ClassRoom

Mapping Triangle Shapes

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The triangle inequality has a familiar cadence to it and most students can recite it spontaneously. In this article, we mathematise our understanding of possible triangle shapes, using the limits of values which the angles first, and then the sides, take. It's a great way for students to explore different ways of expressing their conceptual understanding.

Triangles are of different shapes. The shape of a triangle is determined by its angles, or, alternatively, by the ratios of its sides. We shall focus on the angle aspect now. To fix its shape, it is enough if two angles of a triangle are specified. So we could have a 2-dimensional 'map' where every point stands for a possible triangle shape and every possible shape is represented by a point in the map. For convenience, we could take the greatest and least angles of the triangle to be the variables. Let us denote the angles of the triangle as α, β and γ, satisfying the relation $\alpha \geq \beta \geq \gamma$. We could represent α and γ on the X-axis and Y-axis, respectively, of a plane graph.

Now α cannot be less than 60°, and γ cannot be greater than 60° (can you see why?), i.e.,

$$60^\circ \leq \alpha < 180^\circ \text{ and } 0^\circ < \gamma \leq 60^\circ.$$

(The intermediate angle β has the limits $0^\circ < \beta < 90^\circ$.)

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Thus only points in the α,γ plane lying within these limits can represent possible triangle shapes. Refer Rectangle ABCD in Figure 1.

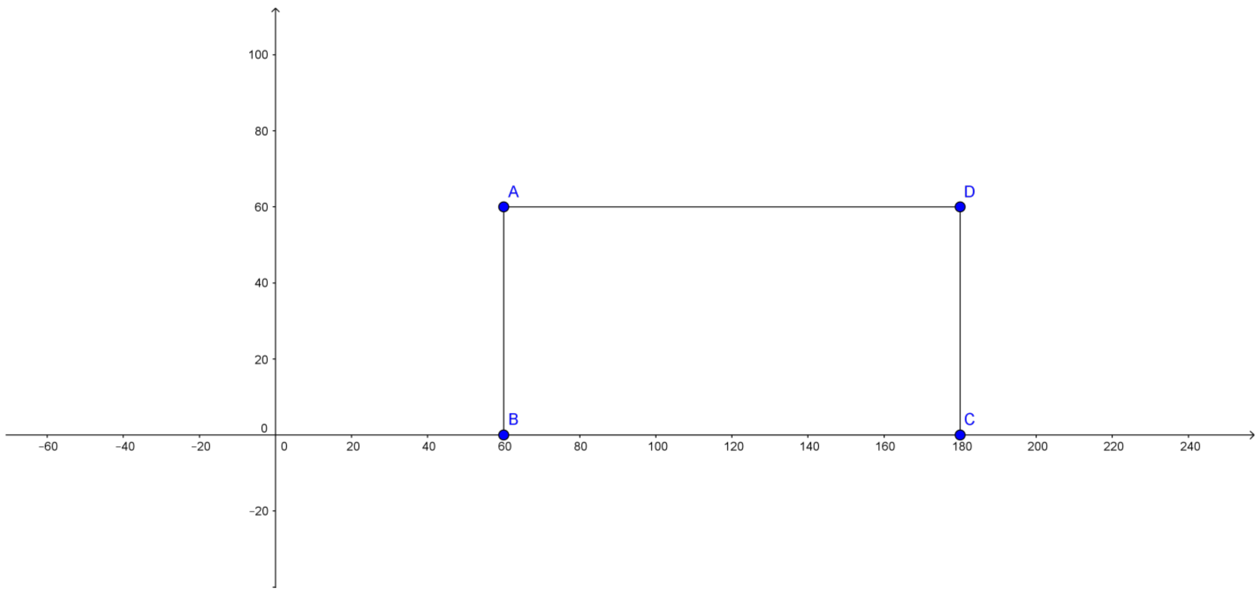

Figure 1.

Actually we have the following additional restrictions on γ for given α. The maximum value of γ for a particular value of α is given by the relation γ = (180° – α)/2 = 90° – α/2, while the minimum value is given by γ = 180° - 2α. These two relations define two straight lines in the α, γ plane, *AC* and *AE*, respectively, intersecting at p*A*nt A (see Figure 2).

