# Learning Curve

Issue XIV, March 2010

Azim Premji Foundation

A Newsletter from
Azim Premji Foundation

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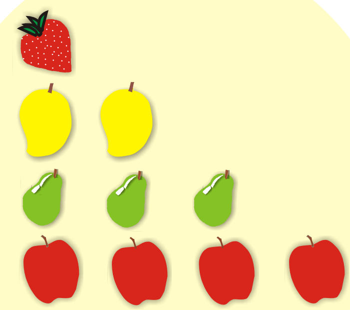
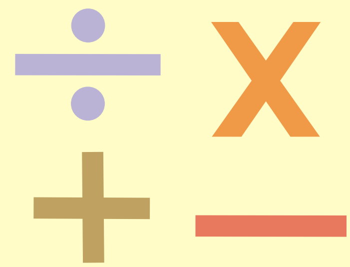
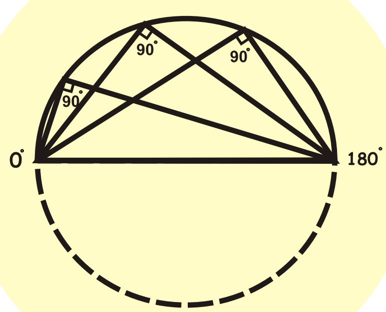
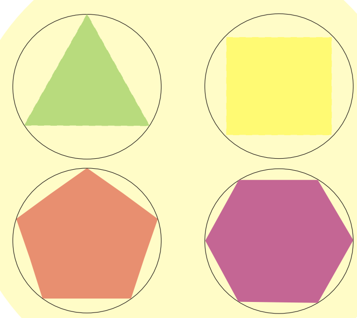
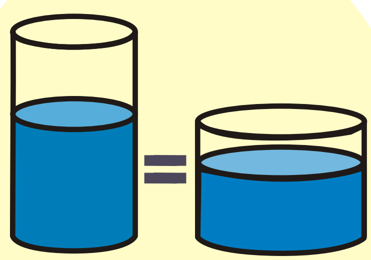




## Special Issue on School Mathematics





In this issue



Also inside: Trivia, Logico-Math Brain Teasers and more....

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# From The Editor

It is a pleasure to bring you this Special Issue of Learning Curve on Mathematics. It is the third in the series to be devoted to a particular theme or subject, the previous ones being on Science and Language learning.

Putting together this special issue on Mathematics has been a precious learning experience for us. Unlike the issues on Science and Language, one realized that one must address all the extreme emotions that surface at the very mention of Math: fear, hate, terror and absolute love and ecstasy too. Topics that we thought were obvious turned out to be not so obvious. In the very process of discussing and finalizing the menu, our own perspective about the subject grew visibly.

As always, we reached out to a distinguished panel of contributors and as usual we have run up a huge debt of gratitude. Each one of the contributors responded to our unreasonable deadlines with tolerance and enthusiasm. One contributor, unfailingly humorous at all times told me that I might as well sign my letters as "naggingly yours" instead of the customary "sincerely yours". Prof H. Subramanian ignored ill health and wedding duties in the family to submit his article in time. With that one gesture he has forever a special place in our hearts. It is only such support that enables us to bring out Learning Curve.

We are hopeful that every article will strike a chord in you; for instance while one article discusses the very nature of Mathematics the other traces the history of the subject; similarly while one describes the pedagogy of the subject the other shares insights and the practical perspective of the teacher. And so on.

We found it immensely enriching to read the essays of students to understand what goes through their minds and souls as they comes to terms with Math?

Nearly 20 years ago in Delhi, I accompanied my daughter for admission to kindergarten class. The head mistress after the usual ice breakers, asked the child, whether one is bigger or two is bigger. The child immediately said one. To which the kind lady opened both her palms and asked the child whether she wanted one chocolate or two. The child took two and the lady repeated her question. Is one bigger or two? One repeated the child firmly. To the eternal credit of the head mistress, she promptly admitted the child. Over the years nothing seems to have changed because last week as I flipped through the cartoon strip in a newspaper I saw a person presumably a teacher ask a child, "If I had four apples in this hand and four in the other hand, what would I have?" To which the child answers very correctly "Really big hands!"

Your views on this edition of Learning Curve are most important. While your encouragement has been the tonic, your candid feedback helps us to continuously improve subsequent editions.

**S. Giridhar**
Head - Programs & Advocacy

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## The Broad Picture

# 1 A Culture of Enjoying Mathematics

**Shashidhar Jagadeeshan**

### Introduction

It appears that whether we like it or not, Mathematics pervades all aspects of our lives. Whether you are a farmer or a techie, a comfortable relationship with Mathematics, and competency at the level at which one uses it, is a requisite in an equitable society. Some will argue that even if the content of Mathematics learnt at school is forgotten, students will retain the ability to think clearly and logically (an essential life skill) because of their exposure to mathematical reasoning. The tacit assumption here is that learning Mathematics will not only help us in our daily lives but will also enhance the quality of our life. How ironic that for a vast majority their experience with Mathematics is so contrary to this assumption. Enough has been written bemoaning the state of Mathematics education the world over, and the term 'Mathphobia' has become part of common parlance. A major reason for school dropout is the inability to cope with Mathematics; it seems to be a universal phenomenon that many students fear and dread Mathematics. Sadly, this feeling often persists into adulthood.

There have been many attempts to reform Mathematics education, and huge sums of money have been dedicated to this cause. Unfortunately, the motives for reform are suspect and, in my opinion, this is part of the problem. Advanced nations want to improve their citizens' Mathematics competency out of a fear that citizens of rival nations are outperforming them. Emerging nations want to improve their Matheducation so that they can create a 'knowledge society'. Humans empowered with knowledge are seen as a great asset in the market place. Reforms based on these motivations do not seem to have made much impact in the long run on Mathematics education (although there was a brief 'golden age of basic science' in the US thanks to the Sputnik fear).

If we are to make any headway in addressing both problems, that of poor mathematical competency and that of Mathphobia, we need to explore several questions first. What is the nature of Mathematics and how do our particular biases impact curriculum design? What is the relationship that students and teachers share with Mathematics? What are the myths or beliefs that studentst and teachers have about Mathematics? And perhaps most importantly, what are the factors that motivate humans to learn? In this article I hope to begin such an exploration by first describing the various ways in which Mathematics is viewed and experienced, and how these views might affect curriculum if applied in isolation. I then go on to look at curriculum design and pedagogy and see if we can truly create a culture of enjoying Mathematics not just for a elite few but for all.

### The Blind Men and Mathematics

We are all familiar with the famous Jataka tale about the blind men and the elephant. Each one makes tactile contact with a different part of the elephant, and comes up with descriptions ranging from a wall to a rope! Mathematics too suffers from partial perceptions. Perhaps the mystery, depth and richness of Mathematics is revealed in the fact that it can be seen in so many different ways. Let us look at some of these perceptions and how they impact curriculum design and pedagogy.

**Mathematics as accountancy:** For a large majority of people, Mathematics is synonymous with accountancy. Perhaps it is not unreasonable to say that the bulk of humanity uses Mathematics to compare prices, make sure they are not being cheated of the correct change, perhaps calculate interests, discounts and rebates; some may even calculate areas and volumes. The more advanced may use it in book keeping. It is also true that many discoveries in arithmetic probably came from the need to keep records of land and accounts of trade. Examples that come to mind are preliminary trigonometry and mensuration, motivated by the need to calculate land holdings in the Nile Valley. Perhaps the motivation for the discovery of the Hindu-Arabic numeral system came more from the need to do large calculations in astronomy than from the need to do book keeping. But surely this discovery, considered one of the 'greatest intellectual feats of humans,' has had its major impact in the field of commerce. For most people arithmetic and number manipulation is synonymous with Mathematics.

If this is one's only experience with Mathematics, then one will design the curriculum and teach Mathematics as if it were a science of algorithms to do mechanical calculations and lose many students in this drudgery. Sofia Kovalevskaia expresses this wonderfully: "Many who have never had the occasion to discover more about Mathematics confuse it with arithmetic and consider it a dry and arid science. In reality, however, it is a science which demands the greatest imagination."

**Mathematics as problem solving and mental gymnastics:** One of the major features of Mathematics is problem solving, and many who discover the thrill of problem solving at a young age become professional mathematicians when they grow up. However, if this aspect of Mathematics is distorted, and seen in the wrong perspective, it becomes a source of fear and aversion toward Mathematics. Since talent in problem solving appears at a very young age, children are often classified as 'brilliant' or 'dull' based on this single ability. When an education system equates a child's self worth with their mathematical problem solving ability, it does great harm both to the ones who are adept at problem solving, and to those who are not. Those who find problem solving difficult, and who then go on to be labeled as 'stupid' (either by society or by themselves), develop a deep fear and aversion to all of Mathematics. This self image is often linked to their self esteem, leading to feelings of insecurity and shame. All of us have encountered perfect strangers who have a great need to confess how bad they are at Mathematics. On the other hand, those who are very adept at problem solving and Mathematics are automatically labeled 'intelligent', and run the risk of becoming one-dimensional human beings with poor social skills. I leave it as an easy exercise to name your favorite mathematician as an example to illustrate this point!

There is no doubt that a large part of mathematical theory is motivated by the desire to solve difficult problems. Fermat's Last Theorem is a famous example. However, not all mathematical problems are of the same order. Some problems are indeed very profound and like the tip of an iceberg, reveal deep aspects of Mathematics. Many problems (an endless plethora) are simply mental gymnastics often created by working backwards from solutions, requiring some inane trick to solve.

"*Those who find problem solving difficult, and who then go on to be labeled as 'stupid' (either by society or by themselves), develop a deep fear and aversion to all of Mathematics. This self image is often linked to their self esteem, leading to feelings of insecurity and shame.*"

These problems form the core of most of our competitive exams and are used as a sieve to weed out applicants. Any system that uses this gymnastic ability as a yardstick to decide how to distribute access to resources such as education and jobs will surely create a skewed society. The effects of these are already being seen today at our institutions of higher learning. Students who have been put through a grind of mindless problem solving are burnt out and have no motivation to learn anything new. Such students have a very narrow view of Mathematics and very few will choose Mathematics research and teaching as a career. I have heard senior professors and administrators bemoaning the fact that it is very hard to find competent people to teach Mathematics at many new prestigious institutions in India. Imagine the fate of the many thousands of students who have, after several years of preparation, failed to get access to so-called quality education. With damaged psyches and bruised self confidence, what kind of learning can take place? Further the erroneous identification our society has made between intelligence and mathematical ability has led to a dismal state of education for those interested in pursuing the humanities, because disproportionate funds are made available to science education. Many students with no real interest in the sciences and perhaps very gifted in other areas still pursue science.

**Mathematics as the 'language of the universe' and as a useful tool in modern society:** With Galileo, Mathematics has begun to be seen as the language of the universe. Those who seek to unravel the mysteries of the universe see Mathematics as a sixth sense needed to comprehend the universe. We marvel with the physicist Eugene Wigner who gave a lecture in 1959 titled 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences.' Wigner ends his lecture by saying, "The miracle of the appropriateness of the language of Mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning." For many who pursue the more theoretical aspects of science, it is this 'miraculous' aspect that they most appreciate in Mathematics.

> *Mathematics with its brilliant ability to model phenomenon has had a far reaching impact in all aspects of our lives and on many fields of study, ranging from biology to economics.*

Mathematics with its brilliant ability to model phenomenon has had a far reaching impact in all aspects of our lives and on many fields of study, ranging from biology to economics. This modeling ability has also made Mathematics an extremely useful tool for a wide variety of people ranging from businessmen to engineers. The vast majority of us use computers these days with no clue as to how they work, and similarly Mathematics is used as a tool by practitioners who have no clue as to why the tool works. A view of Mathematics that demands that its utility be demonstrated at all times will also have an adverse effect on Mathematics curricula and the teaching of Mathematics. The trouble is, very little of school Mathematics can be shown to the student to be applicable in a real sense. Most often the examples are rather contrived and meaningless. Furthermore, an attitude that says, "I will learn something only because it is useful" comes in the way of true learning.

A utilitarian approach to Mathematics with benefits to be reaped in the future, while doing mechanical and meaningless calculations in the present, is not going to inspire students. It makes it boring, and Mathematics loses its playful and joyful aspect. As Julian Williams put it "The average student needs emotional and intellectual satisfaction now, not just in five or ten years' time, when they become adults!"

**Mathematics as truth and beauty:** We now enter esoteric descriptions of Mathematics! All pure mathematicians worth their salt will declare that the reason that they do Mathematics is because it is beautiful. If they are Platonists then they will further declare that they are in search of 'mathematical truth', something to be discovered rather than invented. Who better to express this than G. H. Hardy, the advocate for all pure mathematicians? “A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.” He goes on to say, “The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly Mathematics.”

In my opinion however, the strongest motivation for pursuing Mathematics is experienced at an emotional level. All mathematicians, no matter what their view on the nature of Mathematics, will agree that in the process of creating Mathematics they experience a sense of 'illumination spreading throughout the brain'. Alain Connes, a Fields medalist (the highest honour one can achieve in Mathematics), describes this sensation as follows: "But the moment illumination occurs, it engages the emotion in such a way that it's impossible to remain passive or indifferent. On those rare occasions when I've actually experienced it, I couldn't keep tears from coming to my eyes."

The view of Mathematics that it is akin to a creative art form, and that only those who have tasted the heady joy of discovering Mathematics truly understand it, has the strongest appeal to most people (including myself) who study Mathematics for its own sake, and not only for its applications or other aspects as discussed above. From time to time mathematicians have lamented the fact that it is because both teachers and students do not truly understand this nature of Mathematics that we have distorted its curriculum and teaching.

However, a view that says that all mathematical experiences that should be similar to experiences of art or music also has its limitations. Beauty in art and music is relatively easily accessible to most human beings, but to see the beauty of Mathematics requires a special connection to it and a fair degree of training. A large part of school Mathematics often does not have a rich enough structure to reveal its beauty; in fact in these years it is problem solving that draws most children to Mathematics. If Mathematics is an art form, then why force all children to learn Mathematics? If we make Mathematics optional based on early reactions to it, are we being responsible to children? Since aesthetics is a matter of taste, must we then allow teachers to fashion their curricula according to their taste? This will surely not satisfy the taste of all their students, let alone help them use Mathematics as a foundation to learn other subjects or earn a livelihood.

Then there is the question of why society should support mathematical activity. Most artists need patrons or find buyers for their art work. Mathematicians do not sell their theorems for a living! Frankly it is because policy makers see Mathematics as a useful tool that most people in the field of Mathematics are able to feed themselves! They either teach Mathematics or 'do' Mathematics which is considered useful for a living. A tiny minority is supported for doing Mathematics for its own sake.

**Mathematics for all?**

If we insist that Mathematics be part of the core curriculum for all students then we must also make it a fundamental right that all students enjoy learning Mathematics! A renowned Carnatic musician once told me that Carnatic music enjoyed a huge revival some years ago, thanks to the efforts of several young musicians who created and nurtured a broad 'rasika' base. These young people revived the 'sabha culture' in Chennai and hundreds of other small towns and villages throughout South India. Musicians young and old, amateur and accomplished, now have an appreciative audience and can earn a decent livelihood.

Can we create a culture of enjoying Mathematics? Surely this is the only holistic solution to the problems discussed so far. This can happen only if all stake holders really get a feel for the joy and thrill of doing, using and learning Mathematics. This seems a utopian dream given the current state of affairs - unimaginative curricula, poor infrastructure, poorly prepared teachers ("Mathematicians are not interested in teaching children, and teachers are not interested in doing Mathematics," says Paul Lockhart) and a culture of fear and anxiety as far as Mathematics is concerned. But like all revolutions change must begin at both the individual/grassroots and the systemic level.

*"If we insist that Mathematics be part of the core curriculum for all students then we must also make it a fundamental right that all students enjoy learning Mathematics."*

At a systemic level we need to delink mathematical ability from intelligence. We need to help each child discover what they really love, but also learn to love the things they do. We should stop using arcane mathematical problem solving skills as the main criterion for access to resources such as education and jobs. We have to urgently develop a broader base for assessing the aptitude and skills of our young. I am not suggesting a watering down of standards, instead I am asking for a broad based system of evaluation which takes into account the multiple facets of human intelligence, the capacity to be accountable and the more elusive quality of being a sensitive and responsible human being. A radical shift in this area will wipe out cultural anxiety towards Mathematics.

At the level of curriculum, we need to be clear about our goals for Mathematics education. At the minimum we would like everyone to be competent at numeracy, have sufficient critical understanding of data gathering and presentation so that they do not buy into false propaganda, and have the reasoning ability to detect fallacious arguments. For a smaller number of people, the goal would be one of competence in the use of Mathematics as a tool; for an even smaller fraction, to create new Mathematics (rarely are creative mathematicians the programmed outcome of a system-they turn to Mathematics in spite of any system, as they cannot but do Mathematics!).

The Mathematics curriculum framework outlined in the NCF 2005 is an excellent document and goes a long way in giving very clear guidelines. However, there is an urgent need for a think tank of mathematicians, teachers and educational psychologists to create material keeping these goals in mind. We need to teach numeracy in creative ways so that these skills are mastered and retained. Since they will be used in daily practical situations, these skills should be evaluated through projects and games that simulate relevant situations, rather than through stressful exams. Since Mathematics often builds on itself, concepts need to be revisited but in creative rather than repetitive ways. The whole curriculum should be infused with the philosophy that Mathematics is the 'science of pattern recognition.' We need also to pay special attention to how pattern recognition is assessed. I have had several students who would not appear good at conventional textbook Mathematics, but nevertheless have a very strong spatial sense and are adept at recognizing patterns and solving logical puzzles. Children should have sufficient experience with solving meaningful problems and experience the thrill of having an insight. There is no ready made material available in the market that reflects all these demands. As I have said earlier, it is urgent that we set aside resources to create or at least put together such material in a coherent manner and train teachers to use them effectively.

At the level of the classroom, it is extremely important that a teacher creates a true learning space. For such a space to be created there has to be a relationship of trust and affection between the student and teacher. The teacher must really enjoy doing Mathematics so that his students feel inspired. More importantly, he must help them understand their own fears and resistance to learning, and enable them to take ownership for their own learning. Twenty years of education at Center For Learning has taught us that all these are not romantic pipe dreams, but very much within the realms of possibility.

**Suggested Reading**


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# 2 The Nature of Mathematics and its Relation to School Education

**Amitabha Mukherjee**

## Background

Mathematics, among all school subjects, enjoys a unique – and paradoxical – status. On the one hand, it is regarded as an essential ingredient of school education. It is taught as a compulsory subject right from Class I to Class X. Moreover, it is often regarded as a kind of touchstone: an educated person is one who knows Mathematics. On the other hand, it is the most dreaded of school subjects, leading to a widespread sense of fear and failure among children. Even adults who have gone through school successfully can be heard to declare: "I could never follow Math in school." (When some of us started the School Mathematics Project at the Centre for Science Education and Communication, Delhi University, in 1992, our aim was to address this fear. For a more recent articulation, see the Position Paper of the National Focus Group on the Teaching of Mathematics, URL http://www.ncert.nic.in/html/pdf/schoolcurriculum/position_papers/Math.pdf)

The above dichotomy raises a number of questions. Some of these are: what is Mathematics and why should we teach it in school? Does the problem with school Mathematics have something to do with the nature of Mathematics, or the way it is taught, or both? Can everyone learn Mathematics up to a point? What Mathematics should we teach in school? How should we teach it?

To attempt to provide answers to all the above questions would be ambitious, even foolhardy. In this article I will focus on some changes that have taken place in the thinking about school Mathematics over the last five decades, and their impact as felt in India in the last few years.

## Mathematics for all

Any contemporary discussion on school Mathematics must take into account the context of Universalisation of Elementary Education (UEE). Today, UEE seems to be an attainable target rather than a distant dream. The next milestone of Universal Secondary Education (USE) will surely form a major part of the educational agenda in the coming decade. Thus when we talk of school Mathematics we are talking of something that is addressed to all children.

Can everyone learn Mathematics? The answer, fifty years ago, would perhaps have been a clear NO. Even now, we hear adults talk of children who 'will never be able to learn Mathematics'. How does this face up to the concerns of UEE/USE? Taking a categorical position, the Position Paper mentioned earlier asserts that:

> *Our vision of excellent mathematical education is based on the twin premises that* ***all students can learn Mathematics and that all students need to learn Mathematics.*** *It is therefore imperative that we offer Mathematics education of the very highest quality to all children.*

The question which then arises is: what kind of Mathematics teaching can meet the needs of all students? To be able to address this, we need to achieve some clarity about the goals of Mathematics education.

> *"Given that all children are going to be learning Mathematics up to Class VIII and perhaps Class X, the main aim of school Mathematics teaching cannot be to produce Mathematicians."*

## The aim(s) of School Mathematics Education

Given that all children are going to be learning Mathematics up to Class VIII and perhaps Class X, the main aim of school Mathematics teaching cannot be to produce Mathematicians. It cannot, for that matter, be to help produce scientists and engineers, in spite of the special and important place that Mathematics occupies with respect to these disciplines. What then are the goals of school Mathematics education? The Position Paper says:

***Simply stated, there is one main goal— the mathematisation of the child's thought processes.***

In other words, the aim is to learn to think about the world in the language of Mathematics, and to develop the kind of thinking that is special to Mathematics. On the other hand, a look at curricula and textbooks in force in the country during much of the last five decades suggests otherwise. It would seem that 'university education', or perhaps 'IIT education' , has dominated the content and style of school Mathematics. No wonder a majority of past and present school goers have no love for the subject!

### What is Mathematics, anyway?

If mathematisation of thinking is the main goal of Mathematics education, we need to have some agreement on what constitutes Mathematics. If you ask people at random the question "What is Mathematics?" You will most likely get spontaneous answers "Addition, Subtraction, Multiplication and Division". (On second thoughts or if pressed, people usually add algebra and geometry.) Now these operations on numbers undoubtedly form an important part of Mathematics, but they alone cannot serve to define Mathematics or mathematical thinking. I will not attempt to give a definition; instead, I give you some examples of mathematical thinking.

"The door is between me and the wall."
"There are around fifty toffees in the jar."
"This glass is tall but thin. It will take less water than the wide mug."
"Nineteen and fifteen is ... twenty and one less than fifteen ... that's thirty-four."
"The station is about fifteen minutes if you take the road, but there's a short cut which will get you there in ten minutes."

At first sight, it may seem that the first statement carries no evidence of mathematical thinking. For a pre-school child, however, articulating spatial relationships such as 'above', 'below', 'between', 'beyond' is an important part of mathematisation.

Mathematisation of thought is not an absolute, one-time event. Through school and beyond it, children and even adults continue to Mathematise. On the other hand, our curricula may contain a lot of things that students learn without any accompanying processes, and hence without contributing to the real learning of Mathematics. Here are some examples, which, unless backed up by appropriate classroom processes, could end up being learnt by rote.

"To divide something by $m/n$, you multiply by $n/m$."

"The LCM of $a$ and $b$ is $a$ times $b$ divided by the HCF of $a$ and $b$."

"All triangles with the same base and height have the same area."

### The Problem of Abstraction

Young children learn about the world by handling objects. Their introduction to Mathematics therefore, is through the same route. Yet Mathematics, even in Class I, necessarily involves abstraction. Consider a statement from the lowest level of school Mathematics:

"Two and two make four."

This is a statement about two and four, which are abstract entities. The wheels of a bicycle, a pair of socks and two apples have something in common: a property which we can call 'two-ness'. "Two apples and another two apples taken together make four apples" is a statement about the physical world, which can actually be tested – unlike the above abstract statement.

Martin Hughes in his 1986 book "Children and Number" records many conversations with children, which show that children have a "surprisingly substantial knowledge about number" before they start school. However, this knowledge is not couched in the formal language of the Math classroom. A child may correctly count the number of bricks in a box, and predict that if there are eight bricks in it, two more bricks added will make ten bricks in all. Yet the same child has no clue when asked the abstract question: "How many is eight and two?"

Such experiments have subsequently been done by many others, with similar findings. The implication for the classroom is that activities with concrete objects should come before the transition to the formal, abstract language in which mathematical content is usually framed. Moreover, the transition from the informal to the formal should be specifically addressed in our classroom practices.

### The Construction of Mathematical Knowledge

Since the basic objects of Mathematics are abstract, we may wonder if they have an existence which is objective and independent of the human mind, or if they are constructs of the mind. This is an issue which philosophers have been debating since at least the time of the philosopher-Mathematician René Descartes (1596-1650). Are numbers, for instance, 'out there', or do they exist only in our minds? The various positions on this are summarised, for example, by Bertrand Russell in his very readable little book "Introduction to Mathematical Philosophy". I will sidestep this discussion for the moment to consider a slightly different aspect of the issue, one which is more directly relevant to the classroom.

It is generally agreed now, following the work of Piaget, Vygotsky and others, that children do not acquire knowledge passively. Rather, each learner actively constructs knowledge for herself. The process of knowledge construction involves interacting with the external world as well as with other people. Thus it does not matter whether mathematical entities have an objective existence or not: we all have to go through the process of constructing them for ourselves.

Although Piaget was not really concerned with school Mathematics, his work bears directly on the learning of Mathematics at the early stages. Constance Kamii has argued, for example, that young children do not discover arithmetic, they re-invent it. At first sight this may seem contrary to the claim that pre-school children have a substantial knowledge of Mathematics, or at least number. However, there is no real contradiction if we remember that children are exposed to many contexts for mathematical knowledge before they enter school.

### Is Mathematical Knowledge Unique?

Before we turn to the implications of these considerations for the classroom, we have to address the issue of what Mathematics to teach. Should our curricular choices be dictated by the structure of mathematical knowledge alone? If so, is this structure unique and universal? If this question is posed to a professional Mathematician, the likely answer will be an emphatic YES. However, we must remember that members of the Mathematics research community are a self-defined, closed social group. As argued earlier, the aim of school Mathematics education cannot be to secure for learners membership of this elite group.

Researchers in many countries, including India, have documented many different traditions in Mathematics. Some of these are found in tribal and other isolated communities, while others – labelled 'street Mathematics' – can be seen to co-exist with the formal Mathematics taught in schools. Masons, plumbers and other artisans are often found to use their own, trade-specific, forms of Mathematics.

At a deeper level, the kind of Mathematics that engages the community of mathematicians at any place and time is determined by the other social groups to which the mathematicians belong. Considerations of race, language, nationality and religion cannot be ruled out, even though mathematicians may like to believe they are above and beyond such influences. The picture of Mathematics as a subject that has evolved linearly, largely in the West, from Euclid through Newton to the present day, is one that is increasingly challenged these days.

### Implications for the Pedagogy of Mathematics

The above considerations naturally lead to some conclusions on how Mathematics should be taught. Since this volume carries an article on the Pedagogy of Mathematics, I will be brief.

1. Children should be provided contexts in which the learning of Mathematics can take place. These contexts have to be 'realistic' but not necessarily real.
2. In the early classes there should be plenty of opportunity for children to handle concrete objects.
3. Special attention should be paid to the transition to the formal, symbolic mode. Early teaching of algorithms is to be discouraged.
4. Learning basic skills is important, but thinking mathematically even more important.
5. Learners should not be given the impression that mathematical knowledge is a finished product.
6. Overall, the teacher should play the role of a facilitator with each learner engaged actively in the processes of learning Mathematics.

## Conclusion

It may appear that issues related to the nature of Mathematics belong to the realm of philosophy, and have little relevance to the teaching of Mathematics in elementary classes. However, as argued above, there is in fact a profound connection. It is important, therefore, for people involved in school Mathematics – teachers, school heads, teacher educators, etc. – to engage at some level with the kind of issues discussed here. How best this can be done remains an open question.

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## Logico - Math Brain Teasers

There are 10 smart people who are participating in a game. The 10 players are lined up in a straight line one behind the other so that the last person can see all the 9 others in front of him, the ninth person can see the 8 others in front and so on while the first one can not see any one. The sequence of the 10 players in the line is decided by the Game Master. There are adequate number of Black caps and White caps available. The Game Master will place one cap on the head of each of them. He will then ask each one starting from the last (who can see all others) the colour of the cap on his own head. The player in answer can say either Black or White and nothing else. The person/s who gives the correct answer would be given a prize. The answers can of course be heard by all. The players are allowed some time to discuss and plan their strategy before participating in the game (no tricks like tone change or loudness change etc in answering are permitted). What strategy can they adopt to make sure that the maximum number of them can get a prize? And how many can definitely hope to be get the prize with this strategy?

*(Hint: The answer may change if there are 11 prisoners instead of 10)*

Use this space for calculation 🙂

# 3 Pedagogy of Mathematics

**Hriday Kant Dewan**

Before we begin certain issues entwined with the word "Pedagogy" need to be pointed out and elaborated. This word is commonly used as a convenient hold all but because of this and in some contexts inspite of this it still can not be completely discussed by itself. To explore its implications some other key elements need to be specified.

**Can Pedagogy stand by itself?**

The first pre-requisite is the need to know the discipline being considered well. We need to know what it contains and its nature. This means to think about pedagogy of Mathematics, we need to first know what Mathematics is, what it includes, how it functions and then go to other questions. The first level answer to what it contains is: arithmetic and its generalization (i.e. algebra), geometry, statistics, analysis of number system and other such categories. It can be described as abstracting, organizing and generalizing of human experience related to quantity, shape and their transformation. Subsequently it becomes the basic language for building abstract and general ideas in all disciplines. Knowledge and constructs in Mathematics have gone far beyond the initial need of the human society for quantification, measurement and spatial visualization. As an abstract language, it links ideas and concepts in different domains. As it has grown, it has also sought to nurture commonalities across different domains of human experience.

The second pre-requisite is the need to articulate within Mathematics what we are going to transact. The manner in which tables can be memorized is different from the way in which students can be helped to understand how to solve word problems or understand the idea of a variable. Pedagogy is not an epistemic category and cannot help you choose what you want to transact even though it may relate to and even be governed by these choices sometimes and vice verse. This relationship, where it can be seen, is striking and crucial. For example, you cannot help children rote learn tables in a so-called constructivist manner nor have children explore open ended patterns in a classical behaviorist framework.

**How do we construct what is to be transacted?**

The multifaceted linkages of Mathematics and its abstract nature prompt the NCF to suggest mathematization of child's understanding as a key goal for Mathematics teaching.

This means there needs to be an attempt to help the child abstract logically formulated general arguments, go into organizing her experiences deeply and equip the child to transcend individual events and chance occurrences. In a sense move towards a more general and rational view point. The Mathematics syllabus for the elementary classes

> *"The Mathematics syllabus for the elementary classes has to revolve around understanding and using numbers and the system of numbers, understanding comparisons and quantifying them, understanding shapes and spatial relations, handling data etc."*

has to revolve around understanding and using numbers and the system of numbers, understanding comparisons and quantifying them, understanding shapes and spatial relations, handling data etc. In order to understand what aspects of these we need to transact and how we would transact it, the area of Mathematics needs to be understood in a wider perspective. We need to have a broad picture and the entire scope in our minds. This would then need to be narrowed to the classroom and specific choices. For clarity on these we would require a statement in our mind about the reason for these choices.

The capability of solving problems can be considered in many ways. One very obvious way is to get the child to almost copy solutions. The problems given follow the examples. There are no other types of questions leave alone finding ways to address them. A good problem solving task requires being able to locate and find a variety of clues within the problem, find the formulation to solve it and then fulfill the steps. The expectation is development of the ability to solve just one kind of problem in the same way. The way Math is described here is not a system of handling operations but rather the ability to construct and understand algorithms.

This logically leads us to the other question "why do we teach Mathematics?" If children fail to learn to abstract and are not able to follow the logic, do we really need to teach these aspects of the subject to them? Is there a cultural bias to Mathematics or can there even be a genetic bias that implies only some children can learn it? Is abstraction in Mathematics, Science, Philosophy, other subjects including History and Music a non-universal ability? Or is there something peculiar about abstraction in Mathematics? We can all enjoy the rhythm of a beat but to appreciate what is known as classical music or classical dance requires an experience or situations that do not appear to be universally available. Is the ability to generalize and play with numbers and space similar?

In this situation what then should constitute the universal elementary or secondary school curriculum?. What is it that we can expect and want all children to learn such that they do not end up thinking of themselves or being described as incapable? The question asked can be, is it not sufficient for them to know counting numbers and operations on them and a bit of decimal fractions and commonly used fractional numbers? Do we need to insist on making Math abstract and apparently so complex that many cannot follow it? Is the fact that children do not understand a certain kind of Math and are terrorized by it, a result of the way it is taught or is it due to the kind of content covered? Is terror the nature of the subject itself? So there is a complex interplay between the questions – What is Mathematics? And what area of Mathematics is needed and can be transacted in elementary classes? In this we need to also consider whether all of it need to be universally learnt at this stage. We have to spell out (a) why is it needed for that age group, that background and in that historical context for children and (b) can it be learnt by students of that background at that stage given the circumstances of schools and teachers. The choices made need to be able to go through these filters.

It is obvious that it is neither easy to construct these fillers with comprehensive information and arguments and nor is it easy to reach a consensus on implementing and discussing them given the hiatus these abilities seem to provide in the social and economic status accessibility.

As is evident from above content, 'what is pedagogy?' is difficult to address on its own. Its scope and concerns are not articulated very precisely and there is not enough consensus on how it may be defined. There is, however, a common sense understanding that guides the way it is used generally.

### What is Pedagogy?

Pedagogy is broadly used to imply the way a subject will be transacted. Described thus there are many obvious components of the word pedagogy. They include classroom transaction and processes, nature and type of teaching-learning materials, assessment system, teacher student relationship, the nature of student engagement, the classroom arrangement etc. This is of course influenced by (and for some people includes) the chosen set of content, information, skills and concepts to be transacted and acquired. Pedagogy needs to worry about the inclusion of all the learners in the learning engagement. This implies the need for an awareness and sensitivity towards diversity and a concern about choices and context in the syllabus. If you carefully consider the manifestations of pedagogy in the classroom, then we know that it is also concerned with the way teachers are prepared, how they are dealt with administratively, the school building, the classroom, the social, economic and political undercurrents existing due to the diversity in the classroom and among teachers. There may also be other systemic and contextual aspects that may influence how transaction takes place. This then becomes a really extended set.

We would here, limit ourselves to some of the aspects. In these a few of the clearly discernible aspects mentioned above will be reiterated as issues that critically influence pedagogic consideration. These include:

(a) Aims of teaching Math
(b) Nature of Mathematics and its key principles
(c) The teacher and her perspective

became definitions and operations. The itemized view of Mathematical ideas implied the narrowing of space for the child to formulate and articulate her own ideas and logic.

Since 'doing' was reduced to a largely mechanical repetition and therefore the 'doing' that stems from exploration, building arguments, developing articulation and definitions to get feedback on them was conspicuously absent. This is not to say that children need to bring out and re-discover the entire human knowledge or they have to discover things by themselves. The knowledge that human society has gathered over time has to be shared, but in a manner that they preserve their freshness of thinking, curiosity and keenness to learn. It cannot mean imposing the hegemony of existing knowledge.

**Two views on how to teach Mathematics**

In analyzing how Mathematics is taught there are two contrasting views under which programs can be classified. We see classrooms constructed as a combination of these in some proportion. One view is that if you have students practice a lot of sums using algorithms and shortcuts, they eventually start understanding how the algorithm works and may get a sense of why it works. In any case they learn the steps clearly and are able to use it in any context. The nature of questions would, however, be varied.

The other view is that learning Mathematics is about developing an understanding of how the subject is constructed, its basic elements and working out the logical steps that lead to the algorithm and short-cuts in some areas. The child here is expected to be able to develop multiple strategies for problems and also use the algorithms if she finds it appropriate. The argument would not be that this is the best algorithm and has to be learnt by everyone but choose if appropriate. Students can also know, discover and discuss the nature of shortcuts and apply them if they so desire.

There are many examples given for the need for having children learn more than just algorithms. The simplest is addition of two digit numbers and the evidence that very often children introduced to these mechanically, end up viewing them as adding two independent one digit numbers placed in different columns.

