ProblemCroner

TWO Problem Studies

In this edition of Adventures in Problem Solving, we continue the theme of studying two problems in detail. Problem 1 is adapted from Problem 6 of the Regional Mathematics Olympiad (India) held in October 2016, and Problem 2 is from the Tournament of the Towns which we had featured in the previous issue (see https://en.wikipedia.org/wiki/Tournament\of\the\Towns). The problem studied here appeared in 1997; it was also featured in an Olympiad in South Africa. As usual, we state the problems first, so that you have an opportunity to tackle them before seeing the solutions.

Problem 1

ABC is an equilateral triangle (Figure 1). Points $P_1, P_2, \ldots, P_{10}$ are taken on side BC, in that order, dividing that side into 11 equal parts. Similarly, points $Q_1, Q_2, \ldots, Q_{10}$ are taken on side CA, in that order, dividing that side into 11 equal parts, and points $R_1, R_2, \ldots, R_{10}$ are taken on side AB, in that order, dividing that side into 11 equal parts. Count the number of triangles $P_iQ_jR_k$ such that the centroid of $\triangle P_iQ_jR_k$ coincides with the centroid of $\triangle ABC$.

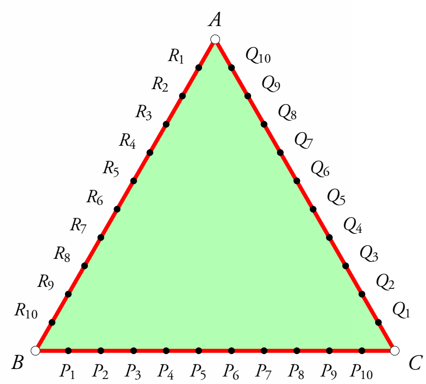


Figure 1

Problem 2

$ABCD$ is a square (Figure 2), and K is any point on side BC. The internal bisector of $\measuredangle KAD$ cuts side CD in point M. Show that $AK = BK + DM$.

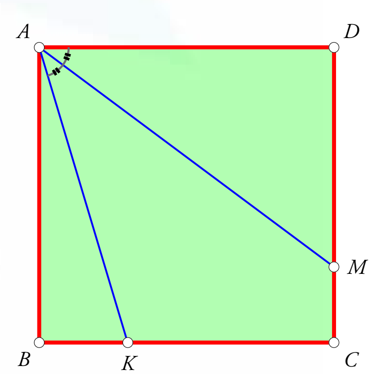


Figure 2

Solution to Problem 1.
This problem is easier than it looks. Also, the answer is what you would intuitively expect!

We consider a more general question. *Suppose that P, Q, R are points on the sides BC, CA, AB respectively of a given $\triangle ABC$. Suppose further that the centroid of $\triangle PQR$ coincides with the centroid of $\triangle ABC$. What, if anything, can be said about the placement of P, Q, R on the respective sides?*

Let us regard A as the origin of the coordinate system. Since P, Q, R lie on sides BC, CA, AB respectively, real numbers u, v, w exist such that (here we are using the name of each point to also denote its coordinates):

$$P = uB + (1-u)C, \quad Q = vC + (1-v)A, \quad R = wA + (1-w)B. \tag{1}$$

Since $A = 0$, this yields:

$$P = uB + (1-u)C, \quad Q = vC, \quad R = (1-w)B. \tag{2}$$

As the centroids of the two triangles coincide, we have

$$\frac{A+B+C}{3} = \frac{P+Q+R}{3}, \tag{3}$$

which yields:

$$B + C = uB + (1-u)C + vC + (1-w)B. \tag{4}$$

This yields, on simplifying:

$$(w-u)B = (v-u)C. \tag{5}$$

Now $(w-u)B$ is a point on ray $\overrightarrow{AB}$, while $(v-u)C$ is a point on ray $\overrightarrow{AC}$. The only point shared by these two rays is A. Hence $w-u=0$ and $v-u=0$, which yields $u=v=w$. Therefore: *P, Q, R divide the sides BC, CA, AB in the same ratio.*

Applying this result to the RMO problem, we see that $\triangle P_iQ_jR_k$ has the same centroid as $\triangle ABC$ if and only if $i=j=k$; i.e., if and only if the triangle is of the form $P_iQ_iR_i$. It follows that there are 10 such triangles. □

Remark. The shape of $\triangle ABC$ has absolutely nothing to do with the problem! So the information that the given triangle is equilateral is of no relevance. The same result would be true regardless of what shape the triangle has.

