Low Floor High Ceiling Tasks

SUMS OF CONSECUTIVE NATURAL NUMBERS

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We continue our Low Floor High Ceiling series in which an activity is chosen – it starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that students are pushed to their limits as they attempt their work. There is enough work for all, but as the level gets higher, fewer students are able to complete the tasks. The point however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

As we developed this series, we realised that most of our activities began with an investigation. Mathematical investigation refers to the sustained exploration of a mathematical situation. It distinguishes itself from problem solving because it is open-ended.

In mathematical investigations, students are expected to pose their own problems after initial exploration of the mathematical situation. The exploration of the situation, the formulation of problems and their solution give opportunity for the development of independent mathematical thinking, and in engaging in mathematical processes such as organizing and recording data, pattern searching, conjecturing, inferring, justifying and explaining conjectures and generalizations. It is these thinking processes which enable an individual to learn more mathematics, apply mathematics in other disciplines and in everyday situations and to solve mathematical (and non-mathematical) problems. Teaching anchored on mathematical investigation allows for students to learn about mathematics, especially the nature of mathematical activity and thinking. It also makes them realize that learning mathematics involves intuition, systematic exploration, conjecturing, reasoning, etc., and that it is not about memorizing and following existing procedures.

Keywords: numbers, consecutive, sum, pattern, digits

While we have developed questions based on the investigation we carried out, we would urge you to get your students to engage in their own investigations. The following questions may help:

- What did I observe?
- What did I know?
- What did I discover?
- What was challenging?
- Can I check this in another way?
- How many solutions?
- What happens if I change ……….?
- What else did/can I learn from this?

Here are our questions for this investigation on the Sum of Consecutive Natural Numbers. As usual, they go from Low Floor to High Ceiling:

1. Investigate the numbers from 1 to 50 to pick out numbers which can be written as the sums of a series of consecutive natural numbers. For example, 3 = 1 + 2; 12 = 3 + 4 + 5, and so on.

2. Are there numbers which can be written as sums of two or more consecutive natural numbers in more than one way, for example, can the same number be written as the sum of two or three consecutive natural numbers?

3. Find a pattern for numbers which:
   i. Can always be written as a sum of two consecutive numbers
   ii. Can always be written as a sum of three consecutive numbers

4. If we add (2n + 1) consecutive natural numbers and the sum is N, investigate factors of N.

5. If we add (2n + 2) consecutive natural numbers and the sum is N, investigate factors of N.

6. Based on your investigations, can you generalise about the kinds of numbers which can be written as a sum of two or more consecutive natural numbers?

7. Given such a number, investigate in how many ways it can be written as a sum of two or more consecutive natural numbers.

8. Which numbers cannot be written as the sum of two or more consecutive natural numbers?

Additional question: Predict the number of terms in any SCNN given N and its odd factor 2n + 1.

Sums of Consecutive Natural Numbers (SCNN)
This is a great example of how visual representation can give insight into proofs.

The number 18 can be written as the sum of two or more consecutive natural numbers (SCNN) in two ways:

$$3 + 4 + 5 + 6 \text{ and } 5 + 6 + 7$$

SCNN can be illustrated by columns of tiles as shown in Figure 1. Investigate the numbers from 1 to 50 to determine which numbers can or cannot be written as SCNN

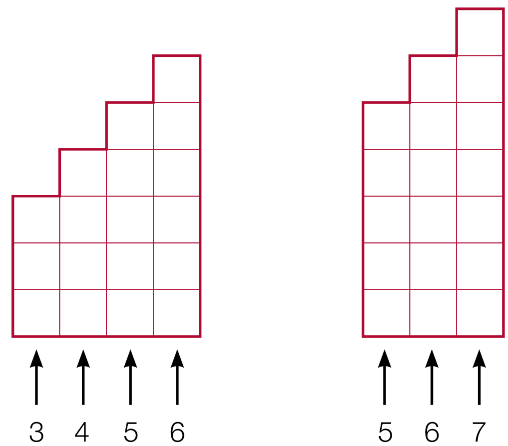

Figure 1 from ref. 1

i. When we add two consecutive numbers, we get an odd number.
   n + (n+1) = 2n+1. The converse statement: any odd number ≥ 3 can be written as the sum of two consecutive natural numbers.

ii. When we add three consecutive natural numbers, we get a multiple of 3.

$$(n - 1) + n + (n + 1) = 3n$$

The converse statement: any multiple of 3 which is ≥ 6 can be written as a sum of three consecutive natural numbers. Putting (i) and (ii) together, we see that an odd multiple of 3 can be written both as the sum of two consecutive natural numbers and as the sum of three consecutive natural numbers.

