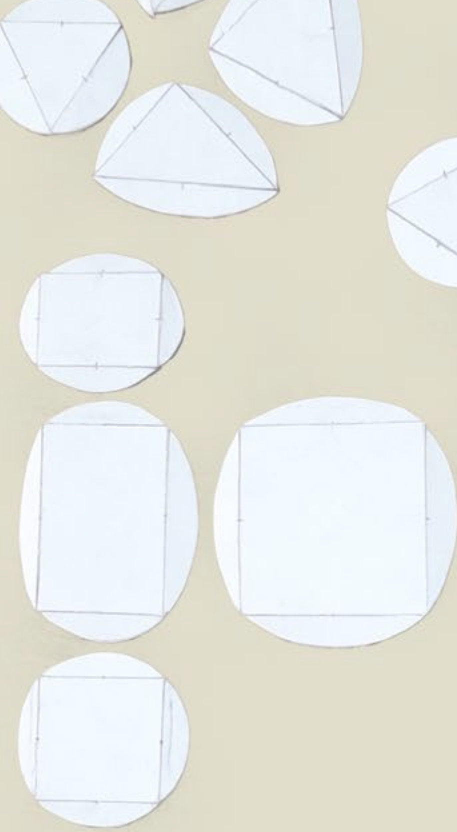
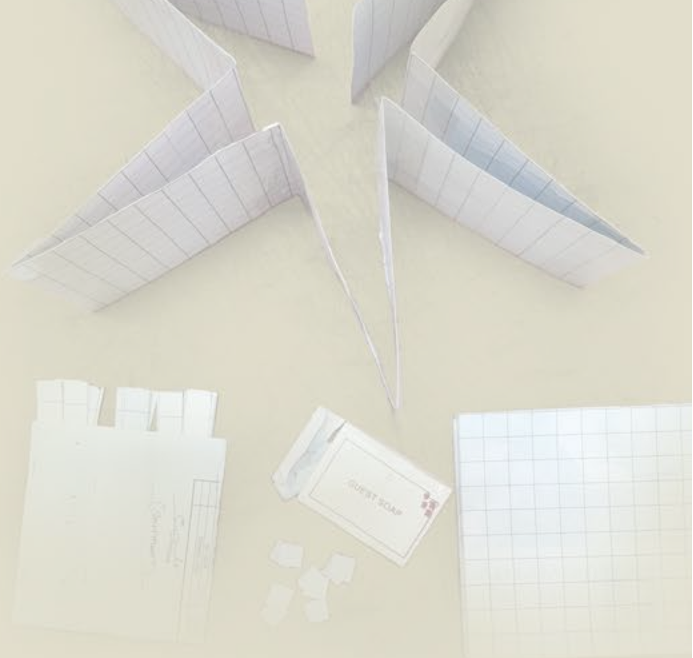
Issue 25 | July 2026
Triannual | Bengaluru
An Azim Premji University publication
RNI No.: KARENG/2018/76619

Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

ISSN 2582-1873





Mat(h)erials – a term coined by
Swati Sircar to describe materials
used for teaching mathematics.
Usually upcycled from discarded dry
waste, *mat(h)erials* – like precious
gems- need the perfect setting in order
to show off their natural beauty.
They cannot gather dust in a hallowed
'math lab' nor can they languish in a
lonely 'math corner'. They need to be
explored, thought about and most
importantly talked about. They are not
an end in themselves, they are to be used
thoughtfully, with careful facilitation,
planning and assessment of
children's understanding.

From the Editor's Desk . . .

Dear Readers,

The halfway mark is always a good opportunity to pause and reflect and in this second issue of 2026, we have much to look back on and look ahead to. The digital edition of At Right Angles https://atrightangles.azimpremjiuniversity.edu.in/ is up and active with new articles every week- in fact, articles of the July 2026 issue are already available on this platform. It's *the* go-to place for all those who love engaging with primary school mathematics.

The cover of the July 2026 issue features a paraphrased line from the 90's hit *Material Girl* – which may not be familiar to most of our readers. But those steeped in the pedagogy of mathematics will be familiar with *mat(h)erials* (materials used to teach mathematics). Surely, these exist in abundance all around us, and their use in the classroom is strongly recommended by policy, teacher trainers, and school administrators. Will this guarantee improvement in the learning of mathematics? Here are a couple of points from Anurag Behar's column in Mint*.

> *Last week in a school I watched a teacher run an "activity" on fractions. About twenty children folded paper into halves and quarters with evident enjoyment. There was movement, colour, buzz —all signs of a progressive classroom. At the end of the class, I asked a few children what a fraction was. They looked at the folded paper in their hands and then at me and had nothing to say. The activity had happened; the learning had not.*
>
> *The teacher was not at fault. He was sincerely and diligently doing what he had been told was good teaching. The fault lay much deeper and earlier, in an intellectual error that has quietly influenced Indian school education for two decades.*

He goes on to say

> *From the significantly true claim that 'learning is an active process of construction', we slid to a very different set of claims that 'teaching must therefore be minimally directive' —that the teacher should not explain, that direct instruction is authoritarian, that children must discover ideas for themselves through activities, that practice and memorization are the enemies of understanding.*

This is exactly the caution with which we present our two **Feature** articles by Nandita Jayaraj and Swati Sircar. The facilitation skills of the teacher are the most important part of the use of manipulatives. Communication is paramount! This theme runs throughout this issue in *Concrete to Concept* by Sonia Kundu, *Kaprekar Constant* by Suyash Tiwari and *Measuring the Gap* by Jivesh Panchbai.

In the **Joy of Mathematics**, we focus on problem-solving: *Dancing Digits*, *Star Problem* by Sumit Pandey, and *Pentominoes* by Jayasree and Mohan. Again, not just to get an answer or to play aimlessly, but to examine the process and relish and develop the art of problem-solving. The Review features something very different this time! Do check it out. And we conclude with *Math Talk* by Padmapriya Shirali- featuring math conversations that showcase good teaching.

We conclude with a quote from Behar:

> *Good teaching is not the application of any doctrine, old or new. It is judgment: knowing when to explain and when to let children struggle, when an activity deepens understanding and when it merely fills time, which things must be automatized through practice so that the mind is free for thought.*

Join us in the journey to build classrooms 'where the mind is without fear and the head is held high!'

Sneha Titus
Chief Editor, At Right Angles
AtRightAngles.editor@apu.edu.in

* https://shorturl.at/Zxaer

Editorial Office
Azim Premji University, Survey No 66,
Burugunte Village, Bikkanahalli Main Road,
Sarjapura, Bengaluru, Karnataka - 562125
Email: publications@apu.edu.in
Website: www.azimpremjiuniversity.edu.in

Design
Zinc & Broccoli
Bengaluru, Karnataka

Print
Repromen
Bengaluru, Karnataka

At Right Angles is a publication of Azim Premji University which provides quality mathematics learning resources for school teachers. It intends to facilitate more experiential and meaningful teaching-learning processes, not just inside the classrooms but also in the broader context of school processes. To celebrate purposeful and passionate teaching, At Right Angles showcases practical insights grounded in the realities of India and its diverse communities.

Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Azim Premji University
A publication of Azim Premji University

Features

Classroom

The Joy of Mathematics

Review

Pullout

Manipulatives are not Magic: Ground Realities in the Mathematics Classroom

Nandita Jayaraj

For decades, numerous initiatives have been trying to transform mathematics education through the "math lab" approach. Is this a worthy quest? How is this transformation measured? And why is it that little turnaround has been reported?

When the geologist Jean de Heinzelin was slicing out sections of rock from the shore of Lake Edward in Congo in the 1950s, he was not surprised to come across the occasional human bone or human-made object. After all, the region—called Ishango—was known to have been occupied by prehistoric human settlements. What he definitely did not expect was to unearth the (possibly) earliest known example of a mathematical instrument.

We still don't know for sure what the mysterious patterns on the "Ishango bone" signify, but some experts are convinced they are evidence that our ancestors from 20,000 years ago were already playing with numbers, arithmetic operations, progressions and prime numbers. Whatever the actual story of the Ishango Bone, it is a striking example of how physical objects can profoundly shape our early understanding of mathematics.

The latest ASER report (2024), a nationwide rural household survey that reached nearly 6.5 lakh children across 605 rural districts in India, once again highlighted that there continues to be grave problems with how mathematics is being taught and assessed at our schools. Less than 50% of Class 8 students were able to do basic arithmetic. This is concerning because it is established that proficiency in mathematics is indicative of children's later life success in employment as well as in independent living.

Usha Menon, educationist and founder of Jodo Gyan, wrote in an article: "Children today… need not become calculators like some people had to in the nineteenth century. They need to become people who can pose new problems to solve in order to face today's challenges, whether of climate change or of increasing inequalities. They need to learn to think quantitatively and not just numerically, to think relationally and not just procedurally."

Getting under the skin of mathematics

In recognition of this, governments, NGOs, schools and teachers have led various initiatives over the past decades to improve the situation. One of the strategies deployed to achieve our foundational numeracy goals has been the inclusion of Teaching Learning Material (TLM) such as manipulatives in the classroom. Examples of math manipulatives or *mat(h)erial*—as they are known in Azim Premji University's Math Space—are many. There is the ganitmala, a chain of beads used to learn whole numbers; Dienes blocks or Flats-Longs-Units used for basic arithmetic operations; and arrow cards used to understand place value.

Keywords: *Teaching Learning Materials (TLM), exploration, play, attitudes, transformation, pedagogical benefit.*

When there is physical material in the classroom, children are inclined to pick them up and play with them. In doing so, such interactions can help children develop intuitions about number and mathematical relationships. Garima Bhatt, who is a mathematics teacher at the Azim Premji School in Uttarakhand, sometimes leaves the material with her students in class. "At least in the primary grades, the joy of materials should not be taken away and students should not depend on the teacher to learn everything," she said. Often, she enters class only to be accosted by eager children who have been playing games with the objects. "I realised that the children had looked at the instructions at the back of the box or interpreted the rules through the pictures."

"You really don't have to teach them," reaffirmed Swati Sircar, who co-created and runs the university's Math Space. "Once we left a 2-metre scale fixed to the ground of a classroom in Uttarkashi," she recalled. When they returned, they found children stretched out on the ground excitedly measuring their heights. "The children have access to the objects even when mathematics class is not happening. And they learn even faster when there are older kids in the room." She remembered a Montessori school teacher's amazement. "Our kids have been adding and subtracting their parents' phone numbers!" the teacher raved to her, marvelling at their comfort with ten-digit numbers.

Though it is very likely that manipulatives were being used in mathematical learning in pre-colonial times (using pebbles, seeds, shells, sticks, etc.), the teaching of mathematics became largely restricted to the blackboard and textbooks in the past century. The influence of the educationist Maria Montessori resulted in Montessori schools cropping up around the country in the 1900s. These schools used manipulatives to teach math and other basic skills in early education, but they were the exceptions. A slew of programmes by the government in the later years of the 20th century, such as the District Primary Education Programme (DPEP), reintroduced the idea of teaching math in more tactile ways. This has been reiterated with every new report about primary education, notably the National Curriculum Framework 2022 and the National Education Policy. Consequently, more and more schools in India have installed math labs in the hope that this will tackle the prevalent attitude of 'math phobia' among students.

It doesn't always work

More than 75% of the schools which participated in the ASER survey had received TLM and/or funds to make or purchase TLM for foundational literacy and numeracy. While these are encouraging signs, those who work on the ground observe that merely having a math lab in the school will not do. It all comes down to how they end up being used. Can pupils use a particular manipulative in such a way that it connects with their existing knowledge, and hence is meaningful to them? Is the manipulative used in such a way that it requires reflection or thought on the part of students?

"The main thing is that all students in a class should have access to some concrete material, so that it's not just the teacher demonstrating," stressed Swati. Considering the high strength of a typical Indian classroom, it is not realistic to aim for one set for each student, but according to her, 7 to 8 sets for a class of 30 is adequate. While this may be achievable by the more elite private schools, for the majority of primary schools in India, even this ends up being a costly affair.

The problem is exacerbated because six- to ten-year-olds aren't exactly the gentlest of handlers. "Some students end up breaking things while playing with them. At times, they take the small cards or tokens home without permission," said Garima. Wary of such instances, many schools end up limiting access of students to math labs, sometimes to such an extent that Swati dubs them "math temples". "The students are taken there in a line, not allowed to touch anything, they just do one round of *darshan* and then go back to class," she lamented.

A thrifty way out of this obstacle is to use material that is easily available, cheap, and convenient to fix or repair when broken. Making and distributing low-cost or zero-cost math kits has been one of the main activities of Swati's Math Space during its 12-year history. Students from the university, predominantly the BSc-BEd pursuers, are signed up to conceptualise and create mat(h)erial using recycled material sourced from the campus ecosystem such as cardboard, broomsticks, discarded packaging material, etc. "The moment material goes to kids, you should expect wear and tear, some will get lost, etc. Instead of fretting over that, you can just replace them!" said Swati.

Using locally sourced material has another advantage. Studies show that familiarity with a resource helps in learning. This familiarity can be achieved by using material that students come across in their context. For example, a Hungarian classroom may use chestnuts to learn counting, but chestnuts may be completely unfamiliar to a student in a town in Telangana. The young student may end up distracted by this strange object and unable to relate to the teacher's storytelling. Groundnuts may be a better example.

"If a learner is very conscious of various attributes of the resource it is unlikely to facilitate its use as a representation of a specific mathematical structure," wrote researcher Jenni Back in an article published by the University of Cambridge. "One resource that I frequently use in work on geometry is loops of string and I find that unless I let learners of any age have a chance to play freely with the string for a while before setting a mathematical task, they will be distracted by their desire to play and explore various properties of the loop of string."

Right across the corner

An alternative to a math lab is a "math corner"—an interactive learning space within the classroom itself. "See, mathematics in the primary stage doesn't need electrical connections, water inlet and outlet, and any fancy stuff," pointed out Swati. "You really don't need to take kids to the lab every math period. The material can come to the child, instead." In the math corner model, the material is kept in closer and more constant contact with the students. This alternative is also suitable for resource- and space-limited settings.

Shivasharanappa, who teaches math at the Azim Premji School in Kalaburgi, mentioned that his students lose 8 to 10 minutes in transit to and from their math lab which is located on a different floor of the school. Garima's classroom, on the other hand, uses the math corner model, and hence her students can save this precious time.

Sometimes, Garima feels that a dedicated room for the math lab would make her students take the activities more seriously and keep the manipulatives safe. Moreover, "the place will have its own vibe, like the school library or science lab." That said, Garima realises that this comes with a risk. "It should not be the case that the material is kept protected, but out of reach of the children."

These days, the concept of the math lab has become quite popular, even fashionable. It was not so in the 1990s, when SN Gananath started Suvidya Education Resource Centre in Bengaluru. On the surface, this is a pleasant trend, but Gananath cautions that actual impact can only come if there is a constant and continuous engagement with teachers—preferably face-to-face, but if not, at least online. He noticed that when his organisation was invited to schools to deliver one-off workshops, the response was mixed. "They will all clap and everyone is happy, but we need more. It can take 3 to 4 years for teachers to change their views and develop confidence in the approach," he stated.

According to Gananath, for the math lab approach to be effective, a fundamental paradigm shift in the user mindset is essential. What he means by this is that most students, teachers and teacher-trainers have internalised the belief that mathematics is about procedures and computation. "It's not that they are lazy. They are not aware of the philosophy of why mathematics is taught in schools, so it is difficult for them to appreciate, accept and implement new ideas."

When sustained engagement happens, the results are tremendous. Gananath fondly remembered teachers from rural Karnataka who came for Suvidya's workshops thirty years ago, took the message to other districts, and still practise what they learnt then. "One teacher would travel from Kolar to Bengaluru for monthly workshops three whole years. She had actually been appointed as an English teacher but after this, she requested the Principal to change her subject to mathematics. Her fear of math disappeared and she started teaching it!"

Educationists and primary school math teachers seem to largely agree on one thing: manipulatives *do* work in stimulating curiosity, exploration, and observation among young learners. However, for this to happen, teachers need to play first, reminded Sneha Titus, editor of At Right Angles and co-creator of Math Space. "When the teacher seizes learning opportunities and teaching moments from this confident yet playful approach, it builds mathematical skills of documentation, reasoning, conjecture, argument—all of which lead to proof."

This is when the "magic" happens.

References

1. Royal Belgian Institute of Natural Sciences. (2024, March). *The Ishango bone* [Brochure]. Royal Belgian Institute of Natural Sciences.
2. Baur, J. (2026). *The Ishango bone: Evidence for intentional arithmetic design in the Upper Palaeolithic?* (Version 4). arXiv. https://shorturl.at/MltnU
3. ASER Centre. (2025). *ASER 2024 national findings.* ASER Centre. https://shorturl.at/WzxIT
4. Geary, D. C. (2011). *Consequences, characteristics, and causes of mathematical learning disabilities and persistent low achievement in mathematics. Journal of Developmental & Behavioral Pediatrics, 32*(3), 250–263. https://shorturl.at/ZSW15
5. Menon, U. (2021, December 27). *What is foundational numeracy?* Jodo Gyan. https://shorturl.at/xh8cm
6. Back, J. (2013, October 8). *Manipulatives in the primary classroom* [PDF]. NRICH, University of Cambridge. https://shorturl.at/6ASwI

NANDITA JAYARAJ is a freelance science writer, author and co-founder of TheLifeofScience.com. She is currently also a science communication consultant at Azim Premji University. You can reach her at nandita.jayaraj@apu.edu.in

And I am a *Mat(h)erial* Girl!

Swati Sircar

The term *Mat(h)erials* refers to a playful contraction for Teaching Learning Materials (TLM) for mathematics. Since there will be readers who are not familiar with all or some of the *mat(h)erials* mentioned in this article, a picture guide is given in Tables 1-4.

Based on working with many people – teachers and teacher educators – and organizations working with school level mathematics for almost two decades, it seems to me that we may have passed the point in history when we were debating why *mat(h)erials* are needed to teach mathematics which is by nature abstract. I have observed that there is a growing consensus that these are needed for primary (i.e., till Class 5) but the jury seems to be out on their use in the middle school stage (i.e., for Classes 6-8).

Interestingly, the new set of NCERT textbooks embed a lot of *mat(h)erials* within chapters and include illustrative sample sheets at the end of each textbook for Classes 3-8. The National Curriculum Framework for Foundational Stage (NCF-FS, 2022) and National Curriculum Framework for School Education (NCF-SE, 2023) both emphasize the use of *mat(h)erials*. In fact, NCF-SE specifically mentions algebra tiles (a set of tiles which are used to represent commonly used algebraic terms). Note that algebra tiles are not needed until the middle school stage!

Very encouragingly, these recommendations seem to have been taken to heart in the new NCERT textbooks where the following striking changes have been observed. Chapters involving Numbers and Operations, Fractions, and Integers show active use of a range of *mat(h)erials*. These *mat(h)erials* are used not just to introduce each kind of number but also for operations. Chapters involving geometry, spatial understanding, measurement, and mensuration utilize and suggest the use of appropriate *mat(h)erials*.

Mat(h)erials: What do they look like?

This is a list of *mat(h)erials* mentioned in this article. Links in the references provide details of how they may be made. Some of these mat(h)erials have been reviewed in past issues of At Right Angles and links to the reviews are provided in these cases.

Keywords: *TLM, mat(h)erials, math lab, manipulatives, concrete, exploration, precision, optimization, assessment, low-cost, free play.*

For whole numbers and decimals:

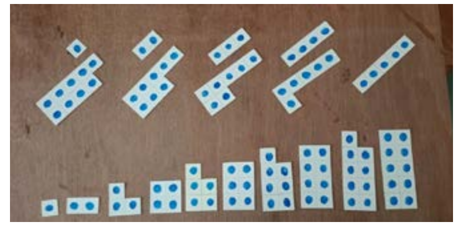
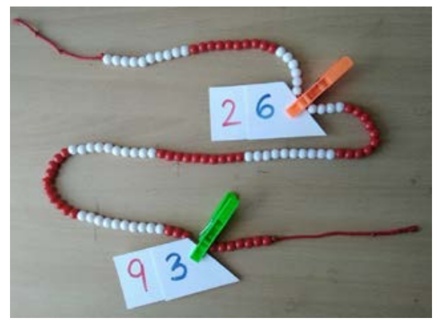
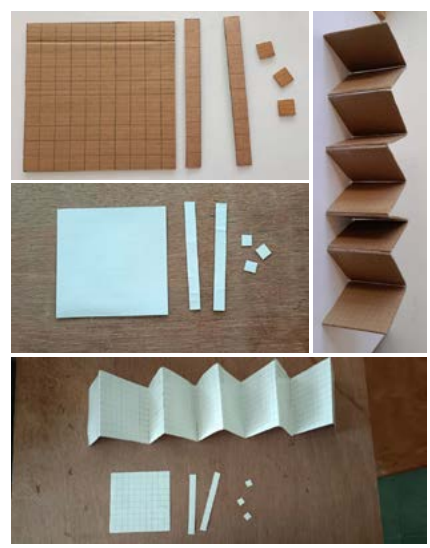
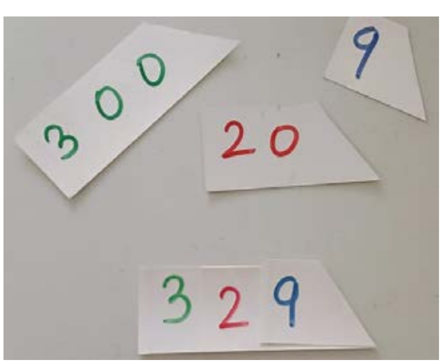
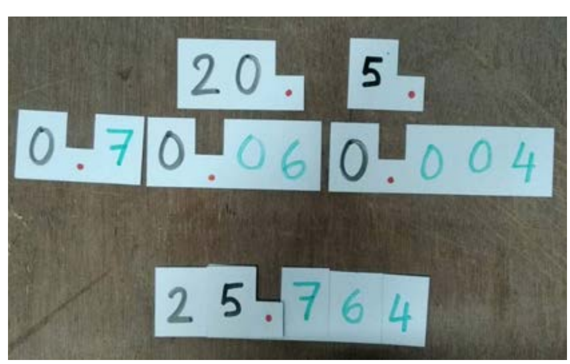
Counters: anything that can be counted, can be buttons, seeds, pebbles, etc. – review [11] 	Ten-frames: a two-by-five frame to represent numbers – review [12] 
Ganitmala: a hundred beads mala in two colours for whole numbers and operations – review [13]	
Bundle-sticks: anything that can be bundled, can be twigs, sticks from broom, ice-cream sticks, dried sketch pens [14]	
Flats-Longs-Units: 2D base-10 blocks with a unit or small square, a long (ten) which is a rectangle 10 times longer than the unit and a flat (hundred), which is a bigger square, 100 times the small one or 10 times wider than the long – ten big squares are joined to represent a thousand – review [15] • Big FLU for younger students in Foundational stage – unit is 2 cm × 2 cm • Small FLU for older students in Preparatory and Middle stages – grid-size ≈ 1cm [15] • Decimal FLU – big square (10 cm × 10 cm) is 1, rectangle (10 cm × 1 cm) is 0.1, small square (1 cm × 1 cm) is 0.01 [16]	
Arrow Cards: [17], can be extended for decimals – review [17]   	

Table 1

For Fractions: [18]

<table>
<tr><td colspan="2">Fraction Circles and Sectors: the whole is a circle, split in equal sectors with unit fractions including ½, 1/3, ¼, 1/5, 1/6, 1/8, 1/9, 1/10, 1/12, 1/15
</td></tr>
<tr><td>Fraction Strips: the whole is a 21 cm strip with unit fractions including 1/7 and 1/14</td><td>Fraction Squares: the whole is a 21 cm long square from A4 sheet [19]</td></tr>
</table>

Table 2

For Algebra:

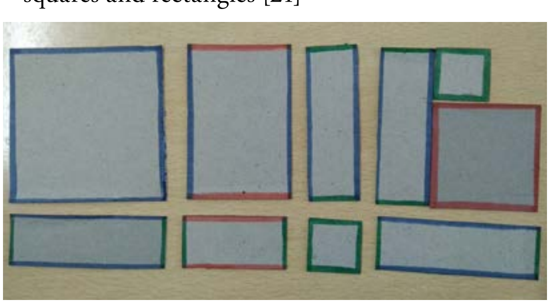
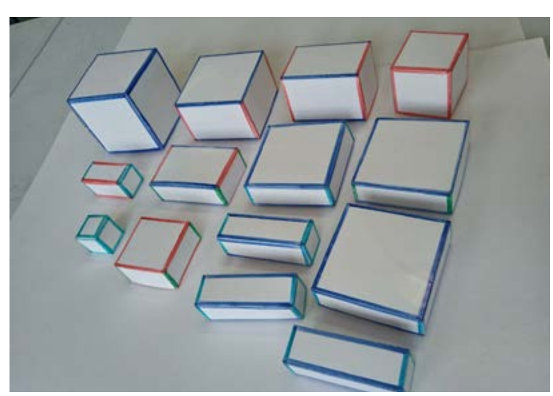
<table>
<tr><td>Algebra Tiles: generalisation of FLU, big square is x^2 while rectangle is x, small square is 1 with possible extension to xy, y and y^2 – review [20]</td><td></td></tr>
<tr><td>Algebraic Identities: models unpacking algebraic identities
Quadratic: identities like $(a \pm b)^2$, $a^2 - b^2$ etc. – squares and rectangles [21]
</td><td>Cubic: identities like $(a \pm b)^3$, $a^3 \pm b^3$ – cuboids [22]
</td></tr>
</table>

Table 3

For Geometry and Mensuration:

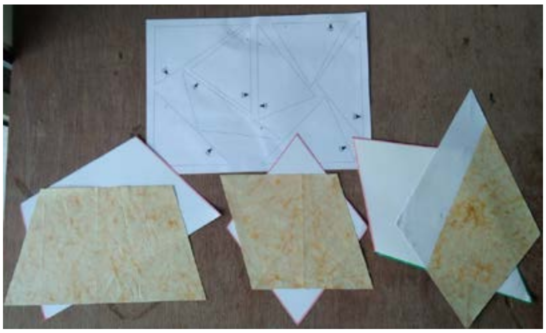
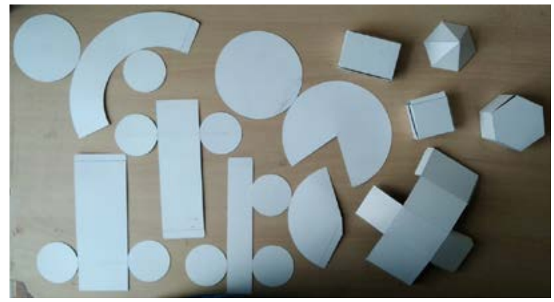
Cutouts of triangles and quadrilaterals – a polygon may be cut in several pieces and rearranged to find the area [23] 	Nets of Solids to understand surface area and much more – of cuboids, prisms, pyramids, cylinders, cones, etc. [24] 
Straw Models are made of straws by stapling them – for angles and polygons [25] 	Angles with intersecting circles: two or three intersecting circles in different colours to introduce and to understand angles as 'turn' and 'gap', and to get a sense of measure of an angle [26] 

Table 4

Details of the various mat(h)erials and learning sheets included in the NCERT math textbooks are shown in Table 5.

Class 3	(i) Rectangular and isometric dot sheets (ii) Number name and numeral cards (iii) Nets of various solids (iv) 2D base 10 blocks (v) Fraction sectors and strips (vi) Tangram
Class 4	(i) 2D shapes (ii) Dienes' blocks (iii) More nets (iv) Arrow cards (v) Fraction sectors (vi) Centimetre scale (vii) Number tokens (1, 10, 100, 1000)
Class 5	(i) 2 types of square grids and the isometric dot sheet (ii) Fraction sectors (iii) Tangrams (iv) Regular polygons and more (v) Number tokens (vi) Nets of 2 platonic solids
Class 6	(i) Integer snake-ladder game (ii) Rectangular dot sheets and the square grid (iii) Tangram (iv) Fraction wall (v) Square tiles split along a diagonal
Class 7	(i) Tangram (ii) Square dot sheet, etc.
Class 8	(i) Binairo game (ii) Square dot sheets (iii) Isometric grid

Table 5

Some of these manipulatives may be unfamiliar to the reader, but the data in Table 5 clearly shows that *mat(h)erials* are now perceived as an integral part of the pedagogy of mathematics, right up to Class 8!

