Features

Complex ……
NAPOLEON'S THEOREM
…… Made Simple

Part 2

SHAILESH SHIRALI

In an earlier issue of At Right Angles, we had studied a gem of Euclidean geometry called Napoleon's Theorem, a result discovered in post-revolution France. We had offered proofs of the theorem that were computational in nature, based on trigonometry and complex numbers. We continue our study of the theorem in this article, and offer proofs that are more geometric in nature; they make extremely effective use of the geometry of rotations.

Napoleon's Theorem states the following. Let ABC be an arbitrary triangle. With the three sides of the triangle as bases, construct three equilateral triangles, each one outside $\triangle ABC$. Next, mark the centres P, Q, R of these three equilateral triangles. Napoleon's theorem asserts that $\triangle PQR$ is equilateral, irrespective of the shape of $\triangle ABC$. (See Figure 1.)

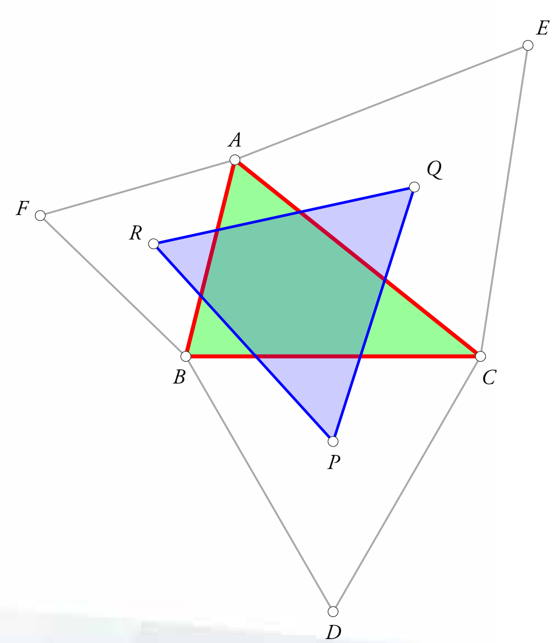


Figure 1

Keywords: Napoleon, equilateral triangle, rotation, parallelogram, basic proportionality theorem

In Part 1 of this article, we had considered computational proofs of Napoleon's theorem. In the trigonometric proof, we derived an expression for the length of one side of $\triangle PQR$ in terms of the sides and the angles of the $\triangle ABC$ (i.e., in terms of a, b, c, A, B, C). After going through the computations, we discovered that the resulting expression is symmetric in the parameters of the parent triangle. This fact suffices to prove that triangle PQR is equilateral.

Now we study an extremely elegant pure geometry proof of Napoleon's theorem; it makes very effective use of rotational geometry. In the literature, it is ascribed to an Irish mathematician, MacCool [1].

Before proceeding, we make a comment about rotations. Figure 2 shows a segment AB being subjected to two different rotations, both centred at a point O. The first one is through an angle of $+30°$ (the positive sign tells us that the rotation is in a *counterclockwise direction*); it takes segment AB to segment A_1B_1. The second one is through an angle of $-30°$ (the negative sign tells us that the rotation is in a *clockwise direction*); it takes segment AB to segment A_2B_2. Note that segments AB, A_1B_1 and A_2B_2 have equal length.

Now we get back to the proof of Napoleon's theorem. Consider a rotation through an angle of $-30°$, centred at B (see Figure 3; the rotation is in a clockwise direction). Our interest is in what this rotation 'does' to points R and P, i.e., where it takes these two points. Since $\measuredangle ABR = 30°$ and $\measuredangle DBP = 30°$, it follows that the image R_1 of R lies on side AB, and the image P_1 of P lies on side BD.

We argue as follows. The steps of the reasoning are laid out in itemised form at the right side of the diagram.

To see why $BR_1/BA = 1/\sqrt{3} = BP_1/BD$, you will first need to understand why $BR/BA = 1/\sqrt{3} = BP/BD$. But this follows from the basic geometry of an equilateral triangle. We leave the details for you to fill in.

