in the classroom

Visual Connect in Teaching

Paper Folding And The Theorem of Pythagoras

Can unfolding a paper boat reveal a proof of Pythagoras' theorem? Does making a square within a square be anything more than an exercise in geometry at best? Art and math come together in delightful mathematical exercises described in this article.

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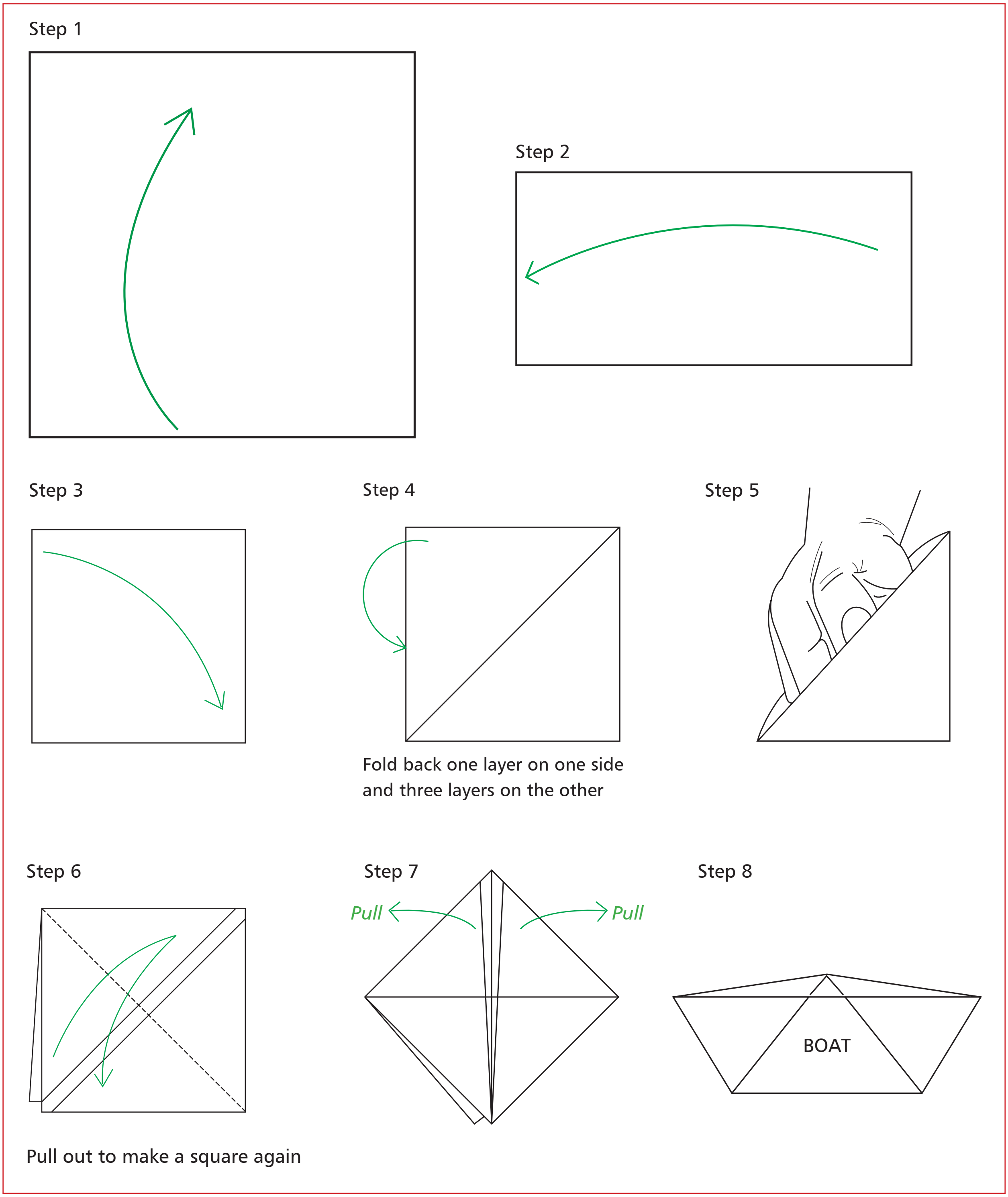
Pythagoras' theorem is one of the most popular theorems in geometry. Reams of paper have been used to write different proofs of this theorem but in this article we cut and fold paper to demonstrate two different proofs.