There is also a lack of appreciation of the fact that when we multiply any number by a 2 or 3 digit number, the product from the 'tens' digit is not placed directly under the product from the unit place number. It is shifted by putting a cross under the units place. For example:

```
   17
x  23
 ----
   51
  34x
 ----
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we are not always asked to seek a reason for the shift. There are similar examples from division as well.

Some people argue that the concepts of carry over or borrowing require an understanding of place value and therefore, unless we have children develop reasonable capability in place value they will not be able to do additions and subtractions that require such steps. The essential point here is that the focus is on learning the structure of the subject and the concepts. Once that happens the applications would gradually be learnt by the students. So in these while the eventual goals may be agreed upon, the approach is strikingly different.

**Concrete to abstract: What does it mean?**

Another aspect of pedagogy is related to the role and nature of materials in the classroom. We generally believe that abstract concepts are acquired through a process of creating, experiencing and analyzing concrete situations. There has been an increasing stress on putting in more and more concrete materials in the Mathematics classrooms. The idea of so-called Math lab has been supported and advocated widely. The feeling is that children understand concepts through the experiences in Mathematics laboratory. This needs to be considered carefully.

It is evident that the idea of using concrete materials and contexts for helping children learn is important. These serve as a temporary model to represent abstract concepts. For example 5 stones are a concrete model for 5 and so 5 chairs. A triangle cut out from card board is a model for triangle as it can portray some key properties of the triangle. It must be recognized that these artifacts do not fully represent the concepts of 5 or the triangle. They are only scaffolds for us to communicate what these terms mean in the initial stages. Gradually learners have to move away from these concrete scaffolds and be able to deal with mathematical entities as abstract ideas that do not lend themselves to concrete representations.

A quadrilateral is closed figure bounded by 4 straight lines. A line is a one dimension infinite string that has no thickness. The point is that an actual line and hence a quadrilateral cannot be represented by even a drawing on the board leave alone by a concrete representation. So while it is important to begin with concrete experiences, gradually the child must articulate using her own language and move on. Mathematics going through the stage of using pictures and then tally marks etc. has to transit to symbols. This is an essential component of learning to do Math. The learning of Mathematics has to culminate in being able to deal with mathematical ideas on their own without any scaffolds. Therefore, when we advocate the Math laboratory for senior schools there is both a pedagogic as well as an epistemic question about whether this is the appropriate direction to proceed in.

The idea of laboratory in Science is to have the students explore some phenomena. She would make observations related to it and then based on the observations attempt to deduce some kind of causal connections. Utilizing many such experiments and data from earlier experiences, the student can attempt generalization and building hypothesis that can be checked by further experimentation. The epistemological touch stone for ideas in Science can be arguably experimental observations and validations. This unfortunately is not true for Mathematics and therefore using the Math lab to have children deduce or prove mathematical statements by measurements or through models, is an epistemic and also a pedagogic error. The attempt at this stage has to be to enable the child to deal with abstract ideas.

Unlike the rich experience of language that the child comes to school with, ideas of Mathematics are not so richly experience based. All children are able to deal with numbers and arithmetic that they need in daily life. They are also able to organize the space around them and carry out spatial transformation to the extent they need. This knowledge is profound and complex. It shows the innate capability of the child to acquire these ideas. All children in any society are able to deal with these ideas. The problem comes when we attempt to transact Mathematics and want them to de-contextualize and abstract the number, shapes transformation, operations and why all these work. The discipline of Math is to be able to talk about abstractions and how relations between abstract quantities can be understood and developed. In the primary classes Social Science and Science are also largely experience based and there is recognition that abstract concepts should not be imposed at this stage. Even in the upper primary classes it is possible to make Science replete with concrete experiences and use the available experiences of the child as well as the experiences provided in the classroom to help her construct a framework of concepts. Mathematics does not allow this easily.

A lot of Mathematics pedagogy depends upon how the teacher engages with children. The classroom atmosphere has to be such that children can participate, articulate their ideas, make mistakes and talk about them without fear. Such an atmosphere will determine the relation children have with Mathematics. There is no one method or one technique that we can recommend for teachers to follow. She has to follow the classroom and create processes that facilitate engagement and dialogue that move forward gradually but can also return to an earlier point and develop again in a different way. The key aspect of Math classroom has to be the recognition that children will develop mathematical ideas and concepts through assimilation with their own previous ideas and experiences and modify them in the process of interactions. Each of us develop our own way of solving problems. It may require exposure to a lot of algorithm and methods but with an openness to create and examine more. They should be able to absorb available ideas and accommodate them in their conceptual framework. The models that anyone of us use or the artifacts a student constructs can help her understand the problem and develop a strategy but would not help everyone. They will be different for each of us.

You cannot help a person learn Mathematics by giving her short-cuts or imposing on her your way of solving problem. Your way may appear very simple, neat and elegant to you but that may not be so for her. We categorize and use ideas in our own ways and use steps that we can think of. It is a doubly difficult task to understand the problem and then also discover the underlying logic of the process you have used to construct the solution.

(d) How children learn Mathematics

(e) The attitude to the subject in society

This will help us derive specific expectations and purposes for different class and age groups. This is what constitutes the syllabus. The first two components have to be informed by the so-called subject, its nature, purpose for human society and for the students for whom the transaction program is being developed. One has to keep in mind the person who is going to transact the learning so as to understand what the aims, expectations and learner backgrounds demand from her. The third is: is there any specific understanding that we need about how this subject is learnt? This will help us construct classrooms that aid learning. The fourth is the prevalent attitude in society about Math- be it teaches, students or parents. All these contribute critically to the pedagogy of the subject.

**Teaching Mathematics: Some Approaches**

Discussing teaching-learning of any subject requires a basic understanding of how children learn. That should form the basis of our program particularly if each different component of the subject has a character that gives a specific tinge to its learning. The experience of these components for a particular child and the nature of the expectations from her can also be very different in comparison to the other children. For many years, Mathematics learning, like all other learning was considered to be linear and through repeated practice. Whatever was to be learnt had to be broken up into small components and given to children to practice bit by bit. The MLL (Minimum Learning Level) was a crucial example of this approach. In this the pedagogy was claimed to be competence directed.

There is also an expectation from the text book and other materials that for each small element termed as 'competency', there would be one page or one section entirely devoted to it. It was expected that once the child has gone through this she would automatically and surely have developed that part of the competency and needs now to go on to learn the next part. The MLL document itself used the word competency in many different ways. It was used loosely to describe information recall, procedure following, applying formula and in some cases concepts and problem solving as well. As a result of this, it is not clear how the word competency in the MLL document should be unpacked. The on-ground discourse on competency has also not moved forward. In this case Mathematics given its so-called hierarchical nature, its learning seems to be still analyzed in the same framework and conceptualized as bit by bit and through practice of procedures and remembering facts.

Another element that pedagogy is crucially dependent on is the presentation of the teaching learning material (workbook and textbook) and what it expects the child to do and how it suggests the class be organized and assessment made. The material needs to be clear on whom it is addressed to and therefore what it should contain. If the material is for the child then it has to have appropriate spaces, font size, suitable illustrations designed for children and appropriate language.

The textbooks and Mathematics classrooms before the advent of MLL and after the advent of MLL have remained essentially similar due to the fact that students are still being asked to practice algorithms and learn to numerate quickly. Articulation by the child, inclusion of the language of the child and allowing the child to explore and create new approaches to engage with mathematic situations are still not expected and not even accepted in materials. They follow the "consider the given solved example and do some more", approach to Mathematics learning. We may also point out that the mention of a specific competency to be acquired meant the earlier mixed exercises that at least exerted the mind of the child in someway, also got limited to practicising just one option. It was at this time recognition for design, need for illustrations and color in the books emerged so at least the books were different. The principles informing the illustrations, design and other aspects however did not include the need to create space for the child to actively engage her mind.

In the absence of clear articulation, word competency was focused on explanation and telling short-cuts and facts. The key words ‘learning by doing’ and ‘competency’, in the context of Mathematics were inadequately explored and insufficiently addressed. Addition was a mere operation and acquiring it was the capability of adding single digit,2 and more digit numbers with no carry over and then with carry over as column additions. In the quest to make Math a doing subject, competency based fractional numbers

Mathematics will be learnt when the student will develop her own strategy, use the concepts and the algorithm in the way she wants. This clearly implies that children must have the opportunity to do lots of problems and solve them in many different ways.

We must expose the learner to these different varieties and develop not only the capacity to construct their own answer but also look and attempt to analyze and comprehend somebody else's answers. They need to be unafraid of making mistakes and confident of articulating their understanding. The implications in the classrooms are that children will work on their own, in groups make presentations on the solutions they have found and construct new problems as well as new generalizations. The classroom has to be such that the child is involved and engaged at each moment.

There has been a lot of talk about constructivism and teaching-learning processes. There have been arguments suggesting that teaching-learning process should be constructivist. This is sometimes interpreted to mean that children should be allowed to follow their own paths and decide what they want to do. It must be emphasized here that like the use of materials in Mathematics the space for the child to articulate her own understanding and building upon it needs to be interpreted in the context of an organized sharing of knowledge with the child and the nature of the discipline. Once the basis of deciding the Mathematics curriculum is arrived at then the classroom and the school has to help the child develop capability in the areas considered important. The teacher cannot ask children what should be done. At best she can construct options that are in conformity with the goals and objectives set out in the program for them to choose from. The notion of constructivism itself and its relationship to Mathematics teaching-learning needs to be explored and analyzed more carefully.

**Assessment in Mathematics**

An important part of any pedagogical statement is assessment. While there are general principles. The general key principles of assessment such as (a) the purpose of assessment (b) the participation of student in the assessment process (c) the mechanism of assessment (d) the way feedback would be provided to the child.

The manner in which assessment is done at present instills a feeling of fear and purposelessness for most children. Except for those few who are confident of doing well, the others usually want to get over it quickly and scrape through somehow. No one sees a relation between the examination, performance in examination and learning. In Mathematics examinations, particularly, the nature of the tasks given and the manner in which they are assessed lead to children being afraid of not just the examination but even the process of engaging with Mathematics. The entire assessment process is aimed to exhibit what the child does not know rather than to discover what she knows. Concepts of formative, summative evaluation and other such terms do not spell out the purpose, importance and implications of good assessment processes. In recent years we have talked about continuous and comprehensive evaluation, no examination assessment and have argued for the teacher providing extra support to children who lag behind outside the class.

The revocation of the examination, the non-detention policy and the idea of outside the classroom support may appear to be conceptually nice but it is not operationally possible.

Education is a dialogue between school, teachers and the children. If this dialogue is not facilitated with trust, and openness is disallowed it would result in serious distortions in the classroom processes. In Mathematics specifically it is important for the child and the teacher to know what she knows and also have a sense of areas that she is struggling with. The progress of the child needs to be based on what she was able to do earlier. We need to grade the performance of the child in that period rather than grade her against other children. Assessment and expectation is an important part of the requirement to make an effort. The fear of examination cannot take away purpose that assessment serves.

The way society looks at Math is a combination of awe, fear and a passport to success. There are strong beliefs about those who are able to learn Mathematics being more intelligent and have a greater chance and capability to succeed in life.

Mathematics is looked upon as a filter that would separate those who would be moving towards higher intellectual pursuits and those who would take up less intellectual roles in society. The anxiety of occupying the intellectual and technical roles leads parents and teachers to put pressure on students to learn. There is sub-conscious beginning of sorting by declaring many students incapable of learning and therefore helping them by some short-cuts to pass the examinations.

The fear of assessment and subsequent doors that are assumed to open on leaning Mathematics lead to a tense atmosphere in the classroom. The general feeling in the society that it is difficult and has to be such that it can only be done by a few, prevents any attempt to allow children to slowly develop their ability.

It is difficult to conclude this discussion but it is clear that in considering pedagogical aspects of Mathematics it is not merely methods, classroom arrangements and presentations styles that we are talking about. We have to comprehensively look at education and the entire education process, place that in the context of Mathematics, children, parents and teachers along with their aspirations, to move forward on the understanding of pedagogy.

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# 4 Computational & Mathematical Concepts in Arts and Science

**Krishnan Balasubramanian**

RD Burman's *'mere nainaa saavan bhaadon',* sung by Kishore Kumar and a 1982 song *'tere mere beech mein kaisa hai ye bandhan'* from the movie *Ek Dhuje keliyey* were both hits in their times and perhaps of all times. Then again, we have the song *'jaane kahaan gaye vo din',* in 1970 Raj Kapoor's *mera naam joker.* Besides being great hits do they have anything else in common? Well they are all similar in tune or melody, as they have a similar form in some mathematical ways and they all derive from the same Hindustani *Raga 'Shivaranjani'.* But do we recognize a pattern or a mathematical form of melody and rhythm? One of the central themes of this article is the mathematical and Computational Concepts in Arts and Science. Quantifying similarity as in patterns of nature is part of non-numerical branches of computational science such as artificial intelligence. The field is not only of aesthetic interest such as machine or cognitive perception of music but also of pragmatic value such as in the field of computer aided drug discovery and molecular similarity.

Who said aesthetics is only for artists and not scientists or engineers? Is it not amazing that there could be some latent mathematical form in aesthetics? Is it possible that science and art can really come together? This article essentially deals with - the role of mathematical concepts originating from symmetry, asymmetry and duality with examples of ancient temples of Chola and Pallava of Tamil Nadu to the Orissa temples.

Consider Fig 1. This is from *Kailasa Nadha* temple located in Kancheepuram, a temple town about a 2 hour hop from Chennai.

Figure 1 - *Kailasa Nadha* temple in Kancheepuram

The temple stands out as a monumental piece of sculptural aesthetics of the Pallava Empire. A recurring mathematical pattern that emerges from many of such temple sculptures is the union of concepts of symmetry, asymmetry and duality. For example there is overall symmetry of the flanking dancers and yet the niches have some asymmetry and duality built into them as if there needs to be a global balance beyond the usual symmetry. By duality I mean on one side you have an image of a young person and the other side that of an old. Sometimes you find the image of a demon and the other side an angel.

In a more general sense I explored the role of symmetry, asymmetry and duality in India's versatile and rich culture, religion and philosophy. While we understand symmetry in Math as invariance under point group operations such as rotation, reflection, inversion, and improper rotation, concept of duality will be developed as a juxtaposition of contrasting images, as a flanking or union of images of “demon” and “devil” or “fire” and “water” or “shiva” and “shakthi” so as to bring equilibrium or global symmetry.

> *How does one quantify such a qualitative or aesthetic feature? One way is to develop a set of rules under which two species can be related or even 2 ragas may become related. Once the rules are in place one can define Euclidian types of distances between species or molecules and then use statistical methods such as clustering and principal component analysis.*

Moving on to music, mathematical and computational ideas were introduced in music theory using mathematical and computer generation of *ragas*. A *raga* as we all know is the backbone of melody and it is the most fundamental part of Indian music. So how many *ragas* are really there? And are there new, yet to be discovered *ragas*? The answers to these intriguing questions are in combinatorics of *raga* formation and enumeration. This can be done systematically using polynomial generating functions called *raga* inventory by considering various kinds of *arohan* (ascent) and *avarohan* (decent).

The coefficients of various terms in the *raga* inventory polynomial enumerate the various types and numbers of *ragas*. Then a computer code was developed to construct various *ragas*. Finally it was shown that there are 262,144 non-*vakra* or non-kinky *ragas*. This means the ascent and descent have uniformly increasing and decreasing frequencies. We have created a list of such non-kinky *ragas*. Good news is ragas like *Shivaranjani, Bhoopali, Malkauns, Charukesi,* and so on are covered by the enumeration. But still *ragas* such as *Darbari Kanada and Sri* would not be included as they are *vakra ragas*.

Moving on to quantifying similarity in the context of molecular architecture and drug design - how does one quantify such a qualitative or aesthetic feature? One way is to develop a set of rules under which two species can be related or even 2 ragas may become related. Once the rules are in place one can define Euclidian types of distances between species or molecules and then use statistical methods such as clustering and principal component analysis.

This would show an interesting amalgamation of concepts from computer science, Mathematics, quantum mechanics, and biology that lead to some very unique perspectives. For example, how a complex proteome of a living organism such as rat's liver cell can be characterized using complex algebra and how we can develop algorithms for characterizing such complex 2d-gel patterns of complex array of thousands of proteins in a cell.

Let us now consider Einstein's theory of relativity and the nature of chemical bond in very heavy elements and newly discovered super heavy elements. Significant portion of my research deals with the applications of relativity to the chemical bonding of molecules containing very heavy atoms. Well, we have all heard that all that glitters is not gold, did you ever wonder that gold glitters because of relativity! Yes, you heard it right! But for Prof Einstein the beauty of gold would have been buried into a blackish orsilverfish look. The yellow glitter of gold is attributed to relativity due to increased speed of electron in the gold atom. As you admire all that glittering golden jewelry, pay a tribute to Einstein for his landmark papers on relativity. As we keep discovering new elements, many of them yet to be named such as 114 and 115 discovered at Lawerence Livermore National Lab, relativity becomes more and more important as the speeds of electrons of these elements increase resulting in a parabolic relativistic effect as the atomic number increases.

Look at the breadth of topics we have discussed - aesthetics, sculptures, music, duality, similarity, relativity, quantum mechanics, bioinformatics, and of course quantum chemistry. How in the world did I manage to write about the theory of Hindustani and Carnatic music to Pallava architecture to relativity to drug design-all in one article? As I deeply ponder over this question, I transcend back in time in India, to Pilani and vividly recollect thebeautiful Saraswathy mandir and the BITS clock tower. That is the beauty and versatility of the education that we have received as students of BITS. The greatest boon that one can get to face the modern world filled with a plethora of interdisciplinary topics is multidisciplinary and broad education that Pilani offered to us and continues to offer.

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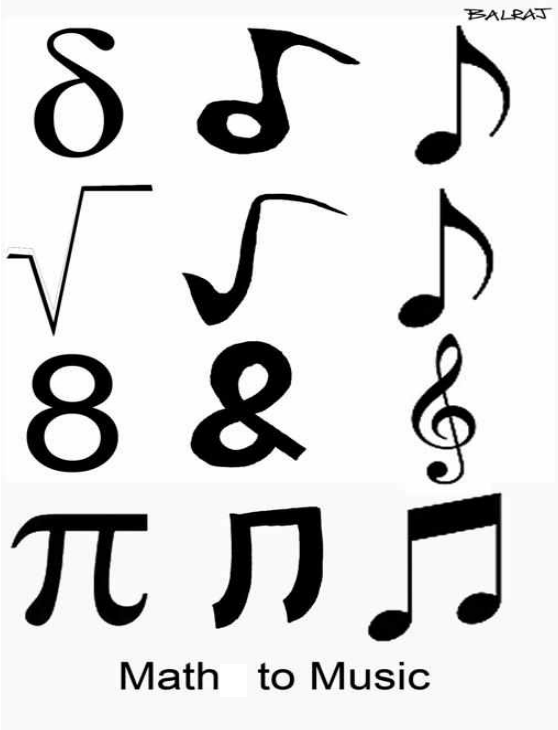

# 5 Culture in the Learning of Mathematics

**K. Subramaniam**

At a recent workshop in Ahmedabad, we asked primary school teachers to talk about what their students do outside school, and whether it involves any Mathematics. The teachers spoke a lot. Their pupils, who came from poor urban homes, helped their parents sell vegetables. They made and sold kites, packets of bindi, agarbathis and many other things. They knew the price of vegetables for different units, knew how much profit they would make from selling a 'kori' (unit of 20) of kites. Kites had to be assembled from paper sold in packets and sticks sold in bundles – all in different units. Problems arose naturally while making decisions about how much raw material to buy, how much to make and sell, how much time to spend, and so on. Children, together with older siblings or adults, were finding their own ways of getting around these problems. And all the time, they were dealing with numbers and Mathematics.

*"Almost no school curriculum gives any place to such 'everyday' Mathematics. At best there may be an attempt to add some contextual details to enhance children's interest. Thus the Mathematics that children learn to do inside and outside school remain separate and disconnected"*

It was not the children telling us these things, it was the teachers. We asked them how they discovered that the children knew so much. They replied that when the children absent themselves from school, they visit their homes to find the reason. They talk to the parents and often find that the child was helping them – perhaps hawking vegetables while the mother went on an errand. We were happy that the teachers took pains to ensure attendance, but we also felt a little uneasy with this reply. When we opened the worksheets prepared for the children – this chain of schools used their own worksheets rather than a regular textbook – we did not find anything of what we had just heard about the children's lives. It struck us that teachers found out about the children's activities outside of the school, and not in the Mathematics classroom.

After some discussion with the teachers, we realized that they held strong beliefs about what counts as 'proper' Mathematics. A problem used in a Dutch study, 'If a polar bear weights 350 kg, about how many children weigh the same as a polar bear', was for them not a good problem, because it did not have all the data needed to solve it. The problems that the children were solving outside school often had incomplete data, did not have a precise single answer, and the children used informal methods of solving them. So the teachers did not think that the children were really doing Mathematics. There seemed to be an invisible wall separating the Mathematics in school and the thinking and figuring that the children did in the context of economically productive activities.

This story is not an unusual one. In many poor urban households, children participate in economic activities. In a different social or geographical context, if one looks carefully, one will discover that here too children have opportunities to engage with Mathematics outside school. Almost no school curriculum gives any place to such 'everyday' Mathematics. At best there may be an attempt to add some contextual details to enhance children's interest. Thus the Mathematics that children learn to do inside and outside school remain separate and disconnected. Of course, the larger issue here is of the relation between the school curriculum and life outside school. Since Mathematics is an abstract branch of knowledge, one may think that there is little to be said about its connection with culture and everyday life. Yet, many researchers have studied the relation between 'everyday' and school Mathematics leading to important insights.

Advocates of constructivism, following Piaget, stress the fact that children don't enter schools with empty minds waiting to be filled – they have already acquired complex knowledge in the domains that overlap with school Mathematics and science. Psychologists studying cognitive development have constructed a detailed picture of the spontaneous conceptions that children acquire. The first wave of constructivism was however criticised for focusing largely on individual learning. The criticism came from a broad range of perspectives that were more sensitive to the influences of culture and society. The implications of these critiques are still being worked out by researchers and thinkers in the Mathematics education community. Here we will look at some of the ideas and possibilities that have emerged from this debate.

The pioneering studies of street Mathematics by Terezinha Nunes and her colleagues, the anthropological studies by Geoffrey Saxe of the Mathematics of the Papua New Guinea communities, the studies by Farida Khan in the Indian context, and many other studies have revealed how Mathematics arises spontaneously in the context of everyday activity. These studies have also shown how 'everyday' Mathematics differs from school Mathematics. In everyday contexts, calculation is 'oral', and mostly uses additive build-up strategies. When an adult from the Mushari tribe in Bihar was asked to give the cost of ten melons if each melon costed Rs 35, he did not 'add a zero to the right' to straight away get 350. Instead, he first calculated the cost of 3 melons as Rs 105. Nine melons were Rs 315 and so ten melons were Rs 350. Exactly the same procedure was used to solve the same problem by a Brazilian child vendor in Nunes' study. The 'add zero to the right' strategy is a part of 'written' Mathematics, and is uncommon in everyday Mathematics. Proportion problems are usually solved in the everyday world through a build-up strategy rather than by using a 'unitary method' or the 'rule of three'. For example, consider the problem 'if 18 kg of catch yield 3 kg of shrimp after shelling, how much catch do you need for 2 kg of shelled shrimp?' A fisherman in Nunes' study calculated it as follows: we get 1½ kg of shelled shrimp from 9 kg of catch, so ½ kg from 3 kg of catch. Nine plus three is twelve. So 12 kg of catch would give you 2 kg of shelled shrimp.

Since these procedures were oral, sometimes respondents forgot to complete a step of the calculation, but the errors were usually small and the answers reasonable. Nearly always, the calculation model was accurate. In contrast, school children often use the wrong operation for a problem and produce unreasonable answers. Culture and cognition seem to work together in everyday Mathematics to create a robust sense of appropriate modelling. When children are presented with a problem that they can understand, and are encouraged to find their own way of solving them, we see that their spontaneous solution procedures are often like those of everyday Mathematics. These findings have important implications for teaching and learning Mathematics. One can, for example, re-conceptualize learning trajectories so that the problems, concepts and procedures of everyday Mathematics provide the springboard for more powerful mathematical concepts. The rich contexts that are familiar to children provide valuable scaffolding while solving a problem, verifying that its solution is reasonable and looking at a problem from different angles.

If we see cultural knowledge as merely a vehicle to deliver formal Mathematics that is otherwise 'difficult-to-swallow', then we may be adopting a view which is too narrow. We cannot simply mine what is present in the culture as a resource to push a particular curricular agenda. Putting cultural knowledge alongside formal knowledge leads us, as educators, to reflect more deeply about their relation. We need to not only take from the culture sources of mathematical thinking, but also give back to the culture what it values highly. In the long run, if a form of knowledge is to survive and flourish, it must have deep roots in the culture. We don't understand well the meeting points between disciplinary knowledge and knowledge that is dispersed as part of culture. Is such culturally dispersed knowledge incommensurable with the academic knowledge of the universities, as some thinkers in education have argued (Dowling, 1993)? Can the familiar dichotomies of folk vs formal knowledge, or traditional vs. modern knowledge capture the relationship between the two kinds of knowledge? In some domains of knowledge, cultural dispersion and transmission through formal institutions have both had a strong presence over long periods. A good example is classical Indian music. Another example is traditional medical knowledge, which is now reproduced through modern educational institutions. Both music and medicine as formal systems preserve a connection to the diversity of cultural forms – to popular music or to the many local and specific healing traditions. Much of the knowledge that we seek to impart in school has no comparable cultural presence or diversity of forms of expression.

Mathematics may have deep roots in our culture that we are still to become aware of. Among some members of the Mushari community, there is an impressive knowledge of mathematical puzzles or riddles and their solutions. These puzzles are called 'kuttaka', which is the name of a mathematical technique, whose oldest description is found in the Aryabhatiyam of the 5th Century CE. The 'kuttaka' is an important and powerful technique, which led to important developments in Indian Mathematics. Brahmagupta, in the 6th Century CE referred to algebraic techniques in general as *'kuttaka ganitha'*. The Mushari puzzles, which involve the solution of equations, may preserve a connection to this deeper tradition of Mathematics. It is intriguing that such knowledge exists among a community which is very low in the social hierarchy. We need a better understanding of the cultural transmission of mathematical knowledge between communities at different social strata. Culture can support the reproduction and circulation of mathematical knowledge not just through work, but also, as the puzzles indicate, through play. The revival of traditional art forms like music and their reshaping through digital technologies point to the possibilities of connecting art and Mathematics that are still to be explored.

Viewing the relation between 'everyday' and formal Mathematics through a different lens shows that political considerations are also relevant. As several writers have argued, with the growing dependence on mathematical science of modern technological societies, there is an increasing withdrawal of Mathematics to more hidden layers distant from everyday life. Not only is the complex Mathematics that underlies technological devices inaccessible to a lay person, but even everyday commerce may become emptied of mathematical thinking.With regard to everyday finance, which is relevant to nearly everybody, technology seeks to make Mathematics redundant. Calculators, EMI tables for loans, and other aids function as black-boxes that replace reasoning and calculation. This results in deskilling, and also takes attention and interest away from the underlying Mathematics. In a small study that we did, we found profound lack of awareness among educated users about how the credit card system operates and such critical issues as the effective rate of interest. Thus the increasing mathematization of society is accompanied by the growing de-mathematization of its citizens. Since Mathematics is entrenched as an essential part of the school curriculum, it begins to serve a different social function – that of weeding out large numbers from obtaining any access to the Mathematics and science that decisively shape modern society.

The emergence of small-scale production activities as a part of the informal sector, offers to poorer households a means of subsistence and resistance against the harsh impact of changes in the organized economy. One cannot resist drawing a parallel in the light of the discussion on de-mathematization. Against the increasing trend of de-mathematization, the emergence of Mathematics on the street or in the workplace is a counter trend that resists the complete exclusion of the under-privileged from Mathematics. Of course such emergence by itself has no power to provide access to significant Mathematics. But the institution of education can amplify this possibility; bringing everyday Mathematics into the curriculum may prepare the way for bringing more Mathematics to wider sections of society.

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# section B

## Some Perspectives

# 6 Number: The Role of Pattern and Play in its Teaching

**Shailesh Shirali**

### Pre-history

The concept of number is crucial to Mathematics, yet its origin may forever be hidden from us, for it goes far back in time. Human beings must have started long back to use the tally system for keeping records – livestock, trade, etc – but we may never know just when. A remarkable discovery made in 1960 in Belgian Congo of the Ishango bone, dated to 20,000 years BP, suggests that the seeds of Mathematical thinking may go still further back than thought; for, carved on the bone are tally marks grouped in a deliberate manner, seemingly indicative of a Mathematical pattern (there is even a hint of a doubling sequence: 2, 4, 8). However, until further evidence is uncovered, the matter must remain as speculation. See the Wikipedia reference for more information.

Tally counting as a practice may well be as many as 50,000 years old; even today we use it to count, in various contexts, e.g. in a class election.

### Base ten number system

The notation we use today – the base ten or decimal system – has its origins in ancient practices. Long back the Babylonians used a system based on powers of 60, and traces of that practice remain to this day – we still have 60 seconds in a minute, 60 minutes in an hour, 60 minutes in a degree (for angular measure). Later the Egyptians developed a system based on powers of 10, in which each power of ten from ten till a million was represented by its own symbol. But this system differs from ours in a crucial way – it lacks a symbol for zero.

A system of arithmetic without a symbol for zero suffers from two difficulties. The first is that there is confusion between numbers like 23, which represents 2 tens and 3 units, and 203, which represents 2 hundreds and 3 units. Without the zero symbol some way has to be found to indicate that the 2 means “2 hundreds” and not “2 tens”. This can be done, but it is quite cumbersome. But a greater difficulty is that computations become significantly harder, and it becomes that much more difficult to progress in arithmetic.

The Greeks did not have a symbol for zero, and it is not surprising that they did not develop arithmetic and algebra the way they developed geometry, which they took to great heights. It was in India that the symbol for zero came into being (probably as early as the 5th century), along with the rules for working with it. Not coincidentally, arithmetic and algebra grew in a very impressive manner in India, in the hands of Aryabhata, Brahmagupta, Mahavira, Bhaskaracharya II, and many others.

On the other hand the ancient Indians did not progress anywhere as far in their study of geometry. But it is striking that one area where the methods of algebra and analysis enter into geometry in a natural way, namely trigonometry, did originate in India (in the work of Aryabhata, 5th century AD).

> *“Concepts are caught, not taught”. It is only by actual contact with collections of objects that concepts form in one's brain.*

### Abstraction and the number concept

Embedded in our brains is an extraordinary ability: the ability to form concepts; the ability to abstract common features and shared qualities from collections of objects or phenomena. It is this ability that lies behind the creation of language, and it is this that enables us to “invent” numbers. To understand what this means, think of a number, say 3. Is 3 a thing? Can it be located somewhere? No, it cannot; but our brains have the ability to see the quality of “threeness” in collections of objects: three fingers, three birds, three kittens, three puppies, three people – the feature they share is the quality of threeness. This ability is intrinsic to the very structure of our brains. Were it not there, we would never be able to learn the concept of number (or any other such concept, because any concept is essentially an abstraction).

Even in something as simple as tally counting – creating a 1-1 correspondence between a set of objects and a set of tally marks – our brains show an innate ability for abstraction: by willfully disregarding the particularities of the various objects and instead considering them as faceless entities.

Realization of this insight has pedagogical consequence; for, as has been wisely said, “Concepts are caught, not taught”. It is only by actual contact with collections of objects that concepts form in one's brain. How exactly this happens is still not well understood, but I recall a comment which goes back to Socrates *( the teacher's role is akin to that of a midwife who assists in delivery).*

The invention of algebra represents one more step up the ladder of abstraction. To illustrate what this means, let us examine these number facts: 1+3 = 4, 3+5 = 8, 5+7 = 12, 7+9 = 16, 9+11 = 20. We see a clear pattern: the sum of two consecutive odd numbers is always a multiple of 4. This statement cannot be verified by listing all the possibilities, for there are too many of them – indeed, infinitely many. But we can use algebraic methods! We only have to translate the observation into the algebraic statement (2n-1) + (2n+1) = 4n; this instantly proves the statement. Such is the power of algebra and also the power of abstraction – and this ability too is intrinsic to our brains.

**Number patterns**

Another feature intrinsic to the brain is the desire and capacity for play. Most mammals seem to have it, as we see in the play patterns of their young ones – and what a pleasing sight it can be, to watch kittens or puppies or baby monkeys at play! But human beings have a further ability: that of bringing patterns into their play. When our love of play combines with the number concept and with our love of patterns, Mathematics is born. For Mathematics is essentially the science of pattern.

It is crucial to understand the element of play in Mathematics; for one is told, repeatedly, of the utility ofMathematics, how it plays a central role in so many areas of life, and how it is so important to one's career. But the element of play gets passed over in this viewpoint; the subject becomes something one must know, compulsorily, and the stage is set for a long term fearful relationship with the subject.

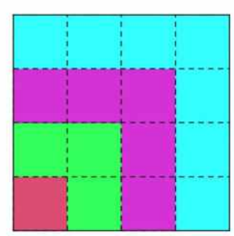
From the earliest times – in Babylon, Greece, China, India – there has been a playful fascination with number patterns and geometrical shapes one can associate with numbers. From this are born number families – prime numbers, triangular numbers, square numbers, and so on.

Let us illustrate what the term “pattern” means in this context. We subdivide the counting numbers 1, 2, 3, 4, 5, 6, 7, 8,... into two families, the odd numbers (1, 3, 5, 7, 9, 11, ...), and the even numbers (2, 4, 6, 8, 10, 12, ...). If we keep a running total of the odd numbers here is what we get: 1, 1+3 = 4, 1+3+5 = 9, 1+3+5+7 = 16, 1+3+5+7+9 = 25. Well! We have obtained the list of perfect squares!

There is a wonderful way we can show the connection between sums of consecutive odd numbers and the square numbers; it is pleasing to behold and incisive in its power at the same time. All we have to do is to examine the picture below: this property is closely related to one about the triangular numbers: the sequence 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ... formed by making a running total of the counting numbers: 1, 1+2 = 3, 1+2+3 = 6, 1+2+3+4 = 10, etc. They are so called because we can associate triangular shapes with these numbers.

There is just one red square; when we put in three green squares around it, they together make a 2 by 2 square; hence we have 1 + 3 = 2 times 2.

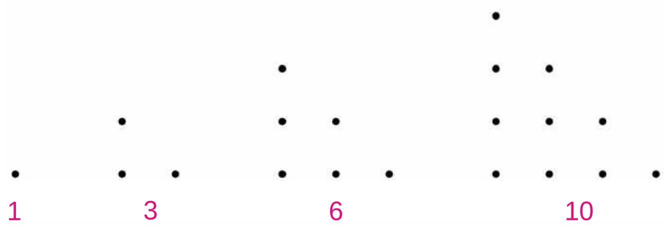
Put in the five purple squares and now you have a 3 by 3 square; hence 1 + 3 + 5 = 3 times 3.



Put in the seven blue squares, and now you have a 4 by 4 square; hence 1 + 3 + 5 + 7= 4 times 4. And so on!

There are two striking properties that connect the triangular numbers with the square numbers (1, 4, 9, 16, etc), and they can easily be found by children: (I) the sum of two consecutive triangular numbers is a square number; e.g., 1+3 = 4, 3+6 = 9, 6+10 = 16, ...; (II) if 1 is added to 8 times a triangular number we get a square e.g., (8 × 3) + 1 = 25, (8 × 6) + 1 = 49, (8 × 10) + 1 = 81.

Why is there such a nice connection? A lovely question to ponder over, isn't it?




Here is another pattern. Take any triple of consecutive numbers; say 3, 4, 5. Square the middle number; we get 16. Multiply the outer two numbers with each other; we get 3 times 5 which is 15. Observe that 16 – 15 = 1; the two numbers obtained differ by 1. Try it with some other triple, say 7, 8, 9: 8 squared is 64, 7 times 9 is 63, and 64 – 63 = 1; once again we get a difference of 1. Will this pattern continue? Yes, and it is easy to show it using algebra; but think of what pleasure discovering this can give a young child playing with numbers!

We find a similar but more complex pattern with the famous Fibonacci sequence, which goes 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …; here, each number after the first two is the sum of the preceding two numbers (e.g., 8 = 5 + 3). Repeat the computation with this sequence. With the triple 2, 3, 5 we get: 3 squared is 9, and 2 times 5 is 10; the squared number is 1 less than the product of the other two. With the triple 3, 5, 8 we get: 5 squared is 25, and 3 times 8 is 24; now the squared number is larger by 1. With 5, 8, 13 we get: 8 times 8 is 64, and 5 times 13 is 65; once again the squared number is smaller by 1. And so it goes – a curious alternating pattern.

We see the same thing if we study collections of four consecutive Fibonacci numbers; for example, 1, 2, 3, 5. The product of the outer two numbers is 5, and the product of the inner two is 6; they differ by 1. Take another such collection: 3, 5, 8, 13. The product of the outer two is 39, that of the inner two is 40; once again, a difference of 1. And again the alternating pattern continues. Astonishingly, even nature sees fit to use the Fibonacci numbers. If we keep records of the numbers of petals in various flowers, we find that the number is generally a Fibonacci number. Study the spirals in which pollen grains are arranged in the center of a sunflower; there are spirals running in clockwise and anticlockwise directions; you will find that the number of spirals of each kind is a Fibonacci number. Nature is just as fond of patterns as we are!

Many years back I used a textbook called "Pattern and Power of Mathematics". It is a nice title for a textbook, for patterns are what the subject is all about, and it is this that gives it its astonishing power. But – more important – it is this feature that makes us study the subject in the first place.

**Large numbers, small numbers**

There are numbers, and then there are large numbers. Children naturally like large numbers, and many of them discover on their own that there is no last number: however large a number one may quote, one only needs to add 1 to it to get a larger number. So the number world has no boundary! There are some who make a similar discovery at the other end – with small numbers; I recall a student telling me, many years back, how she could make an unending sequence of tinier and tinier fractions, simply by halving repeatedly; she could not believe that such tiny numbers could exist! She had made this wonderful discovery herself, and was very excited by it.

The ancient Indians loved large numbers, and here's a problem that shows this love. If I ask you to find a squared number that is twice another squared number, you would never succeed, because there aren't any such pairs of numbers. (Why? – there's a nice story behind that, but we cannot go into that now.) So we change the problem a little bit: I ask for a squared number which exceeds twice another squared number by 1. Now we find many solutions; e.g., 9 and 4 are squared numbers, and 9 – (2 × 4) = 1. Here are some more solutions:

$$289 - (2 \times 144) = 1,$$
$$9801 - (2 \times 4900) = 1.$$

If we replace the word "twice" by "5 times" we find solutions to this too:

$$81 - (5 \times 16) = 1,$$
$$(161 \times 161) - (5 \times 72 \times 72) = 1,$$

and so on.

In the 7th century, Brahmagupta asked if we could find solutions with "5 times" replaced by "61 times". The smallest solution in this case is very large indeed – yet Brahmagupta found it:

$$(1766319049 \times 1766319049) - (61 \times 226153980 \times 226153980) = 1.$$

Feel free to verify the relation.

I think the date is significant: the Indians were asking such questions thirteen centuries back! The love of play has been there in all human cultures, for a long time. There's no holding it back.

But now a strange thing happens. What began as play takes wing, and flies away a mature subject, with an inner cohesiveness and structure that is strong enough to find application in the world of materials, living bodies, and finance – the “real world”. Such flights have happened two dozen times or more in history, and no one really knows how and why they happen; but they do. Maybe it is God's gift to us. (But we do not always use it as intended; the power of Mathematical methods also finds application in the design of bombs and nuclear submarines and other instruments of killing.)

### Closing note

There are so many topics in which we can bring out the theme of pattern and play in Mathematics:

- Magic squares (arranging a given set of 9 numbers in a 3 by 3 array, or 16 numbers in a 4 by 4 array, so that the row sums, column sums, diagonal sums are all the same); not only do these bring out nice number relationships, but in the course of the study one learns about symmetry.

- Cryptarithms (solving arithmetic problems in which digits have been substituted by letters; for example, ON + ON + ON + ON = GO; many simple but pleasing arithmetical insights emerge from the study of such problems);

- Digital patterns in the powers of 2 (list the units digits of the successive powers of 2; what do you notice? Now do the same with the powers of 3; what do you notice?)

These examples are woven around the theme of number, but the principle extends to geometry in an obvious way. Here we study topics like rangoli and kolam; paper folding; designs made with circles; and so on.

Alongside such activities, teachers could also raise questions relating to the role of Mathematics in society, for discussion with students and fellow teachers; e.g., questions relating to the use of Mathematics for destructive purposes, or more generally, "When is it appropriate to use Mathematics?"; or the question of why society would want to support mathematical activity. After all, most artists find patrons or buyers for their art work, but mathematicians do not sell theorems for a living! Is it that policy makers see Mathematics as a useful tool, and thus enable people in this field to sustain themselves, by teaching or doing useful Mathematics? The notion of usefulness takes us back to the question of appropriateness of usage. Such questions are not generally seen as fitting into a Mathematics class, but there is clearly a place for them in promoting a culture of discussion and inquiry.

We need not try to make a complete listing here – it is not possible, because it is too large a list, and ever on the increase. Instead, we wish only to emphasize here that pattern and play are crucial to the teaching of Mathematics, for pedagogic as well as psychological reasons.

A great opportunity is lost when we make Mathematics into a heavy and serious subject reserved for the highly talented, and done under an atmosphere of heavy competition. It denies the experience of Mathematics to so many.

**Suggested Reading**

1. http://en.wikipedia.org/wiki/Ishango_bone
2. “Number, The Language Of Science” Tobias Dantzig
3. “The Number Sense: How the Mind Creates Mathematics” Stanislas Dehaene
4. http://en.wikipedia.org/wiki/History_of_Mathematics

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# 7 Mathematics and the National Curriculum Framework

## Indu Prasad

Mathematics is one of the oldest fields of knowledge and study and has long been considered one of the central components of human thought. Some call it a science, others an art and some have even likened it to a language. It appears to have pieces of all three and yet is a category by itself.

According to the National Curriculum Framework (NCF) 2005, the main goal of Mathematics education in schools is the 'mathematisation' of a child's thinking. Clarity of thought and pursuing assumptions to logical conclusions is central to the mathematical enterprise. While there are many ways of thinking, the kind of thinking one learns in Mathematics is an ability to handle abstractions and an approach to problem solving.

The NCF **envisions** school Mathematics as taking place in a situation where:

1. Children learn to enjoy Mathematics rather than fear it
2. Children learn "important" Mathematics which is more than formulas and mechanical procedures
3. Children see Mathematics as something to talk about, to communicate through, to discuss among themselves, to work together on
4. Children pose and solve meaningful problems
5. Children use abstractions to perceive relationships, to see structures, to reason out things, to argue the truth or falsity of statements
6. Children understand the basic structure of Mathematics: arithmetic, algebra, geometry and trigonometry, the basic content areas of school Mathematics, all of which offer a methodology for abstraction, structuration and generalisation
7. Teachers are expected to engage every child in class with the conviction that everyone can learn Mathematics

On the other hand, the NCF also lists the **challenges** facing Mathematics education in our schools as:

1. A sense of fear and failure regarding Mathematics among a majority of children
2. A curriculum that disappoints both a talented minority as well as the non-participating majority at the same time.
3. Crude methods of assessment that encourage the perception of Mathematics as mechanical computation - problems, exercises, methods of evaluation are mechanical and repetitive with too much emphasis on computation
4. Lack of teacher preparation and support in the teaching of Mathematics
5. Structures of social discrimination that get reflected in Mathematics education often leading to stereotypes like 'boys are better at Mathematics than girls. However the difficulty is that computations become significantly harder, and it becomes that much more difficult to progress in arithmetic.

> *The importance of systematic reasoning in Mathematics cannot be over-emphasised, and is intimately tied to notions of aesthetics and elegance so dear to mathematicians.*

The NCF, therefore, **recommends:**

1. Shifting the focus of Mathematics education from achieving 'narrow' goals of mathematical content to 'higher' goals of creating mathematical learning environments, where processes like formal problem solving, use of heuristics, estimation and approximation, optimisation, use of patterns, visualisation, representation, reasoning and proof, making connections and mathematical communication take precedence
2. Engaging every student with a sense of success, while at the same time offering conceptual challenges to the emerging Mathematician
3. Changing modes of assessment to examine students' mathematisation abilities rather than procedural knowledge
4. Enriching teachers with a variety of mathematical resources.

A major focus of the NCF is on removing fear of Mathematics from children's minds. It speaks of liberating school Mathematics from the tyranny of the one right answer found by applying the one algorithm taught. The emphasis is on learning environments that invite participation, engage children, and offer a sense of success.

### Methods of Learning

The NCF says that many general tactics of problem solving can be taught progressively during the different stages of school: abstraction, quantification, analogy, case analysis, reduction to simpler situations, even guess-and-verify exercises, is useful in many problem-solving contexts. Moreover, when children learn a variety of approaches (over time), their toolkit becomes richer, and they also learn which approach is the best. Children also need exposure to the use of heuristics, or rules of thumb, rather than only believing that Mathematics is an 'exact science'. The estimation of quantities and approximating solutions is also an essential skill. Visualization and representation are skills that Mathematics can help to develop. Modelling situations using quantities, shapes and forms are the best use of Mathematics. mathematical concepts can be represented in multiple ways, and these representations can serve a variety of purposes in different contexts.

For example, a function may be represented in algebraic form or in the form of a graph. The representation 'p/q' can be used to denote a fraction as a part of the whole, but can also denote the quotient of two numbers, 'p' and 'q.' Learning this about fractions is as important, if not more, than learning the arithmetic of fractions. There is also a need to make connections between Mathematics and other subjects of study. When children learn to draw graphs, they should also be encouraged to think of functional relationships in the sciences, including geology. Children need to appreciate the fact that Mathematics is an effective instrument in the study of science.

The importance of systematic reasoning in Mathematics cannot be over-emphasised, and is intimately tied to notions of aesthetics and elegance so dear to Mathematicians. Proof is important, but in addition to deductive proof, children should also learn when pictures and constructions provide proof. Proof is a process that convinces a skeptical adversary; school Mathematics should encourage proof as a systematic way of argumentation. The aim should be to develop arguments, evaluate arguments, make and investigate conjectures, and understand that there are various methods of reasoning.

The NCF also speaks of mathematical communication – that it is precise and employs unambiguous use of language and rigour in formulation, which are important characteristics of mathematical treatment. The use of jargon in Mathematics is deliberate, conscious and stylised. Mathematicians discuss what appropriate notation is since good notation is held in high esteem and believed to aid thought. As children grow older, they should be taught to appreciate the significance of such conventions and their use. This would mean, for instance, that setting up of equations should get as much coverage as solving them.

### Organization of the Curriculum

The NCF recommends the following for different stages of schooling:

1. Pre-Primary: At the pre-primary stage, all learning occurs through play rather than through didactic communication. Rather than the rote learning of number sequence, children need to learn and understand, in the context of small sets, the connection between word games and counting, and between counting and quantity. Making simple comparisons and classifications along one dimension at a time, and identifying shapes and symmetries, are appropriate skills to acquire at this stage. Encouraging children to use language to freely express one's thoughts and emotions, rather than in predetermined ways, is extremely important at this and at later stages.

2. Primary: Having children develop a positive attitude towards, and a liking for Mathematics at the primary stage is as important as developing cognitive skills and concepts. mathematical games, puzzles and stories help in developing a positive attitude and in making connections between Mathematics and everyday thinking. Besides numbers and number operations, due importance must be given to shapes, spatial understanding, patterns, measurement and data handling. The curriculum must explicitly incorporate the progression that learners make from concrete

