THE GOLDEN RATIO Unexpectedly

Part 1

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A **Kepler triangle** is a right-angled triangle whose sides are in Geometric Progression, which requires that its sides are in the ratio $1 : \sqrt{\varphi} : \varphi$ where $\varphi = (1+\sqrt{5})/2$ is the Golden Ratio.

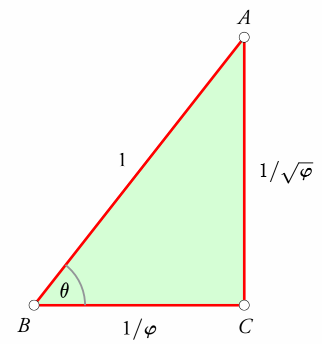

Figure 1

Clearly, if the length of the hypotenuse is 1 (see Figure 1), the sides will have lengths

$$\frac{1}{\varphi}, \quad \frac{1}{\sqrt{\varphi}}, \quad 1.$$

If θ is the larger acute angle of the triangle, then $\tan\theta = \sqrt{\varphi}$. This angle is the acute solution to the equation $\tan^2\theta\ \cos\theta = 1$. (For, since $\varphi^2 = \varphi + 1$, we get $\tan^4\theta = \tan^2\theta + 1$, i.e., $\tan^4\theta = \sec^2\theta$, which yields $\tan^2\theta = \sec\theta$ since θ is acute. Hence $\tan^2\theta\ \cos\theta = 1$.)

For clarity, we note that we are using here the symbol φ to represent the value $1.618\ldots$, which means that $\varphi^2 = \varphi + 1$. The other value which could equally well be called the Golden Ratio is the reciprocal of this number, which is also $\varphi - 1$, and which is here given the symbol G.

Keywords: Kepler triangle, golden ratio, incircle, geometric progression

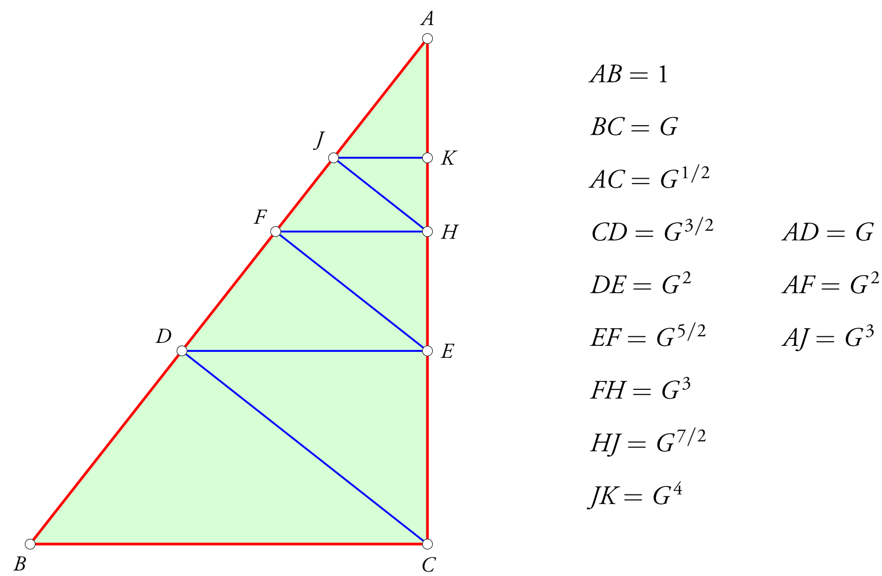

Figure 2

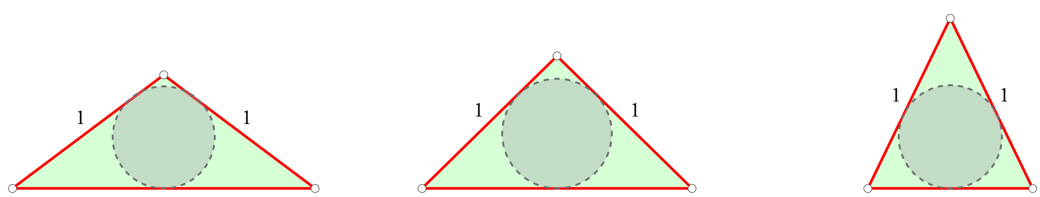

Figure 3

Dropping perpendiculars progressively from the right-angle vertex to the hypotenuse and then back to a leg (see Figure 2) produces lengths which are powers of $G^{1/2}$; continuing the process generates similar triangles, so any length of the form $G^{n/2}$ can be achieved in this manner.

Now consider an isosceles triangle whose equal sides have length 1. Clearly there are lots of possible 'shapes' for it, and presumably therefore different areas for its incircle (see Figure 3). It seems reasonable to ask which of these isosceles triangles has the largest incircle.

An intuitive response might be that the required isosceles triangle is going to be equilateral; certainly if that turned out to be the case, it would fit one's sense of what is 'right'. And, indeed, it is when the isosceles triangle is equilateral that *the largest proportion of its area* is included in its incircle. But if the incircle itself should be as large as possible, we need to make the isosceles triangle not equilateral but in the form of a double Kepler triangle, produced by placing two Kepler triangles alongside each other, with their longer legs coinciding (see Figure 4).

