Formulas for Special Segments in a Triangle

In many programs of study, the material on the formulas relating the sides and special segments in a triangle does not appear as part of the study of mathematics in high school. On the other hand, in many programs of study the background required to understand this subject is studied already by the ages of 13-15. This situation gives us the opportunity to teach the relationship formulas at an early stage, even before the studies of geometry have begun in the precise manner at the higher level of difficulty.

In this paper we propose a structure and a method for teaching the relationship formulas that has been tried with a group of students. Teaching the relationship formulas by this manner will present the students with many uses for material that has already been studied, and will expose them to new methods for solving problems in geometry and algebra.

This paper presents material that is suitable for students aged 13-15. The material includes: (1) Obtaining three formulas that relate special segments in the triangle to the sides of the triangle; (2) Using these formulas for proving three geometrical theorems; (3) Examples of problems in geometry that can be solved algebraically using these formulas; (4) Didactic recommendations for teaching this material.

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Suitable for students in classes 9 and 10, this article uses formulas relating the sides of a triangle with its special segments such as the median and altitude. Students have an opportunity to derive new geometric relationships using familiar algebraic identities. The pedagogy strategy used is of worksheets with scaffolded questions which enable students to derive these relations for themselves. Answers to all questions are explained in the first part of the article. The article provides an interesting way to devise extension activities which enable students to both practise and build on learnt concepts.

The teaching of geometry at school is comprised of several stages. In the last stage (usually starting from the ages of 15-16), the study of geometry focuses on the logical structure of the topics and on proofs of theorems, followed by application of the knowledge gained to proving and solving problems and tasks.

In the preceding stage (usually at the ages of 13-14), some isolated subjects are studied, theorems are presented without proof or with a partial proof only. During this stage, the emphasis is laid on solving geometrical problems of calculation. Usually during this stage, the following topics are studied: segments, angles, angles between parallel lines, the sum of the angles in the triangle, congruence and similarity of triangles, the Pythagorean Theorem, etc. (Note: the topics of the similarity of triangles and the Pythagorean Theorem are studied at the ages of 13-14, as an extension of the topic of ratios and proportions.) In parallel, as part of the

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studies of algebra, the following topics are studied: solution of equations of the first degree, algebraic identities, operations with algebraic fractions, simplification of algebraic expressions.

From the history of mathematics we know that great mathematicians discovered by chance the famous theorems in geometry by performing mathematical manipulations of different formulas.

Solving a task using different methods

Some mathematical tasks can be solved using different methods – by using mathematical tools from the same field, by using tools from a different field or by combining tools from several fields. The larger the toolbox available to the student, the higher is his/her chance of successfully dealing with the mathematical tasks, and the more capable is he or she of finding the solution by the shortest and simplest method. Using a wide variety of mathematical tools, one can discover unorthodox solutions or proofs which accentuate the beauty of mathematics, increase motivation and the joy of both teaching and learning the subject [1], [2], [3], [4].

Use of formulas

As early as possible, in their primary education in mathematics students learn to use formulas, such as: calculation of the area of a triangle, calculation of the volume of a box, velocity calculations, etc. As students progress in their studies, formulas are added, such as: the Pythagorean Theorem, the sum of an arithmetic progression, calculation of the weighted average and standard deviation, the Laws of Sines and Cosines, etc. The question is whether the use of a formula to calculate a particular value constitutes knowledge and a technical skill of substituting values in a formula – as expected from low-achieving and intermediate-achieving students, or a tool that can be used to develop new formulas, to find proofs to theorems and to solve unique problems, as can be expected from the advanced and excelling students. These students are able and deserve to rise to a higher mathematical level, on which they have the ability to develop new formulas, and subsequently to know how to use them as a tool that allows them to deal with different tasks (see [5]). The use of formulas has a significant importance in the age of computerized technology, since it allows the student to investigate and deal with various tasks throughout all the fields of mathematics in a dynamic manner, as well as in other areas of daily life.

In this paper we shall present use of material that is acquired usually before the ages of 14-15, as a sufficient basis for the development of formulas that relate sides and special segments in a triangle, which allow one to prove new properties in shapes and to perform different calculations.

