Three Means

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Given two positive numbers u and v, their arithmetic mean (AM) is a, such that u, a, v are in arithmetic progression; this requires that

$$2a = u + v.$$

The geometric mean (GM) is g, such that u, g, v are in geometric progression, which means that

$$g^2 = uv.$$

The harmonic mean (HM) is h, such that the reciprocals of u, h, v are in arithmetic progression, and so

$$\frac{2}{h} = \frac{1}{u} + \frac{1}{v}, \qquad \therefore \;\; h = \frac{2uv}{u+v}.$$

Interestingly, this can be rearranged as

$$h \cdot \frac{u+v}{2} = uv,$$

from which it follows that the GM of u and v is also the GM of their AM and HM.

For each of these three means, there is a simple and well-known geometric construction that illustrates it, but I was curious to see whether one could find a single diagram that illustrated all three at the same time.

Keywords: *Arithmetic mean, geometric mean, harmonic mean, visualisation, geometry*

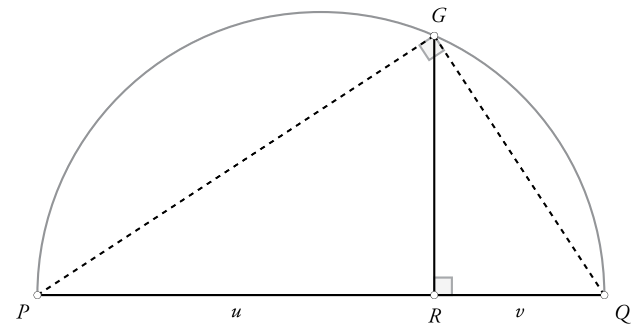

Figure 1

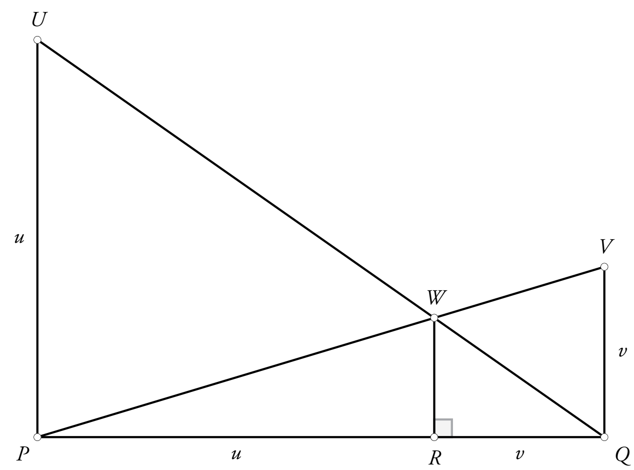

Figure 2

We begin by recalling two of the basic diagrams.

Figure 1 depicts a semicircle with diameter $u + v$; PQ is a diameter of the semicircle, and the perpendicular RG is erected at the point R so that $PR = u$ and $QR = v$. Then $\triangle PGR \sim \triangle GQR$ and so $GR^2 = PR \cdot RQ$, i.e. the length RG represents the geometric mean of the lengths PR and QR.

In Figure 2, PQ is the common perpendicular to PU and QV; UQ and VP are joined and meet at W, and R is the foot of the perpendicular to PQ from W. Using the proportional intercepts theorem for $\triangle QUP$, and then again for $\triangle PVQ$, we find that $PR : RQ = u : v$, and also

$$\frac{WR}{u} + \frac{WR}{v} = 1, \qquad \therefore \quad \frac{1}{WR} = \frac{1}{u} + \frac{1}{v},$$

which means that the length of RW is half the length of the harmonic mean of the lengths PU and QV.

Now, consider a line segment PQ with length $u + v$, and a point R on that segment such that $PR = u$, $QR = v$ (Figure 3). Erect perpendiculars PU and QV so that $PU = PR = u$, $QV = QR = v$, and erect the perpendicular to PQ at R. Draw the semicircle on PQ as diameter (centre O), meeting the perpendicular through R at G, and let W be the point of intersection of UQ and PV.

From our earlier remarks, it is clear that since $\measuredangle PGO$ is a right angle subtended by the diameter PQ, RG is the geometric mean of PR and QR, and therefore of u and v; moreover the vertical WR coincides with the vertical GR, i.e. W does indeed lie on RG.

We now add in the circle centred on W and passing through R, meeting RG again in H (Figure 4). RH, being twice RW, will be the harmonic mean of PR and QR.

