Addendum to “A 20-30-130 triangle”

In the March 2016 issue of AtRiA, we posed the following problem: *Triangle ABC has* $\measuredangle A = 130^\circ$, $\measuredangle B = 30^\circ$ *and* $\measuredangle C = 20^\circ$. *Point P is located within the triangle by drawing rays from B and C, such that* $\measuredangle PBC = 10^\circ$ *and* $\measuredangle PCB = 10^\circ$. *Segment PA is drawn. Find the measure of* $\measuredangle PAC$. (See Figure 1.)

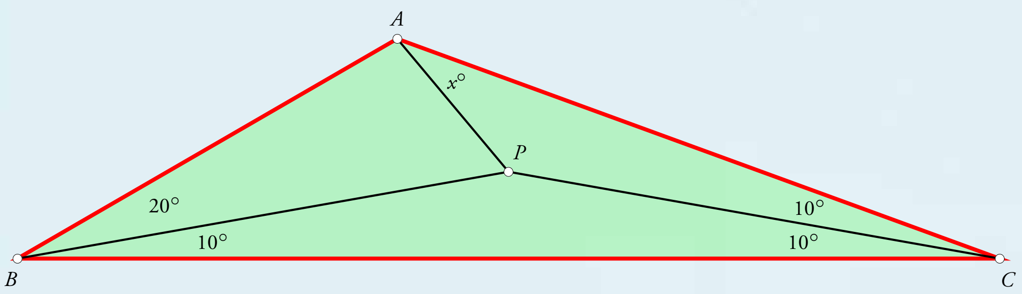

Figure 1

We had offered a trigonometric solution, making use of the sine rule and numerous standard trigonometric identities. At the end of the article we posed the question of finding a pure geometry solution.

We are happy to say that a reader (and contributor of several articles in earlier issues), **Ajit Athle**, has sent in a very elegant pure geometry solution—just as we had hoped! Here are the details.

Construction: Extend BA to E such that $AE = CE$ (see Figure 2; this is equivalent to saying: let the perpendicular bisector of segment AC meet BA extended at E); then $\measuredangle EAC = \measuredangle ECA = 50^\circ$, and $\measuredangle AEC = 80^\circ$.

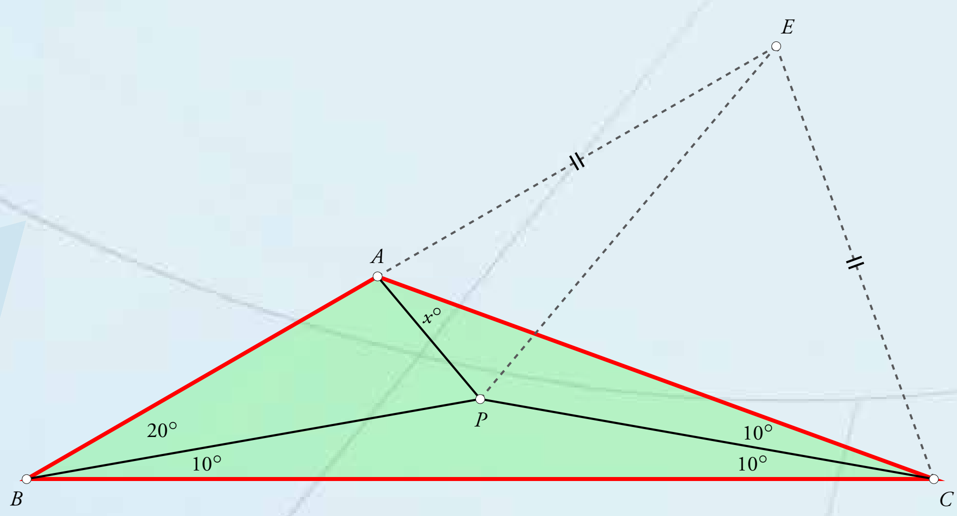

Figure 2. Solution to the 20-30-130 triangle problem by Ajit Athle

Keywords: *angle chasing, isosceles, equilateral, circum-circle, pure geometry*

Since $\measuredangle BPC = 2\measuredangle BEC$ and also $PB = PC$, it follows that P is the circumcentre of $\triangle EBC$. From this it follows that $\measuredangle EPC = 2\measuredangle EBC$, i.e., $\measuredangle EPC = 60°$.

