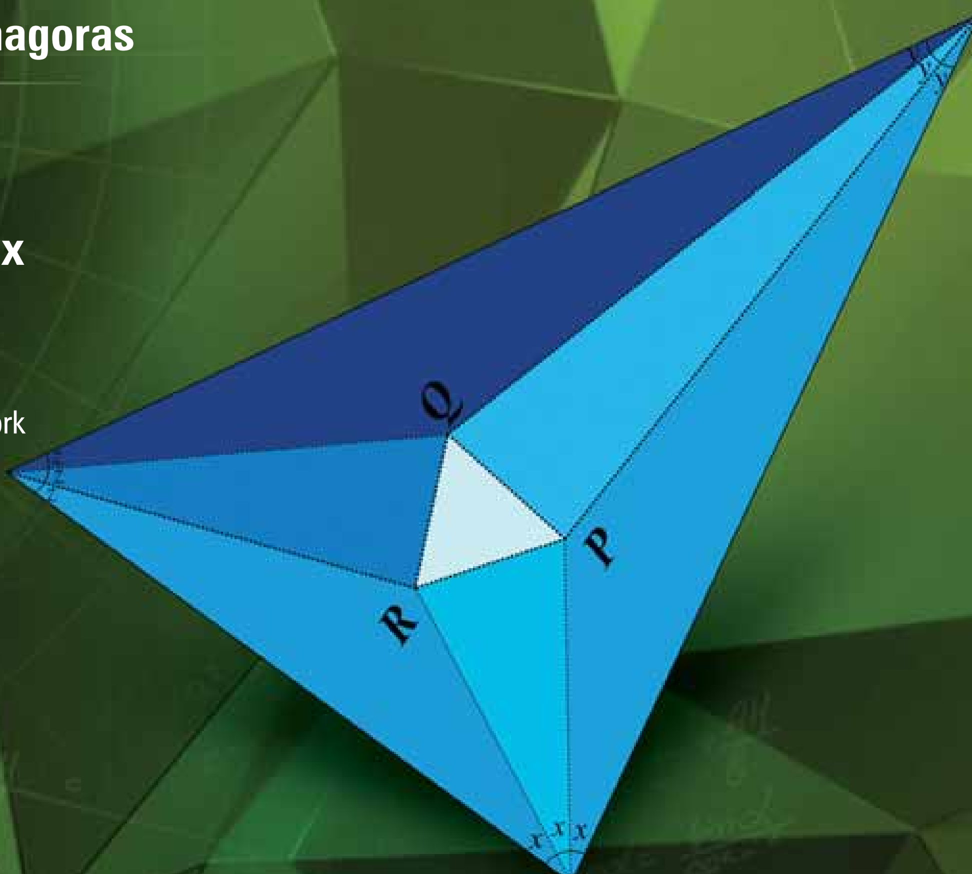
Azim Premji University

A publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 1, No. 1
June 2012

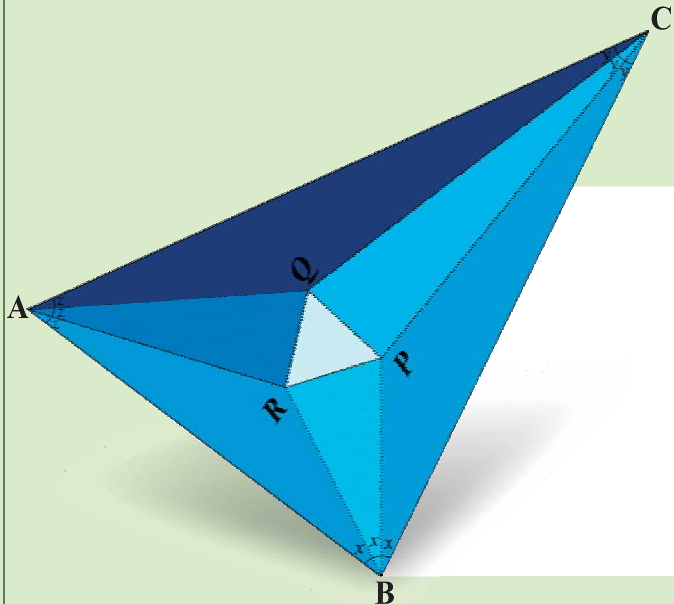


Pullout: Teaching Fractions through paper folding

Notes on Cover Image

Morley's Miracle

The cover displays an extremely famous theorem - Morley's theorem, discovered by Frank Morley in 1899. It states the following (with reference to a given triangle ABC): In any triangle ABC, draw the two internal trisectors of each angle. Let the trisectors nearest to side BC meet at P, let those nearest to side AC meet at Q, and let those nearest to side AB meet at R, as shown in the figure. Then, regardless of the shape of ABC, triangle PQR is always equilateral.



The theorem is one of the many results in Euclidean geometry discovered in recent centuries, discoveries missed by the Greeks. Numerous results pertaining to triangles and circles were discovered during the eighteenth and nineteenth centuries when the centuries old subject of Euclidean geometry was undergoing a revival; but of these, Morley's theorem is perhaps the most remarkable and most beautiful. It is sometimes called Morley's Miracle. It can be proved in many different ways, but to find even one proof is a great challenge! Perhaps the one most easily found is the trigonometric proof; it is direct, and based on the triple angle identity for the sine. There are also proofs based on 'pure geometry'.

Today, Euclidean geometry is undergoing yet another revival, thanks to the entry of powerful dynamic geometry software like GeoGebra and the easy availability of computers.

From The Chief Editor's Desk . . .

Greetings to all readers. With this issue we start the publication of a new mathematics magazine which will serve as a resource for math teachers and students in school.

Many of us involved in math education have long felt the need for such a national level magazine — a publication addressed directly to teachers and students, which offers a space to read about interesting topics in mathematics, share one's observations and experiences, read about new resources, hear about upcoming events, and participate in problem solving. At small, local levels there do exist a few such publications, but they do not seem to reach the national mainstream, or play a part in a nation-wide debate on math education.

What value does such a forum have? Oh, it is vitally important! Mathematics education in India is in a deep crisis today, at almost every level. There are many reasons behind this, but one of them, certainly, is the absence of a forum for mathematics teachers where they can talk about their subject, share ideas, articulate doubts and anxieties, and learn from one another. Likewise, for students who love mathematics, there is not much of a space to relate with others of like mind.

About a year back the Community Mathematics Centre ('CoMaC') of Rishi Valley School approached the Azim Premji Foundation ('APF') to find out whether they would consider a joint venture: a school level mathematics magazine! APF expressed immediate interest, and soon an editorial group was formed, comprising members from APF, CoMaC, some schools and colleges, and a few other organizations engaged in mathematics and science education. It was decided by this group that the magazine would be for high school and middle school, and would be meant for teachers as well as students. It would be brought out thrice a year, and each issue would have a 'Features' section, with articles on math topics; a 'Classroom' section, for teachers; a 'Technology' section, with articles on math software; a 'Reviews' section with reviews of books, software and YouTube clips; a 'Problem Corner'; a detachable 'Pullout' for primary school teachers; and miscellaneous material. The result is the magazine that you now hold in your hands.

We named it At Right Angles — a phrase suggestive of mathematics but also with connotations of thinking 'out of the box' and of trying out bold, new ideas. It is our hope that our magazine will help stimulate such thinking and such action. We even have a nickname for it: AtRiA. This is because 'atria' has a dual meaning: in architecture the word 'atrium' means 'large open space', and the plural of 'atrium' is 'atria'. Indeed, Wikipedia has the following: "Atria are a popular design feature because they give their buildings a feeling of space and light."

It is our hope that At Right Angles will play just such a role for mathematics education: provide a large open space, in which we can talk about matters related to mathematics and mathematics education, matters we wish to share with others — those that have given us pleasure, and those which have given us pain . . . .

A magazine of this sort needs regular contributions from its readers; these are essential for its survival and good health! So we urge readers to write articles for us. The guidelines for submissions are on the back inside cover of the magazine. We also hope that readers will convey to us what they think of the various articles. Suggestions, views, comments, criticisms, . . .—all are welcome, so do please send them in!

The inaugural issue has a lot of material on the Pythagorean theorem and themes related to this theorem; this will continue into the second issue as well. It seems fitting that we do this, because the theorem of Pythagoras has an incredibly rich history, with roots in every part of the world, and it is certainly the best known result in all of mathematics.

But there are also other articles: a lovely 'proof without words' from the late Prof A R Rao; articles on paper folding, on Pythagorean triples, on the use of spreadsheets, on a classification of quadrilaterals, on the use of math portfolios in teaching; a review of a well known book on Fermat's Last Theorem; a pullout on teaching fractions at the primary level; . . . . Enjoy your reading!

— Shailesh Shirali

Design & Print

SCPL Design
Bangalore - 560 062
+91 80 2686 0585
+91 98450 42233
www.scpl.net

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley. It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

At Right Angles
A Resource for School Mathematics

Azim Premji University
A Publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

Contents

Features

This section has articles dealing with mathematical content, in pure and applied mathematics. The scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practicing mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

Tech Space

"Tech Space" is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and data analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

At Right Angles
A Resource for School Mathematics

Azim Premji University
Publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

Contents contd.

In the Classroom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The Classroom' is meant for practicing teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

Reviews

Pullout

Problem Corner

feature

The Most Famous Theorem

The Theorem of Pythagoras

Understanding History – Who, When and Where

Does mathematics have a history? In this article the author studies the tangled and multi-layered past of a famous result through the lens of modern thinking, looking at contributions from schools of learning across the world, and connecting the mathematics recorded in archaeological finds with that taught in the classroom.

Shashidhar Jagadeeshan

Imagine, in our modern era, a very important theorem being attributed to a cult figure, a new age guru, who has collected a band of followers sworn to secrecy. The worldview of this cult includes number mysticism, vegetarianism and the transmigration of souls! One of the main preoccupations of the group is mathematics: however, all new discoveries are ascribed to the guru, and these new results are not to be shared with anyone outside the group. Moreover, they celebrate the discovery of a new result by sacrificing a hundred oxen! I wonder what would be the current scientific community's reaction to such events.

This is the legacy associated with the most 'famous' theorem of all times, the Pythagoras Theorem. In this article, we will go into the history of the theorem, explain difficulties historians have with dating and authorship and also speculate as to what might have led to the general statement and proof of the theorem.

Making sense of the history

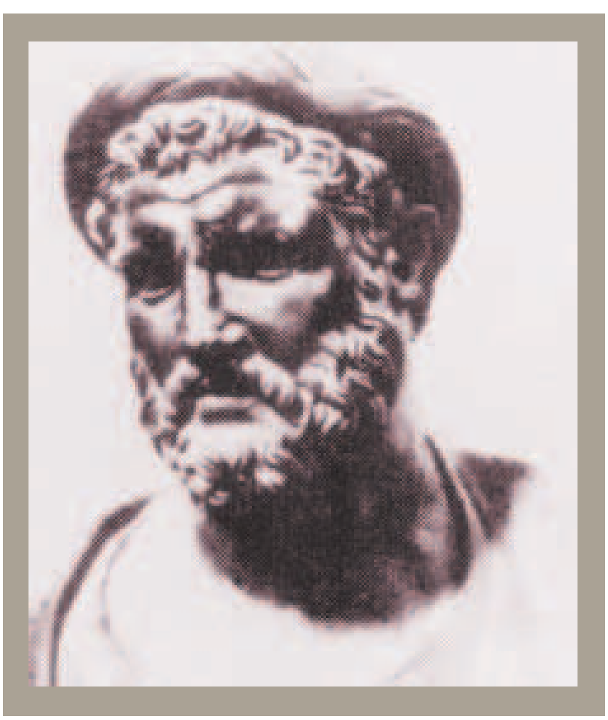
Often in the history of ideas, especially when there has been a discovery which has had a significant influence on mankind, there is this struggle to find out who discovered it first. This search is very often coloured by various biases and obscured by the lack of authentic information and scholarship. The Pythagoras Theorem suffers from the same fate. In this article I hope to give a summary of current understanding and, not being an expert historian, would like to state right at the beginning that I might have left out some major contribution.

Before proceeding further let us, at the cost of redundancy, recall the Pythagoras Theorem as stated by Euclid (I.47 of the *Elements*) and refer to it from now on as PT (P here can stand for Pythagoras or 'Preeminent'!).

Pythagoras Theorem (PT)

In right-angled triangles, the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle.

Obviously the first challenge for historians is the name.