Studying the relationship formulas at the ages of 14-15 shall give the students the following advantages:

a) Turning learned material into a useful tool both at the present stage and later during their studies.

b) Deepening knowledge in various fields in geometry and algebra and the ability to implement this knowledge.

c) Acquaintance with the method for solving geometrical problems related to the triangle by the algebraic method.

In order to develop the formulas relating sides and special segments in the triangle, one requires knowledge in the following topics: Similarity of triangles (definition and condition of similarity by two angles), the Pythagorean Theorem, the expansion of the square of a sum or difference, the difference of squares formula, operations with algebraic fractions, and simplification of algebraic expressions.

The proposed program for developing and using the relation formulas is based on this knowledge only and is composed of the following parts:

1. Obtaining the formulas of relations between special segments in the triangle and the sides of the triangle.
2. Proving two tests that permit one to determine if the triangle is an isosceles triangle:
   a. The test of two medians of equal length.
   b. The test of two angle bisectors of equal length.
3. Obtaining Heron's formula for calculating the area of a triangle.
4. Solving problems of calculation in a triangle by an algebraic method.

Obtaining the formulas of relation between special segments in the triangle and the sides of the triangle.

Let ABC be some triangle, the lengths of whose sides are

$BC = a$, $AC = b$ and $AB = c$.

$AH = h_a$ is the altitude to the side a.

$AL = l_a$ is the bisector of the angle $\measuredangle BAC$.

$AM = m_a$ is the median to the side a (see Figure 1).

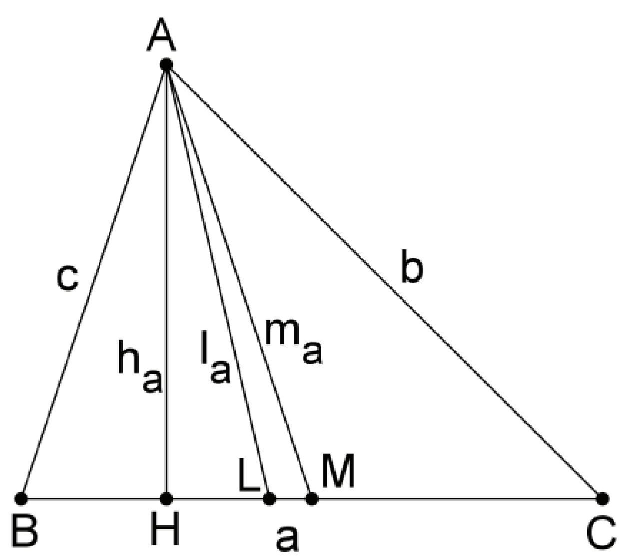

Figure 1.

The following well-known formulas relate the segments h_a, l_a and m_a to the segments a, b and c:

$$h_a^2 = \frac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{4a^2}; \tag{1}$$

$$m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4}; \tag{2}$$

$$l_a^2 = bc\frac{(b+c)^2 - a^2}{(b+c)^2}. \tag{3}$$

We hereby present the methods for obtaining these formulas based on the material studied by students at the ages of 13-14.

Obtaining the first formula for the altitude in the triangle, which is the starting formula for proving the formulas 1 to 3.

Given is the triangle $\triangle ABC$, in which AH is the altitude to the side BC (see Figure 2).

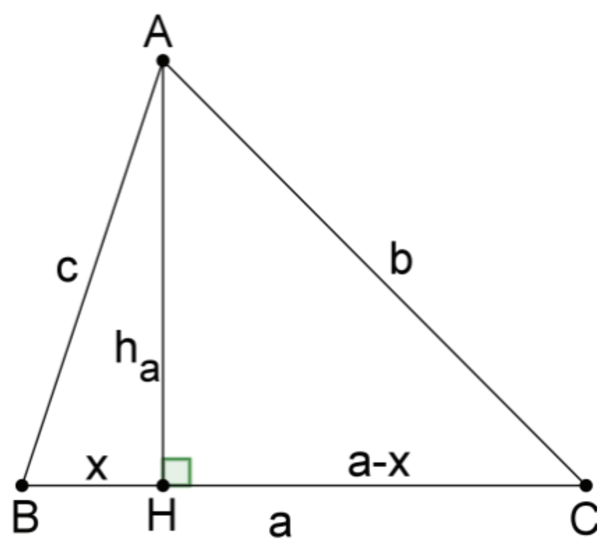

Figure 2.

