Classification of Quadrilaterals

The Four-Gon Family Tree

in the classroom

A Diagonal Connect

Classification is traditionally defined as the precinct of biologists. But classification has great pedagogical implications — based as it is on the properties of the objects being classified. A look at a familiar class of polygons — the quadrilaterals — and how they can be reorganized in a different way.

A Ramachandran

Quadrilaterals have traditionally been classified on the basis of their sides (being equal, perpendicular, parallel, ...) or angles (being equal, supplementary, ...). Here we present a classification based on certain properties of their diagonals. In this approach, certain connections among the various classes become obvious, and some types of quadrilaterals stand out in a new light.

Three parameters have been identified as determining various classes of quadrilaterals:

1. Equality or non-equality of the diagonals
2. Perpendicularity or non-perpendicularity of the diagonals
3. Manner of intersection of the diagonals. Here four situations are possible:
    a. The diagonals bisect each other.
    b. Only one diagonal is bisected by the other.
    c. Neither diagonal is bisected by the other one, but both are divided in the same ratio.
    d. Neither diagonal is bisected by the other one, and they divide each other in different ratios.

	Diagonals equal		Diagonals unequal	
	Perpendicular	Not-perpendicular	Perpendicular	Not-perpendicular
Both diagonals bisected	Square	General rectangle	General rhombus	General parallelo-gram
Diagonals divided in same ratio (not 1:1)	Isosceles trapezium with perp diagonals	Isosceles trapezium	General trapezium with perp diagonals	General trapezium
Only one diagonal bisected	Kite with equal di-agonals	Slant kite with equal diagonals	Kite	Slant kite
Diagonals divided in different ratios, neither bisected	General quadrilateral with equal and perp diagonals	General quadri-lateral with equal diagonals	General quadrilateral with perp diagonals	General quadrilat-eral

Table 1. Quadrilateral classes based on properties of the diagonals

	Diagonals equal		Diagonals unequal	
	Perpendicular	Not-perpendicular	Perpendicular	Not-perpendicular
Both diagonals bisected	Cyclic (4)	Cyclic (2)	Non-cyclic (2)	Non-cyclic (0)
Diagonals divided in same ratio (not 1:1)	Cyclic (1)	Cyclic (1)	Non-cyclic (0)	Non-cyclic (0)
Only one diagonal bisected	Non-cyclic (1)	Non-cyclic (0)	Either (1)	Either (0)
Diagonals divided in different ratios, neither bisected	Non-cyclic (0)	Non-cyclic (0)	Either (1)	Either (0)

Table 2. Cyclic/non-cyclic nature of quadrilateral & the number of reflection symmetry axes (shown in parentheses)

These parameters allow us to identify 16 classes of quadrilaterals as listed in Table 1. The meaning of the phrase ‘slant kite’ though not in common parlance should be clear.

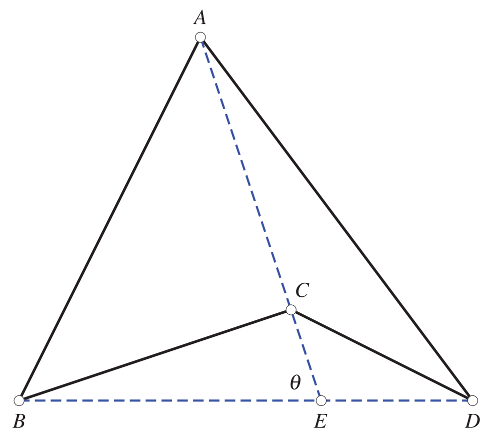
Certain relations among these classes are obvious. Members of columns 2 and 3 are obtained from the corresponding members of column 4 by imposing equality and perpendicularity of diagonals, respectively, while members of column 1 are obtained by imposing both of these conditions.

Sequentially joining the midpoints of the sides of any member of column 1 yields a square, of column 2 yields a general rhombus, column 3 a general rectangle, and column 4 a general parallelogram.

Table 2 gives the cyclic/non-cyclic nature and number of reflection symmetry axes of each type. An interesting pattern is seen in both cases.

Quadrilaterals with equal diagonals divided in the same ratio (including the ratio 1:1) are necessarily cyclic, as the products of the segments formed by mutual intersection would be equal. Quadrilaterals with unequal diagonals divided in the same ratio and those with equal diagonals divided in unequal ratios are necessarily non-cyclic, as the segment products would be unequal. Quadrilaterals with unequal diagonals divided in different ratios could be of either type.

The quadrilateral with the maximum symmetry lies at top left, while the one with least symmetry lies at bottom right. A gradation in symmetry properties is seen between these extremes.