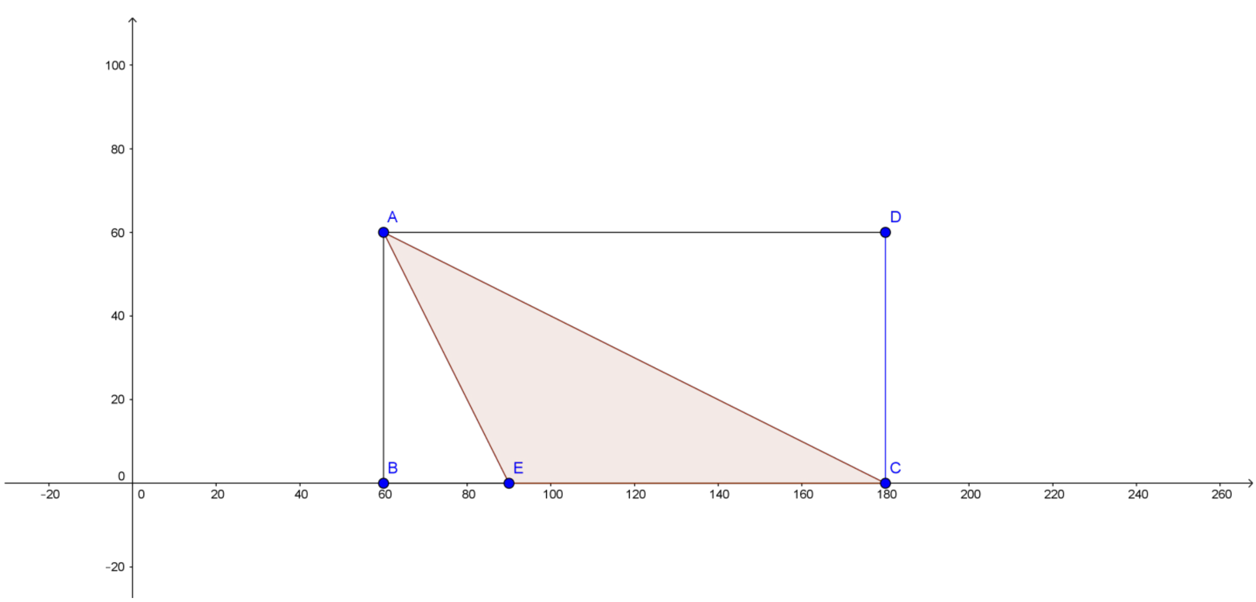

Figure 2.

So we now have the triangular region *AEC* which is the required 'map' of triangular shapes. Point *A* represents the equilateral triangle shape. Points on line segment *AC*, excluding the endpoints *A* and *C*, stand for isosceles triangles of the form $\alpha > \beta = \gamma$. Points on line segment *AE*, excluding the endpoints *A* and *E*, stand for isosceles triangles of the form $\alpha = \beta > \gamma$. Points in the interior of the triangular region *AEC* represent scalene triangle shapes.