Here is a fun fact! The page numbering of the Class 7 textbook is inspired by a specific representation that is available as a virtual manipulative in Mathigon Polypad (more later)! The textbook did compile the numbers at the end of the corresponding chapter but nudged the students to decode it (Figure 1). So, these textbooks intend to let students discover rather than be told. And they want students to observe the mathematics that is all around them – including the page numbers in their textbooks!

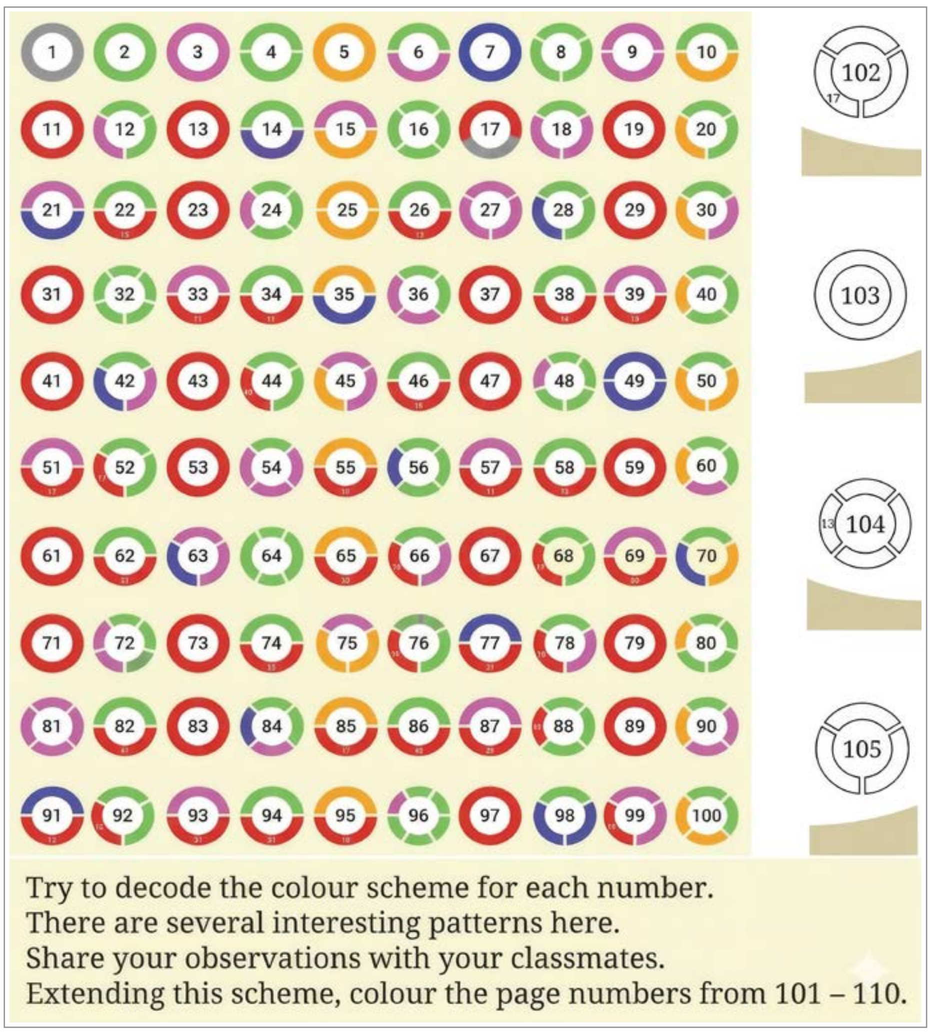


Figure 1

Mat(h)erials have been given official recognition! It remains for us to discuss how they may be meaningfully and mindfully sourced and used.

This article is based on the experience of equipping eleven schools with *mat(h)erials*. Eight of these are Azim Premji Schools (APS) (see reference). Of the remaining three, one is an NGO school in Pokhrama, Bihar catering to the students from the village, another is an alternative school in north Bengaluru catering to students from all sections of society and the last is a high-end school in east Bengaluru.

Most of these schools do not have a spare room that they can dedicate for a math lab. But the teachers of each class took charge of the appropriate *mat(h)erials* and have been using them regularly. They bring these to class as and when needed. Some keep them in a cupboard in the classroom itself. Students, especially those in Class 1-2, eagerly participate in tidying away and storing the mat(h)erials properly at the end of the class.

Mat(h)erials: Who gets them?

Usually, if a school has some *mat(h)erial*, then it's the teacher who displays them and who is in full control demonstrating their use to the entire class. In a marginally improved situation, the teacher may allow some students to come to the front of the class and demonstrate their use. Just imagine what goes through the mind of a student in Class 3 who doesn't get a chance to explore and demonstrate that day. How much will this child focus on learning *vs* feeling left out? And that too for practical constraints. Instead, what if we can provide one set of *mat(h)erials* for every four students? Then everyone in the class gets access and can engage directly. But wouldn't that be too expensive? Except for one of the eleven schools, the *mat(h)erials* were made on site. In eight of the schools, intensive work was done for a week each for two years, along with some long-distance resource sharing and online supervision.

Mat(h)erials: Who makes them?

Most of the things can be made in a school by the teachers, with the help of students from middle to high school. When students of Class 6-8 engage in making *mat(h)erials*, they experience the precision that mathematics demands, as well as the optimization of raw material and effort. Many facts fall into place conceptually. It also becomes an assessment of measurement as well as various skills and attitudes that they should develop, for example, drawing a line with a given length, drawing lines which are parallel or perpendicular to given lines, drawing a given angle. There are rules in *mat(h)erial* making, but there are reasons behind them. I have seen that students find greater motivation and accountability when they hear that these will be used by several batches of students, junior to them (foundational and preparatory – Class 1-5) or by their peers! Art teachers can be of great help. (And they do get some things in return which can be utilized for art classes later!)

Usually, we create suitable containers for each *mat(h)erial* so that it becomes easy to use and then put away. These are also made using locally available material and are low-cost.

Some manipulatives require a bit of skill, an eye for detail, etc., but nothing beyond the capabilities of a Class 8-9 student in terms of construction. Step by step guidance on how to make each item – or its packet – can be provided on request [10].

Mat(h)erials: Math lab or Corner or Cupboard?

Most *mat(h)erials* are not big, can be easily carried around, especially if they are segregated in user-friendly packets. Unlike a science/EVS class or a physical education class which needs a laboratory or an open space, there is absolutely no need to bring students of Class 1-5 to a separate room to work on these TLM. In fact, valuable time is lost by moving from one class to the other with no meaningful gain. In my opinion, a math lab is therefore, not needed at the foundational and preparatory stages.

However, there are some models that may be needed only occasionally and can be too big or delicate to move around at the middle or higher stages. For example, I personally feel that it helps to have a big cube with 100 cm long sides to represent ten lakh (or a million), compared to a smaller one with 10 cm sides for a thousand and a tiny one with 1 cm sides for one or unit. By extending this to a set of cuboids, the powers of ten: one, ten, hundred… ten-lakh may also be seen. Figure 2 gives an idea of how big a lakh is, and how useful this representation can be at the beginning of Class 6. But it would not be easy to bring these to class. For such cases, it may help to have a separate room where students can come occasionally.

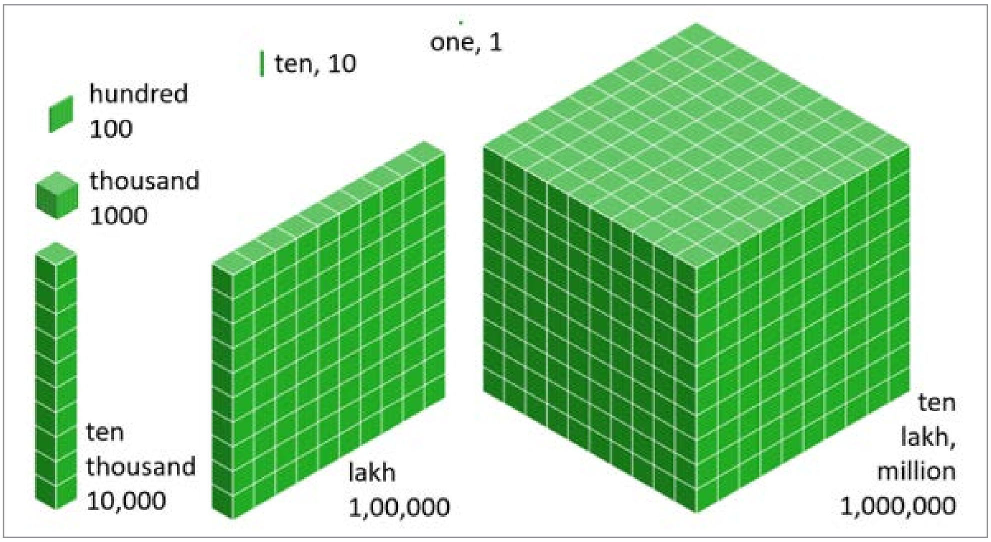


Figure 2

If such a room is possible, then it can be utilized not only to store and display *mat(h)erials* but also as a place where these can be repaired or made as and when the need arises. One must assume (or rather, hope!) that there will be some wear and tear when anything gets used regularly – it is a positive sign when students examine and play with such artefacts regularly! When teachers make their own *mat(h)erials*, they will know how to duplicate or repair these. However, the same repair or replication can also happen in an art room, teachers' room or any other suitable place.

Mat(h)erials: What are they made of?

Most of the items are made of (i) thick chart paper (ii) corrugated cardboard (iii) boxes with card type thickness, e.g., cornflakes or detergent boxes. A few other materials such as beads, rope, flex (laminated banner or signage material), plastic straws are also used for some specific mat(h)erials.

Different sets are kept in separate packets which are usually made out of used A4 sheets or newspapers, various plastic packets and other packaging material usually available at a school or brought in from home by students – a practical strategy for recycling and leading by example.

Mat(h)erials: How do we plan and organise?

These are the must have *mat(h)erials* for each stage.

(Colour Code: Blue = individual sets, Pink = one set for the whole class, otherwise multiple sets)

Mat(h)erials	Foundational Stage (pre-primary, Class 1-2)	Preparatory Stage (Class 3-5)	Middle Stage Class (6-8)
Numbers and Place Value	1. Counters – round and square 2. Ten-frames 3. Ganitmala 4. Arrow cards 5. Bundle sticks 6. Flats-Longs-Units (FLU) – big	1, 2, 4 and 7. Flats-Longs-Units (FLU) with thousand – small	8. Arrow cards (till crores) 9. FLU small
Fractions and operations	NA	10. Fraction sectors 11. Fraction strips 12. Fraction squares	
Decimals and operations	NA	13. Decimal FLU 14. Decimal arrow cards 15. Centimetre graph paper	
Measurement and Mensuration	16. Toothpicks/match sticks/ paper clips 17. Balance scales	18. Squares of size matching toothpicks/ matchsticks/paper clips 19. Measuring tapes 20. 17 with weights 21. Calibrated bottles/jars	22. Straw models for perimeter 23. Cutout models for area 24. Nets of solids for surface area
Patterns and Algebra	25. Counters in at least two colours 26. Toothpicks/match sticks in ideally two colours		27. Algebra tiles 28. Algebraic identities – quadratic and cubic
Geometry and Spatial Understanding	29. 2D shapes cutouts 30. 3D solids 31. Geoboard	23-25 and 32. Tangrams 33. Polyominoes	34. Nets of solids 35. Straw models 36. Angles with intersecting circles
Who can make	Teachers, students in higher grades – Class 8-9	Mostly teachers and senior students, preparatory students can help by cutting some things	Mostly by students themselves, can be done as group projects

Mat(h)erials: Do these have digital avatars?

This article will be incomplete unless we talk about virtual or digital manipulatives, especially post pandemic. There are many websites that provide such manipulatives. These have certain advantages over physical ones, since things can be explored much faster compared to if done offline. My favourite website is Mathigon Polypad: truly a mathematical playground. There are many opportunities of learning by discovering and observing patterns. But students may need some guidance especially in documenting what they observe and then drawing conclusions based on that. Math Learning Centre, Toy Theatre and a few others also have decent collections of digital manipulatives. We may review them in a later issue.

Some schools have computer labs. For such schools, it makes sense to use online manipulatives instead of physical ones. My personal opinion is that this should be done only from the middle stage. Children at the Foundational stage (pre-primary and Class 1-2), should use only physical manipulatives. In the Preparatory stage, (Class 3-5), they should use mostly physical manipulatives – though one can initiate students into online ones possibly around Class 4 or 5. This is mainly to avoid screen addiction. While students do benefit from exposure to such sophisticated tools, the adults must decide when, how much time and how such tools should be used. These tools are better used with adult supervision as students could get overly engrossed in using them.

Mat(h)erials: What can you do?

This article is intended to give the reader a sense of what exists and what are the possibilities – in a classroom, in a school. We hope to bring a series on how you can make *mat(h)erials* in your school, possibly with the students. We will share 'how to make', (especially multiple sets), as well as 'how to use'. Meanwhile, here are some questions that a teacher willing to enrich his/her teaching with *mat(h)erials* could reflect on:

1. Why do you want to make *mat(h)erials*?
2. In your personal opinion, do these need to be housed in a math lab? Why?
3. How many sets do **you** need of each *mat(h)erial* that you would consider making? (It will help to consider the number of students in each class and whether the same sets can be shared among different classes.)
4. Who will help you in making these *mat(h)erials*?
   [Consider your fellow teachers including art teacher, selected students from middle to high school (Class 6-10) who are good with their hands.]
5. How will your class change physically once you have these *mat(h)erials*?
   [Consider arrangement of chair-table, benches.]
6. Will your pedagogy and assessment change? How?

References:

1. Why mat(h)erials?
   http://publications.azimpremjiuniversity.edu.in/1852/1/18_Why%20TLM%20%E2%80%93%20Its%20Aims%20and%20Uses.pdf
2. Mat(h)erials: https://sites.google.com/apu.edu.in/mathspace/matherials
3. NCERT textbooks: https://ncert.nic.in/textbook.php
4. NCF-FS:
   https://ncert.nic.in/flipbook/NCF/National_Curriculum_Framework_for_Foundational_Stage_2022/mobile/index.html
5. NCF-SE: https://www.ncert.nic.in/pdf/NCFSE-2023-August_2023.pdf
6. Mathigon polypad: https://polypad.amplify.com/p
7. Math Learning Center: https://www.mathlearningcenter.org/apps
8. Toy Theater: https://toytheater.com/
9. Azim Premji Schools: https://azimpremjifoundation.org/what-we-do/education/azim-premji-schools/
10. Contact Math Space at mathspace@apu.edu.in
11. Counters: https://shorturl.at/70xz9 Review https://shorturl.at/7EI9l
12. Ten-Frame: https://shorturl.at/mviMk Review https://shorturl.at/Aiz2r
13. Ganitmala: https://shorturl.at/LHDyW Review https://tinyurl.com/38rcjbcf
14. Bundles-Sticks https://shorturl.at/HISoe

15. Flats-Longs-Units https://shorturl.at/2GFli Review https://tinyurl.com/2wehbacv
16. Decimal Flats-Longs-Units https://shorturl.at/unslv
17. Arrow Cards https://shorturl.at/QSXxn Review: https://shorturl.at/okMQS
18. Fraction kit: https://tinyurl.com/hffpn6bh
19. Fraction Squares: https://tinyurl.com/4wt7kmeh
20. Algebra Tiles: https://shorturl.at/3wirZ Review https://shorturl.at/lj6DH
21. Algebra Identities: Quadratic https://shorturl.at/8is1e
22. Algebra Identities: Cubic https://shorturl.at/8dvJc
23. Cutouts: https://tinyurl.com/2bb65cvt
24. Nets of Solids https://shorturl.at/V9ac4
25. Straw Models: https://tinyurl.com/h9t7hc7a
26. Angles with Intersecting Circles: https://shorturl.at/3fbU4

SWATI SIRCAR is a consultant with Azim Premji University. Mathematics is the second love of her life (the first being drawing). She is a B.Stat-M.Stat from Indian Statistical Institute and an MS in math from University of Washington, Seattle. She has been doing mathematics with children and teachers for almost 20 years and is deeply interested in anything hands on – origami in particular. She initiated Math Space, a mathematics laboratory at the Bengaluru campus of Azim Premji University. It caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. Swati may be contacted at swati.sircar@apu.edu.in

Concrete to Concept: Teaching Mathematics that Sticks

Sonia Kundu

Let us begin with a simple yet significant question: *Can we, as educators, create a meaningful path? How can children move from counting objects to understanding written algorithms?*

Often, the word "algorithm" evokes images of rote memorisation and mechanical steps. But imagine a child who not only understands the concept behind an algorithm but also enjoys discovering it, eventually developing the ability to use it independently and efficiently because of true conceptual understanding. I have realised that this path isn't always straightforward. What truly supports this journey? Is working with Teaching Learning Materials (TLMs) enough, or do we need more? In this article, I share the trajectory that I adopted, which was based on the following principles:

- Beginning with children's familiar contexts before introducing formal notation;
- Using concrete materials not merely as aids, but as representations of place value;
- Allowing children's informal strategies to surface before introducing the standard algorithm;
- Moving gradually from concrete materials to representational tools and then to symbolic notation;
- Treating "carrying over" not as a rule to be memorised, but as the result of regrouping ten ones as one ten.

These principles guided my planning and helped me to incorporate thoughtfully designed materials and real-world examples into the lesson. By facilitating engaging conversations, I hoped to nurture deeper learning and empower children to connect ideas meaningfully rather than mechanically.

Let me take you through the journey of my students as they transitioned from foundational number sense in Class 1 to applying the algorithm of adding 2-digit numbers with carry-over in Class 2.

By the end of Class 1, my students had already developed a strong sense of number, using bundles (of ten) and units to count up to 99. They were also adept at counting-on strategies, such as solving problems like 16 + 3 by keeping the larger number in mind and counting forward. At that stage, their learning objective was to add numbers only up to 20, and they practised addition using pairs of appropriately selected numbers.

As they began Class 2, I knew we had to deepen their conceptual understanding of the algorithms of addition, of numbers between 10 and 99 (chosen such that the sum was always less than 100), but we had to do this in a way that remained meaningful and rooted in the child's lived experiences. So I designed a sequence of real-life situations, woven into their context.

Keywords: *addition, regrouping, child's context, understanding, application.*

Relating to Familiar Contexts: A Morning at the Shop

During our morning circle, I shared a story:

"Yesterday I broke my piggy bank because I wanted to buy things such as noodles, butter, etc. I went to Chauhan ji's shop to buy butter. There were three others at the shop, and he kept ignoring me!"

I then asked my students, "Why do you think he was ignoring me?"

Their responses were full of insight and humour:

- "Maybe he wanted you to be late to school."
- "Maybe the other people were his relatives."
- "Maybe you didn't have money."
- "Maybe you had too many coins!"

At this point, I revealed: "Yes! I had only one-rupee coins."

This led to a spontaneous conversation about their own experiences with coins, delays, and shopkeepers.

Roleplay and simulation with the use of concrete materials

To bring the idea to life, I created a roleplay. I brought empty packets of butter, biscuits, and noodles into the classroom and distributed them among small groups. Each of these items had been chosen because they were familiar to the students and their costs were between ₹10 and ₹99. Each packet had a price tag (noodles ₹10, butter ₹57, biscuit ₹5).

I had also brought in about 75 one-rupee coins (as specimens), and asked five of the students to perform a scene in which they had to find the total of the bill. The children threw themselves into the roleplay, adding their own stories, which added empathy and humour to the storyline, and showcased their deepening understanding of money and counting strategies.

After dividing them into groups I asked them to find the total. (I kept the coins with myself, wanting to see how they grappled with the mental mathematics.) I observed animated discussions and the adoption of different strategies. Some groups used direct addition with the counting on strategy, starting from the price of the butter. A few groups split the calculations - one child adding tens, another adding ones, while others rounded up, thinking of 57 as 60 and then adding only 2 more instead of 5 for the price of the biscuit (they added 5 to 57 in 2 steps: 57 + 3 + 2). A few compared answers between peers, leading to small debates like, "No, you forgot the noodles," or "Let's add again slowly." This peer interaction helped them self-correct without immediate teacher intervention.

Towards Efficient Strategies

The roleplay led us to a critical thinking moment. I asked, "What can I do next time to save time at the shop?" They said, "Bring notes instead of coins."

I asked, "If I only have 10-rupee notes, how many notes and how many coins should I take instead of the 75 one-rupee coins?" Most answers were accurate "7 ten-rupee notes and 5 one-rupee coins." One child said: "Or 50 + 20 + 5!" (taking 50 as 5 tens and 20 as 2 tens) I encouraged them to think further as I observed their flexible thinking and decomposition of numbers. The transition from counting and adding units to regrouping the numbers into tens and units, was planned not through direct instruction, but through exploration and shared reflection. As students worked with real-life prices, they naturally began regrouping numbers by separating each cost into tens and ones, collecting tens together, and then adding them separately from the ones. This hands-on process helped them understand place value in a meaningful way and paved the way for column addition of larger numbers.

Starting with physical currency helped children move from concrete understanding to abstract reasoning. The classroom discussion allowed them to articulate and revise their thinking. And most importantly, the learning continued after the class ended, showing how joyful and meaningful learning can stick.

The following day, I wanted to deepen the students' understanding of place value by using structured representational materials. I introduced the 2D-Base 10 blocks popularly known as Flats, Longs, Units or FLU where a long strip of ten squares stood for ten units, and each small square represented one unit. I began by simply handing them the long strip and asking them to observe it. "What do you notice?" I asked. Several children immediately pointed out, "There are ten of the small squares in one strip!" This was the perfect bridge.

To make the connection more hands-on, I gave each student a random number of unit squares and asked them to count these. Instinctively, they began grouping the small squares into sets of ten, and as they completed each group, I handed them a long strip to replace it. This natural transition, moving from many small units to fewer tens, helped the idea of *one ten instead of ten ones* to surface naturally.

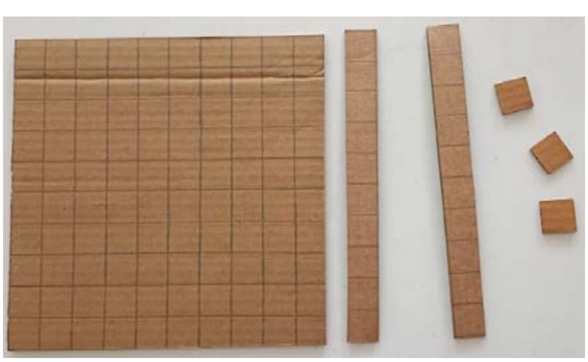


Figure 1: Flats-Longs-Units (FLU)
One Long = ten Units; one Unit square = one

The visual arrangement helped many students see the structure of tens and ones. By seeing the quantity laid out clearly in front of them, they could make sense of the structure of numbers and the logic of regrouping. I asked them to represent each of the prices using this material, which helped them to shift from concrete (coins and shopping) to representational (blocks). I was preparing them for more abstract work ahead and allowing them to engage with place value through their own discovery. Initially, the activity was only on counting and regrouping into tens and ones. There was no expectation to find totals using the material that day.

Why FLU? I used FLU materials instead of Dienes Blocks because I wanted students to work with resources that were easily available, accessible, and practical in our classroom context. While Dienes Blocks are effective for teaching place value, I had limited sets for the whole class. The hand-made FLU materials ensured that every child could actively participate and manipulate the materials independently. They were also cost-effective, easy to prepare, and could be made by any teacher using simple classroom resources.

The next day, we used the FLU kits for addition of 2-digit numbers. I intentionally selected numbers where the addition of ones did not exceed nine (like 31 + 13; 47 + 21). This allowed students to attend to relationships between ones and tens without the added complexity of carrying over. When they recorded their answers, students themselves noticed patterns: ones were added with ones,

Figure 2: Students using Flats Longs Units.
(They wrote with whiteboard markers, which could be erased and did not deface the table.)

tens with tens, and column addition evolved! The algorithm was not introduced by me; it evolved through their own observations.

Only after this understanding was secure did we move to situations where regrouping of ones went beyond nine. While adding 32 and 29, the students arrived at 11 units. At that point, a child referred to a familiar story of Bhola (https://bit.ly/4f2cODg)connecting the idea of regrouping to making groups of ten objects. There was a brief pause. Then, with visible excitement, one child stood up and said, “Yes, you’re right, we can keep one ten in place of these ten ones, as we did earlier while arranging and counting the strips and units.”

In that moment, the idea of “carrying” a ten transformed from a mechanical rule into an efficient algorithm that made sense to the entire class. The FLU representations of 32 and 29 sharpened their attention to a key realization: when ones accumulate beyond nine, they don’t just increase, they reorganise, forming a new ten. What the traditional algorithm calls “carry” was no longer mysterious; it became a visible, logical consequence of grouping.

Students recognised that whenever the sum of ones exceeds nine, regrouping naturally creates a ten and shifts its value to the tens place. The written algorithm, therefore, stopped being an imposed procedure to follow and instead became a meaningful record of their own reasoning. They had been making groups since UKG, but this experience allowed them to connect that familiar action to symbolic notation. In doing so, they did not simply learn how addition works, they understood why it works, and that shift turned a step-based method into genuine mathematical sense-making.

Conclusion

As described in the article, the lesson moved from the students’ lived experience with money and shopping, to representing numbers as tens and ones. Adequate time was spent in replacing ten ones with one ten, using structured materials -in this case the Flats, Longs, Units. At the end of the unit, we moved to understanding the written addition algorithm as a record of this regrouping process.

My teaching trajectory which began with the child's familiar contexts, mindful use of concrete materials, listening to and building on the child's informal strategies, and then graduating to representational tools and symbolic notation made me resolve to use this method for the algorithm of subtraction too!

SONIA KUNDU is a foundational stage teacher since 2018, committed to strengthening early literacy and numeracy. She worked with Azim Premji Foundation, gaining insight into grassroots education realities, and now teaches at Heritage Experiential Learning School, Noida, where she enthusiastically fosters joyful, meaningful, child-centred learning experiences for young learners. Sonia may be contacted at soniakundu1911@gmail.com

In search of Kaprekar's Constant

Suyash Tiwari

I got an extra period in class today. It was towards the end of the day, so I decided not to continue the regular classroom discussion. I was searching through the book *"Apne Haath Ganit"* for a one-period activity that would engage the children. I came across a topic called "Kaprekar's constant." From the name, it sounded like a high school topic, but I verified the Kaprekar constant routine with a few numbers, and it worked! This part was actually simple enough to design an exploration for Class 3 students, which is what I was looking for.

Prerequisites

- They should know how to add and subtract 3-digit numbers.
- They should know how to arrange the digits of a number in ascending and descending order.

Classroom Discussion

As I entered the class, the students were drained after a full day of study and wanted some playtime or a computer class during this extra period. Amid the chaos, I ignored their requests and started writing "Kaprekar's constant" on the board. The children began pronouncing it and asking what it was.

Suyash Tiwari

I said, "Let's first sit quietly for two minutes, and then I will tell you about it." Everyone settled down, and I began: "As a person who does great work in science is called a scientist, what do we call a person who does great work in Mathematics?"

Students

Students started guessing—"Matheist," "Mathintist," and other difficult-to-spell words.

I replied, "We call them mathematicians, and today we are going to talk about a great mathematician from India—Kaprekar ji—and his magical constant number."

> "D.R. Kaprekar was a mathematics teacher in a government school in Devlali, Maharashtra. He liked playing with numbers very much and found many beautiful patterns in numbers that were previously unknown.
>
> (Source: NCERT Ganit Prakash Class-6)"

Keywords: *Kaprekar's Constant, investigation, exploration, repeated operation, addition, subtraction.*

Everyone murmured, “What is the magical constant number?”

I continued, “Let’s do the same work Kaprekar ji did. Take out your notebooks, and together we will find the magical constant number.”

They all eagerly followed, taking their “swords”—their pens—in hand.

“You are all used to 3-digit numbers. But today, let’s start with any 4-digit number. Yes, Maira, tell me a 4-digit number, in which all four digits are not the same.”

“1762.”

Maira

Good, here all the digits are different. Now I will tell you what to do and you all write down your answers in your notebooks.

Start with the 4-digit number		1	7	6	2
Arrange the digits in descending order. This is the first number.		7	6	2	1
Now arrange the same digits in ascending order. This is the second number.	-	1	2	6	7
Find the difference between the first and second numbers.		6	3	5	4
Repeat the process with the new number.					
Descending		6	5	4	3
Ascending	-	3	4	5	6
Difference		3	0	8	7
Descending		8	7	3	0
Ascending	-	0	3	7	8
Difference		8	3	5	2
Descending		8	5	3	2
Ascending	-	2	3	5	8
Difference		6	1	7	4

Table 1: The repeated sequence of operations to arrive at Kaprekar's constant

“Let’s try doing the steps once more—what do you notice?”