- $[ABCD] = [\triangle ABC] + [\triangle ADC]$
- $[\triangle ABC] = [\triangle ABE] - [\triangle CBE]$
- $[\triangle ADC] = [\triangle ADE] - [\triangle CDE]$
- $[\triangle ABE] = \frac{1}{2} AE \cdot BE \cdot \sin\theta$
- $[\triangle CBE] = \frac{1}{2} CE \cdot BE \cdot \sin\theta$
- $[\triangle ADE] = \frac{1}{2} AE \cdot DE \cdot \sin\theta$
- $[\triangle CDE] = \frac{1}{2} CE \cdot DE \cdot \sin\theta$

Fig. 1 Square brackets denote area: [△ABC] denotes area of △ABC, etc

The present scheme also suggests a common formula for the areas of these figures. The area of any member of column 4 can be obtained from the formula $A = \frac{1}{2} d_1 d_2 \sin\Theta$ where d_1, d_2 are the diagonal lengths and Θ the angle between the diagonals. The formula simplifies to $A = \frac{1}{2} d_1 d_2$ for column 3, $A = \frac{1}{2} d^2 \sin\Theta$ for column 2, and $A = \frac{1}{2} d^2$ for column 1.

The general area formula $A = \frac{1}{2} d_1 d_2 \sin\Theta$ is applicable in the case of certain other special classes of quadrilaterals too.

Non-convex or re-entrant quadrilaterals are those for which one of the (non-intersecting) diagonals lies outside the figure. the applicability of the formula to these is demonstrated in Figure 1. The computation shows that

$$
\begin{aligned}
[ABCD] &= [\triangle ABC] + [\triangle ADC] = [\triangle ABE] - [\triangle CBE] + [\triangle ADE] - [\triangle CDE] \\
&= \frac{1}{2} \sin\Theta \, (AE \cdot BE - CE \cdot BE + AE \cdot DE - CE \cdot DE) \\
&= \frac{1}{2} \sin\Theta \, (AC \cdot BE + AC \cdot DE) = \frac{1}{2} \sin\Theta \, (AC \cdot BD)
\end{aligned}
$$

Reflex quadrilaterals with self-intersecting perimeters such as the ones shown in Figure 2 can be considered to have both the (non-intersecting) diagonals lying outside the figure. The applicability of the area formula to such figures is demonstrated in Figure 3. Here the area of the figure is the *difference* of the areas of the triangles seen, since in traversing the circuit *ABCDA* we go around the triangles in opposite senses. Hence in Figure 3,

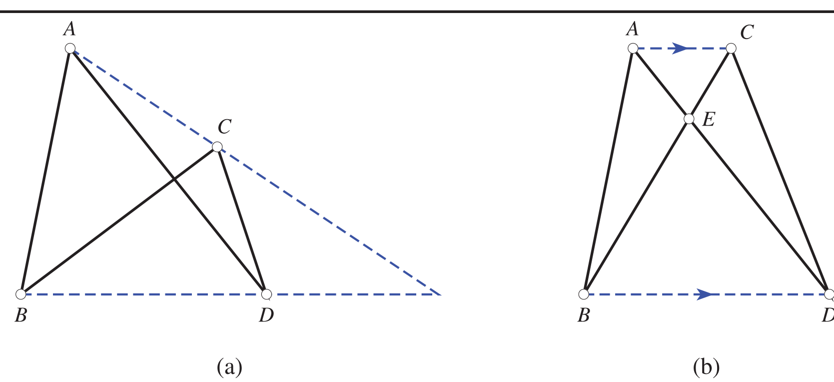


Fig. 2 Reflex quadrilateral: AC and BD are diagonals lying outside the reflex quadrilateral ABCDA, whose perimeter is self-intersecting

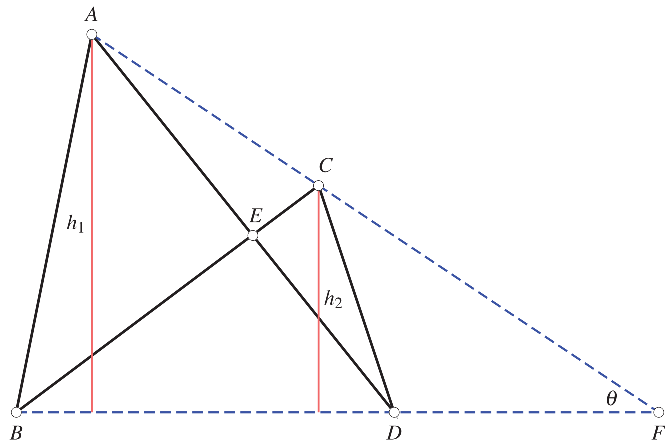


Fig. 3

$$
\begin{aligned}
[ABCDA] &= [\triangle ABE] - [\triangle CDE] = [\triangle ABD] - [\triangle CBD] \\
&= \tfrac{1}{2} BD \cdot h_1 - \tfrac{1}{2} BD \cdot h_2 = \tfrac{1}{2} BD \cdot (h_1 - h_2) \\
&= \tfrac{1}{2} BD \cdot (AF \sin \Theta - CF \sin \Theta) = \tfrac{1}{2} BD \sin \Theta \cdot (AF - CF) \\
&= \tfrac{1}{2} BD \cdot AC \sin \Theta.
\end{aligned}
$$

