3 4 5 …

And Other Memorable Triples

Part IV

Features

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In Parts I, II and III of this series of articles we identified triples of consecutive positive integers with striking geometrical properties; for each triple, the triangle with the three integers as its sides has some special geometrical property. The triples $(3, 4, 5)$, $(4, 5, 6)$ and $(2, 3, 4)$ all turned out to be special in this sense.

The 'special property' possessed by $(3, 4, 5)$ is well known: there is just one triple of consecutive integers with the feature that the triangle with these integers as its sides is right-angled; namely, $(3, 4, 5)$. But the triple $(3, 4, 5)$ has a further feature which can easily be missed: the area of the triangle with sides $3, 4, 5$ is $(3 \times 4)/2 = 6$, an integer. This property is not possessed by either of the triples $(2, 3, 4)$, $(4, 5, 6)$. Indeed, the area of the triangle with sides $2, 3, 4$ is

$$\sqrt{\frac{9}{2} \times \frac{1}{2} \times \frac{3}{2} \times \frac{5}{2}} = \frac{3\sqrt{15}}{4},$$

while the area of the triangle with sides $4, 5, 6$ is

$$\sqrt{\frac{15}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2}} = \frac{15\sqrt{7}}{4};$$

Keywords: *Triangles, consecutive integers, area, integers, perfect squares, Heronian, induction*

and both of these numbers are not even rational. What about $(5,6,7)$? The semi-perimeter now is 9, so the area is

$$\sqrt{9 \times 4 \times 3 \times 2} = 6\sqrt{6},$$

which too is not rational. The same is true of the triples $(6,7,8)$ and $(7,8,9)$. So the triple $(3,4,5)$ scores above all these triples in this regard.

The following question now poses itself naturally: *Is there another triple of consecutive integers with the property that the triangle with the three integers as its sides has integral area?*

A triangle whose sides and area are both integers is known in the literature as a **Heronian triangle**. Thus the triangle with sides $3,4,5$ is Heronian; so are the triangles with sides $5,5,6$ (made by joining two 3-4-5 triangles along the side of length 4, so its area is $2 \times 6 = 12$); sides $5,5,8$ (made similarly); and sides $4,13,15$ (with area 24). The above question may thus be framed: *Is there another triple of consecutive integers which yields a Heronian triangle?* (See Box 1.)

A Heronian query

The area of the $3,4,5$ triangle is 6, which is an integer.

Are there other triangles whose sides are consecutive integers and whose area is also an integer?

Box 1

Stated this way, the answer is easily found, simply by trying out more possibilities in sequence (and, obviously, making use of a computer). The next such triple we find after $(3,4,5)$ is $(13,14,15)$. The semi-perimeter is $(13+14+15)/2 = 21$, so the area of the triangle is

$$\sqrt{21 \times 8 \times 7 \times 6} = 84,$$

an integer. We see that the triple $(3,4,5)$ is not unique in possessing the stated property.

Now we ask: How many such triples are there? Remember that our question is limited to triples of consecutive integers. The answer comes as a pleasant surprise: *There are infinitely many such triples.* Moreover, they come in a beautiful pattern which enables us to enumerate them, as we now show.

Let $n > 2$ be an integer such that the triangle with sides $n-1$, n and $n+1$ has integer area. The semi-perimeter s is given by $2s = 3n$, so by Heron's formula the area of the triangle is

$$\sqrt{\frac{3n}{2} \times \frac{n-2}{2} \times \frac{n}{2} \times \frac{n+2}{2}} = \frac{n\sqrt{3(n^2-4)}}{4}. \tag{1}$$

It follows that n must be such that $n\sqrt{3(n^2-4)}/4$ is an integer.

We first show that under this requirement, n must be even. Suppose that n is odd. Then n^2 leaves remainder 1 under division by 4, hence $3(n^2-4)$ leaves remainder 3 under division by 4. But no square integer is of this form. It follows that if n is odd, then $\sqrt{3(n^2-4)}$ is an irrational quantity, implying that $n\sqrt{3(n^2-4)}/4$ cannot be an integer. Hence n must be even.

In the paragraph below, it is assumed that n is an even integer. We shall suppose that n is such that $\sqrt{3(n^2-4)}$ is an integer and show that under this supposition, the area too is an integer.

Suppose that $\sqrt{3(n^2-4)}$ is an integer; then $\sqrt{3(n^2-4)}$ is an *even* integer (since n itself is even), hence $n\sqrt{3(n^2-4)}$ is a multiple of 4, implying that $n\sqrt{3(n^2-4)}/4$ is an integer. That is, the area of the triangle is an integer, as claimed.