Pythogoras (Approx 572 BC to 475 BC)

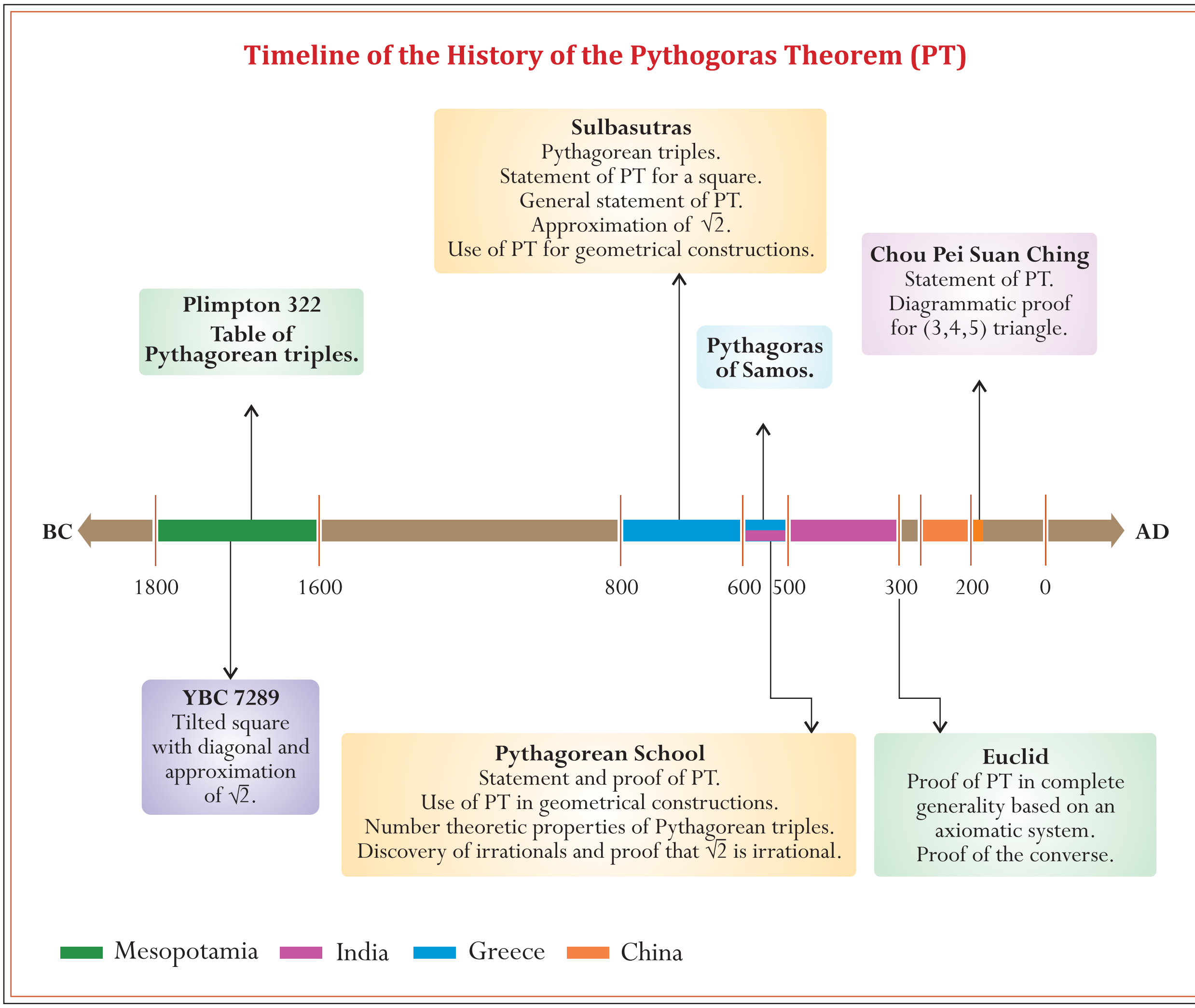
Why Pythagoras? Greek scholars seem to be in agreement that the first person to clearly state PT in all its generality, and attempt to establish its truth by the use of rigorous logic (what we now call mathematical proof), was perhaps Pythagoras of Samos. We actually know very little about Pythagoras, and what we do know was written by historians centuries after he died.

Legend has it that Pythagoras was born around 572 B.C. on the island of Samos on the Aegean Sea. He was perhaps a student of Thales, a famous Greek philosopher and mathematician who was born half a century before Pythagoras. It is believed that Pythagoras travelled to Egypt, Babylon and even to India before he returned to Croton, a Greek settlement in south-east Italy. Here he seems to have gathered a group of followers forming what we call the Pythagorean sect, with beliefs and practices as described in the introduction. It is believed that many Greek philosophers (Plato, for instance) were deeply influenced by Pythagoras, so much so that Bertrand Russell felt that he should be considered one of the most influential Western philosophers.

We will return to the Pythagorean School after we take a detour and look at contributions outside the Greek world (this is often difficult for many Eurocentric historians to swallow!)

The problem of dating!

As students of history we must realise that the greatest challenge historians of antiquity face is that of giving accurate dates to events. There are many reasons for this, including the fact that many cultures were oral, records of events were burnt, languages of some cultures have yet to be deciphered and very often, as mentioned earlier, our only knowledge about people and events are from historians referring to them many years later. So I have tried to use a very conservative and broad timeline.



Let us now try and understand the history of PT chronologically.

The Mesopotamian contribution

You may recall from your school history that one of the oldest known civilizations (Mesopotamia or Babylonia) existed in the geographical region between the Tigris and Euphrates rivers. Records of this civilization date back to 3500 BC. They used the sexagesimal system (base 60) and used mathematics for record keeping and astronomy. They also seemed interested in number theory and geometry.

We know this because they left records of their work on thousands of clay tablets, five hundred of which seem mathematical in nature. There are two main sources that tell us about the Mesopotamian contribution to the PT. These are clay tablets with wedge shaped markings on them. Historians date these to the period of Hammurabi between 1800 BC and 1600 BC.

The tablet known as YBC 7289 (tablet number 7289 from the Yale Babylonian Collection) shows a tilted square with wedged shaped markings. The markings show calculations for the approximation of $30\sqrt{2}$. This would not have been possible

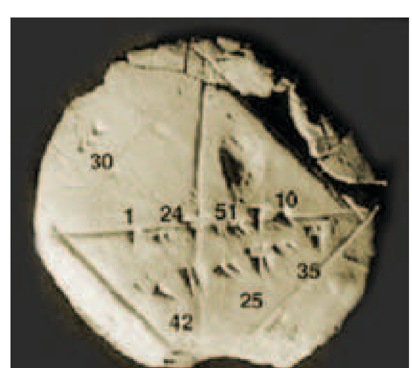


Fig. 1, YBC 7289

Fig.2, Plimpton 322

without the knowledge of the PT or at least the special case of the isosceles right-triangle (see Figure 5).

The second is referred to as Plimpton 322, a slightly damaged clay tablet measuring 13 9 2 cm and a part of the University of Columbia collection. It contains a 15 4 table of numbers. The table is thought to be a list of Pythagorean triples. Pythagorean triples are integers (*a, b, c*) which satisfy the equation $a^2+b^2=c^2$ (see articles on Pythagorean triples). For example (3, 4, 5) is a Pythagorean triple.

Let us look at the entries in the tablet. The tablet contains errors in Rows 2, 9, 13 and 15 and the original entries were in base 60, but the table below is in base 10 with the errors corrected. Here '*s*' stands for the shortest side of a right-triangle, '*d*' for the hypotenuse and '*l*' for the other side (see Figure 3).

Let us look at entries in the first row. It is not hard to check that $169^2-119^2=120^2$. That is, (119, 120, 169) is a Pythagorean triple. Similarly we can verify that (s, l, d) form Pythagorean triples in each row (if you are sceptical – go ahead and do the computations!) So it seems clear that the tablet was a list of Pythagorean triples.

However, mathematical historians are left with many questions. What exactly does Column 1 represent? Is there any pattern behind the choice of '*s*' and '*d*'? Is there some general principle at work here?

There are three main interpretations of the purpose of these tablets. The first is that Plimpton 322 is a trigonometric table of some sort. Column 1 is Csc^2 A, where angle 'A' ranges from just above 45° to 58°.

$(d/l)^2$	s	d	
$(169/120)^2$	119	169	Row 1
$(4825/3456)^2$	3367	4825	Row 2
$(6649/4800)^2$	4601	6649	Row 3
$(18541/13500)^2$	12709	18541	Row 4
$(97/72)^2$	65	97	Row 5
$(481/360)^2$	319	481	Row 6
$(3541/2700)^2$	2291	3541	Row 7
$(1249/960)^2$	799	1249	Row 8
$(769/600)^2$	481	769	Row 9
$(8161/6480)^2$	4961	8161	Row 10
$(75/60)^2$	45	75	Row 11
$(2929/2400)^2$	1679	2929	Row 12
$(289/240)^2$	161	289	Row 13
$(3229/2700)^2$	1771	3229	Row 14
$(106/90)^2$	56	106	Row 15

Table 1 - Plimpton 322 in modern notation

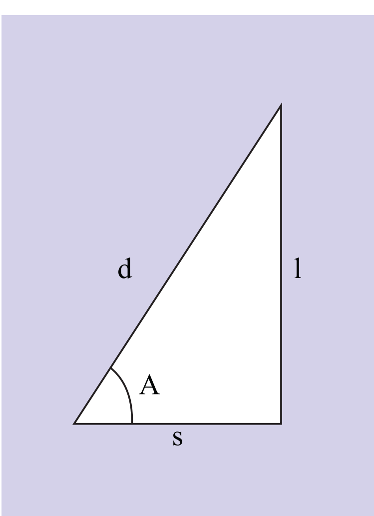


Fig. 3

Row#	p	q	x	y	S	d
1	12	5	144/60	25/60	119	169
8	32	15	128/60	101250/ 216000	799	1249
15	9	5	108/60	20/36	56	106

Table 2

The second and third interpretations are more involved and we will require another table to understand them! (See Table 2; the roles played by '*x*' and '*y*' will become clear after reading the following two paragraphs.)

Let us examine the first, last and middle rows of Plimpton 322 to see if any pattern emerges. Neugebauer and others have proposed that '*s*' and '*d*' are generated by a pair of positive integers '*p*' and '*q*', which are of opposite parity and relatively prime. The relationship between *s*, *d*, *l*, *p* and *q* is as follows: $s=p^2-q^2$, $d=p^2+q^2$, $l = 2pq$. In fact, if we are given any two integers *p*, *q* relatively prime, with one of them even, we can generate all Pythagorean triples. Quite a remarkable feat, don't you think? The article by S Shirali (elsewhere in this issue) explores various ways of generating Pythagorean triples and in the next issue this method will be explored in detail.

Of course one can now ask, is there a pattern in the choice of *p* and *q* (and also *x* and *y*)? They are so-called 'regular' numbers (numbers of the form $2^a 3^b 5^c$, where *a*, *b*, *c* are integers). Can you see the connection between 2, 3, 5 and the sexagesimal system? Moreover, there is a pattern on how *p*, *q*, *x*, *y* change as we move from Row 1 to 15; but this is quite technical, so for more details we refer the reader to [AA] and [RE].

The third interpretation, first put forward by Bruins in 1949, is called the 'reciprocal' method. Here the table is believed to be generated by a pair of rational numbers '*x*' and '*y*' such that $xy = 1$ (So *x* and *y* are a pair of reciprocals.)

Here $\frac{s}{l} = \frac{x-y}{2}$ and $\frac{d}{l} = \frac{x+y}{2}$

It is impossible to say for sure which interpretation is correct. But scholars feel that apart from Plimpton 322 there is no other evidence of knowledge of trigonometry in Mesopotamia, and that the second interpretation is not in keeping with the approach to mathematics found in the other tablets. Many scholars favor the 'reciprocal' method as they feel that it is not only mathematically valid, but is also historically, archeologically and linguistically consistent with the style and conventions of ancient Babylonian mathematics.

It may amuse readers to know that scholars like Robson [RE] feel that the author of Plimpton 322 was a teacher, and the tablet is a kind of 'question bank' which would *"have enabled a teacher to set his students repeated exercises on the same mathematical problem, and to check their intermediate and final answers without repeating the calculations himself."*

These two tablets, along with evidence from tablets found in Susa and Israel from the Babylonian period, clearly demonstrate that they were well versed with the PT and were also adept at using it.

Contribution from India

The history of India and Indian mathematics poses many challenges to historians. The difficulties range from giving a balanced and accurate picture to dating various events. At the same time, there seems to be a great deal of interest today in the contributions of the Indian subcontinent to mathematics. This is particularly so after the discovery of the Kerala School of mathematics, which came very close to discovering calculus long before Newton and Leibnitz. Mathematics in India was inspired by astronomy, record keeping, religion and perhaps sheer curiosity.

