ClassRoom

INEQUALITIES in Algebra and Geometry Part 2

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This article is the second in the 'Inequalities' series. We prove a very important inequality, the Arithmetic Mean-Geometric Mean inequality ('AM-GM inequality') which has a vast number of applications and generalizations. We prove it using algebra as well as geometry.

The AM-GM inequality for two numbers

Let a and b be non-negative numbers. Their arithmetic mean m is the quantity $(a+b)/2$, and their geometric mean g is the quantity $\sqrt{ab}$. For example, if the numbers are 2 and 8, then the arithmetic mean is $(2+8)/2 = 5$ and the geometric mean is $\sqrt{2 \times 8} = 4$. If the numbers are 1 and 25, then the arithmetic mean is 13 and the geometric mean is 5. Observe that in both these instances, the arithmetic mean (AM) exceeds the geometric mean (GM). We shall show that this invariably happens; that is, it is invariably the case that $m \geq g$.

Theorem 1 (AM-GM inequality for two numbers)**.** *For any two non-negative numbers a and b, it is always the case that*

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Moreover, the equality sign holds if and only if $a = b$.

Proof. The most straightforward approach is to rewrite the inequality in various equivalent forms as shown below:

$$\begin{aligned} \frac{a+b}{2} &\geq \sqrt{ab} \quad \text{for all } a, b \geq 0 \\ \iff a+b &\geq 2\sqrt{ab} \quad \text{for all } a, b \geq 0 \\ \iff a - 2\sqrt{ab} + b &\geq 0 \quad \text{for all } a, b \geq 0. \end{aligned}$$

The inequality in the last line is clearly true since

$$a - 2\sqrt{ab} + b = \left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0.$$

Keywords: AM-GM inequality, harmonic mean, root-mean-square, area, perimeter, maximise, minimise

This proves the AM-GM inequality for two numbers. Observe how the inequality has reduced to the well-known fact that a squared number is non-negative.

Equality holds if and only if $\sqrt{a} - \sqrt{b} = 0$, i.e., if and only if $a = b$, as claimed. $\square$

Alternative formulation of the same idea. The proof presented above can be recast in a different manner as follows. Let $m = (a+b)/2$ be the AM of a and b, and let $g = \sqrt{ab}$ be their GM. Then a and b are equidistant from m, and we can write

$$a = m + c, \quad b = m - c$$

for some number c which could be either positive or negative or zero (note that $c = 0$ corresponds to the case when $a = b$). This yields:

$$ab = (m+c)(m-c) = m^2 - c^2 \leq m^2, \text{since } c^2 \geq 0.$$

Since $g^2 = ab$, we get:

$$g^2 \leq m^2, \quad \text{i.e.,} \quad g \leq m,$$

since $g \geq 0$, $m \geq 0$. The equality sign will hold if and only if $c = 0$, i.e., if and only if $a = b$. It follows that the GM is less than or equal to the AM, with equality if and only if the two numbers are identical. $\square$

Isoperimetric property of the square. In Part-1 of this article (November 2016 issue of *At Right Angles*) we proved the isoperimetric property of the square, namely: *Among all rectangles sharing the same perimeter, the square has the largest area. Among all rectangles sharing the same area, the square has the least perimeter.* Let us now show how this property follows from the AM-GM inequality.

Consider a rectangle with sides a and b. Its area is ab, and its perimeter is $2(a+b)$. Hence the side of the square with equal perimeter is $2(a+b)/4 = (a+b)/2$. Therefore the area of the square whose perimeter is the same as that of the given rectangle is equal to $(a+b)^2/4$. Now we have, by the AM-GM inequality:

$$\sqrt{ab} \leq \frac{a+b}{2}, \quad \therefore \ ab \leq \frac{(a+b)^2}{4},$$

with equality if and only if $a = b$. This proves that among all rectangles sharing the same perimeter, the square has the largest area. The proof of the second assertion ("among all rectangles sharing the same area, the square has the least perimeter") follows in exactly the same way. (Please do fill in the details.) $\square$

Here is a nice example of a result which follows from the AM-GM inequality:

Proposition 1. *The sum of a positive number and its reciprocal cannot be less than 2. Moreover, the only positive number for which the sum of the number and reciprocal equals 2 is 1.*

Stated in symbols: if x is any positive number, then

$$x + \frac{1}{x} \geq 2,$$

with equality if and only if $x = 1$. For proof, we apply the AM-GM inequality to the two positive numbers x and $1/x$. Their geometric mean is $\sqrt{x \cdot 1/x} = 1$, hence their arithmetic mean cannot be less than 1. In other words:

$$\frac{x + 1/x}{2} \geq 1, \quad \therefore \ x + \frac{1}{x} \geq 2.$$

Moreover, equality holds if and only if $x = 1/x$, i.e., $x = 1$. (The other solution of the equation $x = 1/x$ is $x = -1$, but this does not need to be considered since x is supposed to be positive.) $\square$

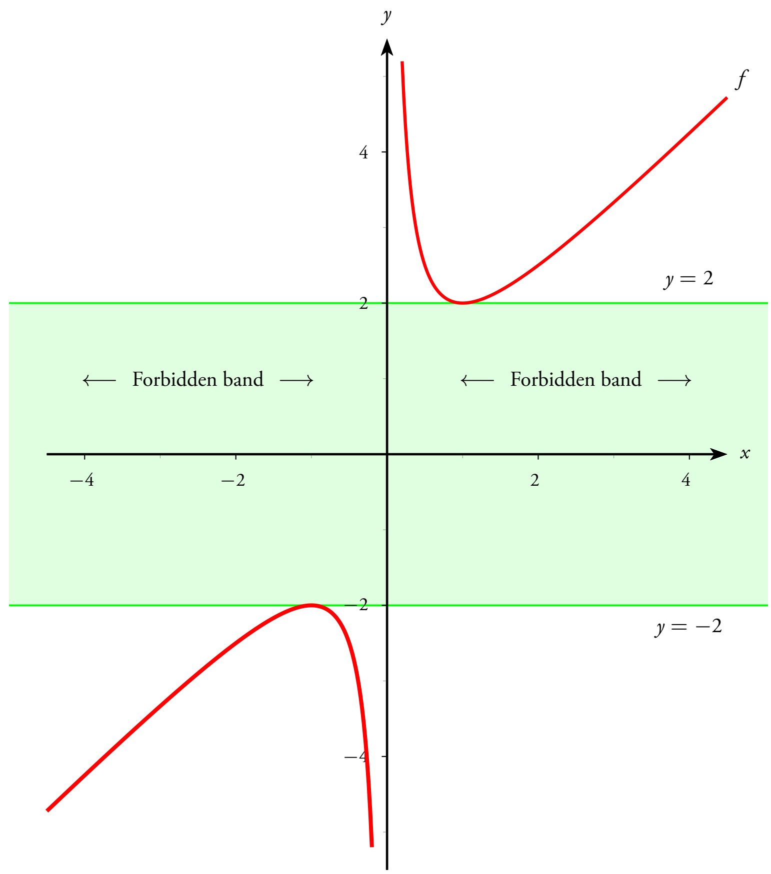


Figure 1. Graph of $f(x) = x + 1/x$

An extension, and a graphical representation. Proposition 1 can be extended in the following manner: *The sum of a nonzero real number and its reciprocal cannot lie between* -2 *and* 2. That is, if $x \neq 0$ is any real number, then it is **not** possible that

$$-2 < x + \frac{1}{x} < 2.$$

The inequality shows itself in striking form when we draw the graph of the function $f(x) = x + 1/x$ over the real numbers (see Figure 1). There is a "forbidden band" between the lines $y = -2$ and $y = 2$, within which the graph of the function never enters.

The graph itself is a hyperbola with centre $(0, 0)$ and asymptotes $x = 0$ and $y = x$.

Applications of the AM-GM inequality for two variables
We now demonstrate the utility and versatility of the two-variable AM-GM inequality.

Proposition 2. *Let a, b be positive real numbers. Then:*

$$(a+b) \cdot \left(\frac{1}{a} + \frac{1}{b}\right) \geq 4,$$

with equality if and only if $a = b$.