This in turn implies that $\triangle EPC$ is equilateral, and hence that $\measuredangle PEC = 60°$. From this we infer that $\measuredangle AEP = 20°$. Again, $EA = EP$ (both sides are equal to EC), i.e., $\triangle EAP$ is isosceles. Hence $\measuredangle EAP = 80°$. Since $\measuredangle EAC = 50°$, it follows that $\measuredangle PAC = 30°$, i.e., $x = 30$. □

The fact that P is the circumcentre of $\triangle EBC$ suggests an alternate way of presenting this proof. Namely: draw the circle centred at P and passing through B and C. Let it intersect the extension of BA at E. (See Figure 3.)

Then we have $PE = PC$ and $\measuredangle EPC = 2\measuredangle EBC = 60°$, hence $\triangle EPC$ is equilateral, so $\measuredangle PCE = 60°$ and $\measuredangle ACE = 50°$. We also have $\measuredangle EAC = 50°$ (since $\measuredangle BAC = 130°$); therefore $EA = EC = EP$. The rest of the solution is the same as earlier; we get $x = 30$. □

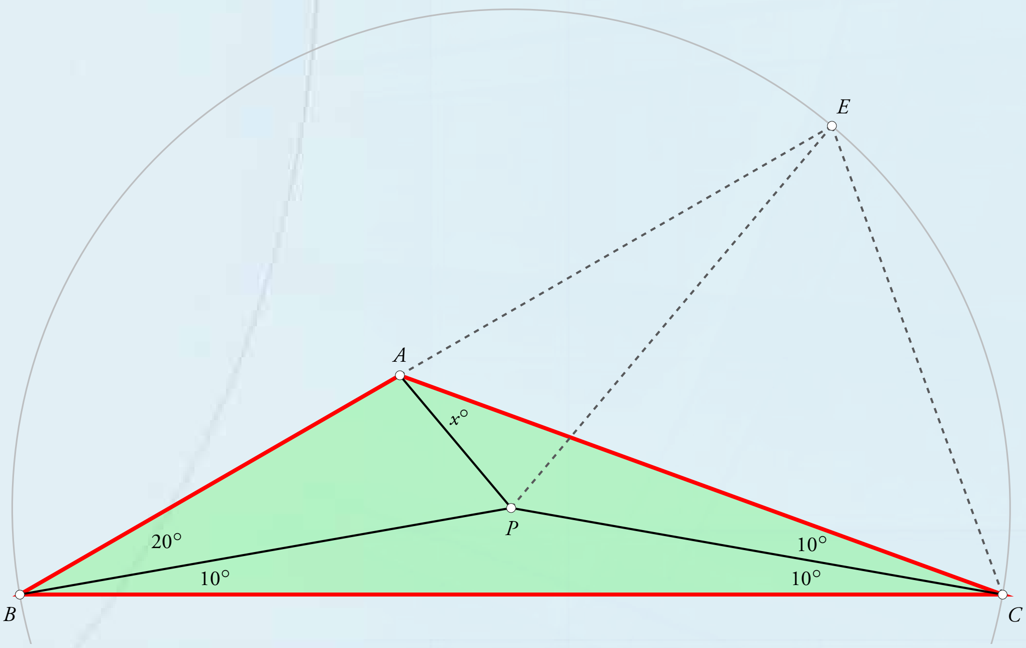

Figure 3

Remark. In hindsight, the idea of trying a circle centred at P and passing through B and C should have suggested itself to us right away; after all, we have $PB = PC$ as per the given data.

But, as they say, hindsight is the best sight of all!

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.