Features

The 3-4-5 Triangle: Some Observations

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In Figure 1 we see a right-angled 3-4-5 triangle ABC in which $AB = 3$, $BC = 4$ and $AC = 5$. The incircle (centre I) has been drawn; also the angle bisector AIP through vertex A, and a fourth tangent PQ to the incircle.

Since $\tan A = 4/3$, we get:

$$\tan\frac{A}{2} = \frac{\sqrt{(4/3)^2+1}-1}{4/3} = \frac{5/3-1}{4/3} = \frac{2/3}{4/3} = \frac{1}{2}.$$

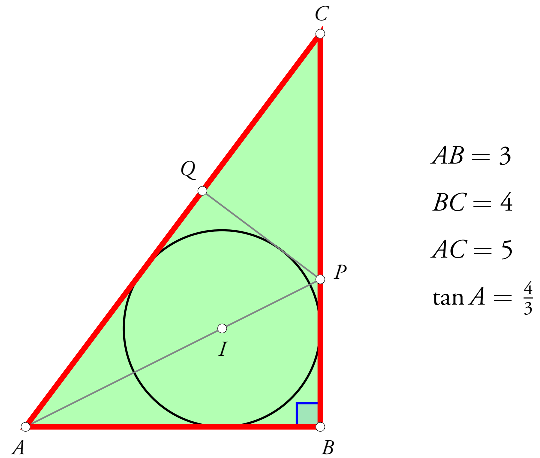

Figure 1

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Therefore $PB = 3/2$ and $CP = 5/2$. (Note: We could also have found the length of PB using the angle bisector theorem, which tells us that $CP : PB = 5 : 3$.)

Since $\triangle APQ \cong \triangle APB$ (angle-side-angle or ASA congruence; for: $\measuredangle PAQ = \measuredangle PAB$; $\measuredangle PQA = \measuredangle PBA$, both being right angles; and AP is a shared side), we have $PQ = 3/2$.

Note that $\triangle CPQ \sim \triangle CAB$, the similarity ratio being $PQ/AB = 1/2$.

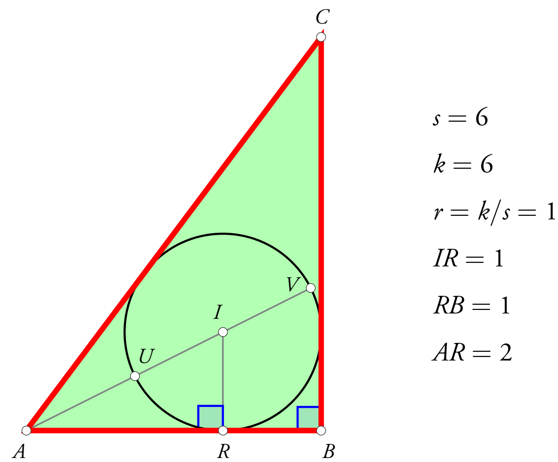

Figure 2

For any triangle, the radius r of its incircle is given by the formula $r = k/s$ where k is the area and s is the semi-perimeter of the triangle. In the case of the 3-4-5 triangle this gives an in-radius of $r = 6/6 = 1$ unit.

Let line AI intersect the incircle at points U and V as shown, and let R be the point of tangency of AB; then $IR = 1$, $RB = 1$, so $AR = 2$. But $AI = \sqrt{5}$ (by Pythagoras), so $AV = \sqrt{5} + 1$, which means that the ratio $AV : UV$ is

$$\frac{AV}{UV} = \frac{\sqrt{5}+1}{2} = \text{the Golden Ratio } \varphi.$$

We may therefore describe the point U as a *Golden Point* of AV.

Now we consider triangle PAB, where PA is the bisector of angle A. We already know that $PB = 3/2$. Draw the incircle of $\triangle PAB$; let its centre be J, and let its radius be x. Let T be the point of tangency of the circle and PB. (See Figure 3.)

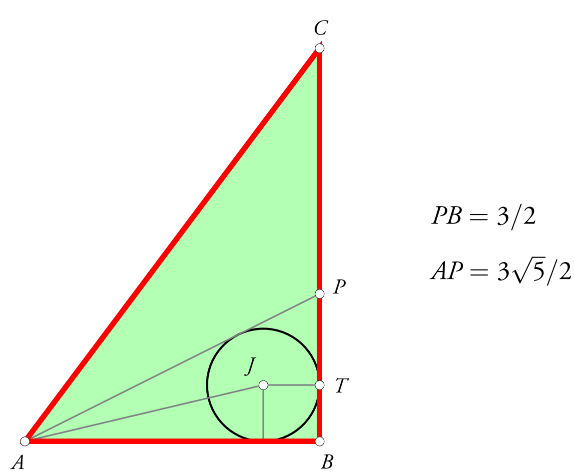

Figure 3

Since $AB = 3$ and $PB = 3/2$, we have (Pythagoras) $AP = 3\sqrt{5}/2$. Hence the semi-perimeter of $\triangle PAB$ is

$$\frac{1}{2}\left(3 + \frac{3}{2} + \frac{3\sqrt{5}}{2}\right) = \frac{3\left(3+\sqrt{5}\right)}{4}.$$

The area of $\triangle PAB$ is $1/2 \times 3 \times 3/2 = 9/4$. Hence the radius x of its incircle is

$$x = \frac{9/4}{3(3+\sqrt{5})/4} = \frac{3}{3+\sqrt{5}} = \frac{3\left(3-\sqrt{5}\right)}{4}.$$

Hence the ratio PB/PT is

$$\frac{PB}{PT} = \frac{3/2}{3/2 - 3(3-\sqrt{5})/4} = \frac{2}{2-(3-\sqrt{5})} = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}.$$

In other words, T is a Golden Point of PB.

