Preamble

Problems, The Life Blood of Mathematics

problem corner

Many mathematicians take great pleasure in problem solving, and 'Problem Corner' is where we share interesting problems of mathematics with one other: talk about experiences connected with memorable problems, show the interconnectedness of problems, and so on.

It has been said that "problems are the lifeblood of mathematics." This short, pithy sentence contains within it a great truth, and it needs to be understood.

What is a problem? It refers to a task or situation where you do not know what to do; you have no way already worked out to deal with the situation, no 'formula'; you have to discover the way afresh, by thinking on the spot.

In this sense, a problem is not the kind of exercise you meet at the end of a chapter. On completing a chapter on quadratic equations one may be assigned a list of ten or twenty quadratic equations to solve. But these are 'drill exercises' — they must not be called 'problems'. *A problem is essentially non-routine.* You have to throw yourself at it in order to solve it.

In the history of mathematics it has happened time and again that problems posed by mathematicians — to themselves or to others — would lie unsolved for a long time. Perhaps the most famous instance of this is that of Fermat's Last Theorem ('FLT'), whose origin lies in a remark casually inserted by Fermat in the margin of a mathematics book he was reading; the eventual solution to the problem came after a gap of three and a half centuries! (See the Review of FERMAT'S ENIGMA elsewhere in this issue for more about this story.) And each time this happens, in the struggle between mathematician and problem, the winner invariably is mathematics itself; for in the encounter are born fresh concepts and ideas, fresh ways of organizing and looking at old ideas, fresh notation. In the case of FLT, number theory developed enormously as a result of this encounter, and a whole new field was born, now called *Algebraic number theory*.

The problem corner is a very important component of this magazine. It comes in three parts: Fun Problems, Problems for the Middle School, and Problems for the Senior School. For each part, the solutions to problems posed will appear in the next issue.

To encourage the novice problem solver, we start each section with a few solved problems which convey an idea of the techniques used to understand and simplify problems, and the ways used to approach them.

We hope that *you* will tackle the problems and send in your solutions. We may choose your solution to be the 'official' solution! 'Visual proofs' are particularly welcome — proofs which use a minimum of words.

Another instance where this happened was in the struggle to solve polynomial equations. Quadratic equations (i.e., equations of degree 2, like $x^2 + 3x + 2 = 0$) were mastered a long time back, perhaps as early as the seventh century (though there was no concept of negative numbers back then); cubic equations (degree 3) were solved by several mathematicians independently over the twelfth to fifteenth centuries; and biquadratic equations (degree 4; also called 'quartic equations') were solved soon after. Naturally, attention then turned to the quintic equation (degree 5). Here researchers hit what seemed to be a wall; no matter what approach was tried they could not cross this barrier. Eventually the matter was resolved but not in the way that everyone expected; it was shown by a young Frenchman named Evaristé Galois that in a certain sense the problem was not solvable at all! In the process was born one of the gems of higher algebra, now called *Galois theory*.

It is not difficult to see why a struggle of this kind will bring up something new. Take any real problem, tackle it, struggle with it and do not give up, no matter what happens; and examine at the end how much you have learnt in the process. What you find may surprise you … It is remarkable that this happens even in those instances where you do not get the solution. But for that, it is essential not to 'give up' ….

In the problem section of CRUX MATHEMATICORUM, which is one of the best known problem journals, there occur these memorable words: *No problem is ever closed*, and the editor adds that solutions sent in late will still be considered for publication, provided they yield some new insight or some new understanding of the problem. We are happy to adopt a similar motto for our three problem sections.

Submissions to the Problem Corner

The Problem Corner invites readers to send in proposals for problems and solutions to problems posed. Here are some guidelines for the submission of such entries.

(1) Send your problem proposals and solutions by e-mail, typeset as a Word file (with mathematical text typeset using the equation editor) or as a LaTeX file, with each problem or solution started on a fresh page. Please use the following ID: **AtRiA.editor@apu.edu.in**

(2) Please include your name and contact details in full (mobile number, e-mail ID and postal address) on the solution sheet/problem sheet.

(3) If your problem proposal is based on a problem published elsewhere, then please indicate the source (be it a book, journal or website; in the last case please give the complete URL of the website).

Notation used in the problem sets

For convenience we list some notation and terms which occur in many of the problems.

Coprime
Two integers which share no common factor exceeding 1 are said to be coprime.

Example: 9 and 10 are coprime, but not 9 and 12. Pairs of consecutive integers are always coprime.

Pythagorean triple ('PT' for short)
A triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$.

Primitive Pythagorean triple ('PPT' for short)
A triple (a, b, c) of coprime, positive integers such that $a^2 + b^2 = c^2$. Thus a PPT is a PT with an additional condition — that of coprimeness.

Example : The triples (3, 4, 5) and (5, 12, 13). The set of PPTs is a subset of the set of PTs.

Arithmetic Progression (AP for short)
Numbers $a_1, a_2, a_3, a_4, \cdots$ are said to be in AP if $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots$ The number $d = a_2 - a_1$ is called the **common difference** of the AP.

Example : The numbers 3, 5, 7, 9 form a four term AP with common difference 2, and 10, 13, 16 form a three term AP with common difference 3.