We denote: $BC = a$, $AC = b$, $AB = c$, $AH = h_a$, $BH = x$, $HC = a - x$. *By the Pythagorean Theorem*, in the right-angled triangle $\triangle ABH$ there holds: $h_a^2 = c^2 - x^2$, in the right-angled triangle $\triangle ACH$ there holds: $h_a^2 = b^2 - (a-x)^2$.

Therefore, $c^2 - x^2 = b^2 - (a-x)^2$, and *from the square of the difference formula,* we have $c^2 - x^2 = b^2 - a^2 + 2ax - x^2$, from where we have for x: $x = \dfrac{a^2 + c^2 - b^2}{2a}$.

We substitute the obtained expression for x in the formula for h_a^2, to obtain:

$$h_a^2 = c^2 - \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 \qquad (4)$$

The formula (4) is the first formula that expresses the square of the altitude of the triangle by the lengths of its sides and is also the basic formula for proving the formulas (1) to (3) .

Obtaining a second formula (Formula (1)) for the altitude in a triangle.

By using the difference of squares formula $a^2 - b^2 = (a-b)(a+b)$ on the right-hand side of (4), we obtain:

$$h_a^2 = \left(c - \frac{a^2 + c^2 - b^2}{2a}\right)\left(c + \frac{a^2 + c^2 - b^2}{2a}\right),$$

from which, *by adding the fractions* in each pair of parentheses, and *by using the abridged multiplication formula* $(a \pm c)^2 = a^2 \pm 2ac + c^2$, we obtain:

$$
\begin{aligned}
h_a^2 &= \frac{2ac - a^2 - c^2 + b^2}{2a} \cdot \frac{2ac + a^2 + c^2 - b^2}{2a} \\
&= \frac{b^2 - (a^2 - 2ac + c^2)}{2a} \cdot \frac{(a^2 + 2ac + c^2) - b^2}{2a} = \frac{b^2 - (a-c)^2}{2a} \cdot \frac{(a+c)^2 - b^2}{2a} \\
&= \frac{(b - a + c)(b + a - c)}{2a} \cdot \frac{(a + c - b)(a + c + b)}{2a} \\
&= \frac{(a + b + c)(a + b - c)(a + c - b)(b + c - a)}{4a^2}.
\end{aligned}
$$

Obtaining the formula for the median (Formula (2)).
In Figure 3 it is given that

$AM = m_a$ is the median to the side BC, $AH = h_a$ is the altitude to the side BC, $BC = a$, $AC = b$ and $AB = c$, and hence:

$$BM = MC = \frac{a}{2}$$

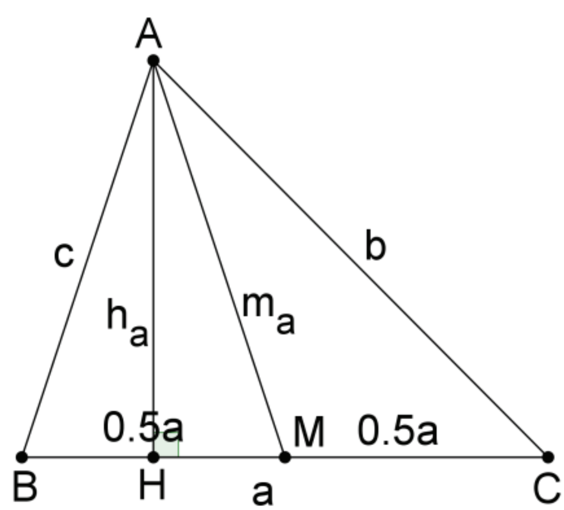

Figure 3.

