ClassRoom

Characterisation of a Right Triangle

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Let us start by making a few remarks on the notion of ***characterisation*** in mathematics, a theme that is central to the subject. The notion has relevance in other settings as well, but we will restrict ourselves to its meaning in mathematics. Given any set S which has been specified in some well-defined manner, we may want a test by which we can decide membership of this set. That is, we want to fill in the blanks in the following sentence in some appropriate and meaningful way:

$$\text{Entity } x \text{ belongs to } S \iff \_\_\_\_ \ \_\_\_\_ \ \_\_\_\_.$$

To be of any interest, the test must not be a mere restatement of the defining property of the set. If this requirement is met, we call this a *non-trivial characterisation* of the set. Some of the most interesting and nicest results of mathematics are non-trivial characterisations of one kind or another.

Here are two simple examples which illustrate the theme.

Right triangles: A non-trivial characterisation of the set of right triangles is Pythagoras's theorem: *A triangle is right-angled if and only if the square of one of the sides is equal to the sum of the squares of the other two sides*. The beauty of this result is its compactness and its surprise value: there is no obvious reason whatever why the result should be true. (But the surprise is spoiled to some extent by the great fame of this result!)

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Prime numbers: Do there exist non-trivial characterisations of the set of prime numbers? This is an enormously interesting and deep question which has occupied the attention of mathematicians for over two millennia. Ancient Chinese mathematicians came close when they stated that a positive integer $n > 1$ is prime if and only if $2^n - 2$ is divisible by n. This turns out to be *almost* correct! (The correct statement is: If a positive integer $n > 1$ is prime, then $2^n - 2$ is divisible by n. The converse statement is false.) The answer to our question is: Yes, there do exist such characterisations, but to understand them requires a substantial buildup of concepts and we do not dwell on them for now.

Against this background, we offer the following surprising characterisation of right-angled triangles. It is adapted from Problem 1 of the European Girls' Mathematical Olympiad (EGMO), 2013 [1].

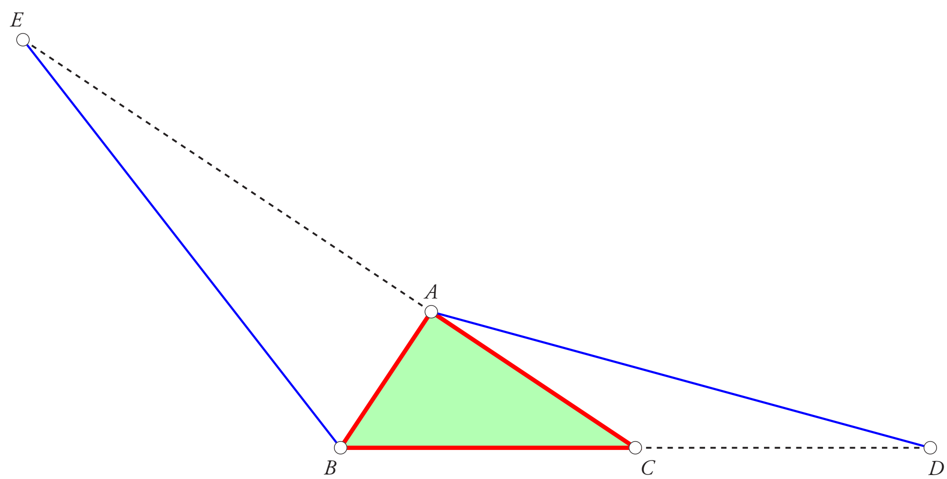

Figure 1.