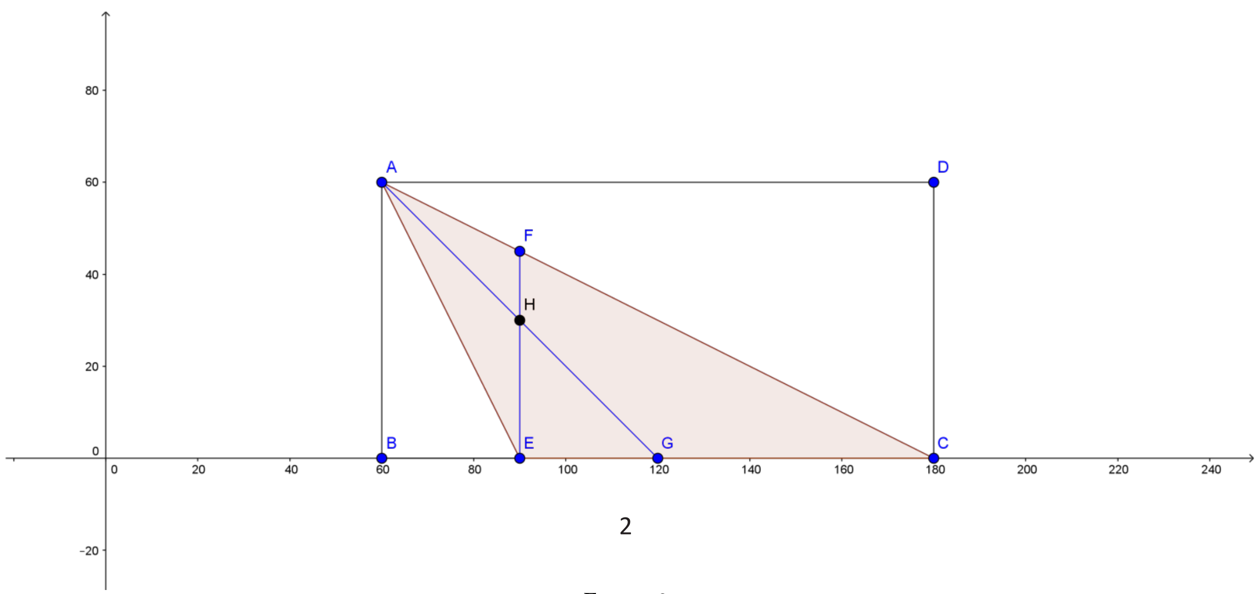

Figure 3.

In Figure 3 we see another line segment marked *EF*. Points on this line segment, excepting *E* itself, represent right-angled triangles. Point *F* itself represents the right-angled isosceles triangle (with angles 45°,45°,90°). Points in the interior of ΔAEF stand for acute-angled scalene triangles, while points within ΔFEC represent obtuse-angled scalene triangles. Also shown in Figure 3 is the line segment *AG* with a slope of -1. If we move along this line, starting from *A*, α increases while γ decreases to the same extent, leaving β unchanged. Hence points on this line segment, except *G* itself, represent triangles with angles in arithmetic progression. Point *H*, where this line intersects line *EF*, represents the 30°, 60°, 90° triangle, the only right-angled triangle with angles in arithmetic progression.

Let us now try a similar exercise taking the sides into consideration. We can take the side of intermediate length to be of unit length, the shortest of length φ and the longest of length ψ, with the proviso $\varphi \leq 1 \leq \psi$.

Since we have two variables, we can again think of a 2-D map, taking ψ on the *X*-axis and φ on the *Y*-axis. Now what are the limits on the values these can take? Clearly

$$0 < \varphi \leq 1 \quad \text{and} \quad 1 \leq \psi < 2.$$

(If ψ ≥ 2, it would be longer than the sum of the other two sides.) So our 'map' is confined to the square area *PQRS* (Figure 4).

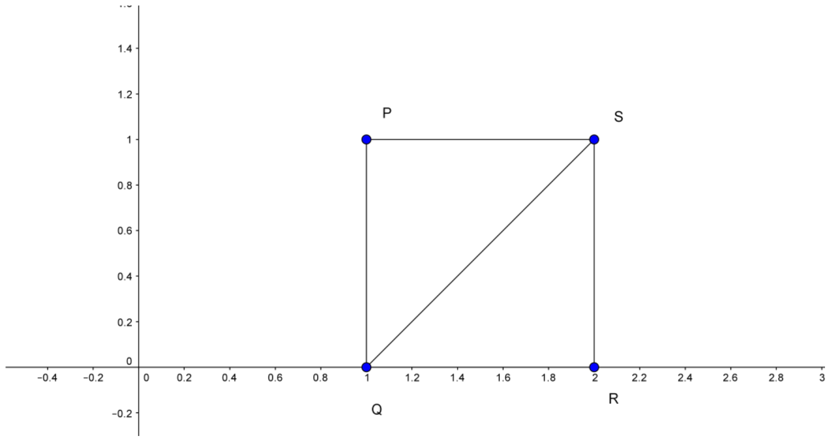

Figure 4.

Now there is a further constraint in the values ψ can take for a given φ value: ψ cannot equal or exceed φ + 1 at any point. So the line given by the equation ψ = φ + 1 or φ = ψ - 1 is a limiting line for the map (line *QS* in Figure 4). Our map is now confined to the triangular area *PQS*, excluding points on line *QS* itself.