We consider the triangle $\triangle ABM$, whose sides are $AB = c$, $BM = \frac{a}{2}$ and $AM = m_a$; the segment AH is an altitude in this triangle, and therefore from (4) for $AH^2 = h_a^2$ there holds:

$h_a^2 = AB^2 - \left(\frac{BM^2 + AB^2 - AM^2}{2BM}\right)^2$. After substitution we obtain:

$$h_a^2 = c^2 - \left(\frac{\left(\frac{a}{2}\right)^2 + c^2 - m_a^2}{2 \cdot \frac{a}{2}}\right)^2 \quad or: \quad h_a^2 = c^2 - \left(\frac{\frac{a^2}{4} + c^2 - m_a^2}{a}\right)^2. \qquad (5)$$

By comparing (4) and (5), it follows that

$$\left(\frac{a^2+c^2-b^2}{2a}\right)^2 = \left(\frac{\frac{a^2}{4}+c^2-m_a^2}{a}\right)^2.$$

Now observe that $a^2+c^2-b^2$ and $\frac{a^2}{4}+c^2-m_a^2$ have the same sign; for if $\measuredangle B$ is acute, then both $a^2+c^2-b^2$ and $\frac{a^2}{4}+c^2-m_a^2$ are positive (from $\triangle ABC$ and $\triangle ABM$, respectively); and if $\measuredangle B$ is obtuse, then both $a^2+c^2-b^2$ and $\frac{a^2}{4}+c^2-m_a^2$ are negative. Hence the equality sign is preserved if we take square roots of both sides in the above equality. On doing so and cancelling common factors, we obtain the desired formula, $m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4}$.

Obtaining the formula for the angle bisector (Formula (3)).
In Figure 4 it is given that:

$AL = l_a$ is the bisector of the angle $\measuredangle BAC$, $AH = h_a$ is the altitude to the side BC, $BC = a$, $AC = b$ and $AB = c$. Note that AH is also the altitude to the side BL in the triangle $\triangle ABL$.

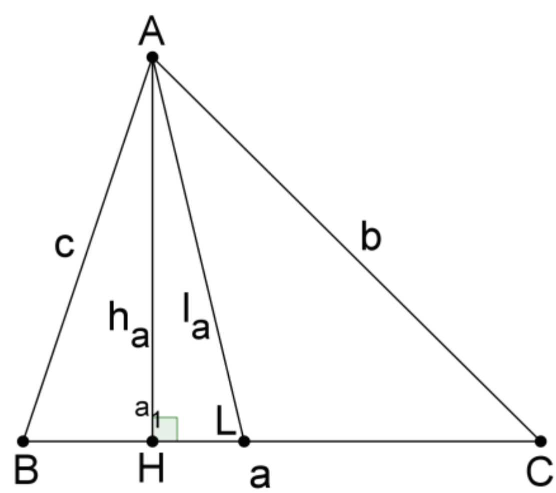

Figure 4.

The lengths of the sides of the triangle $\triangle ABL$ are: $AB = c$, $AL = l_a$ and $BL = a_1$.

From equation (4) for the altitude h_a in the triangle $\triangle ABL$ there holds:
$h_a^2 = AB^2 - \left(\frac{BL^2+AB^2-AL^2}{2BL}\right)^2$ and after substitution we obtain:

$$h_a^2 = c^2 - \left(\frac{a_1^2+c^2-l_a^2}{2a_1}\right)^2. \qquad (6)$$

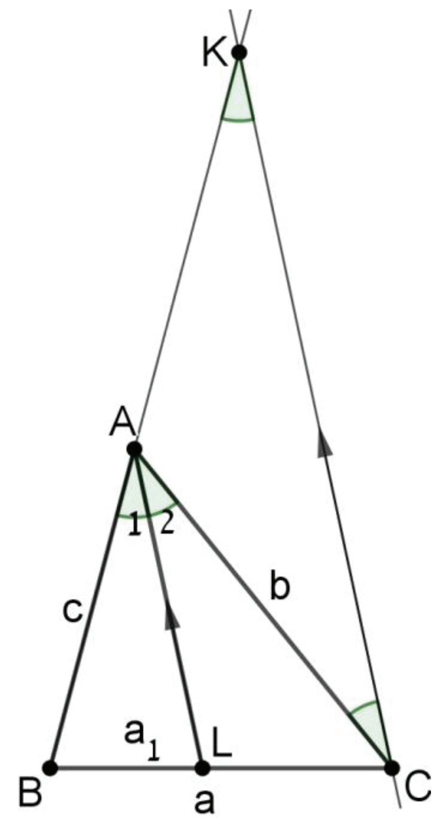

Figure 5.