Proof. Multiplying out, we get:

$$(a+b)\cdot\left(\frac{1}{a}+\frac{1}{b}\right) = 1+1+\frac{a}{b}+\frac{b}{a}$$
$$= 2+\frac{a}{b}+\frac{b}{a}.$$

Invoking Proposition 1, we have: $\frac{a}{b}+\frac{b}{a}\geq 2$, with equality if and only if $\frac{a}{b}=1$, i.e., if and only if $a=b$. The stated inequality thus follows. □

Proposition 3. *Let a, b, c be positive real numbers. Then:*

$$(a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9,$$

with equality if and only if $a=b=c$.

Proof. Multiplying out, we get:

$$(a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)$$
$$\geq 3+2+2+2=9,$$

with equality if and only if $\frac{a}{b}=1$ and $\frac{b}{c}=1$ and $\frac{c}{a}=1$, i.e., if and only if $a=b=c$.

Generalisation. The obvious generalisation of Propositions 2 and 3 is the following, which we state without proof:

Proposition 4. *Let $a_1, a_2, \ldots, a_n$ be n positive real numbers. Then:*

$$(a_1+a_2+\cdots+a_n)\cdot\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\geq n^2,$$

with equality if and only if $a_1=a_2=\cdots=a_n$.

Proposition 5. *Let a, b, c be positive real numbers such that $abc=1$. Then:*

$$(1+a)(1+b)(1+c)\geq 8,$$

with equality if and only if $a=b=c=1$.

Proof. The AM-GM inequality applied to the pairs $\{1,a\}$, $\{1,b\}$ and $\{1,c\}$ yields:

$$\frac{1+a}{2}\geq\sqrt{a},\quad \frac{1+b}{2}\geq\sqrt{b},\quad \frac{1+c}{2}\geq\sqrt{c}.$$

Hence we have:

$$1+a\geq 2\sqrt{a},\quad 1+b\geq 2\sqrt{b},\quad 1+c\geq 2\sqrt{c}.$$

Multiplication yields:

$$(1+a)(1+b)(1+c)\geq 8\sqrt{abc}.$$

Since $abc=1$, the desired result follows. Equality holds if and only if $a=b=c=1$.

This proposition can be extended quite easily. For example, if a, b, c, d are positive real numbers such that $abcd=1$, then $(1+a)(1+b)(1+c)(1+d)\geq 16$. Equality holds if and only if $a=b=c=d=1$. □

Proposition 6. *Let a, b, c be real numbers. Then:*

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Equality holds if and only if $a = b = c$.

Proof. This is Problem 5 from the set given in Part-1 of this article (November 2016 issue).

Let us apply the AM-GM inequality to the pairs $\{a^2, b^2\}$, $\{b^2, c^2\}$ and $\{c^2, a^2\}$. We get:

$$\begin{aligned} a^2 + b^2 &\geq 2ab, \\ b^2 + c^2 &\geq 2bc, \\ c^2 + a^2 &\geq 2ca. \end{aligned}$$

Adding the three inequalities we get $2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca$. On dividing by 2, we get the desired inequality.

For equality to hold, we must have equality in each of the three inequalities listed above. This obviously requires that $a = b = c$. Hence the claim. □

Remark. As noted in the November 2016 issue, in the article on Napoleon's theorem, the following assertion is true in the domain of real numbers: *If $a^2 + b^2 + c^2 = ab + bc + ca$, then $a = b = c$.* But in the domain of complex numbers, a totally different conclusion holds, namely: *If $a^2 + b^2 + c^2 = ab + bc + ca$, then the points corresponding to a, b, c are the vertices of an equilateral triangle.*

Proposition 7. *Let a, b, c be positive real numbers. Then:*

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

Equality holds if and only if $a = b = c$.

Proof. The method used to prove Proposition 5 (above) works here as well. We leave the details for you to fill in.