Theorem 1 (EGMO-2013-1)**.** *Given any $\triangle ABC$, extend BC to D and CA to E so that* $\overrightarrow{BD} = 2 \cdot \overrightarrow{BC}$ *and* $\overrightarrow{CE} = 3 \cdot \overrightarrow{CA}$. *(See Figure 1.) Join AD and BE. Then we have the following result:*
$\measuredangle BAC = 90^\circ \iff AD = BE.$

Remark. Before plunging into the proof, we note that there appears to be a basic lack of symmetry about the result; given that it is $\measuredangle A$ which is ultimately going to be the right angle, the condition surely should not discriminate between vertices B and C. (For example, in the same setup, Pythagoras's theorem asserts that $a^2 = b^2 + c^2$; note that the condition is symmetric in b and c, i.e., it does not discriminate between the vertices B and C.) But it appears to do just that. The resolution of this is the following. We find that if in Theorem 1, we completely swap the roles of B and C, we get (as anticipated) another correct statement. That is, the following is true.

Theorem 2 (EGMO-2013-1)**.** *Given any $\triangle ABC$, extend CB to F and BA to G so that* $\overrightarrow{CF} = 2 \cdot \overrightarrow{CB}$ *and* $\overrightarrow{BG} = 3 \cdot \overrightarrow{BA}$. *(See Figure 2.) Join AF and CG. Then we have the following result:*
$\measuredangle BAC = 90^\circ \iff AF = CG.$

Proof. Given the remarks made earlier, it suffices to prove either Theorem 1 or Theorem 2. We choose to prove Theorem 2 and we do so using vector algebra.

Let A be the origin, and let $\vec{u} = \overrightarrow{AB}$, $\vec{v} = \overrightarrow{AC}$. Then we have $\overrightarrow{BF} = \overrightarrow{CB} = \vec{u} - \vec{v}$ and $\overrightarrow{GA} = 2\,\overrightarrow{AB} = 2\,\vec{u}$, so:

$$\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF} = \vec{u} + (\vec{u} - \vec{v}) = 2\,\vec{u} - \vec{v},$$

$$\overrightarrow{GC} = \overrightarrow{GA} + \overrightarrow{AC} = 2\,\vec{u} + \vec{v}.$$

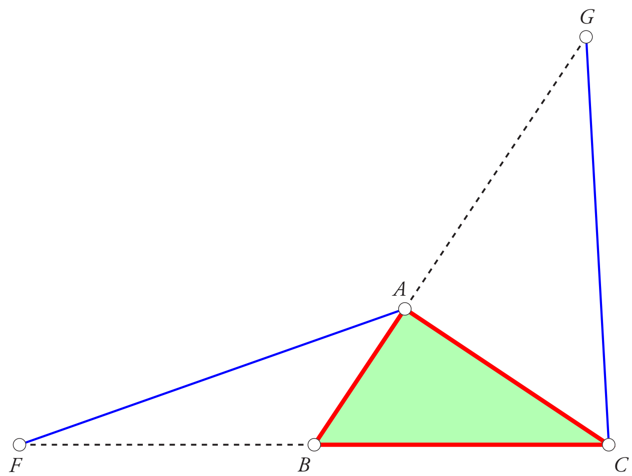

Figure 2.

Hence

$$AF^2 = 4\,\vec{u}\cdot\vec{u} - 4\,\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v},$$
$$CG^2 = 4\,\vec{u}\cdot\vec{u} + 4\,\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}.$$

It follows that

$$AF = CG \iff \vec{u}\cdot\vec{v} = 0 \iff \vec{u}\perp\vec{v},$$

i.e.,

$$AF = CG \iff \measuredangle BAC = 90^\circ.$$

This proves the desired result. However, the proof yields a bit more. For, we have:

$$AF^2 - CG^2 = -8\,\vec{u}\cdot\vec{v}.$$

Since $\vec{u}\cdot\vec{v}$ is positive when $\measuredangle A$ is acute and negative when $\measuredangle A$ is obtuse, we are able to make a more complete statement:

$$AF < CG \iff \measuredangle A < 90^\circ,$$
$$AF = CG \iff \measuredangle A = 90^\circ,$$
$$AF > CG \iff \measuredangle A > 90^\circ$$

A geometric proof. You may notice that there is something odd about the above proof. Though it is simple in terms of the algebra involved, it does not tell us *why* the result is true; it does not yield any understanding of the result. At the end of the proof, one submits to the force of its logic but is left with no understanding of "what is going on." (It is, surely, reasonable to expect that of a proof.) So it seems worthwhile to be on the lookout for a proof that yields some geometric insight into the configuration.