Historians believe that early Indian civilizations date to the third or fourth millennium BC. Our main interest is in the *Sulbasutras,* which literally means the 'rule of cords.' They are a series of texts (*Vedangas*) which accompany the Vedas and give detailed instructions on how rituals are to be performed and sacrificial altars (*Vedis*) constructed.

The most important Sulbasutras are attributed to Boudhayana, Manava, Apastamba and Katyayana. Boudhayana is believed to have lived around 800 BC and Apasthamba around 500 BC.

What is of significance is that in the Sulbasutras we find a general statement of the PT as follows [see PK]: *"The cord* [equal to] *the diagonal of an* *oblong makes* [the area] *that both the length and width separately* [make]. *By knowing these* [things], *the stated construction is* [made]. *"* {Apastamba Sulbasutra 1.4 and Boudhayana Sulbasutra 1.12}

This may be the earliest general statement (at least for right triangles with rational sides) of the PT, perhaps predating 800 BC. However, we must mention that there is a reference to this result in China clearly before Pythagoras, but whose exact date is unknown. We will discuss this in the section on the Chinese contribution.

We also find in the Sulbasutras (see [PK]) the application of the PT to a square (isosceles right triangle): *"The cord* [equal to] *the diagonal of a* [square] *quadrilateral makes twice the area. It is the doubler of the square."* {Apastamba Sulbasutra 1.6, Boudhayana Sulbasutra 1.9 and Katyayana Sulbasutra 2.9}

We will discuss in a later section how perhaps the recognition of PT for the special case of the isosceles right triangle led to the discovery of the general theorem.

The Sulbasutras also contain Pythagorean triples, approximation of square roots and the use of PT for many geometrical constructions. Why don't you try your hand (using a straightedge and compass) at some of the constructions found in the Sulbasutras? For example, try constructing a square whose area is equal to the sum of two given squares. Or, try constructing a square whose area is equal to that of a given rectangle.

What emerges clearly is that the Sulbakaras (authors of the sutras) had a very good understanding of PT and its applications, both to extracting roots and to geometrical constructions. We must, however, acknowledge that there is no evidence that the notion of *proving* mathematical statements was part of their framework.

Chinese contribution

We are all aware that China has been home to a very ancient civilization that developed along the rivers of Yangtze and Huang Ho more than 5000 years ago. The Chinese were interested in many areas of mathematics, again perhaps driven by astronomy, the need to have accurate calendars and sheer intellectual interest.

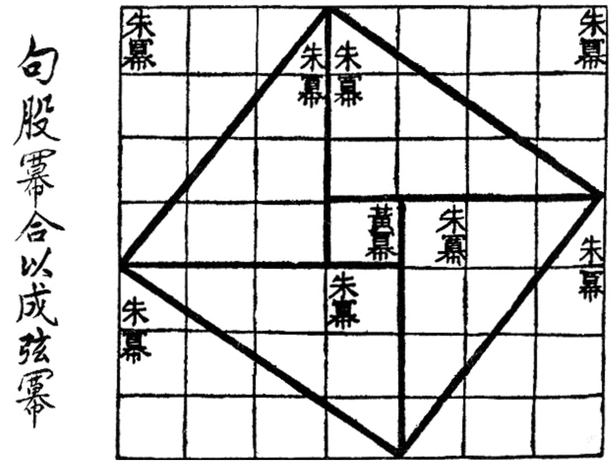


Fig. 4 - Chou Pei Suan Ching

As far as the PT is concerned, our main source of information is the Chou Pei Suan Ching (The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven). The exact date of this book has been debated for a long time. It refers to a conversation between the Duke Zhou Gong and his minister Shang Kao around 1000 BC, discussing the properties of a right triangle, with a statement of the PT and a diagrammatic proof given. It is not clear if such a conversation did take place. However, scholars believe that earlier results were put together in the form of a book, from 235 BC to 156 BC, and were edited by Zhang Chang around 156 BC. Further, a famous mathematician Zhao Shuang wrote commentaries on the Chou Pei, adding original material of his own, including the well-known diagrammatic proof (see Figures 4, 7 and 8).

The PT in Chinese literature is referred to as '*kou ku'* (see [JG]). We will discuss the diagrammatic proof in the section on how the Greeks might have arrived at a general proof for the PT. What is clear is that the Chinese were not only aware of PT long before Pythagoras, but had many applications for it and came up with a pictorial demonstration for the (3, 4, 5) case which can be generalized.

Returning to the Greeks

Having traversed the globe, let us return to the Greek contribution. There is no doubt that the Greeks were the first to bring in the notion of proof in mathematics. The Pythagorean School seems to have definitely had a proof for PT at least for rational sides. Stephen Hawking argues in [HS] that they had perhaps an incomplete proof for the general theorem, because Euclid took great pains to give a new and complete proof in the Elements. The Pythagorean School were also the first to prove rigorously that the square root of 2 is irrational. While earlier civilizations did come up with approximations for $\sqrt{2}$ there is no clear evidence that they were aware of irrationals.

Before we conclude the Greek contribution we should mention Euclid's role (approx 300 BC). As you perhaps know, he is the author of *Elements*, a collection of 13 books containing 465 propositions from plane geometry, number theory and solid geometry. He was the first person to create an axiomatic framework for mathematics with rigorous proofs. Once again we know very little of Euclid, except that he worked in the great library of Alexandria during the rule of Ptolemy I (323 – 283 BC). Euclid gave two rigorous proofs of PT: one is the 47^{th} proposition of Book I and the other is the 31^{st} proposition of Book VI.

Proposition VI.31 is a generalization of PT, for while Proposition I. 47 refers to squares constructed on the three sides of a right-angled triangle, Proposition VI.31 refers to any figure constructed similarly on the sides of a right-triangle. For example, if semicircles are constructed on the sides of a right-triangle, then the area of the semicircle on the diagonal is equal to the sum of the areas of the semicircles on the other two sides. He was also the first to give a rigorous proof of the converse of PT (proposition 48 of Book I).

Here is a lovely compliment (sourced from [HS]) a fellow Greek, Proclus, pays to Euclid several centuries later: *"If we listen to those who wish to recount ancient history, we may find some of them referring this theorem (PT) to Pythagoras and saying that he sacrificed an ox in honour of his discovery. But for my part, while I admire those who first observed the truth of this theorem, I marvel more at the writer of the Elements, not only because he made it fast by a most lucid demonstration, but because he compelled assent to the still more general theorem by the irrefragable arguments of science in the sixth Book. For in that Book he proves generally that, in right-angled triangles, the figure on the side subtending the right angle is equal to the similar and similarly situated figures described on the sides about the right angle."*

What motivated the discovery of PT?

It is a matter of great curiosity as to how human beings all over the world discovered a result such as the PT. There are two main threads of speculation.

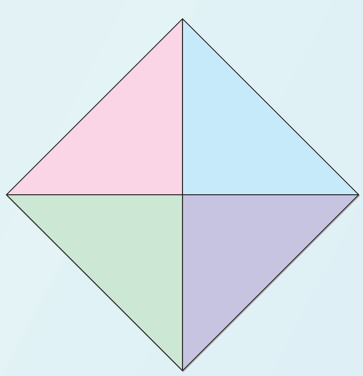


Fig. 5

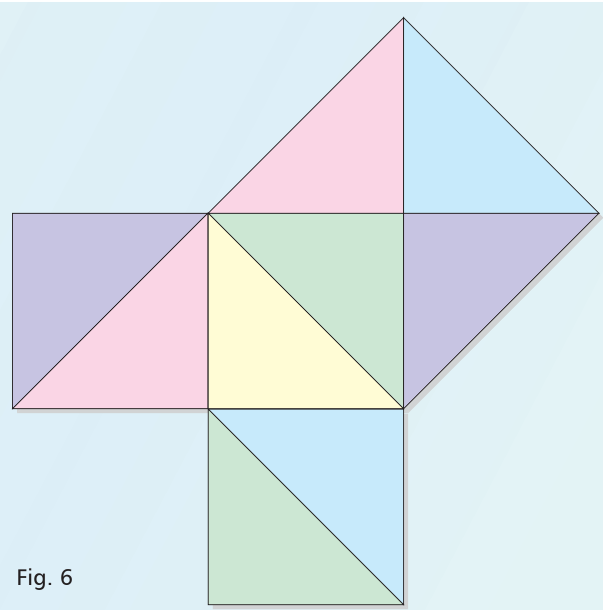


Fig. 6

The first thread looks at the special isosceles right triangle. Historians believed that tiles as shown in Figure 5 would have been the inspiration for the mathematically curious.

If that figure is not self evident to you, what about Figure 6?

The second thread looks at the triple (3, 4, 5). There is evidence that the Egyptians knew the relation $3^2 + 4^2 = 5^2$. However, there is no evidence that they knew that a triangle with side lengths 3, 4 and 5 units was right-angled. Stephen Hawking and others speculate that the next jump in ideas took place when there was a realisation, much along the lines of the Chinese, that a right triangle with legs of length 3 and 4 has hypotenuse of length 5, essentially proving PT for this special case.

Let us see how this is done. Start with a right-angled triangle of side lengths 3 and 4. You then wrap around 4 such right-triangles to form a 7   7 square (see Figures 7 and 8).

Now look at the inner square that is sitting on the hypotenuse of each of the four triangles.

There are two ways to see that it is 25 square units. One is that the original square is 49 square units and it is made up of two 3   4 rectangles and the inner square. The other is that the inner square is made of two 3   4 rectangles and a unit square. Hence we have shown that a triangle with legs of size 3 and 4 units has a hypotenuse of size 5 and the PT holds for this triangle.

This method can be generalized to other right-angled triangles with sides of integer lengths. For example, take a right-angled triangle whose legs are of length 5 and 12. Then using the method above, one will get a 17   17 square with an inner square of size 13   13. This shows that the hypotenuse of such a triangle is 13 units. And using the fact that $5^2 + 12^2 = 13^2$, we have once again a specific example of the PT. The above figure can also be used to establish PT for any right-angled triangle. Can you use algebra and prove it for yourself? This is essentially how Bhaskara proved PT in the eleventh century AD.

It is not clear if Pythagoras and others used the method I have just asked you to prove. As mentioned earlier, since Euclid gave a completely different proof, historians believe that Pythagoras might have used the method of similar triangles to establish PT. However since they dealt only with rational numbers, this proof would have been incomplete.

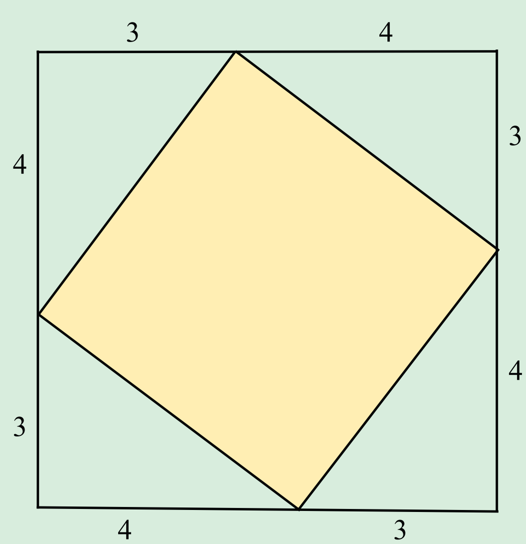


Fig. 7

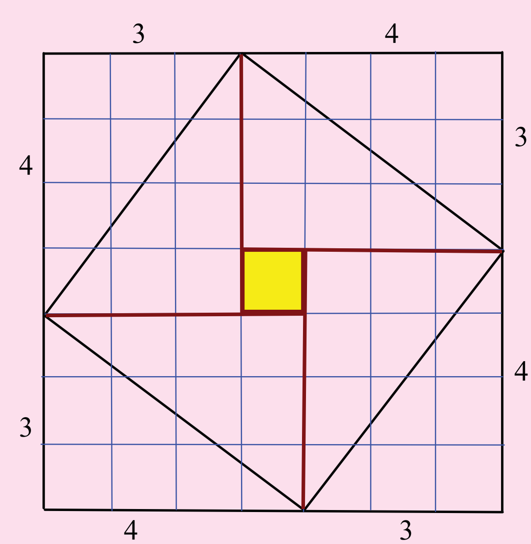


Fig. 8

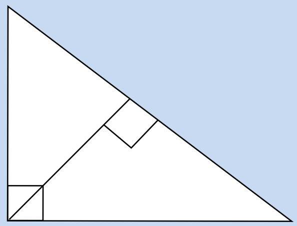


Fig. 9

Are you familiar with the proof of PT using similar triangles? If not, why don't you give it a try? Figure 9 will help you. It is considered the shortest proof of the Pythagoras theorem!