We will get 6174 again sir!” they responded.

Will you get 6174 again if the difference was 4716?

"Yes, because the same digits are there: 7, 6, 4, and 1; so we are finding the difference between the same numbers."

"Ok, great!"
"Now, try the same process with another 4-digit number. Let's check whether we arrive at a similar situation. This time, each one of you take your own number and follow the process."

Within a minute, everyone was working like little mathematicians—following the routine of descending order, ascending order, and subtraction. After ten to fifteen minutes, many students came up excitedly:

"We got the same number 6174 again!"

"Yes, you all must have got the same number. We call it Kaprekar's constant—6174. We call it constant because it does not change."

By then, the period was over.

I gave them two numbers to try at home. For some advanced students, I added a challenge: "Is there any similar constant for 3-digit or 5-digit numbers? If you find one, we will name it after you."

The Next Day

"I found the 3-digit constant—495!"

"I also found it!"

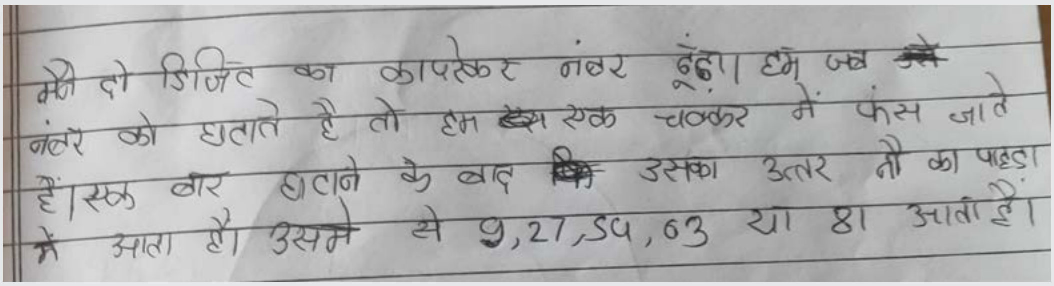
I was not expecting this level of exploration from Class 3 students. Tripti, Maira and Jigyanshu found the 3-digit number. They independently followed the routine on multiple numbers and found the constant. Even I was not aware of the 3-digit constant. I appreciated their work.

"But I couldn't find a 5-digit constant. It does not stop at a single number, but I did see that the same set of numbers repeat."

Jigyanshu

I saw the problem and tried it for myself. Yes, it was not a single number but a set of numbers start appearing in cycles after a few steps. We decided to do further work on this. We found on the internet that for 2-digit and 5-digit there is no fixed number but that there are certain fixed cycles. I told Jigyanshu to work on 2-digit numbers and I myself took 5-digit numbers.

Following were the observations of Jigyanshu for 2-digit number:


मैंने दो डिजिट का कापरेकर नंबर ढूंढा। हमें जब
नंबर को घटाते है तो हम एक चक्कर में फंस जाते
हैं। एक बार घटाने के बाद उसका उत्तर नौ का पहाड़ा
में आता है उसमे से 9,27,54,63 या 81 आता है।

I tried to find if there is a 2-digit Kaprekar number. When we subtracted a 2-digit number from its reverse ordered digits number, we fell inside the trap of a cycle. After the first subtraction step, the numbers that appear are in the tables of 9. Among them, 9, 27, 54, 63 and 81 specifically appear.

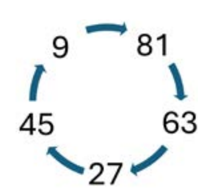
Figure 1: Jigyanshu's work

When I started working on 5-digit numbers, I found an outcome similar to Jigyanshu's observation. After two to five steps, the number starts to fall inside a cycle. But here, there was more than one type of cycle. I used a program to try it on about 25 different 5-digit numbers and found that all the numbers I tried would end in one of the following three cycles.

Figure 2: Three cycles of 5-digit Kaprekar sequence

Key findings

- Like Kaprekar's constant 6174 for 4-digit numbers, 3-digit numbers arrive at a constant 495 when subjected to the same process.
- For 2-digit and 5-digit numbers the process ends in a cycle, not a single constant.
- All the numbers (constants or in the cycle) are divisible by 9.
- Both the constants and the repeating cycles were arrived at in a maximum of 7 steps.
- The starting number should have at least two different digits. (For a number such as 3222, the first difference is a 3-digit number 999, but on taking this number as 0999, we could arrive at the Kaprekar's constant at the fourth step.)



Cycle of 2-digit Kaprekar sequence

Takeaways for teachers

- **Increasing love for numbers** - Generally, students don't ask for more numerical problems but, in this class, the students were asking – "Give us another number, we will find the constant." I was able to get them to practise the new skill of subtraction of 4-digit numbers with regrouping without any external persuasion.
- **Developing persistence and problem-solving skills** - Students find repetitive work boring, but they all persistently solved and found the constant. Some students had to go for 7 steps to find the constant, which is challenging for Class 3 students.
- **Introducing mathematicians- not theory but practical** - Starting with the new word 'mathematician', they got to know details of Kaprekar's life – a simple rural teacher who persisted in his explorations for the sheer love of mathematics. I decided not to tell them directly about Ramanujan, but to start with Srinivasa Ramanujan's famous magic squares (https://bit.ly/4dxCEhg).
- **Confident learning by doing** - After one day, I asked – "Who can come up and talk about Kaprekar constant?" Many hands were raised as they had practised it on their own and were confident that they could explain the process. Also, I gave interested students a chance to explore other constant numbers on their own.

Editor's Note

What started as a simple search for an activity to keep students engaged during a free period, developed into a learning experience for both students and teacher. This teacher became an explorer along with his students and was able to challenge students with different abilities to push their limits and learn from each other. Truly a valuable use of free time!

SUYASH TIWARI has been working with the Azim Premji Foundation since 2023 and is a preparatory stage school teacher at Azim Premji School, Dhamtari. He tries to include art and application-based activities in class to make mathematics more meaningful for his students. Suyash may be contacted at suyash.tiwari@azimpremjifoundation.org

Measuring the Gap

Jivesh Panchbai

Measurement is a basic skill, and measurements determine daily decisions. So, it is not surprising that the NCF (National Curriculum Framework) 2023 continued to give it importance even at primary school. As students, we have been taught the measurement of length by cramming the standard values (such as, 1 m = 100 cm). But current pedagogy begins with the child's perspective – using non-standard units such as handspan and foot length and then reasoning out the need for standard units and associated vocabulary, symbols and conventions. This article is based on Jivesh's first experience of introducing the concept of measurement of length to his Class 3 students.

When I started teaching, I was confident that a story related to the history of measurement would set the stage to achieve my learning objectives. For non-standard units, I would explain that in ancient times people lacked measuring tools and used units like 'Paatha' (a unit used in hilly regions to measure roughly 2 kg of weight). I emphasized that technology later led to precise, standardized tools. Over time, I realized that while this historical context might aid sixth graders, it often confuses third graders. More importantly, I learned that non-standard units are not just relics of the past; they remain relevant today. We don't always carry a ruler, but we often estimate using hand spans, foot spans, cubits, etc.

Planning based on Learning Outcomes

These are the first few Curricular Goals and Competencies related to the measurement of length for the preparatory stage as per the NCF 2023.

Curricular goals	Competency	Learning Outcome
Understand measurable attributes of objects and the units, systems, and processes of such measurement, including those related to distance, length, weight, area, volume, and time, using non-standard and standard units.	**C-3.1** "Measures in non-standard and standard units and evaluates the need for standard units" **C-3.2** "Uses an appropriate unit and tool for the attribute (like length, perimeter, time, weight, volume) being measured"	Understands the importance of standard units over non-standard units; (e.g., in case of length measurement, can tell the situations where they would face difficulty using non-standard units) Knows and uses appropriate standard units and tools to measure length (centimetres and metres)

Table 1: Source https://ncert.nic.in/pdf/focus-group/NCF-SE_2023EN.pdf

Note: The Learning Outcomes were retrieved from the Elementary School Learning Outcomes (2017) found at https://www.ncert.nic.in/learning-outcome.php?ln=en

At the start of the unit, I checked the prior knowledge of the students. I found that 95% of the class could compare lengths, sort objects from shortest to longest/highest, and arrange objects in increasing or decreasing order of length.

Keywords: *measurement, length, standard and non-standard units, common errors, remediation.*

We started with a hands-on activity. Each pair of students was given an activity sheet (shown in Figure 1) and a box of closed safety pins. They were first required to measure the length of their tables using their handspans and to enter their findings in Table I. Then they had to measure the same length using the safety pins and enter their findings in Table II. I tried to pair shorter students with taller counterparts for this activity which used non-standard units.

Student	**Length of the Table**
Seema	__ handspans
Ahom	__ handspans
Teacher	8 handspans

Table I: Table length in handspans

Student	**Length of the Table**
Seema	__ safety pins
Ahom	__ safety pins
Teacher	22 safety pins

Table II: Table length in safety pins

Figure 1: Activity Sheet

Student	**Length of the Table**
Seema	11 handspans
Ahom	10 handspans
Teacher	8 handspans

Table I: Table length in handspans

Student	**Length of the Table**
Seema	22 safety pins
Ahom	22 safety pins
Teacher	22 safety pins

Table II: Table length in safety pins

Figure 2: Activity Sheet filled in by students

The students were able to observe that though they were measuring the same table there was a difference in the number of handspans from person to person. However, there was no difference in the number of safety pins, provided the measurement was done correctly. (I used the opportunity provided by some errors to illustrate how mistakes could be made.)

In both parts of the activity, we used non-standard units (NSU). But the difference was that the NSU was uniform in the case of the safety pins and non-uniform in the case of the handspans. The class was transitioning from NSU to SU via uniform units.

In both the activities, after measuring, if the end of the object's length was less than the length of a handspan or a whole safety pin, most students became confused. They wrote measurements such as "Five and a half safety pins."

Consider Figure 3. The lengths of pens A and B were both measured as "Three and a half safety pins," but when compared, the two lengths were not actually equal.

Figure 3

At that point, I had to leave the question in the students' minds. Safety pins don't have sub-units, and hence accuracy is compromised but this was something I could address only after standard units were brought in.

Standard Units

Now it was time for standard units, and we started with a "Long" shown in Figure 4.

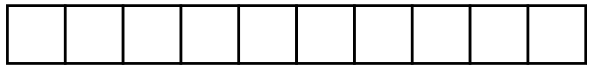


Figure 4: Flat or Long from the FLU (Flats Longs Units by Swati Sircar)

I was lucky to have square-grid notebooks with exactly 1 cm grids. Usually, the grid size is larger in such notebooks. So, I made the Longs with sheets from unused pages. Alternatively, Dienes' Blocks can be used since that's a perfect 1 cm grid by design. If such blocks are not available, then one can easily make 10 cm × 1 cm strips from card/boxes/stiff paper. Using the 1 cm × 1 cm squares for each cell helps the transition to standard units very well.

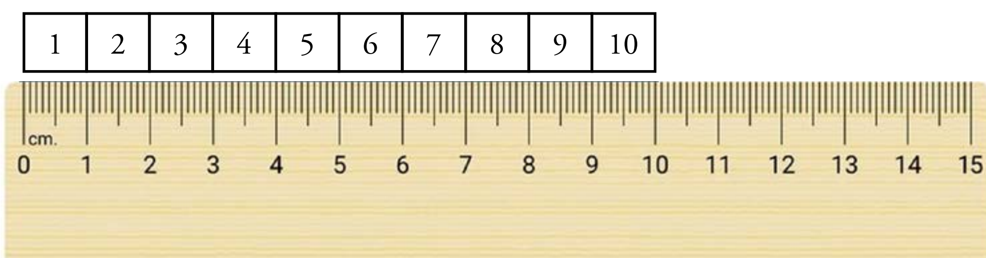
We asked the students to measure different objects such as crayons, sharpeners, erasers and pencils using this teaching learning material (TLM). They were instructed to measure by counting the gaps between the lines.

The Long was then marked with numbers (1 to 10) and students were asked to measure objects with the Long and also with a ruler.

1	2	3	4	5	6	7	8	9	10

Figure 5: Modified Long or Ten

Later, the ruler is connected with the Long for an Aha! moment in Figure 6.



Source: https://shutr.bz/49BNGjf

Figure 6: The Long becomes the 10 centimetre scale

Challenges that I encountered

1. Students faced difficulties with the non-standard units. This part took longer than I expected.
2. The students were confused when the ruler was introduced, especially whether measurement began from 1 cm or from the 0 cm mark.
3. Students did not understand that the object's length was not increased when it was shifted as shown in Figure 7.

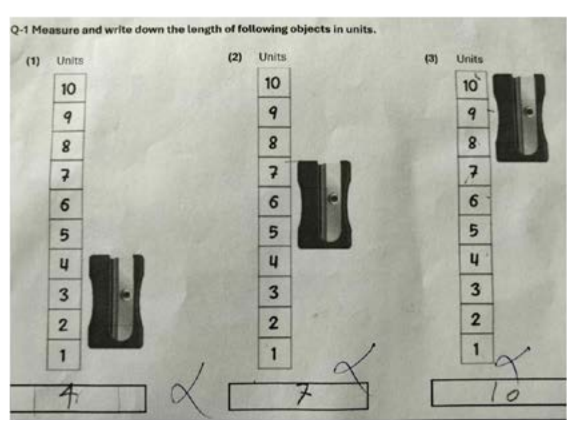


Figure 7: The confusion about length of an object

I realised that an error had crept in due to how the squares on the long were numbered.

Editor's note: What was needed at this point was the following explanation:

- The square marked 1 starts at 0 on the scale and ends at 1 on the scale. So, the length of a side of this square is the same as the gap between 0 and 1 on the scale.
- The square marked 2 starts at 1 on the scale and ends at 2 on the scale. So, the length of these 2 squares is the gap between 0 and 2 on the scale.
- Similarly, the length of squares 1-5 is the gap between 0 and 5 on the scale.
- And the length of all 10 squares is the gap between 0 and 10 on the scale.
- So, we have to start from 0 on the scale.

It was also important to discuss how the length of something doesn't change if we shift it along the scale. For example, check Figure 8.

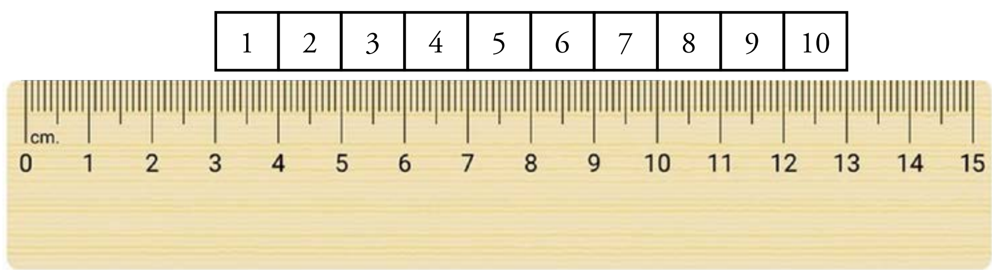


Figure 8

It is the same strip of 10 squares. So, the length has to be the same. But now it matches the gap between 3 and 13 on the scale. Note that there are 13 such intervals (0 to 1, 1 to 2… 12 to 13) between 0 and 13 on the scale and 3 such intervals between 0 and 3. Now, the 10 intervals between 3 and 13 overlap with the strip. So, we ignore the first 3 intervals (or take them out) from the 13 intervals. Therefore, we have to subtract 3 from 13 to get the length of the strip, i.e., the length of the strip is 13 cm – 3 cm = 10 cm.

Conclusion

For an adult who was used to years of measuring lengths, I found it a challenge to predict the difficulties that students have with the skill. This was why I estimated a shorter time to complete this unit. But I am glad that I was able to zoom in on specific difficulties and give students both practical as well as pen and paper exercises that addressed the hard spots that were thrown up in class. Very often, it is difficult to find a worksheet that addresses the difficulties which have been noted in a specific group of students. The accompanying worksheet was designed by Kshama Chakravarthy, with this objective. The notes below the questions explain the thinking behind them. Helpful suggestions for creating more such questions are also given.

JIVESH PANCHBHAI works at the Azim Premji School, Uttarkashi. He completed his M.Sc. in 2018 and his B.Ed. in 2022 from HNB Garhwal University. He aims to make joyful resources and activities for his Class 3 students and to help them master the art of solving word problems. He may be contacted at jivesh.panchbhai@azimpremjifoundation.org

Measurement

Kshama Chakravarthy

1. **Nidhi and her mother are trying to measure the length of the same table through handspans. Figure 9 shows how they measured. Why do you think that they got different measurements?**

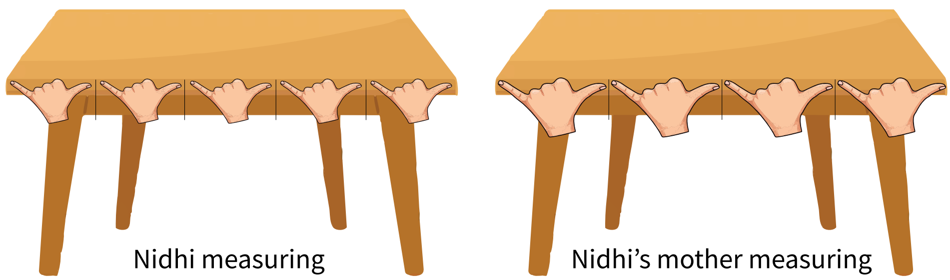


Figure 9

a. They are not measuring it accurately

b. Nidhi's hands are smaller than her mother's

c. Nidhi's hands are bigger than her mother's

d. One of them counted incorrectly

Note: This question is based on non-standard units and the distractors are based on the doubts which are normally observed in class. It reinforces that a handspan (or similar non-standard non-uniform units) is not reliable or "universal." The real-life context allows students to check for themselves how the same length can be measured (accurately) to provide different answers.

2. **Four friends are measuring the length of the same pencil using erasers. The image below shows how each of them measured the pencil. Who measured it correctly?**

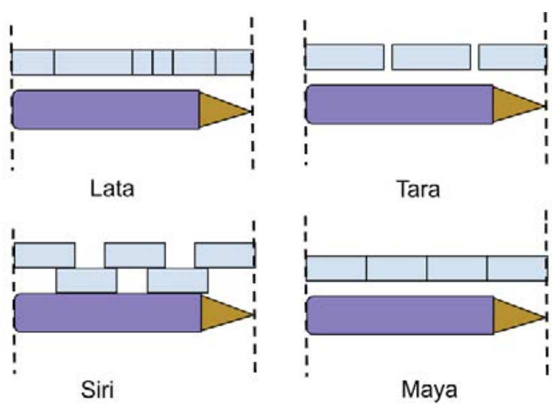


Figure 10

a. Only Maya

b. Lata and Maya

c. Only Tara

d. All of them

Note: This question checks the understanding of **measuring length using non-standard units** (in this case, erasers). It is immaterial that erasers of different sizes are used, the question is not asking the length of the pencil. It focuses on measurement with non-standard units. Measuring with non-standard units requires 3 major points to be kept in mind – (1) The repeating unit must be the same length, (2) There should be no overlaps, (3) There should be no gaps. Of course, the start and end points must align for both the object (pencil) and the unit (eraser). Multiple questions can be designed keeping these common errors in mind.

Worksheet

3. What is the length of each item, in terms of paper clips? Write your answer next to each item.

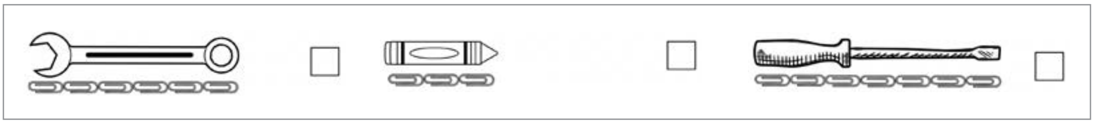


Figure 11

4. How many blocks long is each object?

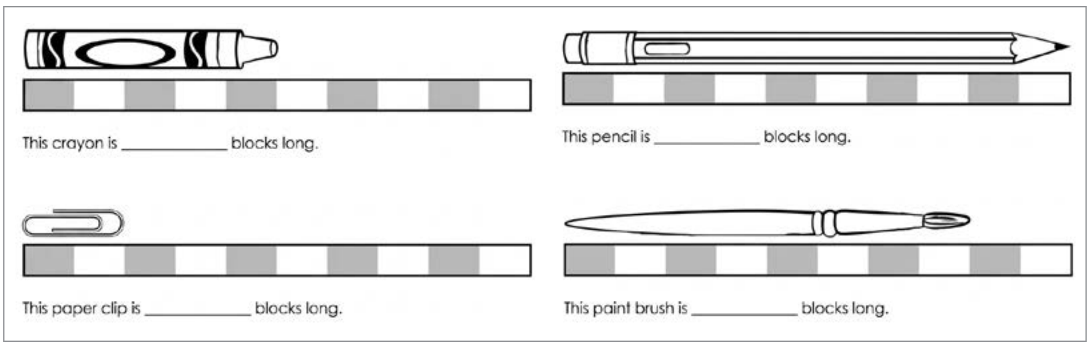


Figure 12

Note: Questions 3 and 4 are typical practice questions involving measurement using a non-standard uniform unit of measurement (a paper clip or block in these cases). Note the progression from discrete objects in Question 3 to a continuum in Question 4 – getting towards a scale – inspired by the old NCERT Class 3 Chapter 4 p.48.

5. Sunil placed 6 blocks along the length of a pencil to measure it. How many MORE blocks does he need to measure it?

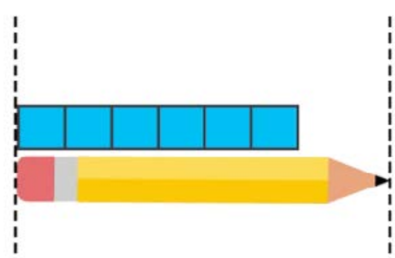


Figure 13

a. 2

b. 6

c. 4

d. 8

Note: Inspired by the old NCERT Class 3 textbook (Chapter 4, p.47), this question requires the skill of estimation. Students choosing option b may not have understood the question correctly and counted the number of blocks they see in the image, while those choosing option d may have counted the total number of blocks that measure the length of the pencil, failing to read what the question asked.

Question 6 again requires estimation.

Worksheet

6. **Estimate the length of each item in terms of paper clips. A few clips are placed to help you estimate.**

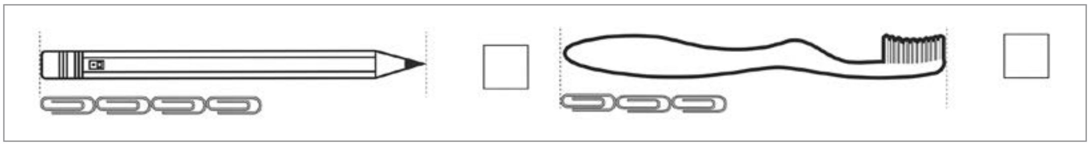


Figure 14

7. What is the length of the pencil in each case?

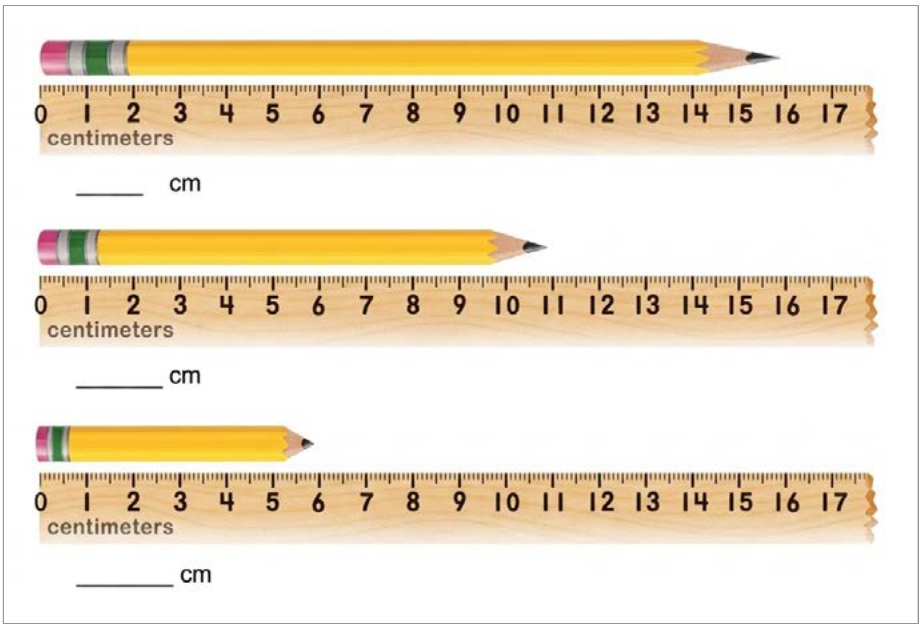


Figure 15

8. What is the length of the colour pencil when measured using a ruler?

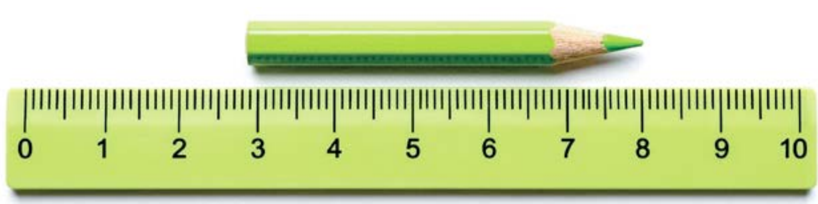


Figure 16

a. 8 cm

b. 6 cm

c. 5 cm

d. 3 cm

Note: Questions 7 and 8 test a student's true understanding of measurement using a ruler. Questions on measuring length using a ruler always tend to place the object at the 0 mark. This may lead to alternative conceptions in the student's mind and generalisations about how the object is measured. The most common answer to question 8 is option a, where they blindly look at the pencil's endpoint without noticing the start point which is crucial for determining the length. Students who pick Option b count the number of lines between the start and end point, while those choosing Option d look only at the start point.

Worksheet

9. Ayra has a broken ruler with her, with which she wants to measure the length of her pencil. She places it as shown below. What is the length of the pencil?

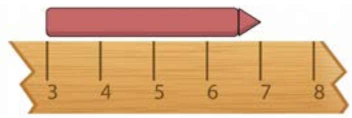


Figure 17

a. 4 cm

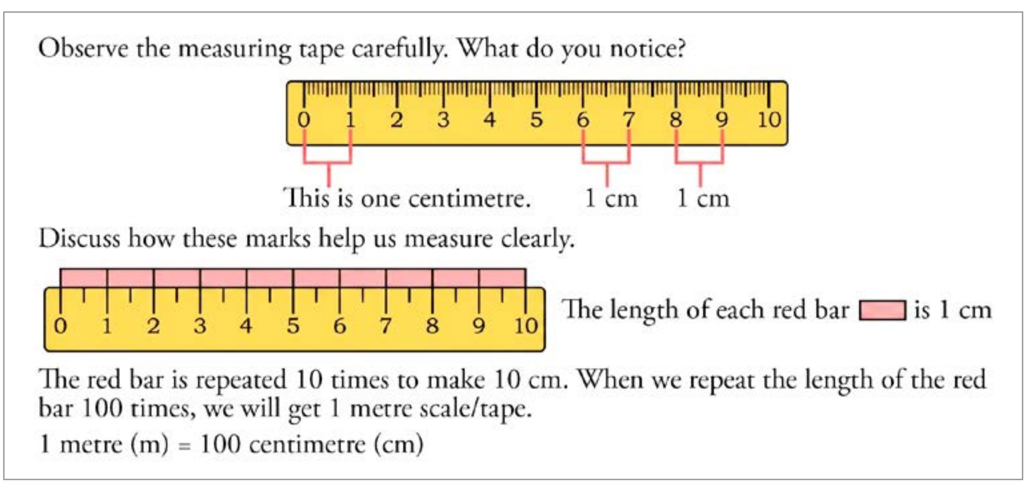
b. 5 cm

c. 7 cm

d. The length cannot be measured

Note: Question 9 is similar to the previous one, but the ruler is broken here, meaning some numbers at the start are missing. Students choosing option b are likely counting the number of lines from the start point to the end point, clearly missing that the distance between 2 lines counts as a unit. A student who chooses option c is looking only at the end point, because when we place the object at 0, its length is the same as the end point. The student generalizes this without realising that the crayon has been “shifted”. Students choosing option d believe that the length cannot be measured from a point in between, and that it always needs to be measured from 0 (or that a broken ruler cannot show the right measurements!).