To see why, we need first to understand something about the incentre of a triangle, which is at the same distance r from each of its sides (see Figure 5). The figure shows a triangle and its

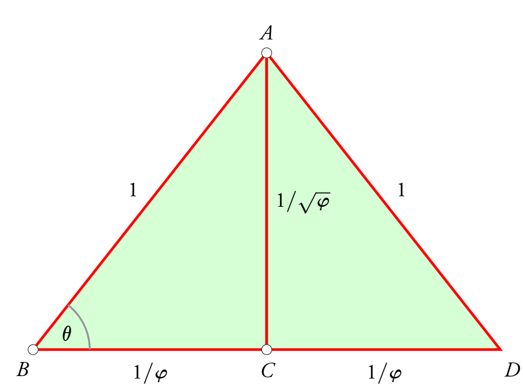

Figure 4

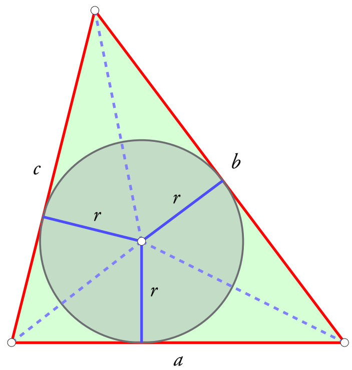

Figure 5

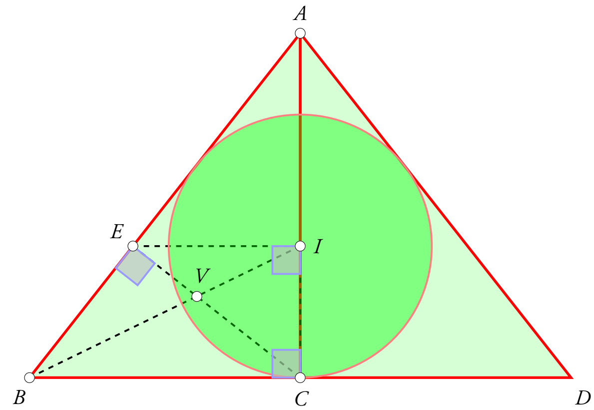

Figure 6

incircle, and also lines connecting the vertices of the triangle to the incentre; they demonstrate how the area of the triangle can be seen as the sum of the areas of three smaller triangles. Of course the radii are perpendicular to the sides and therefore serve as heights on those bases for the smaller triangles.

Thus the area of the whole triangle is $ra/2 + rb/2 + rc/2 = rs$ where s is the semi-perimeter of the triangle, and we deduce that for any triangle the length r of its in-radius is given by the formula

$$r = \frac{\text{area of triangle}}{\text{semi-perimeter of triangle}}.$$

In the case of an isosceles triangle whose equal sides have length 1 and with base angle x, we have:

$$\text{area of triangle} = \cos x \cdot \sin x = \frac{1}{2}\sin 2x,$$

$$\text{perimeter of triangle} = 2 + 2\cos x.$$

Hence the radius of the incircle is given by

$$r = \frac{\sin 2x}{2 + 2\cos x}.$$

Setting the derivative of r equal to zero gives

$$(2 + 2\cos x)2\cos 2x - \sin 2x(-2\sin x) = 0,$$

$$\therefore\ 4(1 + \cos x)\left(2\cos^2 x - 1\right) + 4\sin^2 x\cos x = 0,$$

$$\therefore\ (1 + \cos x)\left(2\cos^2 x - 1\right) + \cos x\left(1 - \cos^2 x\right) = 0,$$

$$\therefore\ 2\cos^2 x - 1 + \cos x(1 - \cos x) = 0,$$

since $1 + \cos x \neq 0$. The equation in the last line yields

$$\cos^2 x + \cos x - 1 = 0,$$

so that $\cos x = G$. The verification that this indeed yields a maximum value of r is left to the reader.

This particular isosceles triangle ($\triangle ABD$, with $AB : AD : BD = 1 : 1 : 2/\varphi$) has the additional interesting property that if AC is the altitude to its base, then the perpendicular to AC through the incentre I of the triangle meets side AB at the foot E of the perpendicular CE to that side from the midpoint C of the base BD; see Figure 6.

Moreover, I is a Golden Point of AC (i.e., divides it in the Golden Ratio). Further, the point V where BI meets CE is a Golden Point of both CE and BI. Readers might like to tackle these proofs for themselves.

MARCUS BIZONY is now retired, having been Deputy Head (Academic) at Bishops in Cape Town, and before that Head of Mathematics; he taught at Bishops for over thirty years. He was Associate Editor of *Learning and Teaching Mathematics*, a journal published by the Association for Mathematics Educators of South Africa. He was also convenor of the panel that sets the South African Olympiad questions for juniors (Grades 8 and 9). He may be contacted at marcusbizony@gmail.com.