Features

An Eye on Eyeball

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Euclidean Geometry is fascinating. It has captured our imagination for centuries. Many beautiful theorems have been discovered and proved, and myriad mind-boggling problems have been posed and solved, yet we haven't got tired of it. To the creative mind, geometry is a source of immense pleasure and contentment. We look for some more in a little-known result in plane geometry called "The Eyeball Theorem" and uncover some of its geometrical features.

The Eyeball Theorem

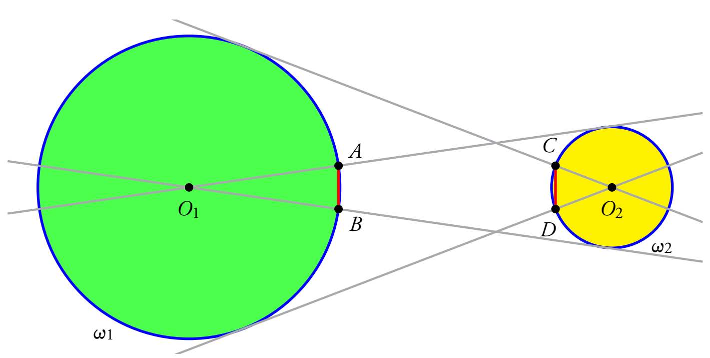

Figure 1

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Consider two non-overlapping circles ω_1 and ω_2 in the plane; neither circle is contained in the other. Let O_1 and O_2 be their respective centres. Draw tangents to ω_2 from O_1 and to ω_1 from O_2. Let the tangents to ω_2 intersect ω_1 at A and B. Let the tangents to ω_1 intersect ω_2 at C and D. The **Eyeball Theorem** now states that $AB = CD$. (See Figure 1.)

There are several ways to prove the assertion. To start with, let us mark a few more points in the configuration. Let X and Y be the respective points of intersection of O_1O_2 with AB and CD; see Figure 2. Let P_1 and P_2 be the points of contact of the tangents from O_1 to ω_2, and let Q_1 and Q_2 be the points of contact of the tangents from O_2 to ω_1.

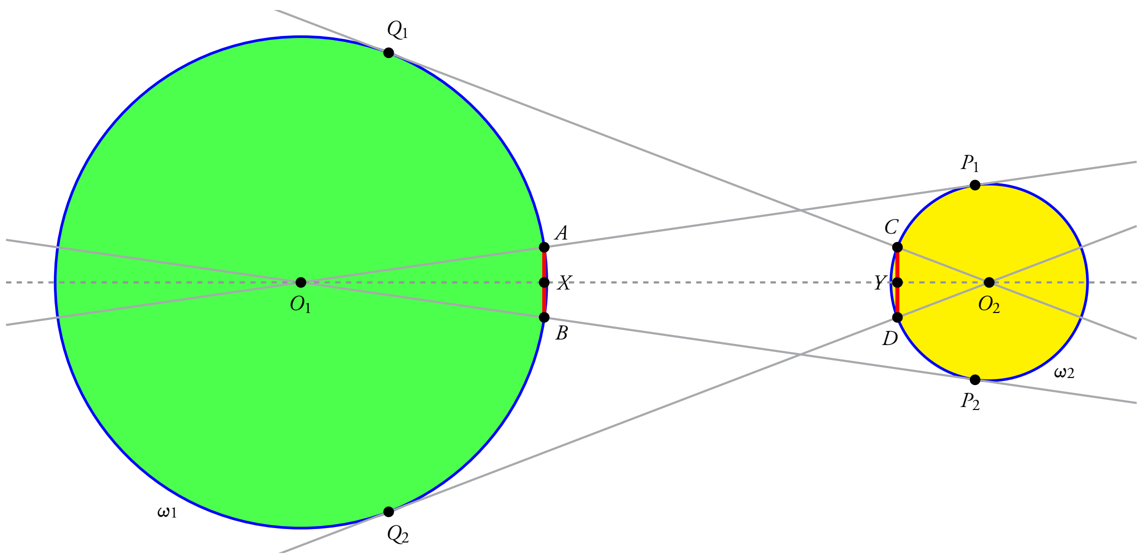

Figure 2

Here is a simple argument which shows that $AB = CD$. Let r_1 and r_2 be the radii of ω_1 and ω_2 respectively. Observe that line O_1O_2 is an axis of symmetry of the configuration. Therefore $AB = 2AX$ and $CD = 2CY$. The triangles O_1XA and $O_1P_1O_2$ are similar. Hence:

$$\frac{AX}{P_1O_2} = \frac{O_1A}{O_1O_2}, \quad \therefore\ AB = 2AX = \frac{2r_1r_2}{O_1O_2}. \tag{1}$$

A similar argument leads to

$$CD = \frac{2r_1r_2}{O_1O_2}, \tag{2}$$

and we see that $AB = CD$.

