feature

Primitive Pythagorean Triples

How Many Primitive Pythagorean Triples in Arithmetic Progression?

A simple investigation and a convincing proof based on a novel connection between two topics — the Pythagorean Theorem and Sequences — taught in middle and high school.

COMAC

Everyone knows that (3, 4, 5) is a Pythagorean triple ('PT'); for, the numbers satisfy the Pythagorean relation $3^2+4^2=5^2$. Indeed, it is a Primitive Pythagorean triple ('PPT') since the integers in the triple are coprime. (See the Problem Corner for definitions of unfamiliar terms.)

But this triple has a further property: *its entries are in arithmetic progression* for, 3, 4, 5 forms a three-term AP with common difference 1. Naturally, our curiosity is alerted at this point, and we ask:

Is there any other PPT whose entries are in AP?

Surprisingly, no other such triple exists. Let us show why.

Suppose there does exist a PPT with entries in AP. Let the common difference of the AP be d, and let the PPT be $(a-d,\ a,\ a+d)$; here a and d are positive integers with no common factor exceeding 1. (If a and d had a common factor exceeding 1 then this factor would divide all three of the numbers $a-d,\ a,\ a+d$, and the triple would no longer be primitive.)

By definition the numbers $a-d,\ a,\ a+d$ satisfy the equation

$$(a-d)^2+a^2=(a+d)^2.$$

Expanding all the terms we get: $2a^2-2ad+d^2 = a^2+2ad+d^2$, and hence:

$$a^2 = 4ad.$$

Since $a>0$ we may safely divide by a on both sides; we get:

$$a = 4d.$$

So d is a divisor of a. Since a and d are supposed to have no common divisor other than 1, it must be that $d=1$. Hence $a=4$, and the triple we seek is (3, 4, 5). Therefore:

$(3,4,5)$ *is the only PPT whose numbers are in AP.*

amicable numbers

Take the number 220; its proper divisors (i.e., its divisors excluding itself) are:

1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110,

and the sum of these numbers is 284 (please check!). Now we do the same for the number 284. Its proper divisors are: 1, 2, 4, 71, 142, 284,

and the sum of these numbers is 220. How curious! – the sum of the proper divisors of 220 is 284, and the sum of the proper divisors of 284 is 220.

Pairs of positive integers with such a property are called amicable numbers. The Greeks knew of this pair of numbers. They named them 'amicable', saying to themselves that true friendship between people should be like the relationship between a pair of amicable numbers!

Such number pairs are not easy to find, even if one uses a computer. Here is another such pair, found in the ninth century by the Arab mathematician ibn Qurra: {17296, 18416}.

More such pairs of numbers are known now. It has been noticed that in all these pairs, the numbers are either both odd or both even. Whether there exists any pair of amicable numbers with opposite parity is not known.

Question. We pose the following to you: How would you check that 17296 and 18416 form an amicable pair? What is the easiest way to carry out such a check?

We'll reveal the answer in a future issue