INEQUALITIES in Algebra and Geometry Part 4

ClassRoom

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This article is the fourth in the 'Inequalities' series. This time, we present a novel proof of the general AM-GM inequality, based on iteration. Following this, we present some applications of the inequality.

The arithmetic mean-geometric mean inequality (generally referred to as the AM-GM inequality; said to be part of the daily diet for aspiring mathletes, and routinely used in many branches of mathematics) is well-known. New proofs come up once in a while. The following iterative proof is highly unusual and will be of interest to some readers.

Statement of the theorem

The arithmetic mean A and the geometric mean G of n given positive numbers $a_1, a_2, \ldots, a_n$ are defined as follows:

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{\sum_{i=1}^{n} a_i}{n}, \tag{1}$$

$$G = (a_1 a_2 \cdots a_n)^{1/n} = \left(\prod_{i=1}^{n} a_i\right)^{1/n}. \tag{2}$$

Theorem (AM-GM inequality). *Let $a_1, a_2, \ldots, a_n$ be positive numbers with arithmetic mean A and geometric mean G. Then $A \geq G$. Moreover, the equality $A = G$ holds if and only if $a_1 = a_2 = \cdots = a_n$.*

Proof of the theorem

For convenience, we use the short forms AM for arithmetic mean and GM for geometric mean.

We assume for convenience that the a_i's are indexed in increasing order, so that $a_1 \leq a_2 \leq \cdots \leq a_n$; this implies that $a_1 \leq A \leq a_n$. If $a_1 = A$ or $a_n = A$, then it means that all the a_i's are equal. In this case we have $A = G$, so there is nothing to prove; so we may assume that $a_1 < A < a_n$. Note that in this case, the quantity $X = a_1 + a_n - A$ is positive.

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We now replace the numbers a_1 and a_n by A and X, respectively. The n numbers we now have are: $A, a_2, a_3, \ldots, a_{n-1}, X$. The AM of these numbers is exactly the same as that of the numbers in the earlier list, because $A + X = a_1 + a_n$. However, their GM is strictly greater than the earlier value, because A and X lie strictly between a_1 and a_n (i.e., $a_1 < A, X < a_n$), implying that $AX > a_1 a_n$. To see why, observe that the inequality $AX > a_1 a_n$ is equivalent to

$$A\left(a_1 + a_n - A\right) - a_1 a_n > 0,$$

i.e., to

$$(A - a_1)(a_n - A) > 0,$$

and this is clearly true since $a_1 < A < a_n$.

So the replacement preserves the AM but results in an increased value for the GM.

The procedure is now iterated: at each stage, we replace the least and greatest numbers in the latest list by the AM of the collection and the value of

$$\text{least number} + \text{greatest number} - \text{AM},$$

respectively, and we continue this as long as the numbers in the collection are not all equal.

After each iteration we obtain a list of numbers in which the number of entries equal to A has increased. Therefore, after no more than $n - 1$ iterations we reach a stage when all entries are equal to A. So the iteration definitely comes to an end after a finite number of steps.

Let G_i represent the geometric mean at the i-th stage; then $G_0 = G$ and

$$A = G_{n-1} \geq G_{n-2} \geq \cdots \geq G_1 \geq G_0 = G,$$

so $A \geq G$, as required. $\square$

An example using numbers. It helps if we show the working of the algorithm using actual numbers. Let us start with the list $1, 2, 3, 4, 10$ and see what the algorithm accomplishes.

We display the working in the form of a table as shown below. In the second column, we always display the list in sorted form, i.e., in non-decreasing order.

Step	Latest list	Min element	Max element	AM	X	GM
1	$1, 2, 3, 4, 10$	1	10	4	7	$240^{1/5}$
2	$2, 3, 4, 4, 7$	2	7	4	5	$672^{1/5}$
3	$3, 4, 4, 4, 5$	3	5	4	4	$960^{1/5}$
4	$4, 4, 4, 4, 4$	4	4	4	4	$1024^{1/5}$

Observe the steady increase in the values of the GM, while the AM stays fixed. At the end, when all the numbers are equal, we have AM = GM.

Some applications of the AM-GM inequality

In this section, we invert the usual procedure. Rather than start with a problem from some Olympiad collection or the other, we try to *create* interesting inequalities by applying the AM-GM inequality to various lists of numbers. It can become quite a nice game to play! Here are some inequalities that we obtain as a result.

Result 1: Start with the list of numbers $1, 2, \ldots, n$, where n is any positive integer ($n > 1$). Their arithmetic mean is

$$\frac{1+2+\cdots+n}{n} = \frac{n(n+1)}{2} \times \frac{1}{n} = \frac{n+1}{2},$$

and their geometric mean is

$$(1 \times 2 \times \cdots \times n)^{1/n} = (n!\,)^{1/n}.$$

It follows that

$$(n!\,)^{1/n} < \frac{n+1}{2}. \tag{3}$$

The inequality is strict, since the numbers in the list $1, 2, \ldots, n$ are not all equal (indeed, they are all unequal).

Result 2: Start with the list of n numbers $1, 1, \ldots, 1, 2$, with $(n-1)$ repetitions of 1 and a solitary 2; here n is any positive integer ($n > 1$). Their arithmetic mean is

$$\frac{1}{n}\left(\underbrace{1+1+\cdots+1}_{n-1 \ \text{repetitions}}+2\right) = \frac{n+1}{n} = 1 + \frac{1}{n},$$

and their geometric mean is

$$(1 \times 1 \times \cdots \times 1 \times 2)^{1/n} = 2^{1/n}.$$

It follows that

$$2^{1/n} < 1 + \frac{1}{n}. \tag{4}$$

The inequality is strict, since the numbers in the list $1, 1, \ldots, 1, 2$ are not all equal.

