The EXPONENTIAL SERIES an Addendum

SHAILESH SHIRALI

In the accompanying article, the author (Gaurav Bhatnagar) presents a heuristic derivation of the well-known series for the exponential function,

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots . \end{aligned} \tag{1}$$

Here, we depict the same steps graphically and present the material in a different way. This article may thus be regarded as an addendum to the article mentioned above.

The intention is to find a smooth function f, mapping the set of real numbers $\mathbb{R}$ into itself, and having the following properties: (i) the function equals its own derivative, i.e., $f(x) = f'(x)$ for all x; (ii) $f(0) = 1$. It is remarkable that these two simple requirements completely fix the function.

Note the following consequence of these properties: since $f(x) = f'(x)$, we obtain, by repeated differentiation:

$$f(x) = f'(x) = f''(x) = f'''(x) = \cdots . \tag{2}$$

Keywords: Exponential series, derivative, polynomial, partial sum, convergence, analytic function

Thus, all its derivatives are identical to itself. Since $f(0) = 1$, a particular consequence of this is:

$$f(0) = f'(0) = f''(0) = f'''(0) = \cdots = 1. \tag{3}$$

That is, at $x = 0$, every derivative equals 1 (including the zeroth derivative which is the function itself).

Remarkably, condition (3) by itself (which contains within itself infinitely many requirements) allows us to make substantial inroads into the problem: it allows us to find a power series for the function $f(x)$. What we do is to consider, at each stage, only a finite number of these requirements, reading from the left. Moreover, we confine our search to the family of polynomials, these being the simplest functions of all. Here is how we argue the case:

- Let condition (3) be replaced by: $f(0) = 1$. What is the simplest polynomial that satisfies this condition? Clearly, it is the constant polynomial

$$f(x) = 1 \quad \text{for all } x \in \mathbb{R}. \tag{4}$$

  (We must explain our use of the word 'simplest'. The meaning here is: *having the smallest possible degree*. In our particular case, this requirement is equivalent to: *with as few terms as possible*.)

- Now let condition (3) be replaced by: $f(0) = 1$ and $f'(0) = 1$. What is the simplest polynomial that satisfies these conditions? A moment's thought will reveal that it is the polynomial of degree 1 given by

$$f(x) = 1 + x \quad \text{for all } x \in \mathbb{R}. \tag{5}$$

  (To see why, note that since there are two conditions to be satisfied, we need two unknown coefficients to play around with. Now consider the general linear polynomial, $f(x) = a + bx$. By substituting the given conditions we obtain $a = 1 = b$; this yields $f(x) = 1 + x$, as stated.)

- Next, let condition (3) be replaced by these: $f(0) = 1$ and $f'(0) = 1$ and $f''(0) = 1$. (Note how we are gradually adding more and more conditions.) What is the simplest polynomial that satisfies these conditions? Calculation reveals (arguing as suggested above) that it is the polynomial of degree 2 given by

$$f(x) = 1 + x + \frac{x^2}{2} \quad \text{for all } x \in \mathbb{R}. \tag{6}$$

  (Exercise: Please fill in the algebraic details in the above derivation.)

- Next, let condition (3) be replaced by these: $f(0) = 1$ and $f'(0) = 1$ and $f''(0) = 1$ and $f'''(0) = 1$. What is the simplest polynomial that satisfies these conditions? Calculation reveals that it is the polynomial of degree 3 given by

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \quad \text{for all } x \in \mathbb{R}. \tag{7}$$

  (Exercise: Please fill in the algebraic details in the above derivation.)

- From the above, it is an easy step to guess — and then to prove using the principle of mathematical induction — that the simplest polynomial $f(x)$ satisfying the $n + 1$ conditions

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad f'''(0) = 1, \quad \ldots, \quad f^{(n)}(0) = 1, \tag{8}$$

  is the following polynomial of degree n:

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R}. \tag{9}$$

  Filling in the details makes for a good exercise.

Having reached this far, it is easy to see that the power series given by

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{for all } x \in \mathbb{R} \tag{10}$$

satisfies the requirements stated at the start, (2) and (3). It may be shown that the power series in (10) converges for all real values of x. (We shall not attempt to prove this statement. It belongs to a branch of mathematics called *real analysis*, in which criteria are worked out under which a power series converges. To learn more about analysis, which typically is studied in college rather than in school, you could refer to a standard text such as [1], chapters 5 and 7.) Hence, (10) defines a proper, well-behaved function ("analytic function" is the technical term; again, we shall not try to give a technical definition here; please consult [1]). Indeed, it may be shown that this is the *only* function which satisfies requirements (2) and (3).

