ClassRoom

CONGRUENCY AND CONSTRUCTIBILITY IN TRIANGLES

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Every criterion of congruency in triangles is linked to a method of constructing a unique triangle from given data, using ruler, compass and protractor. Typically we first draw a 'base' and then draw rays or arcs from its ends fitting the given data. The point of intersection of these is the third vertex of the required triangle.

For instance, take the 'SSS' rule. If we are given the three side lengths in a triangle we can construct a unique triangle with those side lengths. Of course, they have to comply with the basic existence requirements for a triangle – that the sum of any two sides exceeds the third side, and that the difference of any two sides is less than the third side.

Let us see what happens if the first condition is not fulfilled – i.e., taking a, b, c to be the sides, let us assume that $b + c < a$. Draw a base of length a. Taking its two ends to be the centres, draw circles of radii b and c, respectively. Note that the circles do not intersect at all, so that no third vertex can be located (Figure. 1). If $b + c = a$, the circles just touch each other and the vertices of the triangle are collinear. We get a flat triangle, with angles of 180°, 0° and 0° respectively. Triangles of this kind are called "degenerate" by mathematicians.

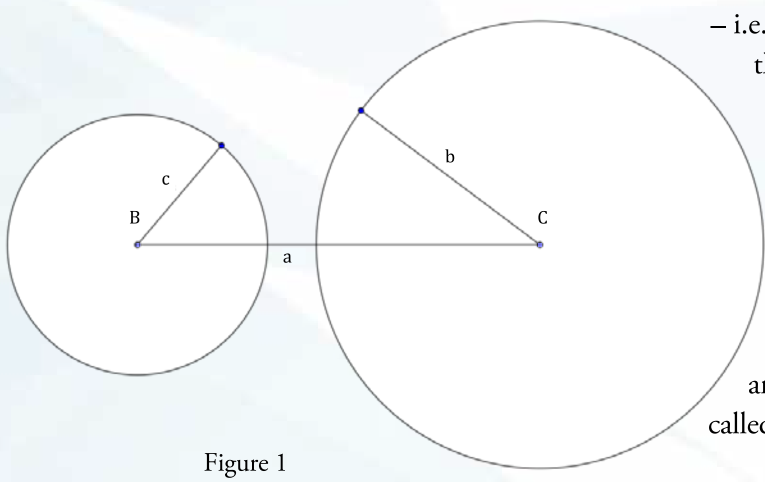


Figure 1

Keywords: Constructibility, congruency, triangle, inequality

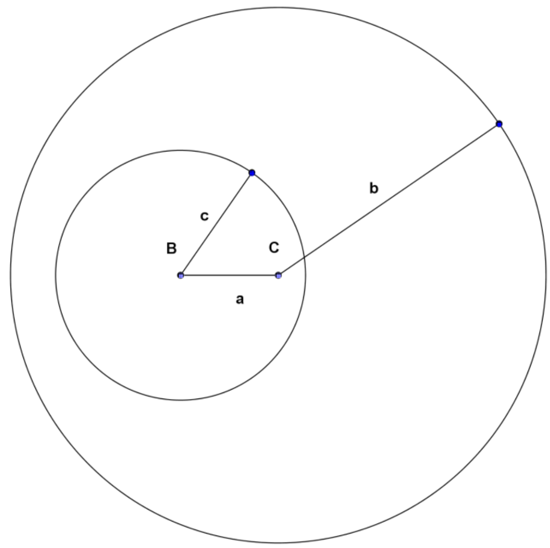


Figure 2

What happens if the second condition is not met? Again, let us take *a*, *b*, *c* to be the sides, and assume that $b - c > a$. As before, draw a base of length *a*, and taking its ends as centres draw circles of radii *b* and *c*. Now one of the circles is fully inside the other and so no third vertex can be located (Figure. 2). If $b - c = a$, the circles touch each other internally, yielding a third vertex that is collinear with the other two. This possibility too yields a degenerate triangle.

In the case of the 'SAS' rule, there is no restriction on the data except that the given angle must be less than 180°. In the case of the 'ASA' rule too, the only restriction is that the given angles add up to less than 180°. Of course it must be specified to which angle the given side is opposite.