Endnote

I hope in the course of reading this article you have got a sense of the rich history and depth behind the Pythagoras Theorem and how challenging ancient history is. You probably have also realized that it is a quirk of fate that has named the most famous theorem after Pythagoras. It well might have been the '*Mitharti siliptim* (Square of the diagonal) *Theorem'* from Mesopotamia or the '*Sulba Theorem*' from India or the '*Kou ku Theorem'* from China! So, what is in a name?

References

1. [AA] Abdulaziz, Abdulrahman A, *The Plimpton322 Tablet and the Babylonian Method of Generating Pythagorean Triples,* 2010 (source: arXiv:10004.0035v1[math.HO]).
2. [DW] Dunham, William, *Journey Through Genius: The Great Theorems of Mathematics,* John Wiley & Sons, New York, 1990.
3. [EH] Eves, Howard, *An Introduction to the History of Mathematics* (fourth edition), Holt, Rinehart and Winston, New York, 1976.
4. [HS] Hawking, Stephen, *God Created the Integers: the Mathematical Breakthroughs that Changed History,* Running Press, Philadelphia 2005.
5. [JG] Joseph, George Gheverghese, *The Crest of the Peacock: Non-European Roots of Mathematics,* Penguin Books, New York 1991.
6. [ME] Maor, Eli, *The Pythagorean Theorem: a 4,000-year History,* Princeton University Press, Princeton 2007.
7. [PK] Plofker, Kim, *Mathematics in India: From 500 BCE to 1800 CE*, Princeton University Press, Princeton 2009.
8. [RE] Robson, Eleanor, *Words and Pictures: New Light on Plimpton 322*, American Mathematical Monthly, 109, 105-120, 2002.
9. [YS] Yao, Shing-Tung, *The Past, Present and Future of Mathematics in China and India,* (source: homepage.ntu.edu.tw/~d95221004/math_history.pdf)

Sources for Figures

- Figure 1(YBC 789): http://en.wikipedia.org/wiki/YBC_7289#History
- Figure 2 (Plimpton 322): http://en.wikipedia.org/wiki/Plimpton_322
- Figure 4(Chou Pei Suan Ching): http://en.wkipedia.org/wiki/Chou_Pei_Suan_ Ching

SHASHIDHAR JAGADEESHAN received his PhD from Syracuse University in 1994. He has been teaching mathematics for the last 25 years. He is a firm believer that mathematics is a very human endeavour and his interest lies in conveying the beauty of mathematics to students and also demonstrating that it is possible to create learning environments where children enjoy learning mathematics. He is the author of Math Alive!, a resource book for teachers, and has written articles in education journals sharing his interests and insights. He may be contacted at jshashidhar@gmail.com

feature

Proof Without Words

Insight into a Math Mind: Prof. A R Rao

Elegance, they say, cannot be defined, merely demonstrated. Mathematics — and mathematicians — can have incredible style. Read on to find out how.

Shailesh Shirali

It is always a pleasure to read a beautifully crafted proof; and when the proof is of the kind which uses a minimum of words — 'Proofs Without Words' as they are called — the pleasure is doubled. Here is one such proof, from the late Professor A R Rao, about whom we shall say more at the end of this article. It is a solution to the following problem: *Show that the perimeter of a quadrilateral inscribed in a rectangle is not less than twice the diagonal of the rectangle*.

Thus, for the rectangle *ABCD* in Figure 1 and its inscribed quadrilateral *PQRS*, we must show that $PQ + QR + RS + SP \geq 2AC$. Prof Rao's solution makes extensive use of reflections. Figure 2 shows the constructions used.

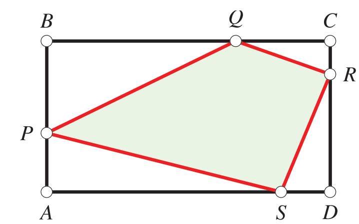


Fig. 1, Quadrilateral in a rectangle

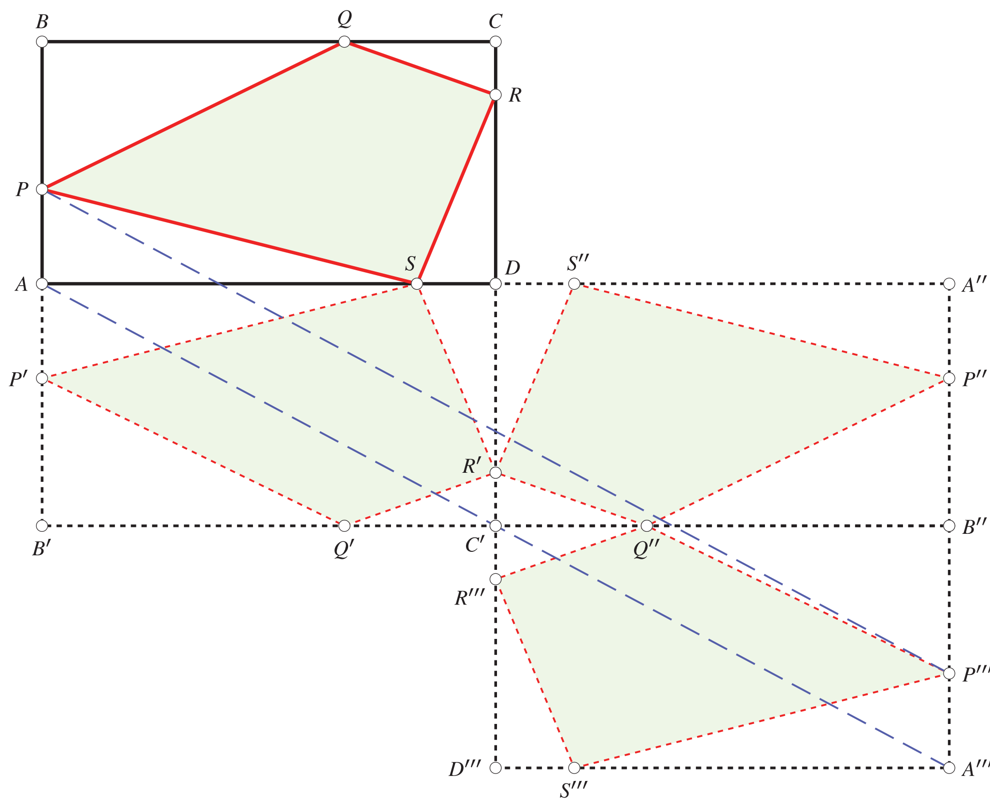


Fig. 2, Prof. A R Rao's proof - Can a diagram say a thousand words?

We first reflect the entire figure in line *AD*; the resulting rectangle is *AB′C′D*, and the resulting quadrilateral is *P′Q′R′S*. Then we reflect the new figure in line *CD*; the resulting rectangle is *A″B″C′D*, and the resulting quadrilateral is *P″Q″R′S″*. One final reflection is needed: we reflect the new figure in line *B″C′*. The resulting rectangle is *A‴B″C′D‴*, and the image of the quadrilateral is *P‴Q″R‴S‴*.

Since the length of a segment is unchanged by a reflection, the perimeter of *PQRS* is the same as the length of the path *P–S–R′–Q″–P‴* (for: *SR* = *SR′*, *RQ* = *R′Q″* and so on). The endpoints of this path are *P* and *P‴*. Since the shortest path joining two points is simply the segment which joins them, we can be sure of the following:

$$PS + SR' + R'Q'' + Q''P''' \geq PP'''.$$

Therefore, the perimeter of *PQRS* is not less than *PP‴*.

Since reflection preserves length, *AP* = *P‴A‴*; and as the two segments are parallel to each other, figure *APP‴A‴* is a parallelogram; thus, *PP‴* = *AA‴*. So, the perimeter of *PQRS* is not less than *AA‴*.

But *AA‴* is simply twice the diagonal of *ABCD*! It follows that the perimeter of *PQRS* is not less than twice the diagonal of *ABCD*. And that's the proof!

Professor A R Rao (1908–2011), Teacher Extraodinaire . . .

Fig. 3, Source of photo: http://www.vascsc.org

Prof A R Rao, 23 September 1908 to 11 April, 2011. This is a picture from his 98th birthday celebrations. Naturally, it is a maths lecture! Study the 4 × 4 magic square shown (Prof Rao seems to be telling us just that!) and work out how it has been formed.

Professor A R Rao was an extraordinary teacher and professor of mathematics whose life spanned an entire century. He taught in various colleges in Gujarat, and then worked as Professor Emeritus for the last three decades of his life at the Vikram A. Sarabhai Community Science Centre in Ahmedabad. He established a Mathematics Laboratory at this Centre, filled with all kinds of puzzles, games, mathematical models and teaching aids. He was amazingly creative with his hands, and just as creative in his ability to come up with beautiful solutions to problems in geometry. He had a deep love for Euclidean and projective geometry, and also for combinatorics and number theory. He was the recipient of numerous awards for popularizing mathematics. He was deeply revered and loved by large numbers of mathematics teachers and students across the country.

For a moving account of his life and career, reference (1) is highly recommended. Reference (2) is from the website of the Vikram Sarabhai centre. Reference (3) is about a remarkable theorem in geometry (the 'Pizza Theorem') for which Professor Rao found a beautiful proof using only concepts from high school geometry.

References

1. C R Pranesachar, Professor A R Rao. Can be downloaded from: http://www.ramanujanmathsociety.org/mnl/BackIssues/mnl-v21-jun11-i1.pdf
2. http://www.vascsc.org/about_arr.html
3. Shailesh Shirali, *A Pizza Saga*. Can be downloaded from: http://www.ias.ac.in/resonance/May2011/p437-445.pdf

SHAILESH SHIRALI is Head of the Community Mathematics Centre which is housed in Rishi Valley School (AP). He has been in the field of mathematics education for three decades, and has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books addressed to high school students, and serves as an editor for the undergraduate science magazine Resonance. He is currently engaged in outreach projects in teacher education in his locality. He is also a keen nature enthusiast and enjoys trekking and looking after animals. He may be contacted at shailesh.shirali@gmail.com.

Shape, Size, Number and Cost

A Sweet Seller's Trick

Analyzing Business through Math

How much math can there be in two bowls of gulab jamun? Prithwijit De models, estimates, calculates and presents a convincing argument on which of the two products earns the sweetseller a greater profit.

feature

PRITHWIJIT DE

Gulab jamuns are a popular sweet in India, often sold singly or two to a bowl. However, I was recently intrigued when the sweet shop across the road from my institute's canteen introduced a bowl of three gulab jamuns at a competitive price. The sweet lovers in my office immediately changed loyalties but the price conscious stayed with the canteen.

What motivated this competitive strategy from the shop across the road? Being a mathematician, I naturally had to solve the problem and in typical fashion, I called my canteen walla 'Mr. X' and the sweet shop owner 'Mr. Y'. Here is my mathematised version of the situation.

Mr. X sells two pieces of *gulab jamun* at ₹p_1 per piece, in a cylindrical cup of cross-sectional radius R. The pieces are spherical in shape and they touch each other externally and the cup internally. Mr. Y, a competitor of Mr. X, sells three pieces of *gulab jamun* in a cup of the same shape and size used by Mr. X, and he charges a price of ₹p_2 per piece. Snapshots of their offerings are displayed in Figure 1, and top-down views are shown in

Fig. 1

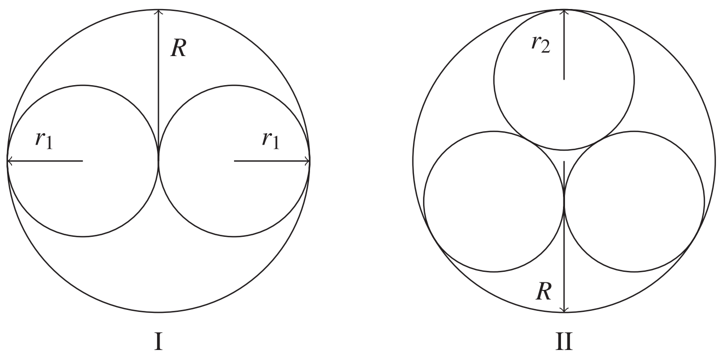


Fig. 2

Figure 2. Certainly p_2 has to be less than p_1 since price is directly proportional to size if the material is kept the same. A person buying from Mr. X would be paying $2p_1$ and a person buying from Mr. Y would be paying $3p_2$.

Listen to some of the conversations I overheard while working on a particularly difficult problem in differential geometry: "Mr. X is charging ₹6 for a gulab jamun! So one bowl with two gulab jamuns from his shop costs only ₹12!" "A bowl at Mr. Y's shop costs ₹15 but I get 3 pieces. So it's only ₹5 per gulab jamun and the bowls are of the same size." "Yes …but the 3 gulab jamuns in the same bowl are smaller! I get 2 bigger gulab jamuns at a cheaper price! " As you can see we have serious and weighty discussions in my office.

Thinking deeply about this, I finally reduced the problem to two main questions.