Editor’s Note: The current NCERT textbooks have modified the sequence a bit. Class 3, Chapter 10 provides a lot of examples of measuring with NSU and introduces just the metre. So, the need for scale doesn’t arise till Class 4 when centimetre gets introduced. Class 4, Chapter 6, p.84 introduces centimetre (cm) and measuring with the cm scale in a similar manner with the red bars taking the place of the strip with 10 squares (Figure 9). Students get to measure with broken scale (i.e., not start from 0) on p.88.


Observe the measuring tape carefully. What do you notice?

Discuss how these marks help us measure clearly.

The length of each red bar is 1 cm

The red bar is repeated 10 times to make 10 cm. When we repeat the length of the red bar 100 times, we will get 1 metre scale/tape.

1 metre (m) = 100 centimetre (cm)

Figure 18

References

1. NCF 2023 - https://ncert.nic.in/nep.php?ln=en

2. Math Space: FLU - https://bit.ly/3QfYDQX

KSHAMA CHAKRAVARTHY is an educator. She holds a master's degree in Mathematics from IIT Madras and a master's in Education from Azim Premji University. With over 15 years of experience in math education, she has worked in areas like content development, teaching, and teacher training, as well as conducting student interviews and creating assessments. Passionate about nurturing young minds, Kshama loves spending time with toddlers and enjoying nature. She can be reached at kshamagc@gmail.com

Star Problem

Sumit Kumar Pandey

Recently my students and I were exploring the chapter Prime Time in the new NCERT (National Council of Educational Research and Training) textbook Ganita Prakash for Class 6. The chapter covers the concept of different sets of numbers including prime numbers, composite numbers, co-prime numbers, etc.

As prescribed by the textbook, we worked on multiples using the suggested game "Idli-Vada". Then for factors we used the Jump Jackpot number line activity. As the students were familiar with the concepts of factors and multiples from Class 5, these activities provided a quick review and reinforcement for them. We then moved on to prime numbers and using the 10 by 10 hundreds grid, found the prime numbers using the "Sieve of Eratosthenes" method described in the chapter.

As assessment, I administered a class test, and the results were quite satisfying, indicating that the students were able to understand the concept of prime numbers. The students responded well on questions such as –

- Is 1 prime, composite or neither? Explain why.
- Rohit says, *"All even numbers are composite."* Is it correct or not? Explain your answer using examples.

A few samples of their answers are given below –

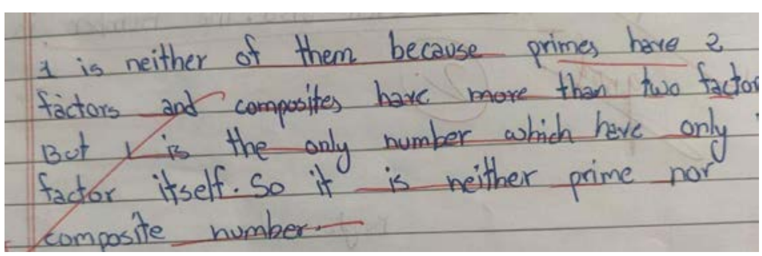

1 is neither of them because primes have 2 factors and composites have more than two factors But 1 is the only number which have only factor itself. So it is neither prime nor composite number.

Figure 1

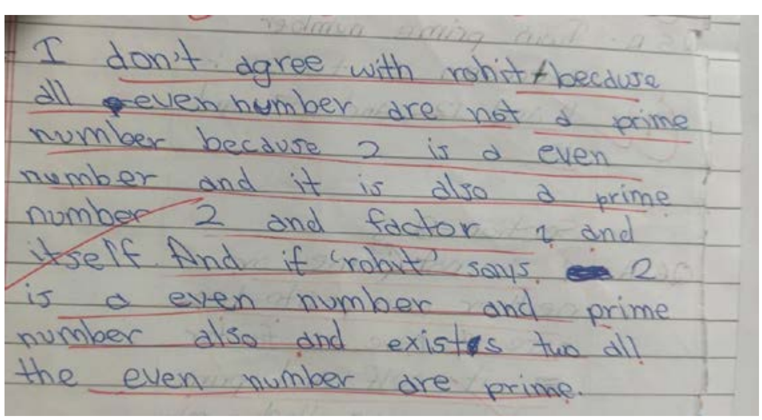

I don't agree with rohit because all even number are not a prime number because 2 is a even number and it is also a prime number 2 and factor 1 and itself And if 'rohit' says 2 is a even number and prime number also and exists two all the even number are prime.

Figure 2

Keywords: *Prime numbers, composite numbers, common factors, procedural understanding, conceptual understanding, art.*

Co-prime exploration

After understanding prime numbers, we moved on to co-prime numbers. NCERT suggests that students should be encouraged to read the textbook and to ensure this, I got the students to read page 115, where a modified version of the Jump Jackpot game is given.

In this activity, treasures are kept on two numbers. Jumpy gets the treasures only if he is able to reach both the numbers with the same jump size. Jump sizes of 1 are not allowed. The question asked where Grumpy should place the treasures so that Jumpy could reach at most one of them. The example given – of placing the treasures on 4 and 9, rather than on 12 and 36, illustrated a winning strategy for Grumpy.

After students played a few rounds of this game, the concept of co-prime numbers was explained. Co-prime numbers were defined as *"two numbers having no common factor other than 1"*. Many students connected Grumpy's winning strategies to this definition but a few of them were not sure about it.

The next day, we worked on a different activity for which some guidelines were given in the textbook under the heading 'Co-prime Art'. The activity was based on what will be called star-problem sheets; these sheets contain a circular number line with a finite number of dots. I gave the following (more detailed) instructions.

- Use the number of dots on the circle as your first number.
- Then choose any second number (let us call it *m*), that is smaller than the first number.
- Connect the dots by going round the circle starting with the first number and moving in equal steps of size *m*.
- Repeat this process till
  Case 1: The matching pattern covers all the dots without skipping any.
  Case 2: The matching stops or repeats before covering all the dots.
  Statement: In Case 1, the numbers are co-prime. If the jumps stop, or move in a repeating cycle (Case 2), then the numbers are not co-prime.

The students understood the instructions and started working in groups. A few work samples are given below.

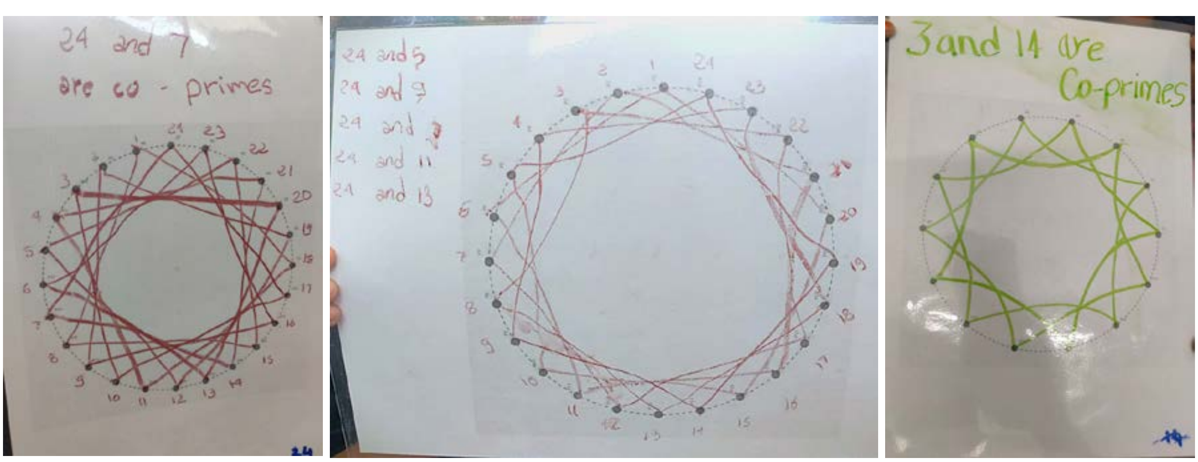


Figure 3: A few work samples

But I soon realised that though they were able to do the activity, they did not know the reasoning behind it. In the remaining part of this article, an exploration has been designed, which prepares students to build further on the exercise that they had undertaken.

The star-problem activity gives students an opportunity to explore co-prime numbers through movement and patterns. This exploration was extended by asking them to carefully record what they notice when the number of dots and the jump size are changed.

For example, consider a circle with **6 dots**. We start from one dot and keep jumping by the same number of steps, this time with arrows indicating the direction of jumps. The pictures below show what happens for different jump sizes.

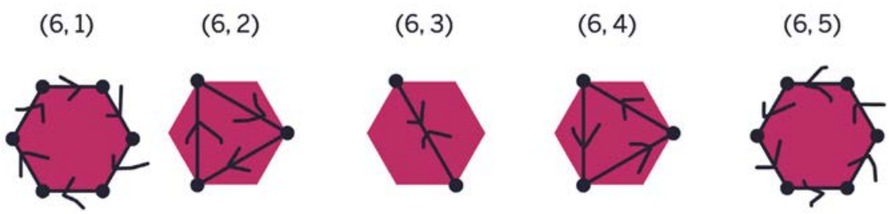


Figure 4

Students can observe that some jump sizes allow the path to visit all dots, while in other cases the path repeats without reaching all dots.

The following observation sheet can be used. (Table 1) The first few rows are filled as examples.

Observation Sheet: Exploring Patterns on a Circle

Choose a number of dots on the circle. Choose a jump size and connect the dots by repeatedly making the same jump, keeping a record of the direction of the jump. Record your observations.

Number of dots on the circle	Jump size	Does the pattern reach all dots?	Number of dots reached	Gap between two connected dots	Similar to another figure in the table?
6	1	Yes	6	1	Jump size 5 gives the same drawing in the opposite direction
	2	No	3	2	Jump size 4 gives the same drawing in the opposite direction
	3	No	2	3	None
7	1				
	2				
	3				
8					

Table 1

After completing more examples, students can discuss:

1. For 6 dots, and different jump sizes.
   - Which jump sizes helped you reach every dot?
   - What is the connection between 6 and such numbers?
   - Which jump sizes make the path repeat the same circuit?
   - What is the connection between 6 and these numbers?
2. Compare:
   - 6 dots with jump size 1
   - 6 dots with jump size 5
3. What is similar? What is different?
4. What patterns did you notice with:
   - 5 dots
   - 7 dots
   - 8 dots
   - 9 dots
5. When the path does not reach all dots:
   - What is the smallest gap between any two points that are connected in the process?
   - Now, try different dots as the starting points. How many different groups of dots are formed in this case?
6. When the jump size is same as the smallest gap between any two points that are connected in the process:
   - When we have 6 dots, what are the different jump sizes for which this happens?
   - When we have 8, 10, or 12 dots, what are the different jump sizes for which this happens?
   - When we have 24 dots, can you predict the jump sizes for which this will happen?
7. Before actually drawing, can you predict what will happen for different jump sizes when the number of dots is 10, 11, or 12?

After the activity, I picked a pair of students who had chosen 15 dots with a jump size of 2. They were confident that 15 and 2 were co-prime as all the dots had been connected. But when asked to explain *why*, the confidence gave way to silence. I then guided their thinking through a second round of recording. This time they numbered the dots. The jump started from the last dot (which had the same number as the number of dots on the circle). They entered their observations in Table 2.

Number of dots on the circle	Jump size	Numbers of the connected dots and gap between connected dots	Numbers of the dots not touched	Observation
15	3	3, 6, 9, 12, 15 Gap 3	1, 2, 4, 5, 7, 8, 10, 11, 13, 14	The connected dots are all multiples of 3. 15 is a multiple of 3 The gap 3 is a common factor of 3 and 15.
15	6	6, 12, 3, 9, 15 Gap 3	1, 2, 4, 5, 7, 8, 10, 11, 13, 14	The connected dots are all multiples of 3. 6 and 15 are multiples of 3.
15	2	2, 4, 6, 8, 10, 12, 14, 1, 3, 5, 7, 9, 11, 13, 15 Gap 1		All the dots are connected. The only common factor between 15 and 2 is 1. The gap of 1 gives the only common factor. 15 and 2 are co-prime.

Table 2

Then, the students began to see that the gap between connected dots reflected the common factor of the two numbers. When two numbers share no common factor, every dot finds its place in the chain. These two numbers would then be co-prime.

Complete the Sentences

Based on your observations:

- A path reaches every dot when ___________________.
- The smallest gap between connected dots gives information about ___________________.
- The number of dots visited depends on ___________________.
- Additional Question: If the jump size is changed to the number of dots minus the jump size, the ___________________.
- The smallest gap between the dots in a pattern equals the jump size if and only if __________ ________________________.

This extension helps students move from simply drawing patterns to noticing numerical relationships. The completed sentences will help students make observations and conjectures.

- A path reaches every dot when the number of dots and the number representing the jump size have no common factors.
- The smallest gap between connected dots gives information about a common factor of these two numbers.
- The number of dots visited depends on the number of common factors between the number of dots and the number representing the jump size.

- Additional Question: If the jump size is changed to the number of dots minus the jump size, the dots visited remain the same but the direction reverses.
- The smallest gap between the dots in a pattern equals the jump size if and only if the jump size is a factor of the number of dots. So, the HCF of two numbers is the same as one of them if and only if the smaller of the two is a factor of the other.

Rather than defining pairs of numbers as co-prime or not co-prime based on doing a prescribed art activity, it would be better to lay the foundation for understanding factors, Highest Common Factor, and co-prime numbers by such explorations. This would help students observe, document, conjecture, and later, even prove.

Acknowledgement: The author acknowledges and is grateful for the significant inputs by Ajaykumar K, Swati Sircar and Kshama Chakravarthy.

Reference

1. https://www.evidyarthi.in/wp-content/uploads/2024/08/cbse-class-6-maths-book-ganit-prakash-chapter-5-pdf-download.pdf

SUMIT KUMAR PANDEY has been a mathematics teacher in Azim Premji School, Barmer, since April 2020. He has been working in the mathematics teaching field for the last 7 years. He is passionate about creating engaging and inclusive learning environments. He enjoys playing cricket, watching movies and also exploring new places by travelling. Sumit may be contacted at sumit.pandey@azimpremjifoundation.org

Dancing Digits

This is the first of a set of problems gleaned from various sources by A. Ramachandran.

```
    _ _ _
+
    _ _ _
---------
  _ _ _ _
---------
```

The challenge is that each of the ten digits 0–9 must be used exactly once across the two 3-digit addends and the 4-digit sum.

Neither three-digit addend nor the four-digit total can begin with 0.

Figure 1

Open-ended problems such as this one are not very common in the primary classes, and it has been noticed that even adults tend to take some time to arrive at a correct solution. When this problem was given to a group of Classes 4 and 5 children with digit cards, most began by guessing. In this part of the article, we present the problem with some pedagogical strategies which are aimed at building structured problem-solving skills.

This digit-card puzzle is best carried out with square cardboard cutouts bearing the digits. At the end of this article, you will find a template for the same. Children can place the cards in the blank spaces to get a valid addition problem.

Pedagogical Strategies

1. Stating the problem and the constraints. This will help students to be clear about what the question is asking. Looking for correct solutions using trial and error will not only clarify acceptable solutions but also close some possibilities.
2. Not stopping at one correct answer. Focusing on the product limits the potential of the problem. When a process which yields multiple solutions is discovered, students understand the power of generalisation.
3. Giving time and opportunity for thought. Such problems may be given for students to solve over several days and are a good opportunity for pair work- leading to productive arguments, conjecture, mathematical communication and reasoning.
4. Asking for all possible solutions. When students are able to close the gate of the solution set, the teaching moments which can be mined from this problem will be maximised.
5. Using the principles of computational thinking which means breaking the task into smaller parts, spotting patterns, following a systematic search, and checking and fixing errors (debugging).

Keywords: *Problem solving, computational thinking, procedural, conceptual, understanding.*

6. Using good trigger questions. When students appear to have arrived at a stalemate, good questions can help them to refocus and redirect their thought processes. Some suggestions are given below:
   - *Where can the 0 go?*
   - *What is the maximum sum of two single-digit numbers?*
   - *What must the thousands digit of the total be?*
   - *When will there be a carry-over when two single-digit numbers are added?*
7. Scaffolding their problem-solving with thought-provoking examples, such as: Suppose the hundreds digits of the 3-digit addends are 3 and 8, and the hundreds digit of the sum is 2. Can you fill the remaining blank spaces with 0, 1, 4, 5, 6, 7, 9 (no repetition allowed).
8. Generating new solutions from one solution using the properties of commutativity and associativity of the addition of numbers. (For example, if $347 + 859 = 1206$, then is it possible to switch around any of the digits of the addends to get the same total?)

Further Reading

Jennie Pennant's article, Developing Excellence in Problem Solving with Young Learners [1], speaks about the stages of problem-solving including 'getting started', 'working on the problem', 'digging deeper', and 'reflecting'. In another article [2] the observations made by Deloache and Brown (1987) described three problem solving strategies adopted by four- to seven-year-olds making a train track circuit.

- Brute force: trying to hammer bits so that they fit
- Local correction: adjusting one part, often creating a different problem
- Dismantling: starting all over again
- Holistic review: considering multiple relations or simultaneous adjustments, for example, repairing by insertion and reversal.

It is interesting to think that these very strategies may be used to solve pen-and-paper problems like the one posed in this article. 'Brute force' in this case would be to insert digits randomly and arrive at a solution with endless trial and error. 'Local corrections' would involve deviating from a blocked path and trying a fix with a minor tweak. 'Dismantling' is to start over and over again – (hopefully, learning from previous attempts). The aim of the exercise would be to help students view the problem and its solution holistically and to relish the process of building their problem-solving muscles.

We look forward to responses which detail students' thinking processes as they tackle this problem. Send them in to AtRightAngles.editor@apu.edu.in A follow up article featuring innovative contributions and a complete solution is in the pipeline!

References

1. https://nrich.maths.org/articles/developing-excellence-problem-solving-young-learners
2. https://nrich.maths.org/articles/mathematical-problem-solving-early-years-developing-opportunities-strategies-and

Acknowledgements: Swati Sircar, Mohan R and Sneha Titus made significant contributions to this article.

2 3 4 5

6 7 8 9

1

0

+

Pentominoes on a Hundreds Grid: An Exploration

Jayasree Subramanian & Mohan R

Place an 'L'-shaped pentomino anywhere on a standard hundreds grid so that it perfectly covers five squares. Add up the five numbers it covers. Then, slide the shape to a completely different spot on the grid and add up those new numbers. Do this again and again.

Take a look at Figure 1. The sums of the numbers covered in these four positions are 121, 211, 311, and 346. Do you notice a pattern?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1: L-Pentomino on a hundreds grid

Every single one of these sums is exactly one more than a multiple of 5. In other words, it appears that no matter where you slide this 'L' pentomino on the grid, the sum of the numbers it covers will always leave a remainder of 1 when divided by 5.

Would this remainder stay the same if the piece were rotated or flipped? What about entirely different shapes? And most importantly—*why* does this happen?

These questions recently sparked a rich mathematical investigation among middle school teachers at the MTAI - IIT Palakkad Teachers Math Circle. The hundreds grid is a familiar, versatile Teaching Learning Material (TLM) widely used in primary classrooms for counting and place value. However, as highlighted in a previous issue of *At Right Angles* (Jagrati Mehra, March 2025), it can also serve as a powerful springboard for advanced problem-solving. This article reports on how a simple activity, such as placing pentominoes on a grid, evolved into a deep exploration of algebra, symmetry, and mathematical proof.

Starting Off: Concrete Experience

A pentomino is a geometric shape formed by joining five identical squares edge-to-edge. Considering rotations and reflections as identical to the original, there are 12 distinct pentomino shapes, each named after the uppercase letter it resembles.

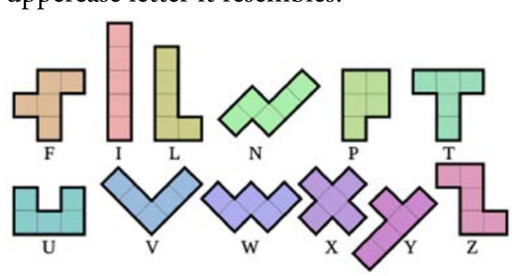


Figure 2: The 12 Pentominoes.
Image source: https://bit.ly/4nRLI4f

Keywords: *pentominoes, hundreds grid, number patterns, explorations.*

During the workshop, teachers were handed printouts of the hundreds grids and physical pentomino cutouts to physically play with and explore (Figure 3). Given the freedom to investigate, most teachers instinctively started with the 'I' pentomino, perhaps assuming its straight-line geometry would be the simplest to decode.

Figure 3: The set up

Placing the 'I' piece horizontally across different rows, they quickly made a conjecture: the sum of the numbers covered by the horizontal 'I' always gives a remainder of 0 when divided by 5.

Can we conclude that this will be true based on the few observations that we make? No - it calls for an explanation or proof. One teacher reasoned that the 'I' pentomino covers five consecutive numbers on the grid and that the sum of five consecutive numbers is always divisible by 5. Another teacher came up with an algebraic proof:

The numbers that the 'I' pentomino covers can be taken as $x-2$, $x-1$, x, $x+1$, $x+2$. Their sum is $5x$ and hence divisible by 5.

But what if the 'I' piece is placed vertically? Now, it covers numbers such as 7, 17, 27, 37, and 47. The teachers noted that the difference between consecutive numbers in this sequence is always 10. The sum, therefore, is simply five times the middle number—again, always a multiple of 5.

Finding the Shortcut: Smarter Arithmetic

Having convinced themselves that the 'I' piece always yields a remainder of 0 when divided by 5, regardless of its orientation or position, the teachers moved on to the other 11 shapes.

To check their conjectures, many initially reached for calculators to sum the five numbers and divide by 5. However, when challenged to find a method "smarter and faster than a calculator," a brilliant pedagogical shift occurred.

One group realized they only needed to look at the units digits. Because any number on the grid can be written as $10a + b$; the tens component ($10a$) is already a multiple of 5 and can be ignored. Shortly after, they refined this even further: instead of summing the units digits, they could just "cast out the fives." They had independently rediscovered a core principle of modular arithmetic: *the remainder of a sum is equal to the sum of the individual remainders after casting out the divisors.* This leap in logical reasoning dramatically simplified their investigation. For example, in order to calculate the remainder when the sum of numbers covered by the blue pentomino in Figure 6 is divided by 5, all they need to do is to find the remainders when each of the numbers covered is divided by 5 (1, 1, 1, 1, and 2 in this case) and find the remainder when their sum 6 is divided by 5. This is much easier than finding the sum $6 + 16 + 26 + 36 + 37 = 121$ and then finding the remainder when that is divided by 5.

From Pictures to Symbols: Proving the Invariance

Through rapid calculation, the teachers established a clear conjecture: The remainder is independent of the piece's position on the grid. But how to prove it?

Multiple approaches emerged, showcasing different levels of abstraction:

- The Visual/Translational Approach: One group argued dynamically. "If you slide a piece one column to the right, the sum of the five numbers increases by 5 (1×5).

The remainder doesn't change. If you slide it down one row, the sum increases by 50 (10 × 5). The remainder still doesn't change."

- The "Non-Verbal" Structural Approach: Another group assigned a value of *x* to the leftmost column of a shape and added one to the assigned number as they moved one column to the right. They arbitrarily assigned the value 0 to *x* and took all numbers in the leftmost column as 0. They then assigned the number 1 to all cells in the second column and the number 2 to all cells in the third column of the shape (Figure 4). Note that they have not marked the 0s below the leading *x* in the shapes where there are cells below the marked *x*. By adding these relative values together, they could find the remainder without needing specific grid numbers.

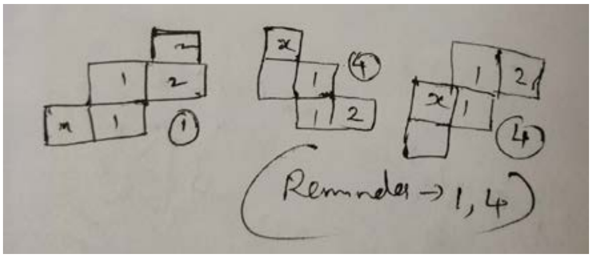
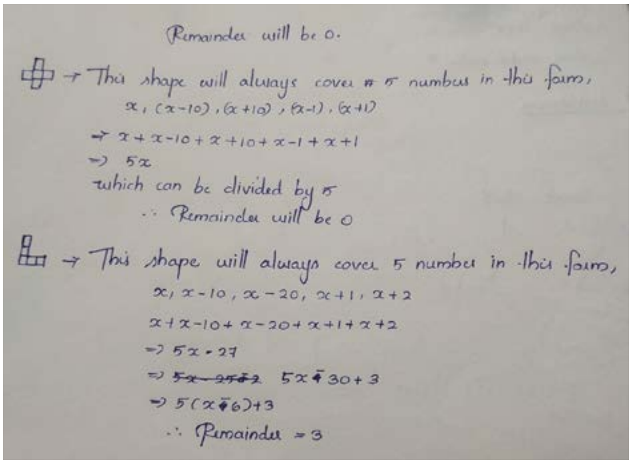


Figure 4: A "non-verbal" approach

- The Formal Algebraic Approach: Many groups used algebra to lock in their proofs. For the 'W' piece, located on the right edge of Figure 4, one teacher explained: "Twice a number, added to twice the next number, plus the next number, gives a remainder of 4." Written symbolically, if the number in the first column is n, the sum is: $2n + 2(n + 1) + (n + 2) = 5n + 4$. Any number of the form $5n + 4$ will always leave a remainder of 4 when divided by 5.

Interestingly, a debate sparked around the algebraic proofs (Figure 5). When setting up the equations, should *x* represent the smallest number in the shape, or the centre square? Using *x* for the number in the centre square often simplified the final equation, but using *x* as the smallest number avoided introducing negative signs (like $x - 10$), which some teachers argued was a much friendlier approach for middle school students.



Remainder will be 0.

→ This shape will always cover 5 numbers in this form,

$x, (x-10), (x+10), (x-1), (x+1)$

$\Rightarrow x + x - 10 + x + 10 + x - 1 + x + 1$

$\Rightarrow 5x$

which can be divided by 5

$\therefore$ Remainder will be 0

→ This shape will always cover 5 number in this form,

$x, x-10, x-20, x+1, x+2$

$x + x - 10 + x - 20 + x + 1 + x + 2$

$\Rightarrow 5x - 27$

$\Rightarrow$ ~~5x − 25 − 2~~ $5x - 30 + 3$

$\Rightarrow 5(x - 6) + 3$

$\therefore$ Remainder = 3

Figure 5: Formal explanations

The Twist: Orientation and "Sundaran" Shapes

While sliding a piece around the grid didn't change its remainder, the teachers made a fascinating discovery: the remainder *does change* if the piece is rotated or flipped. Exploring the eight possible orientations for a single piece (four rotations, each with a flipped counterpart), a beautiful symmetry emerged. The teachers noticed that if a piece yields a certain remainder, reflecting it horizontally or vertically produces a remainder that perfectly complements it to 5. For example, if the original 'L' piece gives a remainder of 1, the flipped 'L' piece (Figure 6) gives a remainder of 4 (since 1+ 4 = 5).

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 6: 'L' Pentomino reflected

Even more intriguing were the pieces that completely resisted this change. The 'I', 'X', and 'Z' pentominoes stubbornly yielded a remainder of 0 in *every* possible position and orientation. Noting a specific, inherent symmetry to these pieces, the teachers affectionately dubbed them *"sundaran"*—meaning "beautiful." When trying to mathematically define what made a shape *sundaran*, they ultimately described it as having "a sort of balance to the left and right, and top and bottom," pushing at the edges of formal rotational symmetry.

Into the Classroom

Activities like these demonstrate to students that mathematics is far more than a set of memorized procedures. It is about the joy of noticing patterns, asking *why* they happen, and finding convincing ways to explain them.

This specific pentomino task is a pedagogical goldmine. It requires very little prerequisite knowledge, allowing students of all levels to jump right into concrete play. Yet, it scales beautifully, guiding learners from basic arithmetic patterns into algebraic expressions, providing a gentle, intuitive segue into modular arithmetic.

Furthermore, the exploration doesn't end when the bell rings. The bounds of the problem can easily be stretched to keep curious minds engaged:

- What happens if we use a different grid, like a standard 7-day calendar?
- What if we explored triominoes or tetrominoes instead?
- What patterns emerge if we divide the sum by 3, or 10, instead of 5?
- What if we looked at the remainder of *products* rather than of sums?