The RHS rule is similar to the SAS rule in that two sides and an angle are given. However the given angle is not the angle included between the given sides, and it is specified to be a right angle. Also, the hypotenuse needs to be longer than the other given side. Other than these, no restrictions need to be placed on the data.

Can there be other such rules of congruency/constructibility where the restriction on the angle being 'included' is relaxed, but other conditions are imposed?

Given the following data, it is possible to construct a unique triangle.

In ΔABC, $\measuredangle A = x > 90°$, $AB = c\ cm$, $BC = a\ cm$, $a > c$.

We first draw a line segment AB of length *c*. We then draw a ray from A making angle *x* with AB. Keeping the point of a compass opened to radius *a* on B, we draw an arc to cut the above ray at C (Figure. 3).

Note that the arc would also cut ray CA extended backwards to yield another triangle, but then ∡ A would not be obtuse.

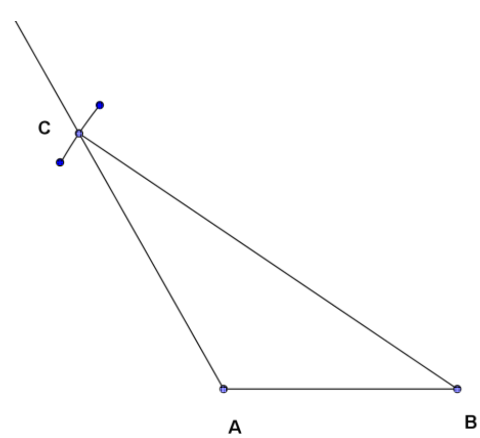


Figure 3

Given the following data, it is possible to construct a unique triangle.

In ΔPQR, $\measuredangle P = x < 90°$, $PQ = r\ cm$, $QR = p\ cm$, $p \geq r$.

We first draw a line segment PQ of length r. We then draw a ray from P making an angle x with PQ. Keeping the point of a compass opened to radius p on Q, we draw an arc to cut the above ray at R (Figure. 4).

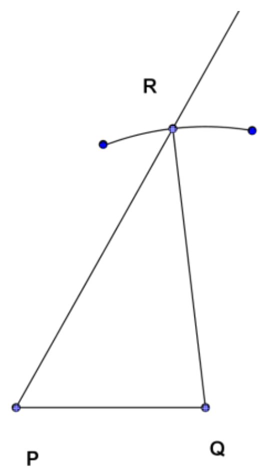


Figure 4

Two cases arise:

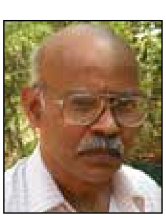
- $p = r$: In this case, the arc from Q cuts the ray from P at point R, yielding an isosceles triangle;
- $p > r$: In this case, the arc from Q cuts the ray from P at a point R 'further downstream'. It would also cut the ray extended backwards to yield another triangle, but then $\measuredangle$P would not be acute. Note that the resulting triangle need not be an acute angled triangle. The angle formed at Q could be acute, right or obtuse, depending on the data given.

Though the case $p < r$ is not under consideration here, we could explore what happens in such a case. There are three possibilities. If $p > r \sin x$, the arc cuts the ray at two places, which means there are two different triangles fitting the given data. If $p < r \sin x$, the arc does not cut the ray at all. If $p = r \sin x$, the arc is tangent to the ray, with a single point of contact.

Note that in both the above situations, we are given two side lengths of a triangle and the magnitude of an angle not included between those sides, but additional constraints have been imposed. We are able to construct unique triangles in both cases.

The above deliberations suggest two congruency situations supplementary to the commonly encountered ones. The first one could be called the "OLA" rule (O – obtuse angle, L – longest side, A – adjacent side). The second one could be called the "AAELO" rule (A – acute angle, A – adjacent side, ELO – equal or longer opposite side).

Alternatively, the two rules suggested above and the RHS rule could be absorbed into a single generalisation: an "AALO" rule [A – angle (which could be acute, right or obtuse), A – adjacent side, LO – longer opposite side (i.e., longer than the given adjacent side)].



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