We now express the length of the segment $BL = a_1$ through the side lengths of the triangle $\triangle ABC$. We carry out the following auxiliary construction (as shown in Figure 5):

Through the vertex C we draw the straight line m that is parallel to the line AL, and which intersects the continuation of the side BA at the point K. This creates two pairs of equal angles: $\measuredangle A_1 = \measuredangle K$ (*corresponding angles in these parallel lines and their secant BK*) and $\measuredangle A_2 = \measuredangle ACK$ (*alternate angles in the same parallel lines and their secant AC*). In addition we have $\measuredangle A_1 = \measuredangle A_2$, and therefore $\measuredangle K = \measuredangle ACK$. Hence it follows that $\triangle ACK$ is an isosceles triangle in which $AK = AC = b$. In the triangles $\triangle ABL$ and $\triangle KBC$ there are two pairs of equal angles: $\measuredangle A_1 = \measuredangle K$, where $\measuredangle B$ is a common angle. Thus, *from the similarity theorem (angle, angle)*, the triangles are similar, $\triangle BAL \sim \triangle BKC$. Hence, *from the definition of similar triangles*, there holds $\dfrac{BA}{BK} = \dfrac{BL}{BC}$. After the lengths of the obtained segments are substituted in the proportion, we obtain: $\dfrac{c}{b+c} = \dfrac{a_1}{a} \Rightarrow a_1 = \dfrac{ac}{b+c}$.

We substitute the expression for a_1 in (6), and obtain

$$h_a^2 = c^2 - \left(\frac{\left(\frac{ac}{b+c}\right)^2 + c^2 - l_a^2}{2\frac{ac}{b+c}} \right)^2. \tag{7}$$

By comparing (4) and (7) we obtain: $\dfrac{a^2+c^2-b^2}{2a} = \dfrac{\left(\dfrac{ac}{b+c}\right)^2 + c^2 - l_a^2}{2\dfrac{ac}{b+c}}$ from which follows:

$\dfrac{a^2+c^2-b^2}{2a} \cdot \dfrac{2ac}{b+c} = \dfrac{a^2c^2}{(b+c)^2} + c^2 - l_a^2$, and hence we have for l_a^2 :

$$l_a^2 = \frac{a^2c^2 + (b+c)^2 c^2}{(b+c)^2} - \frac{c\,(a^2+c^2-b^2)}{b+c} \Rightarrow l_a^2 = c\frac{a^2c + (b+c)^2 c - (a^2+c^2-b^2)(b+c)}{(b+c)^2}.$$

After opening the parentheses (multiplication of polynomials, use of the abridged multiplication formula and collecting similar terms in the denominator of the fraction), we obtain:

$$l_a^2 = c\frac{2b^2c + bc^2 + b^3 - a^2b}{(b+c)^2} = bc\frac{2bc + c^2 + b^2 - a^2}{(b+c)^2} = bc\frac{(b+c)^2 - a^2}{(b+c)^2}$$

Thus we obtained formula (3).

Proving the signs (tests) of an isosceles triangle

Sign 1 (test of an isosceles triangle based on two medians of equal length)

A triangle in which there are two medians of equal lengths is an isosceles triangle.

In $\triangle ABC$ it is given that:

$AD = m_a$ is the median to the side $BC = a$,

$BE = m_b$ is the median to the side $AC = b$,

and also $AD = BE$ (as shown in Figure 6).

Prove that $\triangle ABC$ is an isosceles triangle.

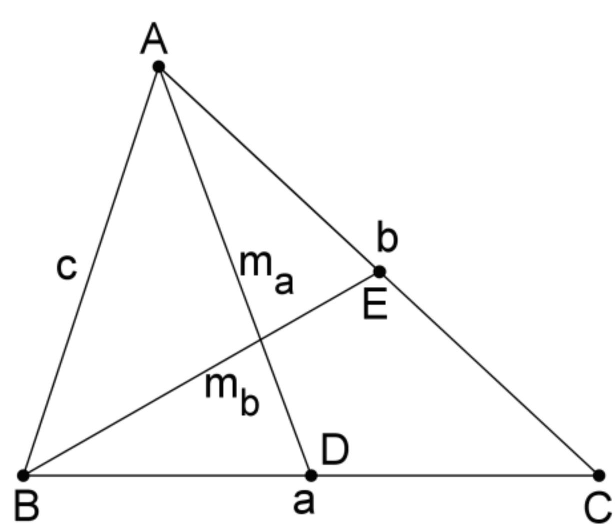

Figure 6.