More interestingly, the left hand side of the algebraic representation of (n-1) + n + (n + 1) = 3n gives us an intriguing line of thought to pursue. Note that the -1 in (n – 1) is compensated for by the +1 in (n + 1).

Now, this can be easily extended if there are 5, 7, 9 ……..in fact, for any odd number of numbers.

Algebraically, suppose there are an odd number, (say 2n + 1), of natural numbers which are added, then this can be written as (m – n) + … + (m – 2) + (m – 1) + m + (m + 1) + (m + 2) + … + (m + n); then every number that is added to the right of m is subtracted to the left of m, so that the figure can be rearranged as in Figure 2, to get a m × (2n + 1) rectangle.

Figure 2 illustrates this for n = 3 and m = 6.

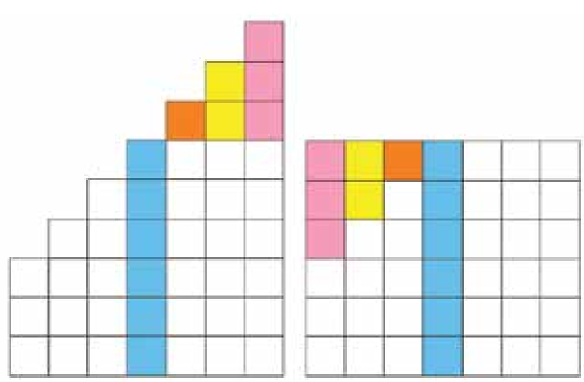

Figure 2

Apart from the neatness of this rearrangement, notice that the sum of 2n + 1 consecutive natural numbers is divisible by the odd number 2n + 1. For example, when we add 5 consecutive numbers say, 8 + 9 + 10 + 11 + 12, the sum is divisible by 5.

What if we add an even number of consecutive natural numbers?

Algebraically, suppose there are an even number, (say 2n + 2), of natural numbers which are added, then this can be written as

$$(m - n) + \dots + (m - 2) + (m - 1) + m + (m + 1) + (m + 2) + \dots + (m + n) + (m + n + 1)$$

Now, pairing the terms from the ends, then the next two, ... finally the middle two, we get:

$$\begin{aligned} (m - n) + (m + n + 1) &= 2m + 1 \\ (m - n + 1) + (m + n) &= 2m + 1 \\ &\vdots \\ (m - 1) + (m + 2) &= 2m + 1 \\ m + (m + 1) &= 2m + 1 \end{aligned}$$

and there are (n + 1) such pairs, so that

$$(m - n) + \dots + (m - 2) + (m - 1) + m + (m + 1) + (m + 2) + \dots + (m + n) + (m + n + 1) = (n + 1) \times (2m + 1)$$

Figure 3 illustrates this for n = 2 and m = 6.

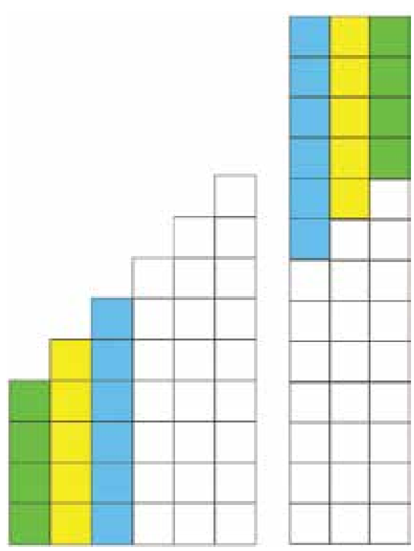

Figure 3

Notice that when 2n + 2 consecutive natural numbers are added, the sum is divisible by (n + 1). Not just that, the sum is divisible by 2m + 1, an odd number. For example, when we add 8 consecutive numbers say, 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11, the sum is divisible by 4 and by 15 (which is 2 × 7 + 1). The rearranged rectangular array has a height of 15 since 4 + 11 = 5 + 10 = 6 + 9 = 7 + 8 = 15. In other words the height of the column is (m – n) + (m + n +1) = 2m + 1, an odd number.

Summing up these two findings we see that any sum of two or more consecutive natural numbers always has an odd factor.

Now, we come to a very interesting question: If a number N has an odd factor, then can it be written as a sum of two or more consecutive natural numbers?

