Problem corner

Problems for the MIDDLE SCHOOL

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A popular pastime for children – and adults - is doing puzzles. Here is a puzzle with a seemingly difficult solution. In how many ways can you trace this figure without lifting your hand from the paper or retracing any path?

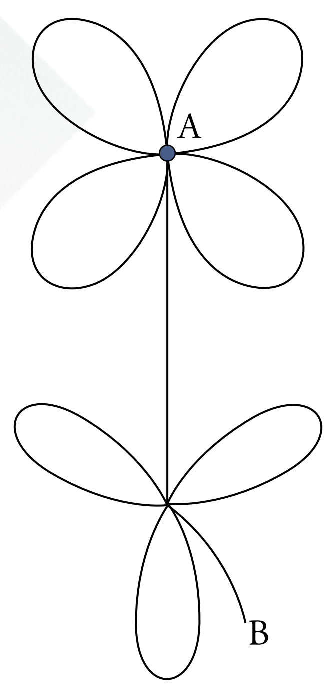

Figure 1: A counting problem
Problem Source: Dr. S. Muralidharan, TCS

Keywords: Counting, permutations, combinations.

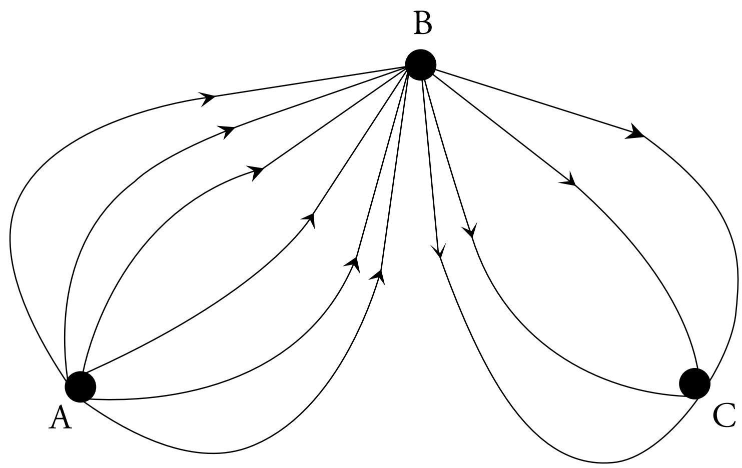

Figure 2: A simpler problem
Source: Mathematical Circles

First of all, is it possible to trace this figure without lifting your hand or retracing any path? It is easily done and can be done starting either from point A or point B. So clearly, there is more than one way of tracing the figure. And depending on the order in which the petals are traced and whether they are traced clockwise or anticlockwise, we can see the number of options increasing. How can we systematically count the number of options? We start with some simpler problems.

How many ways are there to get from A to C? (See Figure 2)

Solution: We can see that to get from A to C, one has to go via B. There are 6 ways to go from A to B and for each of these 6 ways, there are 4 ways to go from B to C. Hence, the total number of ways is $6 \times 4 = 24$. This is called the multiplication principle in counting.

Problem VI-3-M.1

How many two-digit numbers are there whose digits are both odd? How many such numbers have different digits (both odd)?

Problem VI-3-M.2
How many factors does 24 have?

Problem VI-3-M.3

How many 5 digit multiples of 5 can be formed from the digits 1, 2, 3, 4, 5?

We use exactly the same technique as in the example for all the problems. Taking up the first problem, there are 5 odd digits and so there are 5 options for the units place. Having placed one of the odd digits in the units place, there are again 5 options for the tens place. So there are totally 5×5 two-digit numbers whose digits are both odd. On the other hand, if the digits are different, there are only 4 options for the tens place and so there are totally $5 \times 4 = 20$ two-digit numbers whose digits are different from each other and are both odd. Do encourage students to create variations of this problem and strengthen their understanding of the multiplication rule.

Though the second problem was addressed in the previous issue, this is again a great illustration of the multiplication rule. $24 = 2^3 \times 3$, so the factors can be formed with 0 to 3, i.e., zero, one, two or three 2's, and 0 to 1, i.e., zero or one 3's. The total number of factors is therefore $4 \times 2 = 8$.