The beautiful discovery that we now make about this function, as described in the accompanying paper, is that the function defined by (10) is the exponential function; namely, if we define the constant e by $e = f(1)$, so that

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots \approx 2.71828\ldots, \tag{11}$$

then the function is given by

$$f(x) = e^x. \tag{12}$$

It is fascinating at this stage to compare the graphs of the various functions that we have listed till now. In each of the plots below, we show two graphs: the graph of e^x, and the graph of a partial sum of the power series given in (10).

Note that owing to the extreme rapidity with which the exponential curve rises, we have had to use different scales for the x-axis and the y-axis.

Figure 1 displays the graphs of $y = e^x$ and $y = 1$, and Figure 2 displays the graphs of $y = e^x$ (red) and $y = 1 + x$ (blue). And so on for the other graphs.

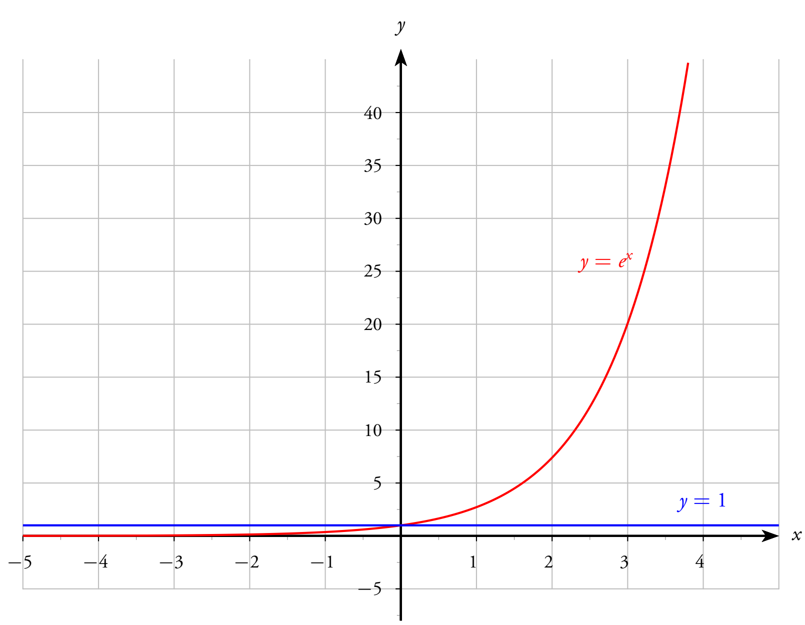
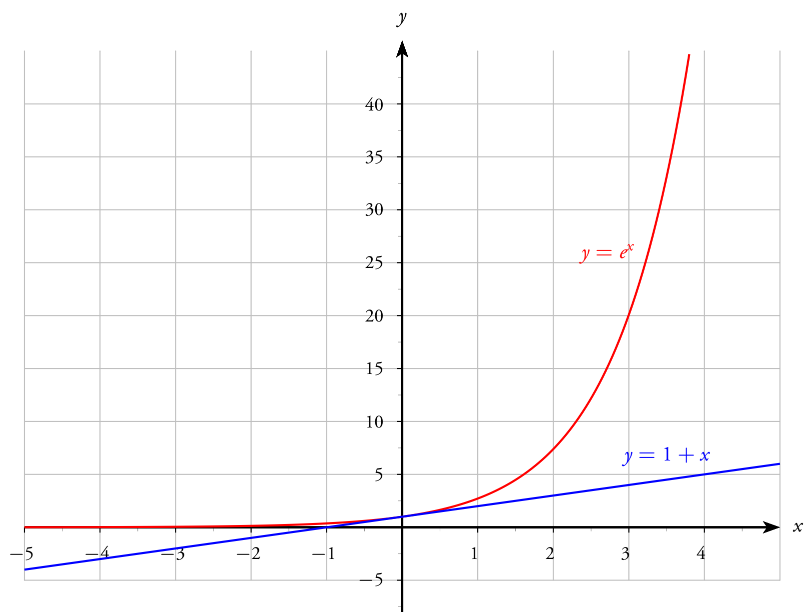


Figure 1. Graphs of $y = e^x$ (red) and $y = 1$ (blue)



Note: The slope of the line $y = 1 + x$ may not "look right" (it should be 1); but this is only because of the different scales used on the two axes.

