feature

Shape, Size, Number and Cost

A Sweet Seller's Trick

Analyzing Business through Math

How much math can there be in two bowls of gulab jamun? Prithwijit De models, estimates, calculates and presents a convincing argument on which of the two products earns the sweetseller a greater profit.

Prithwijit De

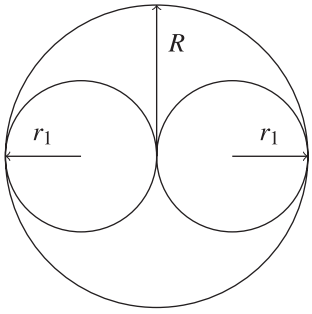
Gulab jamuns are a popular sweet in India, often sold singly or two to a bowl. However, I was recently intrigued when the sweet shop across the road from my institute's canteen introduced a bowl of three gulab jamuns at a competitive price. The sweet lovers in my office immediately changed loyalties but the price conscious stayed with the canteen.

What motivated this competitive strategy from the shop across the road? Being a mathematician, I naturally had to solve the problem and in typical fashion, I called my canteen walla 'Mr. X' and the sweet shop owner 'Mr. Y'. Here is my mathematised version of the situation.

Mr. X sells two pieces of *gulab jamun* at ₹p_1 per piece, in a cylindrical cup of cross-sectional radius R. The pieces are spherical in shape and they touch each other externally and the cup internally. Mr. Y, a competitor of Mr. X, sells three pieces of *gulab jamun* in a cup of the same shape and size used by Mr. X, and he charges a price of ₹p_2 per piece. Snapshots of their offerings are displayed in Figure 1, and top-down views are shown in

Fig. 1

Fig. 2



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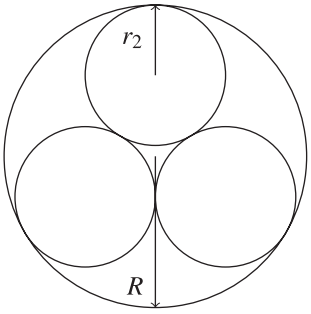

II

Figure 2. Certainly p_2 has to be less than p_1 since price is directly proportional to size if the material is kept the same. A person buying from Mr. X would be paying $2p_1$ and a person buying from Mr. Y would be paying $3p_2$.

Listen to some of the conversations I overheard while working on a particularly difficult problem in differential geometry: "Mr. X is charging ₹6 for a gulab jamun! So one bowl with two gulab jamuns from his shop costs only ₹12!" "A bowl at Mr. Y's shop costs ₹15 but I get 3 pieces. So it's only ₹5 per gulab jamun and the bowls are of the same size." "Yes …but the 3 gulab jamuns in the same bowl are smaller! I get 2 bigger gulab jamuns at a cheaper price! " As you can see we have serious and weighty discussions in my office.

Thinking deeply about this, I finally reduced the problem to two main questions.

1. Which of the two cups contains a greater amount of sweet?
2. Which sweetseller makes a greater profit if the cost of producing unit volume of the sweet is the same?

The sweets are sold in a cylindrical cup of cross-sectional radius R. The pieces are spherical in shape and they touch each other externally and the cup internally.

The radius r_1 of each piece in Figure 2 (I) is $\frac{1}{2}R$. The total volume V_1 of sweet in the cup is therefore:

$$2\left(\frac{4\pi}{3}\left(\frac{R}{2}\right)^3\right) = \frac{\pi R^3}{3}.$$

To find the volume of a piece in Figure 2 (II) we must find the radius of each sphere.

In the two dimensional representation of the configuration (Figure 2 (II)) the spheres become circles and the cylindrical cross-section turns into a circle circumscribing the three circles. The task of finding the radius of the sphere thus reduces to finding the common radii of the inner circles. The triangle formed by joining the centres of the inner circles (Figure 3) is an equilateral triangle whose centroid is the centre of the large circle. If r_2 is the radius of an inner circle then the length of the side of the triangle is $2r_2$ and

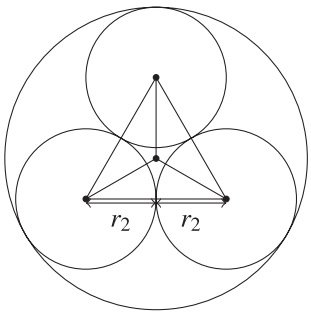

Fig. 3

$$R = r_2 + \left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right)(2r_2) = r_2 + \frac{2r_2}{\sqrt{3}} = \left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)r_2.$$

Thus the volume of a piece is

$$\frac{4\pi}{3}\left(\frac{R\sqrt{3}}{2+\sqrt{3}}\right)^3$$

and the total volume V_2 of sweet offered is

$$4\pi\left(\frac{R\sqrt{3}}{2+\sqrt{3}}\right)^3$$

Now one asks, who is giving more sweet and by how much? Note that

$$(1) \qquad \frac{V_1}{V_2} = \frac{(2+\sqrt{3})^3}{36\sqrt{3}} \approx 0.83, \quad \frac{V_2}{V_1} \approx 1.2.$$

So Mr. Y is giving about 20% more sweet than Mr. X. From a buyer's perspective this is a good deal. S/he may be paying more per cup but the per piece price is still less as $p_2 < p_1$.

If Mr. X charges ₹6, then a bowl costs ₹12. If Mr. Y charges ₹5, then a bowl costs ₹15 but it will have 20.48% more sweet than a bowl from Mr. X and the price per piece is still less at Mr. Y's.

But from a seller's perspective is it really worth it? Assuming that the cost of producing unit volume of the sweet is the same, c (say), in both cases, is it possible for Mr. Y to price a piece in such a way so as to ensure greater profit than Mr. X?

The selling price per unit volume is the cost of one bowl divided by the volume of sweet given, so the profit at each shop is

$$\frac{\text{cost of one bowl}}{\text{volume of sweet given}} - \text{cost per unit volume.}$$

To find out which shop makes greater profit, we therefore study the following inequality:

$$(2) \qquad \frac{3p_2}{V_2} - c > \frac{2p_1}{V_1} - c.$$

This is equivalent to

$$\frac{p_2}{p_1} > \frac{2}{3}\left(\frac{V_2}{V_1}\right)$$

By virtue of (1) this is equivalent to $p_2 > 0.8p_1$.

Thus, if Mr. Y chooses p_2 such that $0.8p_1 < p_2 < p_1$, then he makes greater profit than Mr. X despite reducing the size and price of the sweet.

For instance, if Mr. X charges ₹6 per piece then Mr. Y can set the price anywhere between ₹4.80 and ₹6 per piece in order to beat his rival in the money-making game (see Table 1). The buyer, in all probability, will be happy to pay less per piece and get three instead of two, as *the more the merrier* is likely to be his/her motto.

p_1	Cost/bowl	p_2	Cost/bowl
6	12	4.80	14.40
		5.00	15.00
		5.50	16.50

Remember that the volume of the bowl is the same in both cases; so the buyer will probably go to Mr. Y in order to get 3 gulab jamuns at a slightly higher price. Mr. Y gets more customers and a greater profit on each bowl!

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