Problem Corner

A Minimum PERIMETER PROBLEM

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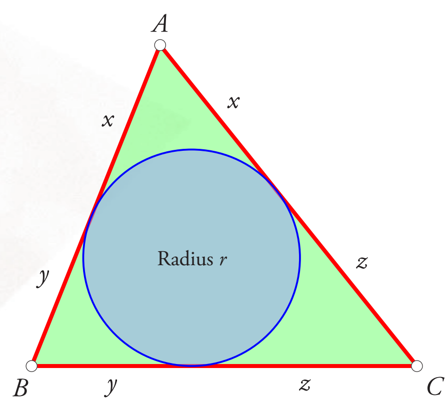
LinkedIn reader Peter Lovasz asks ([1]): *Among all triangles that share a given circle as incircle, which one has the smallest perimeter?* We give two approaches to the solution of this problem.



Solution I. Let the given circle have radius r. Let x, y, z denote the lengths of segments as marked in the figure. (Note that the figure makes implicit use of the theorem that the two tangents drawn from an external point to a circle have equal length.) Then the sides of the triangle are $y+z, z+x, x+y$; so the semi-perimeter is $s = x+y+z$. Therefore by Hero's formula, the area k of the triangle is given by the following relation:

$$k^2 = (x+y+z)xyz. \tag{1}$$

We also have $k = rs$ (this is another well-known formula), i.e.,

$$r(x+y+z) = \sqrt{(x+y+z)xyz};$$

hence:

$$r^2(x+y+z) = xyz. \tag{2}$$

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We now invoke the AM-GM inequality on the non-negative numbers x, y, z (proved and discussed in the article "Inequalities" elsewhere in this issue):

$$(xyz)^{1/3} \leq \frac{x+y+z}{3},$$
$$\therefore \; xyz \leq \frac{(x+y+z)^3}{27}, \tag{3}$$

with equality precisely when $x = y = z$ (i.e., precisely when the triangle is equilateral). Hence, from (2),

$$r^2(x+y+z) \leq \frac{(x+y+z)^3}{27},$$

which gives:

$$(x+y+z)^2 \geq 27r^2,$$

i.e.,

$$x+y+z \geq 3\sqrt{3}\,r. \tag{4}$$

Hence the perimeter of the triangle is not less than $6\sqrt{3}\,r$. Equality holds precisely when the triangle is equilateral. Therefore the equilateral triangle is the minimizing one. □

Solution II. This solution requires a good working knowledge of trigonometry and calculus. For convenience, we list a few relevant formulas at the end of the article (Box 1). We hope you will work your way through the solution!

Let the angles of the triangle be $2\alpha, 2\beta, 2\gamma$; then

$$0 < \alpha, \beta, \gamma < \frac{\pi}{2}, \quad \alpha+\beta+\gamma = \frac{\pi}{2}, \tag{5}$$

and the perimeter p of the triangle is given by

$$p = 2r(\cot\alpha + \cot\beta + \cot\gamma). \tag{6}$$

Hence we must minimise $\cot\alpha + \cot\beta + \cot\gamma$ subject to (5). We have:

$$\begin{aligned} \cot\alpha + \cot\beta &= \frac{\cos\alpha}{\sin\alpha} + \frac{\cos\beta}{\sin\beta} = \frac{\cos\alpha\sin\beta + \cos\beta\sin\alpha}{\sin\alpha\sin\beta} \\ &= \frac{2\sin(\alpha+\beta)}{\cos(\alpha-\beta) - \cos(\alpha+\beta)} \\ &= \frac{2\cos\gamma}{\cos(\alpha-\beta) - \sin\gamma} \geq \frac{2\cos\gamma}{1-\sin\gamma}. \end{aligned} \tag{7}$$

with equality precisely when $\alpha = \beta = \frac{1}{2}(\pi/2 - \gamma)$; the inequality in the last line follows from the fact that the cosine of any angle cannot exceed 1. Hence

$$\cot\alpha + \cot\beta + \cot\gamma \geq \frac{2\cos\gamma}{1-\sin\gamma} + \cot\gamma, \tag{8}$$

with equality precisely when $\alpha = \beta = \frac{1}{2}(\pi/2 - \gamma)$.

Write $g(\gamma)$ for the function $\dfrac{2\cos\gamma}{1-\sin\gamma} + \cot\gamma$; then we find that

$$g'(\gamma) = \frac{2}{1-\sin\gamma} - \csc^2\gamma = \frac{2}{1-\sin\gamma} - \frac{1}{\sin^2\gamma}. \tag{9}$$

From (9), we infer that $g'(\gamma)$ attains a minimum value in $(0, \frac{1}{2}\pi)$ when

$$2\sin^2\gamma + \sin\gamma - 1 = 0, \quad \text{i.e.,} \quad (\sin\gamma + 1)(2\sin\gamma - 1) = 0, \tag{10}$$

i.e., when $2\sin\gamma = 1$; this happens when $\gamma = \frac{1}{6}\pi$. To see why it is a minimum, we check the sign profile of $g'(\gamma)$ at $\gamma = \frac{1}{6}\pi$; it is $-$, 0, $+$, indicating that it is a minimum. (It is easier to perform this check in this situation than to compute the second derivative.) Since

$$g\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{1/2} + \sqrt{3} = 3\sqrt{3}, \tag{11}$$

it follows that for $0 < \gamma < \frac{1}{2}\pi$,

$$\frac{2\cos\gamma}{1 - \sin\gamma} + \cot\gamma \geq 3\sqrt{3} \tag{12}$$

with equality only at $\gamma = \frac{1}{6}\pi$; and therefore that

$$\cot\alpha + \cot\beta + \cot\gamma \geq 3\sqrt{3}, \tag{13}$$

with equality precisely when

$$\gamma = \frac{\pi}{6}, \quad \alpha = \beta = \frac{\pi/2 - \pi/6}{2} = \frac{\pi}{6}.$$

Hence the minimising figure is the equilateral triangle. □

References

1. https://www.linkedin.com/groups/1872005/1872005-6126047981431980033

List of relevant trigonometric formulas

For angles x, y, we have the following relationships:

- $\sin(x + y) = \sin x\cos y + \cos x\sin y$
- $\cos(x + y) = \cos x\cos y - \sin x\sin y$
- $2\sin x\sin y = \cos(x - y) - \cos(x + y)$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$

Box 1

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.