In the particular case when $AC \,||\, BD$, the formula implies that the area is 0. This makes sense, because, referring to Figure 2(b), the area of quadrilateral *ABCDA* is:

$$
\begin{aligned}
[ABCDA] &= [\triangle ABE] - [\triangle CDE] \\
&= ([\triangle ABE] + [\triangle EBD]) - ([\triangle CDE] - [\triangle EBD]) \\
&= [\triangle ABD] - [\triangle CBD] = 0.
\end{aligned}
$$

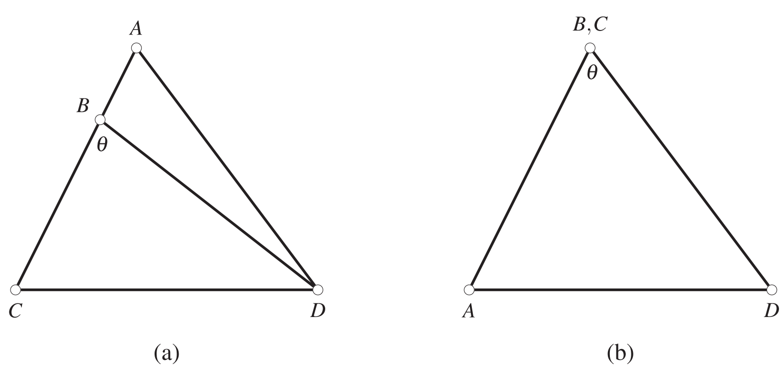


Fig. 4 Two examples of a degenerate quadrilateral ABCD; in (a) vertex B lies on diagonal AC, and in (b) vertices B and C coincide

If one or both the diagonals just touches the other one, the figure degenerates to a triangle, and the formula reverts to the area formula for a triangle, $A = \frac{1}{2} ab \sin C$ (see Figure 4). The formula $\frac{1}{2} AC \cdot BD \sin \Theta$ continues to remain valid.

To conclude, the above scheme integrates the various quadrilateral types normally encountered in classroom situations and highlights a few others. It suggests new definitions such as: *A trapezium is a figure whose diago-*

nals intersect each other in the same ratio. Non-convex and reflex quadrilaterals are also brought in as variations on the diagonal theme. The formula $A = \frac{1}{2} d_1 d_2 \sin \Theta$ is also shown to be the most generally applicable formula for quadrilaterals.

A Ramachandran has had a long standing interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He has been teaching science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at ramachandran@rishivalley.org.

a new approach to solving equations?

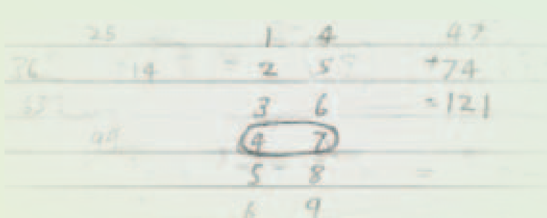
by Shibnath Chakravorty

"Algebra reverses the relative importance of the factors in ordinary language" – Alfred North Whitehead (1861-1947)

Having taught for many years one would think that one cannot be surprised by student responses anymore; one has seen it all! But I discovered that this is not so. I give below an instance of a student's out-of-the-box thinking. During a class the question posed was: *The sum of a two digit number and the number obtained by reversing the order of the digits is* 121. *Find the number, if the digits differ by* 3.

After some explaining, I wrote this pair of equations on the board: $(10x + y) + (10y + x) = 121$, $x - y = 3$. Further processing gave: $y = 7$, $x = 4$. Therefore the answer to the question is 47.

Sheehan, a student of the class who likes to think independently, worked out this problem differently. I reproduce below a copy of his working.



Courtesy: Sheehan Sista, Grade 9, Mallya Aditi International School

He had worked out all the possibilities for this occurrence. They are not many, of course. When I went around looking into their work, I was taken aback by this approach. Later I tried to impress upon him the necessity of solving problems the 'normal' way. The 'board' wanted things done in a particular way. But I must say that it was I who was impressed.

The confidence of these students is high. They are willing to tackle most problems without knowing the 'correct' way to a solution. A positive trait surely. But on the negative side, these students often block out new learning. They sometimes refuse to learn a method as they perhaps feel secure about their own capacity to tackle problems.