Both author and participating teachers have since taken this task into their own classrooms. The results were unanimous: students are entirely capable of uncovering these algebraic truths, proving that when we give learners the right tools and a good mystery, they will eagerly step into the shoes of a mathematician.

Acknowledgements: Thanks to all the teachers who participated enthusiastically in the workshop sharing their work and insights freely. Thanks to MTAI and IIT Palakkad for support in organising the programme. The task was inspired by Liljedahl, P., Chernoff, E. & Zazkis, R. *Interweaving mathematics and pedagogy in task design: a tale of one task*. J Math Teacher Educ 10, 239–249 (2007).

JAYASREE SUBRAMANIAN is a teacher and teacher educator who has a range of experience working with students across age ranges and with teachers. She has a fascination for recreational mathematics and tries to make maths fun and interesting for students. She currently works as the Educational Outreach Officer at IIT, Palakkad.

MOHAN R teaches mathematics at Azim Premji University, Bengaluru. He is interested in mathematics communication and visual aspects of teaching mathematics. He regularly participates in the MTAI related Teachers Maths Circles.

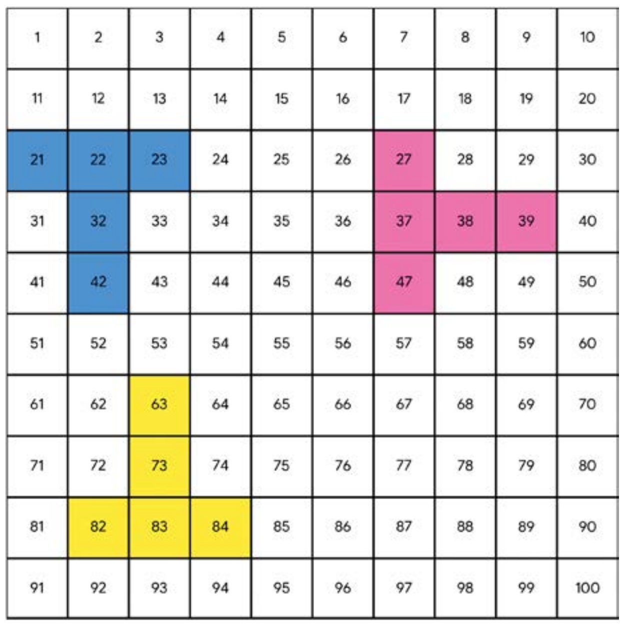
Worksheet

Pentominoes on a Hundreds Grid

Jayasree Subramanian & Mohan R

Materials required: A Hundreds Grid and the set of the 12 pentominoes printed and cut out from thick paper or card stock.

I) a) Place the T-pentomino (The blue T in the figure) in an upright position anywhere on the hundreds grid.



Add the numbers that it covers (21, 22, 23, 32, and 42 in this figure), and find the remainder when the sum is divided by 5.

b) Move the pentomino piece to other locations on the grid, sliding it horizontally or vertically, and place it in three different positions. For each of these positions:

i) Find the sum of the numbers that the T-pentomino covers.

ii) The remainder when this sum is divided by 5.

*To keep things simple, let's call this remainder **R** from now on.*

c) What do you notice about these remainders? Do you think your observation would still be true if the T-pentomino is moved to yet another location? Why do you think this is so? How would you convince a friend of your reasoning?

d) Now rotate the pentomino to other possible orientations so that it covers exactly five cells on the grid (For example: the yellow and pink T's in the figure) and find **R** for these orientations. What do you notice?

Worksheet

e) Based on your observations, formulate a statement about the remainder when the sum of the numbers covered by the T-pentomino at any position and orientation on the hundreds grid is divided by 5. How do you know that your statement is true?

f) Can you come up with an easy way to find the remainder when the sum of numbers covered by the pentomino is divided by 5 without actually calculating the sum?

II) a) Choose a different pentomino piece, say the U-pentomino. Place it at different positions on the hundreds grid and find **R**.

b) Repeat the above for different orientations of the piece. Try rotating the piece and reflecting it about one of its edges and notice how these transformations change **R**.

c) Based on your observation in a) and b) above, come up with a general statement about the **R** values when the U-pentomino is placed at different locations and orientations on the hundreds grid.

III) Explore with another pentomino of your choice, placing it in different positions and orientations on the hundreds grid. What can you say about the different **R** values obtained? How will you convince a friend of your generalisation?

IV) Based on your explorations in I, II, and III, what can you say about how the **R** value changes when you:

i) Slide the same pentomino on the hundreds grid

ii) Reflect it about a line of the hundreds grid

iii) Rotate it about a point in such a way that the piece covers five grid squares in final position as well.

How will you convince your friend that your statement is true?

V) Can you predict the **R** value of any pentomino piece without actually placing it on the hundreds grid and doing the calculation? Justify your method.

VI) State a property shared by those shapes that give an **R** value of 0.

✧✧✧✧✧

Money Matters for Young Learners

A Review of Personal Finance Handbook for Grades 1 & 2 by National Finance Olympiad

Reviewed by Kshama Chakravarthy

A child counting coins at a neighbourhood shop, comparing the prices of two snacks, or watching a parent scan a QR code to make a payment—these are familiar experiences for many children today. Financial ideas also emerge in children's interactions with one another. They negotiate the exchange of toys or stationery with friends, decide whether something is worth trading, and judge whether an exchange is fair. Such everyday experiences involve notions of value, fairness, choice, and negotiation, providing rich opportunities to develop financial thinking from an early age. They also draw upon mathematical ideas such as counting, comparison, arithmetic operations, and estimation. These authentic contexts make money a meaningful theme through which young children can develop mathematical understanding, and it is therefore not surprising that money finds a place in the primary mathematics curriculum across boards.

In recent years, several organisations and publishers in India have developed books and programmes aimed at introducing financial literacy to school-going children. Table 1 lists a few such initiatives and the age groups they target.

Publisher/Organisation	Series/Programme	Intended group
National Centre for Financial Education	Money Smart School Program	Classes 6–10
Pratham Books	*Rupaiya Paisa* series	Ages 9–12
National Finance Olympiad	*Personal Finance Handbooks*	Classes 1–8

Table 1: Examples of financial literacy publications for children in India

According to their X (formerly Twitter) page, the National Finance Olympiad (NFO) is an organisation dedicated to promoting financial literacy among students through structured competitions and learning. The *Personal Finance Handbooks for Grades 1 and 2* developed by the National Finance Olympiad are reviewed here as one example of these emerging initiatives and are of particular interest because they attempt to introduce personal finance concepts in the early primary years. The books aim to make children financially confident and responsible. Self-published by the National Finance Olympiad, their vision statement says that their products *are thoughtfully designed to test students' real world finance acumen and concept application. It aims to make finance easy and fun for students and thus to shape a generation of financially aware citizens* [1].

Keywords: *Finance, money, handbook, activities, needs, wants, budget, financial literacy, primary mathematics, digital payments.*

The handbook includes friendly characters, simple language, day-to-day examples and playful activities that claim to match the child's attention span and learning style. It familiarises students with various finance-related terms like savings, income, budget, and digital payments. There is an accompanying handbook that includes parent-child activities (Figure 1).

The friendly guide, money pal, and the go-to narrator is Gebby the bear. The book, meant for classes 1 and 2, has 8 chapters. Each chapter explains a new concept, followed by a story to convey the idea better, and then a small activity to do (match correctly, cut the picture and paste in the right place, and so on). Answers to all the questions and activities are provided at the end. A brief overview is provided in the table below.

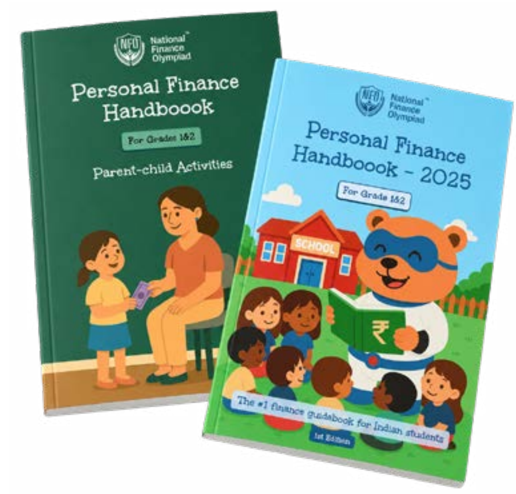


Figure 1: The primary handbook and the parent-child activity book

Chapter Number	Chapter Name	Key Ideas
1	Money Here, Money There!	• Value of money • Notes and coins (denominations) • The concept of change
2	Work Mode: ON!	• Earning through simple tasks • Value of effort • Earned vs gifted money
3	Where Did My Money Go?	• Meaning of spending money • Needs vs wants • Wise spending
4	Every Rupee Counts!	• Meaning of saving • Effective ways of saving
5	Set It, Get It!	• Meaning of goals • Saving for a goal • Ways to achieve the goal
6	Think Twice, Spend Wise!	• The idea of a budget • Spending within limits • Ways to budget
7	Goodbye Piggy, Hello Bank!	• Introduction to bank • Deposit and withdrawal • Piggy bank vs bank
8	Money Here, Money There, Money Everywhere!	• Digital money • Safety mechanisms • Digital vs cash payments

Table 2: Overview of the chapters in The Personal Finance Handbook for Classes 1&2

The breadth of topics covered here is noteworthy. To appreciate what is distinctive about the handbook, it is useful to compare it with how money is addressed in the NCERT mathematics textbooks for classes 1 and 2. The NCERT Foundational Stage Learning Outcomes document says that by Class 2, students should be able to "add up notes and coins to form amounts up to ₹100" [2]. In Class 1, the chapter on money deals with recognising notes and coins and forming different combinations to make a given amount (up to ₹20) (Figure 2).

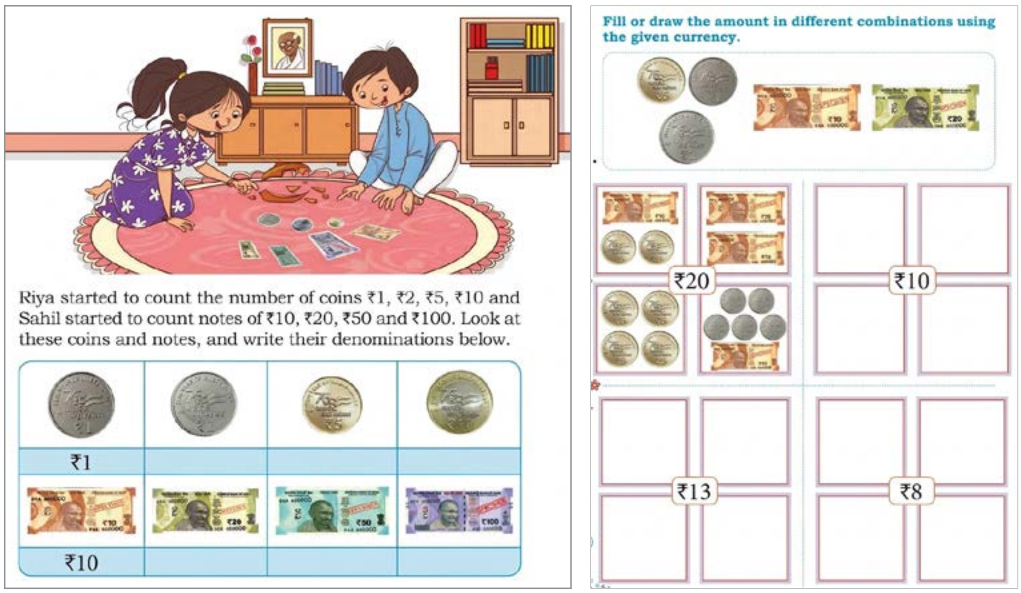


Figure 2: NCERT Class 1, Joyful Mathematics, Chapter 12, pages 115 and 118.
https://ncert.nic.in/textbook.php?aejm1=12-13

The chapter includes pretend play in which children create paper currency and engage in role-play as shopkeepers and customers.

In Class 2, the chapter is called "Fun at the Fair", where a story is built around children going to the fair and playing different games and rides with the money that their parents have given them (Figure 3).

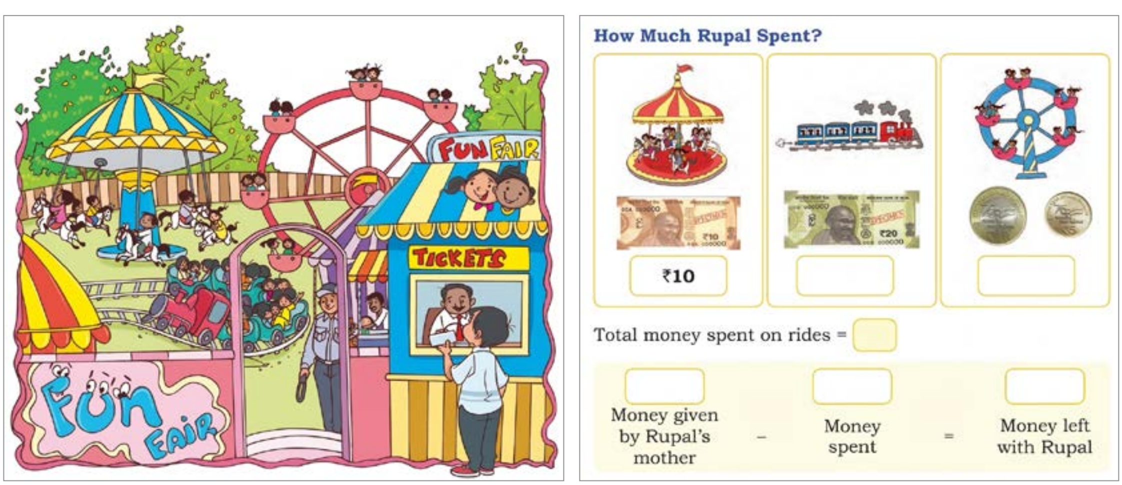


Figure 3: NCERT Class 2, Joyful Mathematics, Chapter 10, pages 113 and 114.
https://ncert.nic.in/textbook.php?bejm1=10-11

This chapter covers different ways of paying the same amount and ends with simple addition and subtraction problems involving money.

Where the handbooks add value beyond the textbook

While the NCERT books introduce children to notes and coins, combinations of amounts, and simple transactions, the handbook extends the discussion to broader ideas of earning, saving, budgeting, and financial decision-making. The following aspects of the handbook are worth noting.

1. The idea of "income" is introduced as money that comes to us- either by working or by getting gifts. The differences between these two kinds of income are also listed. This is followed up by an activity where children should identify if the income mentioned in each situation is earned or gifted (Figure 4). In a typical math textbook, this bit is taken for granted (of course the focus is on the calculations!).


The Big Word: INCOME

(say it like: in-kum)

Money that comes to us – whether by working or by getting gifts – is called income.

Earned Money	Gift Money
Comes because you work or help out	Comes from birthdays, festivals, or special occasions
You earn it by doing a job or chore	You receive it without doing work
Makes you feel proud and happy	Makes you feel happy and loved

15

Figure 4: Chapter 2, pages 15 and 18

2. The distinction between needs and wants. This is one of the most relevant sections in the handbook. In today's "instant" age, everything is available at the click of a button, with little emphasis given to whether it is really needed. With adults getting into this mode of "buy what you see", children are also growing up getting whatever they ask for, sometimes without even having to wait for it! There is a small story to drive home the point, through choices that Ananya makes (Figure 5).

Figure 5: Chapter 3, pages 25 and 27

The story is followed by a useful question (and a response): Does this mean one cannot buy fun things? (Figure 6)

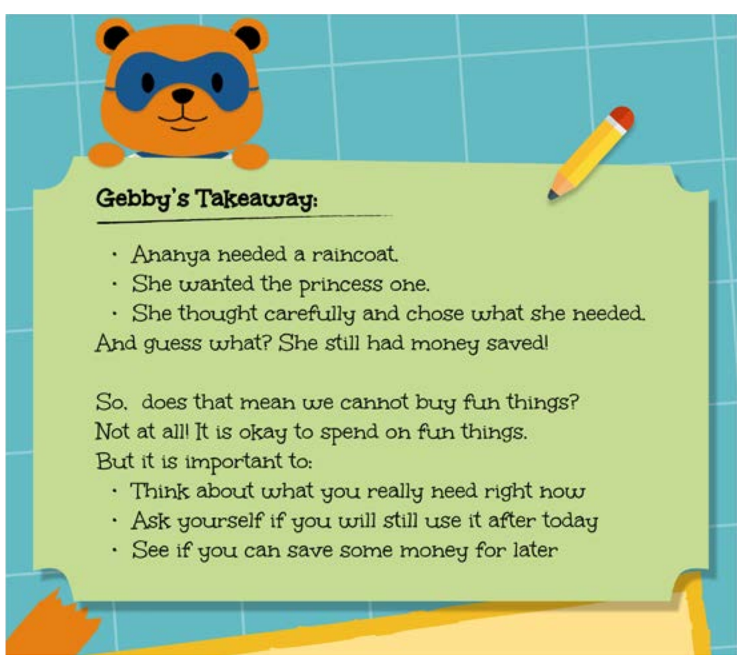


Figure 6: Chapter 3, Page 29

3. An important companion to the handbook is the parent-child activities book, which attempts to extend the ideas shared in the primary handbook beyond reading and classroom discussion. It actively involves the parents (or adults at home) in the child's understanding of the concept, and rightly so, as the topic of money and spending is so central to the home environment.

Areas that need strengthening

1. The chapter on savings talks about "Smart Savers". The idea of saving first and then spending is introduced (Figure 7). The emphasis on being a 'Smart Saver' may inadvertently suggest that saving is always the better choice. It may be useful to acknowledge that people make different decisions about spending and saving depending on their needs and priorities.

Figure 7: Chapter 4, Page 36

2. In keeping with the times, digital money and digital payments are covered in the last chapter. It should be made explicit that young children (6 and 7-year olds in this context) are not expected or advised to use apps or make digital payments on their own (or even under supervision). The first and third points under safety (Figure 8) may convey otherwise.

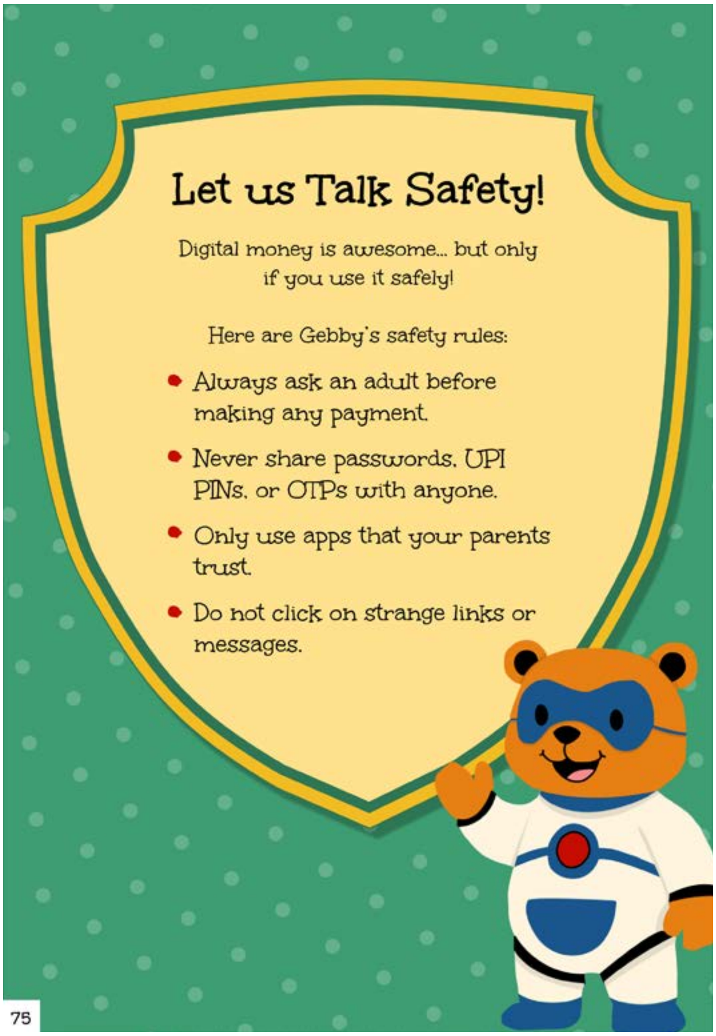


Figure 8: Chapter 8, pages 73 and 75

3. Each activity in the parent-child activity handbook mentions the goal, set-up, and activity flow, with clear instructions on what to do. "Time needed" could be a good addition to this list, for parents to know how much time needs to be devoted for this. Certain activities involve having to head to the store or the bank, so some heads up may help parents/ adults at home plan better.
4. The activity for Chapter 5 requires the child to think of a "goal" (a toy, a trip, or a backpack, bicycle, etc.), list down its cost, how much they can save, and by when they can achieve the goal (Figure 9a). This requires delayed gratification, cost estimation, record keeping, decision making, prediction- which are a lot for children at that age to take in, and higher math skills than they can cope with! The same can instead be tried out in the form of pretend play- A toy costs, say, ₹50, and the child has ₹5. How much more is needed to buy the toy? It is now a simple subtraction problem, or counting backwards by 5 from 50. If the child gets ₹5 every month, how many months are needed to be able to buy the toy, if all the amount is saved? Repeated addition (5 + 5+...) till 50 is reached. How much time is needed if only ₹3 is saved every month? (3 + 3 + 3…) till 50 is reached. How many 3s are needed to reach 50? This helps the child understand the idea by breaking it down into simpler steps instead of being overwhelmed by the task.

5. The activity for Chapter 6 helps the child plan their money wisely using a budgeting rule (The 50 / 30 / 20 Rule)- 50% of income for needs, 30% for wants, and 20% for savings (Figure 9b). Again, a lot to process. Percentages as a concept is way above their level for 6-7 year-olds to understand, and the onus is on the adults to explain. If the idea is that adults plan this entirely, create and compare actual expenses with estimates, calculate the percentages for needs, wants, and savings, then the bigger question is whether parents or adults at home will take time out of their routine and schedule to do this, and then get the child to merely track and fill the sheet. The child is no longer an active participant. Instead, with small amounts like 20 or 50 as the limit, the child can be asked to choose from a few objects within that budget, taking into consideration needs and wants, all done in the form of pretend play.

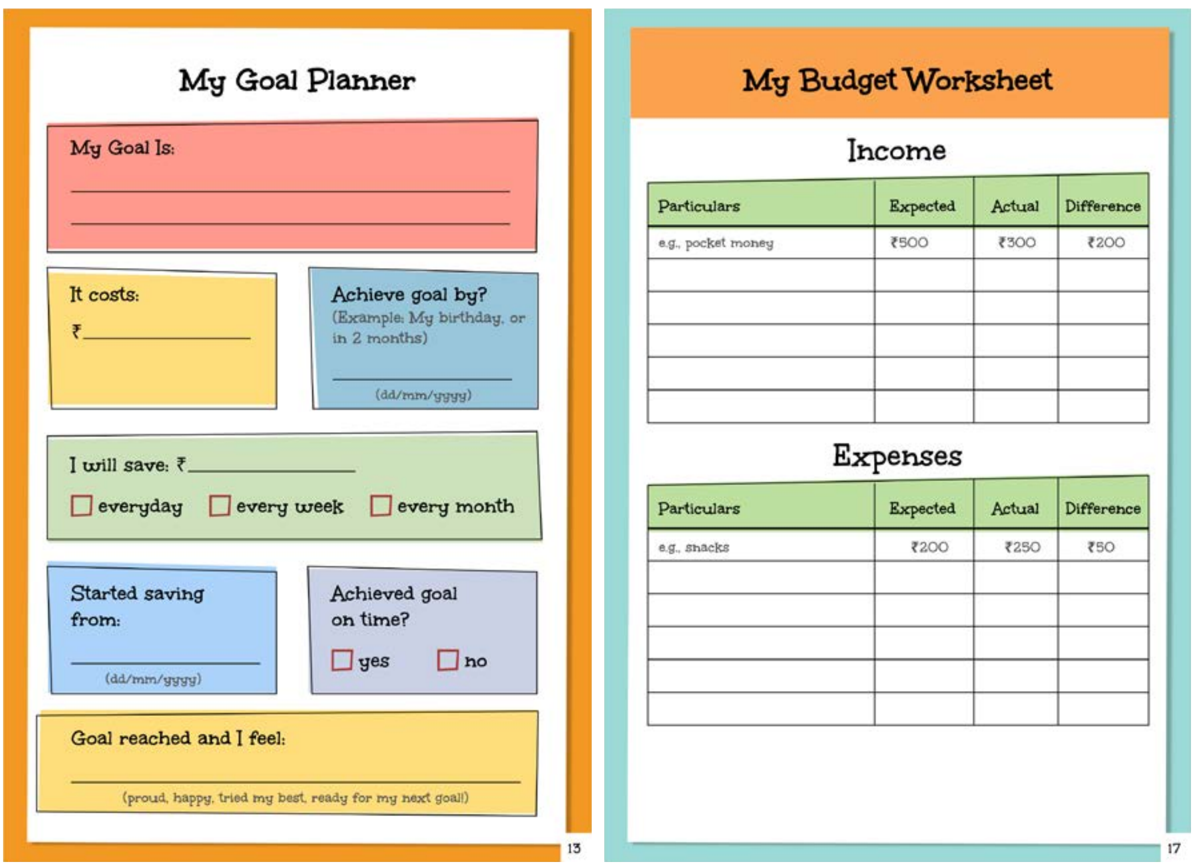


Figure 9a & b: Parent-child activities, pages 13 and 17

In their current form, the goal planner activity and 50/30/20 rule worksheet may be overwhelming and confusing for the child. Sustained saving towards a goal may be difficult for many children of this age, even with adult support. The activities as they exist right now could be moved to the higher grades, where children can be involved directly.

Conclusion

Taken together, the handbook and the parent-child activities book show how financial literacy can be meaningfully integrated into a child's everyday experiences. Children in both urban and rural settings will be able to relate to the ideas and concepts, except maybe digital payments, in case they are not exposed to it widely in their region. Rather than focussing narrowly on financial terminology or basic operations, the books encourage young learners to think critically about choices, resources, and responsibility. However, this needs to be toned down to make it age-appropriate, as discussed in detail in the earlier section.

As an introductory text for grades 1 and 2, the primary handbook offers teachers and parents a useful starting point for nurturing financial awareness from an early age. In its current form though, it serves as a useful supplementary conversation-starter, and not a stand-alone mathematics resource. Teacher notes are definitely needed to provide additional instructions, address sensitive issues around money, provide alternative suggestions to activities, and guide the teacher, so as to do justice to the book. Currently it relies a lot on teacher's adaptation of the material for age, context, and access, which means that the experience is not standardised.

References

1. National Finance Olympiad. (n.d.). *About us.* https://nationalfinanceolympiad.com/about-us
2. National Council of Educational Research and Training. (2025). *Learning outcomes for the foundational stage.* https://shorturl.at/hpTmZ

Images from the Personal Finance Handbook for Grades 1 & 2, reproduced with permission from the National Finance Olympiad (NFO).

KSHAMA CHAKRAVARTHY is an educator. She holds a master's degree in Mathematics from IIT Madras and a master's in Education from Azim Premji University. With over 15 years of experience in math education, she has worked in areas like content development, teaching, and teacher training, as well as conducting student interviews and creating assessments. Passionate about nurturing young minds, Kshama loves spending time with toddlers and enjoying nature. She can be reached at kshamagc@gmail.com

Call for Articles!

At Right Angles is a quality resource dedicated to mathematics education in India's public education system. It is specifically designed for teachers and teacher educators at the foundational, preparatory, and middle school levels.

We invite articles from mathematics teachers, educators, practitioners, parents, and students. If you are looking for a platform to contribute articles that support and enhance the learning experience of mathematics particularly for students approximately in the age group 6-14 years , we welcome your submissions.

Suggested Topics and Themes

Submitted articles should focus on curricular content applicable to Classes 1-8 and could:

- Explain and illustrate themes and topics outlined in the National Curriculum Framework for School Education 2023 (NCF-SE 2023).
- Specifically address challenges discussed in the NCF-SE 2023.
- Be substantiated accounts of the history of mathematics or the history of mathematical thinking.
- Include innovative worksheets or methods to engage students in drill and practice.
- Describe real-life applications of mathematics relevant to the child’s context.
- Describe interdisciplinary activities or projects.
- Review puzzles or games with a practical connection to the syllabus.
- Offer guidance on selecting relevant content, including online resources.
- Develop pedagogical strategies for foundational numeracy as well as computational thinking.
- Assist teachers in implementing differentiated teaching practices.
- Review of Teaching Learning Material (TLM) or describe how to use local context, and local TLM in the math class.
- Provide material to help students bridge gaps in conceptual understanding.
- Address issues in assessment.
- Suggest ways to identify and address misconceptions in mathematics learning.
- Offer a list of problems along with discussions on their solutions and problem-solving strategies that are not commonly found in textbooks.