1. Which of the two cups contains a greater amount of sweet?
2. Which sweetseller makes a greater profit if the cost of producing unit volume of the sweet is the same?

The sweets are sold in a cylindrical cup of cross-sectional radius R. The pieces are spherical in shape and they touch each other externally and the cup internally.

The radius r_1 of each piece in Figure 2 (I) is $\frac{1}{2}R$. The total volume V_1 of sweet in the cup is therefore:

$$2\left(\frac{4\pi}{3}\left(\frac{R}{2}\right)^3\right) = \frac{\pi R^3}{3}.$$

To find the volume of a piece in Figure 2 (II) we must find the radius of each sphere.

In the two dimensional representation of the configuration (Figure 2 (II)) the spheres become circles and the cylindrical cross-section turns into a circle circumscribing the three circles. The task of finding the radius of the sphere thus reduces to finding the common radii of the inner circles. The triangle formed by joining the centres of the inner circles (Figure 3) is an equilateral triangle whose centroid is the centre of the large circle. If r_2 is the radius of an inner circle then the length of the side of the triangle is $2r_2$ and

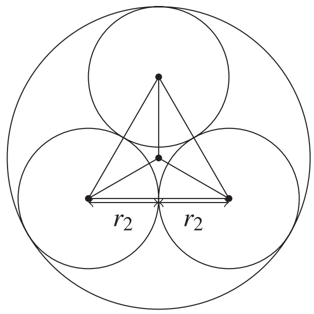


Fig. 3

$$R = r_2 + \left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right)(2r_2) = r_2 + \frac{2r_2}{\sqrt{3}} = \left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)r_2.$$

Thus the volume of a piece is

$$\frac{4\pi}{3}\left(\frac{R\sqrt{3}}{2+\sqrt{3}}\right)^3$$

and the total volume V_2 of sweet offered is

$$4\pi\left(\frac{R\sqrt{3}}{2+\sqrt{3}}\right)^3$$

Now one asks, who is giving more sweet and by how much? Note that

(1) $$\frac{V_1}{V_2} = \frac{(2+\sqrt{3})^3}{36\sqrt{3}} \approx 0.83, \quad \frac{V_2}{V_1} \approx 1.2.$$

So Mr. Y is giving about 20% more sweet than Mr. X. From a buyer's perspective this is a good deal. S/he may be paying more per cup but the per piece price is still less as $p_2 < p_1$.

If Mr. X charges ₹6, then a bowl costs ₹12. If Mr. Y charges ₹5, then a bowl costs ₹15 but it will have 20.48% more sweet than a bowl from Mr. X and the price per piece is still less at Mr. Y's.

But from a seller's perspective is it really worth it? Assuming that the cost of producing unit volume of the sweet is the same, c (say), in both cases, is it possible for Mr. Y to price a piece in such a way so as to ensure greater profit than Mr. X?

The selling price per unit volume is the cost of one bowl divided by the volume of sweet given, so the profit at each shop is

$$\frac{\text{cost of one bowl}}{\text{volume of sweet given}} - \text{cost per unit volume.}$$

To find out which shop makes greater profit, we therefore study the following inequality:

(2) $$\frac{3p_2}{V_2} - c > \frac{2p_1}{V_1} - c.$$

This is equivalent to

$$\frac{p_2}{p_1} > \frac{2}{3}\left(\frac{V_2}{V_1}\right)$$

By virtue of (1) this is equivalent to $p_2 > 0.8p_1$.

Thus, if Mr. Y chooses p_2 such that $0.8p_1 < p_2 < p_1$, then he makes greater profit than Mr. X despite reducing the size and price of the sweet.

For instance, if Mr. X charges ₹6 per piece then Mr. Y can set the price anywhere between ₹4.80 and ₹6 per piece in order to beat his rival in the money-making game (see Table 1). The buyer, in all probability, will be happy to pay less per piece and get three instead of two, as *the more the merrier* is likely to be his/her motto.

p_1	Cost/bowl	p_2	Cost/bowl
6	12	4.80	14.40
		5.00	15.00
		5.50	16.50

Remember that the volume of the bowl is the same in both cases; so the buyer will probably go to Mr. Y in order to get 3 gulab jamuns at a slightly higher price. Mr. Y gets more customers and a greater profit on each bowl!

Acknowledgement

I am grateful to Sneha Titus and Shailesh Shirali for significant improvement in the quality of presentation of the content of this article.

PRITHWIJIT DE is a member of the Mathematical Olympiad Cell at Homi Bhabha Centre for Science Education (HBCSE), TIFR. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. His other interests include puzzles, cricket, reading and music. He may be contacted at de.prithwijit@gmail.com.

feature

Many Ways to QED

The Pythagorean Theorem

Taking note of a collective of contributors

How do I prove thee? Can I count the ways? A look at the wide variety of methods used to prove the theorem of Pythagoras.

C⊗MαC

Not only is the theorem of Pythagoras ('PT' for short) the best known mathematical theorem of any kind, it also has the record of having been proved in a greater number of ways than any other result in mathematics, and by a huge margin: it has been proved in more than three hundred and fifty different ways! (So the relation between the PT and the rest is a bit like the relation between Sachin Tendulkar and the rest ....) There is more: though by name it is inextricably linked to one particular individual (Pythagoras of ancient Greece), as a geometric fact it was *independently* known in many different cultures. (See the article by J Shashidhar, elsewhere in this issue, for more on the history of the PT.) We do not know whether they proved the theorem and if so how they did it, but they certainly knew it was true!

In this article we describe a few proofs of this great and important theorem.

Statement of the PT

Here is how Euclid states it: *In a right triangle, the square on the hypotenuse equals the sum of the squares on the two legs of the triangle*. ('Right triangle' is a short form for 'right-angled triangle'. The 'legs' of a right triangle are the two sides other than the hypotenuse.) Thus, in Figure 1(i) where $\measuredangle A$ is a right angle, we have:

Square *BHIC* = Square *ADEB* + Square *ACFG*.

The way we state the theorem nowadays is: *In a right triangle, the square **of** the hypotenuse equals the sum of the squares **of** the two legs of the triangle*. Note the change: *'on' has been replaced by 'of'*. Therefore: *In Figure 1(ii)*, $a^2 = b^2 + c^2$. This is not merely a change of language. In Euclid's version, it is a statement about areas; in the latter one, it is a statement about lengths. Of course the two versions are equivalent to one another (thanks to the formula for area of a square), and both offer opportunities for generalization; but the second one tells us something about the structure of the space in which we live. Today, this is the preferred version.

Euclid's proof

This has been sketched in Figure 2. The description given alongside gives the necessary steps, and we shall not add anything further here. Note that the reasoning is essentially geometrical, using congruence theorems. *No algebra is used*.

Bhaskara II's proof

The proof given by Bhaskara II, who lived in the 12^{th} century in Ujjain, is essentially the same as the one described in the origami article by V S S Sastry elsewhere in this issue; but the triangles are stacked differently, as shown in Figure 3. Let a right $\triangle ABC$ be given, with $\measuredangle A = 90°$; let sides *BC*, *AC*, *AB* have lengths *a*, *b*, *c*.

The argument given in Figure 3 shows that $(b+c)^2 = 4 \quad (\frac{1}{2}bc) + a^2$, and hence that $a^2 = b^2 + c^2$. Note that all we have done is to 'keep accounts': that is, account for the total area in two different ways.

Bhaskara's proof is a beautiful example of a 'proof without words'. The phrase 'without words' is not be taken too literally; words are certainly used, but kept to a minimum. We shall see many more examples of such visual proofs in future issues of this magazine.

Properly speaking, we must justify certain claims we have made in this proof (and this, typically, is the case for all proofs-without-words); for

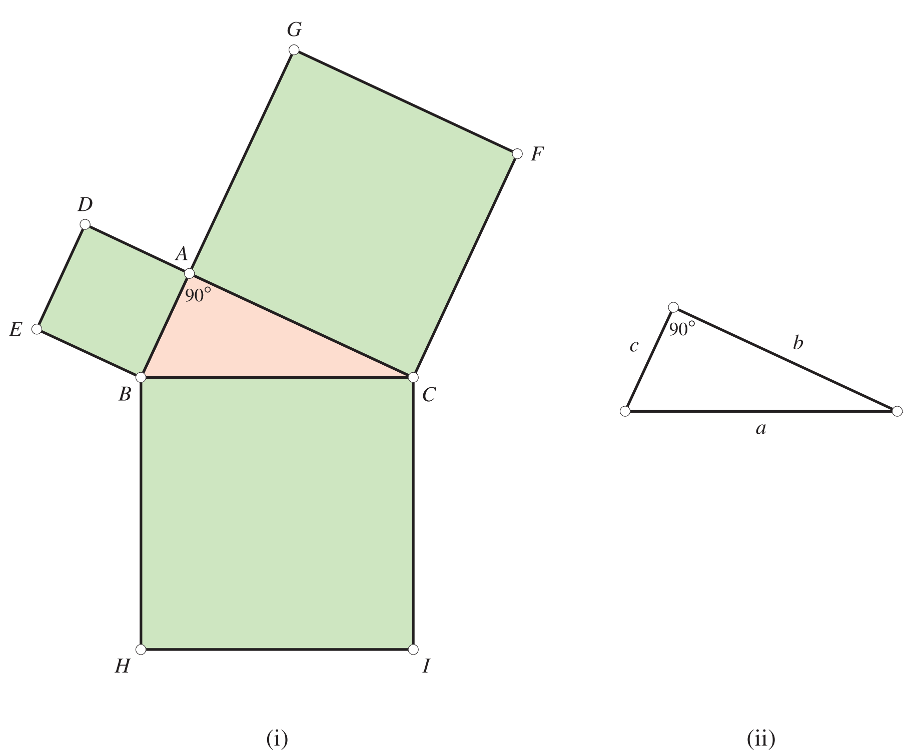
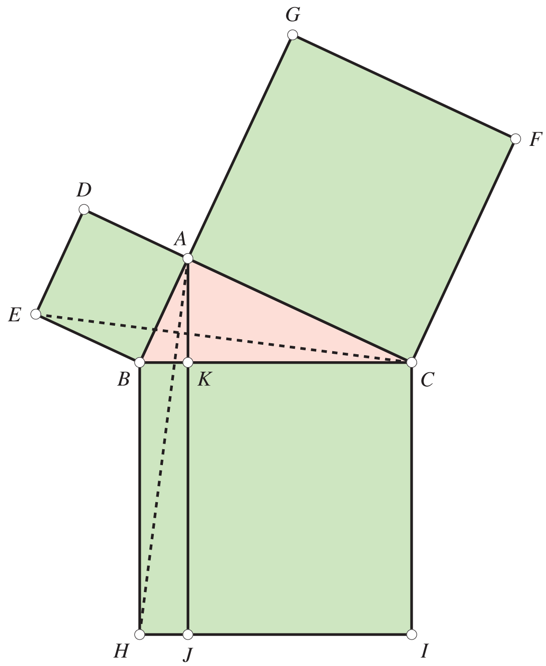
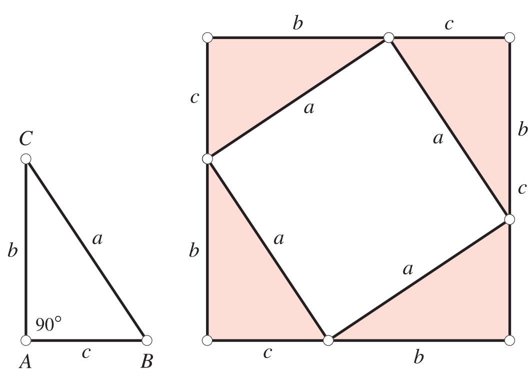


Fig 1:
Two contrasting versions of Pythagoras's theorem.
In (i), Square BHIC =Square ACFG + Square ADEB;
In (ii), $a^2 = b^2 + c^2$



- Draw $AK \perp BC$ and extend AK to meet HI at J.
- **Claim.** Square $ADEB$ and rectangle $BHJK$ have equal area. (Proof: Given below.)
- Draw segments AH and EC.
- $\triangle ABH \cong \triangle EBC$ ('SAS' congruence)
- Area of $\triangle ABH$ is half the area of rectangle $BHJK$, and area of $\triangle EBC$ is half the area of square $ADEB$.
- Hence square $ADEB$ and rectangle $BHJK$ have equal area.
- Similarly, square $ACFG$ and rectangle $KJIC$ have equal area.
- By addition, the PT follows.

Fig. 2 Euclid's proof, which uses standard geometrical results on congruence



Take 4 identical copies of $\triangle ABC$ and arrange them as shown. They make up a square with side $b+c$, but with a square 'hole' in the middle, with side a. The area of each right triangle is $\frac{1}{2}bc$, the area of the hole is a^2, and the area of the large square is $(b+c)^2$.