If the circles touch each other externally, then

$$AB = CD = \frac{2r_1r_2}{r_1 + r_2}, \tag{3}$$

which is the harmonic mean of the radii of the two circles.

This configuration abounds in sets of four or more concyclic points. Let us find as many such sets as we can. The missing ones may be reported by perceptive readers. As the figure is symmetric about O_1O_2, it suffices to look for concyclic sets on one side of O_1O_2, say on the same side of the line as A. See Figure 3, which is the same as Figure 2; we have reproduced it only for the readers' convenience.

Observe that $\measuredangle O_1Q_1O_2 = \measuredangle O_1P_1O_2 = 90^\circ$, which shows that O_1O_2 subtends the same angle at two points P_1 and Q_1 on the same side of it. Therefore the four points O_1, O_2, P_1 and Q_1 are concyclic.

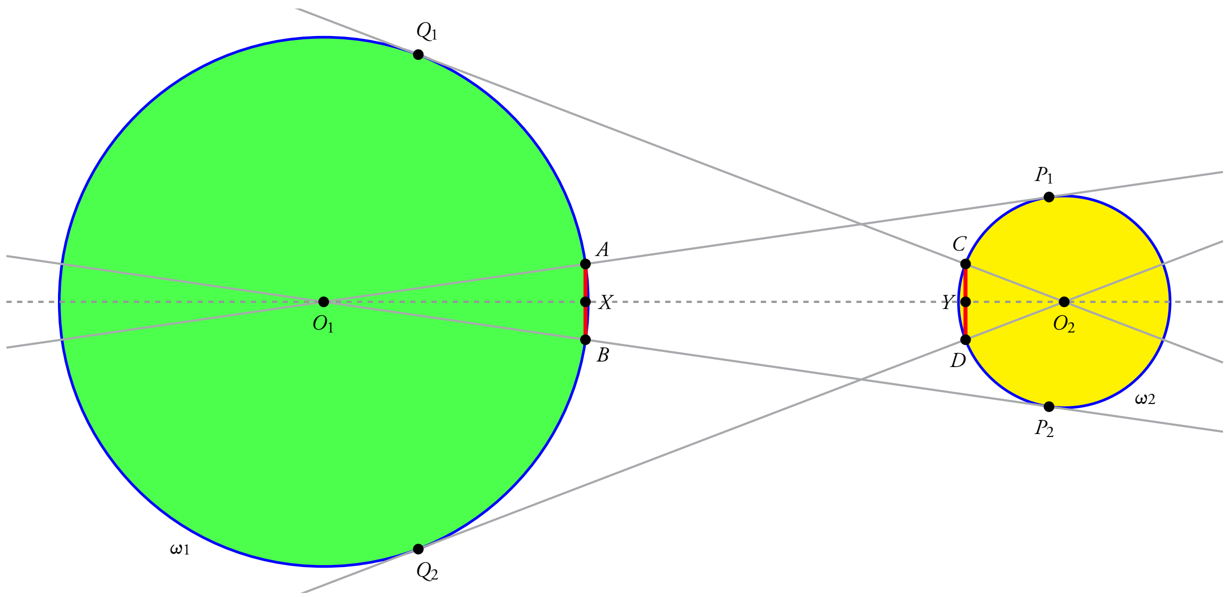

Figure 3

Moreover, O_1O_2 is a diameter of the circle. By symmetry, P_2 and Q_2 lie on the same circle. Thus we have six points on the same circle. Call this circle Ω. We have more in store. Observe that

$$\measuredangle AQ_1O_2 = \frac{1}{2}\measuredangle AO_1Q_1 = \frac{1}{2}\measuredangle CO_2P_1 = \measuredangle CP_1O_1, \tag{4}$$

showing that the points A, Q_1, P_1, C are concyclic. By symmetry the same is true for the points B, Q_2, P_2, D.

Quadrilateral $ABDC$ is a rectangle because $AB = CD$ and segments AB and CD have a common perpendicular bisector, namely, line O_1O_2. Therefore, points A, B, D, C are concyclic. The reader may easily deduce that quadrilaterals AP_1P_2B and CQ_1Q_2D are isosceles trapezoids and therefore their vertices form concyclic sets of points.