In addition to full-length articles, we also welcome shorter pieces that can include a variety of engaging content. These could be reviews of books, mathematics software, or YouTube clips that explore mathematical themes. Other contributions can be ‘proofs without words’, mathematical paradoxes, ‘false proofs’, or creative expressions such as poetry, cartoons, or photographs with a mathematical theme. We also welcome anecdotes about a mathematician or interesting examples of ‘maths in craft, movies, etc’.

Articles may be sent to
atrightangles.editor@apu.edu.in

Please refer to specific editorial policies and guidelines on the inside back cover.

Policy for Accepting Articles

At Right Angles is an in-depth, magazine on matters of consequence to early mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions, and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be indicated in the article.

At Right Angles brings out translations of the magazine in other Indian languages. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published elsewhere, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an ‘Author’s Note’ at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to **At Right Angles**.

While **At Right Angles** welcomes a wide variety of articles, submissions that are found relevant but not suitable for publication in the magazine may be used in other avenues of publication within the University network, with the author’s permission.

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. **Engaging Introduction:** Write in a readable and inviting style, aiming to capture the reader's attention from the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, a figure with an interesting question, or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. **Catchy Title:** Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. **Style:** Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. **Balance:** Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps that depend on hidden calculations.
5. **Accessible language:** Avoid specialized jargon and notation that will be familiar only to specialists. If technical terms are needed, please define them.
6. **Use visuals:** Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. **Concise References:** Provide a compact list of references, with short recommendations.
8. **Exercises and Questions:** Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. **Citation format:** Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. **Abbreviations and Acronyms:** Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. **Labelling visual elements:** Label and number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note: the minimum resolution for photos or scanned images should be 300 dpi).
12. **Precise references to visuals:** Refer to diagrams, photos, figures and tables by their numbers and avoid using references of these kinds: 'here', 'there', 'above', 'below', 'to the left', 'to the right'.
13. **Author Bio:** Include a high-resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. **British Spelling:** Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. **Format for submission:** Submit articles in MS Word format or in LaTeX.

Printed and Published by Rishikesh B S, Registrar, on behalf of Azim Premji University
Printed at Repromen Offset Printers Pvt. Ltd., #46/47, 3rd Cross, Krishna Reddy Colony, Domlur Layout, Bengaluru, Karnataka – 560071
Published at Azim Premji University, Survey No. 66, Burugunte Village, Bikkanahalli Main Road, Sarjapura, Bengaluru – 562125
Editor: Sneha Titus

Azim Premji University

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Azim Premji University

MATH TALK IN THE CLASSROOM

PADMAPRIYA SHIRALI

A publication of Azim Premji University

MATH TALK IN THE CLASSROOM

Keywords: *pedagogical strategies, discussion, reasoning, communication, inclusion.*

Does Math Talk have a place in the teaching of mathematics?

To answer the above question, let us ask ourselves a few more questions. Does a classroom where the teacher asks questions that can be answered with a simple 'yes' or 'no' provide scope for students to share their true understanding of knowledge? Does it throw light on their misconceptions? Is there an opportunity for students to learn from one another? Does it expose students as to how their peers look at the question? The answer is obviously 'no'.

Here are a few important ways in which a student can benefit from a classroom engaged in math talk.

Students will:

- Learn to reason mathematically and articulate clearly, justifying their answers
- Develop self-reliance in testing the correctness of their reasoning
- Engage in stimulating peer interactions and collaborative work
- Develop a deeper appreciation of concepts and processes
- Retain their learning due to this more in-depth engagement

Is math talk possible at the primary level? Are these objectives too ambitious for students who are being introduced to mathematics? If the approach to teaching mathematics in the primary classroom is top-down, teacher-centric, and authoritarian, then it is highly unlikely that children can switch to a different approach in middle school. This may mean that they never develop the confidence to reason, make mistakes, check their own work and learn to try different methods.

Right from Class 1 onwards, teachers can encourage students to demonstrate using objects and informal conversation why 2 and 3 make 5, why 7 is more than 4, etc.

They can also pose questions to get students to express their thinking.

In how many ways can this blank be filled? Why do you think so?

$8 > \_$

Can you show using counters that 2 × 4 is the same as 4 × 2?

Why should multiplying a number by 4 give the same answer as multiplying the same number by 2 and then multiplying the answer you got again by 2?

Many teachers may feel diffident about trying such a format in their classes due to a fear of loss of control over the class, or lack of conviction in such a process as a way of learning. Also, there is a lack of clarity on how to proceed. However, it is possible to conduct classes in which meaningful math talk happens. An illustration of such a discussion from the NCERT book, Class 5, is shown in Figure 1. The discussions listed in this article portray such scenarios.

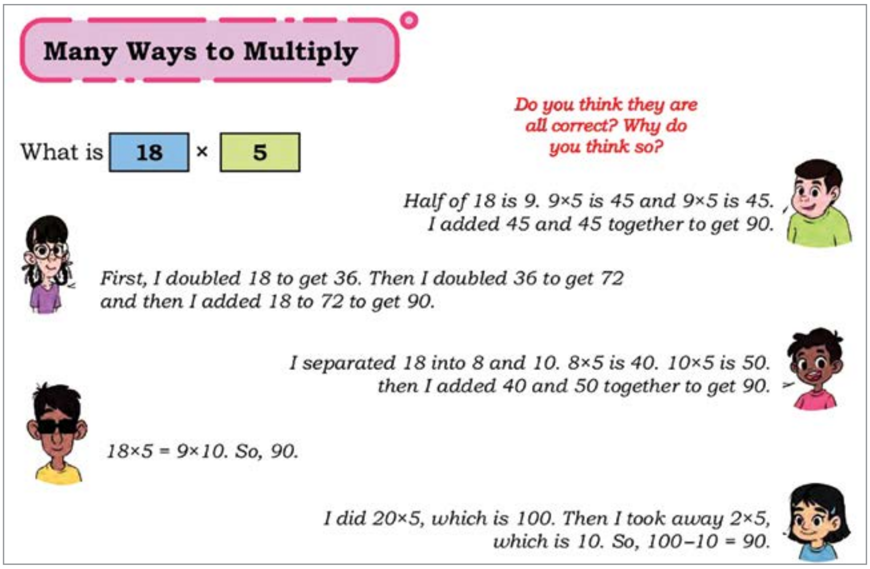


Figure 1: Reproduced with permission from NCERT Class 5 textbook.

What can a teacher do to create such spaces and support productive math talk?

In this article, we will explore a few discussions centred on questions that lend themselves to math talk among peers. These discussions can be organised in different ways.

Figure 2

Source: AI Generated

The teacher can get pairs of students to work together on a problem. They may solve it independently but check with each other if they got the same answer. Or the teacher can get students to work in groups of four sitting around a table, using hands-on material when required. The teacher's prompts will help the students to develop the ability to work together by dividing up the work among themselves.

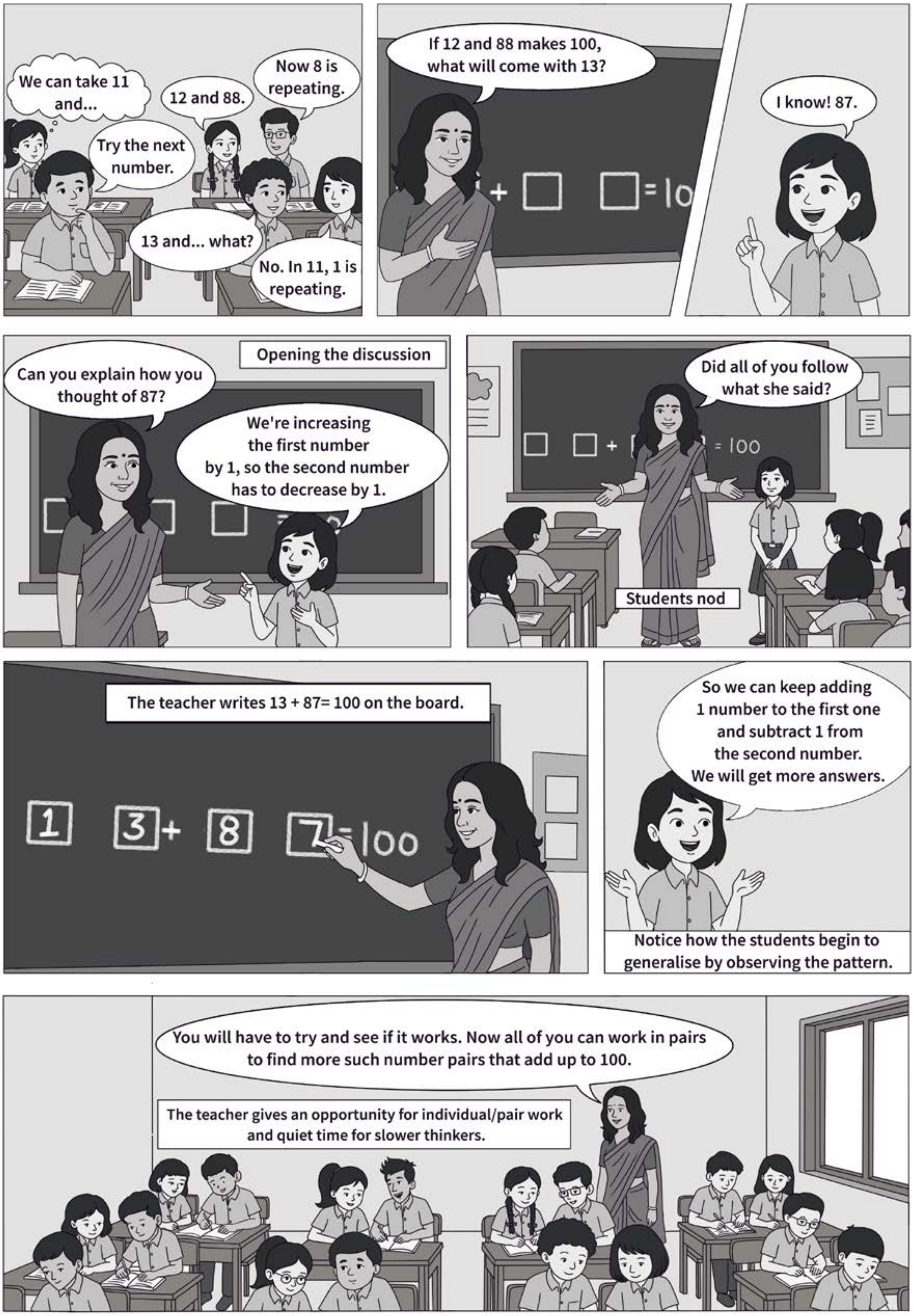
It is good to establish some norms for such a discussion, such as asking them to speak in turns, to listen fully before talking, and so on. This will ensure equal opportunity for all the students to share their thinking and engage in careful listening.

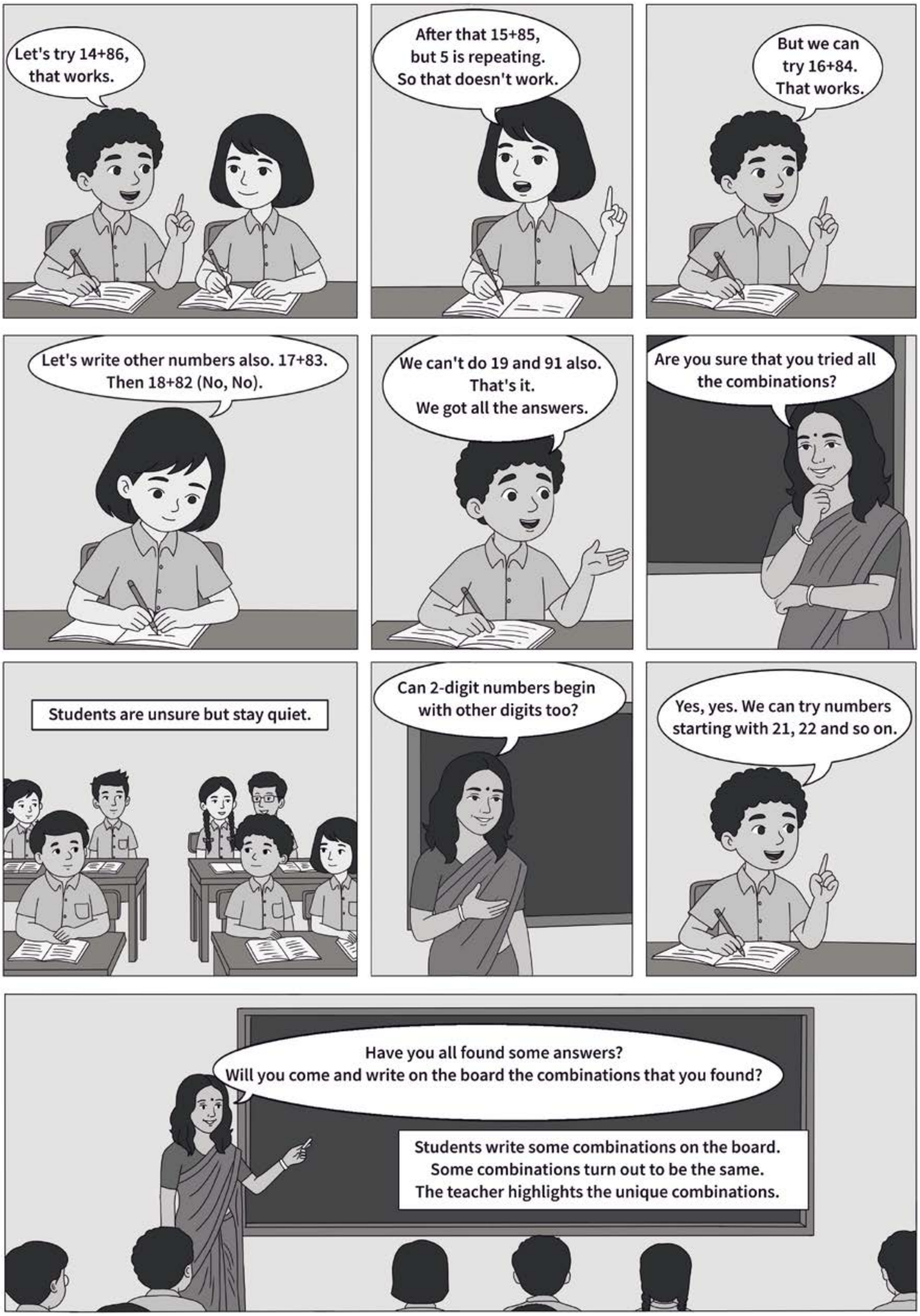
The teacher should pose questions such as *'Can you repeat what he said?'* or *'What did she mean by that?'.* This will develop among the students the skills of rephrasing another student's reasoning and justifying their answers.

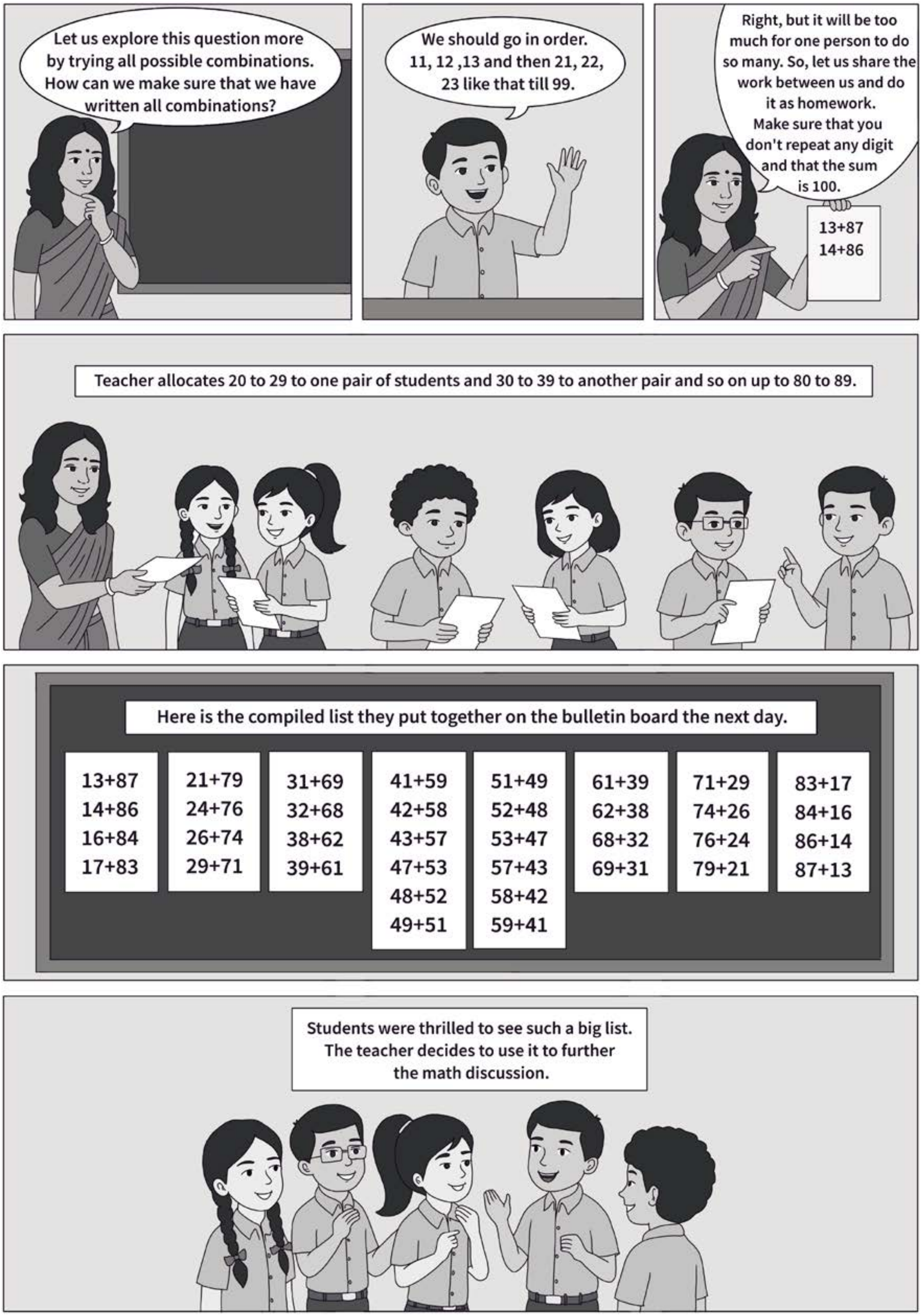
The results of these discussions may be presented to the rest of the class in different ways. The children's written work and drawings can be put up on the bulletin board. If they have come up with a different way to solve a problem, that can be highlighted. Solutions to open-ended problems can be put up. This will help in improving children's appreciation of multiple approaches and make way for peer learning.

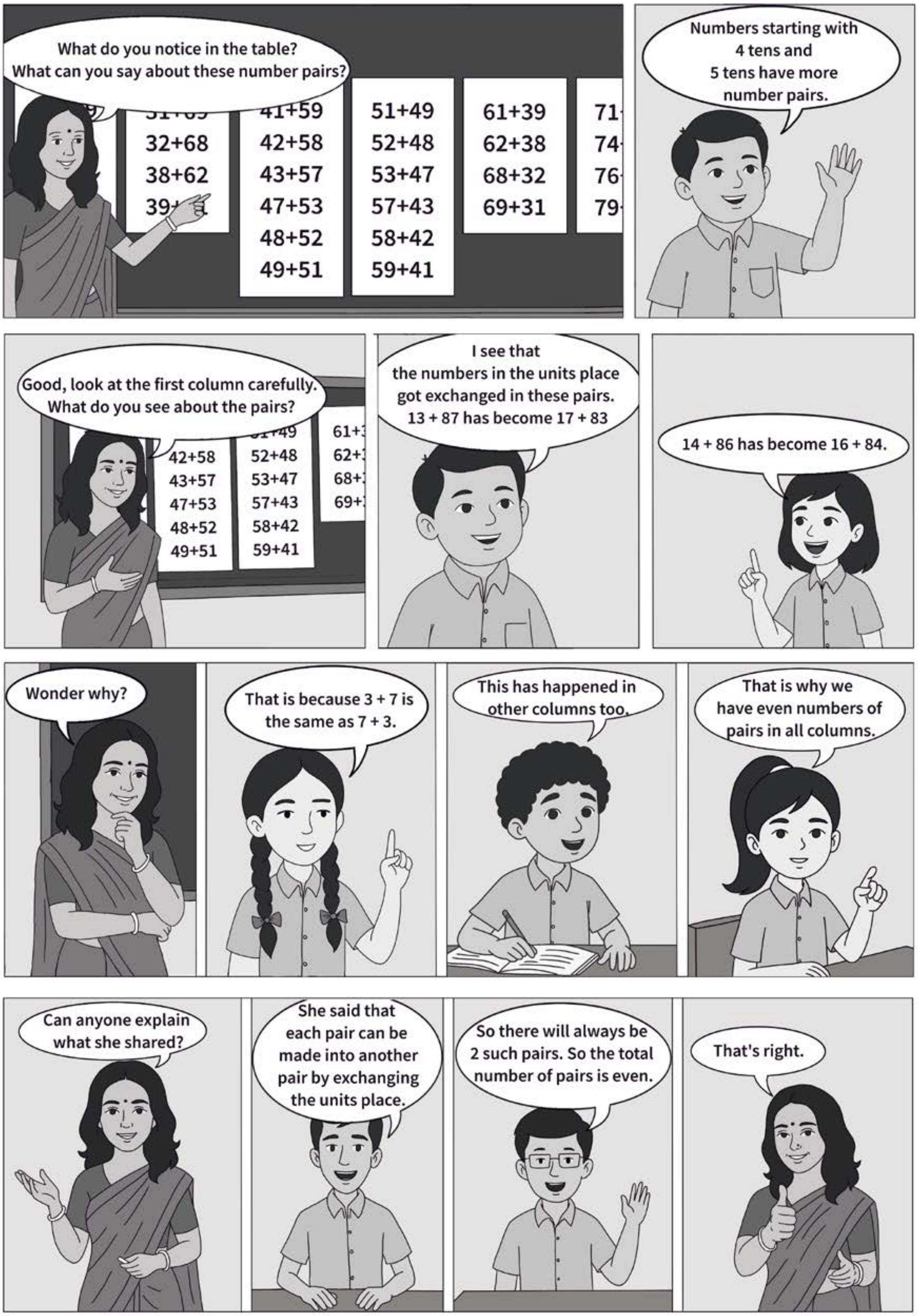
DISCUSSION 1

Grade Level 4, students work in groups of 4.


We can take 11 and...
Try the next number.
12 and 88.
Now 8 is repeating.
13 and... what?
No. In 11, 1 is repeating.
If 12 and 88 makes 100, what will come with 13?
I know! 87.
Opening the discussion
Can you explain how you thought of 87?
We're increasing the first number by 1, so the second number has to decrease by 1.
Did all of you follow what she said?
Students nod
The teacher writes 13 + 87= 100 on the board.
1 3+ 8 7=100
So we can keep adding 1 number to the first one and subtract 1 from the second number. We will get more answers.
Notice how the students begin to generalise by observing the pattern.
You will have to try and see if it works. Now all of you can work in pairs to find more such number pairs that add up to 100.
The teacher gives an opportunity for individual/pair work and quiet time for slower thinkers.


Let's try 14+86, that works.
After that 15+85, but 5 is repeating. So that doesn't work.
But we can try 16+84. That works.
Let's write other numbers also. 17+83. Then 18+82 (No, No).
We can't do 19 and 91 also. That's it. We got all the answers.
Are you sure that you tried all the combinations?
Students are unsure but stay quiet.
Can 2-digit numbers begin with other digits too?
Yes, yes. We can try numbers starting with 21, 22 and so on.
Have you all found some answers? Will you come and write on the board the combinations that you found?
Students write some combinations on the board. Some combinations turn out to be the same. The teacher highlights the unique combinations.


Let us explore this question more by trying all possible combinations. How can we make sure that we have written all combinations?
We should go in order. 11, 12 ,13 and then 21, 22, 23 like that till 99.
Right, but it will be too much for one person to do so many. So, let us share the work between us and do it as homework. Make sure that you don't repeat any digit and that the sum is 100.
13+87
14+86
Teacher allocates 20 to 29 to one pair of students and 30 to 39 to another pair and so on up to 80 to 89.
Here is the compiled list they put together on the bulletin board the next day.
13+87 14+86 16+84 17+83
21+79 24+76 26+74 29+71
31+69 32+68 38+62 39+61
41+59 42+58 43+57 47+53 48+52 49+51
51+49 52+48 53+47 57+43 58+42 59+41
61+39 62+38 68+32 69+31
71+29 74+26 76+24 79+21
83+17 84+16 86+14 87+13
Students were thrilled to see such a big list. The teacher decides to use it to further the math discussion.


What do you notice in the table? What can you say about these number pairs?
32+68
38+62
41+59
42+58
43+57
47+53
48+52
49+51
51+49
52+48
53+47
57+43
58+42
59+41
61+39
62+38
68+32
69+31
Numbers starting with 4 tens and 5 tens have more number pairs.
Good, look at the first column carefully. What do you see about the pairs?
I see that the numbers in the units place got exchanged in these pairs. 13 + 87 has become 17 + 83
14 + 86 has become 16 + 84.
Wonder why?
That is because 3 + 7 is the same as 7 + 3.
This has happened in other columns too.
That is why we have even numbers of pairs in all columns.
Can anyone explain what she shared?
She said that each pair can be made into another pair by exchanging the units place.
So there will always be 2 such pairs. So the total number of pairs is even.
That's right.

Is there anything else that you notice?
Students are quiet.
What if we look at the fourth and fifth lists together?
Oh! They are backward in the fifth list.
What is backward? What do you mean by backward?
What she means is that the number pairs have come in the opposite way.
Can you give me an example?
In the fourth list, 49 + 51 is the last pair. In the fifth list, 51 + 49 is the first pair.
The number pair is same, but the left and right numbers got exchanged. Does that make a difference to the sum?
No. 3 + 5 is the same as 5 + 3.
Oh! After finding the answers up to 50, we could have just written them the other way round.
It is like symmetry!
That was a good observation that you made.

Further questions can be posed

Do you notice if any digit never appears as the unit digit?

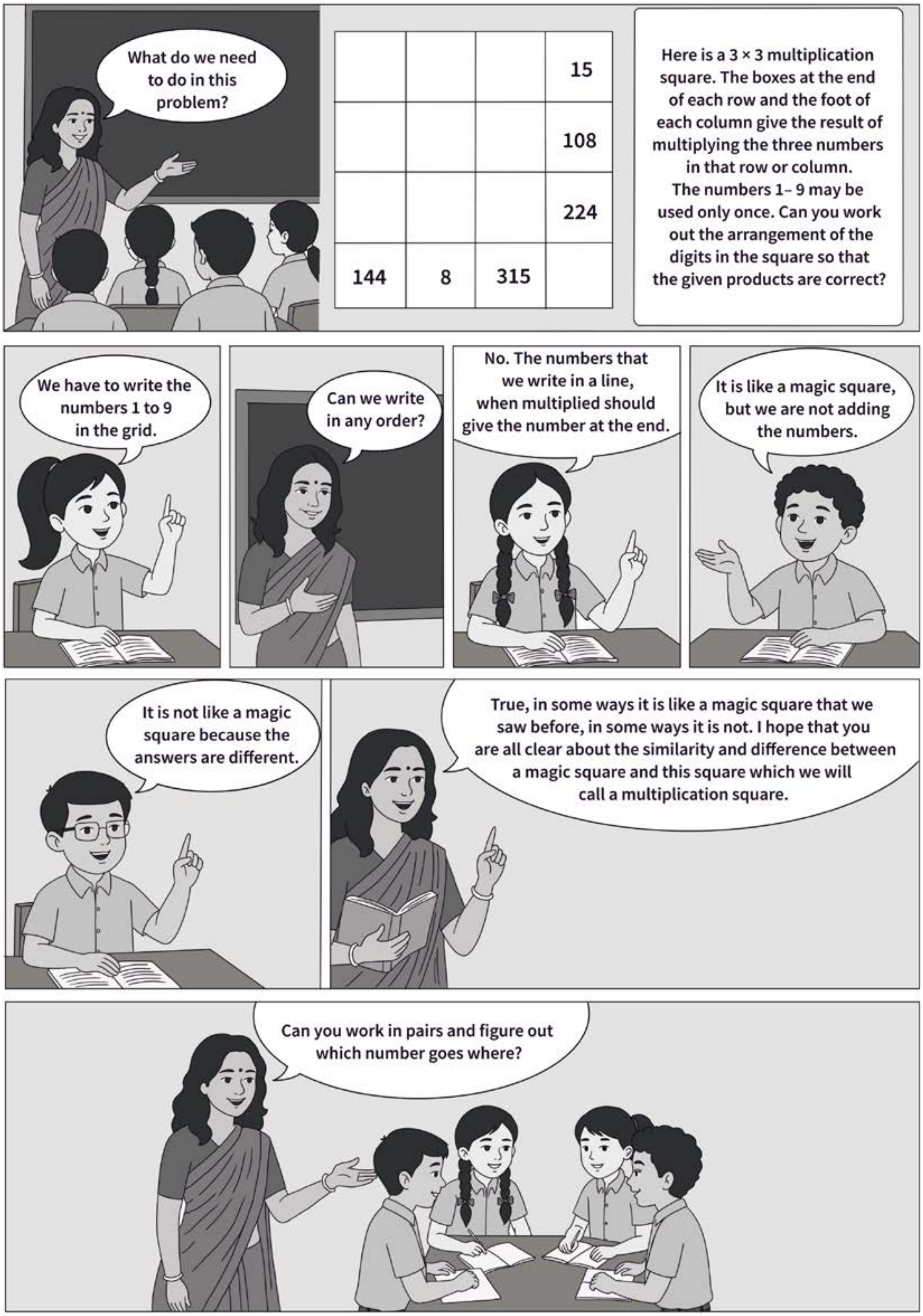
Can you find other ways to find all possible pairs?