Fig. 3 The proof by Bhaskara II. See reference (1) for details

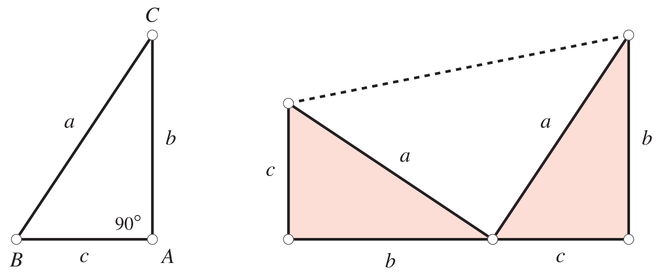
example: (i) why the 'hole' is a square with side a, (ii) why the entire figure is a square with side $b+c$. For this we must show that angles which 'look like right angles' are indeed right angles, and angles which 'look like straight angles' are indeed straight angles. But these justifications are easily given — please do this on your own.

According to legend, Bhaskara did not offer any explanations (we presume therefore that he agreed with the philosophy of a proof without words); he simply drew the diagram and said "Behold! " — assuming no doubt that the reader would be astute enough to work out the details mentally, after gazing for a while at the diagram!

Garfield's proof

The proof given by James Garfield in 1876 is of a similar nature. Garfield was a US Senator at the time he found the proof, and later (1881) became President of the USA. Unfortunately he fell to an assassin's bullet later that same year, and died a slow and painful death.

Garfield's argument is sketched in Figure 4. The trapezium has parallel sides of lengths b and c, and the perpendicular distance between them is $b+c$; its area is therefore $\frac{1}{2}(b+c)^2$. The two right triangles have area $\frac{1}{2}bc$ each. Hence we have: $\frac{1}{2}(b+c)^2 = bc + \frac{1}{2}a^2$. On simplifying this we get $a^2 = b^2 + c^2$.



Take two copies of $\triangle ABC$ and arrange them as shown. On drawing the dashed line, we get a trapezium made up of two right triangles and half a square of side a.

Fig. 4 The proof by Senator Garfield. See reference (2) for details.

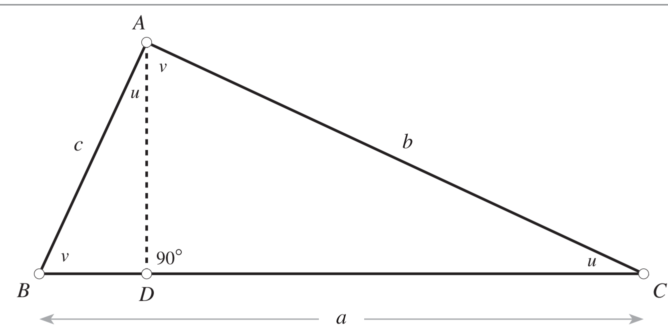


Fig. 5

A proof based on similarity

Next, we have a proof based on similarity of triangles, also given by Euclid in his text ELEMENTS.

Figure 5 depicts the right triangle ABC in which $\measuredangle A = 90°$, with a perpendicular AD drawn from A to the base BC. The two angles marked u are equal, as are the two angles marked v. So we have the similarities $\triangle ABC \sim \triangle DBA \sim \triangle DAC$, and we deduce that

$$\frac{BD}{BA} = \frac{AB}{CB} = \frac{c}{a}, \qquad \frac{DC}{AC} = \frac{AC}{BC} = \frac{b}{a}.$$

These imply that

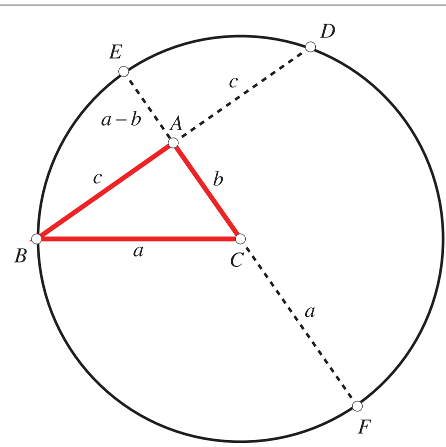
$$BD = c \times \frac{c}{a} = \frac{c^2}{a}, \quad DC = b \times \frac{b}{a} = \frac{b^2}{a}.$$

Since $BD + DC = a$, we get

$$\frac{c^2}{a} + \frac{b^2}{a} = a,$$

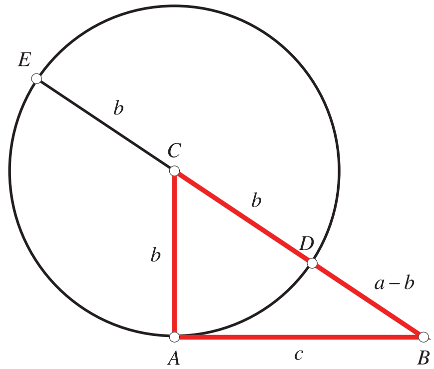
so $a^2 = b^2 + c^2$.

Observe that this proof yields some additional relations of interest; for example, $AD^2 = BD \quad DC$, and $BD : DC = AB^2 : AC^2$.



- Given: $\triangle ABC$, with $\measuredangle A = 90°$
- Circle is drawn with centre C, radius a; it passes through B
- BA is extended to D, and segment AC to E and to F
- Then $CF = a$, $AE = a - b$
- Next, $AD = c$ (for $CA \perp BD$, so A must be the midpoint of chord BD)
- Now apply chord theorem to chords BD, EF: we get $c^2 = (a+b)(a-b)$, and hence $a^2 = b^2 + c^2$

Fig. 6 Proof #63 from the compilation by E S Loomis (reference 3)



- Given $\triangle ABC$, with $\measuredangle A = 90°$
- Draw circle with centre C, passing through A. The radius of the circle is b.
- Let the circle cut ray BC at points D, E.
- $BA = c, \quad BC = a$
- $BD = a - b, \quad BE = a + b$
- $BA^2 = BD \times BE$ by chord theorem
- Hence $c^2 = (a+b)(a-b)$. Expanding we get $a^2 = b^2 + c^2$

Fig. 7 Another proof based on the intersecting chords theorem

A proof based on the intersecting chords theorem

Next, we have a lovely proof based on the intersecting chord theorem ("If UV and LM are two chords of a circle, intersecting at a point T, then $UT \quad VT = LT \quad MT$"), which is a well known result in circle geometry (and, importantly, its proof does not depend on the PT). The construction and proof are fully described in Figure 6.

Another proof based on the intersecting chords theorem

A corollary of the intersecting chords theorem is the following: "If from a point P outside a circle, a tangent PT is drawn and also a secant PQR, cutting the circle at Q and R, then $PQ \quad PR = PT^2$." We may use this to get yet another proof. The details have been given in Figure 7.

Are these proofs really different from one another?

Yes, indeed! Euclid's proof is about the geometric notion of area; it uses standard theorems of congruence, and does not require any algebraic ideas whatever. Bhaskara's proof too uses the notion of area, but requires: (i) the fact that the area of a rectangle with sides x and y is xy (and therefore that the area of a right triangle with legs x and y is $\frac{1}{2}xy$, and the area of a square of side s is s^2); (ii) the formula for the expansion of $(b+c)^2$. Likewise for Garfield's proof. Finally, the proof by similarity and the two proofs based on the intersecting chords theorem have nothing to do with area at all! — they deal with *lengths*, and it is purely by algebraic manipulations that the relation $a^2 = b^2 + c^2$ emerges.

References

1. http://www.robertnowlan.com/pdfs/Bhaskara.pdf
2. http://nrich.maths.org/805 (for the proof by Senator Garfield)
3. Elisha S Loomis, *The Pythagorean Proposition*, NCTM, USA
4. http://www.cut-the-knot.org/pythagoras/index.shtml

Reference (3) contains no less than 371 proofs of the PT, and 96 of these are given in reference (4)!

feature

Fujimoto's Approximation Method

How do you Divide a Strip into Equal Fifths?

How can a simple series of folds on a strip of paper be a mathematical exercise? The article describes not only the how but also the why.....

Shiv Gaur

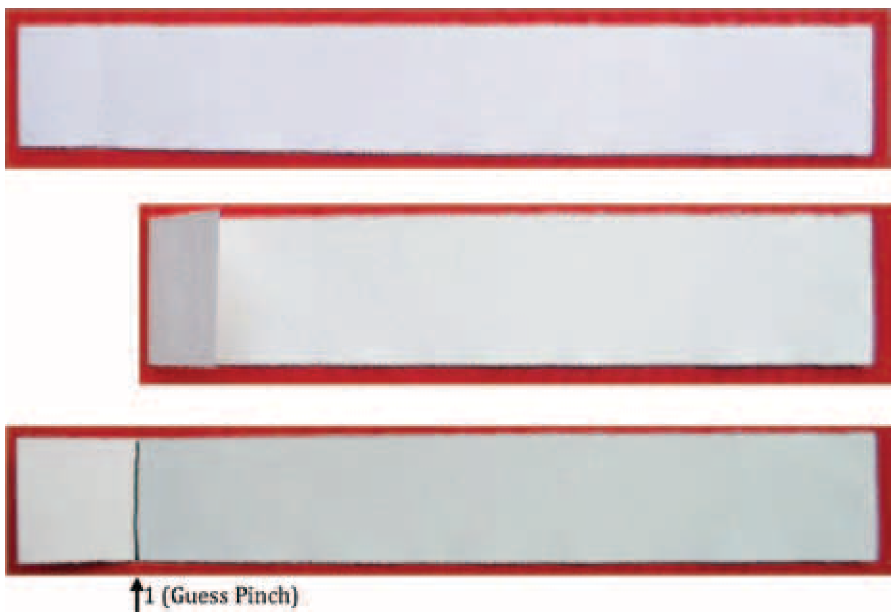
A key principle for learning mathematics is to move from the concrete to the abstract, which implies having "active mathematical experiences" first. Paper folding is one avenue for such experiences. Concepts of perpendicularity, parallelism, similarity, congruence and symmetry are easily experienced through paper folding activities and provide an experiential base for further learning. Paper folding also lends itself readily to explorations, visual proofs and constructions. Angle trisection and doubling of a cube which are not possible with straight edge and compass and the traditional rules of Euclidean constructions are possible using paper folding.

Origami (Japanese-ori: to fold; kami: paper) is the art of paper folding. By a sequence of folds, a flat piece of paper is turned into an animal, flower or a box. In addition to conventional folding (Flat Origami), the art encompasses such genres as Modular Origami (many identical units combined to form decorative polyhedra), and Composite Origami (objects folded from two or more sheets of paper). Origami also provides a highly engaging and motivating environment within which children extend their geometric experiences and powers of spatial visualization.

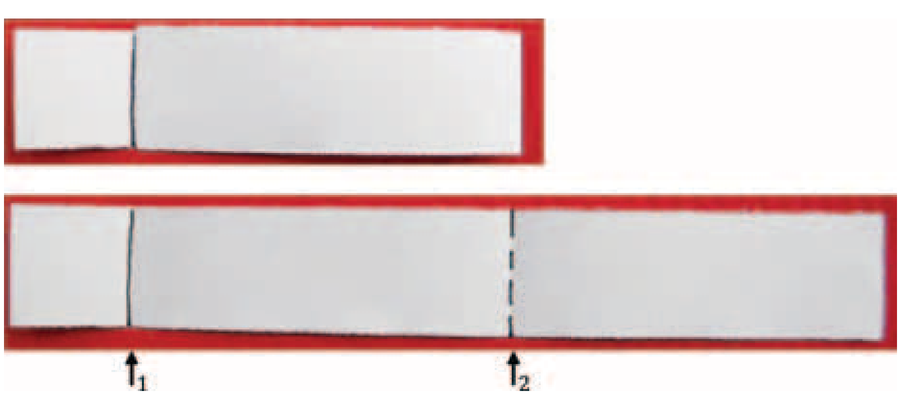
In 1893 T. Sundara Row published his book "Geometric Exercises in Paper Folding" which is considered a classic and still in print. In the development of Origami over the years a number of ideas and techniques have emerged which have mathematical underpinnings, such as Haga's theorems, the Huzita-Hatori Axioms and Fujimoto's Approximation Method to name a few.

In origami it is very common to fold the side of a square piece of paper into an equal number of parts. If the instructions for a particular model ask for it to be folded in half or into quarters or eighths, then it's easy to do so. The difficulty arises if they ask for an equal fifths or any equal 'odd' number of folds. Thankfully there is an elegant and popular method called **Fujimoto's Approximation Method**. Here are the steps for dividing a strip of paper into equal fifths. The photographs are included to show the steps more clearly.