Geometric proof of the AM-GM inequality

We close this article by offering a geometric proof of the AM-GM inequality. It proves more than just the AM-GM; several inequalities get proved simultaneously simply by exhibiting a certain diagram. The geometric facts used are the following:

- *From a point P outside a circle ω are drawn a tangent PT to ω and a line which intersects ω at points Q and R. We now have the equality: $PT^2 = PQ \cdot PR$.* (This follows from the **intersecting chords theorem** which we have used in previous articles of this magazine. Here is its general statement: *In any circle ω, if AB and CD are two chords which intersect at a point P which may lie either inside or outside ω, then $PA \cdot PB = PC \cdot PD$.*)
- *In any triangle, the largest side is the one opposite the largest angle.* This implies in particular that in a right-angled triangle the largest side is the hypotenuse.

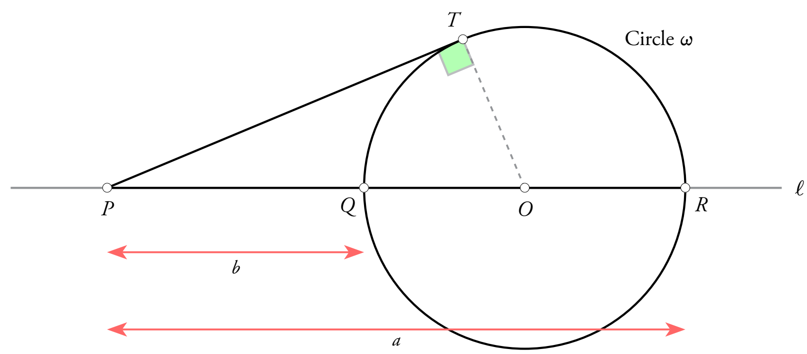


Figure 2

Given two positive numbers a, b ($a > b$), we construct the above diagram (Figure 2).

On a line ℓ we mark three points P, Q, R (in that order) such that $PQ = b$ and $PR = a$. Next, we draw a circle ω on QR as diameter; let its centre be O. We also draw a tangent PT to the circle from P. Note that the radius of the circle is $(a-b)/2$, and also that $PO = (a+b)/2$ and $PT \perp TO$. The length of PT can be computed in two different ways:

- By using the intersecting chords theorem:

$$PT^2 = PQ \cdot PR = ab, \quad \therefore \ PT = \sqrt{ab};$$

- By using the Pythagoras theorem:

$$PT^2 = PO^2 - TO^2 = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab, \quad \therefore \ PT = \sqrt{ab}.$$

  Either way we see that $PT = \sqrt{ab}$.

Now in $\triangle PTO$, which is right-angled at vertex T, the hypotenuse is PO; this is therefore the longest side of the triangle. Hence we have: $PT \leq PO$, i.e.,

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Therefore the geometric mean of a and b cannot exceed the arithmetic mean of a and b.

For equality to hold, $\triangle PTO$ must be *degenerate*, with side TO shrinking to zero length. In other words, we must have $a = b$ for the geometric mean to be equal to the arithmetic mean. $\square$

Extensions. A small modification of the diagram yields a substantial generalisation of the AM-GM inequality.

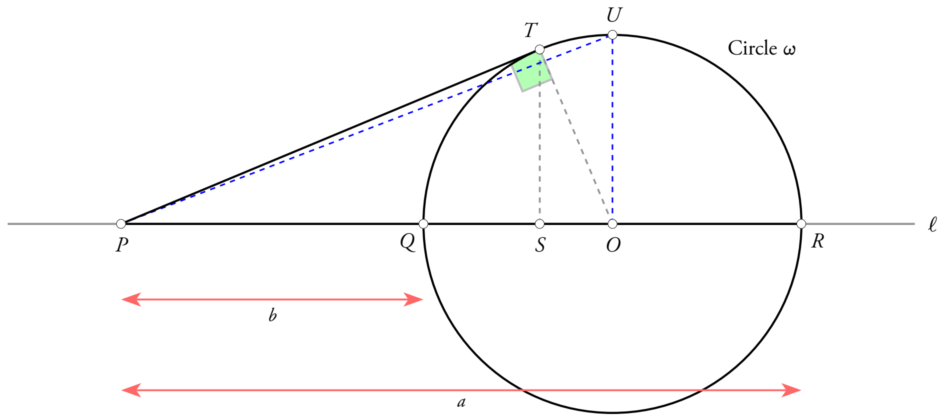


Figure 3

Figure 3 is the same as Figure 2 but with some extra line segments drawn. From T draw a perpendicular TS to ℓ. Also draw a perpendicular to ℓ at O; let it intersect the circle at U. We now have the following string of inequalities:

$$PS \le PT \le PO \le PU,$$

with equality if and only if $a = b$. We already have expressions for the lengths of PT and PO. Let us now do the same for PS and PU. We have:

$$\frac{PS}{PT} = \cos \measuredangle TPS = \frac{PT}{PO}, \quad \therefore\ PS = \frac{PT^2}{PO},$$

i.e.,

$$PS = \frac{ab}{(a+b)/2} = \frac{2ab}{a+b}.$$

We also have:

$$PU^2 = PO^2 + OU^2 = \frac{(a+b)^2}{4} + \frac{(a-b)^2}{4}, \quad \therefore\ PU = \sqrt{\frac{a^2+b^2}{2}}.$$

The quantities

$$\frac{2ab}{a+b}, \qquad \sqrt{\frac{a^2+b^2}{2}}$$

are known respectively as the **harmonic mean** (HM) and the **root mean square** (RMS) of a and b. So we have established that:

$$\mathrm{HM}(a,b) \le \mathrm{GM}(a,b) \le \mathrm{AM}(a,b) \le \mathrm{RMS}(a,b),$$

with equality if and only if $a = b$. Is it not beautiful that we have managed to get four inequalities from a single diagram?

Remark. You may wonder why the HM is a mean, and why the RMS is a mean. In other words, what is 'mean' about the HM and the RMS? Why do we call them 'means'? The underlying logic is revealed when we state the commonalities between the HM, GM and RMS. Each one makes use of a particular function-inverse function pair, (f, f^{-1}). In each case, given the positive numbers a, b, we first apply f to the numbers, thereby getting the f-numbers $f(a), f(b)$ respectively. Then we compute the AM of these two numbers; i.e., we compute the number

$$\frac{f(a)+f(b)}{2}.$$

Lastly, we apply the inverse function f^{-1} to this AM, i.e., we compute the number

$$f^{-1}\left(\frac{f(a)+f(b)}{2}\right).$$

We call this number the f-mean of a and b. Observe how this prescription applies to the HM, GM and RMS:

HM: Let $f(x) = 1/x$; i.e., f maps each number to its reciprocal. (Here, the inverse function of f is f itself; i.e., the function is its own inverse.) So from a, b we get the numbers $1/a$, $1/b$; then we get the AM of these numbers, i.e.,

$$\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{a+b}{2ab}.$$

Lastly we apply f to this number; we get

$$f^{-1}\left(\frac{a+b}{2ab}\right) = \frac{2ab}{a+b}.$$

We have obtained the harmonic mean of a and b.

GM: Let $f(x) = \log_2 x$; i.e., f maps each number to its logarithm to base 2. (In this case, the inverse function of f is $f^{-1}(x) = 2^x$.) So from a, b we get the numbers $\log_2 a$, $\log_2 b$; then we get the AM of these numbers, i.e.,

$$\frac{1}{2}\left(\log_2 a + \log_2 b\right) = \frac{1}{2}\log_2(ab) = \log_2 \sqrt{ab}.$$

Lastly we apply f^{-1} to this number; we get

$$2^{\log_2 \sqrt{ab}} = \sqrt{ab}.$$

We have obtained the geometric mean of a and b.

RMS: Let $f(x) = x^2$; i.e., f maps each number to its square. (In this case, the inverse function of f is $f^{-1}(x) = \sqrt{x}$.) So from a, b we get the numbers a^2, b^2; then we get the AM of these numbers, i.e.,

$$\frac{a^2+b^2}{2}.$$

Lastly we apply f^{-1} to this number; we get

$$\sqrt{\frac{a^2+b^2}{2}}.$$

We have obtained the root mean square of a and b.

Problems for you to solve

We close by offering a small list of problems for you to tackle. Most of them are based on the AM-GM inequality.