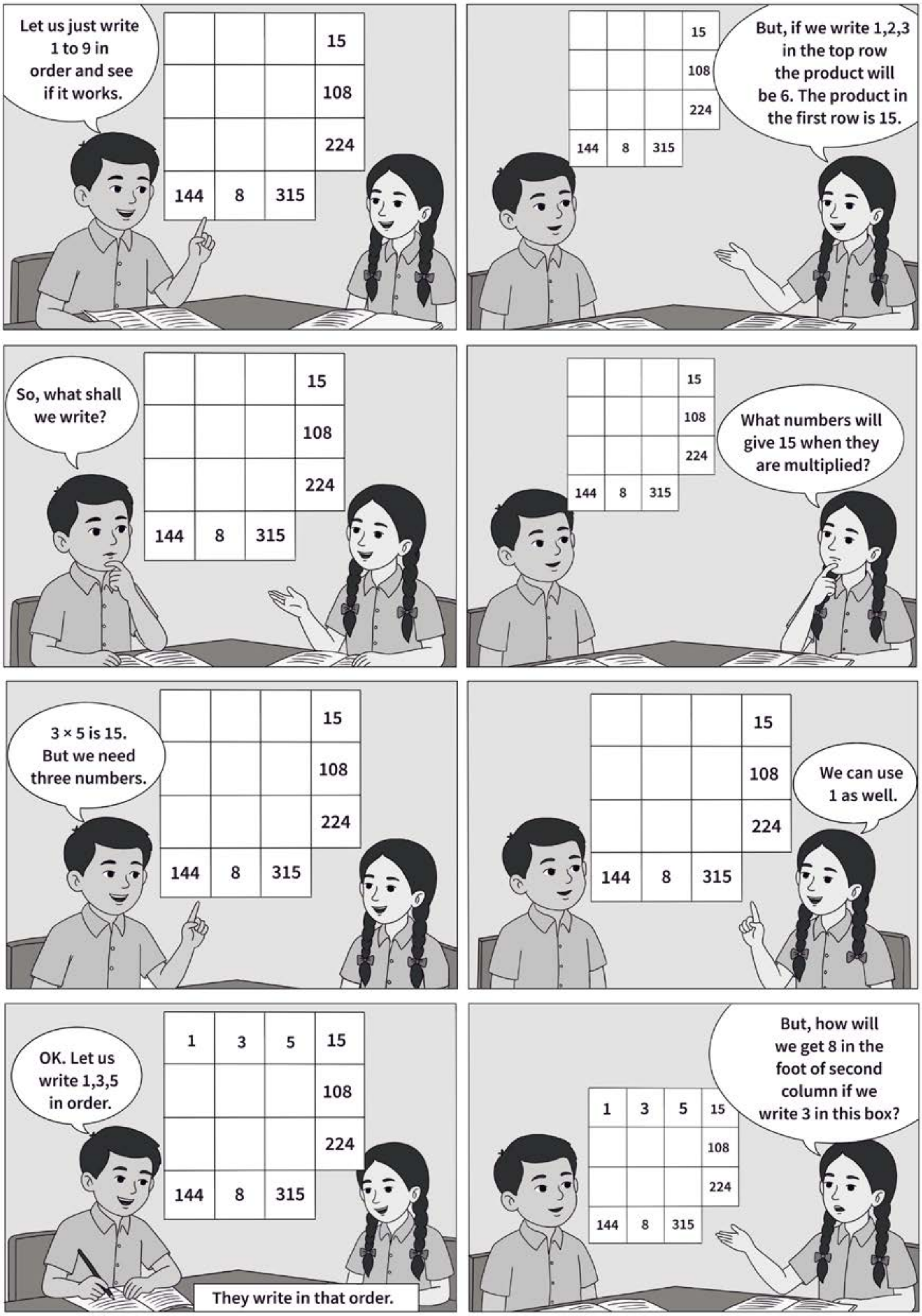
Strategies used by the teacher

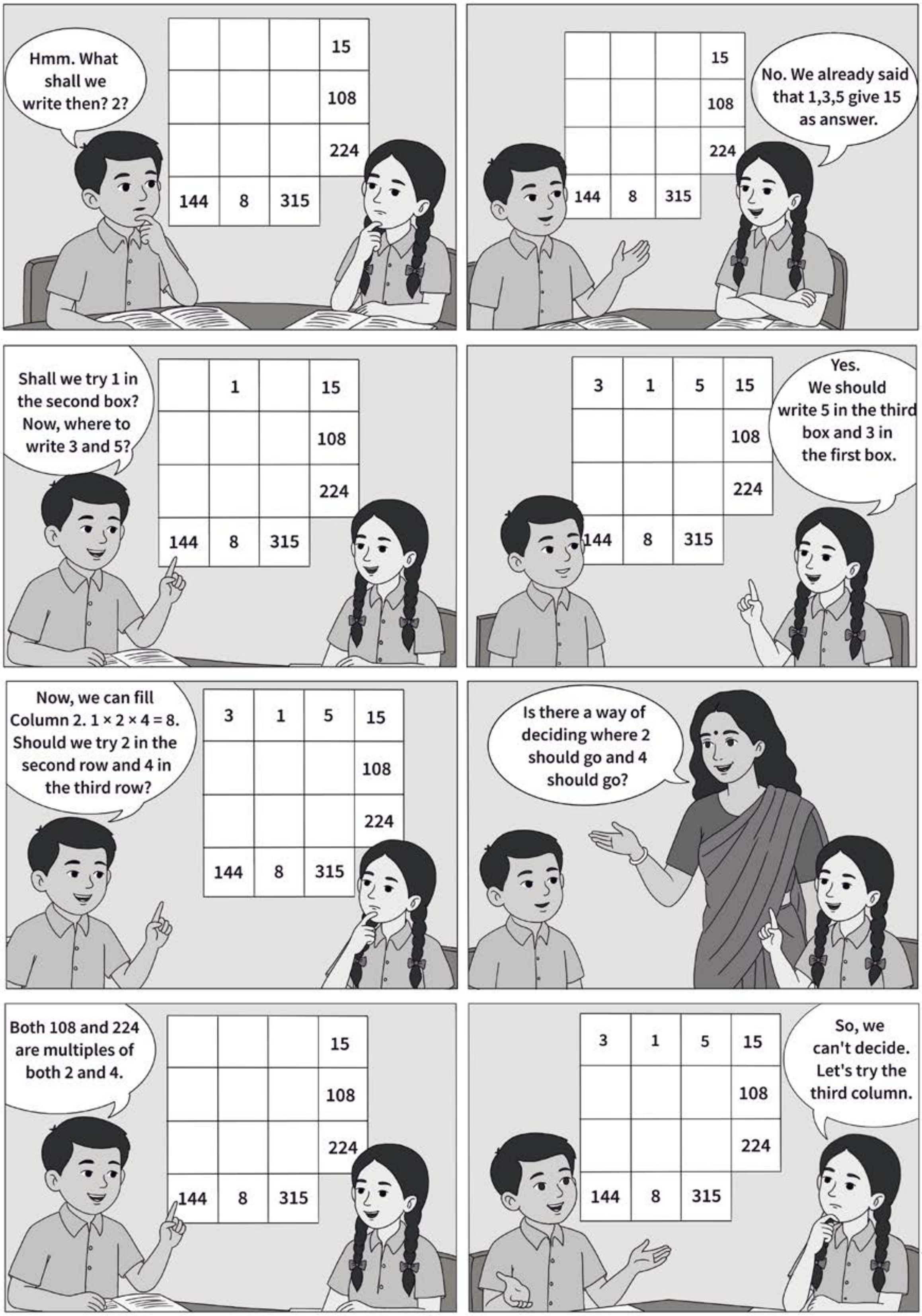
- The teacher used the problem to get students to engage in math talk.
- Students explained the problem in their own words.
- The discussion highlighted the constraints of the problem.
- Together, they reasoned out the answers they got and were provided the opportunity to clarify their understanding of such reasoning.
- Pair work ensured greater participation, particularly among the quieter students.
- The teacher guided the discussion towards a systematic way of finding all possible solutions.
- Homework was an extension of the class work but became an input for individual work and further exploration.
- The students were goaded to search for patterns and find explanations for these. By asking one student to rephrase another's explanation, the class was encouraged to listen to one another. Contributions were valued.
- Usage of appropriate math language was facilitated.
- Arithmetic laws were reinforced and revisited.
- Just one such session could achieve multiple objectives efficiently.
- Did you notice any other strategies that the teacher used?
- In the following discussions, try and pick up some other strategies. What is the teacher doing differently? What is the same? Can you recognise her objectives?

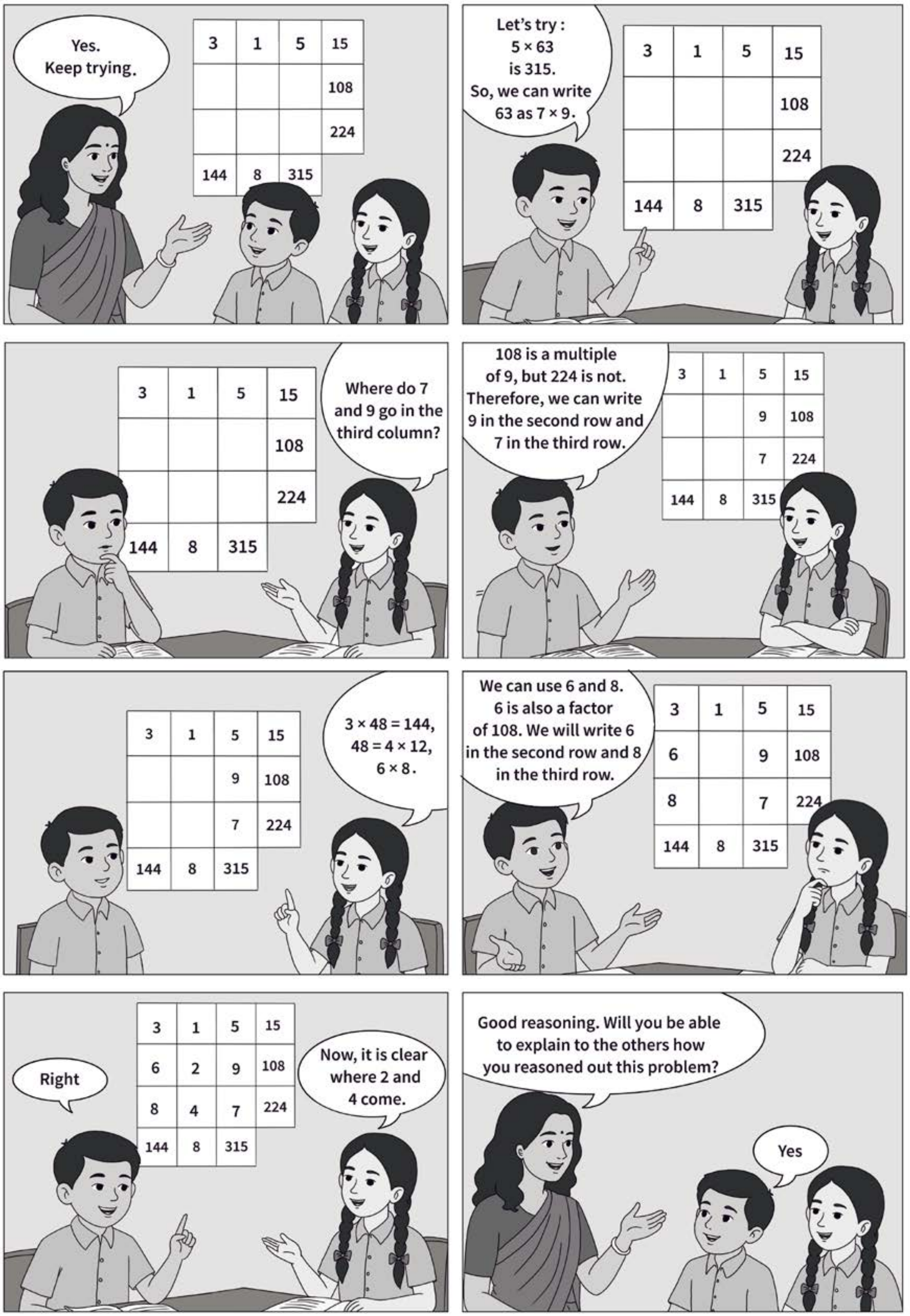
DISCUSSION 2

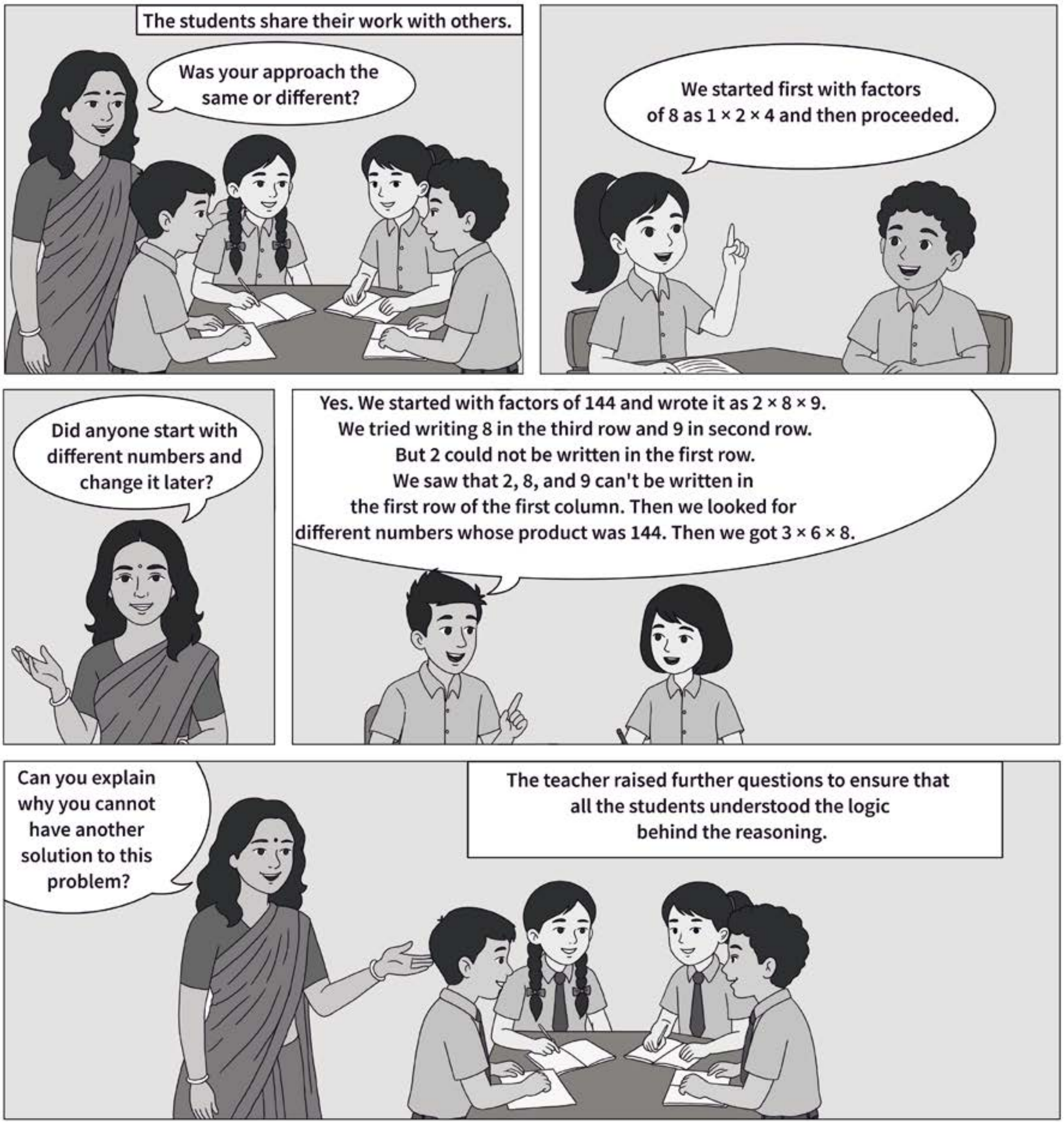
Grade Level 5, students work in pairs.




Let us just write 1 to 9 in order and see if it works.
But, if we write 1,2,3 in the top row the product will be 6. The product in the first row is 15.
So, what shall we write?
What numbers will give 15 when they are multiplied?
3 × 5 is 15. But we need three numbers.
We can use 1 as well.
OK. Let us write 1,3,5 in order.
They write in that order.
But, how will we get 8 in the foot of second column if we write 3 in this box?


Hmm. What shall we write then? 2?
No. We already said that 1,3,5 give 15 as answer.
Shall we try 1 in the second box? Now, where to write 3 and 5?
Yes. We should write 5 in the third box and 3 in the first box.
Now, we can fill Column 2. 1 × 2 × 4 = 8. Should we try 2 in the second row and 4 in the third row?
Is there a way of deciding where 2 should go and 4 should go?
Both 108 and 224 are multiples of both 2 and 4.
So, we can't decide. Let's try the third column.


Yes.
Keep trying.
Let's try :
5 × 63
is 315.
So, we can write
63 as 7 × 9.
Where do 7
and 9 go in the
third column?
108 is a multiple
of 9, but 224 is not.
Therefore, we can write
9 in the second row and
7 in the third row.
3 × 48 = 144,
48 = 4 × 12,
6 × 8.
We can use 6 and 8.
6 is also a factor
of 108. We will write 6
in the second row and 8
in the third row.
Right
Now, it is clear
where 2 and
4 come.
Good reasoning. Will you be able
to explain to the others how
you reasoned out this problem?
Yes



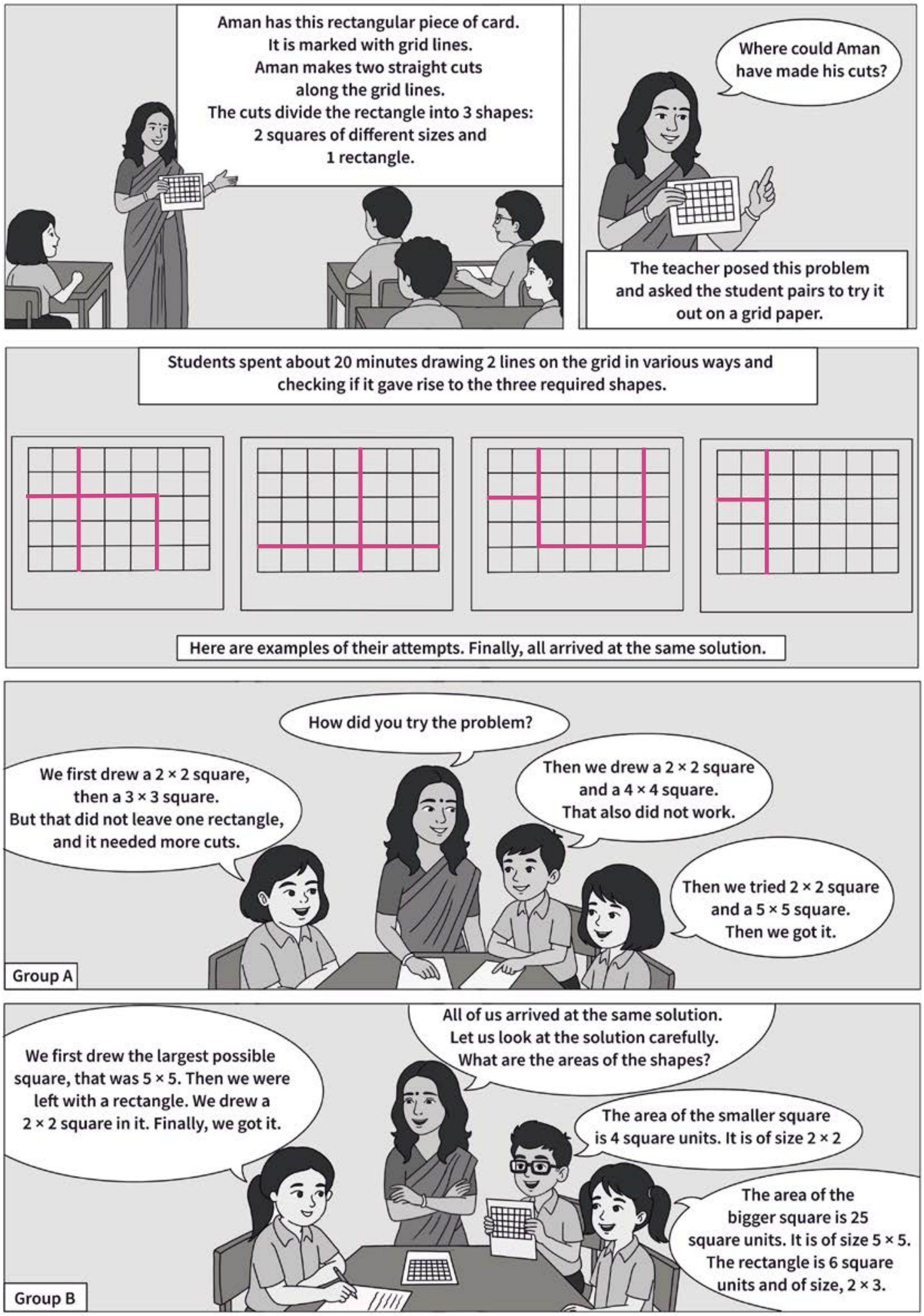
This discussion showcases most of the opportunities mentioned in Discussion 1. In addition, the students saw a connection between this problem and another.

Sometimes, relating to familiar problems can aid in understanding or solving a problem. The teacher left the first part of the discussion to the students and came into it in the middle, building a scaffold by posing questions that helped the students to proceed further.

Though there was only one solution to the problem, the emphasis was clearly on the process and how the answer could be arrived at via different routes.

DISCUSSION 3

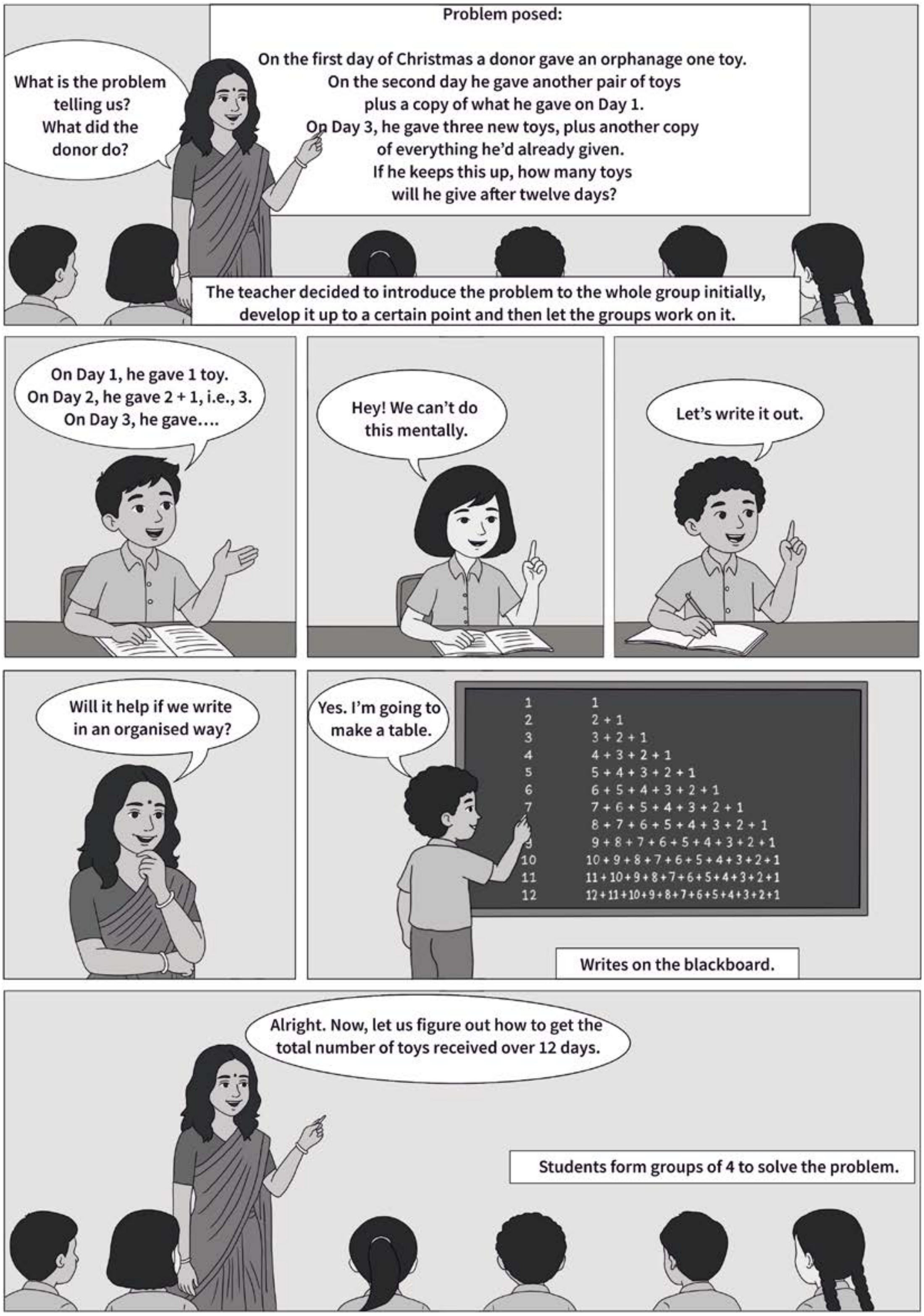
Grade Level 6, students work in groups of two or three.