to abstract while acquiring concepts. Apart from computational skills, stress must be laid on identifying, expressing and explaining patterns, on estimation and approximation in solving problems, on making connections, and on the development of skills of language in communication and reasoning.

3. Upper Primary: Here, students get the first taste of the application of powerful abstract concepts that compress previous learning and experience. This enables them to revisit and consolidate basic concepts and skills learnt at the primary stage, which is essential from the point of view of achieving universal mathematical literacy. Students are introduced to algebraic notation and its use in solving problems and in generalisation, to the systematic study of space and shapes, and for consolidating their knowledge of measurement. Data handling, representation and interpretation form a significant part of the ability to deal with information in general, which is an essential 'life skill.' The learning at this stage also offers an opportunity to enrich students' spatial reasoning and visualisation skills.

4. Secondary: Students now begin to perceive the structure of Mathematics as a discipline. They become familiar with the characteristics of mathematical communication: carefully defined terms and concepts, the use of symbols to represent them, precisely stated propositions, and proofs justifying propositions. These aspects are developed particularly in the area of geometry. Students develop their facility with algebra, which is important not only in the application of Mathematics, but also within Mathematics in providing justifications and proofs. At this stage, students integrate the many concepts and skills that they have learnt into a problem-solving ability. Mathematical modelling, data analysis and interpretation taught at this stage can consolidate a high level of mathematical literacy. Individual and group exploration of connections and patterns, visualisation and generalisation, making and proving conjectures are important at this stage, can be encouraged through the use of appropriate tools that include concrete models as in Mathematics laboratories and computers.

5. Higher Secondary: The aim of the Mathematics curriculum at this stage is to provide students with an appreciation of the wide variety of the application of Mathematics, and equip them with the basic tools that enable such application. A careful choice between the often conflicting demands of depth versus breadth needs to be made at this stage.

On **Assessment,** the NCF recommends that Board examinations be restructured, so that the minimum eligibility for a State certificate is numeracy, reducing the instance of failure in Mathematics. At the higher end, it is recommended that examinations be more challenging, evaluating conceptual understanding and competence.

The NCF's vision of excellent mathematical education is based on the twin premise that all students can learn Mathematics and that all students need to learn Mathematics. It is, therefore, imperative that Mathematics education of the very highest quality is offered to all children.

Problem Posing

1. If you know that 235 + 367 = 602, how much is 234 + 369? How did you find the answer?
2. Change any one digit in 5384. Did the number increase or decrease? By how much?

Source : NCF 2005

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# 8 Important Concepts in Primary School Mathematics

**Kamala Mukunda**

From all the varied research into early mathematical ability and learning that psychologists have done, I have chosen for this article two general topics: the fundamental importance of the mental number line, and the connection (often the lack of connection!) between conceptual and procedural learning in primary school. Both are, I believe, topics of particular interest to people working with children aged 6 to 11 years or so, laying the foundation for arithmetic and Mathematics in years to come.

## The Mental Number Line

Cognitive scientists have established over the past few decades, pretty conclusively, that as human beings we are 'born to numerate'. Some simple but brilliant work with preschoolers shows that they develop and practice basic, key numerical skills before the age of 4 or 5, spontaneously. The development of these skills is reminiscent of the way children learn language: there seems to be an innate module in our brains that clicks into action given a 'minimum' environmental input. The essential accomplishment of the early years is the proper understanding and use of a mental number line (MNL), used in the act of counting. This is no trivial thing! When a toddler or preschooler counts a set of objects, she invokes no less than five crucial principles.

1. There has to be a one-to-one correspondence between each object and a number name. For eg., you cannot assign more than one object the number 'four'.
2. Yet the number names do not belong to the objects in any way; they can be reassigned. On a recount, you can change all the assignments!
3. Number names are always to be spoken in the same, invariant order. In fact, many toddlers have the wrong number order—one, two, three, five, seven, eight, nine, ten!—but they use it invariantly (till they eventually correct themselves, of course).
4. The final number spoken is always the size of the set.
5. Counting is something you can do with any set of objects, from pins to people.

In time, they use this MNL to compare two numbers to say which is larger, and soon begin to do simple addition using a method called 'counting on'. That is, to add 4 and 2, they start with the larger number 4 on the MNL, and move two units to the right to reach 6. Five year olds have been observed to invent this sophisticated method spontaneously, as they learn to combine their skills of comparison with counting on the MNL.

When children begin formal schooling, they typically should have this informal number knowledge available to understand anything new that is taught. But they do not all begin school equal; studies show (as does any teacher's experience) that in first standard, children vary in their levels of number knowledge. Some students have mastered several number facts, which means that they can quickly recall from memory facts such as '4+2=6' without having to actually re-perform the sum. They also are more likely to use strategies such as counting on, more efficiently, when faced with new problems. Other students are at a disadvantage, not having proper representations of counting on the MNL, therefore not having invented certain strategies, and therefore not having enough number facts at their disposal.

> *Why do we "start from the right, work leftward, shift the products one place to the left, add them all up"?*

Several reasons have been proposed to explain these differences, but psychologists are also working on how to close the gap sooner rather than later, to help weaker students build the foundation they need. The most obvious suggestion is to include explicit instruction of the MNL and its properties in first standard curricula, since it is not generally taught that way. Interestingly, one of the strongest correlates of these initial differences in numerical ability is the socioeconomic status (SES) of the child entering school. Children from lower SES are at a significant disadvantage compared to middle or high SES children at number knowledge when they begin school. Unaddressed, this gap only widens with time. Developmental psychologists Robert Seigler and Geeta Ramani offer one interesting reason for the difference: lower SES children do not have access to the kinds of board games other children routinely play with. The main element of many simple board games (such as Snakes-and-Ladders or Ludo) is a series of numbered spaces, linearly arranged. You move your token along a certain number of spaces, counting as you go, one number for each move. Playing this game, they say, gives preschoolers the right stimuli to develop correct understandings of the MNL.

In a recent study, Siegler and Ramani worked with a large number of children from lower- and middle-SES backgrounds in the U.S. They first replicated the general finding that the lower-SES children perform significantly worse on the following simple numerical magnitude estimation task: given a line with 0 at one end and 100 at the other end, place a third number (say, 37) correctly on the line. Second, they provided the lower-SES children with around 30 brief sessions of playing a very basic board game with just ten linearly arranged spaces. The total time of intervention was only around two hours, and yet post-tests revealed a virtual 'catch-up' of these children! This study needs replication in India, of course. But given the importance of the MNL for early arithmetic, and given the simplicity of the intervention, it is definitely worth investigation.

**Marrying Concepts to Procedures**

The schism between number recipes and conceptual understanding is widespread. An example of a number recipe would be the long multiplication method, where we are taught to start from the right, work leftward, shift the products one place to the left, add them all up...Primary school Mathematics is filled with such algorithms. But students cannot tell you (because they do not know) why their algorithms work. Interestingly it seems that it is not enough to teach a concept (using concrete materials and so on), then introduce a procedure, followed by tons of drill, rarely revisiting the concept after the algorithm has been introduced. Teachers seem to invest relatively little time and energy into explaining why the procedure works, by connecting it explicitly to the concrete concept. Why do we "start from the right, work leftward, shift the products one place to the left, add them all up"? Explaining the connection once, or even twice, does not seem to be enough given its considerable complexity.

Now, you might wonder why the connection is so important. It turns out that there are strong arguments in favour of explicitly marrying mathematical concepts with procedures. One is that conceptual understanding is itself strengthened in the process. There should in fact be a continuous back-and-forth process between procedural and conceptual learning. Reflecting on the use of a procedure, asking whether and why certain procedures work and others are wrong reinforces our understanding of the concept. Another good reason is that concept and procedure must be interconnected in order for the procedure to be flexibly applied to new problems. And a third reason is that mistakes in arithmetical algorithms can best be corrected through such an understanding, instead of simply through reminders of "how it should be done."

Psychologist Lauren Resnick did a beautiful in-depth study in 1982 of four students in 2$^{nd}$ and 3$^{rd}$ standard, demonstrating conclusively that there was no correlation between their number knowledge and their computational skills. The two sets of knowledge were neatly isolated from each other in the children's minds. Importantly, she found that the students did not spontaneously figure out the connections between the two. She then developed a 'mapping instruction' method for teaching multidigit subtraction, using blocks and number symbols side by side. The blocks were of size 100, 10 and 1, and at each step a particular arrangement of blocks was presented along with the corresponding step of the algorithm. Written computation was a record of the block arrangement, and block arrangements justified the written computation. The entire duration of her instruction method was just 40 minutes! And with this brief and simple intervention, the students were vastly improved both in correct use of the algorithms as well as in explaining in words why things worked a certain way. Again, we see that a relatively small but well designed addition to our teaching can have a powerful impact, especially in primary school settings.

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## Mathematical Limericks

Limericks are poems which typically consist of five lines and are often humorous or bawdy. Lines 1, 2, and 5 have seven to ten syllables and rhyme with one another while lines 3 and 4 have five to seven syllables and also rhyme with each other.

There was a bright young lad from Madras
Who could not count though in fifth class
His Math teacher tried her best
With tutorials to cram for test
But the bright young lad just could not pass

Entering a discount store is always a tension
Whether to use ratio, proportion or division
Is a price-off percentage of 33
Better than 'buy 2 take 1 free'
Or is it the same? Well, that is the confusion

There was a mathematician who loved his wine
And found the concept of 'Pi' simply divine
He worked hard on 22/7
Mornings and nights, even
To find in the billionth decimal place was a 9

**D D Karopady**

# 9 Mother's Math and the Art of Estimation

**Nat Ramachandran**

Glance at this article and quickly guess the number of printable characters in it.

Mathematics is a definitive science because its underlying elements i.e., numbers and operations are definitive. Contrast Mathematics with language (spoken or written) which is prone to differing interpretations—for instance two people reading the same news article can arrive at different conclusions—but Mathematics, in and of itself, is definitive and logically self-contained. In other words, anywhere and everywhere numeral two will always be less than numeral seven; an additive operation always has a definitive answer; but the words "sky blue" can elicit several chromatic interpretations; John Donne's poetic rendering "And therefore never send to know for whom the bell tolls; it tolls for thee" can evoke varying emotions.

Despite its aura abstract Mathematics is an exception in the real world. The daily course of our lives does not present itself with theoretical problem statements with complete information as we see in our Mathematics examinations. The practical utility of Mathematics lies in blending the science of Math with the art of estimation. Estimation—making an informed guess about something one does not know—is inherently an imprecise act and hence error prone. However by blending the definitive science with an imprecise art, the power and value of Mathematics is enhanced.

When my mother cooked I often used to wonder about how she can add the right quantity of salt in her preparations, despite the varying amount of food that she had to make or when she experimented with a new recipe that gave generic advice ("add salt to taste"). In retrospect I figured that skill came from her intuition and long standing experience in cooking, which gave her the ability to estimate and scale the quantity of salt (and, of course, a pinch of good luck to be right about it every time.) Mothers (and all others) make mathematical guesswork and formulate 'rules of thumb' in instances where information is scant, unknown or the payoffs for collecting information is less compared to the effort.

Jonathan Swift elegantly elaborates in "Gulliver's Travels" on how the small people in the island of Lilliput (who were 1/12th the size of Gulliver) stitched a shirt for Gulliver:

> "Then they measured my right Thumb, and desired no more; for by a mathematical Computation, that twice round the Thumb is once around the Wrist, and so on to the Neck and Waist, and by the help of my old Shirt, which I displayed on the Ground before them for a Pattern, they fitted me exactly."

Swift's Lilliputians, it seems, had good skills in estimation and tailoring.

> *"When my mother cooked I often used to wonder about how she can add the right quantity of salt in her preparations, despite the varying amount of food that she had to make or when she experimented with a new recipe that gave generic advice ("add salt to taste")"*

Consider a cricket batsman's capabilities to blend art and the science of Math: he has to estimate the pace of the ball to figure out when it will reach him; the curvature of the ball to know where it will come (e.g. wide, full-toss, in-swing etc.); and the willow power he needs to exert on the ball at a precise angle so that the ball is swung towards the boundary line. Even a slightly wrong judgment on any of these estimates will mean a missed ball or flying stumps. For a good batsman all these tasks have to be judged within a fraction of a second. Sachin Tendulkar is an excellent judge of balls bowled at him, which makes him an admirable batsman.

Today's school system in India, unduly emphasizes the abstract Math at the expense of the artful and useful aspects of Mathematics. Mathematics must be taught as another tool that can be skillfully deployed in real life. For Math to become a tool its utility in terms of estimation and calculation have to be inculcated in schools. For instance children should be simultaneously taught the abstract notion of area of a rectangle and be asked to make informed and educated guesses about estimating the area of their classrooms.

> *What is the amount of money spent by the Indian population on their breakfast? How many total phone calls are made in India on the day of Diwali?*

While teaching the abstract notions of geometry dealing with right angles and hypotenuse, or simple distance measurement metrics such as meters, centimeters and kilometers, children can be asked to estimate distances in their neighborhood with similar angular features. While teaching children about speed, velocity and distance students could be given hypothetical problems with scant information (e.g. how fast should one walk from school to the bus stand to catch the next bus to their neighboring town?). Such realistic exercises will enable them to apply the abstract concepts, facilitate the skill of making rapid calculations and hone their estimation skills.

Businesses live in a world of uncertainty and incomplete information. Hence they have to constantly make estimates about the number of products they can sell in the future; prospective prices at which they can sell; potential profit margins that they are likely to get. Such estimations are critical for a business to invest (or not to invest) their capital today. Similar exercises of smaller magnitude can be given to children while they are taught the concepts of money, profit margins, mark-up prices, selling price etc. Such exercises make the mathematical concepts easily comprehensible; provide a sense of application for abstract concepts as well as serve as a platform for young minds to think and compute quickly.

Estimation is also an important skill for children to learn in the context that real life does not present itself with complete information as is provided in examination question papers. The utility of estimation and valuation skills lies in its constant application in the ordinary course of life. If done so, children will no more agonize about Mathematics as a subject reserved for thick bespectacled scientists and wooly-haired mathematicians. Making logical and reasonable assumptions are an important component of estimation. Without worrying about errors here are a few estimations for you to ponder over: (1) In an apartment block with fifty residences, how many litres of water would the residents consume in a year? (2) If 1000 apples fit tight in a carton how many lemons could fit in the same carton? (3) What is the amount of money spent by the Indian population on their breakfast? (4) How many total phone calls are made in India on the day of Diwali?

The number of printable characters in this article is 5296—how close was your estimate?

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# 10 What Ails Mathematics Teaching?

**D D Karopady**

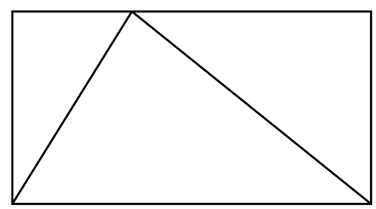
Ask children in school which subject they dislike the most; chances are that 9 out of 10 will say Mathematics. In fact, if you probe a little deeper, you will discover that 7 out of 10 children say they are 'terrified' or 'mortally scared' of the subject. Why children, ask any adult and you will find a similar pattern. I spoke to a random sample of my friends inside and outside the Azim Premji Foundation and asked them 'what words come to your mind when I say Mathematics?' Barring a microscopic minority, the descriptors were very negative – fear of failure, very difficult, getting beaten in school, incomprehensible, no link to life, formulae, memorization, I am stupid since I don't know Math, its only for the intellectual, dry, boring and so on. One response summed it all up – 'Oh my God! Many of the tragic cases of children taking the extreme step after their exams can be traced to Mathematics. The problem is often not limited to that subject alone but can actually result in the child ending up with an aversion to the entire process of 'schooling'.

The reasons for this are not too far to seek; the way the subject is handled in schools by our teachers is a big contributor. In a study carried out among primary school teachers, it was found that most of them are from an 'Arts' background and Math has been one of their weakest subjects too. When they themselves are fearful of the subject, it is but natural that they transmit the same to the children. Unfortunately, the focus of Math teaching is on definitions, memorization, recall, calculation. There is also a lot of effort in providing the 'right context' to the problems and making it 'useful' in real life.

Some where, at the back of the minds of many people, Math is closely associated with and seen to be similar to 'Science'. While Mathematics is certainly used in science, the two are starkly different. Science is grounded strongly in experiments while Mathematics is imaginary and abstract. You require no special equipment or laboratory to do it. If it is handled well, it can be extremely enjoyable. Paul Lockhart in his article 'A Mathematicians Lament' says "Mathematics is an art and it is just that our culture does not recognize it as such. The fact is that there is nothing as dreamy and poetic, subversive and psychedelic as Mathematics. It is as mind blowing as Cosmology or Physics (mathematicians conceived of black holes long before astronomers actually found any). We, as a culture do not know what Mathematics is. The impression we are given is of something very cold and highly technical, that no one could possibly understand – a self fulfilling prophecy if there ever was one."

According to Lockhart, there is no ulterior practical purpose in Mathematics. It is just playing, wondering and amusing yourself with your imagination. He gives a beautiful example with a figure. He says "imagine a triangle inside a box:

Figure 1

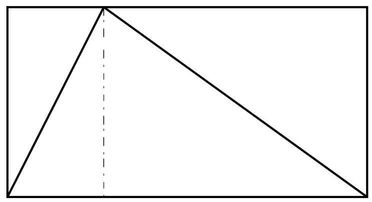


Source: A Mathematicians Lament, by Paul Lockhart

He then goes on to ask "does the triangle take up two – thirds of the box? Imagination is the only way to get at the truth. In this case, one way is to cut the box into two pieces like in figure 2.

> *The idea should be to make the subject a journey of enquiry, discovery and excitement. It is all about patterns and finding ways to look at and explore them, make mistakes, asking oneself further questions, looking for more answers and letting your mind traverse unexplored territory.*

Figure 2



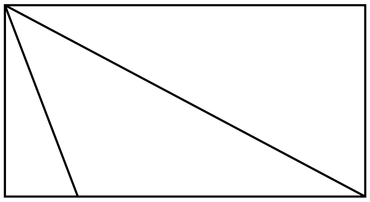
Now, one can see that each piece is a rectangle cut into two by the sides of the triangle. So, there is as much space inside the triangle as outside. That means the triangle takes up exactly half the box. Now, how did the idea of drawing the line come? It is inspiration, experience, trial and error, dumb luck. That is the art of it. The relationship between the two shapes was a mystery till the line made it obvious. I couldn't see, then all of a sudden I could. Somehow, I was able to create a profound simple beauty out of nothing. Isn't that what art is all about?"

Isn't doing this so much more fun than asking children to memorise - Area of a triangle is equal to half its base times its height - and applying it over and over again. Lockhart is not objecting to formulas or to memorizing interesting facts in some context. He says "what is critical is the process of generating options and how that might inspire other beautiful ideas and lead to creative breakthroughs in other problems". He goes on to add "Mathematics is the art of explanation. If you deny students the opportunity to engage in this activity - to pose their own problems, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs - you deny them Mathematics itself. Students are not aliens. They respond to beauty and pattern, and are naturally curious like anyone else. Just talk to them! And more importantly, listen to them!"

So do our Mathematics teachers give children the time to make discoveries, conjectures, or choose engaging problems for them? Are they (teachers) creating an atmosphere of healthy discussion, enquiry and provide space for curiosity to be satisfied? Lockhart says "I doubt that most teachers even want to have such a relationship with the children. It is far easier to use material in some book and follow the 'lecture, test and repeat' method – the path of least effort. If adding fractions is to the teacher an arbitrary set of rules, and not the outcome of a creative process, then of course it will feel that way to the poor students. Teaching is about having an honest intellectual relationship with your students. It requires no method, no tools, and no training".

Now, take the above example of the triangle in the box. What if the triangle was slanted? How does one draw the line? What can be done?

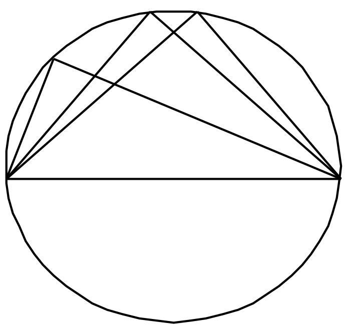
Figure 3



Go ahead and try various possibilities and discover. That is what Mathematics is all about.

Geometry in our schools is made very boring and is invariably reduced to a sequence of – Theorem - - - Proof - - - Rider - - - - solution and then one more Rider. Lockhart has some scathing comments to make about high school Geometry. "The student – victim is first stunned and paralyzed by an onslaught of pointless definitions, propositions and notations and is then slowly and painstakingly weaned away from any natural curiosity or intuition about shapes and their patterns by a systematic indoctrination into the stilted language and artificial format of so-called formal geometric proof, The geometry class is, by far the most mentally and emotionally destructive component of the entire K-12 Mathematics curriculum". He gives a beautiful example of a triangle inside a semicircle and the fact that no matter where you place the tip of the triangle on the circle, it always forms a right angle.

Figure 4



At first sight this seems unlikely. The question then to ask is how can this be true? Here is an opportunity to let students explore this and attempt to find out why this could be so? Or maybe even try and see if this is not true? But all we do is give them a standard 'proof' which they have to remember.

In Geometry, how about asking the students to find out what is the minimum number of colours needed to shade a map so that no two adjacent states have the same colour? Can you think of any special shapes which may require more than these? What special shapes can be managed to be coloured by less number of shades?

As another example, many jig saw puzzles with geometric shapes can be created and that can be a great source of joy and learning for the children just by playing around, turning the pieces around, discovering different shapes in the process and so on.

Children (and for that matter adults) are thrilled to make discoveries. The discovery is all the more exciting if one stumbles upon it accidentally. We get our children to memorise tables. How much more fun it would be for the child to explore patterns in numbers? How exciting would it be to let the child discover this herself?

Take the case of the number 9.

| | |
|---|---|
| 9 x 1 = 9 | 0 + 9 = 9 |
| 9 x 2 = 18 | 1 + 8 = 9 |
| 9 x 3 = 27 | 2 + 7 = 9 |
| 9 X 4 = 36 | 3 + 6 = 9 |
| ..... | |
| ..... | And so on. |

Now, ask the children to discover interesting facts about other numbers like 3, 5, 11, 15 ....

You would have seen lot of people these days on buses and trains busy writing, correcting, rewriting numbers on the newspaper, all the time chewing the pencil. They are at Sudoku, the latest craze. There is no practical relevance or use of it other than fun. The simpler forms of this – magic squares - can be given to children to play with. The simplest is a 3 X 3 grid which needs to be filled up with numbers from 1 – 9 so that the total of each row, each column and each diagonal totals to 15. It is possible to construct several such magic squares of larger size.

**Magic Squares**

Magic Squares have fascinated Mathematicians from the ancient times. The concept seems to have originated in (where else) China around 2800 BC. In India, references to this idea are seen in $11^{th}$ or $12^{th}$ century and some examples have been found in the ancient town of Khajuraho.

A magic square is defined as an arrangement of sequential numbers in a grid, starting in such a manner that each number appears exactly once and the sum of the numbers in each row, each column and each main diagonal is the same. Numbers in many magic squares start with 1. These are called simple magic squares.

The smallest (and most trivial) magic square is a 1 X 1 grid.

The next larger magic square is a 3 X 3 grid with numbers from 1 to 9 in it. The total of each row, column and main diagonal in this case is 15 as shown below. **(It is not possible to have a 2 X 2 magic square. Can you say why?).**

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

From one magic square, it is possible to create multiple squares by transposing the rows and columns.

As we go to larger squares, the interesting possibilities and variations increase. A 4 X 4 square has the total of 34 and can be designed to have a special property where the four corner cells also add up to the same total (shown in the article). There are other interesting variations in larger squares like 'Multiplication Magic Square' where the multiplication of cells in each row, each column and each main diagonal is identical. Yet another special square is one where the addition is the same **and** the multiplication is the same for each row, column and main diagonal.

One other variation is an Anti Magic Square. This is a grid where the cells are filled up with numbers sequentially from 1 onwards but the totals of each row, each column and each main diagonal are **all different and are in a serial sequence**. See the 4 X 4 square below.

| | | | |
|---|---|---|---|
| 15 | 2 | 12 | 4 |
| 1 | 14 | 10 | 5 |
| 8 | 9 | 3 | 16 |
| 11 | 13 | 6 | 7 |

Can you check out the totals to be in a serial sequence for this?

As per the above definition, it is not possible to have an Anti Magic square smaller than 4 X 4. Some consider the following to be an Anti Magic square. However, it does not satisfy the condition that the totals are in a serial sequence.

| | | |
|---|---|---|
| 7 | 6 | 5 |
| 8 | 9 | 4 |
| 1 | 2 | 3 |

You can figure out that this square has been created by starting with the lower left corner moving in an anti-clockwise spiral.

Another variation of the magic squares is the 'Latin Square'. Here, the numbers in the cells are repeated but not in the same row or the same column. The simplest, of course is the 2 X 2 square.

| | |
|---|---|
| 1 | 2 |
| 2 | 1 |

.

Can you try and create the next simplest level of a 3 X 3 Latin Square? It is not too difficult. You would have realized that the 'Sudoku' is a 9 X 9 Latin Square.

There are several other amazing magic figures – cubes, circles, stars besides a large number of innovative variations.

One of my favorite set of magic squares is the Alpha Magic square. Look at the squares below:

| | | |
|---|---|---|
| 5 | 22 | 18 |
| 28 | 15 | 2 |
| 12 | 8 | 25 |

| | | |
|---|---|---|
| 4 | 9 | 8 |
| 11 | 7 | 3 |
| 6 | 5 | 10 |

The contents of the square on the right are derived from the contents of the square on the left. Can you figure out the connection? (Hint: write the contents of the square on the left in English)

The beauty of the magic square lies in discovering patterns in the square and looking for alternate solutions from a given solution. Look at the interesting example below which has 16 cells with numbers from 1 to 16.

| | | | |
|---|---|---|---|
| 1 | 8 | 12 | 13 |
| 14 | 11 | 7 | 2 |
| 15 | 10 | 6 | 3 |
| 4 | 5 | 9 | 16 |

Here, you can see that each row, each column and each main diagonal add up to 34. Do you notice some other patterns?

If you observe closely, you will find that the four upper right hand corner cell numbers (13, 12, 7 and 2) also add up to 34; and for that matter so do the numbers in the four cells at each corner. There are several other cells in the grid which also add up to 34. Can you find them? What other solutions for this exist? Will every grid with even number of cells exhibit the same corner property?

Isn't this a more interesting and fun way to teach Math? There are several other ways to make Mathematics learning an enjoyable process. It is not my attempt to say that every single thing in Mathematics can and should be taught through this route. The idea should be to make the subject a journey of enquiry, discovery and excitement. It is all about patterns and finding ways to look at and explore them, make mistakes, asking oneself further questions, looking for more answers and letting your mind traverse unexplored territory. The process should become more important than the result. The aim should be to take the 'fear' out of Mathematics. This could just lead to the children starting to enjoy not just that subject but the entire process of schooling since the 'scary' part has vanished. The process of course needs to begin with our teachers who themselves need to re-learn the art of 'facilitating' the exploration of beauty in Mathematics.

This article is inspired by and largely based on the article 'A Mathematician's Lament' by Paul Lockhart. The original article can be accessed at http://www.maa.org/devlin/LockhartsLament.pdf. This is a 'must read' for all who dislike Math and can not be missed by all who love the subject.

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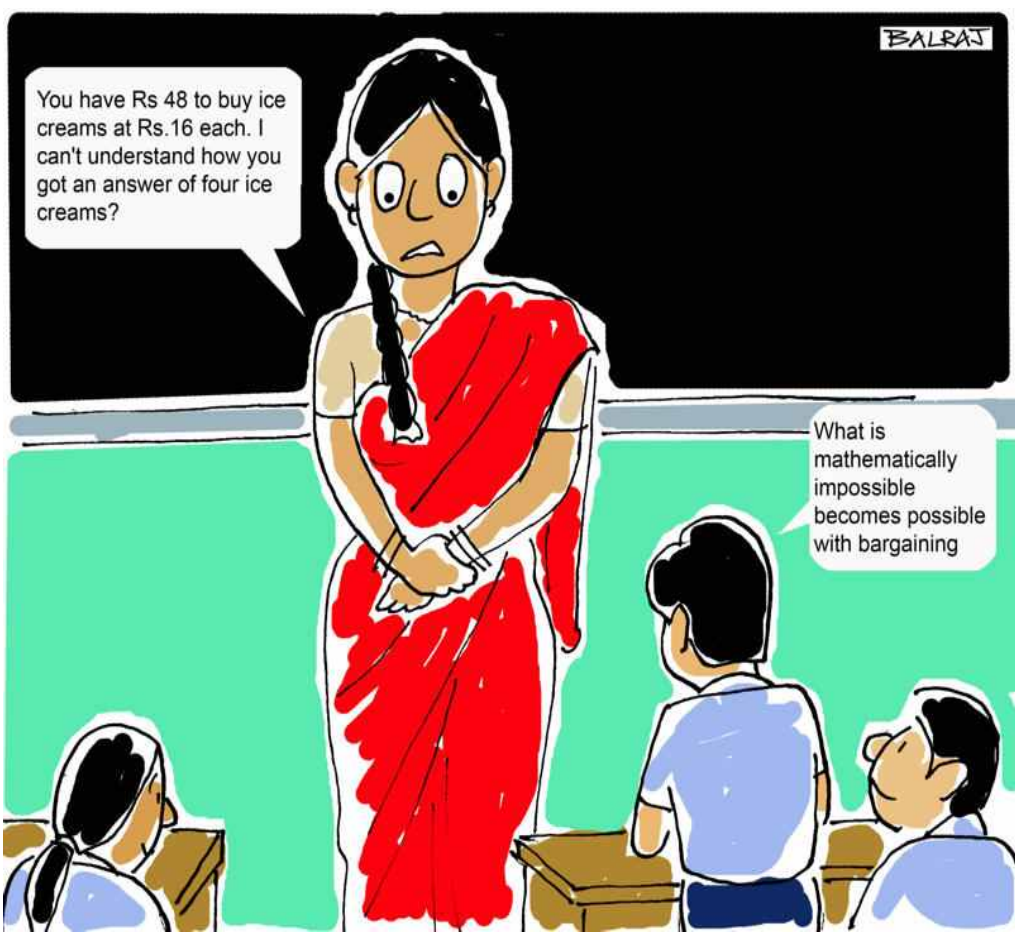

# 11 Ancient India and Mathematics

**Sundar Sarukkai**

One way to approach this very broad topic is to list all that the ancient Indian mathematicians did – and they did do an enormous amount of Mathematics: arithmetic (including the creation of decimal place notation, the invention of zero), trigonometry (the detailed tables of sines), algebra (binomial theorem, solving quadratic equations), astronomy and astrology (detailed numerical calculations). Later Indian Mathematics, the Kerala school, discovered the notions of infinite series, limits and analysis which are the precursors to calculus.

Since these details are easily available I am not going to list them here. What is of interest to me is to understand in what sense these activities were 'mathematical'. By doing so, I am also responding to the charge that these people were not doing Mathematics but something else. This is a charge similar to that addressed to Science in ancient India – the claim here is that what was being done in metallurgy, for example, was not Science but only craftsmanship. Similarly, there is a claim that Indian Mathematics is not really Mathematics since it was not axiomatic, it was related to the world whether in calculation of planet positions or dimensions of the sacrificial pyre, it was not really logic since it was explicitly related to the empirical and so on.

Indian Mathematics was explicitly engaged with the natural world and is in some sense grounded upon the nature of our cognition as well as the nature of the world. It was more about doing and in a sense closer to the constructivist paradigm. A famous example is the Indian mathematicians' pragmatic acceptance of square root of 2 (as something that is used in construction, for example) as against its rejection by the Pythagoreans on idealistic grounds.

Another uniqueness of Indian Mathematics was the form in which it was written. Early Mathematics was often written in poetic form. While it would seem as if the Indian Mathematicians did not use symbols like we see in modern texts, this is not completely true since they used alphabets of Sanskrit to stand for numbers. The implications of writing Mathematics in a poetic form have not been considered in detail and suffice it for me to say here that this approach has important implications for Mathematics education!

There are some things in common between Indian and Greek Mathematics but there are also significant differences – not just in style but also in the larger world view (which influences, for example, the completely different ways of understanding the nature of numbers in the Greek and the Indian traditions). This difference has led many writers to claim that Indians (and Chinese among others) did not possess the notions of Science and Mathematics. The first, and enduring response, to the question of Science and Mathematics in ancient non-Western civilizations is one of skepticism. Did the Indians and Chinese really have Science and Mathematics as we call it now? This skepticism has been held over centuries and by the most prominent thinkers of the west (and is in fact so widespread as to include claims that Indians did not 'have' philosophy, logic and even religion). So even before we begin to understand the nature of Science and Mathematics in ancient India we need to have a response to this skepticism.

*"The west did not have disciplines called the sciences until a few centuries ago. What they had were a variety of disciplines such as physics, chemistry, metallurgy, geology and so on."*

One type of response is to consider the development of the ideas of Science and Mathematics in the west. The west did not have disciplines called the sciences until a few centuries ago. What they had were a variety of disciplines such as physics, chemistry, metallurgy, geology and so on. In the early eighteenth century these disciplines began to get unified under the name of 'Science' and debates during this time illustrate how problematic this unification was since it was very difficult to find common elements in all these disparate disciplines. (In fact, the word 'scientist' was itself coined by Whewell in 1833 and prominent scientists during this time repeatedly wondered as to what was common to all these disciplines.) This debate on what constitutes a science continues even today, and is best manifested in the scientists' reaction (mostly negative!) to calling social science as a science or worse, astrology as science.

There is indeed a genuine problem in placing disparate disciplines such as physics, chemistry, geology, immunology and so on under one category called science. There is little that is common in the practices of these disciplines as well as in the subject matter. This problem leads us to search for a common methodology; something which could be called as the 'scientific method' and which presumably would be found in everything we call science. But the search for this elusive methodology has been long and difficult, and not entirely successful. It has often been simplified to say that scientific methodology is based on the activities of theory and experiment but such a rendering also makes many other human activities scientific.

One way we can understand the nature of science is by viewing it as a title; a title given by a group of people who see themselves as representatives of science. In fact, if we see how national associations of science talk about science we can clearly see this attempt – by these groups to regulate what is science and what is not – as an indication that science is primarily a title.

Given this background, it becomes more obvious that the question of whether Indians and Chinese “had” science and Mathematics is actually a question that can be reasonably asked only by (1) first understanding how different subjects came to be grouped under science or Mathematics and (2) as to why such a question is not posed in the western context. As is well-known, there is very little in common between Aristotle's science and modern science. On the contrary, it was the overthrow of Aristotle's ideas about the natural world that made possible science as we know it today. But in spite of this, we often see scientists talking about Greek science without qualifications but when it comes to science in other cultures – whether ancient or modern – there is often deep skepticism.

The case of Mathematics is slightly different from science although similar questions about the unification of different disciplines remain. That Mathematics was a Greek invention and that it was one of the most influential disciplines which catalyzed other disciplines such as logic has been accepted for a very long time and is still very much a part of 'cultural pedagogy'. (Even today, very influential textbooks, specialized books as well as popular ones continue this myth as if other cultures had no access to these 'subjects'). However, unlike science, there seems to have been less of a confusion about what defines Mathematics. In the case of science, the disciplines came first and then they were put under the category of science. In Mathematics, the situation was quite different since right from the beginning certain kinds of activities were seen to belong to the Mathematical. And this was true for both Greek and Indian traditions.

But the question that is so problematic for science is also partly true for Mathematics. How do we recognize new disciplines such as calculus, differential equations etc. as belonging to Mathematics in the same way that arithmetic and geometry were Mathematics? If geometry is a paradigm example of Mathematics for Euclid, then what is common to the axiomatic system of Euclid and the various new ideas in calculus, topology and other disciplines which are placed under Mathematics? For example, when calculus was created it was not like the Euclidean axiomatic system. Then why is calculus called Mathematics in the same way that Euclidean geometry is Mathematics?

In general, though, it is easier to identify Mathematics in comparison to science. For example, the objects with which Mathematics deals with are very special ones such as numbers, sets, functions and matrices. There is, in general, some commonality in the 'objects of discourse' of Mathematics unlike science since physics deals with the physical world (remember Newton's belief that a primary task of physics was to distinguish real motion from apparent motion), chemistry with organic and inorganic molecules (much of which are synthesized and created in the laboratory), biology with living organisms. In the case of Mathematics, set theory has overlap with arithmetic and algebra, topology with set theory and so on. There is more coherence in the Mathematical objects that occur in these various disciplines.

There are also other common indicators in the different sub-disciplines of Mathematics: the role of operators, the activity of calculation, the creative use of symbols, the creation of new kinds of symbols, the fundamental and essential role of the equality sign (and related to it the inequalities). Most of these characteristics are also closely linked to a very specific way of dealing with language (specifically, semiotics). Thus, Mathematics as a particular kind of 'language' is another common theme that links these various sub-disciplines of Mathematics. These are characteristics which are common to the many sub-disciplines of Mathematics.

They are also common to ancient Indian Mathematics, whether in the fields of arithmetic, trigonometry, algebra or analysis. But discovering these commonalities should not blind us to the unique differences which characterize the cultural imagination inherent in Mathematics. If we take this point seriously, then we might see more clearly that for the ancient Indian practitioners there is no clear distinction (in contrast to the Greeks and later on the western intellectual traditions) between science and Mathematics, just as there is little difference between science and logic. This also leads the Indians and the Greeks to have differing views on the nature of mathematical truth and mathematical objects.

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**Some Great Indian Mathematicians**

1. **Lagadha (c 1300 B.C):** The earliest mathematician to whom definite teaching can be ascribed to, and who used geometry and elementary trigonometry for his astronomy.

2. **Baudhayana (c 800 B.C):** He is noted as the author of the earliest Sulba Sutra which contained several important mathematical results; the now known Pythagorean theorem is believed to have been invented by him.

3. **Yajnavalkya (c 800 B.C):** He lived around the same time as Baudhayana and is credited with the then-best approximation to $\pi$.

4. **Apastamba (c 500 B.C):** He lived slightly before Pythagoras, did work in geometry, advanced arithmetic, and may have proved the Pythagorean Theorem. He used an excellent approximation for the square root of 2 (577/408, one of the continued fraction approximants).

5. **Aryabhatta (476-550 C.E):** His most famous accomplishment was the Aryabhatta Algorithm (connected to continued fractions) for solving Diophantine equations. The place-value system was clearly in place in his work and the knowledge of zero was implicit in Aryabhata's place-value system as a place holder for the powers of ten with null coefficients.

6. **Daivajna Varāhamihira (505-587 C.E):** His knowledge of Western astronomy was thorough. In 5 sections, his monumental work progresses through native Indian astronomy and culminates in 2 treatises on Western astronomy, showing calculations based on Greek and Alexandrian reckoning and even giving complete Ptolemaic mathematical charts and tables.

7. **Brahmagupta 'Bhillamalacarya' (589-668 C.E)**: His textbook Brahmasphutasiddhanta is sometimes considered the first textbook "to treat zero as a number in its own right." Several theorems bear his name, including the formula for the area of a cyclic quadrilateral: 16 A2 = (a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d).

8. **Bháscara (c 600 – c 680 C.E)**: He was apparently the first to write numbers in the Hindu-Arabic decimal system with a circle for the zero, and who gave a unique and remarkable rational approximation of the sine function in his commentary on Aryabhata's work. Bhaskara's probably most important mathematical contribution concerns the representation of numbers in a positional system.

9. **Mahavira (9th-century A.D)**: He is highly respected among Indian Mathematicians, because of his establishment of terminology for concepts such as equilateral, and isosceles triangle; rhombus; circle and semicircle. He asserted that the square root of a negative number did not exist and gave the sum of a series whose terms are squares of an arithmetical progression and empirical rules for area and perimeter of an ellipse.

10. **Sridhara (c. 870 – c. 930 C.E)**: He wrote on practical applications of algebra and was one of the first to give a formula for solving quadratic equations and gave a good rule for finding the volume of a sphere.

11. **Bháscara Áchárya / Bhaskara II (c 1114-1185 C.E)**: His "Chakravala method," an early application of mathematical induction to solve 2nd-order equations, has been called "the finest thing achieved in the theory of numbers before Lagrange." He conceived the modern mathematical convention that when a finite number is divided by zero, the result is infinity.

12. **Madhava of Sangamagrama (1340-1425 C.E)**: He did work with continued fractions, trigonometry, and geometry. Madhava is most famous for his work with Taylor series, discovering identities like sin q = q - q3/3! + q5/5! - ... formulae for $\pi$, including the one attributed to Leibniz, and the then-best known approximation $\pi \approx 104348 / 33215$.

13. **Srinivasa Ramanujan Iyengar (1887-1920 C.E)**: He produced 4000 theorems or conjectures in number theory, algebra, and combinatorics. Because of its fast convergence, formula of an odd-looking Ramanujan is often used to calculate

$\pi$: 992 / $\pi$ = $\sqrt{8}$ $\sum$k=0,$\infty$ (4k! (1103+26390 k) / (k!4 3964k))

14. **Prasanta Chandra Mahalanobis (1893-1972 C.E)**: He is best remembered for the Mahalanobis distance, a statistical measure. He made pioneering studies in anthropometry in India. He contributed to the design of large scale sample surveys

15. **Satyendra Nath Bose (1894-1974)**: As an Indian physicist, specializing in mathematical physics, he is best known for his work on quantum mechanics in the early 1920s, providing the foundation for Bose-Einstein statistics and the theory of the Bose-Einstein condensate

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a. http://fabpedigree.com/james/mathmen.htm; this page is copyrighted (©) by James Dow Allen, 1998-2010.
b. http://www.rbjones.com/rbjpub/index.htm
c. http://www.wikipedia.org

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12

# How Sound are our Mathematics Teachers?

## Insights from the SchoolTELLS survey

**Geeta Gandhi Kingdon and Rukmini Banerji**

**Abstract**

Teachers' pitiful performance on a test of primary school Math – as part of the SchoolTELLS survey – suggests that poor teacher competence is a plausible explanation for children's low Math achievement levels in school. The objectively measured Math skills of teachers correspond well with the subjective perceptions of teachers: about 80% of sample teachers agree to some degree with the statement -"I sometimes have difficulties in addressing mathematical queries and problems of my students". The findings have implications for both recruitment policy and for (pre- and in-service) teacher training curricula.

While there may be many factors behind the low learning achievement levels of primary school children in numeracy skills (ASER, 2005 – 2008), one potentially important factor – the possibility of low teacher competence – has received little attention in research, public debate or in education policy. While anecdotal concern has been expressed about teachers' poor skill levels and their ability to teach the content of prescribed textbooks, to our knowledge there is little systematic evidence on this issue in India

As part of the SchoolTELLS survey (Kingdon, Banerji and Chaudhary, 2008), we tested primary school teachers' cognitive skills in Mathematics (as well as in Hindi language) in 10 districts of Bihar and Uttar Pradesh in the 2007-08 school year. In Math, we measured (a) teachers' knowledge of basic arithmetic operations at the grade 4-5 level, i.e. does the teacher herself know the material that she is meant to teach; (b) teachers' ability to spot children's mistakes, and (c) teachers' ability to explain Math concepts in easy-to-follow simple steps. Assessment tasks for teachers were aligned with the standard Math teaching tasks that teachers in primary school would be required to do in the classroom routinely.

To prepare the teacher tests, we closely examined the material in the primary school Math text books in Uttar Pradesh and Bihar. For example, teachers were given common problems in percentage and calculations of area (Figure 1). These kinds of problems are in the state textbooks at Std 4/5 level.

Teachers were asked to solve the problems (test of knowledge/ability) and to clearly write down step-by-step solutions (test of ability to explain). We also gave teachers tasks that tested their ability to spot mistakes in children's work. For example, we showed teachers 3 examples of children's work in solving a division problem (Figure 2), and asked them to identify which child's solution was the correct one. The tests were marked by senior teachers through Bihar State Council of Educational Research and Training (SCERT) in Patna.

> *Pitiful teacher performance on primary school Math questions suggests that low teacher competence is a plausible explanation for children's low Math achievement levels in school.*

**Questions that test 'Does the teacher know'**

**Percentage problem**

A class has 55 children. Of these 32 have books. What percent of children do not have books?

**Area problem**

To plant a litchi tree you need 25 sq meters. Ramesh has a field that is 80 meters long and 70 meters wide. What is the maximum number of trees that he can plant in his field

Figure 1

A question that tests 'Can the teacher spot mistakes in children's work?'

[the teacher had to identify which of these three workings of the division problem is correct]

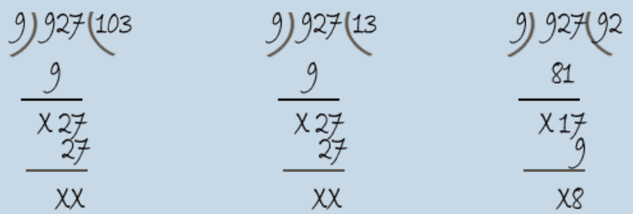

Figure 2

The findings are sobering: Only 25% of teachers could do the percentage sum (Table 1). Bihar teachers had better performance than UP teachers, and government school regular teachers performed significantly better than either para or private-school teachers (though absence rates of regular teachers – not shown here – are also much higher). But even among the best performing group of teachers – Bihar regular teachers – only 43% could do the percentage sum correctly, suggesting large skill deficits to impart primary school Math.Only 28% of teachers could do the area sum (Table 2).

Government school regular teachers' performance was better than para teachers' (and in UP, vis a vis private school teachers). Even so, only 39% of regular teachers in Bihar and 30% in UP could do the area sum correctly. However, the performance of different teacher types (regular, para, private) was more similar to each other in the 'ability to explain' and 'ability to spot mistakes' areas. This meant that in their total Math score they did not differ from each other so much as in the Math 'knowledge' area that was tested only through performance in the percentage and area sums.

Ability to explain in Math was adjudged low because many teachers were not able to show solutions in clear systematic steps. Ability to spot mistakes was better, but still imperfect: 15% of regular teachers and 26% of para teachers could not correctly identify which one of the three children's workings of a simple division sum (927 divided by 9) was correct.

Such pitiful teacher performance on primary school Math questions suggests that low teacher competence is a plausible explanation for children's low Math achievement levels in school.

**Table 1**

E.g. in UP, the absence rate of regular teachers is 25% and that of para and private school teachers is 12% and 17% respectively. Similarly, the mean salary of regular teachers in UP (about Rs. 12,000 per month in Jan. 2008) was about four times the para teacher salary (Rs. 3000 pm) and more than 12 times the private school teacher salary (Rs. 940 pm). This extreme pay-inequality was further exacerbated following implementation of Sixth Pay Commission salary scales in UP in 2009 whereby regular teachers' starting salary rose to Rs. 18,000 per month. Kingdon (2010 forthcoming) estimates that in UP the ratio of regular teacher pay to state per capita GDP is 17: 1, while showing that the average of this ratio for developing countries is 3:1.

**Teachers' performance on the percentage sum question**

| | Bihar | | | | UP | | | All |
|---|---|---|---|---|---|---|---|---|
| PERCENTAGE PROBLEM | Reg. | Para 05 | Para 06 | Priv. | Reg. | Para | Priv. | |
| Not attempted | 14.4 | 12.0 | 26.4 | 37.0 | 16.7 | 23.5 | 28.6 | 20.6 |
| Incomplete | 32.7 | 48.8 | 46.2 | 25.9 | 40.0 | 40.0 | 54.6 | 42.6 |
| Wrong steps & wrong answer | 5.8 | 6.4 | 5.5 | 11.1 | 10.0 | 3.5 | 1.3 | 5.7 |
| Correct steps, wrong answer | 3.9 | 6.4 | 3.3 | 3.7 | 4.4 | 7.0 | 1.3 | 4.6 |
| Only correct answer, no steps | 0.0 | 1.6 | 3.3 | 0.0 | 1.1 | 4.4 | 2.6 | 2.1 |
| Solved correctly | **43.3** | **24.8** | **15.4** | **22.2** | **27.8** | **21.7** | **11.7** | **24.5** |
| % of teachers struggling with the task (rows 1 to 3) | 52.9 | 67.2 | 78.1 | 74.0 | 66.7 | 67.0 | 84.5 | 68.9 |

**Table 2**
**Teachers' performance on the area sum question**

| | Bihar | | | | UP | | | All |
|---|---|---|---|---|---|---|---|---|
| AREA PROBLEM | Reg. | Para 05 | Para 06-07 | Priv. | Reg. | Para | Priv. | |
| Not attempted | 27.9 | 28.8 | 38.5 | 51.9 | 30.0 | 48.7 | 41.6 | 36.6 |
| Incomplete | 19.2 | 25.6 | 26.4 | 7.4 | 18.9 | 19.1 | 26.0 | 21.8 |
| Wrong steps & wrong answer | 5.8 | 4.0 | 1.1 | 3.7 | 7.8 | 3.5 | 2.6 | 4.1 |
| Correct steps, wrong answer | 3.9 | 3.2 | 8.8 | 0.0 | 4.4 | 1.7 | 5.2 | 4.1 |
| Only correct answer, no steps | 4.8 | 5.6 | 3.3 | 0.0 | 8.9 | 4.4 | 9.1 | 5.5 |
| Solved correctly | **38.5** | **32.8** | **22.0** | **37.0** | **30.0** | **22.6** | **15.6** | **27.9** |
| % of teachers struggling with the task (rows 1 to 3) | 52.9 | 58.4 | 66.0 | 63.0 | 56.7 | 71.3 | 70.2 | 62.5 |

Apart from measuring primary school teachers' competence in imparting numeracy skills, we also asked teachers about the extent to which they agreed with the statement “Sometimes I have difficulty in addressing the Math queries and problems of my students”. Table 3 is self reported data. It shows that only about 18% of government school teachers in Bihar and 22% in UP say they disagree with the statement, i.e. about 80% of teachers admit that they have some difficulty in addressing the Math queries and problems of their students. Of these, 25 percentage points of teachers in Bihar and 15 points in UP fully agree with the statement.

**Table 3**
**Percentage of teachers who say they agree with the statement**
**“I sometimes have difficulties in addressing Mathematical queries and problems of my students”**

| | BIHAR | | | | UP | | | |
|---|---|---|---|---|---|---|---|---|
| | Fully agree | Partially agree | Somewhat agree | Dis agree | Fully agree | Partially agree | Somewhat agree | Dis agree |
| Govt. school teachers | 24.5 | 11.0 | 46.8 | 17.7 | 15.2 | 18.3 | 43.1 | 22.3 |
| Private school teachers | 16.7 | 12.5 | 45.8 | 25.0 | 16.9 | 18.5 | 36.9 | 27.7 |

### Implications of the findings

Content knowledge of the material in the primary school textbooks is not tested as a criterion for teacher recruitment for primary schools. This is presumably because it is assumed that teachers' educational qualifications and pre-service teacher training will ensure that they have adequate knowledge and skills for teaching primary grades, or because it is assumed that any deficits in such skills can be plugged later, via in-service training. .

However, these assumptions seem risky and untrue, in light of our finding that (a) competency scores are low even among the 'best' group of teachers – the government-school regular teachers who mostly have BA and MA qualifications as well as pre-service teacher training. The assumption is also risky in light of evidence (not presented here) that teacher competency scores were only weakly related to teacher educational qualifications and pre-service training.

Such weak correlation could arise if there was much variability in the quality of education/training received by different teachers.

While teachers may fear and oppose testing – especially if it is high-stakes (i.e. linked to pay, promotion or contract renewal) – it is inappropriate to subject children to teachers who themselves cannot tackle the textbooks they are meant to teach.

Our findings have important policy implications. Firstly, the skill deficits identified through tests can usefully guide the future pre-service training curriculum of teacher training colleges. Secondly, teacher tests can assist with future recruitment by helping to identify individuals who are competent to teach. Thirdly, teachers should welcome testing as it will reveal their in-service training needs and give them an opportunity to upgrade their skills before they are tested in a high-stakes way. Lastly, there is no provision for subject-specialist teachers in primary grades in most Indian states but it may be useful to make an exception in the case of Math since this is an area of particularly weak skills amongst teachers and it may be a difficult area in which to upgrade the skills of all teachers.

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# section C

## In the Classroom

# 13 Meaningful Teaching of Mathematics

**Devika Nadig and Vijay Gupta**

One of the major issues with teaching of Mathematics has been that students remain procedural; for instance, they know the algorithm of addition or multiplication but do not know whether solving the problem requires them to add or multiply. Therefore, the challenge before us is to move a student from being 'procedural' to 'proceptual' – the latter meaning procedural plus conceptual, where conceptual indicates understanding and application. Proceptual students display five attributes – let us look at each one of them.

**First, proceptual students know the procedure and also understand the concept.** Let us illustrate using the familiar procedure of multiplication:

```
   25
 x 25
 ----
  125
  50x
 ----
  625
 ----
```

Across different states in India students are told either to put a cross in the unit place, or simply leave a blank, or put a zero. Most of the students follow the procedure mechanically without knowing the 'why' of the procedure. This is what we mean by understanding of the concept.

**Second, proceptual students use the most efficient strategy to solve the problem.** In the following sum:

```
  299
 + 21
 ----
  320
 ----
```

If the student realizes that the problem actually is 299+1 is 300, and plus 20 will make it 320, then she would know that this is a far more efficient strategy than going through the procedure of addition.

**Third, student is able to check if their response is reasonable.** In the following division:

212 divided by 2

If the students gets the answer as 16 instead of 106, she should be able to see that 16 is not a reasonable answer, and that the answer should be somewhat more than 100.

**Fourth, proceptual students have a range of known facts.** In the earlier example, that the student knows that 9+1 is 10 and 300+20 is 320 are known facts. Proceptual students have more such known facts. This is one reason why knowing your multiplication tables (with conceptual understanding) is a good idea.

**Lastly, proceptual students use known facts to derive other facts.** In the same example, 299+21 is 320 is now a new fact derived from known facts.

Having understood the many facets of proceptual students, the challenge now is - how should we teach so that every student of Mathematics becomes proceptual. Here are five teaching strategies which we have found effective in our experience of working with teachers.

> *"It is always a good idea to use more than one method to draw/represent concepts so that the teacher can be reasonably sure that the learner has understood and is comfortable with the concept."*

**First, teach from a base of concrete experience.** Mathematics is inherently abstract – for instance, as soon as we write 5 to denote five mangoes the learner is making one degree of abstraction. Therefore, as a first step, learner should become comfortable with concrete before making the leap of abstraction. Every learner will take her own time, and we need to cater for this time.

**Second, learner should draw or represent concepts.** Let us illustrate this strategy through an example. A learner of Mathematics goes through the idea of fractions as one of the first difficult concepts. This strategy urges the teacher to get the learner to draw or represent the idea of fraction in a variety of ways. During our work with teachers we use three different ways of representing fractions. One, we use folding of paper strip to represent fractional numbers. Two, we get them to shade required number of boxes out of the given boxes of equal size to represent a fraction. Third, we get them to mark a given fraction on the number line. The following figures show the three methods of illustrating the fraction 2/5:

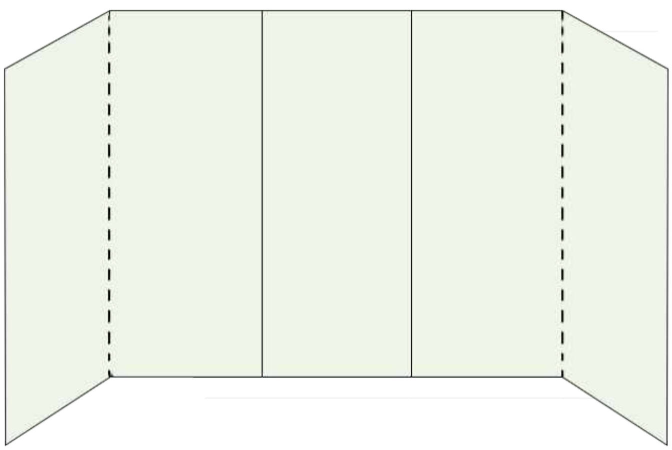

**Figure 1**

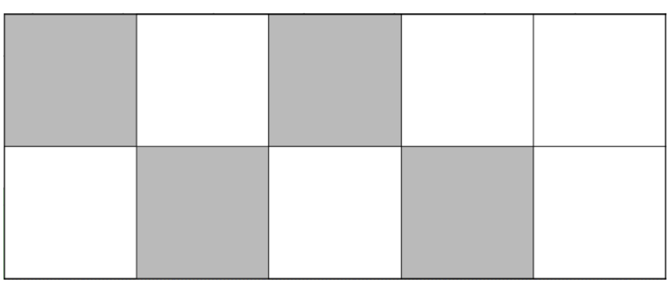

**Figure 2**

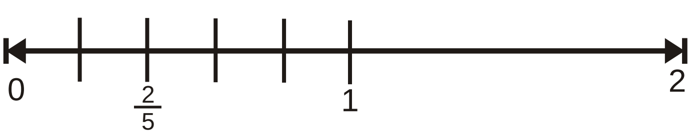


**Figure 3**

It is always a good idea to use more than one method to draw/represent concepts so that the teacher can be reasonably sure that the learner has understood and is comfortable with the concept.

**Third, teacher should verbalize her own strategy while solving a problem.** Let us use the earlier problem of division to illustrate this idea. The teacher, after writing 212 divided by 2 in the usual division format, could articulate her strategy by saying – "I first divide the left most 2 of 212 by 2, and the dividend is 1; I subtract 2 from 2 and get zero; I then copy the next digit of 212 which is one; since 1 is less then 2, 2 cannot divide 1 even once and therefore I write zero as the dividend; I then copy the right most 2 and the new number to be divided is now 12; 2 is able to divide 12 completely 6 times, and therefore the answer we get is 106; as you can see if the number was 200 we would have got 100 as the answer, but since the number is 212, which is a little more than 200, we expect the answer to be little more than 100, and therefore the answer seems to be reasonable". By verbalizing her own strategy, teacher models the thinking process for the learner, and also helps the learner start thinking and articulating her own strategy.

**Fourth, teacher should use alternative solution strategies to solve a problem and get learners to do the same.** Also, get learners to think about which strategy is more efficient. Let us take a simple word problem – "A group of 50 teachers, who are undergoing training, require 10 chart papers for an activity. How many chart papers will be required if 1500 teachers have to undergo training in batches of 50". Now this problem can be solved in a variety of ways. One method could be to work out how many chart papers are required for one teacher (1/5) and then multiply it by 1500 to get 300 as the answer. Second method could be to see that 1500 teachers mean 30 batches of 50 teachers each; further since one batch needs 10 chart papers, 30 batches would require 300 chart papers. Third method could be to use the idea of ratio. If 10 chart papers are required for 50 teachers, how many chart papers would make it the same ratio for 1500 teachers? This would also yield the answer 300. By sharing different methods and discussing their comparative efficiency, we are making the learner become comfortable with concepts.

**Fifth, teacher should use the following two questions frequently to engage the learner in**

**thinking through her response:**

1. How did you do that?
2. How do you know you are right?

While the first question forces the learner to articulate her strategy to solve a problem, the second question makes her defend the answer to be a reasonable one. These questions need to be asked of each student even when the answer is right.

We have found in our work with teachers that many of them are not comfortable with the idea of using alternative solution strategies. They feel that it would confuse the learner, and therefore we should stick to one method. We believe that this kind of thinking is what makes the learner a procedural thinker because what she learns is that there is only method to solve a problem and that method becomes for him an algorithm. Therefore, the primary challenge is to make teachers themselves proceptual thinkers in Mathematics.

Meaningful teaching of Mathematics is all about ensuring conceptual understanding for every learner, sharing your own thinking while solving a problem and getting the learner to share hers, and stretching the mind of the learner by solving a problem using alternative strategies.

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## Logico- Math Brain Teasers

Three players P, Q and R are participating in a game and made to stand one behind the other in a such a way that P can see both Q and R; and Q can see only R while R can see neither. There are a total of 7 caps, 5 of which are Blue and 2 are Red. One cap each is placed on the heads of these three people from among these 7 caps. First, P is asked if she can tell the colour of the cap on her head. She answers 'No'. The question is repeated with Q who too answers in the negative. When the question is asked of R, she says Yes and proceeds to tell the correct colour. Assuming all three can think logically and they can hear the answers given by the others, what is the colour of the cap on R?

*(Hint: Work out all possible combinations)*

Use this space for calculation 🙂

# 14 Concept Attainment Model

**Arun Naik**

A number of critical themes are emerging these days about the types of teaching practice that impact positively on student achievement, confidence and engagement in Mathematics. They focus teaching on the learner and encourage teachers to introduce content in ways that facilitate enhanced cognition.

Concept Attainment is an instructional strategy that uses a structured inquiry process. It is based on the work of Jerome Bruner. In concept attainment, students figure out the attributes of a group or category that has already been formed by the teacher. To do so, students compare and contrast examples that contain the attributes of the concept with examples that do not contain those attributes. Students then separate the examples into two groups. Concept attainment, then, is the "search for and identification of attributes that can be used to distinguish examples of a given group or category from non-examples".

> *Concept attainment requires a student to figure out the attributes of a category that is already formed in another person's (teacher) mind, by comparing and contrasting examples that contain the characteristics (attributes) of the concept with examples that do not contain these attributes.*

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Before we move on, it will be worthwhile to understand – what is a concept? "A concept is a class of stimuli, which have common characteristics". The learning of concepts involves successfully identifying these common characteristics that defines them as a category. On the contrary, in our schools, most of the times concepts are explained, at the most with suitable examples and students are made to remember the concept by rote. Students never get the idea of what attributes formed a particular concept and such clarity remains alien to them. Students never get an opportunity to form their own concepts based on certain attributes of their own.

The Concept Attainment Model (CAM) tries to address this issue and provides ample opportunity for a child to explore the attributes of a concept. It engages students and encourages them to form the concept by using illustrations, word cards or specimens called examples. Also this model ensures that teacher starts from the student's previous knowledge. In this approach students go beyond merely associating a key term with a definition. Hence the concept is learned more thoroughly and retention is improved. This approach can be used effectively for teaching Mathematics, as the study of Mathematics involves the study of many concepts.

Following steps help us use this model effectively in the class room:

1. Select a concept and analyze the attributes
2. Develop examples and non examples for each of the attributes
3. Introduce the process to the students
4. Present the examples sequentially
5. Allow children to create hypothesis and verify their hypothesis by themselves
6. Develop a concept definition
7. Ask for additional examples
8. Discuss the process with the class
9. Evaluate

Here's an example of how this will actually transpire in a classroom:

1. Teacher chooses a concept to be developed, for e.g. Prime numbers.
2. Teacher identifies and defines the attributes – A number divisible only by itself and 1.
3. Teacher develops examples and non examples for the concept; writes them on flash cards. Examples are 2, 3, 7, 11 etc.

   Non examples are 4, 6, 12, 25, 9, 15 etc.
4. Teacher designates an area for writing examples and non examples or use a chart paper with two columns and have two columns YES and NO.
5. The teacher instructs the students, "I have a concept in my mind. I will present examples of the concepts and also non-examples of the concepts one by one. The examples are written under YES column and the non-examples are written under NO column. Look at the examples under the YES column and discuss in your groups what do they have in common. You will have to find out the concept in my mind". Such an engagement soon turns into an investigative game for the students.
6. Teacher presents the first card by saying "this is YES". Place it under 'YES' column. Eg 2.
7. The teacher then presents the next card and says "This is NO". Place under 'NO' column, eg 9.
8. Similarly present two more examples and non examples, each time one example and one non example are presented by the teacher
9. The students compare and contrast those that are in the same group and those that are in the different groups, attempting to determine the rationale that was used for the classification. As the students are comparing and contrasting, they will develop different hypotheses by discussing in their small groups.
10. Teacher asks the students to share their guess. (You may get responses like – even numbers, multiples of two etc). Teacher just accepts their response, does not comment on them at this stage.
11. Now teacher presents next example and non example, 3 and 7 respectively. This will provide students opportunities to test their hypotheses by further examination of the examples.
12. A few more examples are presented and the students are asked to guess. Teacher does not provide any clues, she only presents as many examples until they are able to identify the attribute of the concept.
13. Once students identify all the attributes of the concept they will be in a position to define the concept. Teacher asks them to share it aloud with the whole class. (Students may respond - the number in your mind is not divisible by other numbers.)
14. The teacher can help them arrive at right definition with the help of further questioning and names the concept.
15. Teacher asks students to generate additional sets of examples for the concept.
16. Once this is done, the teacher questions the students on the thinking process they followed. What was the hypothesis they generated? What was the rationale behind it? What were the hypotheses they eliminated and why?
17. Now evaluate the concept attainment by children. The teacher may ask them to classify a given set of numbers into Prime numbers and non prime numbers.

Here are some hints to following CAM:

1. Examples and non examples presented in the initial stage should be such that they lead to many different hypotheses. This will make students check their own hypothesis against the examples presented in the later stage and eliminate their own hypothesis.
2. Sequencing of these examples keeping the above in mind is crucial.
3. Examples can be presented using power point or OHP or flash cards or on the black board.
4. Students can work individually or in groups while working on the examples given by the teacher. CAM can be an individual activity or can also be done in small groups.
5. It is important that the students identify the concept attributes and not the name of the concept.
6. Don't discourage or respond saying 'yes/no' to the guesses children make in the initial stage. Let them

develop their own hypothesis and eliminate the wrong ones based on the examples presented at the later stage.

7. If a concept is defined by sub concepts (for example, polygon is defined in terms of - plane figure, closed figure, many sided etc), using this model may prove to be very tedious. It is good to couple it with a guided discovery.
8. While students are at work, the teacher should meander through the classroom. During this time, the teacher could make anecdotal records or fill in checklists of student actions.
9. Give the students enough time to develop their definitions for each category/formulate the hypothesis.
10. The teacher can even label your examples as YES or NO.

**Conclusion**

The Concept Attainment Model has many advantages over traditional methods of teaching. It develops information processing skills in children. Students become better analytical thinkers, their critical thinking sharpens as they have to describe their thinking, and students also become more articulate in their descriptions (of their thinking). If used judiciously this model can help children learn Mathematics with joy, with more clarity.

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# Dialogue in Math Teaching: Realities and Challenges

**Ekta Sharma**

## Understanding the Problem

**"It is more useful to know how to Mathematise than to know a lot of Mathematics**" said David Wheeler. Perhaps this sums up the entire problem of Mathematics teaching-learning succinctly. Knowing a lot of Mathematics implies being able to do a variety of computations according to a set procedure that is learnt. Mathematization on the other hand involves the ability to apply Mathematics according to the need of the situation- to be able to think mathematically. Descriptive problems in Mathematics could offer this kind of opportunity to mathematize. Let us attempt to unfold this issue by looking at the scope that Mathematics provides for interaction and dialogue and the extent to which this opportunity is used in teaching learning.

Word Problem- "Seema went to the market with some money. She spent Rupees 150 in buying fruits. She now has Rupees 100 left. How much money did she have when she went to the market?"

Very often when this kind of a word problem is given to children, they get confused about which operation to use. Hence if we try and understand why children might be getting confused, we would be in a position to acknowledge the challenges children might be facing in solving descriptive problems.

> *It needs to be recognized that Mathematics is a language in itself with its own set of symbols, and like any other language, it must be made meaningful rather than just decoded.*

Sometimes we find that children use some phrases like "how many/much in all" or "how many/much left" as cues to decide on the operation they will use – such as addition, subtraction etc. Now the problem given above clearly uses a phrase that could lead the children towards subtraction. Besides the phrasing, the context and setting of the problem is the next big challenge for children. Sometimes if the context is irrelevant to the child, it takes away all the fun of solving descriptive problems and makes the whole exercise mechanical.

Here is another example: "There are 86 pages in the Math textbook of class 3 and 75 pages in Hindi textbook of class 3. How many pages are there in both textbooks put together?"

Why would any reasonable person want to know how many pages are there in both textbooks put together. Instead if a more realistic context is given, it might attract the attention of the child who would then attempt to solve the problem meaningfully. Here's another example-"Deepa has a habit of recording her daily expenses. Today she spent Rupees 86 on fruits and Rupees 75 on vegetables. Help Deepa by calculating how much money she spent today."

In the above example, with very little increase in the 'language' content, the whole idea of the problem situation gets transformed into a more meaningful exercise. Clearly worded, such language problems would go a long way in changing the way Math is viewed, taught and learnt at the primary level.

The medium of language and the context in which the problem is set is perhaps the primary challenge children face in solving word problems. However there are other issues too. Many a times, children who are fairly conversant with the language also struggle to solve such problems. This clearly indicates that the 'problem' is more deep-rooted than the difficulty with of the medium of instruction. Let us take some examples of simple algorithm based questions to illustrate this point.

| | | |
|---|---|---|
| 24<br>+ 32<br>-------<br>------- | 24 + 32 = ……. | 24 and 32 add up to.. |
| (CASE 1) | (CASE2) | (CASE 3) |

While the first two questions purely comprise mathematical symbols or are pure mathematical expressions, the third one is a verbal expression of CASE 1 and 2.

CASE 1 depicts the most usual format in which addition problems are presented to learners all over the country. CASE 2 is a minor alteration in the format with the inclusion of a new sign i.e. “equal to”, but when one reads verbally, states the problem more precisely (than CASE 1) in mathematical terms- “ twenty four plus thirty two equals ....). It has been observed that children fairly conversant in addition as given in case 1 and with limited or no exposure to case 2 are either unable to understand how to solve CASE 2 or mix-up the procedure, i.e adding units to tens etc.

This shows that the problem is not entirely with the medium in which a question is asked. Even in the case of a pure mathematical expression children face difficulties in understanding if they are presented in a modified manner.

CASE 3 is a different kind of situation. Here too the language used is not the language that we typically use in our day to day lives. It also involves some abstraction like in case 1 and 2 but at the same time also spells out the operation to be used to solve the problem. It involves a mathematical expression that is written using terms that are mathematical for instance 'add up to' or sometimes '2 and 2 make 4' This kind of terminology does not form a part of the language that is used as a medium which, in this case is English. These are mathematical terms but many a times we tend to overlook this important fact. These terms are a sort of verbal expression of mathematical language. In the absence of introducing children to these terms and phrases as a way of conversing mathematically, problems that involve their use are more confusing for children.

It needs to be recognized that Mathematics is a language in itself with its own set of symbols, and like any other language, it must be made meaningful rather than just decoded. Just knowing symbols and procedures will certainly facilitate computation but it does not help children think mathematically until opportunities for understanding the meaning are provided.

**Key issues**

Why do children find it so difficult to solve word problems? The NCF, 2005 strongly articulates the centrality of meaning in learning. It argues, in no uncertain terms that meaning-making is a necessary pre-condition for learning which is relatively permanent in nature.

Having said that, let us examine whether the way Mathematics is being introduced and taught to children meets this condition. Taking some examples from textbooks used in government primary schools we notice that :

1. The pre-mathematical concepts of big/small, identifying patterns, recognizing similarities and the likes do not get adequate space
2. Even where they do, they are not really made use of in understanding numbers.
3. The numbers and their verbal equivalent for eg. '3' and 'three' are both introduced simultaneously.
4. Connections with day to day life are not very strong and apt
5. The scope for practice has been significantly reduced in textbooks as more space is given to concept introduction. While this is a good thing as it attempts to focus on conceptual understanding, in the absence of alternative practice material like workbooks etc., it adversely impacts children's learning. (Here I suggest that practice also has a place in learning Mathematics). Practice in solving problems with conceptual clarity is indeed essential for learning the subject as Mathematics is all about developing a certain way of thinking and reasoning – this definitely requires practice.

Considering the reality that textbooks almost entirely guide teaching-learning, it would be reasonable to say that the teaching of Mathematics is also characterized by the above mentioned points.

Added to this, the fact that children fare better in algorithm based problems like CASE 1 (shown in previous page) when compared not just to descriptive problems but even slight alterations in form (CASE 2), it is easy to see that the difficulty is rooted in lack of understanding of the nature of Mathematics as a subject. The key issue appears to be the misplaced emphasis on following a set procedure to arrive at the correct answer. There is no emphasis on allowing the children to engage with the problem, to identify different ways of solving a problem, be it a simple algorithm based one, to articulate the procedure used and the reasons/logic behind it.

## Looking ahead

Simply put, just as for learning any language needs to be contextualized, likewise Mathematics, when introduced at the primary level, needs to be put in a context. Teaching-learning should be organized so as to provide realistic problem situations. Children must be encouraged to find multiple solutions in place of just asking them to arrive at the correct answer. In order to do this, the descriptive problems presented should be in a manner that allows dialogue to take place. For example, instead of asking children to do a simple addition, ask them to find out 3 different ways of adding the given numbers; or even give them answers to which they are required to create questions.

There is need to bring this kind of dialogue and interactivity into Mathematics in order to make it a more meaningful learning experience for children.

This is a big challenge as Mathematics has always been used as a discriminatory tool in learning situations. It has evoked fear and anxiety and even led many young learners to take extreme steps. Even teachers feel that some children are just not upto the challenge that Mathematics offers. That girls are not as mathematically oriented as boys is also a common myth.

These beliefs are unfounded and it is important for all of us to recognize that **"Every child can learn Mathematics and all children need to learn Mathematics".**

All that is needed is to teach the subject in a manner that it evokes interest and allows for deriving meaning rather than just confusing the learners. It is time to remember what Bruner said with regard to the suitability of subject matter for learning; **"Any subject can be taught effectively in some intellectually honest form to any child at any stage of development".**

**Mathematics, I believe, is no exception!**

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## Logico-Math Brain Teasers

There are three containers on the table, one is of 800 ml and is full of Apple juice, The others are 500 ml and of 300 ml size respectively. Rajdeep wants to measure exactly 400 ml Apple juice for a recipe. How can you help Rajdeep solve this problem?

Use this space for calculation 🙂

# 16 The Role of Concrete Experiences in Learning Primary School Math

**Meena Suresh**

Traditional patterns of Math education in schools have largely remained unchanged for almost a century now, even when other things like our culture, beliefs and way of life have tremendously changed. Till date, Math learning in schools is synonymous with rigorous mundane practice. Teaching in schools has lacked focus especially in primary classes. Math has largely been a confusing subject with teachers themselves not knowing what Mathematics actually is; they remain blissfully unaware of the pedagogical issues relating to the subject. Teachers are ill prepared because of this lack of understanding and as a result the confusion continues and prevails throughout the students' school days. It is a paradox that even high scorers in school Math are unprepared for college level Math. Mushrooming tuition centers stand a testimony to this fact. This lack of understanding has profound effect even in the tasks that a person does as a common man, i.e. tasks like finding the percentage or calculating taxes. This leads to the discussion is exam Math indicative of mathematical understanding?

NCTM (National Council of Teachers of Mathematics) has identified two categories as standards for learning Math. The standards are (a) thinking Math standard (b) content Math standard. The 'thinking Math' standard focuses on the nature of mathematical thinking (i.e.) problem solving, communication, reasoning and connections. The 'content Math' standard consists of specific Math topics such as number sense, estimations, geometry, measurement, statistics, probability, fractions, patterns and relationships etc. While content of Mathematics is very important, it is equally important that all of the 'thinking Math' areas be woven into the content. The whole experience of learning Math at school should turn out to be one fabric for the learner in which students can paint their experience to discover new ideas and develop mathematical thinking. Mathematical skills are often cumulative in nature, one skill building upon the previously learned one. For eg algebraic concepts cannot be imbibed without proper understanding of basic arithmetic. This being true lack of understanding of basic concepts has a far reaching and cumulative effect on students. It is essential that primary students have clear understanding of the basic concepts and not just memorize formulas and facts at the dictate of a teacher.

Experiential learning provides concrete experience and ensures maximum student involvement in the process. This type of learning focuses on understanding and enhancement of critical thinking, as students learn to plan, act, discuss, communicate and conclude by themselves. This is especially true of Mathematics, as the subject is abstract.

> *In a typical class, the teacher assumes the role of a ring master and gives step wise instruction to the class with little room for any exploration. Instructions like "take one red link, put it on the table. Take two red links, join them and lay them next to one" and "do as I say". A sensitive teacher would put the manipulatives to better use by withholding instructions like 'use red links' or 'put two links next to one' and so on.*

One practical way of providing this experiential learning to students is the use of 'manipulatives'. Manipulatives are objects that can be touched and moved by students. These can be used to either introduce or reinforce particular Math content. While use of manipulatives will ensure active involvement of students, it is the teachers' role to design activities which will help in delivering the content and also achieve requisite levels of 'thinking Math'. Research suggests that manipulatives are especially helpful in learning Math, as they help the learner move from concrete to abstract levels. Class activities should be carefully designed and structured such that they form a bridge between the concrete and the abstract. It is very important to note that just the use of manipulatives will not produce the desired outcome unless experiences using the manipulatives have been carefully designed. This brings us to the practical question of why, when, what, how and with whom manipulative material should be used. While it is beyond the scope of this article to discuss all the questions asked above, it is certain that a teacher can ask the question to oneself and answers are not difficult to find or arrive at.

We shall discuss experiential learning in the perspective of optimal use in class rooms. The experiences using manipulatives should be such that students draw their own conclusions and achieve mathematical insight. The approach should be explorative and teachers should remain guides or facilitators, permitting students at times to even attempt tasks that seem to be ridiculous or non productive. To take the discussion further, let us take a look at what typically happens in a class room.

**Situation 1: Introduction of number sequence using links – links being used as external representation**

In a typical class, the teacher assumes the role of a ring master and gives step wise instruction to the class with little room for any exploration. Instructions like "take one red link, put it on the table. Take two red links, join them and lay them next to one" and "do as I say". The activity gets completed with the teacher having satisfied herself of delivering the content, but, for the student, the activity is abrupt and its for sure the student has not imbibed any of the 'thinking Math' content.

A sensitive teacher would put the manipulatives to better use by withholding instructions like 'use red links' or 'put two links next to one' and so on. This teacher would give minimum instruction and allow students to internalize the concept by allowing colour choices or mixture of colours, while remaining focused that students need to make strings of various lengths ,arrange them and discover for themselves the number sequence and arrive at 'one more gives the next number' .

**Situation 2: Simple addition – counters being used as representation of computational algorithm**

When a teacher attempts teaching addition (eg. 3 + 2 makes 5), students pick 3 red counters and 2 white counters and count all of them to arrive at the answer 5. In this activity, manipulatives were used as representations. Such experiences are a very good starting point; but, soon the teacher should move on to the next level where manipulatives are a reference point to student's 'thinking ways' to solve mathematical problems. Students should explore and find for themselves the commutative or the additive identity (addition of zero) property of addition. Teachers as facilitators should only help in students knowing the symbols and communication associated with it.

**Situation 3: Word problem – place value manipulatives used as external representations of thinking process**

Given word problem: A school has 156 boys and 212 girls. How many students in all?

To solve this problem, students use 'place value manipulatives (The place value manipulative has small squares representing 1s, strips as 10s and big squares of 10s strips bundled together as 100s). Here, the manipulatives are used as an external representation of a thinking process or sequence (i.e.) students imagine the manipulative to be external representation of her thinking process and not use it just for doing the operation. This means that students know what they have to do and use the materials only as a support to arrive at the answer. These experiences are fine but lack mathematical insight. Mathematical insights have to be one's own realization.

**Situation 4: Tables – sticks placed in crisscross pattern used as external representation of a computational algorithm**

Students use 'tables manipulative' (sticks placed in crisscross patterns to learn tables). This experience will remain as an activity where the manipulatives are used as an external representation of a computational algorithm unless students are made to realize the mathematical idea that multiplication is a short form of repetitive addition and also see how numbers grow while multiplying (i.e.) be able to visualize that 6 X 4 is 6+6+6+6 and also how big 6 would grow into. In such experiences, the manipulatives are used to support structural elements of the concepts.

Again, this is an incomplete mathematical perspective. Experiences which encourage discussions and explanations leading to deductions are the best way to support Math learning. (Note: I am tempted to share my experience with a class of young gypsy students, all school drop outs in a school run by a NGO in Kancheepuram. I was stunned to find a student in my class explaining to his mate what they were doing in the class as 'tables'. He explained saying multiplication is just the same as selling chains. He went on to explain – "how do we find the cost of the articles? Let us say if we sell –one chain which costs Rs. 6, then, 4 chains would cost Rs. 24. This comment is significant considering the fact that the students were still grappling with numerals and had no exposure to Math symbols like × or ÷)

**Situation 5: Understanding area – square tiles used as external representation of thinking process and later as a reference point for discussion and exploration**

The manipulatives are used by the teacher to design and experience where by arranging tiles into square and rectangles, the students deduct L X B = Area. This is the bear minimal learning that this experience can offer. This experience attains significance only when students are able to realize that not only square of areas 4, 9 or 16 square units can be formed, but, squares of any area is possible (provided the manipulatives are not a limitation). In this experience, manipulatives are not only used to support conceptual understanding, but, also share the understanding. The discussion that the class has should lead the class to deduce that it is possible to have rectangles of different perimeter with area being constant. Arranging and re-arranging and after several attempts, the students may also arrive at the hidden agenda that for a given area, square shapes are most compact with least perimeter. This would be optimum use of the manipulatives.

From all the above examples cited, it is clear that experiential learning is not just synonymous to an activity using manipulatives. Teachers have to take the students through a series of activities where initially manipulatives are used as representations and then as external representations of a thinking process and later as a reference point for discussion and exploration. It is to be noted that initially, the experiential learning focuses on individual students and gradually shifts to the class as a whole. The way in which a manipulative is used rests wholly on the teacher initially and it is highly teacher dependent. This gradually should become the complete responsibility of students. This is when 'thinking Math' gets incorporated into 'content Math'.

Experiential learning especially in primary classes with manipulatives is good. But, that is only the beginning, just one time use of the manipulatives or using it to show case it once with lot of instructions and unfocussed writing practice will not help in achieving 'thinking Math' standards.

Careful choice of manipulatives combined with intelligent planning by the teacher with optimal instructions, and discussions that follow after using the manipulatives are sure to work wonders with young learners leading to intellectual and independent learning with desired outcomes.

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# The Value of Drill and Practice in Math: A Perspective

Uma Hari Kumar

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The figure gliding over the skating rink at the Olympics, the zeal of Chinese acrobats, melodies flowing like rivulets from Pandit Ravi Shankar's sitar or Bismillah Khan's shehnai - all of these mesmerize you. All these people have something in common to have reached the pinnacle of success — talent and practice.

Having said that, should drill and practice be a part of the Mathematics teaching learning process?

Mathematics is about playing with numbers. The more familiar one is with numbers and what they represent, the easier it is, to see relationships that exist between them. Hence, it is important that children learn to count and are able to identify the number of things in a group either by counting or by patterns. In Mathematics is a type of understanding typically called number sense. It is generally agreed upon that it involves an awareness of number names, values, and relationships. Children with number sense recognize the relative differences in number quantity and how those differences can be represented. Number sense gives meaning both to an automatic Math fact and to a computational procedure. Gersten and Chard roughly compare the importance of number sense in computation to the need for phonemic awareness in reading (Gersten & Chard, 1999). Both are critical building blocks. The ability to recall basic Math facts fluently is necessary for students to attain higher-order Math skills. Garnett describes a typical hierarchy of procedures, or strategies, which rests upon number sense and leads eventually to automatic recall (Garnett, 1992). “The only way to learn Mathematics is to do Mathematics” said Paul Halmos.

“Cognitive psychologists have discovered that humans have fixed limits on attention and memory that can be used to solve problems. One way around these limits is to have certain components of a task become so routine and over-learned that they become automatic.” Whitehurst, 2003)

What do all of above research findings imply for Mathematics?

It would be helpful if some of the sub-processes inproblem solving, particularly basic facts were developed to the point that they are done automatically. If a student constantly has to compute the answers to basic facts, less of that student's thinking capacity can be devoted to higher-level concepts than a student who can effortlessly recall the answers to basic facts. For example, if a child performing multiple-digit division consistently has to use his fingers to subtract or cannot recall multiplication facts during the division process, the attention and memory resources devoted to these procedures reduce the child's ability to monitor and attend to the larger division problem. The result is that the child often fails to grasp the concepts involved in multiple-digit division.

> *“It was in Class 8 that I had to teach Hero's formula - s(s-a)(s-b)(s-c).We made up a song “sun, sun meri aasha, sun meri bhasha, sun meri champa re “where sun stood for s, asha, bhasha and champa stood for a, b, c and meri stood for the subtraction sign. The class rocked.”*

If this fluent retrieval does not develop then the development of higher-order Mathematics skills — such as multiple-digit addition, subtraction, long division, and fractions — may be severely impaired. Lack of Math fact retrieval can impede participation in Math class discussions, successful Mathematics problem solving, and even the development of everyday life skills. Besides all this it can create a lack of self-confidence in the child and also subject her to peer ridicule. Rapid Math-fact retrieval is an asset to perform well in Mathematics achievement tests. Studies in cognitive science also support continual practice, because it develops computational automaticity—it increases retrieval speed, reduces time required for recognition, and decreases interference (Klapp, Boches, Trabert, & Logan, 1991; Pirolli & Anderson, 1985; Thorndike, 1921).

Learning an algorithm is a matter of memorization and practice, but learning the purpose or rationale of an algorithm is not a matter of memorization or practice; it is a matter of understanding. Teaching an algorithm's steps effectively involves merely devising means of effective demonstration and practice. But teaching an algorithm's point or rationale effectively involves the more difficult task of cultivating students' understanding and reasoning. It requires insight and flexibility. Understanding and practical application are sometimes separate things in the sense that one may understand multiplication, but that is different from being able to multiply smoothly and quickly. Many people can multiply without understanding multiplication very well because they have been taught an algorithm for multiplication that they have practiced repetitively. Others have learned to understand multiplication conceptually but have not practiced multiplying actual numbers enough to be able to effectively multiply without a calculator. Both understanding and practice are important in many aspects of Math.

Related to learning theories, teachers apply at least four different approaches in Mathematics teaching:

- Skills Approach: a focus on procedural knowledge in Mathematics.
- Conceptual Approach: a focus on meaningful learning and understanding of facts, rules, formulas and procedures.
- Problem-Solving Approach: focus on development of Mathematical thinking.
- Investigative Approach: A focus on understanding meaningful memorization of facts, rules formulas, procedures and thinking necessary to contact Mathematical inquiry.

All the above approaches require practice of some manner or the other.

Drill can be interesting if the teacher has the ingenuity to repeat in various ways. The number 5 can be shown quantitatively as 5 pebbles/marbles. The concept can be drilled by clapping 5 times, by stomping 5 times, by playing a game which involves the children to group in fives. Here drill is different from writing 5 ten times.

It was in Class 8 that I had to teach Hero's formula--s(s-a)(s-b)(s-c).We made up a song "sun, sun meri aasha, sun meri bhasha, sun meri champa re "where sun stood for s, asha, bhasha and champa stood for a, b, c and meri stood for the subtraction sign. The class rocked. Children came out of their seats and danced. At the end of the class every child knew the formula.

**When does drill and practice appear absurd or does not seem to have an effect on the learning?**

Practicing something that one cannot even begin to do or understand seems absurd and fails. Practicing something that trial and error does not improve, is not going to lead to perfection. There are a number of reasons why a student may not be able to work a problem, and repeating to him things he does understand, or merely repeating things he heard the first time but does not understand, is generally not going to help him. Until the specific area of distress is not identified and clarified the students' needs may not be addressed.

**When does drill and practice seem effective?**

- Drill and practice have to be interesting and joyful to the learner.
- It will be effective only if there is clarity in what is being practiced.
- They must be systematically integrated into the teaching learning process.
- Technology is a valuable tool for repeated practice.

**Can only drill and practice develop the learning process?**

Several studies show that drill and practice must be coupled with periodic reviews to achieve tangible results. In one study Gay (1973) found that students who reviewed arithmetic rules on the first and seventh days after the original teaching presentation learned the rules better than students who reviewed the rules on the first and second days after the original teaching presentation. While most textbooks include review at the end of chapters, research has shown that review should be "systematically planned and incorporated into the instructional program."

Not having enough drill and practice often leads to the students to not master the topic in the specified class. There may be only a small gain and not the mastery of the concept *"...a phenomenon that everybody who teaches Mathematics has observed: the students always have to be taught what they should have learned in the preceding course. (We, the teachers, were of course exceptions; it is consequently hard for us to understand the deficiencies of our students). The average student does not really learn to add fractions in an arithmetic class; but by the time he has survived a course in algebra he can add numerical fractions. He does not learn algebra in the algebra course; he learns it in calculus, when he is forced to use it. He does not learn calculus in a calculus class either; but if he goes on to differential equations he may have a pretty good grasp of elementary calculus when he gets through. And so on throughout the hierarchy of courses. The most advanced course, naturally, is learned only by teaching it. This is not just because each previous teacher did such a rotten job. It is because there is not time for enough practice on each new topic; and even it there were, it would be insufferably dull".*

*- Ralph P. Boas*