On the other hand, if the area is an integer, then $\sqrt{3(n^2-4)}$ obviously must be an integer.

Therefore the exploration reduces to the following question:

Find all even integers $n > 2$ such that $3(n^2-4)$ is a perfect square.

Write $n = 2x$ where x is an integer. Then $3(n^2-4) = 4 \times 3(x^2-1)$. For this to be a perfect square, $3(x^2-1)$ must be a perfect square, hence x^2-1 must be of the form $3y^2$ for some integer y. So the problem further reduces to finding all pairs (x, y) of positive integers such that $x^2-1 = 3y^2$, i.e.,

$$x^2 - 3y^2 = 1. \tag{2}$$

For each x belonging to such a pair, the triangle with sides $2x-1$, $2x$, $2x+1$ has integer area. See Box 2.

An equation that generates the triangles we seek

The pair of positive integers (x, y) where $x^2-3y^2=1$ defines a triangle with sides $2x-1$, $2x$, $2x+1$ (i.e., three consecutive integers, the middle one being even). The area of this triangle is an integer.

Box 2

Equation (2) has a familiar form; it is an instance of the *Brahmagupta-Bhaskara-Fermat equation* (also known in the literature as the 'Pell equation'), $x^2-ny^2=1$, with $n=3$. Readers may recall that we dwelt on this kind of equation in the November 2014 issue of the magazine and studied an algorithm to solve it: the Chakravāla method.

In this article we do not use the Chakravāla. Instead we present a way of enumerating the solutions of the equation which makes use of the irrational quantity $\sqrt{3}$.

We start by noting that $x=2$, $y=1$ is a solution of (2); it is the smallest possible integral solution. Using these two numbers we form the following number α which is an irrational surd:

$$\alpha = 2 + 1 \cdot \sqrt{3} = 2 + \sqrt{3}.$$

We now compute the square of α:

$$\alpha^2 = \left(2+\sqrt{3}\right)^2 = 2^2 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}.$$

Note the numbers which have appeared in the above expression: 7 and 4. If we try out the values $x=7$ and $y=4$ in the expression x^2-3y^2, we find that it equals 1:

$$7^2 - 3 \times 4^2 = 49 - 3 \times 16 = 1.$$

So $x=7$, $y=4$ is a solution of (2). Now we have two solutions to the equation, $(2,1)$ and $(7,4)$.

Next, let us find the cube of α. We have:

$$\begin{aligned}\alpha^3 = \left(2+\sqrt{3}\right)^3 &= \left(2+\sqrt{3}\right)^2 \times \left(2+\sqrt{3}\right)\\ &= \left(7+4\sqrt{3}\right) \times \left(2+\sqrt{3}\right)\\ &= 14 + (7+8)\sqrt{3} + 12 = 26 + 15\sqrt{3}.\end{aligned}$$

Is $x = 26$, $y = 15$ a solution of (2)? Let's check:

$$26^2 - 3 \times 15^2 = 676 - 3 \times 225 = 676 - 675 = 1.$$

It is!

It is natural to try the fourth power now:

$$\begin{aligned}\alpha^4 = \left(2+\sqrt{3}\right)^4 = \left(\left(2+\sqrt{3}\right)^2\right)^2 &= \left(7+4\sqrt{3}\right)^2\\ &= 49 + 2 \times 7 \times 4\sqrt{3} + (16 \times 3) = 97 + 56\sqrt{3}.\end{aligned}$$

We may check that $x = 97$, $y = 56$ is yet another solution of (2):

$$97^2 - 3 \times 56^2 = 9409 - 3 \times 3136 = 9409 - 9408 = 1.$$

We seem to have found a way of generating an unlimited number of solutions of (2)! See Box 3.

Generating Heronian triangles whose sides are consecutive integers

If $\alpha = u_1 + v_1\sqrt{3}$ where u_1, v_1 are positive integers, then positive integral powers of α yield all possible triangles which have sides as consecutive integers and whose area is an integer as well.

Box 3

Two questions pose themselves:

Question 1: Does every positive integral power of α yield a solution to the equation? More specifically, for each positive integer n let

$$\alpha^n = u_n + v_n\sqrt{3}, \tag{3}$$

where u_n and v_n are integers. Is the following relation true for all n?

$$u_n^2 - 3v_n^2 = 1. \tag{4}$$

Question 2: Assuming that the answer to Question 1 is 'Yes', does the list (u_1, v_1), (u_2, v_2), (u_3, v_3), …enumerate *all* the positive integral solutions of (2)?