Figure 2. Graphs of $y = e^x$ (red) and $y = 1 + x$ (blue)

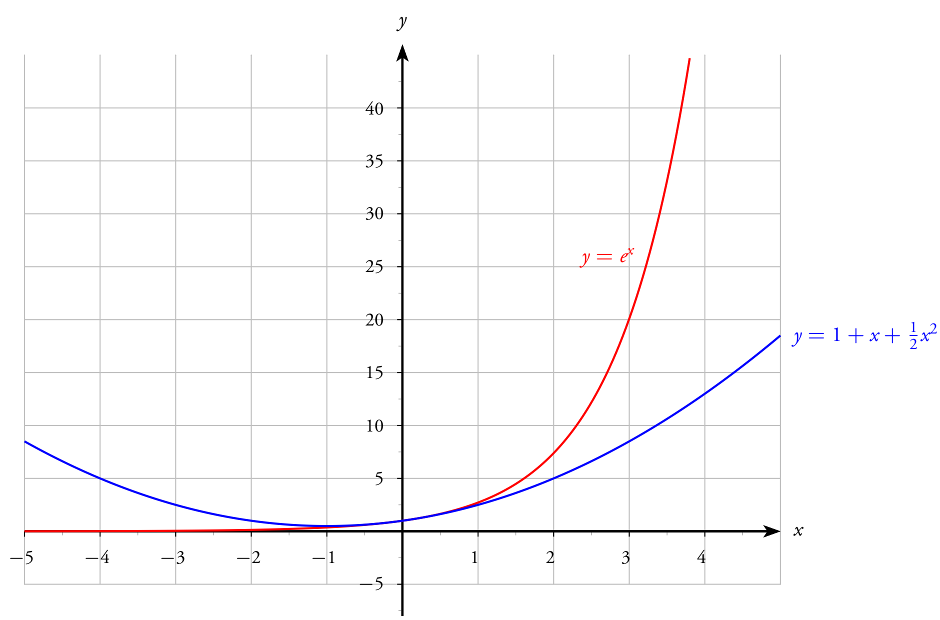


Figure 3. Graphs of $y = e^x$ (red) and $y = 1 + x + \frac{x^2}{2!}$ (blue)

As the degree of the partial sum gets larger, the closeness of the two graphs gets more marked; the range of values of x for which the two graphs seem to coincide and cannot be distinguished visually steadily gets larger.

To reinforce this point, we display in Figure 7 the graphs (with expanded scales) of $y = e^x$ and

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!}.$$

The two graphs are practically indistinguishable from each other over the range of x-values shown (except for a small bit at the left-hand side)!

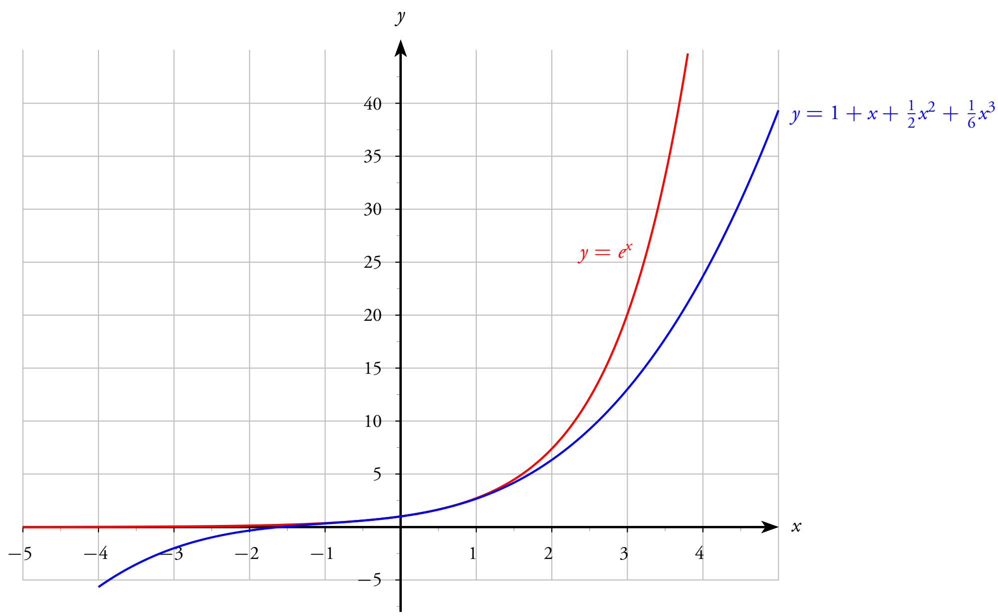


Figure 4 .Graphs of $y = e^x$ (red) and $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ (blue)

