ...

At a function held in December 2011 at the Institute of Mathematical Sciences in Chennai, Prime Minister Shri Manmohan Singh declared 2012 to be 'National Mathematics Year' and 22 December (birth date of the great mathematician Srinivasa Ramanujan) to be 'National Mathematics Day'. He inaugurated a series of year-long celebrations to mark the 125th year of Ramanujan's birth, and said in his speech, "*It is a matter of concern that for a country of our size, the number of competent mathematicians that we have is badly inadequate ... There is a general perception in our society that the pursuit of mathematics does not lead to attractive career possibilities. This perception must change. [It] may have been valid some years ago, but today there are many new career opportunities available [in] mathematics. ...*" He urged the mathematical community to find ways and means to address the shortage of top quality mathematicians and reach out to the public.

These remarks bring home the need to ponder the state of mathematics education in our country: why, for vast numbers of students, mathematics remains a subject of dread, a subject that causes one to 'switch off' at an early age. What are we doing to make this happen?

In contrast we have the extraordinary story of Ramanujan, who was completely in love with mathematics, to a degree that seems scarcely imaginable, and whose life may be described as a passionate celebration of mathematics.

It seems natural to ask what I — as a mathematics teacher — can do to address the situation in the country, and to ask, "Can I bring about a love for mathematics in my students? Can I help children explore this beautiful garden and show them that it is a world in which great enjoyment is possible, even if one is not highly talented at it? "

The answer surely is: Yes. And I do not think it is so very difficult to do. But two things are required at least — a love for one's subject, and a love of sharing with human beings. If these are there, then ways can be found and techniques developed that will bridge most barriers. If as a teacher I have a love of exploration, a love of inquiry, a love of playing with numbers, then surely I will be able to communicate it to children. It seems to me that before I ask for techniques of instruction, I must ask if I have that kind of feeling for the subject and for sharing it with children.

What are the factors which for so many children bring about a fear-filled and alienating relationship with mathematics? It is obvious that a huge contributory factor is a hostile learning environment, in which early contact with fear and comparison as instruments of learning serve to chip away at one's childhood. If there is one area where techniques need to be found, it is this: to find ways of assessment, of feedback and communication, which do away with these traditional and intrinsically violent instruments.

Mathematics education may be in a state of crisis, but this is true of education as a whole, and in a far more serious sense. The world today is in a very grave situation: divisive forces are tearing us apart, and our greed is destroying the earth. We seem to be blind to the fact that our way of life is not sustainable. In what way can we teachers help in bringing some sanity to the world around us, through our teaching and our contact with children? In what way can we convey the beauty of exploration and sharing so that it extends beyond the boundaries of the classroom and spills over into life? In what way can we convey a love for what we are doing, a love which is not bound to the classroom? Let us keep these questions in the foreground, so that they thoroughly permeate our work and everything we do.

— Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.