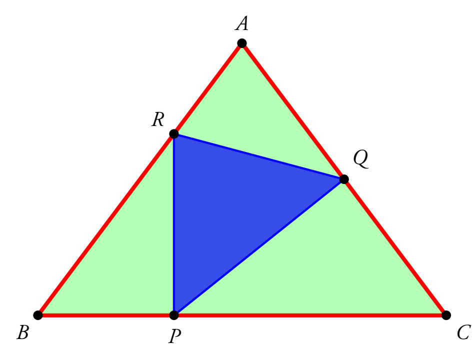


Figure 3

Solution to Problem 2.
We offer three different solutions to this problem! The first one, which uses trigonometry, is perhaps the most straightforward.

Trigonometric solution. Denote $\measuredangle KAM$ by x; then $\measuredangle MAD = x$ as well, and $\measuredangle BAK = 90 - 2x$ (see Figure 4). Let the side of square $ABCD$ have length a units.

We now have:

$$AK = \frac{AB}{\cos(90-2x)} = \frac{a}{\sin 2x},$$

$$BK = a\,\tan(90-2x) = a \cdot \frac{\cos 2x}{\sin 2x},$$

$$DM = a\,\tan x.$$

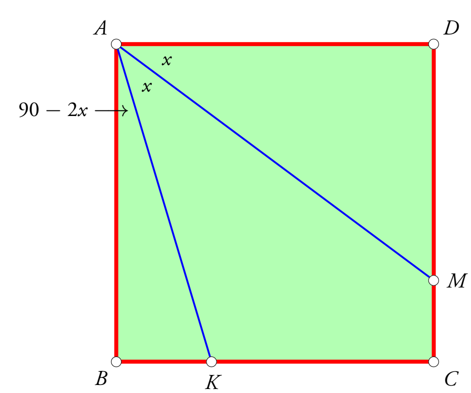


Figure 4

Hence the relation to be proved is the equivalent to the following trigonometric identity:

$$\frac{1}{\sin 2x} = \frac{\cos 2x}{\sin 2x} + \tan x.$$

This in turn is equivalent to the following:

$$\frac{1-\cos 2x}{\sin 2x} = \tan x,$$

$$\text{i.e.,}\quad \frac{2\,\sin^2 x}{2\,\sin x\,\cos x} = \tan x.$$

But this is immediate! Hence the stated relation is true. □

Solution using coordinates. Our second solution makes use of coordinates. Let the side of the square have length 1 unit. Assign coordinates as shown to the right of the figure (see Figure 5).

The slopes of lines AK and AM are respectively:

$$\text{slope of } AK = \frac{0-1}{k-0} = -\frac{1}{k},$$

$$\text{slope of } AM = \frac{m-1}{1-0} = m-1.$$

Since the slope of AD is 0, we get (using the formula for the tangent of the angle between two straight lines):

$$\frac{(m-1)-(-1/k)}{1+(m-1)(-1/k)} = \frac{0-(m-1)}{1+0\cdot(m-1)},$$

which reduces to:

$$\frac{k(m-1)+1}{k-m+1} = -(m-1).$$

This yields an expression for k in terms of m:

$$2k(m-1) = (m-1)^2 - 1,$$

$$\therefore\quad k = \frac{m(2-m)}{2(1-m)}.$$

This is the condition relating k and m, if AM is to be the bisector of $\measuredangle KAD$. Now we must find what condition relates k and m if the condition $AK = BK + DM$ is to hold. We have:

$$AK = \sqrt{k^2+1},$$

$$BK = k,$$

$$DM = 1-m.$$

The condition $AK = BK + DM$ is equivalent to $AK^2 = BK^2 + DM^2 + 2\,BK\cdot DM$, i.e.,