(1) Let a, b, c be positive real numbers. Show that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3.$$

(2) Let a, b, c be positive real numbers. Show that:

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\geq 3,$$

with equality if and only if $a=b=c$.

(3) Let a, b, c be positive real numbers. Show that:

$$\left(a^2b+b^2c+c^2a\right)\cdot\left(ab^2+bc^2+ca^2\right)\geq 9a^2b^2c^2,$$

with equality if and only if $a=b=c$.

(4) Let a, b, c be positive real numbers. Show that:

$$a^3+b^3+c^3\geq a^2b+b^2c+c^2a,$$

with equality if and only if $a=b=c$.

Solutions to problems from November 2016 issue

(1) (a) *Which is larger,* $3^{1/3}$ *or* $4^{1/4}$*?*

Raising both numbers to the 12^{th} power, we get the numbers $3^4=81$ and $4^3=64$ respectively. Since $81>64$, it follows that $3^{1/3}>4^{1/4}$.

(b) *Which is larger,* $4^{1/4}$ *or* $5^{1/5}$*?*

Since $4^5=1024$ and $5^4=625$ and $1024>625$, we conclude that $4^{1/4}>5^{1/5}$.

(2) (a) *Which is larger,* $2^{1/3}$ *or* $3^{1/4}$ *?*

Since $2^4<3^3$, we conclude that $2^{1/3}<3^{1/4}$.

(b) *Which is larger,* $3^{1/4}$ *or* $4^{1/5}$ *?*

Since $3^5<4^4$, we conclude that $3^{1/4}<4^{1/5}$.

(3) *Which is larger:* $1.1+\dfrac{1}{1.1}$ *or* $1.01+\dfrac{1}{1.01}$ *?*

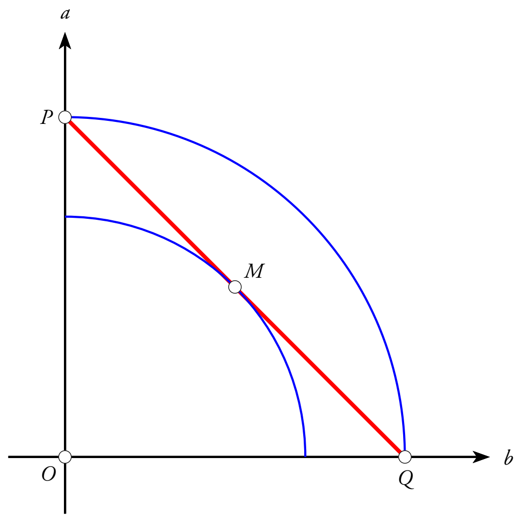
Let $x=1.1$ and $y=1.01$; then $x>y>1$. We have now:

$$\left(x+\frac{1}{x}\right)-\left(y+\frac{1}{y}\right)=x-y+\frac{y-x}{xy}=\frac{(x-y)(xy-1)}{xy}>0,$$

since $xy>1$. Hence $1.1+\dfrac{1}{1.1}>1.01+\dfrac{1}{1.01}$.

(4) *If a, b are non-negative real numbers with constant sum s, what are the least and greatest values taken by a^2+b^2? Express the answers in terms of s.*

We give a graphical solution. The set of pairs (a, b) of non-negative real numbers with constant sum s corresponds to a line segment PQ (Figure 4). The set of pairs (a, b) for which a^2+b^2 has some fixed value k^2 corresponds to a circle centred at the origin O, with radius k. This means that we need to identify the smallest and the largest circles centred at the origin which have some contact with segment PQ. These are clearly the two circles shown. The smaller one touches PQ at its midpoint M, and the larger one passes through both P and Q.



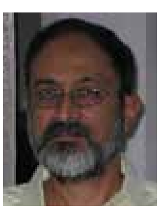
$P = (0, s)$

$Q = (s, 0)$

$M = (s/2, s/2)$

Figure 4

Therefore, given that $a + b = s$ and $a \geq 0$, $b \geq 0$, the least possible value of $a^2 + b^2$ is $s^2/2$ and the greatest possible value is s^2.



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