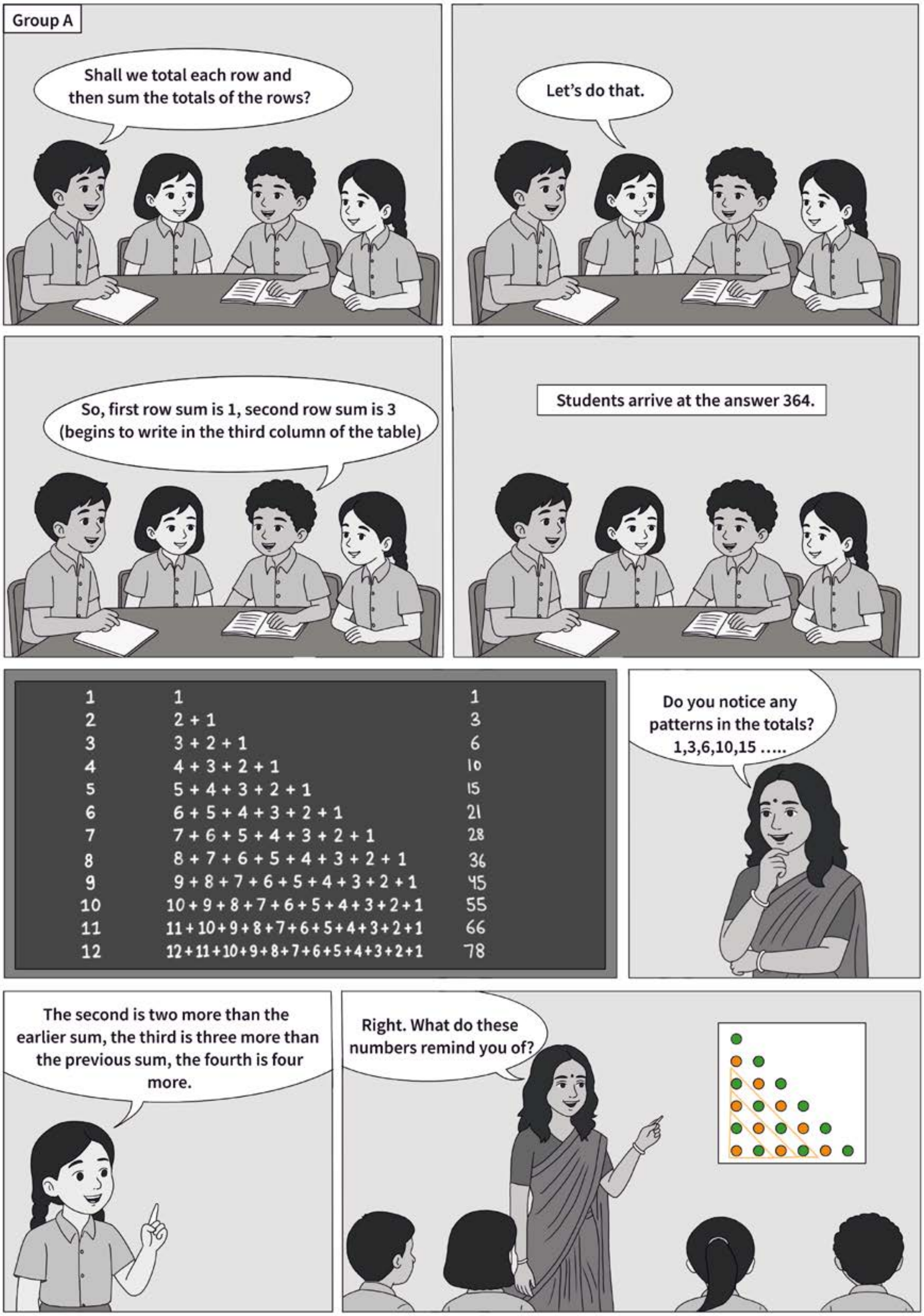


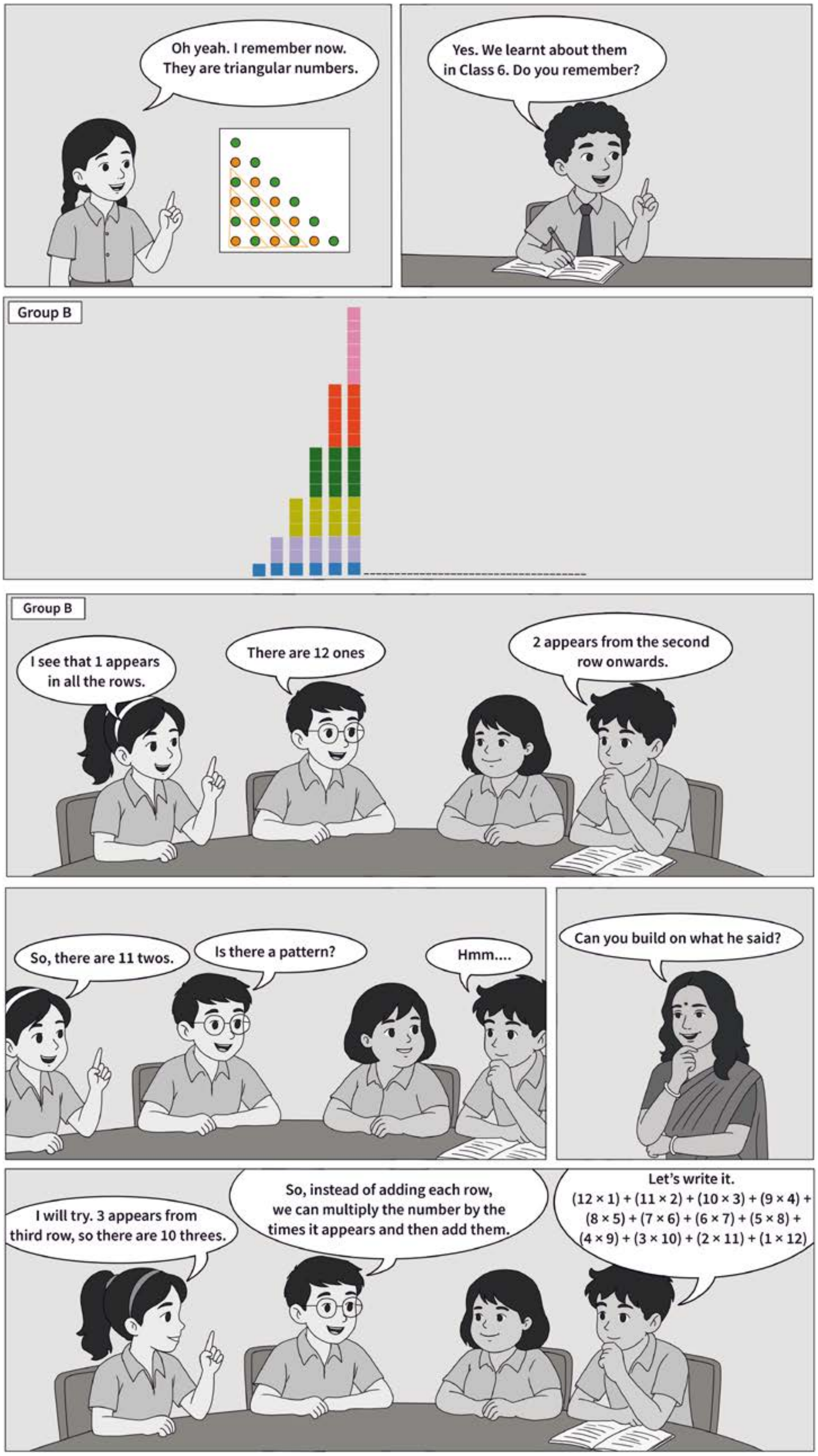
Notice that the teacher did not stop when the class arrived at one solution. What advantage do you see in the extended discussion? What are the various ways through which the teacher encouraged math talk in this discussion?

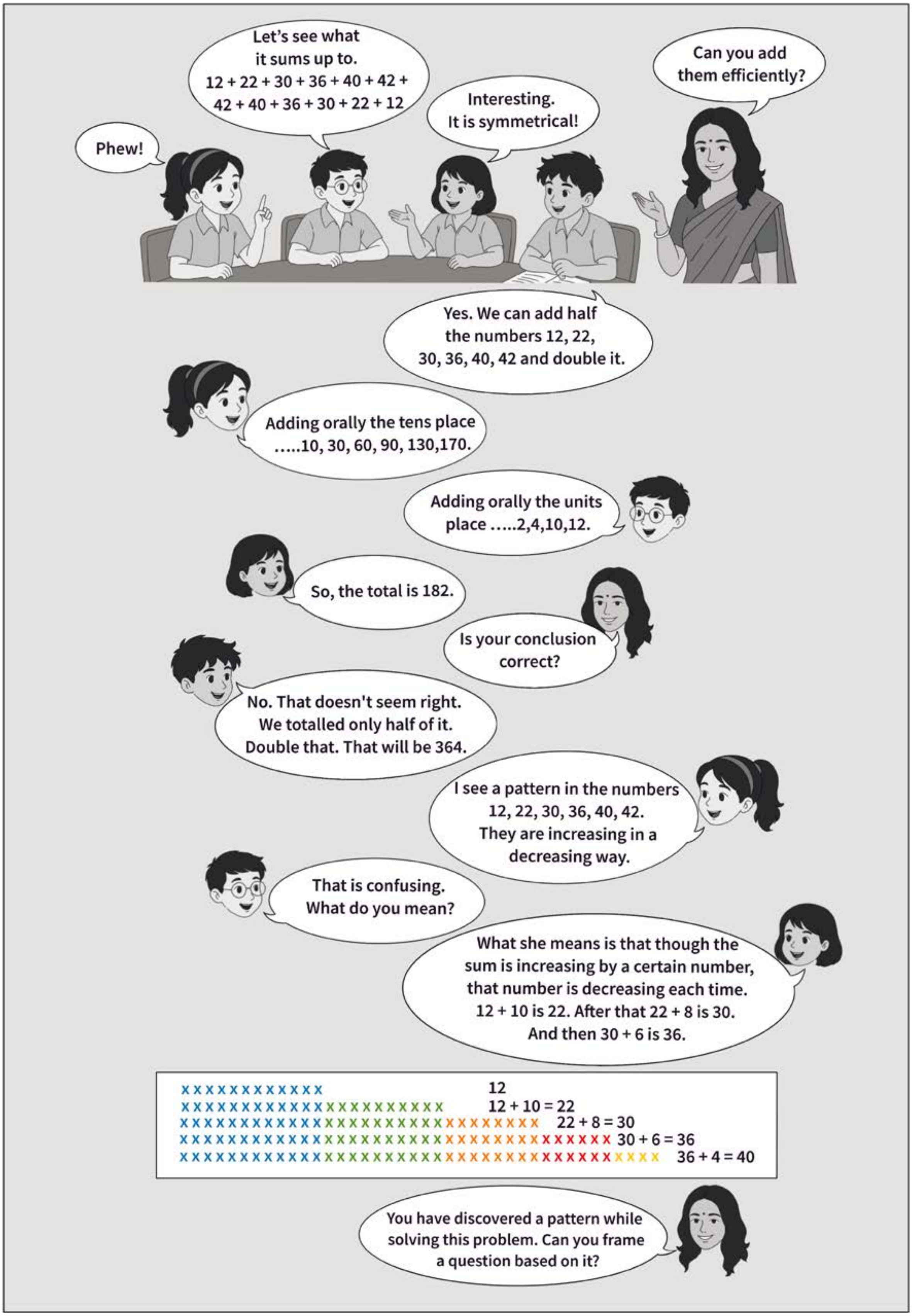
DISCUSSION 4

Grade Level 5/6, Students work in groups of 4.




Group A
Shall we total each row and then sum the totals of the rows?
Let's do that.
So, first row sum is 1, second row sum is 3 (begins to write in the third column of the table)
Students arrive at the answer 364.
1 1 1
2 2 + 1 3
3 3 + 2 + 1 6
4 4 + 3 + 2 + 1 10
5 5 + 4 + 3 + 2 + 1 15
6 6 + 5 + 4 + 3 + 2 + 1 21
7 7 + 6 + 5 + 4 + 3 + 2 + 1 28
8 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 36
9 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 45
10 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 55
11 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 66
12 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 78
Do you notice any patterns in the totals? 1,3,6,10,15 …..
The second is two more than the earlier sum, the third is three more than the previous sum, the fourth is four more.
Right. What do these numbers remind you of?

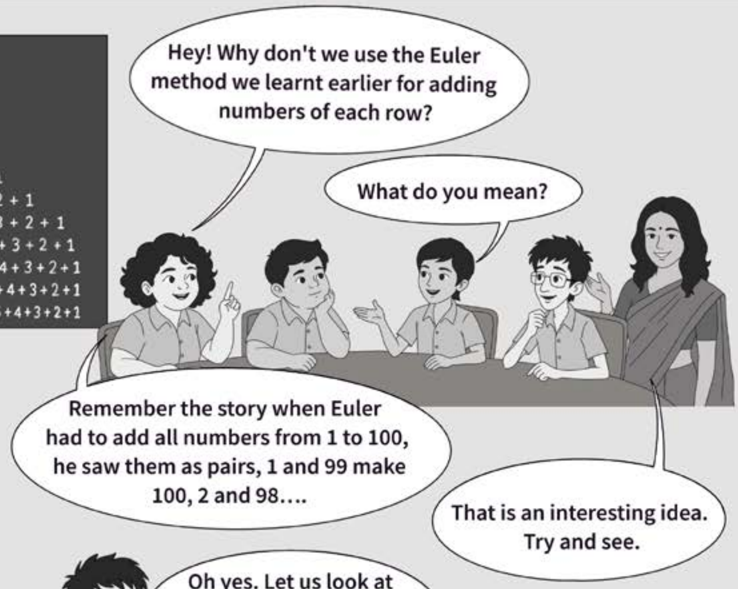

Oh yeah. I remember now. They are triangular numbers.
Yes. We learnt about them in Class 6. Do you remember?
Group B
Group B
I see that 1 appears in all the rows.
There are 12 ones
2 appears from the second row onwards.
So, there are 11 twos.
Is there a pattern?
Hmm....
Can you build on what he said?
I will try. 3 appears from third row, so there are 10 threes.
So, instead of adding each row, we can multiply the number by the times it appears and then add them.
Let's write it.
(12 × 1) + (11 × 2) + (10 × 3) + (9 × 4) + (8 × 5) + (7 × 6) + (6 × 7) + (5 × 8) + (4 × 9) + (3 × 10) + (2 × 11) + (1 × 12)

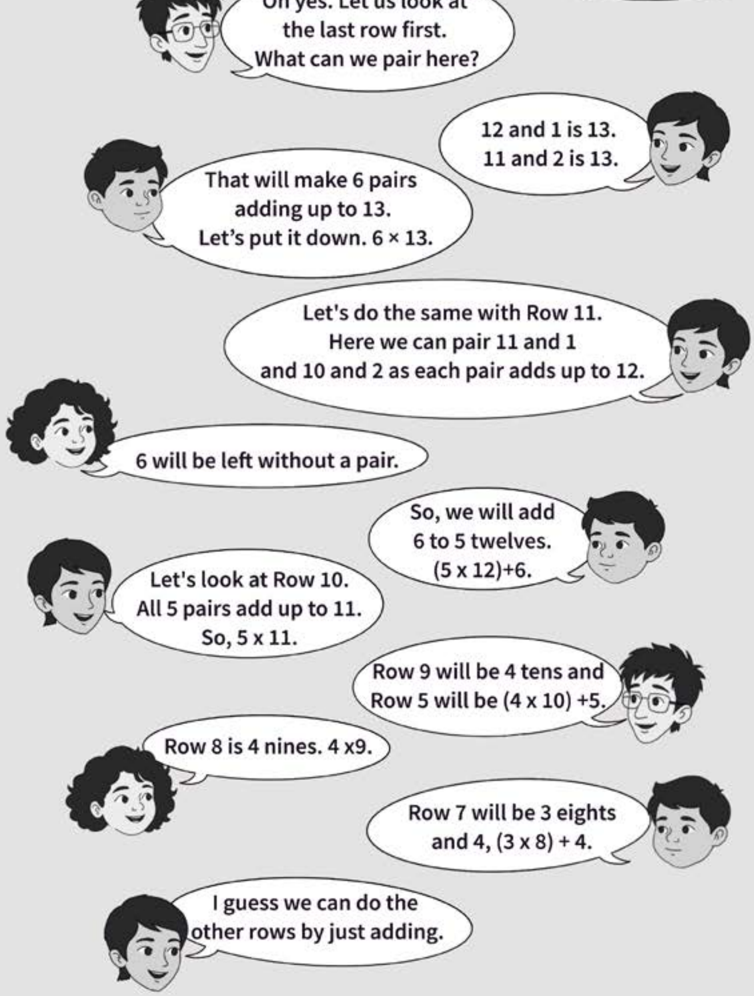


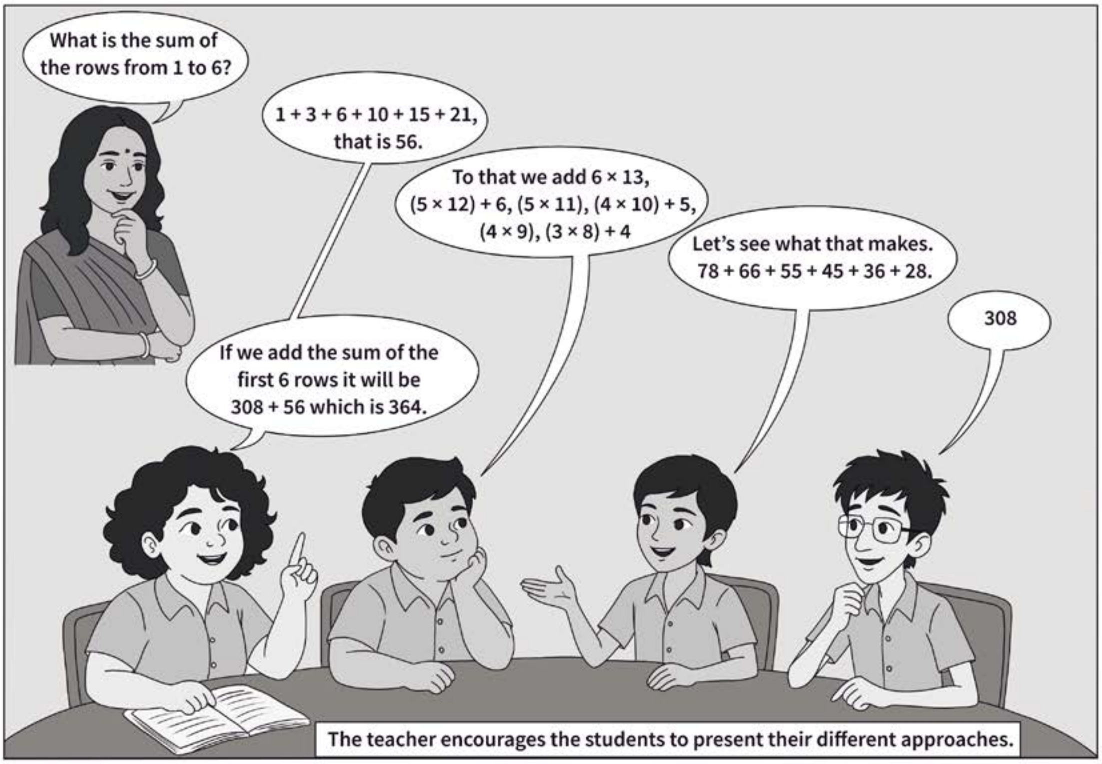
Phew!
Let's see what it sums up to. 12 + 22 + 30 + 36 + 40 + 42 + 42 + 40 + 36 + 30 + 22 + 12
Interesting. It is symmetrical!
Can you add them efficiently?
Yes. We can add half the numbers 12, 22, 30, 36, 40, 42 and double it.
Adding orally the tens place .....10, 30, 60, 90, 130,170.
Adding orally the units place .....2,4,10,12.
So, the total is 182.
Is your conclusion correct?
No. That doesn't seem right. We totalled only half of it. Double that. That will be 364.
I see a pattern in the numbers 12, 22, 30, 36, 40, 42. They are increasing in a decreasing way.
That is confusing. What do you mean?
What she means is that though the sum is increasing by a certain number, that number is decreasing each time. 12 + 10 is 22. After that 22 + 8 is 30. And then 30 + 6 is 36.
12
12 + 10 = 22
22 + 8 = 30
30 + 6 = 36
36 + 4 = 40
You have discovered a pattern while solving this problem. Can you frame a question based on it?

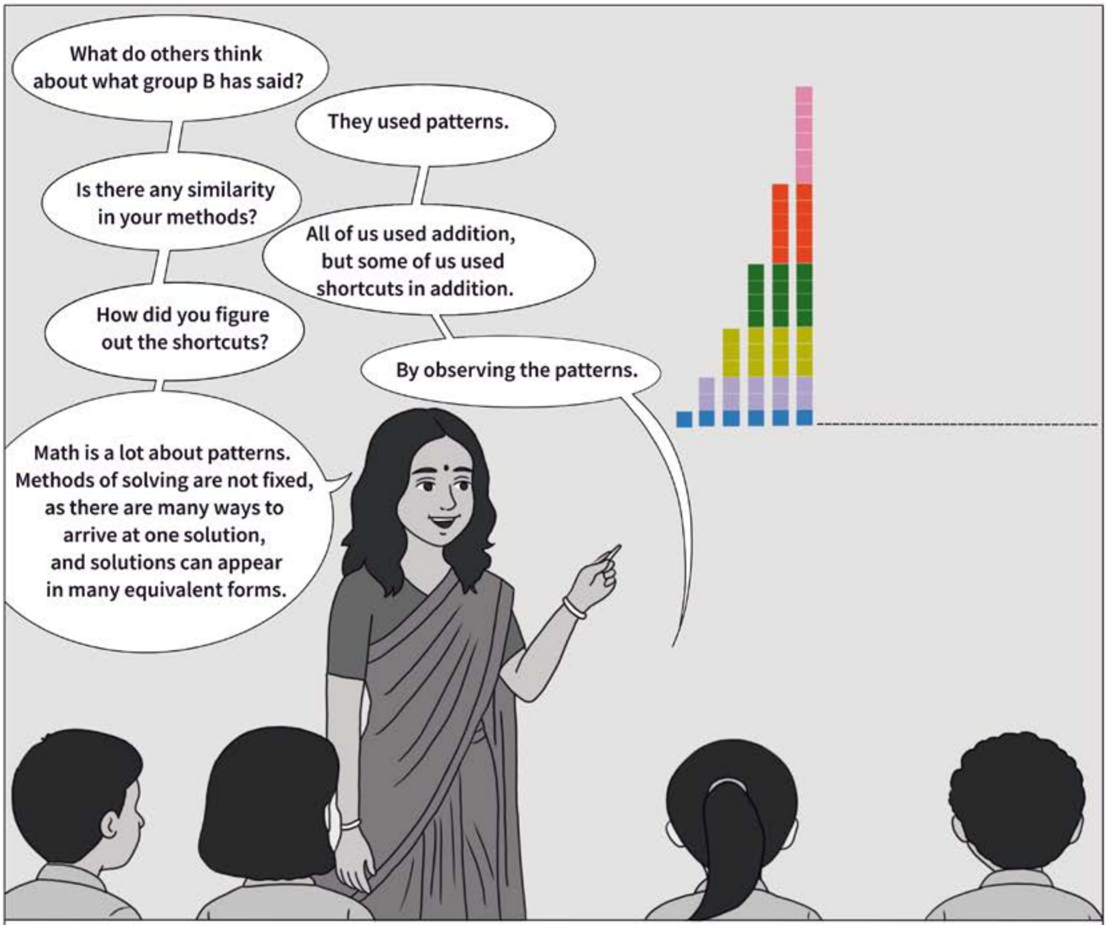
Group C

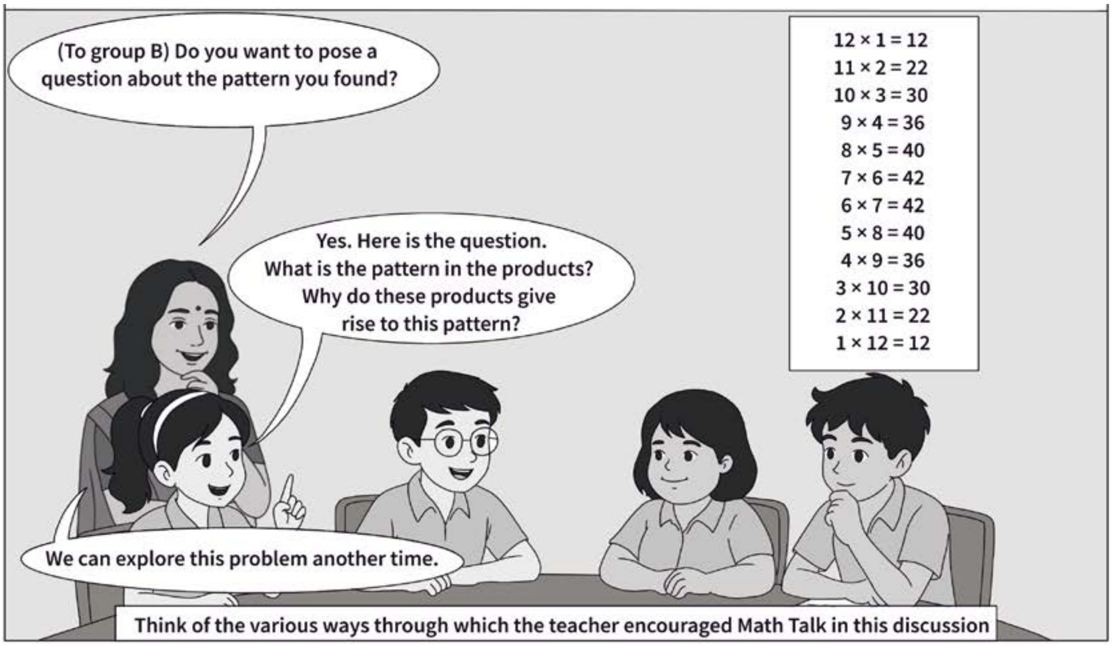
1	1
2	2 + 1
3	3 + 2 + 1
4	4 + 3 + 2 + 1
5	5 + 4 + 3 + 2 + 1
6	6 + 5 + 4 + 3 + 2 + 1
7	7 + 6 + 5 + 4 + 3 + 2 + 1
8	8 + 7 + 6 + 5 + 4 + 3 + 2 + 1
9	9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1
10	10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1
11	11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1
12	12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1






What is the sum of the rows from 1 to 6?
1 + 3 + 6 + 10 + 15 + 21, that is 56.
To that we add 6 × 13, (5 × 12) + 6, (5 × 11), (4 × 10) + 5, (4 × 9), (3 × 8) + 4
Let's see what that makes. 78 + 66 + 55 + 45 + 36 + 28.
308
If we add the sum of the first 6 rows it will be 308 + 56 which is 364.
The teacher encourages the students to present their different approaches.


What do others think about what group B has said?
They used patterns.
Is there any similarity in your methods?
All of us used addition, but some of us used shortcuts in addition.
How did you figure out the shortcuts?
By observing the patterns.
Math is a lot about patterns. Methods of solving are not fixed, as there are many ways to arrive at one solution, and solutions can appear in many equivalent forms.



Think of the various ways through which the teacher encouraged Math Talk in this discussion

Please do share with atrightangles.editor@apu.edu.in any math talk that can be generated by these, or other examples.

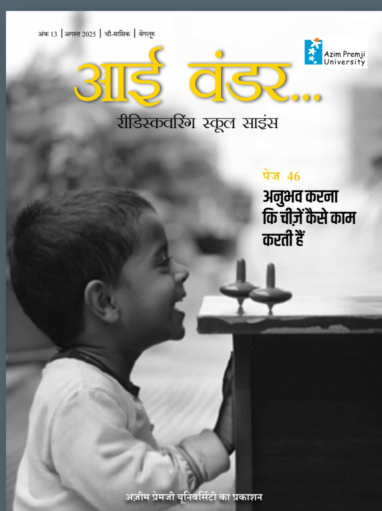
We have showcased interesting questions and even more interesting discussions which allow students to discover things. A natural question would be about how practical it is for the average teacher to incorporate this, first by understanding the nuances of each question and second, by managing her time with portions to complete, deadlines to meet, and so on. In general, whenever we meet teachers, we recommend that they try such investigations and discussions at least once a month. Math talk promotes mathematical thinking and that is an important aim for the teaching of mathematics.


PADMAPRIYA SHIRALI

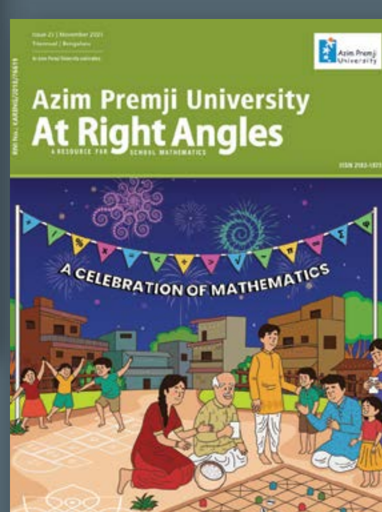
PADMAPRIYA SHIRALI is part of the Community Math Centre based in Valley School (Bangalore) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. In the 1990s, she worked closely with the late Shri P K Srinivasan. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box.' She is currently part of the NCERT textbook development group. Padmapriya may be contacted at padmapriya.shirali@gmail.com

Azim Premji University

Azim Premji University Magazines



Scan here to subscribe to At Right Angles for free!



To know more about our other publications, write to us at publications@apu.edu.in