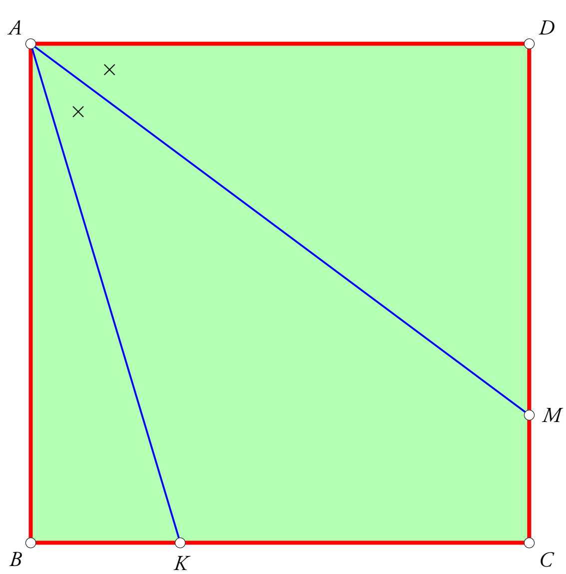
$$k^2 + 1 = k^2 + m^2 - 2m + 1 + 2k(1-m),$$

which simplifies to

$$2k(1-m) = 2m - m^2,$$

$$\text{i.e.,}\quad k = \frac{m(2-m)}{2(1-m)}.$$

Observe that we have obtained exactly the same condition as the one obtained earlier. We conclude that $AK = BK + DM$ if and only if AM bisects $\measuredangle KAD$. □



$A = (0,1),\ B = (0,0)$

$C = (1,0),\ D = (1,1)$

$K = (k,0)$

$M = (1,m)$

$\measuredangle KAM = \measuredangle MAD$

Figure 5

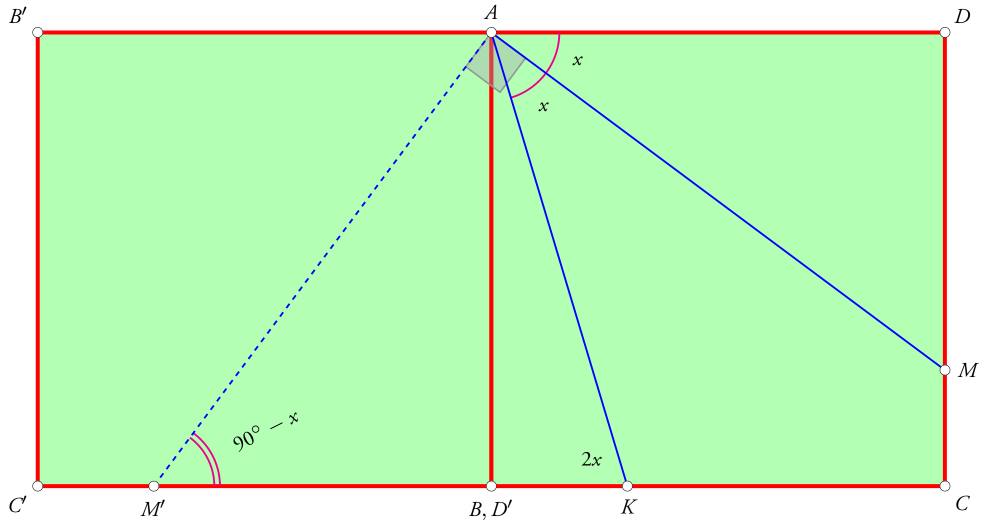


Figure 6

Pure geometry solution. We close by offering a third solution: using pure geometry. As always, such solutions are difficult to find but look extremely easy in hindsight. But if we are able to find such a solution, it constitutes an extremely pleasing discovery; for such solutions are also very elegant. The construction is shown in Figure 6.

We subject the entire configuration (i.e., square $ABCD$ together with segments AK and AM) to a 90° rotation centred at A and oriented in the clockwise direction. (Here we are assuming that the vertices of $ABCD$ have been labelled in a counterclockwise direction.) The rotation takes vertices B, C, D, K, M to B', C', D', K', M' respectively. Note that D' coincides with B. Hence $M'B = MD$. Denote $\measuredangle KAM$ by x, as in the trigonometric solution. An easy computation now reveals that $\measuredangle AKB = 2x$, and $\measuredangle M'AK = 90^\circ - x$. This implies that $\measuredangle KM'A = 90^\circ - x$ as well. Since $\measuredangle KM'A = \measuredangle M'AK$, we must have $AK = M'K$. Hence $AK = BK + DM$. □

COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.