Make a boat and prove Pythagoras' theorem

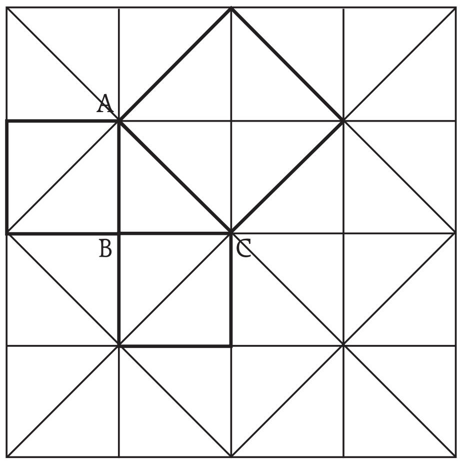
Remember how children float paper boats in running water after heavy rain? There are many types of boats that can be made by folding a single paper sheet. Here, we make the simplest and most common type of paper boat using a square sheet of paper. In case you have forgotten how to fold a boat here are the steps:-



After step 8 you will have a boat. What is its shape? If you look closely you find creases which show many right angled triangles.

Now unfold the boat. Remember we started with a plain square paper. Now look at the creases appearing in the unfolded boat. You will see a pattern which is known mathematically as a *tessellation* ("making tiles"). This particular tessellation consists of squares with creases along the diagonals which divide each square into two right angled triangles. The way we folded the paper ensures that all the squares (and therefore the triangles too) are identical in all respects.

They look like this:



Choose any right angled triangle *ABC*. Here angle *B* is a right angle. Shade the squares on sides *AC*, *BC* and *AC*. Look closely at these squares. All have creases along the diagonals and are divided into right-angled triangles. How do we measure their areas? *Area need not only be measured in terms of unit squares. We can also measure the area by counting the number of identical right angled triangles contained in them*.

The square upon *AB* has 2 right-angled triangles; so AB^2 = 2 right-angled triangles.

The square upon *BC* has 2 right-angled triangles; so BC^2 = 2 right-angled triangles.

The square upon *AC* has 4 right angled triangles; so AC^2 = 4 right-angled triangles.

Hence: $AC^2 = AB^2 + BC^2$ This is the theorem of Pythagoras applied to triangle *ABC*.

Make a square within a square and prove Pythagoras' Theorem

Take a square sheet of paper. Fold along a diagonal and make a sharp crease (Fig. 1).

Fold the bottom right corner towards the diagonal, so that the edge of the sheet lies parallel to the diagonal. Make a crease. You will have folded a right angled triangle (Fig. 2).

Fig 1

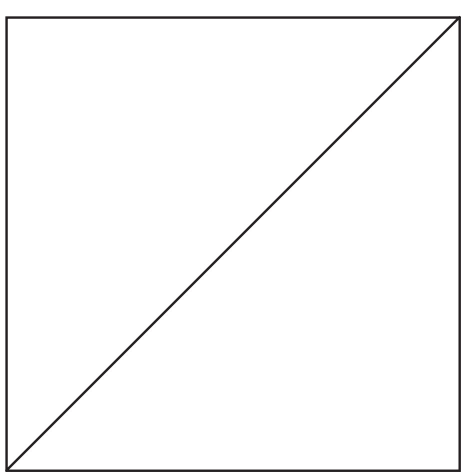
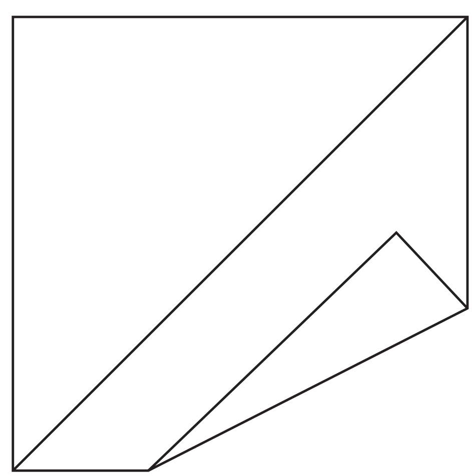


Fig 2



Now fold the next side of the square to the side of the right angle already folded (Fig. 3)and make a crease. Repeat the same with the remaining two corners.(fig. 4 and fig. 5)