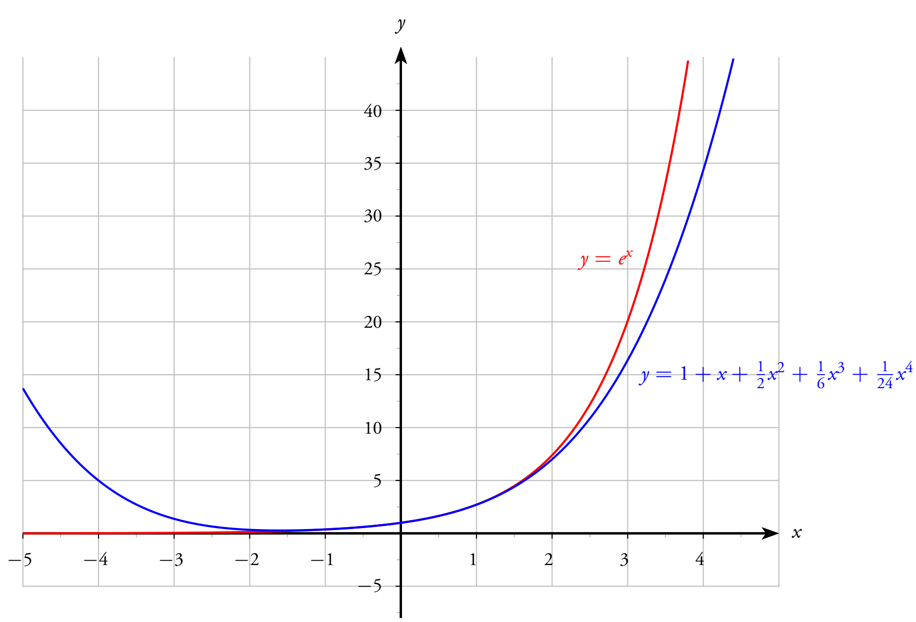


Figure 5. Graphs of $y = e^x$ (red) and $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ (blue)

Another approach is to display the graph of the following function:

$$\frac{1 + x + x^2/2! + x^3/3! + \cdots + x^n/n!}{e^x} \tag{13}$$

for different values of n (say $n = 3, 6, 10, 13$). Figures 8, 9, 10 and 11 do just this.

As n increases, we see that the graph of the function stays closer and closer to the line $y = 1$, for an ever-increasing range of values of x. Note, however, for any fixed value of n, there is only a finite interval of values of x where we see such a marked closeness. Outside this interval, the divergence between the two graphs becomes more visible, particularly at the negative end of the interval (where e^{-x} assumes extremely large values). You can see this happening in Figure 9 as well as Figure 10.

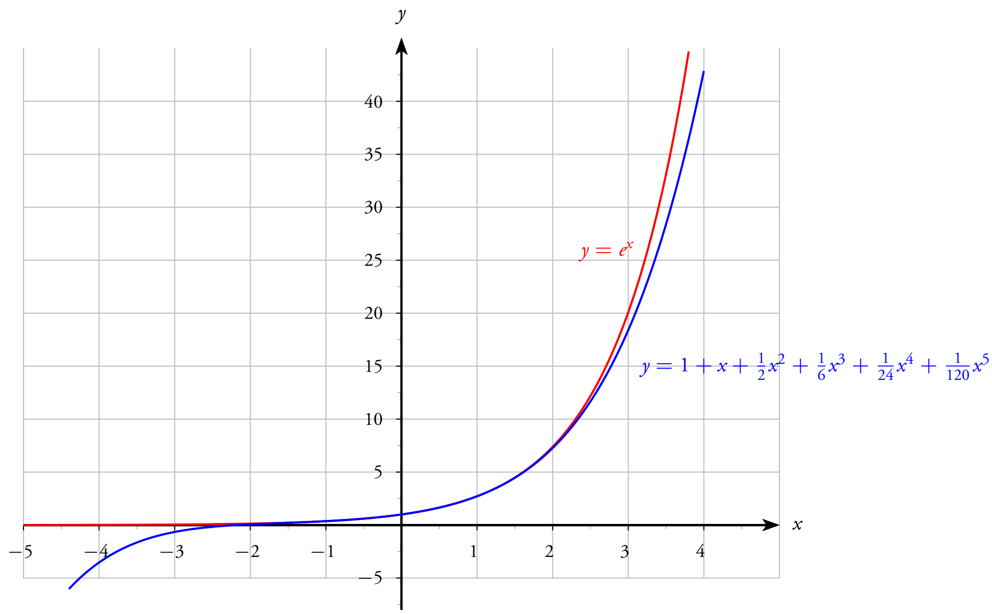


Figure 6. Graphs of $y = e^x$ (red) and $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$ (blue)