The centres of the circles ω_1, ω_2 and $(O_1Q_1P_1O_2P_2Q_2)$ all lie on O_1O_2; so also for the circles (AP_1P_2B) and (CQ_1Q_2D). What can be said about the centre of the circle containing the points A, Q_1, P_1, C? Let us investigate. If O_3 is the centre of this circle, then observe that it is the point of intersection of the perpendicular bisector of AQ_1 and that of CP_1. But the perpendicular bisector of AQ_1 passes through O_1 and that of CP_1 passes through O_2. Now

$$\measuredangle O_3O_1O_2 = \measuredangle O_3O_1P_1 + \measuredangle P_1O_1O_2 = \frac{1}{2}\measuredangle Q_1O_1P_1 + \measuredangle P_1O_1O_2, \tag{5}$$

and

$$\measuredangle O_3O_2O_1 = \measuredangle O_3O_2Q_1 + \measuredangle Q_1O_2O_1 = \frac{1}{2}\measuredangle P_1O_2Q_1 + \measuredangle Q_1O_2O_1. \tag{6}$$

But we also have

$$\measuredangle Q_1O_1P_1 + \measuredangle P_1O_1O_2 + \measuredangle Q_1O_2O_1 = 90^\circ, \tag{7}$$

and

$$\measuredangle Q_1O_2P_1 + \measuredangle Q_1O_2O_1 + \measuredangle P_1O_1O_2 = 90^\circ. \tag{8}$$

It follows that

$$\measuredangle O_3O_1O_2 + \measuredangle O_3O_2O_1 = 90^\circ. \tag{9}$$

Therefore $\measuredangle O_1O_3O_2 = 90^\circ$ and so O_3 lies on Ω. By symmetry, the centre of the circle containing B, Q_2, P_2, D also lies on Ω. (See Figure 4.)

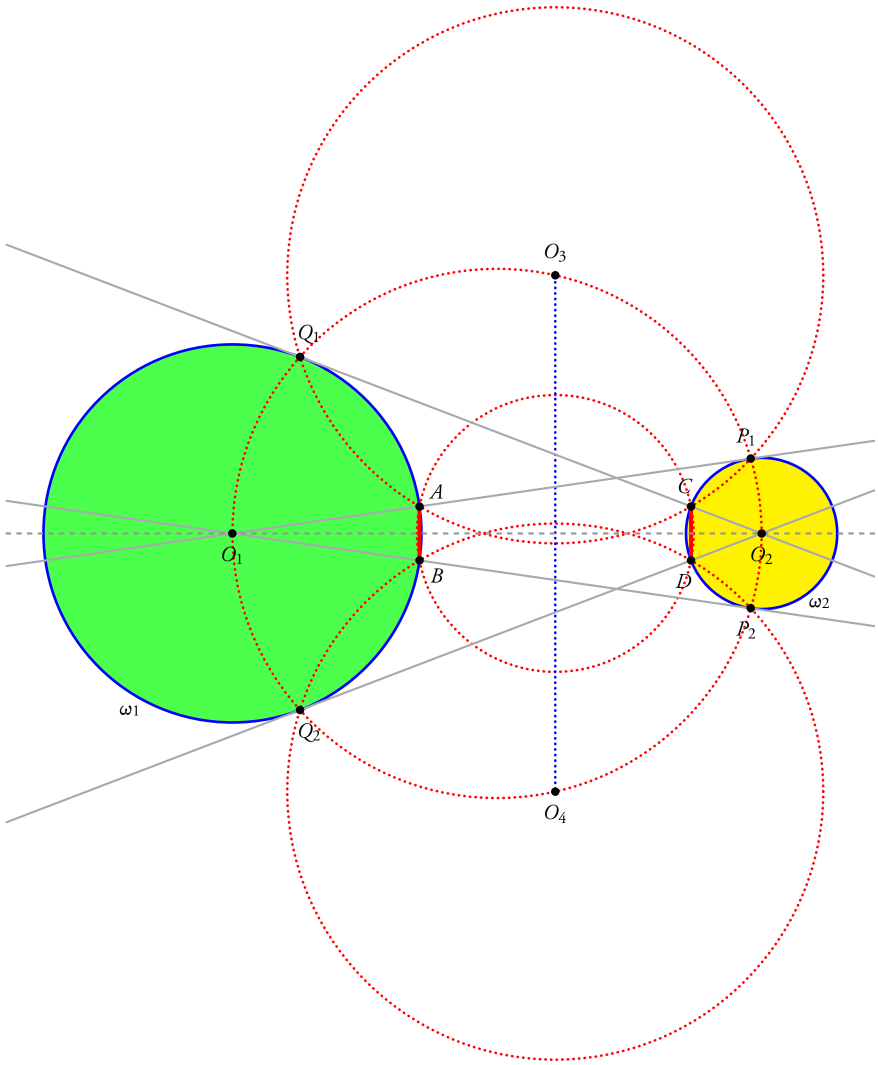

Figure 4

So we have found eight points on a circle. That's exciting, isn't it? Here is something even more exciting. Let O_4 be the centre of the circle passing through B, Q_2, P_2, D. The line O_3O_4 passes through the centre of the circle passing through A, B, C, D. How does one prove it? We leave that as an exercise for you!

References

1. The Eyeball Theorem. http://nrich.maths.org/1935
2. The Eyeball Theorem. http://www.cut-the-knot.org/Curriculum/Geometry/Eyeball.shtml

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