Drill and Practice have to be an integral part of the Mathematics teaching learning process. The creative teacher would so weave it into the tapestry of Math learning that a beautiful design is created. The process of drill would not kill the joy of learning the subject but thrill the learner propelling him to seek more. Hence let us not drill and kill but drill and thrill!

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# 18 2 + 2 = ? A Case for Early Identification of Dyscalculia

**Sulata Shenoy**

A child may have difficulty in learning due to a variety of reasons, including motor impairment, nutritional factors, intellectual disability, emotional-behavioral issues, lack of proper schooling, disturbed home environment, lack of guidance from teachers and parents to name a few. The term Learning Disability or LD refers specifically to the significantly below average performance of a school child in the areas of language processing and expression and/or mathematical ability inspite of adequate educational opportunities and intellectual ability. LD is presumed to be inherent and intrinsic, i.e. due to central nervous system dysfunction but environmental factors can compound the disability.

Children with LD face difficulties not only with academics but also in basic processing of information such as perceptual problems, figure ground discrimination, memory, [visual and auditory], phonological processing deficits and visuomotor problems. They also face a host of psychological problems including low self esteem, behavioral issues, emotional disturbances, self regulatory behavior, social interaction, poor motivation and metacognitive deficits.

The word **Dyscalculia** comes from Greek and Latin which means: "counting badly". The prefix "dys" comes from Greek and means "badly". "Calculia" comes from the Latin "calculare" which means "to count". The word "calculare" again comes from "calculus", which means "pebble" or one of the counters on an abacus. Dyscalculia is a lesser known disability, similar and potentially related to dyslexia and developmental dyspraxia (a disorder that affects the initiation, organization, and performance of action).

Dyscalculia occurs in people who may have difficulties with time, measurement, and spatial reasoning. Current estimates suggest that it affects about 5% of the population.

Dyscalculia was originally identified in patients who suffered specific arithmetic disabilities as a result of damage to specific regions of the brain. Recent research suggests that dyscalculia can also occur developmentally, as a genetically-linked learning disability which affects a person's ability to understand, remember, or manipulate numbers or number facts. The term is often used to refer specifically to the inability to perform arithmetic operations, but it is also defined as a more fundamental inability to conceptualize numbers as abstract concepts of comparative quantities (a deficit in "number sense").

> *"The principle problem here is the way Mathematics is taught to children; this often leads to fear or dislike of Math leading to poor Math performance which is not dyscalculia but similar to it."*

Dyscalculia can be detected at a young age. Experience has proved that dyslexia can be dealt with by using a slightly different approach to teaching, interestingly, so can dyscalculia. However, dyscalculia is the lesser known of the learning disorders and so is often not recognized.

Listed below are seven prerequisite Math skills:

1. The ability to follow sequential directions.
2. A keen sense of directionality, of one's position in space, of spatial orientation and space organization. Examples include the ability to tell left from right, north/south/east/west, up/down, forward/backwards, horizontal/vertical/diagonal, etc.
3. Pattern recognition and its extension.
4. Visualization: the ability to conjure up pictures in one's mind and manipulate them. Eg. three dimensional cube

5. Estimation: The ability to form a reasonable educated guess about size, amount, number, and magnitude.

6. Deductive reasoning: the ability to reason from the general principal to a particular instance, or reasoning from stated premise to a logical conclusion.

7. Inductive reasoning: a natural understanding that is not the result of conscious attention or reasoning, easily seeing the patterns in different situations, and the interrelationships between procedures and concepts. (Sharma 1989)

Before a mathematical concept is learned fully, the student moves through six levels of learning:

1. Intuitive Connections: Student connects or relates the new concept with existing knowledge and experiences.

2. Concrete Modeling: Student looks for concrete material with which to construct a model or show a manifestation of the concept.

3. Pictorial or Representational: Student draws to illustrate the concept. In this way he connects the concrete (or vividly imagined) example to the symbolic picture or representation.

4. Abstract or Symbolic: Student translates the concept into mathematical notation, using number symbols, operational signs, formulas, and equations.

5. Application: Student applies the concept successfully to real world situations, story problems, and projects.

6. Communication: Student can teach the concept successfully to others, or can communicate it on a test. (Sharma 1989)

Curricula in the pre-school and early elementary years should focus on the development of the prerequisite Math-readiness skills. The principle problem here is the way Mathematics is taught to children; this often leads to fear or dislike of Math leading to poor Math performance which is not dyscalculia but similar to it.

Teachers and students need to be aware of and able to accommodate the different learning styles or "Math learning personalities" and the corresponding teaching methods that address each style. It must be emphasised that most children at the beginning levels of Mathematics may reverse numbers, write mirror images of numbers or have difficulty with mathematical concepts. However, when these symptoms persist beyond the grade when most students outgrow them, we must suspect dyscalculia. When a child is not cognitively ready to learn Math concepts, their early introduction will only result in negative experiences and attitudes toward Mathematics, and eventually, Math anxiety. Parents and teachers must wait until the child is developmentally ready. In the mean time, varied informal experiences are to be provided. Gender differences in Math skills have been reported in most cultures. It is hypothesized that these are due to social forces as much as gender-specific brain construction and function. Gender differences can be eliminated by equalizing the activities and experiences of both boys and girls at every level of development leading to neurological sophistication of both genders equally. (Sharma 1989)

**Diagnosing Dyscalculia**

Dysfunction in Math, in individuals with normal mental functioning with discrepancy 1-2 standard deviations below the mean, between their mental age and Math age indicates a clear retardation in mathematical ability:

1. Quantitative dyscalculia is a deficit in the skills of counting and calculating.

2. Qualitative dyscalculia is the result of difficulties in comprehension of instructions or the failure to master the skills required for an operation.

3. Intermediate dyscalculia involves the inability to operate with symbols, or numbers.

**Potential symptoms**

1. Numerical difficulties with counting, recognizing numbers, manipulating Math symbols mentally and/or in writing, sequential memory for numbers and operations, mixing up numbers in reading, writing, recalling, and auditory processing, memory.

2. May transpose (mix up) [21 as 12], interchange similar digits [6 and 9], inappropriately insert, or omit digits, words, and signs or read without acknowledging place value: 5007 as "five hundred seven," or 576 and "five seven six".
3. Difficulty with everyday tasks like checking change and reading analog clocks.
4. Inability to comprehend financial planning or budgeting, sometimes even at a basic level; for example, estimating the cost of the items in a shopping basket or balancing a checkbook.
5. Difficulty with multiplication-tables, subtraction, addition, division, mental arithmetic, etc.
6. Particular problems with differentiating between left and right.
7. Difficulty navigating or mentally "turning" the map to face the current direction rather than the common North=Top usage.
8. Having particular difficulty mentally estimating the measurement of an object or distance.
9. The condition may lead in extreme cases to a phobia or durable anxiety of Mathematics and mathematic-numeric devices/coherences.
10. May be able to read and write numbers but is oblivious to their meaning.
11. Cannot identify a specified number of items.
12. Frequent errors include: mixing up operations like +/-, -/÷ , x/÷ , x/+; mistaken or oversimplification of complex operations; needing written computation over mental calculation, using fingers to assist mental or written computation.
13. Inability to learn and apply the rules for addition, subtraction, multiplication and division resulting in a disability to successfully perform Math operations.
14. Poor memory for counting sequences, operational sequences, Math facts, time, direction, schedules.

**Potential cause**

Scientists have yet to understand the causes of dyscalculia. Investigations indicate that it could be neurological as dyscalculia has been associated with lesions to the supramarginal and angular gyri at the junction between the temporal and parietal lobes of the cerebral cortex,or due to deficits in working memory. Other causes may be short term memory being disturbed or reduced, making it difficult to remember calculations. Children and adults subject to dyscalculia nevertheless tend to be of normal intelligence, but often present an uneven picture in their results on intelligence tests. The majority of children and adults who are subject to dyscalculia have the ability to read and the ability to understand what is read unimpaired, although about 20–30 % of those who are subject to dyscalculia are characterized by having difficulties reading and with Mathematics. They may require extensive mental training to carry out even simple arithmetic tasks.

A child may have only a limited understanding of either numbers as such or numerical symbols. Another form of dyscalculia involves planning difficulties that lead to the child's failure to carry out computations effectively. The child has difficulties with following a clear strategy in solving arithmetic problems, losing track of her mental position among the fundamental mechanics of the mathematical problem. Dyscalculia may also be based on problems with visual perception that lead to difficulties with tasks involving logical thinking as well as in carrying out computations. Eg. learning to read an ordinary clock and understand how the position of the hands is to be interpreted.

Difficulties with Mathematics are associated with the child having general problems with learning, in the area of Mathematics as well as others, learning tends to take longer than normal.

Dyscalculia affects individuals over their life span. Children with dyscalculia fall behind early in primary school, and may develop anxiety or a strong dislike towards Math. In secondary school they are likely to struggle to pass Math and science courses and find their career options reduced. The student can be overwhelmed and this may result in emotional distress. In adult life, they may earn less, and have difficulties managing their everyday finances. For individuals with dyscalculia, Mathematics can be a traumatic experience and emotionally charged because of past failures.

Many people think because LD is considered a central nervous system dysfunction, it can't be changed. However, we now know that the brain is very adaptable (or plastic), especially during childhood. Research has already shown that training programs can increase functioning in brain areas involved in reading. The same is likely for dyscalculia. There is still a lot we do not know about dyscalculia, because research is a good 30 years behind as compared to dyslexia. This situation has started to improve, especially in recent times.

### Mitigative Strategies

Although dyscalculia may be difficult to diagnose, there are strategies that teachers and parents should know about to aid students in learning Mathematics.

1. A child of this category is usually best helped by being allowed to work at a slow pace and by being given simplified learning material.
2. Provide examples and try to relate problems to real-life situations.
3. Encourage student to work extra hard to "visualize" Mathematics problems. Draw them or have her draw a picture to help understand the problem, and make sure that she takes the time to look at any visual information that is provided (picture, chart, graph, etc.)
4. Have the student read problems out loud and listen very carefully. This allows the use of auditory skills.
5. Provide younger students with squared paper and encourage them to use it in order to keep the numbers in line.
6. Provide extra worksheets so that the student is not overwhelmed by too much visual information (visual pollution). Especially on tests, allow scrap paper with lines and ample room for uncluttered computation.
7. Dyscalculia students must spend extra time memorizing Mathematics facts. Repetition is very important. Use rhythm or music to help memorize.
8. Many students need one-on-one attention to fully grasp certain concepts. Have students work with a tutor, a parent, or a teacher after school hours in a one-on-one environment.
9. If possible, allow the student to take the exam on a one-to-one basis.
10. The student might require a chance to do the problem once again when she is wrong. Often mistakes are the result of "seeing" the problem wrong.
11. In early stages, design the test problems to test only the required skills. In their early learning, they must be free of large numbers and unnecessary calculations.
12. Allow more time to complete problems and reassure the student so that he does not succumb to anxiety. Be patient and positive.
13. Assign extra problems for practice and maybe a special teaching assistant or special educator to assist the student.
14. When presenting new material, make sure the student with dyscalculia is able to write each step down.

### Technology And Resources

The technology for remediating and accommodating students with Mathematics disabilities is not as readily available as the technology for reading and writing.

The limited technology can be of help, especially to those who have problems writing numbers down in the correct order. The most common currently available tools include the following:

1. hand-held calculators that can help a learner who has problems writing numbers in the correct order
2. talking calculators that vocalize data resulting in calculations through speech synthesis
3. special-feature calculators that enable the user to select options to speak and simultaneously display numbers, functions, entire equations and results
4. on - screen computer calculator programs with speech synthesis
5. large display screens for calculators and adding machines
6. color coding for maintaining columns
7. big number buttons and large keypads
8. textbooks on CD-ROM and video-taped Mathematics lessons
9. Computer-assisted Instruction (CAI) Mathematics courses (instruction targeted for special students)

Dyscalculia in children can present in a variety of ways.

Initially, the child may have a difficulty or reluctance towards Mathematics. The teacher or parent should immediately be alerted to the possibility of a learning disability in the early grades itself. This is the time when the appropriate intervention is most effective. The thrust of recent research in this area is on early identification. Once a child is identified on the basis of the above, one needs to investigate further. The extent and severity of the learning disability as well as the strength of various contributing factors viz., physical, psychological, socio-emotional, scholastic and familial factors have to be identified so that appropriate actions can be taken to help the child. Our primary focus should be the child and care should be taken to help him so that he can grow up and realize his potential as an adult. Early identification also helps prevent poor school performance and emotional-behavioral problems.

**Summary:** Early intervention in Math is as important as early intervention for Dyslexia. A diagnostic evaluation at any age in a student's Math development should pinpoint the problem areas, provide a plan for Math intervention, and offer recommendations for Math remediation.

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# Fold Paper and Learn Mathematics

## V S S Sastry

Ask any child to fold a paper in a certain way, immediately she will retort “why should I do it?” It is seen as a burden, a chore and a tedious exercise. But ask the same child to fold and make a boat or a bird using paper and give her some guidance and she would immediately start doing that and she would like to learn happily. After doing the model, try telling the girl that after unfolding the model you can see this angle, this area, this line etc. It would definitely have an element of discovery in it. The process now becomes fun.

Folding paper and creating different shapes is an art known as Origami. Certain paper models contain hundreds of folds and a few complex models in fact require two or three differently folded shapes to be attached together.

Math through Origami is an adventure to see patterns as it literally 'unfolds' before the eyes of the child.  This exploration helps to clarify abstract concepts in Math in an enjoyable way. Discarding the exotic models in Origami, let us turn to simple models. I prefer to use those Origami models which do not have more than eight folds for this purpose. Simple and small Origami models are ideal for learning Math.

In our country, Origami has become a folk art but not practiced widely. Nowadays, no one teaches children how to make a boat, a cup, a bird using paper folding. They are lucky if they get to learn from their friends in schools.

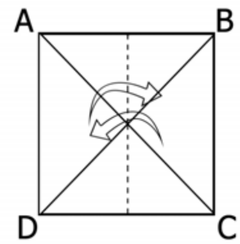
Here I give a couple of examples illustrating Math concepts that can be conveyed by simple Origami models.

> *So the simple peacock actually results in providing explanation for more than ten Math concepts. There are several such models related to the Math curriculum. Hundreds of inexpensive models made by the child would reinforce these concepts.*

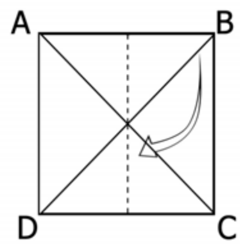
### Paper Peacock

Follow these diagrams and make a peacock. You will see here that the symbols used are very simple and self explanatory.

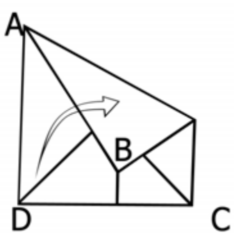
$P_1$



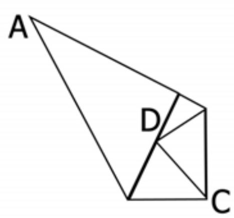

Start with a square paper 10 cm x 10 cm, with diagonal creased.
Fold in half as shown & reverse.




Youngest a middle line fold ‘B’ to the middle line

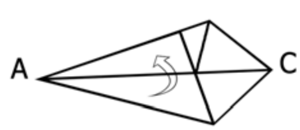



Fold ‘D’ on to the folded edge



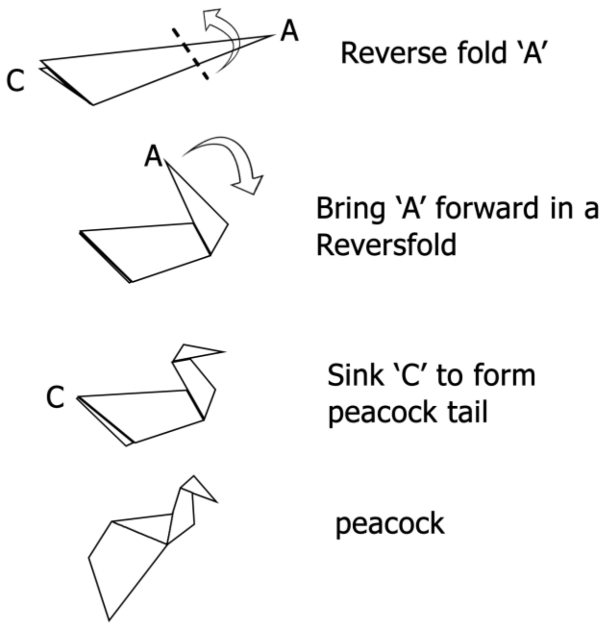

This is the result

$P_2$




Fold along AC

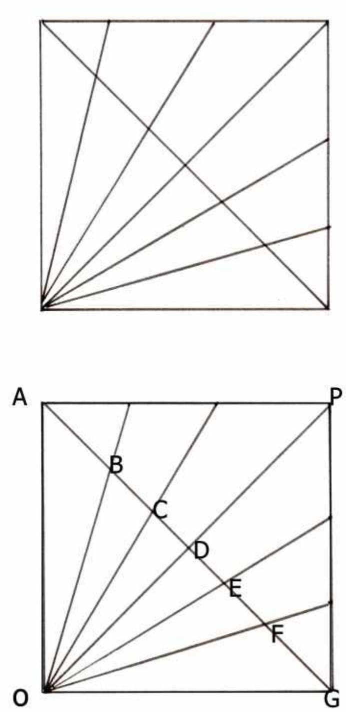

Rotate the model




Having folded a peacock model from a square paper, a child begins to play with it. She may sometimes tear the model. Encourage the child to unfold the model so that the paper is square again. But the folds created in the paper leave their marks. Straight lines, areas, angles etc. Mark these lines with a pencil to highlight them.

Then you will see this...

$P_3$




Here angle ADG=90 degrees which has been divided into 6 parts of 15 degrees each. Now let us get all angles. We now get six angles (15, 30, 45, 60 and 75 besides 90) and we do not need to use a protractor to measure them. This in itself is a wonder for the child.

The same model can be used to illustrate the following:

1. Division of angles
2. Obtuse and acute angles
3. Acute angle triangle
4. Obtuse angle triangle
5. Isosceles triangle
6. Equilateral triangle
7. Isosceles right angle triangle
8. The sum of three angles in a triangle is 180 degrees
9. The sum of internal opposite angles is equal to exterior angle of the triangle
10. Corresponding angles
11. Vertically opposite angles

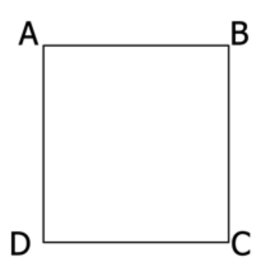
So the simple peacock actually results in providing explanation for more than ten Math concepts. There are several such models related to the Math curriculum. Hundreds of inexpensive models made by the child would reinforce these concepts.

**Heptagon by paper folding**

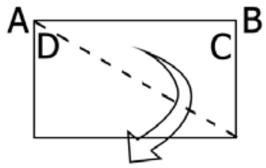
It is very difficult to construct a Heptagon (polygon with seven sides) because the internal angle of a regular heptagon is 128.57 degrees. This cannot be measured with accuracy by any protractor. But you can construct a heptagon by arranging 7 sticks of equal length or by drawing 7 lines on a paper by a process of trial and error.

But using Origami techniques this angle - 128.57 degrees can be obtained very easily. The steps illustrated below will be enough to get this angle from a square paper.

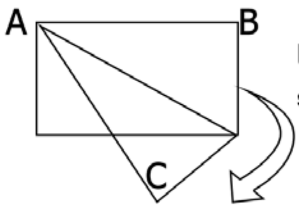
$P_4$



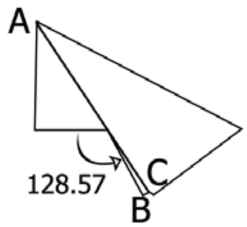

Start with a square paper
Name as A,B,C,D.




Bring CD onto AB




Fold a Diagonal starting from A




The angle shown is $128.57^0$

By inter-locking 7 of such folded sheets we can get actually construct a regular heptagon

At a future date, it is possible to imagine a school, where Origami is taught for fun in lower classes. Children can learn 8-10 models in a year. Using the same models Math could be taught in next class. The learning will not be a burden. Learning Math through paper folding is possible even in higher classes. In fact Karnataka Rajya Vigyan Parishath Bangalore has published a book titled “Desk Book on Mathematics through Origami” to explain Math concepts up to Class Ten.

**Origami**

Origami is an ancient art practiced in Japan. The word comes from 'Ori' which means to fold and 'Kami' which means paper. In olden times, Origami models were used in some rituals. But later it was practiced for pure fun and enjoyment. It is said that paper folding entered Japan from China. There, paper boats are kept in funerary rituals, to carry souls away from this world.

After American intervention in Japan in the $19^{th}$ century, the Japanese arts and crafts began to get noticed in the West where the new word “ORIGAMI” was coined for paper folded art. Earlier it was known only as 'Paper Shapes'. From then on paper folding art developed rapidly. Now, there are several Origami societies and even magicians present paper folding in their shows. Annual Origami shows are organized regularly all over the world. Computer software have been developed, which provide sequence of paper folding instructions to get the required shape.

There were other cultures who folded paper. Southern Spain has a large population of Arabs called the Moors. These people engaged in crafts like ornament making, metal work, stone work etc. They follow Islamic religion. Therefore human figures are not a priority in their art. Their art is full of lines, angles and geometric designs. These artisans do not learn geometric designs through Math-Geometry but through paper folding. They depend on repetition of patterns or 'Tessellations' as it is known in Mathematics. The designs get repeated in all crafts they do. In Moghul times these Moors were invited to decorate buildings in Delhi and Agra. The filigree work for windows executed in Moghul monuments are handiwork of these Moors. In Origami circles, paper folding techniques from Spain are known as the Moorish tradition.

National Council of Teachers of Mathematics America commissioned Olson, a mathematician from Albert University, to compile all that was available on the subject of Math and paper. This mathematician referred to all papers published till then and wrote a 60 page booklet that contains relevant matter in full measure. The Indian reprint is available at Mathematical Sciences Trust Society, C-766, New Friends Colony, New Delhi – 110 066.

**Sunder Row - The Indian Pioneer**

Sunder Row was a Head Master in Royapettah High School Chennai in 1870. He must have been a good Math Teacher. We are told that one day, after his retirement, the old man went to Spencer Department Store. He searched for a suitable present for his grandchild. He found a gift pack that was filled with color papers and a small book, detailing how to fold the papers into different shapes of animals, birds etc. But Sunder Row being a Math teacher saw the lines, angles and areas formed when a folded model is opened and made flat. Soon he began to connect the patterns formed with theorems, lemmae, constructions etc that he had taught throughout his working life.

This Head Master began to write down all that could be done with folds in the paper. He covered Geometry syllabus in full. He could relate many more facts of Math to paper folding. Thus a magnificent book titled "Geometric Constructions in Paper folding" came about and was published in 1893.

This was the first book of its kind in the world and caught the attention of Math educators and teachers around the world. Mathematics Teachers Association of America got to know of this book through a German commentary in a magazine. They commissioned two well known Math Teachers to edit this book for American readers. This edition saw 47 reprints and is still getting reproduced all over the world.

But this seminal book is totally unknown in India - for whose children Sunder Row had first published it. For those who are interested, this book can download free from the web site www.arvindguptatoys.com

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**VSS Sastry** is a serving bank officer with a lifelong passion for Mathematics. He has written twelve books on Mathematics and related activities for children. His book "Origami Fun and Mathematics" (in its third reprint) published by Vigyan Prasar, New Delhi is about using paper shapes to teach Math upto class 10. He has also conducted more than 300 workshops for teachers. His other interests are Aero Modeling, Kirigami and Kites. He can be contacted at vsssastry@gmail.com

## Logico- Math Brain Teasers

Ajay is sent to the river to bring water. He is carrying two buckets, one with a capacity of 11 liters and the other with a capacity of 6 liters. The problem is, he is required to bring back exactly 4 liters of water. How can Ajay do that?

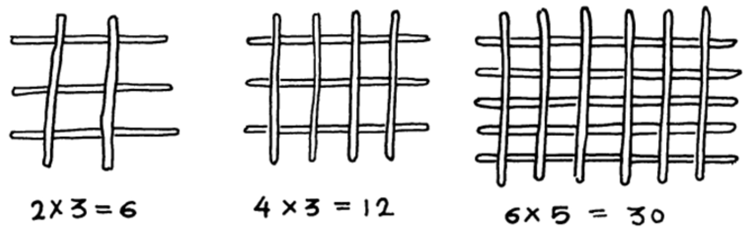
Use this space for calculation 🙂

# 20 Broomstick Tables

**Arvind Gupta**

This article is inspired by the fascinating work of Mr. P. K. Srinivasan of Chennai. Tables are often learnt by rote. This repetitious drill might help quick recall but it kills the whole joy of learning. With only 18 equal length broomsticks children can discover the whole world of tables.

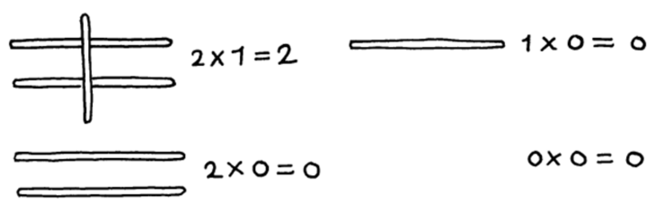
1. Lay one broomstick and place another one across it. At how many points do they meet? Obviously, one. So, 1 x l = 1. If two vertical broomsticks are placed criss-cross over three horizontal broomsticks then they have six junctions. So, 2 x 3 = 6.



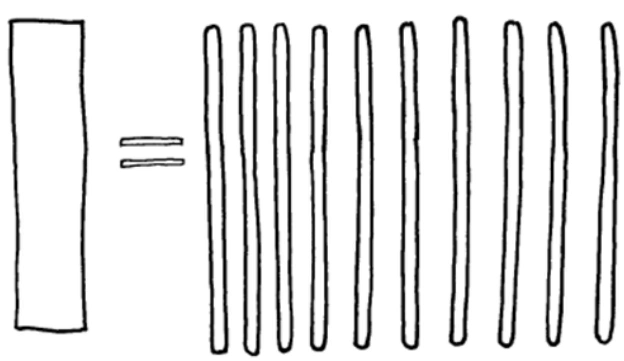

2. Children can make a 0 to 9 matrix on a square ruled copy and make their own table sheet by placing broomsticks criss-cross and counting the number of junctions. So, children who have learnt to count should be encouraged to make their own table sheets.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 2 | | | 6 | | | | | | |
| 3 | | | | | | | | | |
| 4 | | | 12 | | | | | | |
| 5 | | | | | | | | | |
| 6 | | | | | 30 | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |

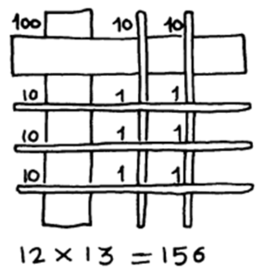
3. This picture shows how the abstract concept of multiplication by zero can be concretised.

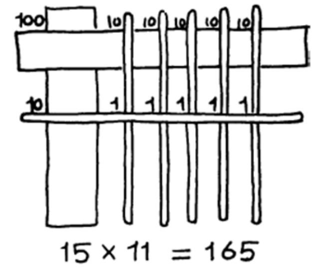



4. Multiplication of two digit numbers would mean counting too many junctions. So, ten broomsticks can be represented by one card strip.



5. Criss-cross of two strips will be 10 x 10 = 100, while that of a strip and a broomstick will be 10 x 1 = 10. Add up the sum of all the junctions to get the multiplication value.




**Arvind Gupta** works at the IUCAA's Children's Science Centre in Pune and shares his passion for books and toys through his popular website http://arvindguptatoys.com. The books referred in the article can be accessed from the mentioned. He can be contacted at arvindguptatoys@gmail.com

# section D

# The Role of Assessment

# 21 Lessons from the International PISA Project

Ross Turner

A simple characterisation of "the good old days": All students in the early years of secondary school studied Mathematics. It was usually taught in a rigorous and challenging way that functioned in part as a sorting mechanism. Most students found it difficult, and eventually gave up studying Mathematics. Many of these ended up hating Mathematics and feeling they were no good at it. And for many, this turned in later life into an attitude of fear and anxiety about Math, and a wish to avoid it where possible. A small proportion of students loved Mathematics, and did well at it. They were part of the relatively small cohort of students who continued through to the end of secondary schooling. Some of these went on with their studies at university level, during which they developed higher level Mathematics skills useful in scientific, technical or quantitative areas.

What has changed? Well, many things of course, of which I mention two. We have a far greater proportion of students continuing their schooling at least to the end of secondary school. And there is far greater demand for quantitative competence in a wide variety of occupations. The level of technical skill including mathematical competence required in the modern workforce continues to expand. The more formal and traditional mathematical skills continue to be important, but it should also be acknowledged that our whole world demands a level of quantitative competence and confidence, to meet even simple day to day challenges in the workplace and in managing our personal circumstances, that is a quantum leap from the simpler demands of the good old days. Increasingly in our world, people who fear or are otherwise incapable of handling mathematical challenges would seem to be at a severe disadvantage. Are there any implications of these developments for schools? Increased school retention rates give schools a key role in equipping their students to better handle these challenges.

But my question is this: to what extent have our teaching and learning practices, our curriculum decisions and our approaches to assessment, changed to accommodate these new demands? The Organisation for Economic Cooperation and Development (OECD) Programme for International Student Assessment (PISA) is a comparative survey designed to assess the extent to which students at the age of 15 in participating countries, who are nearing the end of compulsory schooling, are prepared to meet the challenges of the modern world. PISA aims to measure the extent to which students are able to use the knowledge and skills they have acquired throughout their schooling to meet challenges that involve activation of reading, scientific and mathematical competencies. It is not an assessment of achievement in the school subjects associated with these domains of knowledge; nevertheless the three main assessment domains of PISA (reading, Mathematics and science) do reflect important desired outcomes of student learning in those curriculum areas. PISA uses the term literacy to emphasise its focus on using knowledge in a variety of situations, and to distinguish it from a focus on measuring achievement against some predefined curriculum syllabus.

> *"I must say the evidence at hand is that 15-year-old students around the world find these kinds of demands very challenging indeed. Students seem unaccustomed to thinking creatively in Mathematics classes. They have a great deal of trouble explaining their thinking and reasoning"*

Regular Learning Curve readers will have seen the article by Juliette Mendelovits in a previous issue (Issue XIII, October 2009) focusing on aspects of the PISA Reading domain. Further information about PISA can be obtained from the most recently published framework document (OECD, 2006). Released PISA survey items are also published in book form (OECD, 2009). These and other publications are available from the OECD website (www.oecd.org).

In the case of PISA Mathematics, survey items are designed to measure proficiencies that reflect the skills needed by mathematically literate individuals in a modern society:

1. the ability to recognise the existence of mathematical elements within a situation presenting a challenge;
2. the ability to pare the situation down to its elements, enabling a separation of those mathematical features from the context, and to define a mathematical problem the resolution of which might help to answer the challenge;
3. the ability to call on relevant mathematical knowledge and to apply that knowledge confidently and correctly to forge a mathematical solution to the problem at hand;
4. the ability to relate that solution back to the original situation, thereby ensuring that the solution makes sense and genuinely addresses the challenge – that is, the ability to recognise the extent and limitations of the solution;
5. the ability to communicate the outcomes to others; and
6. the ability to step outside the process and exercise control mechanisms that help direct thought and action to achieve the desired outcome.

Being mathematically literate in the modern world means much more than memorising rules and formulae and mastering a set of procedures. It also means being able to think creatively about challenges, to analyse situations, to link the demands of those situations with relevant knowledge held by the individual, to apply that knowledge in appropriate ways, and to reflect on and evaluate the solution found and the process followed to reach it.

How much teaching and learning time is devoted to presenting students with problems set in authentic contexts? To what extent does India's education system value the investment of effort by individuals and groups of students in grappling with such problems, pulling a problem apart, hunting for the knowledge that would help to make the problem tractable, imposing an analytical structure, applying Mathematics to the problem, developing solutions, evaluating the solutions, communicating the solutions to others, considering ways of implementing the solutions, and talking about that whole exploration and discovery process?

If that investment of time is not made, is it reasonable to expect students to do well at assessments that call on those skills and processes? Could we expect students to go into post-secondary courses with the confidence and competence to interact productively with problems that demand those skills? Could employers reasonably expect their young workers to have those skills?

I must say the evidence at hand is that 15-year-old students around the world find these kinds of demands very challenging indeed. Students seem unaccustomed to thinking creatively in Mathematics classes. They have a great deal of trouble explaining their thinking and reasoning. They frequently find it very difficult to make decisions about what mathematical knowledge and skills might be relevant to solving a particular problem when that information is not given to them directly. Of course the PISA data show the proportions of the student cohort in each participating country demonstrating various levels of proficiency, and in each country some students are in the highest described level. But many more are in lower levels. Furthermore, recent criticism of the Mathematics component of PISA has been based on the claim that the level of Mathematics required to successfully solve the survey items has been too low.

To illustrate a PISA item that starts in a real world situation, and demands some thought, analysis and interpretation rather than simple application of routine procedural knowledge, I present a test item that was used in a previous PISA survey.

PISA Mathematics uses 'authentic' to refer to problems for which there is genuine interest in the solution and for which the purpose of using Mathematics is to solve the problem, in contrast to contrived applications that are presented mainly for the purpose of practising particular skills. The most authentic problem contexts are those for which the context itself influences the solution and its interpretation, and which therefore require students to consciously and actively connect the problem context with the

*Rock Concert*, a Mathematics item from the PISA 2003 field trial shown in the box, presents a context that would be familiar to many 15-year-olds, and provides the opportunity to devise a model for the amount of space that a person might occupy while standing. With the multiple choice format used in this item, this could be done by postulating an area for each person, multiplying it by the number of people given in each of the options provided, and comparing the result to the conditions given in the question. Alternatively, the reverse could be done, starting with the area provided and working backwards using each of the response options, to the corresponding space per person, and deciding which one best fits the criteria established in the question. The student must think clearly about the relationship between the model he or she uses and the resulting solution on the one hand, and the real context on the other, in order to validate the model used and to be sure he or she has chosen the most realistic answer.

**ROCK CONCERT**

For a rock concert a rectangular field of size 100 m by 50 m was reserved for the audience. The concert was completely sold out and the field was full with all the fans standing.
Which one of the following is likely to be the best estimate of the total number of people attending the concert?

| | |
|---|---|
| A | 2 000 |
| B | 5 000 |
| C | 20 000 |
| D | 50 000 |
| E | 100 000 |

The level of the mathematical knowledge required to answer this question is not very high, nevertheless this item was not answered correctly by many students. In fact only about 30 per cent of students internationally selected response option C as the most reasonable answer to this question when it was administered in the field trial for PISA 2003. It would seem that the level of the Mathematics required may not be the main issue here – rather, something more fundamental is going wrong. Students are not sufficiently able to think outside the square, to reason and argue, to discuss and explain their thinking.

The PISA focus on mathematical literacy implies a set of teaching and learning objectives that could form an important part of mathematical instruction at least in the junior and middle years of secondary school, and possibly more widely than that. If school Mathematics programs took these objectives to heart, arguably students would be better equipped to make confident and effective use of the mathematical skills they have learned at school as they negotiate the challenges of their life as citizens.

To illustrate a teaching and learning activity that might be useful in a Mathematics classroom to help students think about and analyze real world situations from a mathematical perspective, I revisit an Idea I first presented to the 2005 annual conference of the mathematical Association of Victoria (Turner, 2005). Suppose that watermelons were made to grow in cuboid shapes as shown in the photograph rather than their more familiar ellipsoidal shape. What possibilities and problems would such a change of shape cause for growers, handlers, sellers, and consumers? What mathematical ideas might come from this simple piece of stimulus? Here are some:

1. It is said that this shape was developed to save space. How might this be true? Consider storage, packing and transport of this shape compared to the more common spherical or ellipsoidal watermelons.

2. When watermelons are cut up into serving slices, we usually use sectors of a circle. What shapes might work for a 'cuboid' watermelon? What would be the advantages and disadvantages of different shapes?

3. What is the relationship between surface area and volume for the 'cuboid' watermelon compared to an ellipsoidal melon? Would one of these shapes have more skin, or rind, per kilogram than the other? What would be the relationship between the relative proportion of skin and flesh for the different shapes?

4. What other mathematical ideas does this watermelon context suggest?

There are some additional steps that would be needed if such an idea were to be used as an assessment item. First, experience in using it as part of a teaching and learning activity will tell you more about its potential in assessment. That experience will tell you what questions you could reasonably ask. In addition, it will tell you what range of responses students might give, which you probably need to know if you are to plan a marking scheme in advance. Here are some possibilities:

Q1. Imagine you have a spherical watermelon with diameter 30 cm. What would be the dimensions of a 'cuboid' watermelon with the same volume? (This could be presented as an open question, or perhaps in multiple-choice form using anticipated calculation errors as distracters.)

Q2. If you were able to grow a 'cuboid' watermelon having dimensions of about 30 cm, about how much more flesh would you expect compared to a spherical watermelon with diameter 30 cm?

Show your calculations, and explain and justify your answer.

The increasing popularity and accessibility of digital photography mean that anyone with a little imagination can come up with interesting stimulus ideas from the world around that can form the basis of significant teaching and learning activities and assessment tasks. This might be an easy stepping stone towards providing more opportunities in the classroom for exploring the relationships between Mathematics and real world phenomena, and to developing and practicing the skills mentioned earlier that are so vital for effective participation in an increasingly quantitative world.

[Readers may be interested to learn that India now participates in PISA. Two states (Himachal Pradesh and Tamil Nadu) are taking part in the implementation of the PISA 2009 survey on a one-year-delayed timeline, following implementation by the 66 participants in the main survey during 2009. Results for these Indian states should be published in late 2011.]

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**Ross Turner** is a Principal Research Fellow at ACER. He manages ACER's involvement as lead contractor to the OECD responsible for implementing the International PISA project. He is also the Deputy International Project Director for PISA, and the leader of the PISA Mathematics assessment domain. He can be contacted at Turner@acer.edu.au

# 22 Response Analysis: Understanding Children from their Frame of Reference

**Abhishek S. Rathore and Falguni Sarangi**

Very often, we are so fascinated with the idea of 'right' and 'wrong' answers that we forget (or refuse) to explore the cause behind a child's response to an item. We are so engrossed in the "wrongness" of a response that we fail to appreciate the beauty of that response. This paper attempts to propose a more rational way of looking at children's responses rather than what we have been doing as part of 'assessing' our children's work. It is an attempt to build perspective on knowing what a child has learnt rather than what she hasn't learnt. In doing so, we also acknowledge the fact that we would never be able to definitely know how a child has understood a concept, as our inferences would be based on a few educated guesses, made on the basis of available evidence. In this attempt we believe that all forms of assessments of children's responses carry some diagnostic information.

Our understanding of Response Analysis is an outcome of an internal study performed on 1500 answer scripts of children. Such a process should, we envision, inspire the teacher and definitely add value to an active classroom teaching-learning environment. In this version of the paper we restrict ourselves to the concept of response analysis while keeping the results of the study at bay.

**Background**

There is more than general consensus that the present education system is examination-driven and this impacts all classes from the Board Examination downwards. Moreover, the examinations are essentially content based and merely test the child's capacity to memorize facts and recall concepts without testing, understanding or application of these concepts. This examination system needs reform and the Government is attempting it in a gradual manner.

Considering the importance given by stakeholders to examinations and assessment, assessment-driven education reforms will be critical. The need to move towards competency-based testing of a child's learning [as opposed to text book and rote memory-based testing] is urgent.

The Learning Guarantee Programme, an assessment led reform programme of the Azim Premji Foundation, aims at changing the classroom teaching-learning processes through bringing about desired changes in assessment tools and practices. In Learning Guarantee Programme, we tried to change the way children's responses were looked at, i.e. we attempted to bring in a more rational way of looking at the responses while 'assessing' children's work. While we do not claim to be the first to attempt this, we do feel that this aspect has not been given its due by practitioners.

> *It is necessary to explore the reasons for a particular response so as to establish "what the child has learnt" for making any further plan of action. Remember, an educated guess is better than just a guess, and similarly, a calculated risk is better.*

**Building conceptual understanding of Response Analysis**

Assessment is a continuous process to ascertain the present learning level of children. Practitioners broadly divide it into two types:

1. Formative or continuous and

2. Summative or end of the academic cycle.

While assessing, we try to establish the learning levels of the children from responses to a few items/questions. Often, these responses of children are categorized into 'right' or 'wrong' or in rare cases 'intermediate' and we conclude that children who have responded correctly to an item have achieved the said competency and those who have not, have not achieved that particular competency.

Such inferences often demean the purpose of assessment and only provide inadequate insights, if any, into the child's learning. We will now try and explore this with the help of an example:

Q. Solve