Clearly, point *P* represents the equilateral triangle as its coordinates are ψ = 1, φ = 1. Points on the line segment *PQ*, except points *P* and *Q*, represent isosceles triangles where the unequal side is shorter than either of the equal sides. Points on the line segment *PS*, except *P* and *S*, represent isosceles triangles where the unequal side is longer than either of the equal sides. Points in the interior of Δ *PQS* stand for scalene triangles, since their ψ and φ values would be different, neither being equal to unity.

Now the question naturally arises: What about right-angled triangles? Now a right-angled triangle in our scheme would have to satisfy the condition $\psi^2 = \varphi^2 + 1$, or $\psi^2 - \varphi^2 = 1$. Now this is the equation for a hyperbola, one arm of which passes through the p*Q*ıt Q (ψ = 1, φ = 0) and intersects line *PS* at the p *T*nt T (ψ = √2, φ = 1); see Figure 5.

Needless to say, this point represents the isosceles right triangle. Points of the hyperbolic arc lying within Δ *PQS* represent other right triangle shapes.

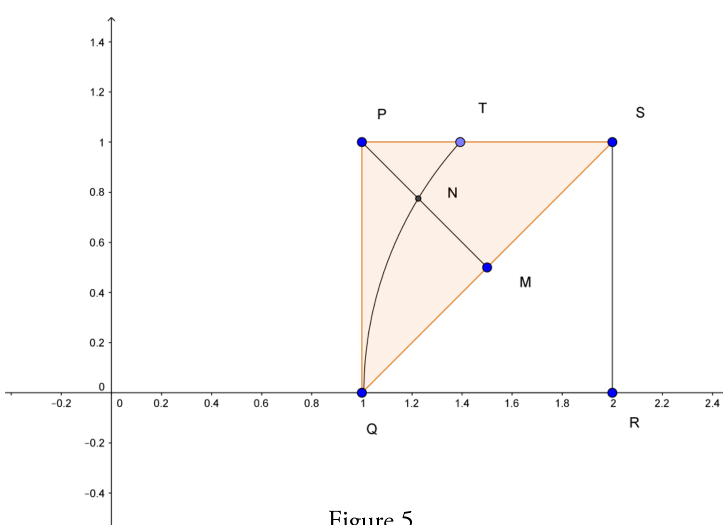

Figure 5.

Points in the interior of region *PQT* have lower ψ value and/or higher φ value compared to points on the arc *QT*. That is, they satisfy the inequality $\psi^2 < \varphi^2 + 1$, which means that the longest side faces an acute angle. So, such points stand for acute-angled scalene triangles. Similar arguments show that points in the region *TQS* stand for obtuse-angled scalene triangles.

Also shown in Figure 3 is the line segment *PM* with a slope of -1. If we move along this line starting from *P*, ψ increases, while φ decreases to the same extent, leaving the perimeter constant. Hence points on this line segment, except *M* itself, stand for triangles with same perimeter as the equilateral triangle represented by point *P*. In other words, the sides of such triangles would be in arithmetic progression, while maintaining an intermediate side length of one unit. Point *N*, where this line intersects arc *QT*, represents a right triangle with sides in A.P. As discussed in earlier articles in AtRiA, such a triangle must be a 3-4-5 triangle.

It is satisfying to see that these two approaches have resulted in 'Maps of triangle shapes' of similar structure. These maps of triangular shapes are themselves triangular. Points close to point *A* in the first case and close to point *P* in the second case represent shapes close to the equilateral triangle shape. Points close to *E* in the first case and close to *Q* in the second case represent triangles where one angle is much smaller than the other two, which are comparable, resulting in a dagger-like shape. Points close to vertex *C* in the first case and close to vertex *S* in the second case represent triangles where one angle is much larger than the other two, resulting in a bow-like shape. Points on line segment *EC* in the first case and *QS* in the second case represent triangles which have collapsed into line segments. R.I.P.

Addendum: The last observation relating an equilateral triangle to a 3-4-5 triangle can be contextualised differently. Let us say we set out to draw an ellipse, choosing as foci two points unit distance apart, and a string of length two units with ends secured at the foci. In the symmetrical position the string and base line together form an equilateral triangle. As we move the string aside, keeping it stretched, we reach a point where the string and base line form a right triangle. This triangle is a 3-4-5 triangle.

A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.