We respond to the challenge by presenting the following proof. But we cast it in a different way: we present it as a theorem about a *parallelogram*. In the wording below, we have tried to label the vertices in such a way that the analysis conducted earlier is compatible with the new diagram. (See Figure 3.)

Theorem 3. *Let EBDG be a parallelogram. Let C be the midpoint of BD, and let EC meet the diagonal BG at A. Join AD. Then we have the following equivalence:*

$$\measuredangle BAC = 90^\circ \iff DA = DG.$$

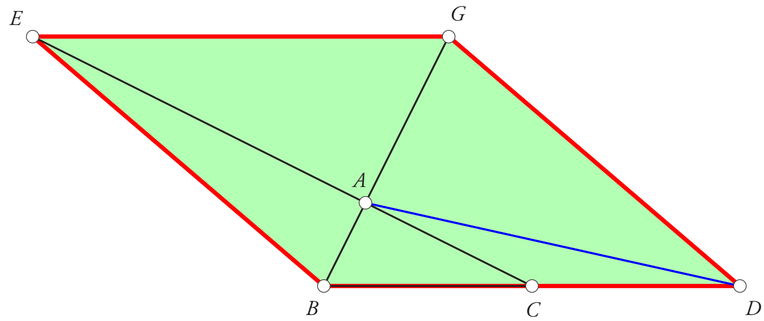

Figure 3. $\measuredangle BAC = 90^\circ \iff DA = DG$

Proof. The proof can be accomplished using vector algebra, along the same lines as above. We leave the details to the reader. For now, we present a geometric proof. The proof is anchored on the rotational symmetry of a parallelogram, namely, the half-turn symmetry about the common midpoint of its two diagonals.(See Figure 4.)

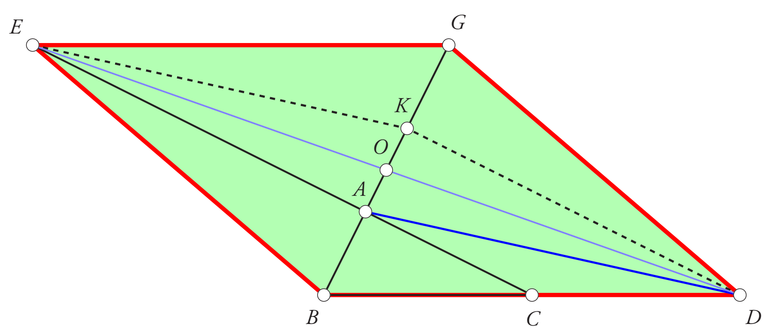

Figure 4. Proving that $\measuredangle BAC = 90^\circ \iff DA = DG$

We subject the parallelogram $BDGE$ to a half-turn about its centre O and let K be the image of A under this transformation. Note that D and E swap places under the same transformation (since O is the midpoint of DE). Join EK and KD. Observe the following: (i) O is the midpoint of AK; (ii) $EADK$ is a parallelogram, since AK and DE bisect one another; (iii) $BA = AK = KG$ (i.e., A and K are points of trisection of diagonal BG).

The proof now rolls on its own! Suppose $\measuredangle BAC = 90^\circ$. Then $EA \perp BK$, from which it follows by rotational symmetry of the parallelogram that $DK \perp AG$. It follows that $\triangle DKA \cong \triangle DKG$, and hence that $DA = DG$.

Conversely, if $DA = DG$, then $\triangle DAG$ is isosceles. We also have $KA = KG$, hence $\triangle DKA \cong \triangle DKG$ ('SSS' congruence), from which it follows that $DK \perp AG$ and hence that $EA \perp BA$, i.e., $\measuredangle BAC = 90^\circ$.

This proof seems much more satisfying!

References

1. European Girls' Mathematical Olympiad 2013, https://www.egmo.org/egmos/egmo2/

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