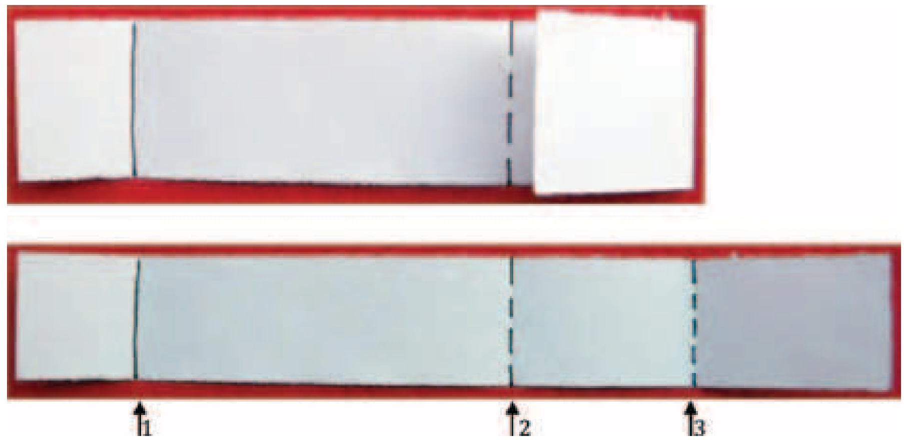
Step 01: Make a **guess pinch** where you may think $\frac{1}{5}$ might be, say on the *left side* of the paper.



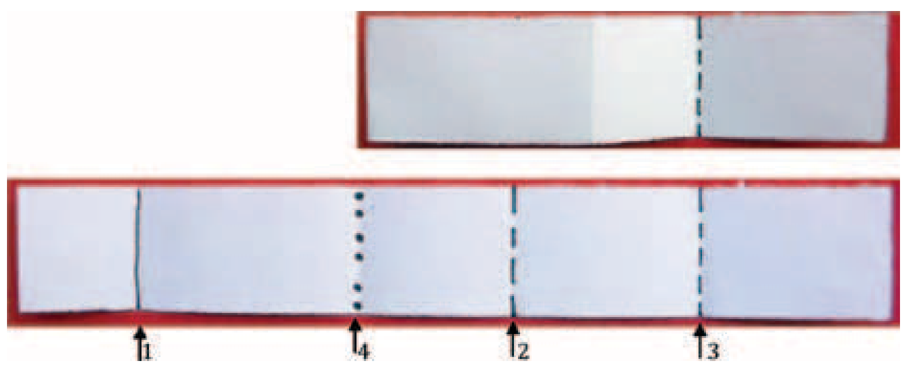
Step 02: To the right side of this guess pinch is approximately $\frac{4}{5}$ of the paper. Pinch this side **in half**.



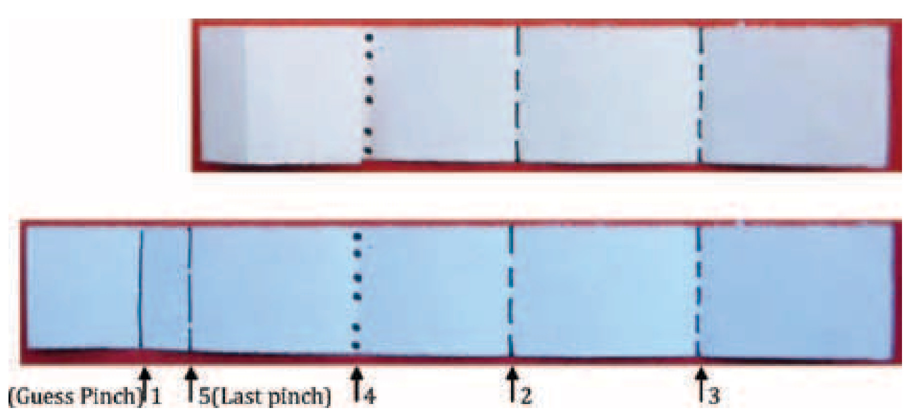
Step 03: The last pinch is near the $\frac{3}{5}$ mark from the left side. To the right side of this is approximately $\frac{2}{5}$ of the paper. Pinch this side **in half**.



Step 04: Now you have a $\frac{1}{5}$ mark on the right. To the left of this is approximately $\frac{4}{5}$. Pinch this side **in half**.



Step 05: This gives a pinch (the dotted line) close to the $\frac{2}{5}$ mark from the left. Pinch the left side of this **in half**. This last pinch will be **very close** to the actual $\frac{1}{5}$ mark!



The set of 3 long dashed lines is the **last pinch mark** very close to the **guess pinch** and a better approximation of $\frac{1}{5}$ than the original guess pinch.

Iteration of the last four steps starting with the **last pinch** as the **new guess mark** helps in finding the fifth mark. The closer the "guess pinch" is to the actual fifth, the fewer the number of iterations.

Why does this work?

Naturally, the question arises: *Why does this work?* Regarding the strip as 1 unit in length, the initial "guess pinch" can be thought of as being at distance

$$\frac{1}{5}+e$$

from the left side, where e represents the initial error. (This could be positive or negative, depending on which side we have erred.) Now with each subsequent fold the error gets halved!

For, in steps 2, 3, 4 and 5, we find that the distances of the latest pinch folds from the left side are, respectively (please verify this):

$$\frac{3}{5}-\frac{e}{2},\quad \frac{4}{5}+\frac{e}{4},\quad \frac{2}{5}-\frac{e}{8},\quad \frac{1}{5}+\frac{e}{16}.$$

The sign of the error alternates between plus and minus. The crucial part is that the last error is $\frac{1}{16}$ of the original one! So each round of this procedure brings down the error by a factor of 16.

Observe that we have traversed pinch marks which cover all the multiples of $\frac{1}{5}$.

Another question is, what would be the procedure for folding a paper into n equal parts, where n is a given odd number?

The general idea in the Fujimoto algorithm is to make an approximate $\frac{1}{n}$ pinch, say from the left hand side. The crease line can be viewed as a fraction of the paper, either from the left side or the right side. Since n is odd, just one of the two fractions will have an even numerator. To get the next crease line, we fold in half that part of the strip from that edge of the paper which corresponds to the *even* numerator, to the latest crease line. Eventually we will reach a pinch mark which provides a new, more accurate approximation for $\frac{1}{n}$ of the paper, since the error gets reduced by half each time the paper is folded in half.

Now try on your own to get a similar method for folding into equal thirds.

Fujimoto's method provides an insight into why clumsy origami folders manage to do a fairly good job of models with intricate folds!

Appendix:
Who is Shuzo Fujimoto?

Fujimoto described this approximation method in a book written in Japanese and published in 1982 (S. Fujimoto and M. Nishiwaki, *Sojo Suru Origami Asobi Eno Shotai* ("Invitation to Creative Origami Playing"), Asahi Culture Center, 1982). Read what has been written about Fujimoto at this link: http://www.britishorigami.info/academic/lister/tessel_begin.php.

A B.Ed. and MBA degree holder, SHIV GAUR worked in the corporate sector for 5 years and then took up teaching at the Sahyadri School (KFI). He has been teaching Math for 12 years, and is currently teaching the IGCSE and IB Math curriculum at Pathways World School, Aravali (Gurgaon). He is deeply interested in the use of technology (Dynamic Geometry Software, Computer Algebra System) for teaching Math. His article "Origami and Mathematics" was published in the book "Ideas for the Classroom" in 2007 by East West Books (Madras) Pvt. Ltd. He was an invited guest speaker at IIT Bombay for TIME 2009. Shiv is an amateur magician and a modular origami enthusiast. He may be contacted at shivgaur@gmail.com

Using Fractions, Odd Squares, Difference of Two Squares

How to Generate Pythagorean Triples -1

feature

Exploring different generative methods

Generating Primitive Pythagorean Triples can introduce students to number theoretic properties, enhance logical reasoning and encourage students to find answers to their 'whys'.

SHAILESH SHIRALI

The relation $a^2+b^2=c^2$ is so familiar to us that we often quote it without saying what a, b, c represent! And this, no doubt, is because the Pythagorean theorem is so well known. We know that if a, b, c are the sides of a right angled triangle, with c as the hypotenuse, then $a^2+b^2=c^2$. We also know, conversely, that if a, b, c are positive numbers which satisfy this relation, then one can construct a right angled triangle with legs a, b and hypotenuse c. Because of this association, we call a triple (a, b, c) of positive integers satisfying this relation a ***Pythagorean triple***, PT for short. But such triples have additional properties of interest that have nothing to do with their geometric origins; they have *number theoretic properties*, and we will be studying some of them in this and some follow up articles.

The most well known PT is the triple (3, 4, 5). Since its numbers are coprime — i.e., there is no factor exceeding 1 which divides all three of them — we call it 'primitive', and the triple is called a ***primitive Pythagorean triple*** or PPT for short. In this article we explore some ways of generating PPTs.

Note. Throughout this article, when we say 'number' we mean 'positive integer'. If we have some other meaning in mind, we will state it explicitly.

What is a 'number theoretic' property?

Before proceeding we must state what we mean by a 'number theoretic' property. Below, we list six such properties about numbers. On examining them you should be able to make out what is meant by the phrase 'number theoretic property'. (We have not justified the statements; we urge you to provide the proofs.)

1. *The sum of two consecutive numbers is odd.*
2. *The sum of two consecutive odd numbers is a multiple of 4.*
3. *The sum of three consecutive numbers is a multiple of 3.*
4. *An odd square leaves remainder 1 when divided by 8.*
5. *The square of any number is either divisible by 3, or leaves remainder 1 when divided by 3.*
6. *The sum of the first n odd numbers is equal to n^2.*

In contrast, here are some statements which are true for any kind of quantity, not just for positive integers: *For any two quantities a and b we have*:

$$a^2 - b^2 = (a-b)\cdot(a+b),$$

$$a^3 - b^3 = (a-b)\cdot(a^2+ab+b^2),$$

$$a^2 + b^2 \geq 2ab.$$

These statements are true even if a and b are not integers. But statements (1) – (6) presented earlier have no meaning if the numbers involved are not integers.

Generating PPTs

We have already given(3, 4, 5) as an example of a PPT. How do we generate more such triples? Below we describe four ways of doing so. The first three are presented without justification; we do not show how we got them, but they are fun to know! In the case of the fourth one, we derive it in a logical way.

Method #1: Using Odd Squares

This method is often found by students who like to play with numbers on their own, and it is perhaps the simplest way of generating Pythagorean triples.

Select any odd number $n > 1$, and write n^2 as a sum of two numbers a and b which differ by 1 (here b is the larger of the two numbers); then (n, a, b) is a PT; indeed, it is a PPT.

Examples

- Take $n = 3$; then $n^2 = 9 = 4 + 5$, so $a = 4$, $b = 5$. The triple is (3, 4, 5).
- Take $n = 5$; then $n^2 = 25 = 12 + 13$, so $a = 12$, $b = 13$. The triple is (5, 12, 13).
- Take $n = 7$; then $n^2 = 49 = 24 + 25$, so $a = 24$, $b = 25$. The triple is (7, 24, 25).
- Observe that each triple generated here has the form $(n, a, a+1)$ where $2a + 1 = n^2$.

Exercises.

(1.1) Justify why this procedure yields PTs.
(1.2) Justify why these PTs are PPTs.
(1.3) Find a PPT which cannot be generated by this method.

Method #2: Using Unit Fractions With Odd Denominator

Of all the methods one generally sees, this one is perhaps the strangest!

Let n be any odd number. Compute the sum $\frac{1}{n} + \frac{1}{n+2}$ and write it in the form $\frac{a}{b}$ where a, b are coprime. Then (a, b, b+2) is a PPT.

Here are some PPTs generated using this method.

- Take $n = 1$; then $n + 2 = 3$, and $\frac{1}{1} + \frac{1}{3} = \frac{4}{3}$. We get the PPT (4, 3, 5).
- Take $n = 3$; then $n + 2 = 5$, and $\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$. We get the PPT (8, 15, 17).
- Take $n = 5$; then $n + 2 = 7$, and $\frac{1}{5} + \frac{1}{7} = \frac{12}{35}$. We get the PPT (12, 35, 37).
- Take $n = 7$; then $n + 2 = 9$, and $\frac{1}{7} + \frac{1}{9} = \frac{16}{63}$. We get the PPT (16, 63, 65).

Exercises.

(2.1) Justify why this yields PTs.

(2.2) Explain why these PTs are PPTs.

(2.3) Find a similar method that uses the even positive integers.

(2.4) Find a PPT which cannot be generated by this method.