Corollary. Here is an interesting result which follows from the above inequality. It is obvious that $2^{1/n} > 1$. Hence the following is true for all positive integers $n > 1$:

$$1 < 2^{1/n} < 1 + \frac{1}{n}. \tag{5}$$

In this double inequality, we let n increase without bound (i.e., $n \to \infty$). The number at the extreme left is a constant (equal to 1), while the numbers at the extreme right tend to 1 (in the limit). Therefore, by the so-called ***sandwich principle*** or ***pinch principle***, the following statement is true:

$$\lim_{n\to\infty} 2^{1/n} = 1. \tag{6}$$

Result 3: Similarly we may prove: for all positive integers $n > 1$,

$$1 < 3^{1/n} < 1 + \frac{2}{n}, \tag{7}$$

and hence:

$$\lim_{n\to\infty} 3^{1/n} = 1. \tag{8}$$

Result 4: Now consider of the list $1, 1, \ldots, 1, 1+x$, with $n-1$ repetitions ($n > 1$) of 1 and a solitary $1+x$, where $x > -1$. (This restriction is needed to avoid having a negative value for $1+x$.) The arithmetic mean of the numbers is

$$\frac{1}{n}\left(\underbrace{1+1+\cdots+1}_{n-1 \ \text{repetitions}}+(1+x)\right) = \frac{n+x}{n} = 1 + \frac{x}{n},$$

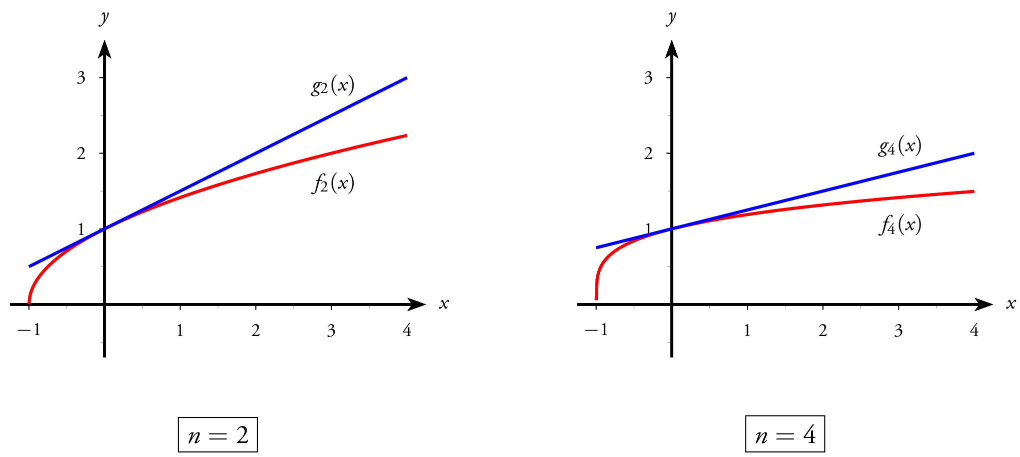

Figure 1

and their geometric mean is

$$(1 \times 1 \times \cdots \times 1 \times (1+x))^{1/n} = (1+x)^{1/n}.$$

It follows that

$$(1+x)^{1/n} \leq 1 + \frac{x}{n}. \tag{9}$$

The inequality is true for all integers $n > 1$ and for all real numbers $x > -1$. Equality holds precisely when $x = 0$. The inequality is strict provided that x is non-zero.

It is interesting to look at this relation graphically. Figure 1 displays the graphs of the following two functions for $x \geq -1$:

$$f_n(x) = (1+x)^{1/n} \qquad \text{(red)},$$
$$g_n(x) = 1 + \frac{x}{n} \qquad \text{(blue)},$$

for $n = 2$ and $n = 4$. Observe that in both cases, the graph of g (shown in blue colour) is a straight line tangent to the graph of f (shown in red colour) at the point $(0, 1)$. The tangent line lies entirely above the curve, touching it only at the indicated point.

Result 5: A particularly interesting inequality is obtained by considering the list of $n+1$ numbers:

$$1,\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ \ldots,\ 1+\frac{1}{n}, \tag{10}$$

where n is any positive integer. (So there are n repetitions of the number $1 + \frac{1}{n}$.) Their arithmetic mean is

$$\frac{1}{n+1}\left(1 + n \cdot \frac{n+1}{n}\right) = \frac{n+2}{n+1} = 1 + \frac{1}{n+1},$$

and their geometric mean is

$$\left(1 + \frac{1}{n}\right)^{n/(n+1)}.$$

We therefore get the following inequality which is true for all positive integers n:

$$1 + \frac{1}{n+1} > \left(1 + \frac{1}{n}\right)^{n/(n+1)},$$

that is,

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n}. \tag{11}$$

This establishes, as a mere corollary, the result that the following sequence

$$1, \quad \left(1+\frac{1}{2}\right)^{2}, \quad \left(1+\frac{1}{3}\right)^{3}, \quad \left(1+\frac{1}{4}\right)^{4}, \quad \ldots, \tag{12}$$

is *strictly increasing*. This result is needed in the proof of the claim that the sequence of numbers (12) has a limit. Some of you may know that the limit is the very well known number e whose approximate value is 2.71828.

Closing remarks. In the latter part of this article, we have tried to show how one can find mathematical results on one's own. This is in fact how mathematics is created! We invite you to find some interesting inequalities of your own by applying the AM-GM inequality to various lists of numbers.

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