Proof

We use the relation (2) between the length of the median and the lengths of the triangle's sides. From this formula:

$$m_a^2 = \frac{b^2 + c^2}{2} - \frac{a^2}{4} \text{ and } m_b^2 = \frac{a^2 + c^2}{2} - \frac{b^2}{4}.$$

From the data it follows that $\frac{a^2 + c^2}{2} - \frac{b^2}{4} = \frac{b^2 + c^2}{2} - \frac{a^2}{4}$, and hence:

$2a^2 + 2c^2 - b^2 = 2b^2 + 2c^2 - a^2 \Rightarrow 3a^2 = 3b^2$, which means that $a = b$, and therefore the triangle is an isosceles one.

Note: of course, the first test can also be proven using other ways, without using the relation formula; for example, using the properties of a parallelogram. However, this material is studied at a later stage, when we focus on the logical structure of the material and on proofs.

Sign 2 (test of an isosceles triangle by two equal angle bisectors)

A triangle in which there are two angle bisectors of equal lengths is an isosceles triangle.

In $\triangle ABC$ it is given that:

$AD = l_a$ is the bisector of the angle $\measuredangle BAC$,

$BE = l_b$ is the bisector of the angle $\measuredangle ABC$,

and also $AD = BE$ (as shown in Figure 7).

Prove that $\triangle ABC$ is an isosceles triangle.

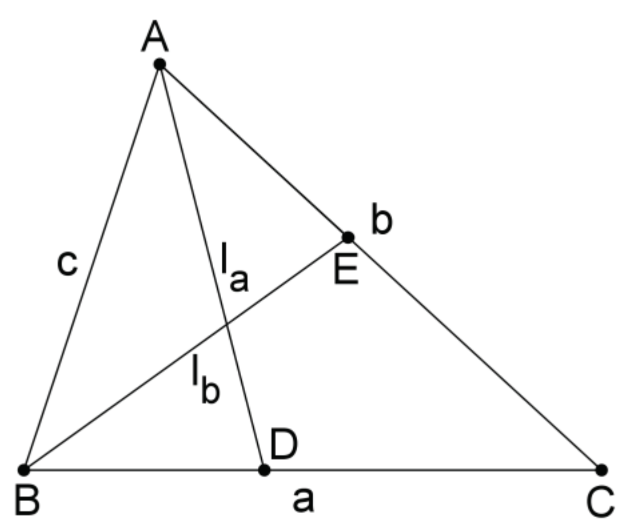

Figure 7.

Proof

From formula (3) that relates the angle bisector to the side lengths of the triangle, there holds: $l_a^2 = bc\dfrac{(b+c)^2 - a^2}{(b+c)^2}$ and also $l_b^2 = ac\dfrac{(a+c)^2 - b^2}{(a+c)^2}$.

From the data we have that: $ac\dfrac{(a+c)^2 - b^2}{(a+c)^2} = bc\dfrac{(b+c)^2 - a^2}{(b+c)^2}$, and hence:

$$a(b+c)^2\left[(a+c)^2 - b^2\right] = b(a+c)^2\left[(b+c)^2 - a^2\right].$$

By using the formulas for the difference of squares and the square of a sum, we have:

$$a\left(b^2+2bc+c^2\right)(a+c-b)(a+c+b) = b\left(a^2+2ac+c^2\right)(b+c-a)(b+c+a)$$
$$\left(b^2+2bc+c^2\right)\left(a^2+ac-ab\right) = \left(a^2+2ac+c^2\right)\left(b^2+bc-ab\right),$$

and after multiplying polynomials and collecting similar terms we obtain:

$$3a^2bc - 3ab^2c + a^2c^2 - b^2c^2 + ac^3 - bc^3 + a^3b - ab^3 = 0,$$

grouping together

$$3abc(a-b) + c^2\left(a^2-b^2\right) + c^3(a-b) + ab\left(a^2-b^2\right) = 0,$$
$$(a-b)\left[3abc + c^2(a+b) + c^3 + ab(a+b)\right] = 0.$$

The factor in the square brackets is always positive, therefore there holds $a - b = 0$, or $a = b$, and the triangle is an isosceles one.