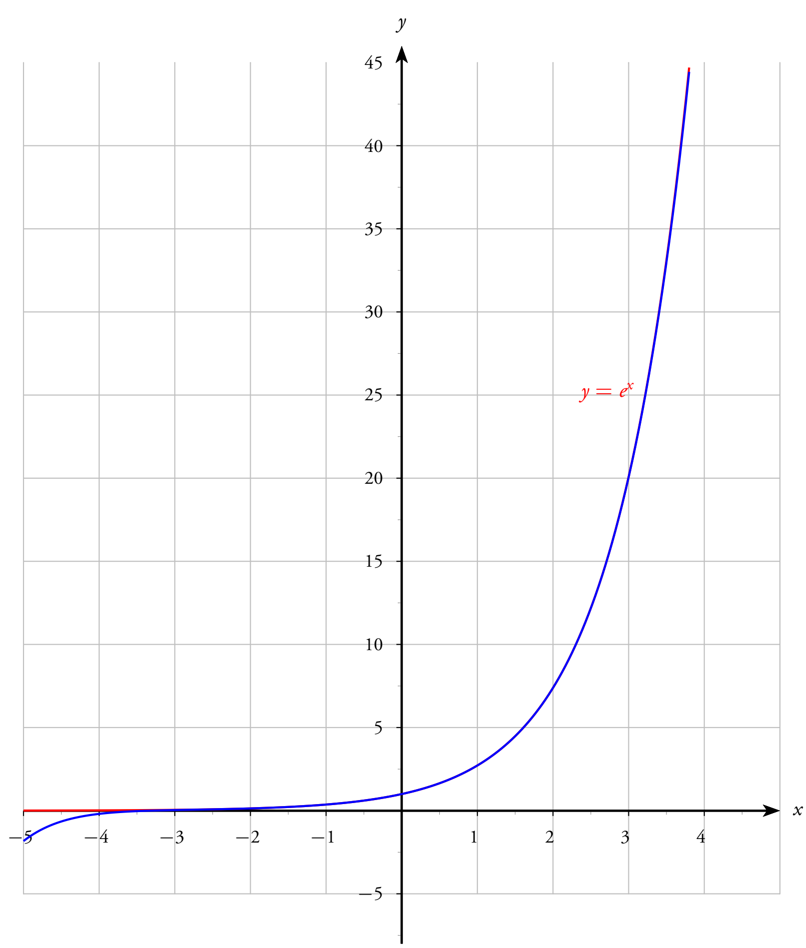


Figure 7. Graphs of $y = e^x$ (red) and $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!}$ (blue)

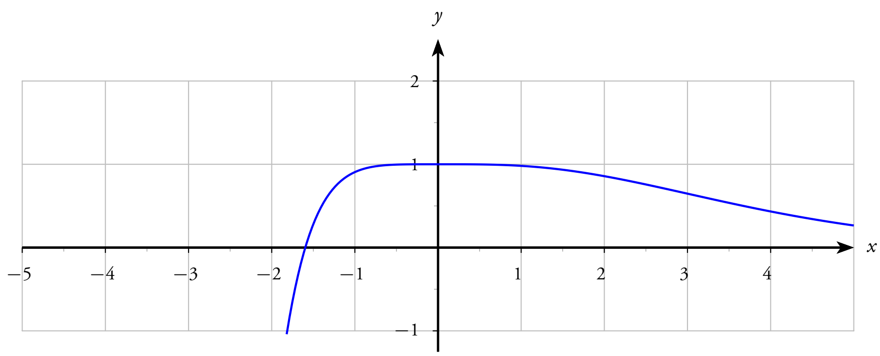


Figure 8. Graph of $y = e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$ for $n = 3$