In problems such as the third one above, we assume that the given digits can only be used once. The first part simply has the constraint that the units digit has to be 5. Then there is 1 option for the units digit, 4 for the tens digit, 3 for the hundreds digit, 2 for the thousands digit and 1 for the ten thousands digit. This makes the total number of 5 digit multiples of 5 made from 1, 2, 3, 4, 5 equal

$$1 \times 2 \times 3 \times 4 \times 1 = 24.$$

We are now ready to tackle the flower tracing problem. Suppose we start from A. Then the 4 petals can be traced in $4 \times 3 \times 2 \times 1 = 24$ different orders. At each petal, we have the choice of tracing it clockwise or anti-clockwise. So we have 2 choices at each petal and so 24×2^4 ways to finish the first flower. Coming to the second flower, we have $3 \times 2 \times 1 = 6$ different orders to draw the petals and 2^3 choices for the direction, clockwise or anti-clockwise, for each petal. So there are 6×2^3 ways to draw the second flower and that means $24 \times 16 \times 6 \times 8 = 18432$ ways to draw the two flowers starting from A and ending at B! And twice that if you start from B and end at A!!

Problem VI-3-M.4

i. How many multiples of 5 can be formed from the same digits, i.e., 1, 2, 3, 4, 5?

ii. How many multiples of 5 can be formed from the digits 0, 2, 7, 5?

We assume that these digits cannot repeat in any of these numbers.

Now a layer of complexity is added with each question because the number of digits in the multiples of 5 is not specified.

i. With the given digits,

- There is 1 single digit multiple of 5
- There are $4 \times 1 = 4$ two digit multiples of 5
- There are $3 \times 4 \times 1 = 12$ three digit multiples of 5
- There are $2 \times 3 \times 4 \times 1 = 24$ four digit multiples of 5
- And 24 five digit multiples of 5

Now, we **add** all these options and get the total number of multiples of 5 from the digits to be 65. Notice that the operation of addition is used when we have several alternative ways in which we can comply with the conditions.

ii. In this case, the numbers can end in 0 or 5. So, we again have to add the possible options:

- Numbers ending in 0: 1 single digit + 3 two digit + 6 three digit + 6 four digit = 16
- Numbers ending in 5: 1 single digit + 2 two digit + 2 three digit (with 0 in tens place) + 2 three digit (no 0) + 2 four digit (with 0 in tens place) + 2 four digit (0 in hundreds place) = 11
- The total number of multiples of 5 made with the digits 0, 2, 7, 5 is 27.

This problem becomes simple once the options are enumerated and students should be able to see that there is no scope for double counting.

Problem VI-3-M.5
Source: Mathematical Circles

The map of a town is depicted in Figure 3. All its streets are one-way, so that you can drive only "east" or "north". How many different ways are there to reach point B starting from point A?

Systematic counting is again called for and again we start with some simpler problems.

Problem VI-3-M.6

n people meet in a room and shake hands all around. How many handshakes are there in total?

Problem VI-3-M.7

How many diagonals does a n- sided polygon have?

How do these problems differ from the previous problems? Look at the handshake problem. If there were 2 people X and Y in the room, common sense tells us that there would be only one handshake. If we go by the multiplication principle, there are 2 options for the first handshake (either X or Y to initiate the

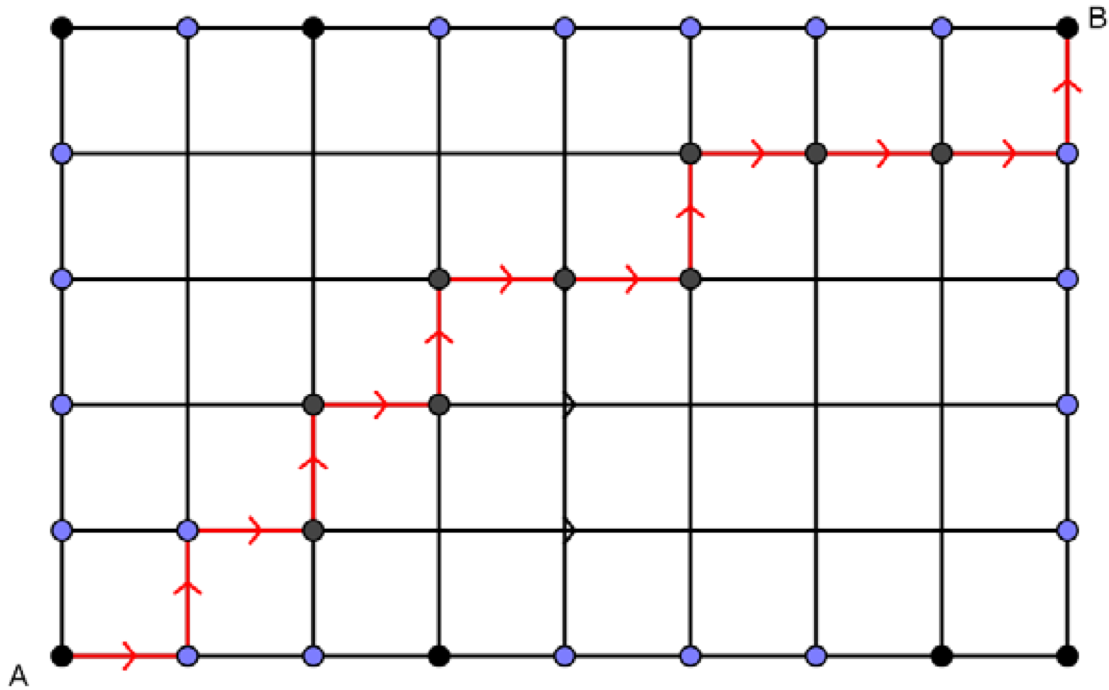

Figure 3 One possible route from A to B

handshake) and then 1 option for the second handshake. But here, a handshake between X and Y is the same as a handshake between Y and X. So we halve the number of handshakes.