We shall show that the answer for both questions is 'Yes'. In other words, the procedure we have described does yield every possible positive integral solution to (2).

Answer to Question 1

Suppose that $x = a$, $y = b$ is a solution to the equation $x^2 - 3y^2 = 1$, that is, $a^2 - 3b^2 = 1$. Consider the following product:

$$\left(a + b\sqrt{3}\right) \cdot \left(2 + \sqrt{3}\right) = (2a + 3b) + (a + 2b)\sqrt{3}. \tag{5}$$

We now show that $x = 2a + 3b$, $y = a + 2b$ is a solution to the equation. To do so, we must verify that $(2a + 3b)^2 - 3(a + 2b)^2 = 1$. Here is the verification:

$$\begin{aligned}(2a + 3b)^2 - 3(a + 2b)^2 &= \left(4a^2 + 12ab + 9b^2\right) - 3\left(a^2 + 4ab + 4b^2\right) \\ &= a^2 - 3b^2 = 1 \quad \text{(as was required)}.\end{aligned}$$

So $x = 2a + 3b$, $y = a + 2b$ is a solution to (2), as claimed.

Inductively, it follows that $x = u_n$, $y = v_n$ as defined by (3) is a solution to equation (2) for every positive integer n. So relation (4) is true for every positive integer n. This answers Question 1.

Do you see how this is a proof by induction? What we have shown is: If $x = u_n$, $y = v_n$ is a solution to equation (2), then so is $x = u_{n+1}$, $y = v_{n+1}$. This is the inductive step. The anchor had already been established, i.e., checking that $x = u_1$, $y = v_1$ is a solution.

Remark. Let us denote the solution $x = a$, $y = b$ by the pair (a, b). Then the generation of the solution $x = 2a + 3b$, $y = a + 2b$ from the solution $x = a$, $y = b$ may be written in the form of a *map* which we call f:

$$(a, b) \stackrel{f}{\longmapsto} (2a + 3b, a + 2b). \tag{6}$$

Starting with the solution $(2, 1)$, we may use f repeatedly to generate infinitely many solutions:

$$(2, 1) \stackrel{f}{\longmapsto} (7, 4) \stackrel{f}{\longmapsto} (26, 15) \stackrel{f}{\longmapsto} (97, 56) \stackrel{f}{\longmapsto} (362, 209) \stackrel{f}{\longmapsto} (1351, 780) \cdots. \tag{7}$$

The x-values of these pairs are the following:

$$2,\ 7,\ 26,\ 97,\ 362,\ 1351,\ \ldots. \tag{8}$$

These yield the following triples of consecutive integers with the property that a triangle with those three integers as side lengths has integer area (recall the rule: associated with the pair (x, y) is the triangle with sides $2x - 1$, $2x$, $2x + 1$ and area $3xy$):

$$\begin{array}{ccc}(3, 4, 5), & (13, 14, 15), & (51, 52, 53), \\ (193, 194, 195), & (723, 724, 725), & (2701, 2702, 2703).\end{array}$$

We can continue applying the rule and get indefinitely many such triples.

Remark. The sequence of x-values,

$$2,\ 7,\ 26,\ 97,\ 362,\ 1351,\ \ldots,$$

has a striking and beautiful pattern which you may have spotted:

$$\begin{aligned}26 &= 4 \times 7 - 2, \\ 97 &= 4 \times 26 - 7, \\ 362 &= 4 \times 97 - 26, \\ 1351 &= 4 \times 362 - 97,\end{aligned}$$

and so on. So if c and d are two successive x-values, with $d > c$, the next one is $4d - c$. We invite the reader to show that this pattern continues indefinitely. (Hint: Use induction.)

Answer to Question 2

Now we tackle the more ambitious question: show that the above procedure captures every solution to (2). The key to the analysis is to find a way of reversing the move from (a, b) to $(2a + 3b, a + 2b)$; in other words, to find the inverse map $f^{-1} = g$, say. Since the map f is based on multiplication by $\alpha = 2 + \sqrt{3}$, the inverse map g must involve division by that number, which is the same as multiplication by $1/\alpha$. But we have:

$$\frac{1}{\alpha} = \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3},$$

so division by $2 + \sqrt{3}$ is the same as multiplication by $2 - \sqrt{3}$. Next, observe that:

$$\left(a + b\sqrt{3}\right) \cdot \left(2 - \sqrt{3}\right) = (2a - 3b) + (2b - a)\sqrt{3}. \tag{9}$$

We infer that if (a, b) is a solution to the equation $x^2 - 3y^2 = 1$, then so is $(2a - 3b, 2b - a)$. This is easy to verify:

$$\begin{aligned}(2a-3b)^2 - 3(2b-a)^2 &= \left(4a^2 - 12ab + 9b^2\right) - 3\left(4b^2 - 4ab + a^2\right)\\ &= a^2 - 3b^2 = 1.\end{aligned}$$

We depict this map as follows:

$$(a, b) \xmapsto{g} (2a - 3b, 2b - a). \tag{10}$$

Note that f and g are a pair of inverse maps.