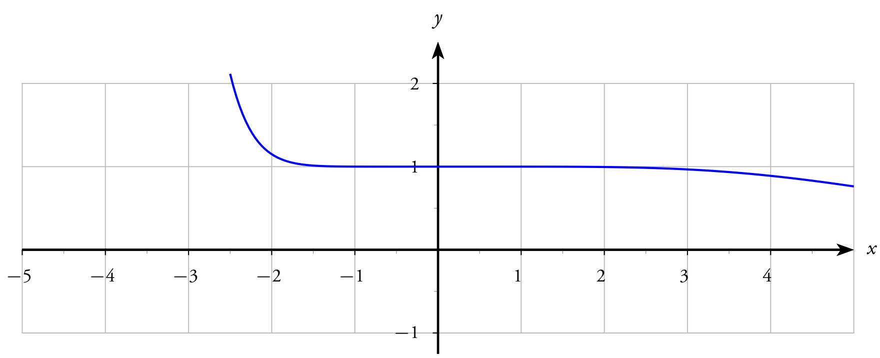


Figure 9. Graph of $y = e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$ for $n = 6$

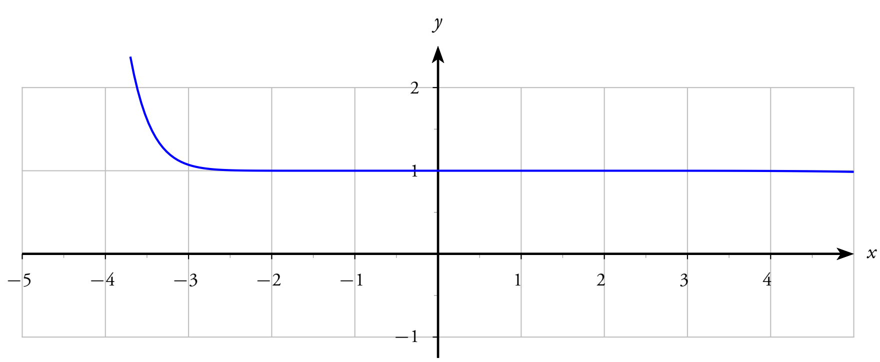


Figure 10. Graph of $y = e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$ for $n = 10$

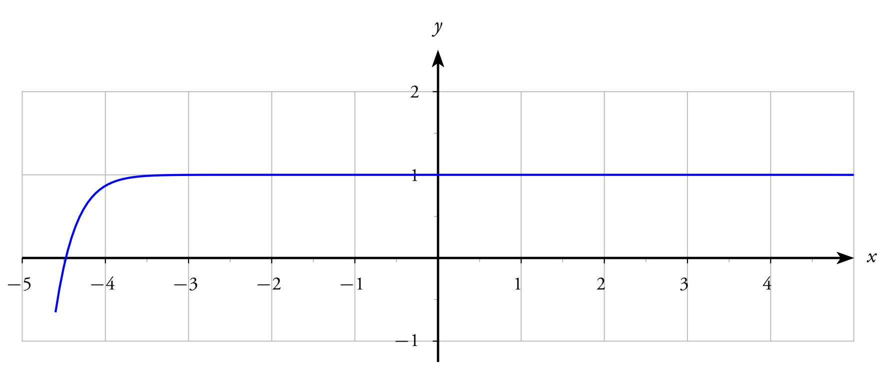


Figure 11. Graph of $y = e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$ for $n = 13$

Closing remark

The power series for the exponential function:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{14}$$

has become a familiar sight even at the high-school level, and perhaps students do not give it a second look when they encounter it; the price of familiarity! But we see that even in this familiar setting, looking at the matter through graphs rather than through equations throws fresh light on the topic and it does so in an extremely pleasing manner. Is it not truly remarkable that by adding more and more monomials, we begin to get closer and closer to an exponential function?

Exercises

(1) Show that the polynomial $f(x)$ with the smallest possible degree that satisfies the conditions $f(0) = 1, f'(0) = 1$ is

$$f(x) = 1 + x.$$

(2) Show that the polynomial $f(x)$ with the smallest possible degree that satisfies the conditions $f(0) = 1, f'(0) = 1, f''(0) = 1$ is

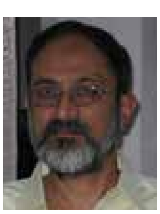
$$f(x) = 1 + x + \frac{x^2}{2}.$$

(3) Show that the polynomial $f(x)$ with the smallest possible degree that satisfies the conditions $f(0) = 1, f'(0) = 1, f''(0) = 1, f'''(0) = 1$ is

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

References

1. Richard Courant and Fritz John, *Introduction to Calculus and Analysis*, Volume 1. Interscience Publishers, John Wiley & Sons, 1965.



SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.