Similarly, if there are 3 people X, Y and Z in the room, then the possible handshakes are: X and Y, X and Z, Y and Z. Again, 3 options for the initiator, 2 options for the receiver and 3x2 is divided by 2 to avoid replication, giving 3 handshakes in all.

What if there are four people in the room? There are 4 options for the initiator, 3 options for the receiver and we divide 4×3 by 2 to get the total. We can extend this pattern to get $5 \times 4 / 2 = 10$ handshakes if there are 5 people in the room. If there are n people in the room, the number of handshakes is $\frac{n(n-1)}{2}$.

The diagonal problem would be almost the same as the handshake problem with one difference. Now, we have to subtract the number of lines which are not diagonals but sides of the polygons. You should be easily able to see that the number of diagonals would be $\frac{n(n-1)}{2} - n$.

Problem VI-3-M.8

In how many ways can a team of 3 students be selected from a group of 5?

This problem differs slightly from the previous one because there are more duplications. The team ABC is identical to the team ACB, BAC, BCA, CAB and CBA. We easily see that the number of repetitions has 3 options for the first place, 2 for the second and 1 for the third, i.e., number of repetitions when a team of 3 is picked is $3 \times 2 \times 1 = 6$.

So the number of 3 member teams that can be constituted is $5 \times 4 \times 3$ divided by 6, i.e., 10 possible teams of 3 students each.

Problem VI-3-M.9

When 5 coins are tossed simultaneously, how many outcomes can there be?

The options are

i. 5 HEADS: 1 way

ii. 4 HEADS, 1 TAIL: Out of the 5 spots, we select 1 spot for the TAIL - can be done in 5 ways.

iii. 3 HEADS, 2 TAILS: Out of the 5 spots, we select 2 spots for the TAILs -, can be done in $\frac{5 \times 4}{2}$ = 10 ways.

iv. 2 HEADS, 3 TAILS: Out of the 5 spots, we select 3 spots for the TAILs - can be done in $\frac{5 \times 4 \times 3}{3 \times 2}$ = 10 ways.

v. 1 HEAD, 4 TAILS: Out of the 5 spots, we select 4 spots for the TAILs - can be done in $\frac{5 \times 4 \times 3 \times 2}{4 \times 3 \times 2}$ = 5 ways

vi. 5 TAILS: 1 way.

Totally, 32 ways!

Now, we are ready to address Problem VI-3-M.5.

Let each point on the grid be connected to the point on the north or the east by a street of length 1. So to traverse from A to B, we need to cover 13 streets, which run either East (E) or North (N). Also, out of these 13 streets, exactly 5 have to be N and 8 have to be E. The route shown is ENENENEENEEEN. How many such routes are there? The answer is simple- we just have to choose 5 spots out of the 13 for the Ns.

This can be done in

$$n \frac{13 \times 12 \times 11 \times 10 \times 9}{5 \times 4 \times 3 \times 2 \times 1} = 1287 \text{ ways.}$$

Students may ask if we can choose 8 spots out of 13 for the E instead, they will be delighted to find that this is equivalent to the above.

Combinatorics provides delightful problems which develop a student's skills of visualization, pattern recognition and logical thinking. Do encourage students to show their reasoning with sketches or in words, instead of just aiming for the correct answer.

Happy problem solving! Do send in interesting discoveries to AtRiA.editor@apu.edu.in

Sources

We provide a list of sources for interesting problems on combinatorics. Most of them can be attempted by students at the middle school level.

https://nrich.maths.org/public/search.php?search=permutations

https://www.khanacademy.org/math/precalculus/prob-comb/combinatorics-precalc/e/permutations_1

https://www.youtube.com/watch?v=siFBqH-LaQQ An unforgettable talk by Manjul Bhargava.

References

Shirali, S. (2002). *Adventures in Problem Solving*. India: Universities Press. Chapter 9, The Art of Counting

Fomin, D. Genkin, S. Itenberg, I. (1998). *Mathematical Circles*. India: Universities Press. Chapter 2, Combinatorics -1.