From the fact that $BR_1/BA = BP_1/BD$, we deduce (using the basic proportionality theorem) that

$$R_1P_1 \parallel AD, \qquad \frac{R_1P_1}{AD} = \frac{1}{\sqrt{3}}. \tag{1}$$

Since $RP = R_1P_1$, it follows that:

$$\frac{RP}{AD} = \frac{1}{\sqrt{3}}. \tag{2}$$

In just the same way, we consider a rotation through an angle of $+30°$, centred at C. Then, if the rotation takes Q and P to Q_2 and P_2, respectively, it follows that Q_2 lies on side AC, and P_2 lies on side CD; and arguing as earlier, we conclude that

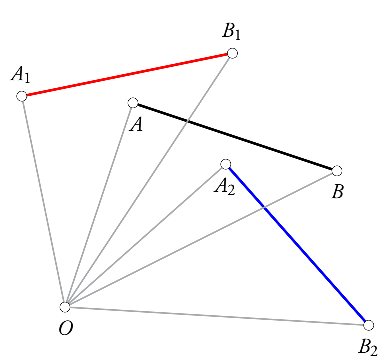
$$Q_2P_2 \parallel AD, \qquad \frac{Q_2P_2}{AD} = \frac{1}{\sqrt{3}}, \tag{3}$$

and

$$\frac{QP}{AD} = \frac{1}{\sqrt{3}}. \tag{4}$$

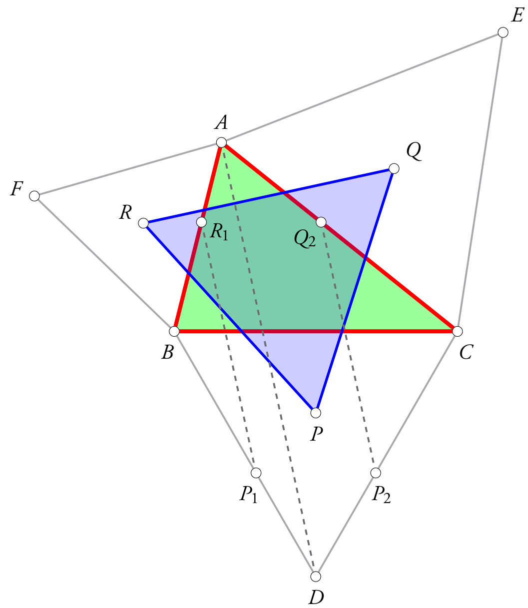
From (2) and (4), we conclude that $RP = QP$.

At this stage, we can proceed in two different ways. One way is to say that the same argument can be repeated for another pair of sides of $\triangle PQR$ and to conclude that equality therefore holds for the lengths of that pair of sides of $\triangle PQR$, and to



- $\measuredangle AOA_1 = +30° = \measuredangle BOB_1$
- $\measuredangle AOA_2 = -30° = \measuredangle BOB_2$
- Segments AB, A_1B_1 and A_2B_2 have equal length

Figure 2



(a) $R_1P_1 = RP$

(b) $BR_1 = BR$

(c) $BR_1/BA = 1/\sqrt{3}$

(d) $BP_1 = BP$

(e) $BP_1/BD = 1/\sqrt{3}$

(f) $BR_1/BA = BP_1/BD$

(g) $R_1P_1 \parallel AD$

(h) $R_1P_1/AD = 1/\sqrt{3}$

Figure 3

conclude from this that all three sides of the triangle have the same length. From this it follows that $\triangle PQR$ is equilateral. (We do not actually have to repeat all the steps of the argument. All we need to say is that since the argument worked for this particular pair of sides, it will also work for another pair of sides. Note that this is an appeal to symmetry.)

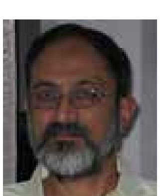
Another way is to say that $RP = QP$ and $\measuredangle RPQ = 60^\circ$; this is so because R_1P_1 is parallel to Q_2P_2, and we had obtained these two segments by rotations of segments RP and QP through 30° and 30° respectively, the first one through a rotation of -30° (i.e., 30° in a clockwise direction), and the second one through a rotation of $+30^\circ$ (i.e., 30° in an anticlockwise direction). So the two rotations are in *opposite directions*. After the two rotations, the resulting segments are parallel to each other, which means that *prior* to the rotations they must have been inclined at an angle of $(+30)^\circ - (-30)^\circ = 60^\circ$ to each other. This suffices to prove that $\triangle PQR$ is equilateral. □

This proof is to be admired for its elegance and its compactness! It shows just how much can be accomplished using arguments belonging only to elementary geometry.

In Part 3 of this series, we will consider generalisations and further aspects of Napoleon's theorem.

References

1. M. R. F. Smyth, "MacCool's Proof of Napoleon's Theorem", Irish Math. Soc. Bulletin 59 (2007), 71–77, http://www.maths.tcd.ie/pub/ims/bull59/M5903.pdf



SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.