Note: the theorem stating that **a triangle in which the lengths of two angle bisectors are equal is an isosceles triangle** is known in literature as the **Steiner-Lehmus Theorem**. This theorem is also called the **"Internal bisectors problem"** and **"Lehmus's Theorem"** [6-8].

Since the original proof by Steiner and Lehmus, dozens of different proofs have been suggested for this theorem both using geometrical tools and using tools from other fields in mathematics, or by a combination of different tools [9-11]. This is an example of a case in which both a theorem and its converse are true, however the proof in one direction is easy and immediate, and the proof in the other direction is much more difficult.

Obtaining Heron's formula for the area of a triangle

Heron's formula for the area of a triangle whose side lengths are a, b and c is $S_{\triangle ABC} = \sqrt{p(p-a)(p-b)(p-c)}$, where $p = \dfrac{a+b+c}{2}$ is half the perimeter of the triangle.

Proof

We use formula (1) for the altitude of a triangle in the following form:

$$h_a^2 = \frac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{4a^2},$$

we multiply both sides of the formula by $\dfrac{a^2}{4}$ and obtain:

$$\frac{a^2h_a^2}{4} = \frac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{16}.$$

Hence:

$$\left(\frac{ah_a}{2}\right)^2 = \frac{a+b+c}{2}\cdot\frac{a+c-b}{2}\cdot\frac{a+b-c}{2}\cdot\frac{b+c-a}{2}. \qquad (8)$$

Since:

$$p-a = \frac{a+b+c}{2} - a = \frac{b+c-a}{2},\ p-b = \frac{a+b+c}{2} - b = \frac{a+c-b}{2},\ p-c = \frac{a+b+c}{2} - c = \frac{a+b-c}{2}$$

and

$$\frac{ah_a}{2} = S_{\triangle ABC}.$$

After substituting in (8), we obtain the equality: $S^2_{\triangle ABC} = p\,(p-a)\,(p-b)\,(p-c)$, and hence:

$$S_{\triangle ABC} = \sqrt{p\,(p-a)\,(p-b)\,(p-c)}.$$

Didactic recommendations: examples for applying the material during lessons.

Activities for obtaining the auxiliary formula (4)

1) In each of the two right-angled triangles shown in Figure 2, express the square of the side AH (h_a^2) using the lengths of the other sides. Use the Pythagorean theorem.
2) Equate the expressions for h_a^2 that you obtained in order to solve the equation and obtain an expression for x.
3) Obtain the formula (4).

Activity for obtaining formula (1)

1) Factor the right-hand side of formula (4) using the formula
$a^2 - b^2 = (a-b)(a+b)$.
2) Using the formulas
$(a-b)^2 = a^2 - 2ab + b^2$ and
$a^2 - b^2 = (a-b)(a+b)$, factor each of the factors you obtained into two additional factors.

Activity for obtaining formula (2)

1) Observe Figure 3 and the data given beside it, and determine the lengths of the sides of the triangle ABM.
2) Write down the formula (4) using the sides of the triangle ABM and its altitude AH.
3) Equate the expression for h_a^2 that you obtained in the previous section with the expression for h_a^2 that appears in formula (4) and solve the equation obtained for m_a^2 .

Activity for obtaining formula (3)

1) Observe Figure 4 and the data given beside it, and determine the lengths of the sides of the triangle ABL.
2) Write down the formula (4) using the sides c, a_1 and l_a of the triangle ABL and its altitude AH.
3) In Figure 5 it is given: the triangle ABC, whose sides are a, b, c; the angle bisector AL of the angle $\measuredangle BAC$, whose length is l_a, the straight line CK that is parallel to AL (K belongs to the continuation of the side BA). Find the following in the figure (show your work): (a) Two similar triangles; (b) An isosceles triangle.
4) Write down a proportion that contains the four sides with one end at the point B of the similar triangles found in the previous section. Using this proportion, express a_1 using the lengths of the other three sides.
5) Using the obtained formula for a_1, and the formula from Section 2, express h_a^2 using a, b, c and l_a .
6) Equate the expression for h_a^2 that you obtained in the previous section with the expression for h_a^2 that appears in formula (4) and solve the equation obtained for l_a^2 .