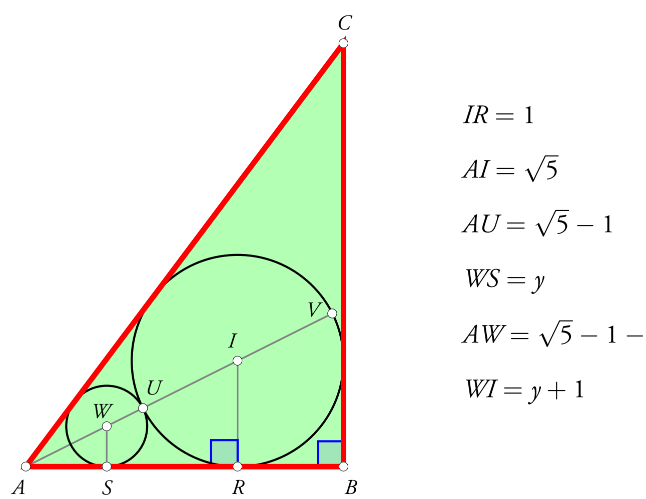

Figure 4

Let a circle be fitted into the region between A and the incircle of $\triangle ABC$, with its centre at W (see Figure 4). Let y be the radius of this small circle.

By using the properties of similar triangles, we get: $y/AW = IR/AI$, i.e.,

$$\frac{y}{\sqrt{5}-1-y} = \frac{1}{\sqrt{5}},$$

which yields:

$$y = \frac{\sqrt{5}-1}{\sqrt{5}+1} = \frac{3-\sqrt{5}}{2}.$$

Hence

$$WI = y + 1 = \frac{5-\sqrt{5}}{2},$$

and therefore

$$\frac{AI}{WI} = \frac{\sqrt{5}}{(5-\sqrt{5})/2} = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}.$$

In other words, W is a Golden Point of AI.

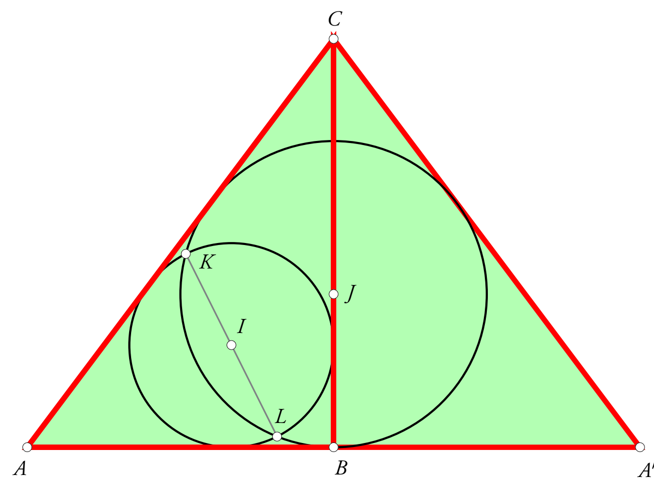

Figure 5

Consideration of similar triangles shows us that if a third circle were fitted in between A and the circle centred at W, we would get a new Golden Section, and so on.

If we "double" the 3-4-5 triangle to make an isosceles triangle (with equal sides 5 and base 6), and consider the incircles of the two triangles (see Figure 5), their common chord is a diameter of the smaller circle; i.e., the common chord KL passes through I.

To see why, let J denote the centre of the larger incircle; both I and J lie on the internal bisector of $\measuredangle BAC$. The radius of this larger incircle is

$$\frac{\text{Area of } \triangle AA'C}{\text{Semi-perimeter of } \triangle AA'C} = \frac{(6 \times 4)/2}{(5+6+5)/2} = \frac{3}{2}.$$

The radius of this larger incircle is $3/2$, while the radius of the smaller circle was earlier established as 1. That means that the radius of the large incircle is $3/2$ times that of the smaller one. Now consideration of the perpendicular from I to AB together with JB shows us that AJ and AI are in the same ratio as the radii of the larger and the smaller incircle; that is:

$$AJ = \frac{3}{2}AI, \quad \therefore \quad IJ = \frac{1}{2}AI = \frac{\sqrt{5}}{2}.$$

Now focus attention on the triangle whose vertices are K, I, J. We have:

$$KI = 1, \quad KJ = \frac{3}{2}.$$

We observe that $KJ^2 = KI^2 + IJ^2$, which indicates that $\measuredangle KIJ$ is a right angle. By symmetry, so is $\measuredangle LIJ$. Hence KL is a diameter of the smaller incircle.

This implies that points K, I, L lie in a straight line.

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