Fig 3

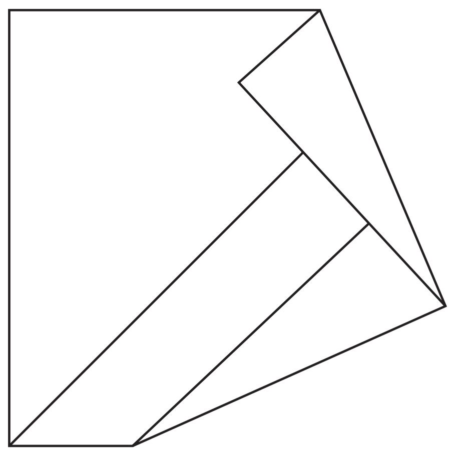
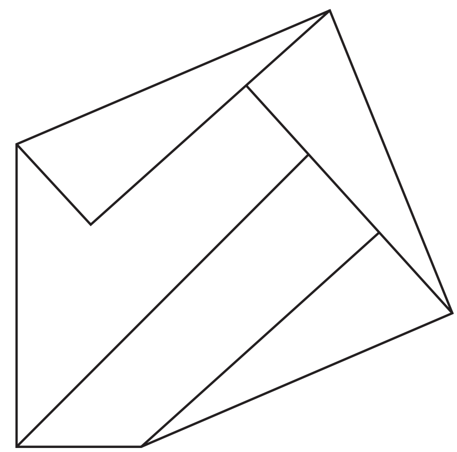


Fig 4



Now you have a square with a square hole in the middle. (Fig. 5). Crease all the sides and unfold(Fig.6). Let $AP = a$, $AQ = b$ and $PQ = c$. In triangle APQ, angle PAQ is a right angle. $PQRS$ is a square with side $PQ = c$. Hence area of $PQRS = c^2$.

Fig 5

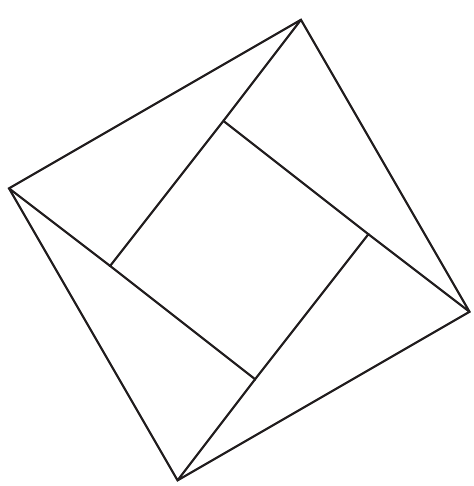
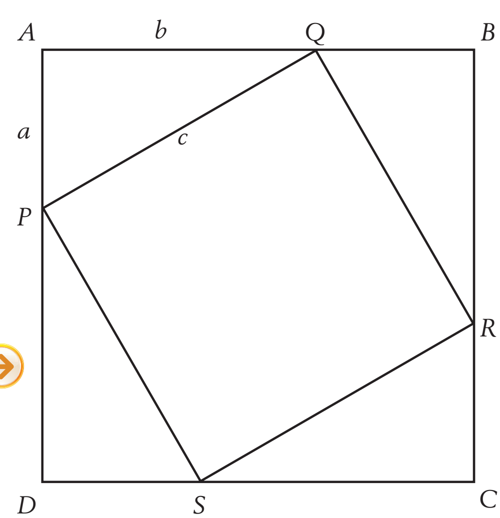
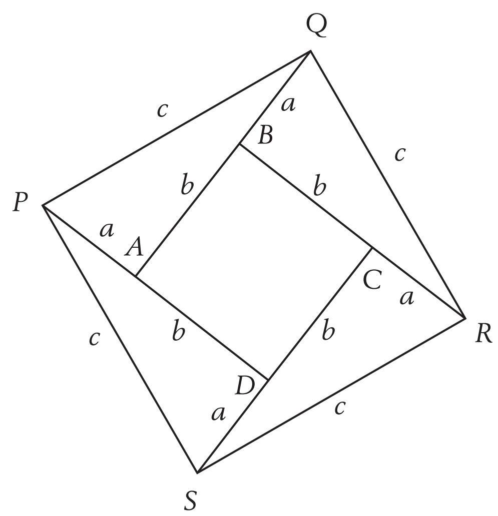


Fig 6





Fold back triangles *PAQ, BQR, RCS, SDP* inside, as before.

Now in the square *PQRS* standing on *PQ* we have identical triangles *PAQ, BQR, RCS, SDP,* and a small square *ABCD*.

Square *PQRS* = Triangle *PAQ* + Triangle *QBR* + Triangle *RCS* + Triangle *SDP* + Square *ABCD*

$= \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \text{square } ABCD$

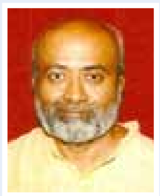
$= 4 \quad \frac{1}{2}ab + AB^2$

$= 2ab + (b - a)^2.$

Hence $c^2 = 2ab + b^2 + a^2 - 2ab$

and so $c^2 = a^2 + b^2$.

Hence: $PQ^2 = AP^2 + AQ^2$. This is the theorem of Pythagoras applied to triangle *APQ*.



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