Now we ask: *Given that (a, b) is a positive integral solution to $a^2 - 3b^2 = 1$, under what conditions on a and b will it be true that $(2a - 3b, 2b - a)$ is strictly smaller than (a, b)? In other words, under what conditions does it happen that both the following are true?*

$$0 < 2a - 3b < a, \qquad 0 < 2b - a < b. \tag{11}$$

The two conditions are equivalent respectively to:

$$1.5b < a < 3b, \qquad b < a < 2b, \tag{12}$$

and these together imply that:

$$1.5b < a < 2b. \tag{13}$$

Now if $a^2 - 3b^2 = 1$ and $b > 1$, then we certainly have:

$$a^2 = 3b^2 + 1 < 4b^2, \qquad \therefore\ a < 2b,$$

and:

$$3b^2 = a^2 - 1 < a^2, \qquad \therefore\ 9b^2 < 3a^2 < 4a^2, \qquad \therefore\ 3b < 2a,$$

i.e., $1.5b < a$. So if $b > 1$, then $0 < 2a - 3b < a$ and $0 < 2b - a < b$. Therefore:

> *If (a, b) is a solution to $x^2 - 3y^2 = 1$ and b exceeds 1, then the solution $(2a - 3b, 2b - a)$ is* ***strictly smaller*** *than (a, b).*

For example, the rule g applied to the solution $(26, 15)$ yields $(7, 4)$, which is smaller than $(26, 15)$; and if we apply g to $(7, 4)$, we get $(2, 1)$, which is smaller still.

If we apply the rule g to $(2, 1)$, we get $(1, 0)$. Though this solution satisfies the relation $x^2 - 3y^2 = 1$, and $(1, 0)$ is certainly smaller than $(2, 1)$, we do not accept it as a solution as we want solutions in positive integers only.

Now let us start with any solution (a, b) to $x^2 - 3y^2 = 1$ with $b > 1$, and let us apply the rule g to it. As already explained, we will get a strictly smaller solution. If the second coordinate of this new solution exceeds 1, we can apply g to that solution and thus obtain a still smaller solution. And so we can continue, obtaining steadily smaller solutions.

Can this process continue forever? Clearly not; we cannot have a strictly decreasing sequence of positive integers which continues forever (this is the 'descent principle': the positive integers are bounded below, by 1; they cannot go below 1). So at some point we will be forced to a halt.

When will this happen?—precisely when the second coordinate of the solution is 1. If we ever reach that stage, it means that the first coordinate is equal to $\sqrt{3 \times 1^2 + 1} = 2$. This means that we have landed on the solution $(2, 1)$. But then the solution *prior* to that must have been $(7, 4)$, for it is the image of $(2, 1)$ under f (and therefore the preimage of $(2, 1)$ under g). And the solution prior to *that* must have been $(26, 15)$. And so on.

We infer that the solution we started with must be part of the sequence enumerated in (7), in other words, it is equal to (u_n, v_n) for some n. This is another way of saying that the list $(u_1, v_1), (u_2, v_2), (u_3, v_3), \ldots$, that is:

$$(2, 1), \quad (7, 4), \quad (26, 15), \quad (97, 56), \quad (362, 209), \quad \ldots \tag{14}$$

is a complete enumeration of the positive integral solutions of the equation $x^2 - 3y^2 = 1$.

This fully answers Question 2. We have thus succeeded in enumerating every triple of consecutive integers such that the triangle with those three integers as side lengths has integral area.

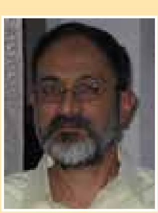
Exercises.

(1) Prove that if x, y are positive integers such that $x^2 - 3y^2 = 1$, then the triangle with side lengths $2x - 1$, $2x$ and $2x + 1$ has area $3xy$.

(2) Show that x is any member of the following sequence,

$$2, \quad 7, \quad 26, \quad 97, \quad 362, \quad 1351, \quad \ldots,$$

then the triangle with side lengths $2x - 1$, $2x$ and $2x + 1$ has integer area. The defining property of the sequence is this: its initial two terms are 2 and 7, and if a, b, c are any three consecutive members of the sequence, then $c = 4b - a$.



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