Find any odd factor 2n + 1 of the given number N and create an array of N = m × (2n + 1) tiles. If we can rearrange the tiles to form steps then N can be written as a sum of two or more consecutive natural numbers. Further exploration with numbers yields that this can be done in mainly two ways:

a. When m > n:

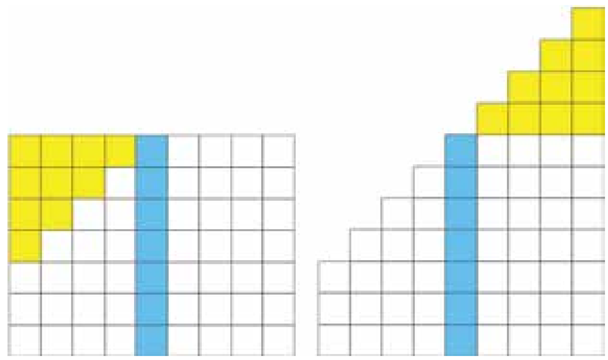

Figure 4

From the (2n + 1) columns, take the 1st n columns (of length m each) and cut off steps n, (n – 1) … 2, 1. Rotate these steps by 180° and place them over the last n columns. This generates the sum of 2n + 1 consecutive natural numbers m – n, m – n + 1, … m + n.

Figure 4 depicts the case m = 7 and n = 4.

b. When m < n:

It should be clear why we can't use the above method.

Cut off n, (n – 1) … (n – (m – 1)) steps from the left end of the array. Rotate by 180° and place them below the remaining rows to get the sum of (n – m + 1) + … + n + (n + 1) + ... + (2n + 1 – (n – m + 1)), i.e., (n – m + 1) + … + n + (n + 1) + ... + (n + m).

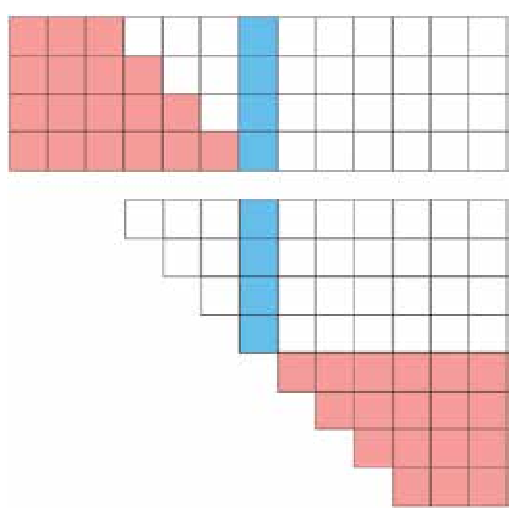

Figure 5

We leave it for the reader to explore what happens when m = n and, in particular, what kind of number N turns out to be in that case.

Given such a number N which has at least one odd factor, in how many ways can it be written as a SCNN?

If $N = 2^a \times p_1^{b_1} \times \ldots \times p_k^{b_k}$ where $p_1, \ldots, p_k$ are odd primes, then there will be $(b_1 + 1), \ldots (b_k + 1) - 1$ odd factors > 1, i.e., $(b_1 + 1) \ldots (b_k + 1) - 1$ possible SCNN

E.g. $N = 360 = 2^3 \times 3^2 \times 5$ has $(2 + 1)(1 + 1) - 1 = 5$ odd factors, viz. 3, 5, 9, 15, 45

$2n + 1 = 3 \Rightarrow m = 360 \div 3 = 120 \Rightarrow N = 119 + 120 + 121$

$2n + 1 = 5 \Rightarrow m = 360 \div 5 = 72 \Rightarrow N = 70 + 71 + 72 + 73 + 74$

$2n + 1 = 9 \Rightarrow m = 360 \div 9 = 40 \Rightarrow N = 36 + 37 + 38 + 39 + 40 + 41 + 42 + 43 + 44$

$2n + 1 = 15 \Rightarrow m = 360 \div 15 = 24 \Rightarrow N = 17 + \ldots + 24 + \ldots + 31$

$2n + 1 = 45 \Rightarrow m = 360 \div 45 = 8 \Rightarrow N = 15 + \ldots + 22 + 23 + \ldots + 30$

It will be a good idea to try different numbers and observe that that there is a unique SCNN for each odd factor.

The reader is advised to predict the number of terms in any SCNN, given N and its odd factor 2n + 1.

Claim: One can explore and see that if the number of terms is divisible by 2^n, then the SCNN will be divisible by 2^{n-1}. From the above, we see that any number which has an odd factor, can be written as a sum of two or more consecutive natural numbers. So which numbers cannot be written as the sum of two or more consecutive natural numbers?

Any number without even one odd factor must be a power of 2. So, the only numbers which cannot be written as the sum of two or more consecutive natural numbers are numbers of the form $2^n \; \forall \; n \in \mathbb{N}$.

We invite responses from our readers for the additional question: Predict the number of terms in any SCNN, given N and its odd factor 2n + 1.

Reference:

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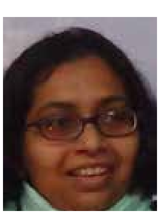

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