Activity for discovering and obtaining a proof for the test (sign) of an isosceles triangle based on two equal medians.

1) Use Formula (2) and write down an expression for the square of the median to the side AC (expression for m_b^2).
2) Equate the expressions for m_a^2 and m_b^2, and simplify the equality obtained.
3) From the result you obtained draw a conclusion concerning the triangle ABC.

Activity for discovering and obtaining a proof for the test (sign) of an isosceles triangle based on two equal angle bisectors.

1) Use Formula (3) and write down an expression for the square of the angle bisector of $\measuredangle ABC$ (for l_b^2).
2) Equate the expressions for l_a^2 and l_b^2, and simplify the equality obtained based on the following instructions:
    a) Divide the two sides of the equality by a common factor.
    b) Transform the proportion to an equality of products.
    c) Open all the parentheses, group all the terms on the left-hand side of the equality, and simplify it.
    d) Factor the left-hand side of the equality obtained into two factors, where one factor is $a - b$ (use the method "Factoring Trinomials by Grouping).
    e) Explain why the second factor (The expression in the large parentheses) is always positive.
    f) Draw conclusion, what is the condition on $a - b$, which assures the existence of the equality. Draw a conclusion concerning the triangle ABC.

Activity for obtaining Heron's formula

1) Multiply both sides of the formula (1) by the expression $\frac{a^2}{4}$.
2) Determine the geometrical meaning of the expression you obtained on the left-hand side of the new equality.
3) Write down the right-hand side of the equality as the product of four fractions, each of which has the denominator 2.
4) Denote the fraction $\frac{a+b+c}{2}$ by p.
5) What is the geometrical meaning of p?
6) Write down each of the expressions $p - a$, $p - b$ and $p - c$ as a fraction which only has the lengths of the sides of the triangle ABC.
7) Write down the right-hand side of the equality as the product of factors that contain p.
8) Express the area of the triangle ABC (S_{ABC}) using expressions that contain p.

Solving calculation problems in a triangle using an algebraic method (using the relation formulas (1) to (3)).

The use of the developed formulas for the calculation of the side lengths of a triangle, as presented in the three examples below is based on the fact that students have the technical skills for solving algebraic equations of the first and the second degree.

Example 1

In the $\triangle ABC$ it is given that:

$AB = 6$, $m_{BC} = 6$ is the median to the side BC, $m_{AC} = 4$ is the median to the side AC.

Calculate the lengths of the sides AC and BC.

$$\left[\text{answer} : AC = \sqrt{\frac{146}{3}}, BC = \sqrt{\frac{56}{3}}\right]$$

Example 2

In the $\triangle ABC$ it is given that:

$BC = 5$, $m_{BC} = \frac{1}{2}\sqrt{209}$ is a median to the side BC, $l_{BC} = 4\sqrt{3}$ is the bisector of the angle $\measuredangle BAC$.

Calculate the lengths of the sides AB and AC.

[answer: $AC = 9$, $AB = 6$]

Example 3

In the $\triangle ABC$ it is given that:

$BC = 10$, $l_{BC} = 6\sqrt{2}$ is the bisector of the angle $\measuredangle BAC$, $h_{BC} = 3\sqrt{7}$ is the altitude to the side BC.

Calculate the lengths of the sides AB and AC.

[answer: $AC = 12$, $AB = 8$]

Notes

Problem 1 is solved by means of double use of the formula (2).

Problem 2 is solved by using the formulas (2) and (3).

Problem 3 is solved by using the formulas (1) and (3).

Summary

Known algebraic formulas for calculating the lengths of certain segments in a triangle were developed, as well as a formula for calculating the area of a triangle – using algebraic manipulations which are within the skill set of students aged 14-15. One should expect that after presenting this method to students aged 14 or more, who had acquired the skill of using simple algebraic formulas, they would be able to develop more complex formulas that may allow mathematical tasks to be solved and proofs for theorems on a high level of difficulty to be found.

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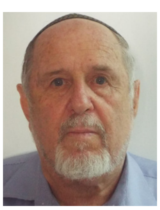

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