```
      21
     -17
R :   16
```

**Typical Inference** – child does not have the understanding of 2 digit subtractions with borrowing. Plan of Action – Will need to practice it again with the child.

Some reflections on this inference and plan of action:

- The above inference elucidates what the child has not learnt rather than what she has learnt. Hence this is incomplete information. So if we want to start with the child, what is a good start point?
- Before making the above inference, we have not examined why the child is making this mistake. This might pan out the wrong road map for our further plan of action. For, who knows whether the problem is with the concept of subtraction or something else?

If we want to further explore the child's response we will have to raise the level of analysis from simply considering a response 'wrong' or 'right'. Any inference made (in any form of assessment) needs a logical basis that particularly emphasizes what the child has learnt rather than what she has not leant. To go back to our initial example:

```
Q.    Solve
        21
       -17
R       16
```

First of all, let us try and gather some information from the response:

Information:

1. At the unit's place, the child is subtracting the numbers in the reverse order (7-1 = 6)
2. At the ten's place, the calculation is correct and child has made no errors according to her frame of reference.

**Analysis Points: What has the child learnt?**

1. The child has developed first-stage understanding of the concept of subtraction because her calculations are correct.
2. The child might have generalized the rule which she has learnt with single digit to two-digit problems, i.e. we always subtract a smaller number from the bigger one. for example: 7-1 = 6.

**Analysis Point: What has the child not learnt?**

1. From the generalization, you can see that the child is not able to differentiate between a 'digit' and a 'number'.
2. Concept of place value therefore needs attention.

With whatever limited information we have, we can infer that with this child, we need to direct our efforts towards building the concept of place value, number and digits and not on subtraction.

Let us take another example:

Q. Arrange the following numbers in ascending order

121,222,117

R. 117,222,121

Information Available – Numbers have been arranged in the wrong order.

Some critical analysis points:

1. The child is not able to arrange the three 3-digit numbers in an ascending order. But who knows, perhaps she can arrange two 2-digit numbers?
2. It is possible that child does not know the meaning of the word “ascending”. If we had framed it “from smaller to bigger”, then she might have solved it.
3. Who knows whether this child can identify a 3-digit number?

It is possible that she has not developed any understanding of ordering numbers.

**Some more Analysis**

Looking at all these analysis points, our inference seem inconclusive, hence where do you start with this child? What this implies is that we have limited information. So for making any inference, we need to gather more information. For instance, we now correlate information gathered during oral assessment; the same child was not able to recognize any 3-digit number correctly. But still, there are questions which remain unanswered: like, "can she identify 2-digit numbers? Can she compare two 2-digit numbers? Can she order numbers?" The point here is that we will need more than one response to analyze the present level of learning of a child, on a particular competency.

With the above two examples, the learning can be summarized as:

1. We need to analyze more than one response to make any inference about a child's learning. Therefore, we need to look at the responses of the linked items in hierarchy of concepts as well. (We shall be talking about linked items in detail in the next section.)

2. It is necessary to explore the reasons for a particular response so as to establish "what the child has learnt" for making any further plan of action. Remember, an educated guess is better than just a guess, and similarly, a calculated risk is better.

3. Every response of a child provides diagnostic information to work with, however irrelevant it may seem.

**An attempt to understand linked items**

It would be wise here to use a familiar frame of reference rather than resorting to a fresh example. We have discussed the problem related to ascending order.

Q. Arrange the following numbers in ascending order

121,222,117
R. 117,222,121

Possible Linked Items:
Q. Use the appropriate sign <, > between the numbers

R 943 > 934
R 498 < 589

Q. Underline the smallest 3 digit numbers
R 1000, 699, 969

Q. Arrange the following numbers from bigger to smaller while earlier, we had only one. The reader can see the difference.

**For whom is this perspective relevant?**

This perspective enables the **teachers** and **practitioners** to reflect upon their classroom teaching-learning processes more rationally, as it equips the practitioners with a process that can reasonably ascertain the learning levels of a child. But one of the contentions that can be raised is: "how is it possible for a teacher to analyze 60 answer scripts thus?" What one needs to understand is that response analysis is nothing more than a perspective and once a teacher has built it, she is no longer dependent solely on written test tools. Her understanding will penetrate the classroom processes. Some carefully framed questions/items can always be asked orally, or some games can be played during day-to-day classroom transactions, to assess the learning. But the perspective here will reflect in the design of that particular item or game.

Here is another example: a teacher is trying to test whether children of Class I have been able to establish the link between numbers (symbolic representation – 10, 11, 12....) and 10 to 20 concrete objects. So she designs a game (assuming that there are 40 children in class I). They are divided into two groups (20 each).

Group1 will have cards from 1 to 20 and each member of group2 will have pebbles.

The task is: group1 shows the card and all members of group 2 have to show the pebbles individually (can be done vice-versa as well). Equipped with the perspective of response analysis, the teacher will not give all the cards 1-20 to Group 1 right away. She will sequence this as she likes - perhaps, in groups of 5. It may be that cards of 6-10 will be flashed first, then 1-5, and then 11-15, – the rest of the sets of 5. Thus, this will help her identify and observe who (and how many) can identify numbers till 5, who can do it till 10 and so on.

This game is a very good activity to make children identify numbers, even if you give all the 20 cards to group 1,with a few modifications, the element of assessment emerges from within it. This change in methodology reflects the shift in perspective. All this equips the teacher to think logically and rationally. This should enable the teacher to think through the classroom processes which, in themselves, have the elements of assessment.

**For the Tool/Question Paper Developers** (who, eventually, are the practitioners) – understanding of the processes can improve the quality of tools and make them more diagnostic in nature. A field test of the tool, followed by such a response analysis will help the practitioner design better tools, because such analysis is only effective with "diagnostic tools". Let us try and understand this:

An example of a set of Diagnostic and Less-Diagnostic Items are mentioned below.

| Sl.no: | Competency Tested (Class I) | Set of Diagnostic items | Set of less Diagnostic Items |
|---|---|---|---|
| 1. | Counting from 20 to 50. | Q. Keep 10 pebbles in front of the child and ask the child to count them. If the child has counted correctly, add 25 more pebbles to this set, and ask the child to count them again. | Q. Keep 35 pebbles in front of the child and ask the child to count them. |
| 2. | Identify and recognize two digit numbers. | Q. Show the below- mentioned numbers to children with the help of flash cards, and ask them to identify the nos:<br>21, 52, 8 | Q. Show the below- mentioned numbers to children with the help of flash cards, and ask them to identify:<br>21, 52, 62 |
| 3. | Arranging two digit numbers in ascending or descending order. | Q1. Circle the biggest number out of the following:<br>25, 52, 39<br>Q2. Arrange the following numbers in increasing order:<br>7, 28, 9, 16 | Q. Arrange the following numbers in ascending order:<br>25, 39, 52 |

If we look carefully at both the sets, we recognize that one set will be able to provide us enough information to assess the child's learning, while the other set of items restricts this assessment somewhat. While (in the column marked "diagnostic item") each item is taking care of the competency mentioned, it is also trying to take care of the immediately preceding competency. For example, if a child cannot count beyond 20, can she at least count till 10 or 20?

Or, if she cannot identify two digit numbers, can she at least identify single digit numbers? Moreover, if you look at the set of diagnostic items, it is also following the same pattern and has lots of linked items.

But that does not mean that the other set of items is not at all diagnostic in nature. It just shows that the other set is less diagnostic. Otherwise, every item has the potential to provide some diagnostic information, provided a child attempts that item. Thus, by the very design of assessment tools, a teacher can save time and effort in detailed analysis later, as good tools allow such (linked and tangential) diagnosis to happen online.

**For Organizations** – Any organization who wishes to bring in qualitative changes in the class room teaching learning process can utilize Response Analysis as a tool.

*[Note – The perspective presented here is an outcome of the organizational learning and experiences put ogether to improve classroom teaching-learning practices.Though not new, the perspective remains unexplored by many of the practitioners and, if pursued, should benefit the larger section of the teaching-learning community.]*

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## Why 1 means "one", and 2 means "two"?

The numbers we all use (1, 2, 3, 4, etc.) are known as "arabic" numbers to distinguish them from the " Roman Numerals" (I, II, III, IV, V, VI, etc). The Arabs popularized these numbers but they were originally used by the early Phonecian traders to count and keep track of their trading accounts.

Have you ever thought why......... 1 means "one", and 2 means "two"? The Roman numerals are easy to understand but what was the logic behind the Phoenician numbers?

### It's all about angles!

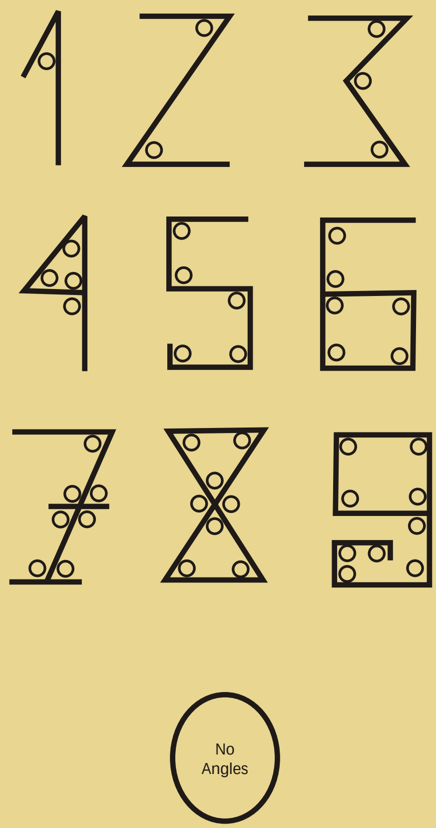
If one writes the numbers down (see below) on a piece of paper in their older forms, one quickly sees why. Notice the angles have been marked with "o"s.

No 1 has one angle.

No 2 has two angles.

No 3 has three angles. etc.

and "O" has no angles



23

# Insights about Student Learning from an Adaptive Learning Math Program

**Sridhar Rajagopalan**

## Introduction

Can we use computer technology to help our children learn better? On one hand, the way this technology has transformed our lives in every sector suggests that the answer is a clear 'yes'. Yet, in spite of a lot of investment over many years, the track record of computers in making a visible, positive impact on student learning – in India and internationally - has not been impressive. Many would say that the hype has been much bigger than actual outcomes.

We believe that the main reason for this has been that educational Information and Computer Technology (ICT) solutions have been too little 'educational' and too much 'ICT solutions'. We approached this problem from the education end – working for over 12 years on student learning (with no ICT offering in that period). For part of this period we ran a school while for another part we assessed the learning levels of lakhs of students across private and government schools testing for understanding and not mere recall of facts or procedures. Following the assessments, student interviews were conducted on important misconceptions that came out which told us why students were thinking in a particular way. Through workshops and diagnostic 'Teacher Sheets' we shared this feedback with teachers. But at a point, we realised that all the voluminous (and fascinating!) educational data we had could probably be used directly to create measurable learning improvement.

## Why ICT is Critical – the Complexities of Learning

We now believe that the complexities of the process of student learning are such that we have to rely on computer technology if we really want every child to learn with understanding. The complexity comes in two ways: one, the process of learning new concepts is inherently complex and probably contains more stages and nuances than a person who has 'learnt' them realises. And two, each student's way of learning a concept, not to mention the pace, tends to be different. Today we can support this constructivist paradigm with hard, detailed data on the learning levels of thousands of children.

## Underlying Principles

Mindspark is a computer-based adaptive learning system which primarily uses questions to help children learn. The questions are of 'finely-graded' meaning such that there are a very large number of questions of gradually increasing levels of difficulty. Questions are specially designed to test understanding and to help students clear misconceptions. (Increasingly, Mindspark student usage data is itself throwing up prevalent misconceptions.) There is very little emphasis on instruction due to the belief that students learn when they have to think – either to answer a question, or do an activity on the computer. This is also done as we see Mindspark as complementary to the teacher and in fact an unobtrusive professional development tool for the teacher himself.

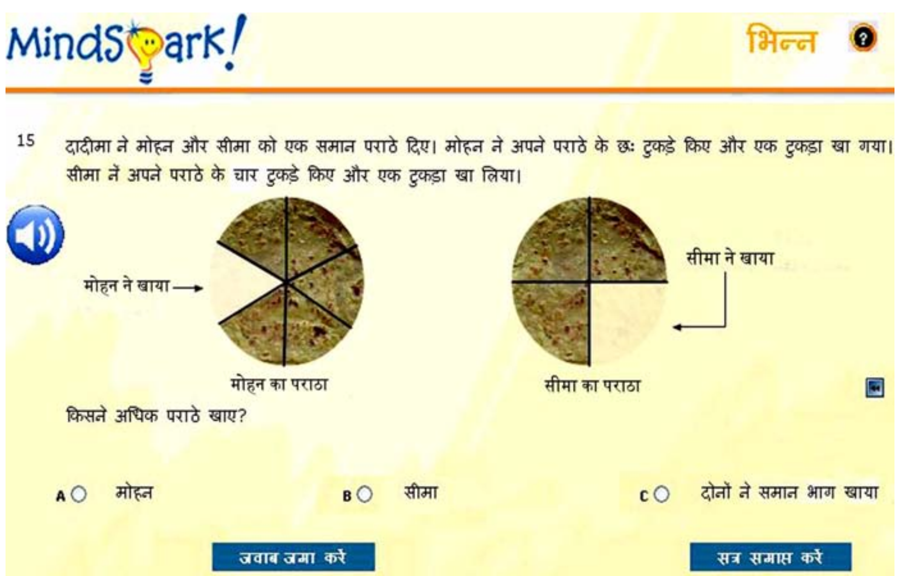



*Fig 1: A question-based adaptive learning software gets students to progressively answer questions of increasing complexity, allowing them to learn at their own pace and get basic concepts cleared before attempting higher ones.*

When a student answers a question incorrectly, she may be provided a simple or detailed explanation, or be redirected to questions that strengthen the basic understanding. These decisions are taken by an adaptive logic which is expected to get better and better with increased student usage. The system is available in private and government school editions and in both, it is used in school with usage time-tabled into the schedule. It thus aims to use not just the interactivity of the computer, but its intelligence; and mimics the diagnostic capabilities of a good teacher.

## The Benefits

There are many benefits of such an ICT-based adaptive learning system. To quote one, when students work one-on-one on problems, their attitude towards Math seems to change – they realise that when they engage with a problem, it is not so difficult to solve! (Perhaps a philosophy that goes beyond Math) Here, however, we shall focus on two important benefits: insights into learning for educationists and teachers and personalised learning that seems to result in learning improvements.

## Insights into Learning for Educationists and Teachers

A New Look at Learning: We are finding that the data from a program like Mindspark may allow such questions to be answered as could not even have been seriously asked before (for example, how much time does it actually take class 4 children to learn fractions?) The way we look at learning, as something we can perceive, but not get a very clear handle of, may change a fair bit with much more detailed information available on learning. Needless to say, 'detailed' does not mean accurate, so such data will need to be subjected to critical and sceptical scrutiny, but statements like this do seem justified from the data: "Ramesh has progressed 12.7% in the topic as compared to 71.3% average progress of the class in the same period of time." Individual student interactions convince us that these data do capture learning reasonably accurately and provide pointers to teachers on specific gaps that can now be addressed.

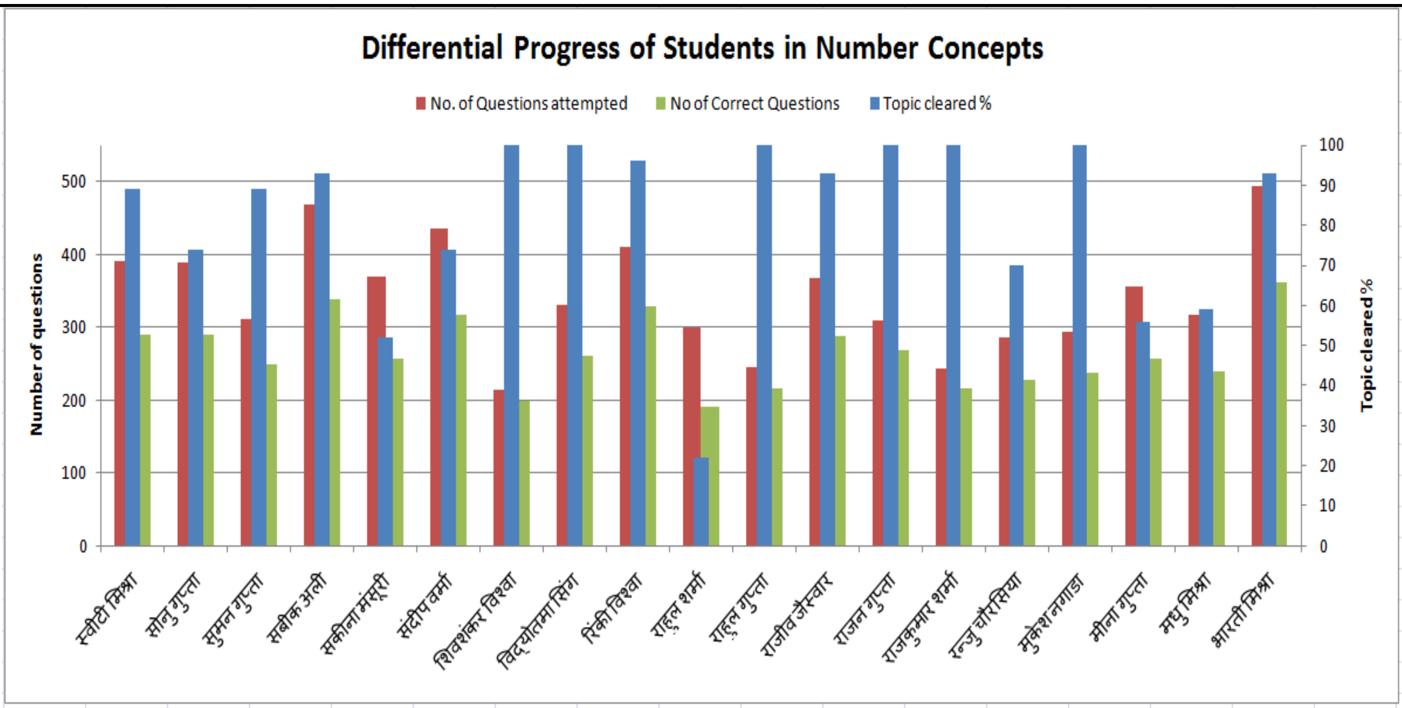



*Fig 2: Though these students spent the same amount of time, the number of questions done, accuracy and overall progress vary a lot.*

- *This is not a statement against whole-class or group learning which should happen in the other Math-learning time.*
- *Throughout this article, names have been changed to protect the privacy of students.*

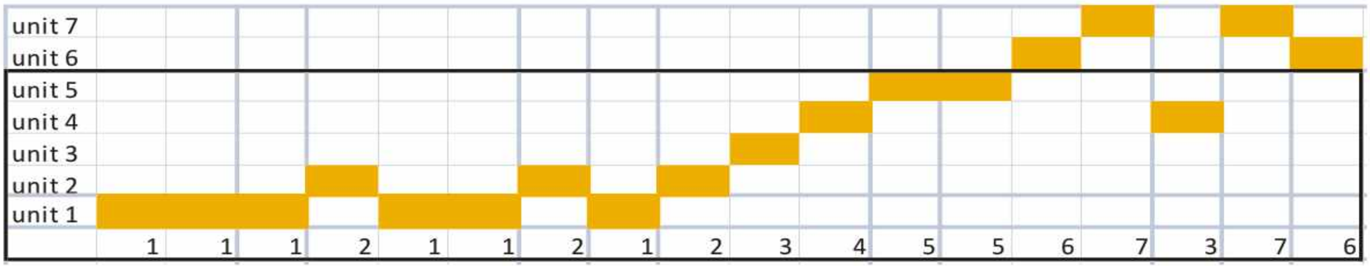



Fig 3: The progress graph of an individual student on a single topic.
The black border represents the topics corresponding to the current class.

| Sr. No. | Cluster Code | Flow No | Attempt Type | Cluster Attempt No | Result | No. of ques attempted | % correct | Last Modified |
|---|---|---|---|---|---|---|---|---|
| **Topic Attempt No:** 1 | (TT Attempt ID:36039) | | | | | | | |
| 1 | FRA003 | 3 | Normal | 1 | FAILURE | 17 | 52.94 | 2009-11-04 11:35:20 |
| 2 | FRA003 | 3 | Remedial | 2 | FAILURE | 23 | 60.87 | 2009-11-06 13:09:52 |
| 3 | FRA002 | 2 | Normal | 1 | FAILURE | 3 | 0 | 2009-11-06 13:11:02 |
| 4 | FRA002 | 2 | Remedial | 2 | FAILURE | 3 | 0 | 2009-11-06 13:11:33 |
| 5 | FRA001 | 1 | Normal | 1 | FAILURE | 19 | 73.68 | 2009-11-06 13:25:47 |
| 6 | FRA001 | 1 | Remedial | 2 | FAILURE | 22 | 59.09 | 2009-11-16 12:53:12 |
| 7 | FRA001 | 1 | Normal | 3 | SUCCESS | 18 | 77.78 | 2009-11-16 13:06:17 |
| 8 | FRA002 | 2 | Normal | 3 | FAILURE | 5 | 20 | 2009-11-16 13:08:32 |
| 9 | FRA001 | 1 | Normal | 4 | SUCCESS | 14 | 100 | 2009-11-16 13:17:25 |
| 10 | FRA002 | 2 | Normal | 4 | FAILURE | 18 | 72.22 | 2009-11-23 11:11:30 |
| 11 | FRA001 | 1 | Normal | 5 | SUCCESS | 16 | 87.5 | 2009-11-23 11:20:00 |
| 12 | FRA002 | 2 | Normal | 5 | SUCCESS | 17 | 76.47 | 2009-11-23 11:30:19 |

fig 4: A table with more detailed information on a different student's progress

Figures 2 to 4 show some of the detailed information available about learning from such systems. Note that each aspect of these graphs and tables can be drilled into till we get to the level of an individual student, an individual concept, or even an individual question. Since overall progress decisions are based on a large number of questions, they seem to correctly represent learning – as can be verified by teachers and researchers talking to the concerned students. The net insight: we need to go really deep into trying to understand how learning takes place, if we have to really improve student learning.

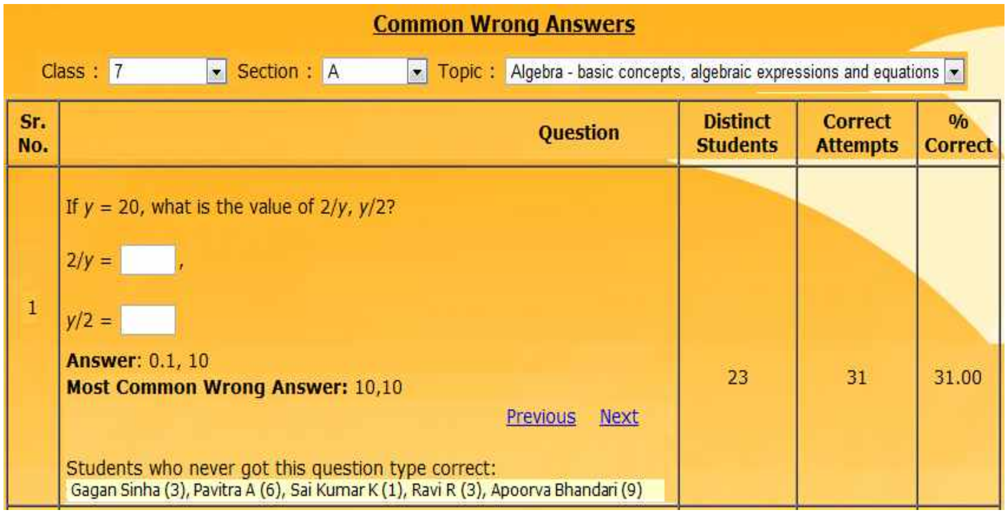
**The Individualised Nature of Student Learning:**

Preliminary data suggests that both learning paths and 'strengths' and 'weaknesses' tend to be fairly individualized. When we looked at the lists of better-performing students in different topics, we found that there was less overlap than we would expect from labels like 'strong in Math' or 'weak in Math'. Similarly individual misconceptions and the time taken to grasp specific sub-concepts seemed quite unique for each child.

In some cases (and we suspect they will be more frequent) students designated as 'slow learners' were found to be having difficulty in specific basic concepts – when these were cleared, these students performed significantly above the class average.

While a number of misconceptions common across students were identified, it was also found that many students had misconceptions and difficulties that seemed unique to the particular student (for example a student who seemed to be mixing up between ¼ and ½ consistently over a certain period of time.)

**Specific Information for Teachers:** we found that by and large, when provided with data which was actionable, teachers took interest and tried acting upon it. Teachers themselves sought certain reports – and these have turned out to be the most useful reports across all schools. Teachers say that information about concepts not learnt by all or even a sub-group of students is extremely useful for them and allows them to immediately address the problem. There are occasionally cases where the student has not learnt in spite of various system strategies. Such cases are referred by the system to the teacher – something teachers have appreciated.


Common Wrong Answers

Class : 7 Section : A Topic : Algebra - basic concepts, algebraic expressions and equations

| Sr. No. | Question | Distinct Students | Correct Attempts | % Correct |
|---|---|---|---|---|
| 1 | If y = 20, what is the value of 2/y, y/2?<br>2/y = ___ ,<br>y/2 = ___<br>**Answer:** 0.1, 10<br>**Most Common Wrong Answer:** 10,10<br>Previous Next<br>Students who never got this question type correct:<br>Gagan Sinha (3), Pavitra A (6), Sai Kumar K (1), Ravi R (3), Apoorva Bhandari (9) | 23 | 31 | 31.00 |

*Fig 5: This report was created iteratively based on a request from some teachers. It lists out the 'common errors' in class or section. A question type would have different questions and a teacher can see all the different questions and the errors children made. The names and number of attempts of students who never got the question type correct (shown in yellow highlight) – became a simple action point for teachers.*

**Subject-specific Insights:** needless to say, there are many specific errors that are detected from the data. Many are known from research, but teachers find this easier to relate to and accept. Some examples from Algebra, class 6 (which, incidentally, tallies with research in this area):

See, for example, Warren, 2003.

- *5 less than t' is considered 5 - t.*
- *20 = 5x4 is considered wrong by many students*
- *Difficulty representing expressions like 'half of 8 less than n' algebraically*
- *Difficulty with questions involving minus signs like the value of -5t or -t + 5 when t equals 2*
- *Confusion between terms like k + 3, 3k, $k^3$, etc.*

## Personalised Learning that seems to lead to Learning Improvements

Programs like this try to tackle the challenging question 'what is needed to help children learn a new concept well?' As most teachers know, the answer seems to be different for different topics, different students and even at different times!

At times, students seem to learn through repeated exposure that increases the familiarity of a concept. At other times, identifying a key misconception and providing more exposure to that concept seems to aid learning. And sometimes, a difficult or non-intuitive concept has to be carefully explained.

We have interesting examples where all these three seem to work. Three cases, one of each type, are shared below:

### Case 1: Counting in 10s:

In a municipal school which was using Mindspark, students of class 5 were struggling with class 1 and 2 concepts (unfortunately, a common situation) Sapna, a Class 5 student, was doing place value questions in Mindspark. She reached a question in which she had to count the total number of beads. She counted all the beads one-by-one – up to 84, in that case!

But after a few such questions, in the next level, the visual would disappear quite quickly, not giving her time to count in ones. On the third or fourth such question, she started off – really fast and furiously – 'dus-bees-tees-chalis-ektaalis-bayalis-tetalis-chawalis-paentalis' ... all in one breath! Then, she took one long breath and wrote '45' (the correct answer) in the blank! The observers burst out laughing - she had 'learnt' to count in tens-and-ones without 'instruction'.

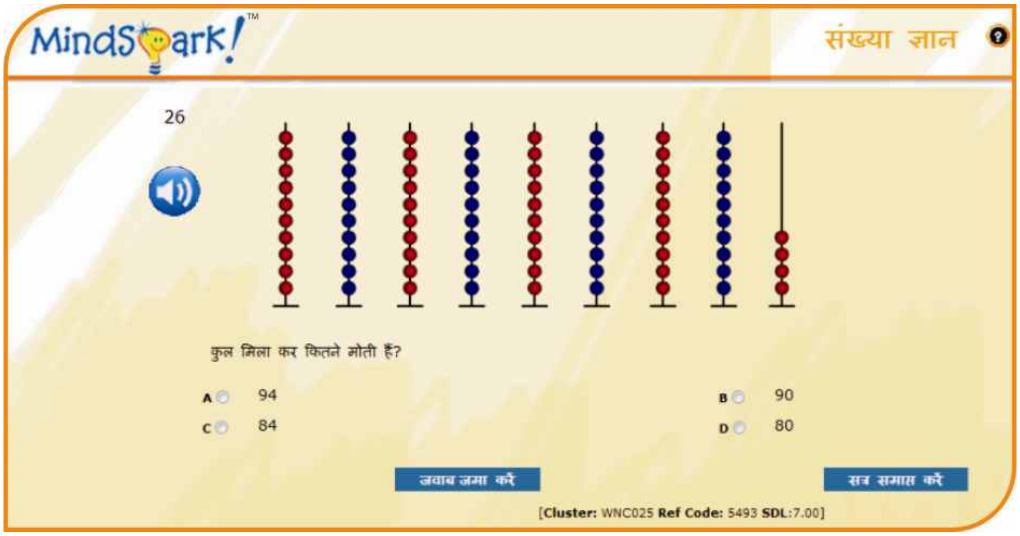
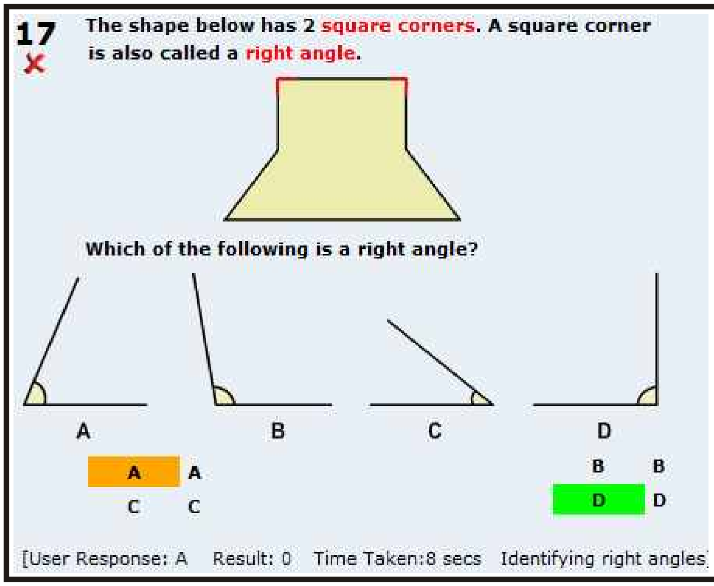



*Fig 6: Simple animations – here, a set of beads that need to be counted before they disappear – helped a child understand counting by tens.*



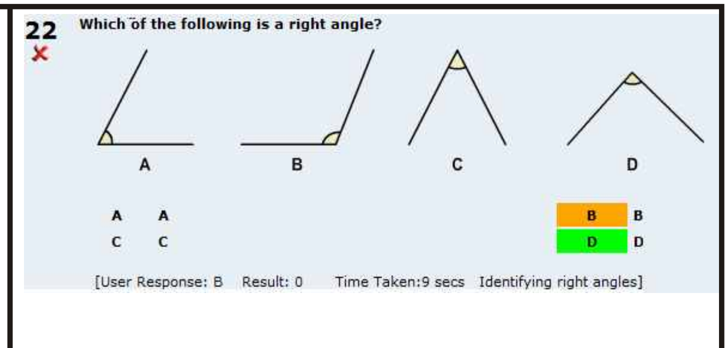
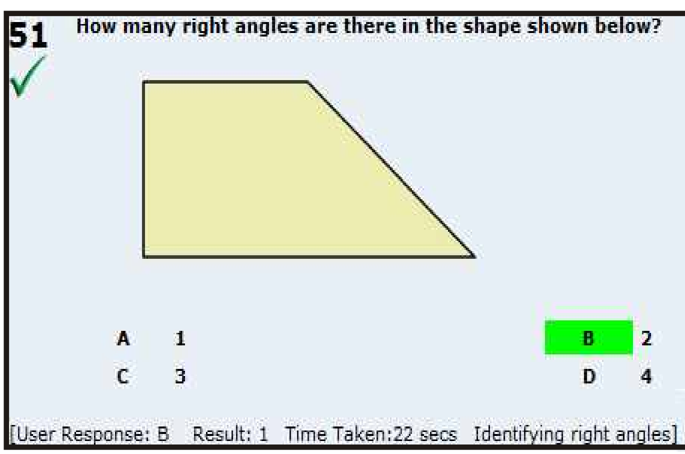




*Fig 7 (left): recognising a right angle in standard orientation*
*Fig 8 (top): recognising a right angle in non-standard orientation*

The student was unable to identify a right angle visually in the beginning either in the standard or non-standard orientation. After a number of questions that both challenge and explain this concept in different ways, the child is clearly able to answer these questions correctly.



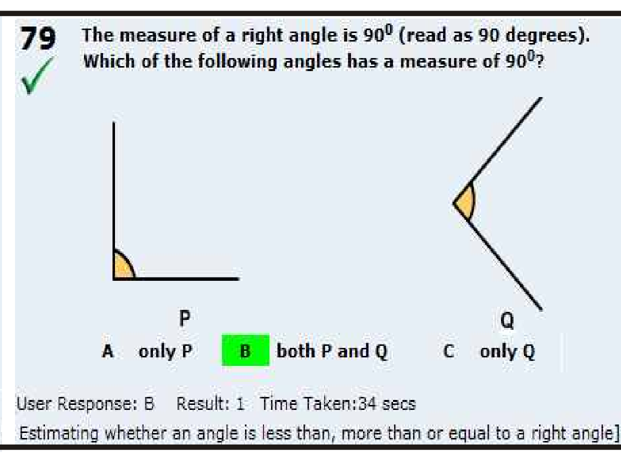




*Fig 9 (far left) and Fig 10: more questions on right angles*

**Case 2: Understanding Angles:**

One of the most interesting analyses is the study of a student's 'trail' – the questions attempted by him or her one by one (sometimes over weeks). Here are some questions in a trail of questions on 'angles' attempted by a class 5 child. The questions were done over a week and the sequence numbers are shown. Though only 4 questions are shown here, the inferences are based on analysing the student's responses to many more questions and the pattern is clear.

**3: Operations with Negative Numbers:**

In some cases, students learn when they are given conceptual reinforcement that addresses a doubt or a problem they have, or gives them an appropriate analogy that helps learning take place. This applies to doubts related to negative numbers, the concept of variables in algebra, etc.

In all these cases, more studies need to be conducted to confirm that the learning is really robust. However, early indications were clearly positive, and suggest that some of these strategies were working to help children learn key concepts they were previously struggling with.

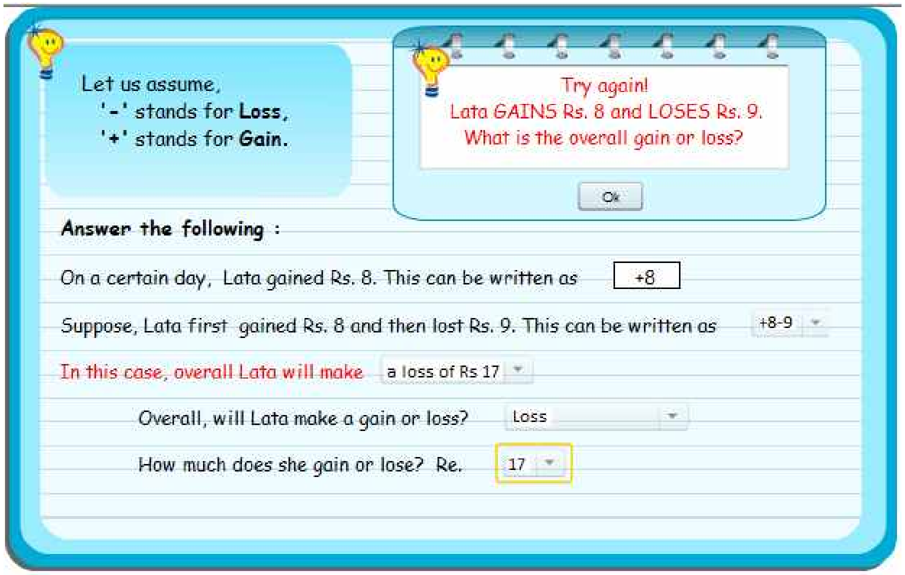



*Fig 10: The above is an interactive remedial item that is only given to students who are not able to handle integer operations. By giving analogies and leading the child through a series of exercises, some children were found to understand the concept (rather than simply remember and apply a rule to handle problems.)*

**The Long Term Vision**

Some people fear that the use of computers in instruction is undesirable as it ignores the human and relationships aspects of learning. There is also a fear that such technologies may further weaken the role of the teacher. Doubts are expressed whether students can really learn through questions.

Though it would not be possible to discuss all these objections in detail, we believe that in the long term, computer learning programs like these will be an important part of education, which will strengthen, not weaken the teacher. The teacher will continue to provide instruction, inspiration, challenge and guidance; the computer will provide individualised practice, remedial, challenge, clearing of conceptual doubts, and to the teacher – pointers on where she should focus. This freeing of the teacher's time, we believe, will allow her to spend more time on many things that are desirable but get left out today – development of social and team skills, project work, real-life activities and the ability to focus more on individual children and their specific needs.

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7. The article 'The Potential of Assessment in Science' (2009) by Agnihotri et al describes this process and also some interesting misconceptions found among students.
8. Many researches on student learning confirm this. Seemingly basic concepts like place value, the notion of respiration or the measurement of learning can be surprisingly difficult to really grasp. See, Kannan and Shukla (2008) and Kamii (2006).

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## Numbers

All numbers are created equal
Yet some are more prime
But then why some are real
And others are truly imaginary?

All numbers are created equal
Yet all numbers are complex.
But then why some numbers
Are more rational than others?

All numbers are created equal
And many numbers are integral
But then why some are positive
And others are negative?

All numbers are created equal
And many numbers have values
But then why there is a number
That's zero like an empty tumbler?

Are there equal numbers?
Or is it only in my slumber
That I see transcendental numbers
And the eternal infinity?

- By Krishnan Balasubramanian

# section E

# Personal Reflections on Mathematics

# 24 Ten Great Mathematicians

**H Subramanian**

**Introduction**.

It is not an understatement to say that there are essentially two subjects to study, viz. language and Mathematics. Everything else becomes dependent on them eventually. All of science, including management and social sciences, borrow on Mathematical formulations for concepts. The reason is simple. A precise and concise expression is made possible within the framework of Mathematics. Actually, the abstract layers of certain structural behaviours are increasingly becoming visible in the context of challenging problems from time to time. In this respect, Mathematics is also taken as another language - a symbolic one. Which Mathematicians can be singled out to have induced this transformation?

On a personal note, let me narrate an event. I was presenting a result as a Ph.D. student in Madurai to a visiting renowned Mathematician Nathan Jacobson from Yale University, USA. He listened and finally asked a two-word question "Then what?" I was totally nonplussed. Later, I found an eye-opener in his question. If a result just closes an issue, it is no big deal. If it leaves a door open for further thoughts, it is a great contribution. Which Mathematicians can be recognized using this screener?

There are several worthy names to consider. It is a hard choice to select just ten persons in the history of Mathematics who have influenced its development and changed the perception of the subject over time. My personal choices are Euclid, Fermat, Newton, Euler, Lagrange, Gauss, Cauchy, Riemann, Hilbert and the nom de plume Bourbaki. There are several geniuses who have missed the hit list. One may wonder why I have not included, say, the German physicist Albert Einstein (1879-1955), the Russian Mathematician Andrey Kolmogorov (1903-1987), the Indian Mathematician Srinivasa Ramanujan (1887-1919) or some ancient Indian Mathematicians. Einstein will find his place as one of the ten great physicists as he changed the perception of physics by his relativity theory. Kolmogorov is recognized as the founder of axiomatic probability theory influencing a lot of stochastic methods; but he does not spell a structural influence in Mathematics. Ramanujan is outright marvelous for his strong intuitive contributions; however this genius does not spur a change of perception of Mathematics. As there will be a separate article on ancient Indian Mathematicians, I am not considering them.

Let me take up the ten cases now. I have chosen to list them in the order of the years they lived. In my opinion, the follower in the list was, at least indirectly, influenced by the work of the earlier ones.

> *It is not the final result of Fermat's Last Theorem by itself that is great; but the techniques that have been developed to get a correct proof for it makes it stupendous.*

### 1. Greek Mathematician Euclid (around 300 BC)

The first axiomatic treatment in Mathematics was started by Euclid. His structure of proofs in geometry based on postulates about points and lines is a major conceptual contribution. His division algorithm for natural numbers is the most used one and the general Euclidean domains in future Mathematics emerged on that principle.

It is important to assess Euclid's achievements in the historic context of what was perceived as Mathematics before his time. It was a folklore understanding that it is all about practical computations in arithmetic and measurements of geometrical objects. Since around 1700 BC, numerical, algebraical and geometrical methods are attributed to Babylonian Mathematics. Later, a level of practical arithmetic and mensuration developed in Egypt and Italy. Much later, approximately in the 7th to 6th centuries BC, we recognize the Greek Pythagorus and his contributions to geometry. Further on, Hippocrates is credited for circular arcs and areas; and, the logical thinking Zeno (remember his paradox!) introduced the concept of divisibility into infinite parts. During 430-349 BC, Plato's directions into philosophy dominated the conceptual frame of thought. Aristotle (384-322 BC) overlapped Plato's time with his say in formal logic and algebraic notation. By early 4th century BC, ideas of irrational numbers, geometric formulation of areas, algebraic procedures and even integration took roots.

It was Euclid's Elements (see Heath [Hea56]), Appollonius's Conics and Archimedes's Analysis that has been recognized as serious Mathematics for centuries. The codification of geometry can be attributed to Euclid. We can say that Euclid, Archimedes and Appollonius made one great age of Mathematics.

It must be said that a relevant and practical Mathematics was presented in abstraction by Euclid. Such abstracting process has gained currency to transform Mathematics to unbelievable levels over time.

Descriptive geometry as well as algebra dominated the thinking during the first few centuries of AD. Menelaus, Ptolemy and Pappus developed synthetic geometry and Diophantus carried the banner of algebra, influenced by the Oriental thought.

From time immemorial to around 12th century AD, the growth in Mathematical thinking can be felt in terms of (i) computational arithmetic transforming to algebra and (ii) measuring geometrical entities (including studies in astronomy and spherical trigonometry) changing to the synthetic version of geometry like the ingredients of projective geometry. What happened during the next 400 years is mostly unknown. It mostly pertains to the contributions by Indians, Arabs and Greeks passing into the western world of Europe.

As it happened later in history, geometry went beyond Euclid's postulates. Geometry, as learnt today, is enmeshed in algebra.The key was the coordinatisation of geometry. The credit for this goes to Rene Descartes (1596-1650) (see Descartes [Des37]). For the next 200 years, significant achievements in Mathematics were focussed mainly on number theory and analysis. But Riemannian geometry (1894) brought another realm of geometric focus. Some other developments in geometry happened in the $19^{th}$ century. Along with some significant revelations in foundations through set theory, a thought provoking model of non-Euclidean geometry surfaced. The Russian Mathematicians Nikolay Ivanovich Lobachesky (1792-1856) and the Hungarian Mathematician Janos Bolyai (1802-1860), during the years 1826-1832, questioned Euclid's parallel postulate and obtained the geometric models without this postulate.

## 2. French Mathematician Pierre de Fermat (1601-1665)

Fermat was the great number theorist. Without going into all his work, especially on prime numbers, let us only touch upon his remarkable scribbling in the margin space of his copy of the book by Diophantus. He mentioned that the space was insuffcient to write his proof about the impossibility of a nonzero integer solution of the diophantine equation $x^n + y^n = z^n$ for $n \geq 3$ unleashed a significant trend thereafter in number theory and algebra to unravel his intended proof. This is familiarly known as Fermat's Last Theorem. The German Mathematician Emst Kummer (1810-1893) gave a false proof of this result based on a mistaken understanding of a certain factorization of $x^n + y^n$ as irreducible and unique. But he corrected his mistake soon and laid the seed for the modern ideal theory in rings.

Many other Mathematicians like Euler, the French Mathematician Adrien-Marie Legendre (1752-1833), the German Mathematician Johann Peter Gustav Lejeune Dirichlet (1805-1859) and Cauchy tried in vain to reconstruct the apparent proof that Fermat might have intended. They succeeded only on the cases $n$ = 3; 4; 5 of Fermat's Last Theorem. With the advent of computers, most sophisticated computer methods were harnessed to verify this result; but they did not succeed. Fermat's work was actually published posthumously in the year 1679. A settlement on this issue took about 400 years. We refer to Edwards [Edw77] and Andrew Wiles [Wil95] to appreciate the vast realm and technicality of Mathematics that it created on this matter.

It is not the final result of Fermat's Last Theorem by itself that is great; but the techniques that have been developed to get a correct proof for it makes it stupendous.

### 3. British Mathematician Sir Isaac Newton (1643-1727)

Newton was the most original contributor among the 17th century Mathematicians. The binomial theorem for rational exponents led to the ideas of infinite series. Obtaining areas by the method of summation of infinite subparts can be attributed to many people, Archimedes onwards to the British Mathematician John Wallis (1616-1703). The idea of differentiation goes back to the French Mathematician Blaise Pascal (1623-1662), who is credited also for the invention of digital calculators. In this background, both Newton and the German Mathematician and philosopher Gottfried Wilhelm Leibniz (1646-1716) attempted the fundamental principle of calculus, viz. the inverse process of differentiation is integration. Leibniz got to calculus independently during 1673-1676 from the works of the British Mathematician Isaac Barrow (1630-1677) and Pascal. Newton got to calculus from the ideas of his teacher Barrow as well as Fermat and Wallis. Both Newton and Leibniz were unable to establish calculus on sound logical basis. Newton included in his work (see Newton [New87]) a short offhand explanation of calculus. This was the greatest of all scientifc treatises; in it can also be found the greatest of scientifc generalizations, viz. the law of gravitation. Newton's aim was to understand nature.

Newton later on explained his calculus on rates of change. In this process, he used the binomial theorem for integral exponents and generalized the same for rational exponents. Leibniz used the idea of infinitesimally small differentials, denoted by $dx$ and $dy$ and tried (unsuccessfully) to explain his method in terms of sums and quotients of these. We are using this notation even today. The method of differentials became the mainspring of Mathematical development during the course of 18th and 19th centuries AD. Leibniz's influence on the European continent was far greater than Newton's. Continuity of functions was an intuitively accepted phenomenon in the earlier rounds of thinking. Fortunately for the progress of Mathematics, Newton and Leibniz took for granted that all functions have derivatives. It was difficult enough to obtain an acceptable notion about it. To the differential calculus thus discovered was added the opposite construction of the integral calculus; and the work of Archimedes was belatedly rediscovered. After sometime, it was realized that certain functions might be represented by power series also. Mechanics, even for Newton, led to considering functions as integrals of differential equations. The infinitesimal geometry of curves, extended to surfaces, demanded the introduction of equations with partial derivatives, necessary also for mechanics of vibrating cords. A study of periodic phenomena led to the consideration of trigonometric series. The original classic by the French Mathematician Jean-Baptiste-Joseph Fourier (1768-1830) (see Fourier [Fou22]) contains the basic ideas in this regard, although with no rigour. Thus each forward step in Mathematics engendered, by a chain of creations, the introduction of new entities which were used as tools for other studies and other creations.

Newton's contribution changed the face of both Mathematics and physics.

### 4. Swiss Mathematician Leonhard Euler (1707-1783)

The Bernoulli family from Switzerland, running through mid 17th century to late 18th century, were fired with enthusiasm for differential calculus. Euler was Johann Bernoulli's student. He, in his treatise (see Euler [Eul48]), brought the concept of function and infinite processes into analysis. Euler was one of the founders of calculus of variations and differential geometry. Both are applications of differential calculus, to cases in which a function depends on another function or curve in calculus of variations and to general properties of curves and surfaces in differential geometry. After Fermat, Euler was the greatest number theorist. The Euler totient function has its claim of importance in number theory. It extended a Fermat result on prime numbers to nonprime numbers also. Euler's formula in radicals for quartic equations is the last such case because Abel proved later that it is impossible to obtain such formulas for fifth and higher degrees. His method of extraction of square roots modulo prime numbers and continuation of certain work by Fermat is another contribution that led Gauss to explore still further. He made strides into continued fractions. In 1779, he also posed a certain conjecture on orthogonal Latin squares; but it took nearly 200 years to prove Euler wrong (see Parker [Par59] and Bose, Shrikhande and Parker [BS59] & [BSP60]). His calculus of quotients of qualitative zeroes and operating with sums of divergent series created furore. Euler also contributed to Mathematical physics.

While Greeks put everything side by side, all aspects of real numbers did not merge with the Greek heritage. While this is so, it seems that the later developments from the beginning of 17$^{th}$ century happened through a perception of compelling need to expand, analyze and rationalize various aspects together. Euler is to be credited for his wide-spread contributions in Mathematics, though he was seen to lack soundness in logical foundations. After Newton and Leibniz, the pioneering 18th century Mathematician was Euler.

## 5. Italian-French Mathematician Joseph-Louis Lagrange (1736- 1813)

Born in Italy, Lagrange had a career in Prussia and later moved to France. In the context of the French Revolutionary period, we note the establishment of Ecole Polytechnique in the year 1794. Lagrange and Pierre-Simon Laplace (1749-1827) were its first teachers. Lagrange's contribution is in a variety of subjects - algebra, number theory, analysis and mechanics. His interpolation formula is one of the most applicable results in numerical analysis. The result in group theory known today as “the order of an element in a finite group divides the number of elements in the group” is attributable to his work extending Euler's congruence (see on the preceding page) that absorbed Fermat's congruence. He continued Euler's work on continued fractions. He discovered a rule for existence of multiple roots of a polynomial based on the greatest common denominator of the polynomial with its derivative. He is the cofounder, with Euler, of calculus of variations. In his works (see Lagrange [Lag88] & [Lag97]) on mechanics and analysis, he uses deductive logic to bring mechanics in the framework of rigorous Mathematical analysis. Lagrange's theory of functions, the graphical representation of complex numbers obtained in 1813-1814 by the Swiss Mathematician Jean-Robert-Argand (1768-1822) and the imaginary period of elliptic functions explained by Abel in the year 1824 led Cauchy to the so-called integral theorem of complex functions in the year 1825. Lagrange is continuation of Euler with a difference. He brought a level of abstraction that paved a smoother path for future Mathematicians.

## 6. German Mathematician Carl Friedrich Gauss(1777-1855)

Gauss was one of the first to feel the need for rigour in Mathematics. In the year 1796, at the age of 19, he studied the Euclidean constructions in geometry; arrived at the notions of constructible numbers; and, created a setting for all algebraic numbers of certain types to be considered. The phrases used today like Gaussian integers, Gaussian elimination method, Gaussian distribution, Gaussian channel acknowledge his remarkable contributions in related areas. In 1799, he proved that every polynomial equation with complex coefficients has a root. In the year 1801, he contributed (see Gauss [Gau01]) to number theory in terms of theory of congruences and quadratic reciprocity; these are enormously significant results. His quadratic reciprocity law enabled many numerical calculations without effort. He conjectured that for infinitely many prime numbers p, p-1 is the least number such that p divides $10^{p-1}$ -1. This conjecture is still unresolved. He provided the basic result, known today as Gauss Lemma, that enables construction of unique factorization domains. He contributed (see Gauss [Gau28]) to differential geometry exploiting the parametric representations.

It is often the case that serious researchers today look backwards to Gauss for inspiration. Another reason for them to do this is to make sure that this giant has not already introduced their ideas.

## 7. French Mathematician Augustin Louis Cauchy (1789-1857)

It was Cauchy who succeeded in introducing clarity and rigour. He was “forced to accept propositions which may seem a little hard to accept; for example, that a divergent series does not have a sum". He introduced, with precision, the necessary definitions of limit, of convergence, and thus made possible, in a short time, great advances in areas which had been finally clarified.

Actually at the beginning of 19th century, Cauchy closed one period in the history of Mathematics and inaugurated a new one which would appear to be less hazardous. He ruthlessly tested the product of three centuries, establishing an order and a rigour unknown before. He rejected as too vague the habitual appeals to "generality of the analysis" and determined the conditions of validity of statements in analysis with rigorous definitions of continuity, of limits, of different sorts of convergences of sequences or series, which he provided. In the early 19th century, Abel described the state that Mathematics was in when he entered it thus: "Divergent series are completely an invention of the devil, and it is a disgrace that any demonstration should be based on them. One can draw from them whatever one wishes when they are used they are the ones that have produced so many failures and paradoxes. Even the binomial theorem has not yet been rigoursly demonstrated. Taylor's theorem, the base of all higher Mathematics, is just as poorly established. I have found only one vigorous demonstration of it that of Cauchy."

In the year 1825, Cauchy proved the well-known integral theorem of complex functions. At the core foundation level, he offered a construction of the real number system. Also constructed the real number system. Much later in the year 1895, the German Mathematician George Cantor (1845-1918) formulated (see Cantor [Can95]) the theory of sets and introduced how to reason in a framework. In doing so, he justified Cauchy's construction of real numbers to be the same as two other versions created by the German Mathematicians Richard Dedekind (1831-1916) and Karl Weirstrass (1815-1897). We refer to the complete works of Cauchy in [Cau74] for his detailed contributions.

We can say that Cauchy and Cantor started the age of reason. It is Cauchy who laid the strong foundation of Mathematics by insisting on logical reasons to prevail at every level.

