ClassRoom

What's the NEXT NUMBER?

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"What's the next number?" is an extremely popular item in so-called mental ability tests which feature in vast numbers of entrance tests in this country and elsewhere. Thus, one may be required to replace the question mark by the most suitable number in each of the sequences shown below:

(i) 8, 7, 16, 5, 32, 3, 64, 1, 128, (?)

(ii) 16, 33, 65, 131, (?), 523,

(iii) 5, 2, 17, 4, (?), 6, 47, 8, 65

These questions have been taken from the National Talent Search (First Level) & National Means-Cum-Merit Scholarship Examination, 2012.

Typically in such questions, the sequence has been generated by the paper-setter according to some pattern, and the student is expected to spot the pattern and then to find the unknown number using that pattern. Such questions make sense, given the fact that patterns are so central to mathematics as well as science, which means that the ability to spot patterns is of great value, in numerous ways. (For example, it is highly valued in a field like cryptography. Some of you may recall seeing, in the film *A Beautiful Mind*, the character played by Russell Crowe (John Nash) displaying an uncanny pattern-spotting ability.)

However, there is an interesting twist to this tale. The underlying question is this: given the initial (say) five terms of a sequence, can we say with any degree of certainty what the next term must be? Let's say we have found a nice pattern in the given initial portion; can we be sure that the sequence has been generated with just that pattern in mind?

Keywords: sequence, pattern, mental ability test, prediction, ambiguity

For example, suppose the first five terms of a sequence $\{f(n)\}_{n\geq 1}$ are

$$1,\ 2,\ 3,\ 4,\ 5$$

and we are asked to guess the sixth term. This looks like a very simple sequence. It *appears* that $f(n) = n$ for all n; if so, it implies that $f(6) = 6$. But is this the only solution possible? Or can it happen that there are multiple patterns possible which match the initial portion of the sequence (i.e., the given terms)? If multiple patterns are possible, then there is no logically justifiable way of predicting the next term. We shall show that this is the case. Indeed, *we shall show that the sixth term can be any number whatsoever!*

Here is a simple way of showing this. Let k be any non-zero number. Consider the function f given by the following expression:

$$f(n) = n + k(n-1)(n-2)(n-3)(n-4)(n-5).$$

The expression $k(n-1)(n-2)(n-3)(n-4)(n-5)$ takes the value 0 when n assumes any of the values $1, 2, 3, 4, 5$. This is true regardless of the value of k. Consequently, for $n = 1, 2, 3, 4, 5$, we have $f(n) = n$. But for $n \neq 1, 2, 3, 4, 5$ we have $f(n) \neq n$, since $k(n-1)(n-2)(n-3)(n-4)(n-5) \neq 0$. The discrepancy between $f(n)$ and n for $n \geq 6$ can be made arbitrarily large by choosing k appropriately. For example, if we take $k = 1$, we get:

$$f(6) = 126, \quad f(7) = 727, \quad f(8) = 2528, \quad \ldots;$$

and if we take $k = 2$, we get:

$$f(6) = 246, \quad f(7) = 1447, \quad f(8) = 5048, \quad \ldots.$$

These values may be compared with the predictions $f(6) = 6, f(7) = 7, f(8) = 8$ which we get if we assume that $f(n) = n$.

Or we may let the function f take the following form:

$$f(n) = n + g(n)(n-1)(n-2)(n-3)(n-4)(n-5),$$

where g is an arbitrary function. It should be clear that by tweaking the expression appropriately, we can arrange for the sixth term to be any number whatsoever.

What this tells us is that if we are given some initial terms of a sequence, there is no logical way of predicting the next term. Indeed, *the next term can be any number whatsoever*. This is so, no matter how striking the pattern which appears to govern the initial terms.

However, there is another way in which the problem can be posed. We can ask: *Given the first few terms of a sequence, what is* ***most likely*** *to be the next term?* Or: *Given the first few terms of a sequence, what is* ***most likely*** *to be the generating formula of the sequence?* It is meaningful to use words like "most likely" only if we assume that the sequence has been generated by some simple rule or pattern. Note that we are asking a conditional question now; *we are imposing a condition of simplicity on the given situation*; we are assuming that the maker of the sequence is a simple person, not inclined to act in a devious manner! With this proviso, it is reasonable to claim that if the first five numbers in a sequence are $1, 2, 3, 4, 5$, then the most likely next number is 6, and the most likely generating formula is: the n^{th} term is equal to n.

The same point of view can arise in another way. It often happens in mathematics and the sciences that it is the simplest function that fits the given data which turns out to be the most satisfactory one. (Not always, but often enough for us to wonder at it.) Or if not the simplest, then a function that is "reasonably simple." More often than not, it happens that nature opts for something simple and elegant. Many great stories can be told around this theme if one dips into the history of science.

The best-known such story is perhaps the one concerning the structure of the solar system. We narrate it very briefly here. To early man, it would have seemed self-evident that we lie at the centre of the universe, with all the heavenly bodies circling around us in geometrically perfect orbits. (Our everyday experience and observation certainly support this view.) During the Greek era, this became formalised as the ***geocentric model***. Now the strength of any model lies in its predictive ability and its ability to account for newly observed phenomena. (Indeed, this is the very purpose of having a model.) In the case of the geocentric model, observers noticed soon enough that there were discrepancies between what this simple model suggests and what is actually seen. To account for this, the model was modified by introducing *epicycles*. Over the centuries, more discrepancies began to be observed, and the response was to introduce more epicycles: more adjustments. This process iteratively continued, until finally an extremely complicated model was obtained: epicycles upon epicycles upon epicycles! And then all of a sudden, late in the 16th century, a new theory emerged—the ***heliocentric theory***. In contrast to the epicycles, it was a very much simpler model, and it explained the observed phenomena beautifully. This model has, of course, survived to the present day.

This account has been extremely brief; perhaps much too brief! We will say more in a future article on this theme, and also showcase more such episodes from the history of science. Hang on for those stories!

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.