Prettily, it seems that the smaller circle is tangent to the semicircle. We prove that this is indeed the case by verifying that the distance between the centres of the two circles is equal to the difference between their radii. Equivalently, we may consider the line OW and extend it till it meets the semicircle at

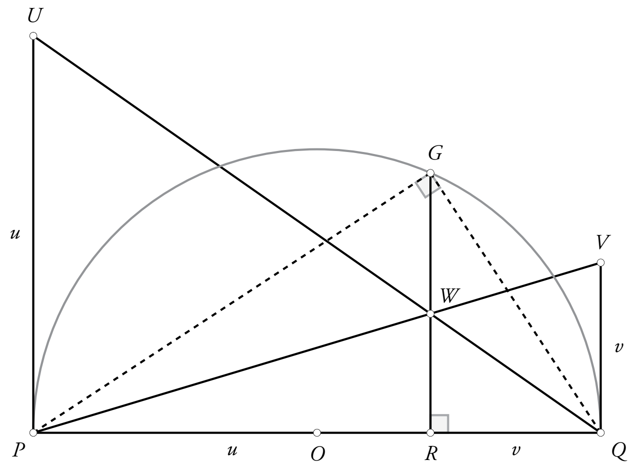

Figure 3

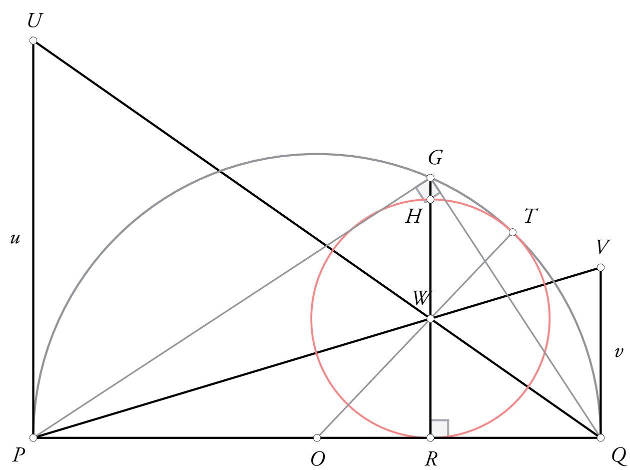

Figure 4

point T. If the length of WT is equal to the radius of the smaller circle, this claim will follow.

Let $WR = h$. We have now: $OR = \frac{1}{2}(u - v)$, so:

$$\begin{aligned} OW^2 &= OR^2 + RW^2 = \frac{(u-v)^2}{4} + h^2 \\ &= \frac{(u+v)^2}{4} - uv + h^2 \\ &= \frac{(u+v)^2}{4} - h \cdot \frac{u+v}{2} + h^2 \\ &= \left(\frac{u+v}{2} - h\right)^2. \end{aligned}$$

Hence the distance between the centres of the principles is equal to the difference between their radii, implying that the circles are internally tangent to each other as claimed.

Now the semicircle and the small circle have a common tangent at T; let this meet line $RWHG$ produced in A (Figure 5). It turns out that RA is the arithmetic mean of PR and RQ.

To prove this, we consider the triangles ORW and ATW: they are congruent (right-angled, vertically opposite angles equal and $RW = WT = h$), so that $WA = OW$; also $OW = OT - h$. This means that $AR = OT - h + WR$, but of course $WR = h$ and so $AR = OT = \frac{1}{2}(u + v)$.

We thus have a very elegant illustration of the three means of the lengths PR and QR in one diagram. Moreover, the standard result that $HM \leq GM \leq AM$ is visually confirmed, for clearly H must lie inside the semicircle, G on it and A outside it.

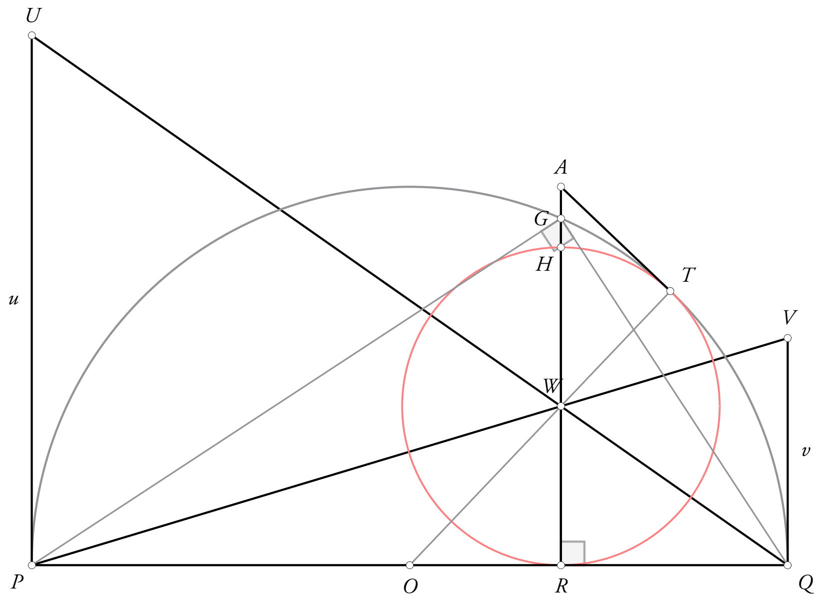

Figure 5

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