## 8. German Mathematician Georg Bernhard Riemann (1826-1866)

During the mid-19th century, Riemann ruled the development of future Mathematics beyond the imagination of those times. One of his masterpieces of work is complex function theory. His geometric intuition in complex analysis in terms of conformal mappings and the so-called Riemann surfaces, his method of handling differential form for arc length, curvature tensor etc. (in the year 1854) involving ideas found useful and essential later in general relativity and his conjecture (in the year 1859) that is still unresolved going by the name Riemann Hypothesis and which has unleashed an unsurpassed level of Mathematical developments - all these qualify him as the undisputed master and genius.

Riemann's Hypothesis remains one of the most intriguing conjectures in all of Mathematics. It is difficult to describe it without going into some Mathematical vocabulary. It states that all the nontrivial zeroes of a certain complex-valued function of a complex variable, described in terms of an infinite series, must have real part equal to ½. The subject matter of this unresolved conjecture has triggered valuable research in analytical number theory and complex function theory. Riemann Hypothesis is still unproved. But several thousands of zeroes have been verified to fall in place by use of computers.

The Riemann Hypothesis, if found true, would have enormous consequences in number theory. For instance, it would establish a better handle on the nature of distribution of prime numbers. No one would have thought about a connection between prime numbers and analytic functions.

We refer to Riemann's works (see Riemann [Rie90]). We also refer to Laugwitz [Lau99] to find how Mathematics changed since Riemann.

## 9. German Mathematician David Hilbert (1862-1943)

The German Mathematician David Hilbert (1862-1943) provided the steering wheel for most of the Mathematical developments during the 20th century by his 23 famous problems presented in his address to Second International Congress of Mathematicians at Paris (see Brouder [Bro76]). The surge in ideas contributed by these problems was huge. Though these challenging problems developed future Mathematics to a great extent, some of his expectations were belied later. A notable one was his own student Godel's result (see Godel [Go40]) in logic proving that arithmetic cannot be simultaneously consistent (meaning both a statement and its opposite cannot be true) and complete (meaning either a statement or its opposite is true); that was contrary to Hilbert's intuition. Hilbert's contribution, in the year 1906, to the theory of infinite dimensional spaces is immense. Hilbert is remarkable for his conceptualization of certain earlier trend-setting problems. In the second half of 19th century, problems in electrostatics and potential theory led to a study of integral equations. In the year 1877, the United States Mathematician George William Hill studied matrices of infinite size relating them to perturbation theory of lunar motion. And, in the year 1900, Ivar Fredholm discovered the algebraic analogue of the theory of integral equations. Henri-Leon Lebesgue laid the ground work with measure theory for the later contributions by Banach and Frechet to generalize Hilbert's work. In the year 1922, the Austrian-Hungarian (later territorially Polish) Mathematician Stefan Banach studied the aspects of “geometrizing" the spaces. And the French Mathematician Maurice-Rene Frechet, in the year 1928, generalized these further. Hilbert's contributions in commutative algebra, diophantine equations, number theory etc. are equally noteworthy.

Hilbert's list of problems kindled so much of research activity during the first five decades of 20th century that this period is known as the golden age of Mathematics.

**10. Nicolas Bourbaki (1935-continuing)**

A group of Mathematicians (mainly French) go together under this pseudonym. Nicolas refers to an ancient Greek hero from whom a French General Charles Soter Bourbaki apparently descended. When Andre Weil was a first year student in Ecole Normale Superieur. He attended a lecture by a senior student, who mockingly presented false theorems and attributed them to various French generals. The last and most ridiculous theorem was named after this Bourbaki. Andre Weil, Henri Cartan, Claude Chevalley, Jean Dieudonne and a few others, all young under 30 years of age, were passionate to bring about changes in future curriculum of Mathematics. Their thought was to publish books from a conceptual and fundamental standpoint. The idea of a forum was born and it was named after Bourbaki, with humour intended.

The first Bourbaki Congress was in the year 1935. The group decided on certain rules for themselves. Membership is limited to 9. Each member would retire at the age of 50. They would meet three or four times a year, each time for a week or two, for a total discussion on the various projects. All members should participate in every project. A member can bring a colleague or even a student as long as these invitees would participate to the same extent that a member is expected to. They should use only axiomatic framework and structural classification of Mathematics to write the books that are agreed to be written. No references could be cited other than a Bourbaki book in line. Everyone would engage in these projects. By turn, each member would prepare a presentation of a topic or a chapter for the book and all others would discuss the material in entirety. Together they would arrive at the final version. There is no authorship and the publishing will go under the name Bourbaki. Bourbaki's aim was to publish high-class text-books quickly. The target size of a book was about 1000 pages running into approximately 10 chapters with the target time set for about 6 months. Their initial list of books is:

(a) Book I. Set Theory

(b) Book II. Algebra

(c) Book III. Topology

(d)Book IV. Functions of One Real Variable Variable

(e) Book V. Topological Vector Spaces

(f) Book VI. Integration

The project moved slowly; only certain chapters were completed by 1942 because of World War II and members going abroad. Roger Godemont, Jean-Pierre Serre, Pierre Samuel, Laurent Schwartz, Samuel Eilenberg were recruited as new members. By 1958, the books were completed. By then, some founding members stepped down and Alexander Grothendieck, Serge Lang, John Tate joined.

Meanwhile Mathematics had grown to a considerable extent, also owing to the inuence of Bourbaki. Members felt that they were not universal Mathematicians to join in all the book projects. However, the decision of Bourbaki was that, even if not universal, their interest to participate in everything was mandatory. No member could stay on the principle of their specialty contribution only. About the rigidity of linear order of arrangement of the topics in books, compromise was made that an organic development would be acceptable without disrupting the unity of Mathematics and structural aspects. The earlier six books needed revisions so new projects could build on them. Bourbaki carried out these and also completed by the year 1980 quite a few chapters in three more books, viz.

(a) Book VII. Commutative Algebra

(b) Book VIII. Lie Groups and Algebras

(c) Book IX. Spectral Theory

Yet another one (Differential and analytic varieties) completed by Bourbaki was just a summary of results and not a full exposition of thoughts. It was just to help the organization of other books.

The various titles of Bourbaki books given here are written in English. Actually, the titles and the work were in French. The first 6 books were later translated into English by Bourbaki.

Bourbaki also publishes several survey articles on advanced topics with their intention to reach out these topics to nonspecialists. These titles emerge after intensive seminars held frequently.

Bourbaki certainly set standards for what a professional Mathematician should know. They brought out books aimed as text-books but they are more like reference books or encyclopedia-cum-treatise-cum-monograph or whatever. What will be the future of Bourbaki? Will their book projects die? Will their seminars and publications take over the main Bourbaki engagement? Will specialists only prevail under Bourbaki banner? The basic starting reason for Bourbaki is the unity of Mathematics. History shows that survival.

## Conclusion

Like I said in the beginning, language and Mathematics are the only two subjects that are characteristically fundamental. Through them, we can handle whatever appropriate expressions are needed to convey ideas in essence (abstraction) and lead these ideas to explore further thoughts (concretisation). They have become essential and irreplaceable in our daily life. The selected Mathematicians in this article have amply fortified this and made future embellishments a distinct possibility.

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## 25 The Hypocrisy in Math

**S C Behar**

There are two different conceptions of Mathematics. In one the emphasis is on quality; in the other a much greater emphasis is on quantity. One believes that Mathematics is to train the mind; the other is based on the assumption that Mathematics is all about quantity. They are not only different, but are in fact divergent, even to the extent of being contradictory. They reflect different perspectives on the universe and life. The irony is that the two different conceptions are made to appear as similar or even identical. The divergences or the contradiction are sought to be camouflaged through clever use of words.

The first conception was presented in ancient times by Plato, when he said, "above all, arithmetic stirs him up who is by nature sleepy and dull, and makes him quick to learn; retentive, shrewd, and aided by art divine, he makes progress quite beyond his natural powers.”. Without listing the names of all those thinkers who believed in and propagated this kind of Mathematics, let me immediately jump to the National Policy on Education, 1986 as amended in 1992, which says, "Mathematics should be visualised as the vehicle to train a child to think, reason, analyse and to articulate logically. Apart from being a specific subject, it should be treated as a concomitant to any subject involving analysis and reasoning”. There's not a word about measurement, quantification and numbers. If Mathematics was conceptualized and propagated in this manner, the history of epistemology would have been different.

The second concept has been the dominant one in actual practice ever since but more particularly in the last five centuries, when it was considered to be the axis and foundation on which advancement in science and technology was built. It is credited with being responsible for a major upward shift in human history, the graph of human progress rising almost vertically. Ever since acquiring this honour, this conception seems to have completely overshadowed the first conception emphasising quality. This conception is reflected in the following excerpts from the National Curriculum Framework 2005: “the narrow aims of Mathematics consist of developing useful abilities in numeracy -- numbers, number operations, measurements, decimals and percentages". Even when this framework goes on to the higher aims, it substantially and materially differs from the policy document. It says that the higher aims are “to develop the child's resources to think and reason **mathematically**”. (Emphasis added). Unfortunately the narrow aims have become the only aims in practice globally as well as historically. While the NPE refers to thinking, reasoning, analyzing etc generally without any qualifications, the framework brings the qualification of making them narrower -just mathematical. The narrower concept within the higher aims continues in a little muted form, when it goes on to refer to ' mathematical communication, ' being precise ' and emphasizes ' rigour in formulation ' ' the use of jargon ', and states that ‘good notation is held in high esteem and believed to aid thought '. These are all statements, which taken together, project an image of Mathematics which is different from the conceptualization in the policy document. One cannot blame the curriculum framework for this notion of Mathematics. This notion is widely spread, practiced and believed. The framework only captures it and presents it, to give due credit, in a more subdued form than the general belief among the students, parents and public at large about Mathematics.

> *“The curriculum, the detailed syllabus, the textbooks, actual classroom teaching and learning processes and the examination only focus on the quantitative aspect of Mathematics. Use of quantitative techniques has become a major criterion for judging the quality of research as also the respectability of the different branches of knowledge.”*

The National Curriculum Framework 2005 has been credited for capturing and presenting the general belief about Mathematics in a more subdued form because in its elaboration it also keeps on referring to the larger or higher aims side by side. It further goes on include, ' to pursue assumptions to their logical conclusion and to handle obstructions, a way of doing things, and the ability, attitude to formulate and solve problems '; it builds abilities of problem-solving and analytical skills and is helpful in preparing children to meet a wide variety of problems in life. It also says, ' proof is important ' and makes children understand ' proof as a systematic way of argumentation '. This duality in the NCF and mixing of broader and narrower aims clearly arises out of a different perspective than the national policy which only talks about broader and higher aims. Interestingly, to camouflage the difference and the divergence, many phrases and statements that are part of the first conception are interspersed as if, both are the same thing.

Unfortunately, in teaching, learning and use of Mathematics only the second conception dominates to the complete neglect of the first one. The curriculum, the detailed syllabus, the textbooks, actual classroom teaching and learning processes and the examination only focus on the quantitative aspect of Mathematics. Use of quantitative techniques has become a major criterion for judging the quality of research as also the respectability of the different branches of knowledge. Indeed, globally and almost universally, in the educated mind, the second conception abbreviated to a science of quantity or a branch of knowledge relating to quantity, is the enduring image.

This has had disastrous consequences and resulted in serious distortions in the direction, progress, emphasis and management of knowledge. It has also resulted in a kind of caste system among the branches of knowledge. More regrettably, it has developed blind spots in human perspective resulting in a kind of duality between quantity and quality in which the former is desirable and therefore to be pursued while the latter is only a consequence of the first and will automatically follow, if the first is achieved. The quality of human life is therefore being solely determined by 'quantities', GDP, Human Development Index and the like. Even richness and poverty are sought to be divided by 'a line'.

There is an urgent need to appreciate that the first conception of Mathematics should be accorded its due place. The educational perspective must change and be permeated by this conception which visualises Mathematics as a critical and important tool for training the mind to think, analyse, and articulate logically. It is amazing how this loftier aim, which the national curriculum framework admits is a higher aim is completely missing from the teaching - learning as it operates today.

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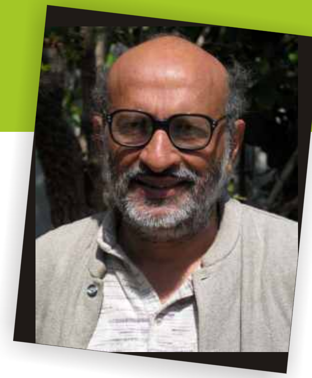
# 26 Maths Missionary – P. K. Srinivasan

**Arvind Gupta**



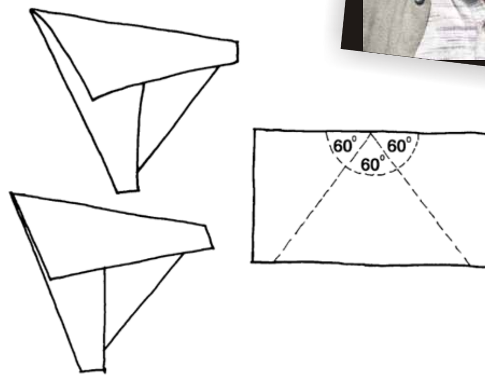
*Skills are Taught*
*Concepts are Caught*
*-PKS*

"From the near to the far, from the concrete to the abstract," is a sound pedagogic approach for learning Math. Before children can understand a thing, they need experience: seeing, touching, hearing, tasting, smelling; choosing, arranging, putting things together, taking things apart. They need to experiment with real things. This is how it's done in Hungary – a small country which has produced some of the world's greatest mathematicians. See this amazing free downloadable video from Teacher TV (http://www.teachers.tv/video/17878).

The greatest proponent in India of learning Math through activities was P. K. Srinivasan (PKS). As a one man Math missionary he did more than anyone else to imbue children with the love for this most beautiful subject Mathematics – the queen of all sciences.

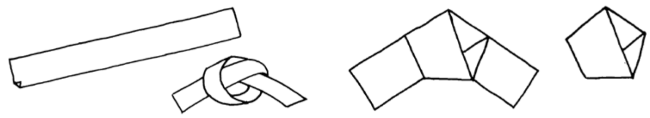
This article is both a tribute as well as a recapitulation of some of PKS's work.

PKS breathed Math. He dreamt Math. More than anything else he rubbed this infectious enthusiasm on everyone who crossed his path. I first met him in 1986 in a workshop organized by the NCERT at the Sri Aurobindo Ashram in Pondicherry. Those were pre-xerox days so PKS summoned a ream of cyclostyling paper, scissors, glue, old newspapers and one lone stapler. PKS gave each teacher one sheet of paper and asked them to fold an angle of sixty degrees? The teachers were at sea! Schooled into drawing angles only with a protractor they didn't know any other way of doing it. After 15-minutes of struggle the teachers gave up. Then PKS folded one straight edge (180-degrees) into 3 equal parts and produced an exact 60-degree angle! The teachers were amazed. It was almost like a revelation – all so elegant and beautiful. He showed them half a dozen different ways of folding 60-degrees. For instance, fold a strip into three equal parts and then into a triangle. All angles of this equilateral triangle would certainly be 60-degrees.




The whole day, teachers folded geometric shapes – a rhombus, a hexagon, an octagon etc. But how do you fold a pentagon? Paper folding by its very nature is binary. As you keep folding and doubling paper you generate 2, 4, 8, 16, 32, 64 ... layers. Weren't these all binary numbers? But how does one fold a pentagon? It is tricky but easy. In 1883, an Indian mathematician T. Sunder Row (Rao anglicized to Row) had shown this in his book Some Geometric Exercises in Paper Folding (still in print by Dover and perhaps the world's first ever book on Origami and Mathematics). How? Cut a long 3-cm wide strip from an A-4 size paper and simply tie a knot! Flatten the knot the trim the long ends to get a regular pentagon. How many times have we tied knots and noticed this?

> *"Mathematics should be visualised as the vehicle to train a child to think, reason, analyse and to articulate logically. Apart from being a specific subject, it should be treated as a concomitant to any subject involving analysis and reasoning". There's not a word about measurement, quantification and numbers!*



In that workshop the teachers folded over 80 shapes, some 2-D and others 3-D. All the 2-D shapes were stuck in an improvised file made by stapling a newspaper! They even folded a protractor with a dozen angles from a square of paper. The teachers were overjoyed. Perhaps they learned more practical geometry in those 2-days than they had done in their 2-year BEd course!

This brings us to the moot point – how out of sync is school Math with the real world. Early Mathematics evolved from the work of the tailor and the tinker – all practical crafts people. Mathematics has deep roots in practice. The very vocabulary of Mathematics is replete with associations of its pragmatic past. Consider for instance, the word "straight line". It comes from the Latin word "Stretched Linen". As any farmer wanting to grow potatoes would simply stretch a string to help him sow his crop in a straight line. Any mason would simply stretch a piece of string to help him lay bricks in a straight line. So, over time "Stretched Linen" became "Straight Line". The "digits" 1 to 10, which we use so commonly come from the Latin word for fingers – the ten little fingers of our hands.

Today school Mathematics is totally cut off from real life. The entire curriculum seems to be overlaid by the mumbo-jumbo of professional mathematicians. In the process the entire beauty and joy of Mathematics has got buried. The horrendous way Mathematics is taught in schools gives children a life time's distaste for this wonderful subject. If children are to appreciate the beauty of Mathematics, it is imperative for children to get a "feel" for Mathematics through practical work.

PKS struggled to infuse life in Mathematics. He cried, he wept and pleaded with one and all that Mathematics was all around them. And when no one listened he wrote a series of 60 odd articles for the Hindu which have become classic. He demonstrated that there was Mathematics in coins, in broomsticks, in matchboxes, in the square copy, in bus tickets, in the calendar in every ordinary thing around us. After considerable struggle these articles were collated by the NCERT into a book "Resource Material for Mathematics Club Activities". This splendid book – perhaps the greatest Maths activity book ever to be produced in India can be downloaded for free (http://gyanpedia.in/tft/Resources/books/pkshindu.pdf). After being out of print for almost a decade the book has just been reprinted by the NCERT.

*"The entire curriculum seems to be overlaid by the mumbo-jumbo of professional mathematicians. In the process the entire beauty and joy of Mathematics has got buried. The horrendous way Mathematics is taught in schools gives children a life time's distaste for this wonderful subject."*

PKS was not always so lucky. In the seventies he wrote two amazing books Number fun with the Calendar and Romping in Numberland. He ran from pillar to post, from one publisher to another without any success. Publishers wanted him to write a high-school Math guide which was directly linked to the school mass market. PKS refused. Often his biggest enemies were his fellow teachers. They hated his popularity with students. Some of them even connived and had him beaten up!

But his students loved him. Some of them never forgot the inspiring way PKS taught them Mathematics. In the mid-eighties, fifteen years after these two books 'Number fun with the Calendar and Romping in Numberland' were written they were published by PKS's ex-student who had made good money in an ice cream business in Chennai! This certainly was a good way of paying *gurudakshina.* These books can be downloaded http://gyanpedia.in/fromtft/Resources/books/calendar.pdf and http://gyanpedia.in/tft/Resources/books/rompinginnumberlandeng.pdf). Alas, despite the plethora of government organizations and private do-gooders there are still no takers for good books in our country!

PKS shared his passions liberally. In the early nineties he sent me a xerox copy of the masterpiece 1001 uses of the 100 squares – by Leah Mildred Beardsley. This landmark book showed possibilities of doing amazingly creative Math activities by using just a square copy – used by children to do their arithmetic sums and available even in far flung villages. This book was a revelation. It can be downloaded from
http://gyanpedia.in/tft/Resources/books/squaresall.pdf

All his life PKS shunned commercial gains. He generously gave his book "Manual for Mathematics Teaching Aids in the Primary School" to the NCERT for free, without any royalties. This gem is out of print for years and needs to be translated into all Indian languages. He was always clad in a white kurta and dhoti spun out of khadi – rough and homespun cotton which symbolized Gandhiji's concept of Swadeshi. He always sported a Gandhi cap as well. His passion for Mathematics was visible as one approached his house in Chennai. The compound gate, walls and grills were laden with equations, identities and proofs-by-sight. This legendary Math teacher passed away in 2005 at theage of 81.

The greatest tribute to PKS will be to translate all his popular books into all languages; to digitize and upload them for the children of the world. There can be no better tribute to this Pied Piper of Math.

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**Arvind Gupta** works at the IUCAA's Children's Science Centre in Pune and shares his passion for books and toys through his popular website http://arvindguptatoys.com. The books referred in the article can be accessed from the mentioned. He can be contacted at arvindguptatoys@gmail.com

# 27 Learning to Add: Are we Subtracting the Importance of the Home Environment?

**Amita Chudgar**

Since my early days in school my association with Mathematics in most of its forms, including Algebra, Geometry, and Statistics has always been very pleasant. In fact, Geometry, followed by Physics, was my favorite subject in school. Admittedly, there were some rough patches but right now they all seem too insignificant to even mention. To this day, I continue my healthy association with Math that I formed when I was young. A vast amount of my work today depends on applied quantitative research and I must say that on most days I can't get enough of it!

As I reflect on this association, and how it came about, I am surprised at how my individual experiences reflect a lot of what researchers from all over the world argue as well. So let me begin with my experiences as a student and now as a teacher of mathematical concepts.

Probably the single most important explanation for my positive relationship with Math is that Math was 'cool' in our house. Though not a mathematician, my mother loves Math too. She would talk passionately about Math and how much she enjoyed it as a student. She would show us simple yet cool things we can do with Math in our daily life, like figuring out the angle at which to slice the apple to make 5 exactly equal pieces! She taught me tricks to learn multiplication tables so that rather than becoming an endless series of numbers to memorize by rote, they are to me a logical sequence that I can construct in my head even today. Sometimes when the material taught in school became too complicated and filled with jargon and endless steps, she would help me out. She would usually break down the problem into several logical parts and in the process she often taught me easier ways to address the same questions.

This love for and excitement about Math also came through the books I got to read as a child and the books I demanded to read. I loved to read books about mathematical tricks, to solve mathematical puzzles, and to read about how Math applies to daily life. In short, in our house Math was fun, relevant, and by no means something to be scared of.

All this mathematical excitement in the house did not turn me into some kind of a Math genius, but it did do something very valuable. It gave me the confidence necessary to pursue complex mathematical concepts in my work and study. It ensured that whenever I encountered daunting mathematical expressions, rather than being turned off by them, I felt comfortable trying to play with those ideas and figure out the logic behind them. Those early positive experiences and interactions freed me to a great extent from 'mathematical apprehension'. Now as a university instructor of applied quantitative research, I regularly teach many PhD students. The biggest barrier for many of my absolutely outstanding students is that their prior interactions with Math have led them to be afraid of it or to see it as something unduly challenging and confusing, rather than relevant and fun. For many of them, once we break that barrier, the rest of the class becomes quite exciting and interesting.

> *"In all the countries we studied we found that the role of the family was potentially more crucial in improving Mathematics achievement, compared to the school. This is not to say that schools (and the curriculum and the teachers which are all subsumed under 'school' in our study) are not important."*

So moving on to the broader patterns identified by research, what does my experience reflect? It highlights the importance of making the Mathematics curriculum more relevant to a child's life; also the need to make Math more fun in the classroom and more importantly the need to ensure that we do not scare children with Mathematics - instead we find ways to share with them the joy and elegance of mathematical thinking. Together it emphasizes the critical role played by both teachers and a well-planned curriculum in enhancing a child's Mathematics competence.

But equally importantly, my experiences reflect the importance of something else: of parental involvement and the home environment. With my colleague, Professor Thomas Luschei of Florida State University, I recently analyzed large-scale data on Mathematics achievement in the 4th grade in 25 diverse countries; we did not have data on India (Chudgar & Luschei 2009). In all the countries we studied we found that the role of the family was potentially more crucial in improving Mathematics achievement, compared to the school. This is not to say that schools (and the curriculum and the teachers which are all subsumed under 'school' in our study) are not important. They indeed are very important too, especially when the country is poor and opportunities are unequal, as in India. But what our study highlighted, as many others have, is that family factors are crucial to a child's learning outcome. Within the family, research shows that indicators like parents' education and the number of books available at home are all strongly related to children's educational outcomes - (Buchmann and Hannum (2001). Separate research has also shown that the mother has a specific effect, and that her education is more strongly associated with the child's learning outcome (Schultz 2002).

In India, according to the 2001 census, close to 40% of Indian adults could not read or write, let alone being educated up to even primary level. More than 50% of adult women are illiterate. The limitations imposed by illiteracy are hard to comprehend for those of us who are fluent in not just one but often multiple languages. But for a parent who cannot even read or write their own name, this may mean a very limited participation with their children's education, textbooks, homework, and teachers in spite of their best intentions. Thus in India today, a vast majority of children currently in school continue to come from households where their parents, especially their mothers, may not be able to engage fully with their school experiences. And while not all the mothers have to enjoy Math or teach their children mathematical tricks, we can only imagine the disadvantage these children of illiterate parents experience as they navigate their school experiences, often with minimal help from adults. In fact, another large-scale data analysis project I undertook using data from India (Chudgar 2009) showed that parental illiteracy may be far more important than even poverty in determining a child's success and failure in the education system.

Improving children's school performance and retention is a multi-faceted problem that requires multi-faceted interventions. From the policy perspective, 'fixing' schools, teaching practices and curricula are all essential: a lot is broken and a lot needs amendment. In particular, research indicates that teachers and teaching practices may be the most important set of factors that policymakers can control to improve student performance (for example Goldhaber, Brewer, & Anderson, 1999). In fact, research from elsewhere shows that teachers may have a role to play in bridging some of the gaps between more and less privileged children (Hanushek & Rivkin, 2004). But as we march ahead addressing problems in the public domain that we can perhaps more readily see and address, I believe that we also need to pay more careful attention to how we can support and empower the parents of the children we want to see succeed. My personal experiences notwithstanding, more than enough evidence points to the importance of family, especially the mother. Data on limited adult literacy levels in India are incontrovertible. Together these numbers point toward yet another area needing attention and intervention as we strive to improve the education our children are receiving.

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Originally from Mumbai, **Amita Chudgar** received her PhD in Economics of Education from Stanford University. She is currently an assistant professor in the College of Education at Michigan State University where she teaches graduate classes in economics of education, international education and quantitative research methods. Her research aims to understand the determinants of school enrollment, retention and school performance in India and in the broader international-comparative context. She can be contacted at amitac@msu.edu

## Logico Math Brain Teasers

There are 100 students in Hostel and there are 100 letter boxes. The letter boxes are all open to begin with. Each student is asked to go and change the status of each box – open it if it is closed or close it if it happens to be open. The first student has to change the status of each box (in his case, all are open so he has to close each of them). The second student has to change he status of every second box (ie change the status of he second, fourth, sixth box etc from closed to opened), the third one has to change the status of every third box (the third, sixth, ninth box etc from opened to closed or closed to opened as the case may be) and so on till all 100 students complete this task. When this is completed, how many of the 100 letter boxes will be closed?

*(Hint: Try it out with a smaller number and see if any pattern emerges)*

Use this space for calculation 🙂

# 28 Confessions of a Victim of Mathematics

## Indu Prasad

My mother has immense faith in her Gods and it is to them that she turns in times of deep trouble. She turned to them on that day - my day of reckoning – my Class X final Mathematics examination. She saw me off to school at 8 a.m. and went straight in to sit before her Gods in supplication. I returned home at 1 p.m. and there she was - still sitting before them having made every kind of promise in return for a passing grade in Mathematics for her youngest child.

You see, Mathematics and I are old combatants. I have been on the losing side of most of our battles. You will understand when I tell you that it was at age thirty-seven that I finally, but finally, discovered how and why $(a+b)^2$ could become $a^2+2ab+b^2$. This was through sheer necessity and no desire. My son needed some help and out of desperation under very strained circumstances turned to me for support. I had no choice and had to make the attempt to understand this – I could almost hear the cackling laugh of my old enemy's delight at having caught me after many, many years.

Like most children, I didn't mind numbers much up until the time that they began to talk about fractions and decimals and area and perimeter and profit and loss. They even tried to fool us into believing that this wasn't about calculations through something called “word problems”. How on earth can a word be a problem? A number is a problem.

I didn't mind shapes much up until they began to ask me areas and angles – why couldn't they have focused on the beauty and weirdness of shapes instead of making us remember measurement formulas and theorems?

I could happily pretend that those x, y, a, b things were actually birds having a meeting until they insisted on converting them into things like

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \ldots\ldots$$

I am forty-four (remember, those are words and not numbers) years old and I firmly believe that Mathematics is the creation of an evil power set out to destroy our peace.

That I actually passed my school final examination is a miracle that I owe to my sainted mother's prayers. My record before that was not pretty – my marks looked more like B. S. Chandrashekhar's batting averages!

> *Our Mathematics teacher in high school was always considered the Most Important Person in the school. Her classes were too important to be interrupted by theatre workshops, sports practice sessions or even any laughter. Nobody smiled, there wasn't an extra movement, no pencil boxes fell, no chits were passed round, no giggling or whispering allowed.*

My greatest joy was to bid Mathematics goodbye after Class X and retreat into a world that I understood much, much more – a world of words. This desire for distance from numbers continues even today. I am a shop-keeper's delight – I never calculate how much I need to pay for my purchases, I always go by what is told to me at the shop. I have colleagues who delight in sending me reams of articles dotted with graphs and statistics – those articles have one place of honor: the dustbin. Presentations that have tables and regressions and formulas leave me glassy-eyed. As for using Microsoft Excel – get a life!

I will give you an example **: why on earth must we say it like this(see table on next page)**

| Class | RW1 | | | Rw2 | | RW3 | |
|---|---|---|---|---|---|---|---|
| | N (Number of children) | % at 0 level | % fully able to read all words | % at 0 level | % fully able to read all words | % at 0 level | % fully able to read all words |
| 2 | 239 | 74.5 | 11.3 | 83.3 | 5.0 | 94.1 | 2.9 |
| 3 | 261 | 63.6 | 19.5 | 70.5 | 11.5 | 80.8 | 6.1 |
| 4 | 230 | 46.1 | 38.2 | 52.2 | 23.0 | 64.3 | 15.6 |
| | RW4 | | | RW5 | | | |
| 2 | 239 | 96.2 | 0.8 | 92.5 | 3.0 | | |
| 3 | 261 | 85.4 | 1.1 | 83.5 | 4.6 | | |
| 4 | 230 | 70 | 6.1 | 60.9 | 16.1 | | |

| N | RW1 | | Rw2 | | RW3 | | Rw4 | | Rw5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| Number of children | % at 0 level | % fully able to read all words | % at 0 level | % fully able to read all words | % at 0 level | % fully able to read all words | % at 0 level | % fully able to read all words | % at 0 level | % fully able to read all words |
| 226 | 10.2 | 76.5 | 17.7 | 61.1 | 27 | 46.9 | 35.4 | 38.1 | 33 | 40.7 |
| 238 | 5 | 70.6 | 11.3 | 55.5 | 22.7 | 45.4 | 31.9 | 34 | 30.3 | 34 |
| 230 | 1.3 | 93.5 | 3.5 | 87 | 9.1 | 80.9 | 15.7 | 72.6 | 6.1 | 78.3 |

**When you can say it like this (the paragraph below):**

*The results of the quantitative assessment showed a significant jump in basic language competency among the children. For example, the baseline indicated that 30 per cent of the children in Class 4 were unable to identify letters. There were very few who could read simple words or sentences. The end line results showed that 97 per cent of the children were able to fully identify and match letters and sounds, and use those letters in constructing words and sentences. Ninety-three per cent of children were able to read words without 'matras' and 87 per cent were able to read words with 'matras.'*

Isn't the paragraph coherent, concise, easy to understand and absorb? Why do people continue to believe that a table (which, by the way, is actually a piece of furniture to keep things on) actually "speaks?"

I believe that most sensible minds happily shut down when they see numbers. This was my effort at self-preservation in my Mathematics class as a student and continues to be so as an adult. I had teachers who mostly had one or more of the following five beliefs:

1. You either "get" Mathematics or do not "get" it
2. Mathematics is all about speed and accuracy
3. Mathematics is very tough and needs extraordinary attention, concentration and memory – it is not for ordinary mortals
4. All these formulae and theorems have been discovered by great men – don't waste your time asking about them – just learn
5. Mathematics is the only area of knowledge worth

learning – the king among subjects (literally!) – everything in life is all about Mathematics.

Our Mathematics teacher in high school was always considered the 'Most Important Person' in the school. Her classes were too important to be interrupted by theatre workshops, sports practice sessions or even any laughter. Nobody smiled, there wasn't an extra movement, no pencil boxes fell, no chits were passed round, no giggling or whispering allowed. One had to fiercely concentrate, frown on forehead, sharpened pencil in hand, no questions asked. Our Mathematics class was a sacred ritual – no less. The mood was always sombre, and to me, funereal. My dearest wish was to have a quick burial of the subject, its text-books and (forgive me, My Lord)….its teachers……..

As an adult (and after much persuasion), I read the biography of Srinivasa Ramanujan - his obsession with numbers, his relationship with Hardy the way he used to, work out complex equations in the sand on Marina Beach and how he wrote much of what was to be his seminal work on bits of paper during stolen moments from his job as a clerk at the Madras Port Trust.

This was a different world – here Mathematics was elegant, was beautiful, was logical and had a story to tell!

The Mathematics I knew was everything but.

So my friends, I have decided that my teachers were right after all – this is not a subject for ordinary mortals. This is for the Ramanujans, the Hardys and that strange and alien species called "lovers of numbers." As for me, I will continue to inhabit a world in which Mathematics is a predatory force from which I am in eternal flight.

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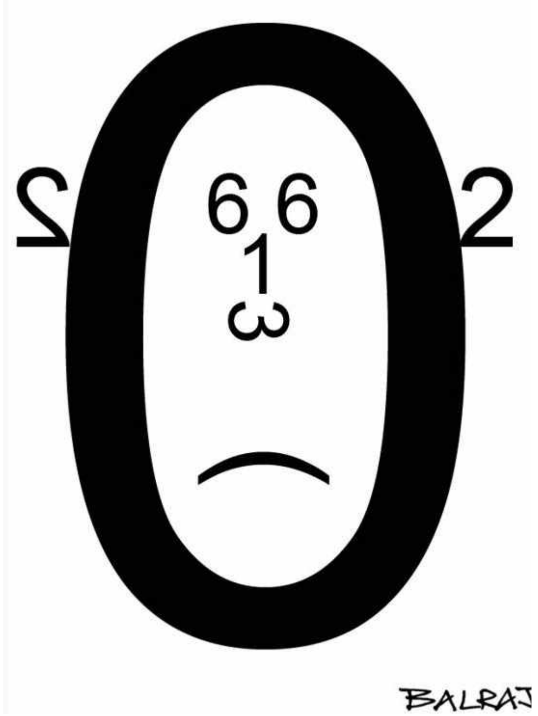

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# The Elusive “Mr.Math”

## Shwetha Ram

When I received an email from Giridhar of Azim Premji Foundation that the forthcoming issue of their newsletter Learning Curve was all about Mathematics, two words instantly popped into my head - Love… and… Hate! And I was pleasantly surprised when I read further on that Learning Curve would like me to narrate my (rather choppy) ride with this intriguingly abominable subject called Math.

My earliest memories of Math were in kindergarten and first grade. I could not understand the concept of numbers, PERIOD! One and two should buckle my shoe, so where did three come from?

While my peers always brought in their completed workbooks, I struggled to comprehend numbers and a vague concept called counting. My poor mother relentlessly tried to make me count with my fingers that I thought were meant only for eating with. Unable to understand the why of it, I often resorted to copying at class tests and I must have been fairly good at that because I was reprimanded very few times (or did my kind teachers turn a blind eye seeing how Math challenged I was?)

Come grade 5, and a sense of grown-up honesty took over. I decided to rough it out and not cheat in exams. Result - I barely managed to scrape through. The mean Math yanked down my overall scores in every exam cycle. I distinctly remember one of my class teachers calling out the final grades at a half yearly exam and announcing that I stood last-but one in my class of 41 students. I smiled to myself at the thought that there was one soul actually worse off than I. My joy was short lived - she hurriedly qualified her statement with “Ram Kumar had to skip two of his exams as he was down with Malaria”, so we could not technically give him a final score!”

By 6th grade, I square rooted to the bottom of my class and seemed stuck there for eternity. By then, my parents and my grand parents had tried every bribe in the book to help me get better, without much result. My weakness in Math seemed contagious, as it soon spread to my performance in other subjects as well. And at this point, school spelled hell for me.

I was never a good with memorizing stuff (I still struggle to remember my phone number), so I didn't know how to approach Math. Should I memorize the complex formulae? Should I remember Geometry theorems by rote? Was there a trick to work through those numbers that my peers seemed to be breezing through?

My uncle who came up with a hare brained scheme – he offered me a holiday in Goa if I got decent marks in Math. But he ended up doing something more sensible around that time. He… Got… Married! We were all in a joint family back then and his lovely bride, turned out to be the best teacher, mentor and companion I could ever ask for. She magically understood my struggle with the subject and devised a workable way out of the abyss. She would take a Math problem and not solve it for me. Instead, she would explain the logic and let me decipher the code and in the process we would solve it together. She introduced a sense of logic to my approach to Math, and that opened a whole new world for me. Seeing things logically meant that I did not have to learn things by rote. All I had to do was to understand and attack the root of the problem and then let logic guide me through. Her mentoring resulted in the beginnings of a tolerance that progressed to a comfortable co-existence with the subject. I finally started enjoying the number game.

> *“Mrs. Srimathi my teacher had what seemed like an unrelenting approach to teaching. She had this weird theory of “do a 500 Math-problem marathon, and for the 501st problem you attack, your pen will do the thinking!” Sounds crazy - but for me it worked like a charm!”*

Alongside, I also started developing a taste for some aspects of the subject Geometry especially turned out to be interesting. Each time I reached a mental block with arithmetic while studying for exams, I would switch gears to doing Geometry. That little break always proved to be a stress buster. When I went back to the tougher side of Math, I would automatically approach the problem with more clarity in my thinking and renewed vigor.

By the time I got to 10th grade, my performance had improved by leaps and bounds. But the Math got MUCH tougher, with Trigonometry, Complex Algebra, Differentiation and Integration thrown into the formidable mix. Was Math going to turn vicious all over again?

Luckily I met a teacher who changed my life. Mrs. Srimathi, my teacher had what seemed like an unrelenting approach to teaching. She had this weird theory of “do a 500 Math-problem marathon, and for the $501^{st}$ problem you attack, your pen will do the thinking!” Sounds crazy - but for me it worked like a charm! In lay man terms, Srimathi simply convinced us that plain hard work would get us results every time; you do not need to be particularly gifted to progress in studies. Can I see raised eyebrows as I narrate the Srimathi formula? Well, it worked for me.

This was a turning point in my life. As my scores improved, the confidence rubbed off on other subjects too. Also, once you taste the sweetness of success, it is very difficult to let go of it!

One also starts to create his/her own study techniques while studying so hard. A personal favorite of mine was that I would set up a ridiculously aggressive study timetable (1000 Integration/Differentiation problems in 4 hours flat!), knowing fully well that the timelines I set were certainly impossible to meet! But this indirectly helped me in stretching myself. A less severe or a practical schedule, I am sure would have made me slack some and not cover as much ground. This technique has come in handy in my professional life too. Stretch goals (goals that are beyond what is expected from ones work desk) are the ones that are best recognized during yearly performance reviews!

I unashamedly admit that I have never been one of those brainy ones. I was always and still am, what you would call an average student. However, I realized that there is one tool that no one can rob me of. That one tool, which I can use any time with guaranteed results. That one tool that will never let me down and that is plain HARD WORK.

Infact, by the time I finished school, it worked so well, that I decided to deal with numbers all my life and ended up choosing a profession in Finance, completing a Chartered Accountancy degree in India and then a CPA degree in the US. These days, when I am asked what my strengths are, I say without a moment's thought - Numbers, Logic and Hard work! So for me Math has been a long story that began with fear and then hate before finally ending in love. Math is akin to an Aubergine! You either love the vegetable, or completely hate it – there is no middle ground. I hope you enjoy your preferred line of study as much as I do pursuing mine.

Mathematically yours,

Shwetha Ram

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**Shwetha Ram** is a senior R&D Finance Analyst at Boston Scientific, California. She has a CA degree from India followed by a CPA degree in the US. Alongside a profession in Finance she also actively pursues singing and volunteering for the San Francisco chapter of Udavum Karangal (a grass roots organization based in Tamil Nadu that runs an orphanage, a school for children and provides shelter to destitute in need of a helping hand). She can be contacted at shwetha1709@yahoo.com

# 30 Math for the Non-Mathematical soul

**Devika Narayan**

Even during our best years, Mathematics and I shared a strained relationship. Today this relationship is so vexed that I have been accused on several occasions of having a 'block', an irrational fear almost phobic in nature with respect to anything even vaguely Mathematical. I admit to this. However I do not see this as baseless paranoia, instead I see it as arising from a painful awareness of my own limitations in this area. This awareness is a burden which prevents any further confrontation with Math, thereby reinforcing the limitation. The problem therefore is not the limitation itself but the intense discomfort or fear that it triggers which disallows any fruitful engagement. The fact is that when faced with a Mathematical threat to my personhood I choose 'flight' over 'fight' every single time.

This was not always the case. I would like to describe the golden years when a brilliant teacher helped me strike a precarious friendship (in urgent need for revival) with Mathematics. I grew up at Centre For Learning, a small community made up of students and teachers who together radically re-image the learning space. In Math class, as we cracked problems together, made mistakes, asked questions, I grew more confident. Slowly, in place of hesitation, nervousness and fear, developed a real sense of awe for the language of Math. Looking back I think it was remarkable how inspite of all my self-doubt and anxiety the classroom was never a threatening place. I felt safe because I knew I wasn't being judged or constantly evaluated. It was because my classroom was always a friendly and responsive environment in which I was able to face up to my fears. The process of wrestling with something that I had no knack for became one in which the reward (i.e. a correct answer) was highly satisfying and most importantly fun. How my teacher achieved this tremendous feat is a real mystery (and let me tell you, the fact that she succeeded in showing me, of all people, how to take pleasure in the Mathematical battle is an incredible accomplishment!). In fact I think this should become the standard way of testing a teacher's expertise. Can you show your most resistant and difficult student the route to enjoyment? The education ministry should make this the basic qualification. What I remember most vividly of my classes is how much I spoke in them – my most recurrent sentence of course being 'but I don't understand'. I followed the steps carefully and cautiously, the wheels in my head turning, and whenever I hit a temporary dead end I interrupted. I interrupted because I wanted to understand. I interrupted because I was encouraged to do so. Math was always a struggle but became one which I wanted to take part in. I find it amazing that even though Math never came easy and was such an effort, I never dreaded or hated class. It felt like a long distance run up a steep hillside-difficult and energy consuming but ultimately rewarding once you arrive at your destination. "Ah okay, now I understand", was what I said once I reached my little hilltop.

> *Looking back I think it was remarkable how inspite of all my self-doubt and anxiety, the classroom was never a threatening place. I felt safe because I knew I wasn't being judged or constantly evaluated. It was because my classroom was always a friendly and responsive environment in which I was able to face up to my fears.*

It has been seven years since I exited that classroom. As far as Mathematics goes I am rusty and terribly out of form. My skills which were once in shape because of persistant practice have now slacked like unused muscle and due to my complete lack of confidence, I revert to my old response of fear and dread.

So why is this particular experience of Math relevant or significant to educators? Does it teach us anything useful about the process of learning? I ask this because, we can safely assume that this is not a unique experience that is wholly mine. Vast numbers of students who are disinclined to Mathematics find it very difficult, frustrating and downright scary. This is constantly reinforced by poor performances in tests, comparisons made by teachers and a general sense of feeling 'stupid' as compared to friends. Teachers have to keep finding innovative and creative ways of reaching out to these students so as to minimize the emotional reaction of panic and anxiety. Math has to be converted into a plaything. Like solving a gigantic jigsaw puzzle or unravelling a big ball of tangled wool. Play with it. Work at it. Admire the precision.

Only if the teacher is able to achieve this, will Math transform from being a predatory monster into a challenging game to have fun with.

I know that when I will have to step up to the challenge again, it will be far from easy and will take tremendous effort and hard work. However, the one thing I will remember from class is how to enjoy the process.

**Devika Narayan** graduated from Centre For Learning in 2006. She is interested in the social sciences, writing, journalism, protest, Delhi, walking, reading books in the sun, and jalebis. Devika is currently pursuing a Masters in Sociology at the Delhi School of Economics. She can be contacted at narayan.devika@gmail.com

**Number Trivia**

- A circle is a round straight line with a hole in the middle.
- A Pangram is a sentence or a paragraph which describes itself, for example the following:
- This Pangram contains four as, one b, two cs, one d, thirty es, six fs, five gs, seven hs, eleven is, one j, one k, two ls, two ms, eighteen ns, fifteen os, two ps, one q, five rs, twenty-seven ss, eighteen ts, two us, seven vs, eight ws, two xs, three ys, & one z.
- Prime numbers are numbers which can be divided completely (without leaving any remainder) only by 1 and itself, like 2, 5, 7, 11, 13 etc. The odd thing about 2 is that it is the only prime number which is 'even'.
- The number 11 is a very interesting number. You will notice that 11 X 11 gives 121. Try the multiplication with larger such numbers. Try with 11111 X 11111 or 111111 X 111111 and so on and you will find an interesting pattern emerging. Try it out.

Take any 4 digit number (with at least one digit different from others) and perform the following:

1. Reorder the digits to form the largest four digit number.
2. Then, reorder the digits to form the smallest four digit number (use a leading 0 if needed)
3. Subtract the smaller of the two numbers from the larger one to get a new 4 digit number
4. Then repeat the steps 1 to 3 above

This process can be repeated till you reach a stage when you constantly end up with exactly the same 4 digit number. This stage is reached in a maximum of 7 iterations (usually less).

This constant number is **6174** and is known as the **Kaprekar Constant** after the Indian who identified it. For example: if you start with number 3524 the steps would be

5432 – 2345= 3087

8730 – 0378= 8352

8532 – 2358 = 6174

Actually, this process can also be carried out with 3 digit numbers and after a few iterations, you end up with a 3 digit number which then repeats itself. Try and work out this number. Interestingly, both these 4 digit and 3 digit constant numbers are divisible by 9.

# 31 Unsung Heroes: What is it that makes them Stand Out?

**Anant Gangola**

What is it that makes a few teachers outstanding in comparison to other teachers? The very thought about this question brings images of several teachers in the depth of any mind and I thoughtfully started analyzing their images and work. The heroes of my story are none other than teachers working in remote village schools run by the Government. This is without any prejudice against the urban or the private schools, but is a mere coincidence that during the last 15-20 years, I have mostly had opportunities to work with rural government primary schools.

My first protagonist is Mahesh Oad, a teacher in an alternative school in a small village called Camp No. 4 in Madhya Pradesh. I distinctly remember the first time I visited the school. All the children of the school surrounded me with their slates and note book asking "Master ji, give us sums". I started providing them sums one by one and they kept asking for more addition, subtraction, multiplication and division sums. They were solving at lightning speed – so fast that even before I finished giving them sums, one of them came running upto me with answers. The children did not seem bored and after a little while I found myself tired. This experience is still afresh in my mind. By now, you might be wondering why I am talking only about children while I wish to really describe the teacher. I am doing so, because I could see the teacher in the eyes of his pupils.

Incidents in this school were extraordinary and nothing similar to other schools I visited previously. Hence my first hero is Mahesh, the teacher who has been doing an exceptional job. I would like to share some simple yet significantly important characteristics about this teacher. Mahesh's acceptance of each and every child as a complete individual personality was something unique and unheard off. Mahesh was taking care to address each child and make non-academic conversation with them. Some examples of this included - "Shobha, how is your calf? Sunil, has your maternal uncle returned? Rekha, who set your hair pleats today?" such informal conversation was making the children so comfortable and the atmosphere was one of no-fear. Mahesh did not see the children as a group but each time he found time to work with each child and called them by their names and did not ever address them as 'Hey Boy or Hey Girl'. This was an important attribute which I feel contributed to his acceptance amongst the children.

The other most important thing that I felt about his teaching method was that Mahesh never considered himself as the only reference person. Whenever any child came to him with a problem he did not try to solve it himself and instead asked the child to go to Seema (another child) and encouraged the child to solve it with her. It seemed like he was never in a hurry to give an answer to the problem himself but patiently suggested a method to the child so that the child either reached the solution herself or sought the help of another mate. Nothing ready-made was available and this resulted in keeping all the children busy with one thing or the other. The arrangement was providing the children an opportunity to learn by themselves; this was helping children use their energy and they thoroughly enjoyed the feeling of joy and discovery each time they solved a problem. The next in the series of outstanding teachers is Saraswati.

> *"He found time to work with each child and called them by their names and did not ever address them as 'Hey Boy or Hey Girl'. This was an important attribute which I feel contributed to his acceptance amongst the children."*

Saraswati's school reflected a different scenario from that of Mahesh's school. Everything seemed very organized. Children were seen sitting in small groups of 5 to 6 each. Saraswati had the days work plan for all the groups. At the beginning of the day Saraswati made the children sit with their groups and distributed them work one by one. Each group was given a different task. After distribution of tasks she went different groups one by one and tried to help them begin their work. She had planned it all out - where to make the each group sit etc.

While some groups were sitting inside the building, the others were in the open ground. Despite this no disorderliness was visible anywhere.

Giving tasks and examining them were both being done by the children amongst themselves; Saraswati looked not like a teacher but like a senior child. One group was working on the parts of a tree. Saraswati did not do it by teaching from any book or by making any sketch but discussed about roots, stem branches and leaves by getting the children to uproot a plant. She then asked them to make sketches in their note books. Saraswati then explained about the parts of a tree from a chapter in the book. She understood every word of what she was teaching and this attribute made her clearly stand out.

My next protagonist is Ramesh who is the lone teacher in a single teacher school. His school is situated in the remote 'Mori Block' of Uttarkashi District in the state of Uttarakhand. Ramesh is the epitome of sensitivity when it comes to his children. His incredible patience and willingness to listen sets him apart. Despite children surrounding him with innumerable queries Ramesh listened to them attentively and responded suitably to each child. He had a democratic way of taking decisions when it came to handling a class. Tasks in his class were never a matter of formality. Instead he actually planned his classes along with the children. I can call this school as 'the children's school', where every child was involved in decision making.

Ramesh's extraordinary talent was his ability to adapt his teaching as per the decisions of the children. As a teacher he was brimming with self- confidence and had a firm grip over his subject.

*She understood every word of what she was teaching and this attribute made her clearly stand out.*

He was also innovating as per the requirement of the situation which make him different from his peers.

In the schools where my protagonists taught, there was a vast difference in the methods used to teach children. When we tried to draw out some fundamental commonalities we found the following- having due respect for children, involving the children in the process of learning, thus making them active participants and not passive learners, encouraging children to ask questions, giving children the freedom of expression and living by democratic values etc. Learning in such schools is not a mechanical process but a visible dynamic process vibrant with interesting activities.

Besides this, there is a distinct commonality amongst these teachers. All three teachers mentioned in this article did not consider themselves as 'learned teachers' instead they believed that they were “learning teachers”. The hunger and urge to learn something new was very much alive in them. As a result, they engaged in constant reading and wished to learn something new each day. Hence rusting over a passage of time is not a real threat with such teachers.

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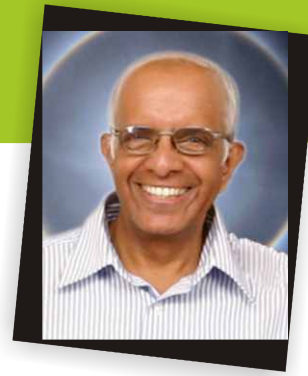
**Anant Gangola** is State Head, Azim Premji Foundation, Uttarakhand. He did his M.Phil. from Indian Institute of Social Sciences and has also done a course on Futurology from Udaipur. He has worked for literacy and social upliftment of tribal community in Neelgarh, Madhya Pradesh and also been the District Project Officer (DPEP), district Raisen, Madhya Pradesh. He was subsequently honoured by the State for his effort. Conferred a fellowship by United Nations Leadership Academy in Jordan, he is also a recipient of Prof. Ramesh Chandra Bhatt fellowship. Anant represented India in the Earth Summit Johannesburg, South Africa. He can be contacted at anant@azimpremjifoundation.org

# 32 Engineers, Accountants And Mathematics

**V S Kumar**



You may have heard the story of the IIT graduate who started business after completing his B Tech, and roped in his Chartered Accountant friend to 'look after the books'. Soon the friend had a problem: he could never balance the cash because the owner, who also had a key to the cash box, would take cash for expenses but never tell him what he took.

One day the accountant accosted the engineer and said "boss, take money when you have to, but put a slip into the box with the amount taken, so I can keep the books", and his friend agreed. The accountant heaved a sigh of relief. The relief did not last long, though. A week later when the accountant opened the cash box he found no cash and the slip from his friend read "I took all the cash that was there".

Prof. Henry Higgins in the film 'My Fair Lady' says "America and England are two countries divided by the same language (the language of English)". Likewise engineers and accountants are two professionals divided by the same subject – Mathematics, or to be precise, Arithmetic. Their understanding of the subject, their approach to it and the purpose they wish to fulfil from it are totally different.

This came home to me strongly when I studied with a large contingent of engineers for my Management diploma. Here they were comfortable with complicated curves, pies and long equations in Operations Research, but all at sea trying to match the liabilities and assets sides of the balance sheet. "I guess it's too simple for me" summed up an Aeronautical Engineer in my class. Actually he was right.

This mental block on the part of engineers about accountancy led to quite a bit of camaraderie between engineers and accountants, with mutual teaching and problem solving. But the differences between engineers and accountants on the subject are a major cause for strife in the business world. What is common between them here is only their scant regard for each other.

Ask any accountant and he'll tell you: "our CEO trying to understand the financial report? Don't make me laugh. He can't grapple with a simple cash flow, what can he get out of the balance sheet?" And the engineer CEO's response will be equally trenchant: "Till date I have run the business on my own figures. Accountants either tell me something I already know or something so full of jargon that only they can understand."

There is a lot in what the engineer-CEO says, actually – and the criticism is too real to be laughed away. The lack of usefulness of an accountant's report often stems from his obsessive need for accuracy, which robs him of the capability to look at the big picture.

Accounting teams of many companies spend night after night, in the company of audit assistants, poring through a maze of schedules and ledgers trying to find an errant voucher or a small discrepancy in the totals. A friend of mine used to call the financials of his company 'masala dosa balance sheets' because he said they were made by people eating masala dosas while doing endless reconciliations.

"The French don't really mind what is said to them, as long as it's pronounced properly" is another classic quote of Prof. Higgins: and accountants are similar. They don't mind what the results are and whether they mean anything to the CEO as long as the figures are tallied to the last decimal – as in this case.

> *"Great, man!" shouted the Chief. "But why is the report in red ink? We are now in black. Go and get the report done in black ink." The accountant shifted uneasily and said "If we buy black ink we would go into red again".*

For the first time after two years, the company had broken even that month, and the accountant rushed into the Chief's room excitedly and showed him the Profit & Loss account with a small profit for the month. "Great, man!" shouted the Chief. "But why is the report in red ink? We are now in black. Go and get the report done in black ink." The accountant shifted uneasily and said "If we buy black ink we would go into red again". You can understand the pique of the engineer.

Equally relevant, though, is the complaint of the accountant about the engineer's refusal to understand his reports. But here, it's not inability to follow as much as unwillingness. The engineer believes – and his ego wouldn't have it any other way – that he knows what he is doing, including the numbers. So when the accountant sends him a report with a markedly different set of figures, the engineer promptly sends it to the trash can and goes back to his make-believe report that is much nicer to him.

In real life I have often enacted this scene with my engineer colleagues across all functions – marketing, manufacturing, HR, CEO.... You name it. I would go into the Sales Conference and announce "the numbers for our new product launch are 30% lower than forecast." Back would come the weary answer from VP-Marketing "of course we are all aware of that. Give me the geography break-up". And when I take this to him, he would say "we know Maharashtra was behind and Tamil Nadu ahead. We need to look at it stockist-wise". And the merry-go-round would go on.

Only much later in my work life as an accountant did I catch on to why this always happened. It was the Engineer-turned Salesperson's way of saying: "look, number-cruncher. You cannot help me solve my problems with your arithmetic and your analysis. Our subject is complex and too deep for you to understand. Lay off".

This was one thing they taught differently in the business school I went to. Here was the Marketing professor for whom class participation (CP) was very important. And since Marketing and I were not exactly on friendly terms, I had resigned myself to an F for CP in his class.

That was until a fellow student showed me the way. He said "use numbers, pal. A lot of them – and you'll find him listening." I found him not just listening but absorbed when, in the next class, I went to the board and produced detailed break-even calculations purely based on assumptions and almost totally irrelevant to the case we were discussing. But the professor was thrilled – you could see he was deciding between A and A- for my grade.

In business, the fun – if you can appreciate the joke – reaches a climax when both the professionals decide to do their own Math and challenge each other. I have been party to hilarious sessions, when the CEO is faced with two sets of figures, one from the accountant and the other from the engineer.

> *"The inherent logic of arithmetic is common to both engineers and accountants who use the subject, albeit with their own tools and techniques. When you therefore see the beauty of the logic and overlook the differential methods, you become capable of appreciating the other professional."*

There will be a bunch of branch managers with profitability reports of their individual branches, each outdoing the previous one in terms of the net profit his branch has earned for the year. And finally the accountant would present the figures for the company – which would show a loss, instead of a healthy total of all the branches' profits. Enough to send the CEO to a rehab centre.

And no branch manager would have fudged any figures to present a rosy picture. Every person in the room would swear by his figure. How could this have happened?

It's very simple, really – a case of convenient forgetfulness on the part of the branch manager. He would take for sales revenue the total billing figure, for instance, and happily forget that it includes sales tax and excise duty or service tax that has to be paid to the Government and cannot be retained. He would include in the revenue reimbursement of expenses by the customer, but not include it in costs. And so on.

The strife between business accountants and engineers, however, ends on a peaceful note, sometimes on a note of conversion, with the engineer embracing Finance as his profession or vice versa. Almost 40% of the class of Management I studied with are full-fledged Finance pros now, and it's great (and a little weird) to see an ex-Textile engineer talking casually about broker margins, treasury bills and so on. Similarly a great boss of mine, a Chartered Accountant, ran a product division of a major engineering firm comfortably, leaving his numbers and his balancing nights behind.

And when you think of it, it's no miracle. The inherent logic of arithmetic is common to both engineers and accountants who use the subject, albeit with their own tools and techniques. When you therefore see the beauty of the logic and overlook the differential methods, you become capable of appreciating the other professional. Which is what these converts have done.

But then there is a downside too – the peace treaty has changed from appreciation to manipulation now, resulting from the unholy alliance between the two 'professionals'. Starting from Enron to the latest Satyam there have been a number of accounting scams, helped a lot by the two people burying their differences and working together – against the shareholder.

Does this imply that the healthy discord between them was a much better option?

**V S Kumar** has over 37 years of experience in Accounts and Financial Management. A cost accountant and an MBA from IIM, Ahmedabad, Kumar's life has been an enjoyable battle with the various functions that make up a corporate organization. Kumar combines a passion for music and numbers laced with lightning quick humour. He can be contacted at vsk3951@yahoo.com

# section F

## Book Review and Resources

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# Two Books Worth a Read

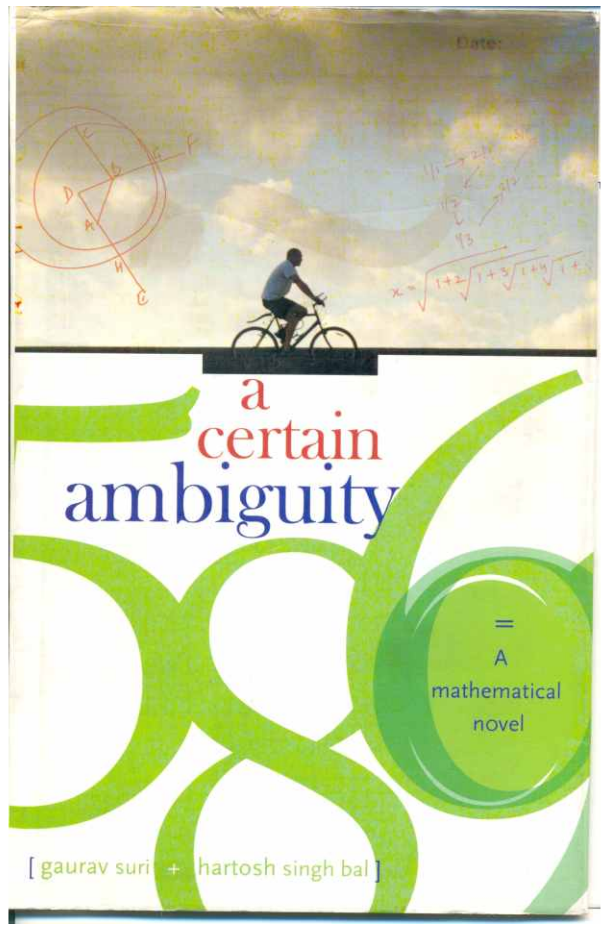
**Neeraja Raghavan**

Here are two attempts: one, to peep into the nature of the subject, Mathematics, and two, to get a glimpse of the workings of a Mathematical genius's mind. I want to share with you my experience of reading two unusual books: **A CERTAIN AMBIGUITY** (By Gaurav Suri and Hartosh Singh Bal) and **THE MAN WHO KNEW INFINITY – A Life of the Genius Ramanujan** (By Robert Kanigel). The former takes the reader on a journey that reveals immense beauty in the subject while the latter leaves the reader astounded at the marvels of one man's mind.

A CERTAIN AMBIGUITY calls itself (or rather, the authors call it) 'a Mathematical novel'. I found this so intriguing (how can a novel be Mathematical?) that I was drawn to reading the book.

And so began a compelling experience. Not having enjoyed Mathematics particularly as a school girl, (and having positively disliked it at the college level), this book left me wishing I had been taught Mathematics in this way. Why did no teacher ever help me see the beauty of numbers? The awesome logic? The aesthetic patterns? Does one have to go and study Math in Stanford University (as the main protagonist of the book does) in order to realize these aspects of the subject? (Surely not, as the second book revealed. But more of that, later.)

The title A CERTAIN AMBIGUITY springs from the dilemma that a grandfather and grandson struggle with throughout the book: Can there ever be absolute certainty in Mathematics or life? Knitting together a well-thought out plot (that has considerable amount of suspense in it) and actual expositions on Mathematics (through lectures delivered in Stanford University and that you wish you could have attended) the book straddles fiction and Mathematics beautifully. I hesitate to tell you that there are lectures in Mathematics strewn throughout the book: for that makes it sound forbidding. Believe me, they are mouth watering lectures! Rigour is maintained throughout the book: and the amazing thing is that it still makes for fascinating reading. I never knew Mathematics could be so attractive. The book also made me wonder: what does it mean to face the extent – and the limits – of human knowledge? A must read, for all Mathematics teachers and particularly, those afflicted by Mathematics phobia.




*Written by Suri and Bal, who have been friends since childhood and have both completed a Masters' in Mathematics (from Stanford and New York University, respectively), the book succeeds largely in realizing the writers' purpose: "Our principal purpose in writing A CERTAIN AMBIGUITY is to show the reader that Mathematics is beautiful. Furthermore, we seek to show that Mathematics has profound things to say about what it means for humans to truly know something." Thus begins the Authors' Preface.*

Most of us know the legendary tale: of how, in 1913, a 25-year-old Indian with no formal qualifications wrote a letter filled with startlingly original theorems to the Cambridge don, G H Hardy. Dimly, we are aware of how Ramanujan turned Mathematics upside down in the next five years.

But we (at least most of us) know little else.

Here is a book about an uncommon and individual mind: whose tragic tale still haunts his countrymen, in more ways than one. While Hardy was avuncular, he was still aloof – the British stiff upper lip - and this young man, who "grew up praying to stone deities; who for most of his life took counsel from a family goddess, declaring it was she to whom his Mathematical insights were owed" returned to India in 1919, depressed, sullen and quarrelsome. He died a year later.

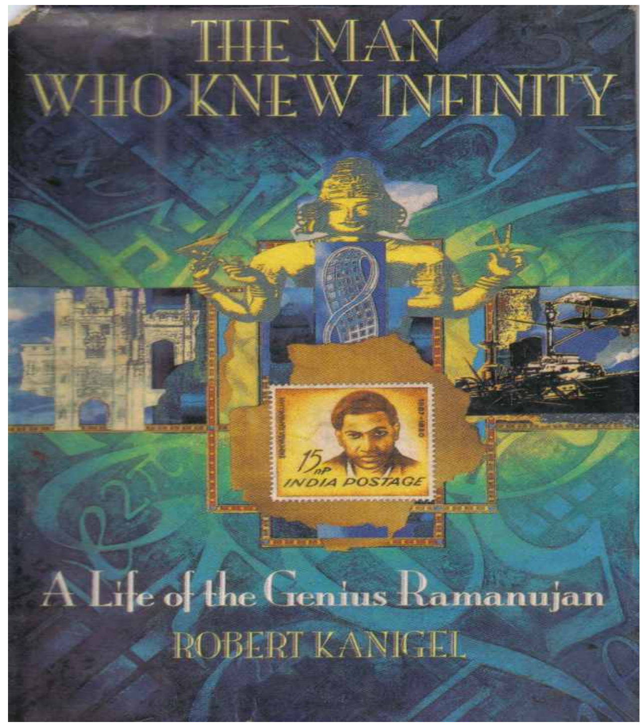
If ever there was an unsung hero, this was one!

In a short life span of 33 years, he accomplished so much that Mathematicians the world over are still trying to fathom some of it. And yet, his twenty year old widow eked out a humble and anonymous existence for most of the next half century, probably not the only one to be unaware of her husband's intellectual prowess.

This is a biography with a difference, for as the author says: "Biographies as do exist either ignore the Mathematics, or banish it to the back of the book. Similarly, scholarly papers devoted to Ramanujan's Mathematics normally limit to a few paragraphs their attention to his life. And yet, can we understand Ramanujan's life without some appreciation for the Mathematics that he lived for and loved? Which is to say, can we understand an artist without gaining a feel for his art? A philosopher without some glimpse into what he believed?"

The book is true to the above intent, in that it takes the reader into some of the problems that he applied number theory to, without dwelling too much on the subtle and powerful Mathematical tools that he used. For, as the writer confesses, Ramanujan's Mathematics is more accessible than some other fields; much of it comes under the heading of number theory, which seeks out properties of, and patterns among, the ordinary numbers with which we deal every day.

Interesting tidbits like the following whet the appetite of a reader (especially if the reader is a teacher) to know more about the curious child Ramanujan: "Quiet and contemplative, Ramanujan was fond of asking questions like, Who was the first man in the world? Or, How far is it between clouds?"




*The book reports ironic tidbits like the following: At the time of Ramanujan's death in April 1920, the editor of the Journal of the Indian Mathematical Society had fallen so far behind their publication schedule that the issue bearing the news was dated December 1919. Into copies of that issue, small olive green slips of paper, bordered in black, were inserted:*

THE LATE MR. S. RAMANUJAN

*We deeply regret to announce the untimely death of Mr. S. Ramanujan, B.A., F.R. S., on Monday, the 26th of April 1920, at his residence in Chetpet, Madras. An account of his life and works will appear in a subsequent issue of this journal. Seven months later, the journal carried two obituary notices.*

The book brings out many (Mathematical as well as real life) paradoxes, not the least of which is the strange alliance between a confirmed atheist (Hardy) and a staunch devotee of the goddess Namagiri of Namakkal. "An equation for me has no meaning," Ramanujan once said, "unless it expresses a thought of God."

While working on this book, Robert Kanigel spent five weeks in the South, traveling to places that had figured in Ramanujan's life. "I rode trains and buses, toured temples, ate with my hands off banana leaves. I was butted in the behind by a cow on the streets of Kumbakonam, shared a room with a lizard in Kodumudi."

Not surprisingly, therefore, this book is conspicuous also in the lack of condescension - which often (maybe even unwittingly?) creeps into the writings of many a Westerner about India/Indians.

Both these books sparked off a desire in me to learn Mathematics well; and shook my former image of the subject (as didactic and dry) substantially. I recommend both books strongly, especially to teachers of Mathematics, for they will get juicy material for use in classes.

**Neeraja Raghavan** is Consultant, Academics and Pedagogy, Azim Premji Foundation, Bangalore. She has been a freelance writer for many years with over seventy articles published in leading newspapers and magazines. In addition, she is the author of three books (CURIOUSER AND CURIOUSER, Full Circle 2004, I WONDER WHY AND I WONDER HOW, Children's Book Trust 2005,2006), co-editor of one (ALTERNATIVE SCHOOLING IN INDIA, SAGE Publications 2007) and editor of a CD titled Understanding Religions (Jain Vishva Bharathi Institute, Ladnun 2004). She can be contacted at neeraja@azimpremjifoundation.org

## Logico-Math Brain Teasers

There are 12 balls in a box. They are identical to each other in terms of size, shape, colour, feel, appearance etc. The weight of one of these 12 balls is slightly different (it could be heavier or lighter) from the others which have identical weight. You are provided a two-pan weighing balance (but no weights). How will you identify the odd ball out and ascertain if it is lighter or heavier than the rest in only three weightings?

*(Hint: Label the balls and try various weighing combinations)*

Use this space for calculation 🙂

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# Digital Resources for Mathematics Teachers

**S N Gananath**

Remarkable progress in the Information and Communication Technologies (both hardware and software) in the past decade or so has made it possible for teachers to access quality digital resources and use them either directly or indirectly in the classroom.

What makes it attractive for teachers are some unique features of these resources such as:

1. Extensive availability
2. Range of resources across topics and grades
3. Innovative ideas
4. Can be used on or offline
5. Resources in Mathematics from other countries / regions can easily be used / customized for our schools as Math is a universal subject
6. A large number of non-text resources like audio/video/applets etc.
7. Very low recurring cost
8. Increased penetration of ICT in schools – both urban and rural (including government schools)

Teachers have taken to this new idea with enthusiasm (is it because of novelty alone? Quite possible). Of course, there are critics who do not approve their usage. There are several studies to understand the effectiveness of using these resources in the classroom. Some of the data has provided useful insights and pointers. However, the results have not been conclusive and more research is needed in this critical area.

Digital resources can surely augment meaningful learning experiences of the child and can be specifically used for the following:

1. Introduction of to a new topic
2. Formation of key concepts
3. Repetitive practice (read drill)
4. Self relearning
5. Group work

and so on .......

Teachers can use them in their classes if they fit into their lesson plan.

The following list gives a rough idea of the range of digital resources (mostly free) available on the Internet. This is only indicative and by no means representative or comprehensive - to be used more as pointers- and the readers are requested to make their own judgments regarding their suitability to their work although I must add that I have found them very useful and enjoyable.

The resources have been grouped into a few (loose) categories to facilitate easy navigation and exploration.

## Websites

### General

The Mathforum@ Drexel University (http://www.mathforum.org)

The Centre for Innovation in Mathematics Teaching (CIMT) (http://www.cimt.plymouth.ac.uk)

Math cats – Fun math for kids (http://www.mathcats.com), Count on (http://www.counton.org)

Illuminations – Resources for teaching maths (http://illuminations.nctm.org) InterActivate (http://www.shodor.org/interactivate)

Gadsden Mathematics Initiative (http://www2.gisd.k12.nm.us/GMIWebsite/IMathResources.html)

Mathematical Interactivities - Puzzles, Games and other Online Educational Resources (http://mathematics.hellam.net)

MathNet – Interactive mathematics in education (http://www.mathsnet.net)

National Library of Virtual Manipulatives (http://nlvm.usu.edu/en/nav/vlibrary.html)

NewZealand Maths (http://www.nzmaths.co.nz)

Primary Resources – Maths
(http://www.primaryresources.co.uk/maths/maths.html)

ProTeacher! Maths lesson plans for elementary schoolteachers
(http://www.proteacher.com/100000.html)

Maths activities
(http://www.trottermath.net/contents.html)

Maths powerpoints
(http://www.worldofteaching.com/mathspowerpoints.html)

Maths is fun – maths resources
(http://www.mathsisfun.com)

Middle school portal for maths and science teachers
(http://www.msteacher.org/math)

Maths games, maths puzzles and maths lessons – designed for kids and fun
(http://www.coolmath4kids.com)

**Numbers**

Magic Squares, Magic Stars & Other Patterns
(http://recmath.org/Magic%20Squares)

Number recreations
(http://www.shyamsundergupta.com)

Broken calculator – Maths investigation
(http://www.woodlands-junior.kent.sch.uk/maths/broken-calculator/index.html)

Calculator chaos
(http://www.mathplayground.com/Calculator_Chaos.html)
Primary School Numeracy
(http://durham.schooljotter.com/coxhoe/Curriculum+Links/Numeracy)

Quarks to Quasars, powers of 10
(http://www.wordwizz.com/pwrsof10.html)

**Algebra**

Algebra puzzle
(http://www.mathplayground.com/Algebra_Puzzle.html)

Algebra tiles
(http://mathbits.com/MathBits/AlgebraTiles/AlgebraTiles MathBitsNew07ImpFree.html)
(http://mathbits.com/MathBits/AlgebraTiles/AlgebraTiles MathBitsNew07ImpFree.html)

Geometry
http://www.cyffredin.co.uk/

The Fractory: An interactive tool for creating and exploring fractals
(http://library.thinkquest.org/3288/fractals.html)

Tessellate
(http://www.shodor.org/interactivate/activities/Tessellate)

MathSphere – Free graph paper
(http://www.mathsphere.co.uk/resources/MathSphereFreeGraphPaper.html)

Paper models of polyhedral
http://www.korthalsaltes.com/

**Problem solving**

Mathpuzzle http://www.mathpuzzle.com/

Puzzling world of polyhedral dissections
http://www.johnrausch.com/PuzzlingWorld/contents.html

Interactive Mathematics Miscellany and Puzzles
(http://www.cut-the-knot.org)

Puzzles and projects
(http://www.delphiforfun.org/Programs/Indices/projectsIndex.html)

10ticks Daily Puzzle Page
(http://www.10ticks.co.uk/s_dailyPuzzle.aspx)

Archimedes' Laboratory – Teachers' resource : Improve problem solving skills
(http://www.archimedes-lab.org/index_teachers.html)

Brain teasers
(http://www.pedagonet.com/brain/brainers.html)

Gymnasium for Brain
(http://www.gymnasiumforbrain.com)

Puzzles and games (www.thinks.com)

**Miscellaneous**

Mathematical imagery (http://www.josleys.com)

The MacTutor History of Mathematics archive (http://www-history.mcs.st-and.ac.uk/history)

Math cartons (http://www.trottermath.net/humor/cartoons.html)

Math Comics (http://home.adelphi.edu/~stemkoski/mathematrix/comics.html)

Mathematical quotation server (http://math.furman.edu/~mwoodard/mqs/mquots.html)

Wolfram Mathworld – The Web's Most extensive Mathematical Resource (http://mathworld.wolfram.com)

Optical illusions and visual phenomena (http://www.michaelbach.de/ot)

Optical illusions gallery (http://www.unoriginal.co.uk/optical5.html )

Teachers' Resources Oline (http://www.cleavebooks.co.uk/trol/index.html)

Interactivate : Activities (http://www.shodor.org/interactivate/activities/#fun)

Maths articles (http://www.mathgoodies.com/articles)

Math words and some other words of interest (http://www.pballew.net/etyindex.html)

Portraits of scientists and mathematicians

(http://www.sil.si.edu/digitalcollections/hst/scientific-identity/CF/display_results.cfm?alpha_sort=R )
Let epsilon < 0 (http://epsilon.komplexify.com)

Grand illusions (http://www.grand-illusions.com)

Portrait gallery - Mathematicians

(http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=2437&bodyId=2241)

Maths teaching ideas (http://www.teachingideas.co.uk/maths/contents.html)

**e-books**

Illustrated maths formulas – Salim
http://www.arvindguptatoys.com/arvindgupta/mathformulas.pdf

Ramanujan – the man behind the mathematician – Sundaresan and Padmavijayam
http://gyanpedia.in/tft/Resources/books/ramanujan.doc

A Mathematician's apology – G.H.Hardy
http://math.boisestate.edu/~holmes/holmes/A%20Mathematician%27s%20Apology.pdf

Puzzle maths – G. Gamov and Stern
http://www.arvindguptatoys.com/arvindgupta/puzzlemath.pdf

1000 uses of a hundred square – Leah Mildred Beardsley
http://www.mediafire.com/download.php?detnojrueje

Geometry comic book – Jeane Pierre Petit
http://www.mediafire.com/?ud0nnnujzyy

Elements – Euclid
http://www.mediafire.com/?ud0nnnujzyy

How children learn mathematics
http://gyanpedia.in/tft/Resources/books/mathsliebeck.pdf

Suggested experiments in school mathematics – J.N.Kapur
http://www.arvindguptatoys.com/arvindgupta/jnkapur.pdf

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**S.N.Gananath** has been involved in the creation, validation and dissemination of resources relating to school mathematics for over two decades with the conviction that joy of learning and teaching can be achieved by innovative, effective methodologies and approaches in any subject and mathematics is no exception! He is the founder director of Suvidya – an Educational Resource Centre now based in Mysore. He can be contacted at sngananath@gmail.com

# 35 Resource Kit

Nidhi Tiwari & Madhumita Sudhakar

*This resource kit is not by any means an exhaustive one. It has been put together with the help of several individuals who work in the field of education, all of whom we would like to sincerely thank for their time and effort.*

**A. Some Popular Publishers of Math Books for Children**

| S.no. | Name of the Publisher | Website |
|---|---|---|
| 1 | Children's Book Trust | www.childrensbooktrust.com |
| 2 | Eklavya | http://eklavya.in/ |
| 3 | Flipkart | www.flipkart.com |
| 4 | Macmillan Publishers | http://international.macmillan.com |
| 5 | National Book Trust | www.nbtindia.org.in |
| 6 | National Council of Educational Research and Training | www.ncert.nic.in |
| 7 | Navnirmiti | http://www.navnirmiti.org/index.html |
| 8 | Pratham Books | www.prathambooks.org |
| 9 | Scholastic India Publishing | www.scholasticindia.com/publishing.as |
| 10 | School Zone Publishing | www.schoolzone.com |
| 11 | The Mathematical Sciences Trust Society | http://www.mstsindia.org/ |
| 12 | Vidya Bhawan Society | www.vidyabhawan.org |
| 13 | Digantar Khel Kud Society | |
| 14 | Homi Bhabha Center for Science Education, Mumbai | www.hbcse.tifr.res.in |

**B. Some Articles on Mathematics published in Sandarbh, by Eklavya, Bhopal***

| Issue no. | Title of the Article | Author |
|---|---|---|
| 1 | I Am Afraid of Mathematics | Ganga Gupta |
| 2 | Something- From the Past | Rohit Dhankar |
| 4 | An article on statistics | Stephen J.Goold |
| 6 | Understanding of Students About Mathematics | Madhav Kelkar |
| 10 | What the Teacher Said and What the Students Understood | Venu Endle |
| 22-23 | Puzzle of a Magical Pond | Vijay Shankar Verma |
| 28 | Wonderful Geometrical Figures | Abhishek Dhar |
| 33 | Relation of Circle with Radius | Jui Dadhich |
| 40 | Pieces of Paper, Algebra and Pythagoras theorem | Prakash Burte |
| 51 | My Journey | P.K. Srinivasan |
| 52 | Teaching Negative Numbers to School Children | Jayashree Subramanian |
| 53 | Series and Infinite Series | Jayashree Subramanian |
| 54 | Making Mathematics Interesting | Pramod Maithil |
| 54 | Multiplication of Negative Numbers | Jayashree Subramanian |
| 55 | Zero + Zero + Zero + Zero + ... | Jayashree Subramanian |
| 57 | Teaching Place Value and Double Column Addition | Constance Kamii and Linda Josesh |
| 62 | My Mathematics Classes and Saurabh | Mohammad Umar |

*Original titles are in Hindi.

## C. Articles from magazines published by Vidya Bhawan ERC

| Title of the Book | Title of the Article | Author |
|---|---|---|
| Construction of Knowledge | About learning Mathematics (Also available in Hindi.) | H.K. Dewan and Ashok Kumar |
| | Teaching of Mathematics at Primary Stage | H.K. Dewan |
| | About learning Mathematics | H.K. Dewan |
| | Mathematics: Materials and Laboratories | H.K. Dewan |
| Material Development for | LMT-01 series , AMT-01 series | |

## D. Articles from "Primary Education vol.2 July-Sep 02"*

| S.no. | Title of the Article | Author |
|---|---|---|
| 1 | A way to explore Children's understanding of Mathematics | Padma M. Saragapani. |
| 2 | Errors as Learning Strategies | R.K. Agnihotri |
| 3 | Reflections on Mathematics Teaching | H.K. Dewan |
| 4 | Common Errors in Primary School Mathematics? | H.C. Pradhan |
| 5 | Does the Child Know any Mathematics | H.K. Dewan |
| 6 | Intercultural Mathematics Education in Peru | Joachim Schroeder |
| 7 | How Mathematical Ideas Grow - an extract | IGNOU's AMT series |
| 8 | Why have a Laboratory for Mathematics? | Rohit Dhankar |
| 9 | Math phobia among Teachers and Children: Glimpses from a Survey | S.N. Gananath & C Srinath |
| 10 | The Metric Mela, a Celebration of Measurement in Karnataka | K.M. Sheshagiri |
| 11 | Teaching Tribal Children Mathematics Through Real Contexts | Binaya Krushna Pattanayak |

*Original titles are in Hindi.

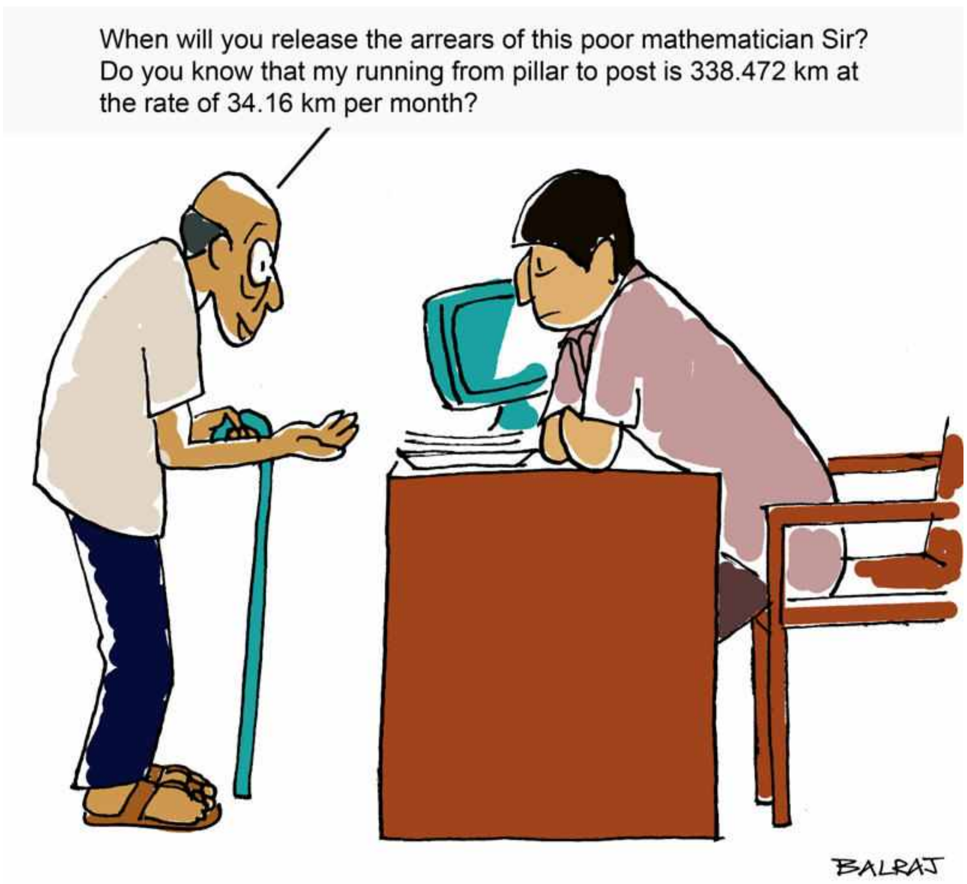
## E. Some NGOs working in the area of Math Education

Eklavya, Bhopal
Homi Bhabha Centre for Science Education, Mumbai
Jodogyan, Delhi
Navanirmiti, Mumbai
Shishu Milap, Vadodara
Suvidya, Mysore
Vidya Bhawan Society, Udaipur

**F.** The site www.arvindguptatoys.com has several books / publications on Mathematics. For example, some of the books are 'Illustrated Maths Formulas', 'Geometry for Kids', 'Math Wonders' etc. We invite readers to visit the site.

**G. Some Web Links to make Math Learning fun:**

1. http://www.mathcelebration.com/index.html
2. http://www.artofproblemsolving.com/
3. http://www.noetic-learning.com/others.jsp
4. http://cte.jhu.edu/techacademy/web/2000/heal/siteslist.html
5. http://www.cimt.plymouth.ac.uk/
6. http://vedicmathsindia.blogspot.com/
7. http://www.vedicmathsindia.org/
8. http://www.teach-nology.com/gold/basicword.html
9. http://www.mathplayground.com
10. http://www.math.com
11. http://www.mathsisfun.com
12. http://www.coolmath4kids.com
13. http://www.mathcats.com
14. http://www-history.mcs.st-and.ac.uk/BirthplaceMaps/Countries/India.html
15. http://www.teachingideas.co.uk/maths/contents.html
16. http://www.playkidsgames.com/mathGames.html




Azim Premji Foundation

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Website : www.azimpremjifountation.org