Method #3: Using Mixed Fractions

In the same way that we used unit fractions we may also use mixed fractions. We write the following sequence of mixed fractions:

$$1\tfrac{1}{3},\quad 2\tfrac{2}{5},\quad 3\tfrac{3}{7},\quad 4\tfrac{4}{9},\quad 5\tfrac{5}{11},\quad 6\tfrac{6}{13},\quad ....$$

The pattern behind the sequence should be clear. Now we write each fraction in the form $\frac{a}{b}$; i.e., we write each one as an 'improper' fraction. We get:

$$\frac{4}{3},\quad \frac{12}{5},\quad \frac{24}{7},\quad \frac{40}{8},\quad \frac{60}{11},\quad \frac{84}{13},\quad ....$$

Examining these fractions, we see quickly that if $\frac{a}{b}$ is a fraction in the sequence, then $(b, a, a+1)$ is a PPT. So we get the following PPTs:

$(3,4,5)$, $(5,12,13)$, $(7,24,25)$, $(9,40,41)$, $(11,60,61)$, $(13,84,85)$,

.... Strangely, we have obtained the same PPTs that we got with the first method.

Exercises.

(3.1) Justify why this yields PTs.

(3.2) Explain why it yields the same PTs that we obtained by Method #1.

(3.3) Explain why these PTs are PPTs.

(3.4) Find a PPT which cannot be generated by this method.

Remark

All these are 'ad hoc' methods; in no case do we give any hint as to how we got the method. In contrast, here is a method which we actually derive. And that is surely so much more satisfactory.

Method #4: Using the Difference of Two Squares Formula

The equation $a^2+b^2=c^2$ looks more friendly when written as $a^2=c^2-b^2$, because on the right side we see a difference of two squares: an old friend! Now if we write the equation in factorized form as

(1) $\quad a^2=(c-b)\cdot(c+b)$

then our chances of success look brighter. Let us solve the equation in this form.

To make progress, let us arbitrarily put $c-b=1$ and explore what happens. The relation implies that b, c are consecutive integers; and from (1) we get $a^2=c+b$. Since a^2 is a sum of two consecutive integers, it is an odd number. So if we take an odd square and express it as a sum of two consecutive integers, it ought to yield a Pythagorean triple. It does — and this is exactly our Method #1!

To put this idea into action, we select a number n and consider the odd square $(2n+1)^2=4n^2+4n+1$. We write it as a sum $b+c$ of two consecutive integers:

$$b=2n^2+2n, \qquad c=2n^2+2n+1.$$

These values correspond to the following identity:

$$(2n+1)^2=(2n^2+2n+1)^2-(2n^2+2n)^2,$$

and they yield the following PT:

$$(2n+1,\ 2n^2+2n,\ 2n^2+2n+1).$$

Here are some PTs generated using this method (you will see that they are all PPTs):

n	1	2	3
PPT	(3, 4, 5)	(5, 12, 13)	(7, 24, 25)
n	4	5	6
PPT	(9, 40, 41)	(11, 60, 61)	(13, 84, 85)

The PPTs generated by this scheme have the feature that the largest two entries are consecutive numbers.

You will naturally want to ask: What was the need to insist that $c-b=1$? None at all! We need not have imposed the condition. Let us examine what happens if we change it to $c-b=2$; this means that b and c differ by 2. Now we get:

$$a^2=2(c+b).$$

From this we see that a^2 is an even number; therefore a is even, and $a=2n$ for some number n, giving $\frac{1}{2}a^2 = 2n^2$. Does this yield a solution? Yes. To put the scheme into action, we select a number n and write $2n^2$ as a sum $b+c$ of two integers differing by 2; we get:

$$b=n^2-1, \quad c=n^2+1, \quad b+c=2n^2.$$

These values correspond to the following identity:

$$(2n)^2=(n^2+1)^2-(n^2-1)^2,$$

and they yield the following PT:

$$(2n, n^2-1, n^2+1).$$

Here are some PTs generated this way (starting with $n=2$ since $n=1$ yields $b=0$):

n	2	3	4
PPT	(4, 3, 5)	(6, 8, 10)	(8, 15, 17)

n	5	6
PPT	(10, 24, 26)	(12, 35, 37)

We see that when n is odd, the method yields PTs whose numbers are all even, so they are not PPTs. But if n is even we do get PPTs.

Observe what we have accomplished: simply by imposing the conditions $c-b=1$ and $c-b=2$, we obtained two distinct families of PTs. It seems reasonable to expect that by changing these conditions to $c-b=3$, $c-b=4$, and so on, we should be able to generate new families of PTs. But we leave the exploration to you. There is much to discover along the way, maybe some which will surprise us, and much to prove ....

Remarks

Methods #1 – #3 yield infinitely many Pythagorean triples, but these constitute only a small subset of the full family of PTs. Method #4 does seem to have the potential to yield the entire family, but we have left the details to you.

In Part II of this article we shall examine how to generate the entire family of PPTs in a systematic and unified manner.

feature

Primitive Pythagorean Triples

How Many Primitive Pythagorean Triples in Arithmetic Progression?

A simple investigation and a convincing proof based on a novel connection between two topics — the Pythagorean Theorem and Sequences — taught in middle and high school.

C⊗MαC

Everyone knows that (3, 4, 5) is a Pythagorean triple ('PT'); for, the numbers satisfy the Pythagorean relation $3^2 + 4^2 = 5^2$. Indeed, it is a Primitive Pythagorean triple ('PPT') since the integers in the triple are coprime. (See the Problem Corner for definitions of unfamiliar terms.)

But this triple has a further property: *its entries are in arithmetic progression* for, 3, 4, 5 forms a three-term AP with common difference 1. Naturally, our curiosity is alerted at this point, and we ask:

Is there any other PPT whose entries are in AP?

Surprisingly, no other such triple exists. Let us show why.

Suppose there does exist a PPT with entries in AP. Let the common difference of the AP be d, and let the PPT be $(a-d,\ a,\ a+d)$; here a and d are positive integers with no common factor exceeding 1. (If a and d had a common factor exceeding 1 then this factor would divide all three of the numbers $a-d,\ a,\ a+d$, and the triple would no longer be primitive.)

By definition the numbers $a-d,\ a,\ a+d$ satisfy the equation

$$(a-d)^2 + a^2 = (a+d)^2.$$

Expanding all the terms we get: $2a^2 - 2ad + d^2 = a^2 + 2ad + d^2$, and hence:

$$a^2 = 4ad.$$

Since $a > 0$ we may safely divide by a on both sides; we get:

$$a = 4d.$$

So d is a divisor of a. Since a and d are supposed to have no common divisor other than 1, it must be that $d = 1$. Hence $a = 4$, and the triple we seek is (3, 4, 5). Therefore:

$(3, 4, 5)$ *is the only PPT whose numbers are in AP.*

amicable numbers

Take the number 220; its proper divisors (i.e., its divisors excluding itself) are:

1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110,

and the sum of these numbers is 284 (please check!). Now we do the same for the number 284. Its proper divisors are: 1, 2, 4, 71, 142, 284,

and the sum of these numbers is 220. How curious! – the sum of the proper divisors of 220 is 284, and the sum of the proper divisors of 284 is 220.

Pairs of positive integers with such a property are called amicable numbers. The Greeks knew of this pair of numbers. They named them 'amicable', saying to themselves that true friendship between people should be like the relationship between a pair of amicable numbers!

Such number pairs are not easy to find, even if one uses a computer. Here is another such pair, found in the ninth century by the Arab mathematician ibn Qurra: {17296, 18416}.

More such pairs of numbers are known now. It has been noticed that in all these pairs, the numbers are either both odd or both even. Whether there exists any pair of amicable numbers with opposite parity is not known.

Question. We pose the following to you: How would you check that 17296 and 18416 form an amicable pair? What is the easiest way to carry out such a check?

We'll reveal the answer in a future issue

feature

Heron's Formula for Area of a Triangle

One Formula - Two Derivations

Cleverly used algebra in an old familiar formula for the area of a triangle in terms of its base and height enables the formula to be restated in terms of the sides of the triangle. An account of the derivation.

C⊗MαC

With what relief a student uses the simple formula 'Area of a triangle = $\frac{1}{2}$base × height'! Though Heron's formula for the area in terms of its three sides has a pleasing symmetry convenient for memorization, it often seems cumbersome in comparison. A look at how this formula is derived will perhaps enable the student to remember and appreciate the formula, not for this reason but for the sheer elegance of the derivation.

The formula is well known: if the sides of the triangle are a, b, c, and its semi-perimeter is $s = \frac{1}{2}(a+b+c)$, then its area Δ is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\,.$$

We present two proofs of the theorem. The first one is a consequence of the theorem of Pythagoras, with lots of algebra thrown in. It is striking to see how heavily the humble 'difference of two squares' factorization formula is used.

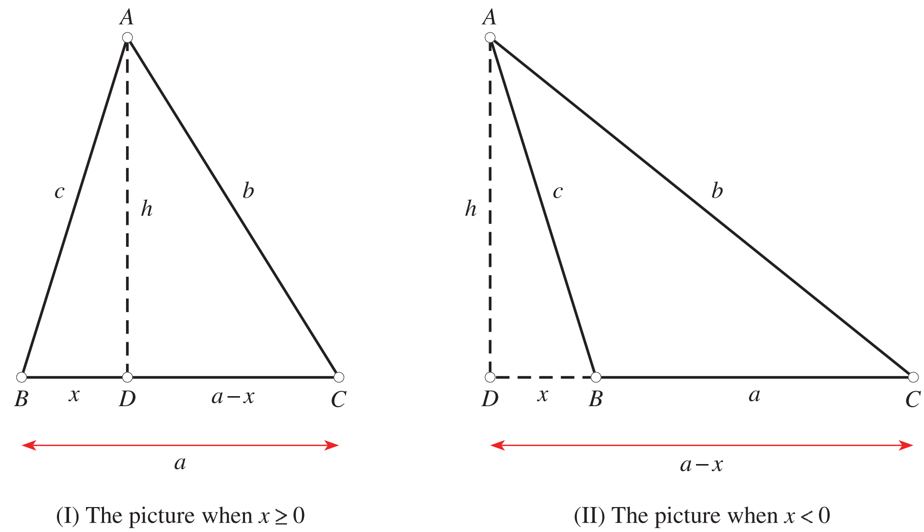


Fig 1: The figure as it looks when $\measuredangle B$ is acute, and when it is obtuse

Proof based on the theorem of Pythagoras

The proof has been described with reference to Figure 1 (I) and Figure 1 (II). Given the sides a, b, c of $\triangle ABC$, let the altitude AD have length h. The area of $\triangle ABC$ is $\frac{1}{2}ah$. To find h in terms of a, b, c we use Pythagoras's theorem. Let $BD = x$, $DC = a - x$. (The notation does not imply that x must lie between 0 and a. Indeed, if $\measuredangle B$ is obtuse then $x < 0$, and if $\measuredangle C$ is obtuse then $x > a$. See Figure 1 (II). Please draw your own figure for the case when $x > a$.) Then:

$$h^2 + x^2 = c^2, \quad h^2 + (a-x)^2 = b^2.$$

Subtract the second equation from the first one:

$$2ax - a^2 = c^2 - b^2, \quad \therefore\ x = \frac{c^2 + a^2 - b^2}{2a}.$$

Since $h^2 + x^2 = c^2$, this yields:

$$h^2 = c^2 - \left(\frac{c^2 + a^2 - b^2}{2a}\right)^2$$

$$= \left(c - \frac{c^2 + a^2 - b^2}{2a}\right) \times \left(c + \frac{c^2 + a^2 - b^2}{2a}\right)$$

$$= \frac{2ac - c^2 - a^2 + b^2}{2a} \times \frac{2ac + c^2 + a^2 - b^2}{2a},$$

$$\therefore\ 4a^2h^2 = (2ac - c^2 - a^2 + b^2) \times (2ac + c^2 + a^2 - b^2).$$

The area of the triangle is $\Delta = \frac{1}{2}ah$, so $16\Delta^2 = 4a^2h^2$, i.e. :

$$16\Delta^2 = (2ac - c^2 - a^2 + b^2) \times (2ac + c^2 + a^2 - b^2).$$

$$= [b^2 - (c-a)^2] \times [(c+a)^2 - b^2]$$

$$= (b - c + a)(b + c - a)(c + a + b)(c + a - b)$$

$$= (2s - 2c)(2s - 2a)(2s)(2s - 2b),$$

$$\therefore \Delta^2 = s(s-a)(s-b)(s-c),$$

therefore $\Delta = \sqrt{(s-a)(s-b)(s-c)}$.

Another proof

We now present an entirely different proof. It is based on a note written by R Nelsen (see reference (1)) and uses two well known results:

- *If* α, β, γ *are three acute angles with a sum of* 90°, *then*

(1) $\tan\alpha \tan\beta + \tan\beta \tan\gamma + \tan\gamma \tan\alpha = 1.$

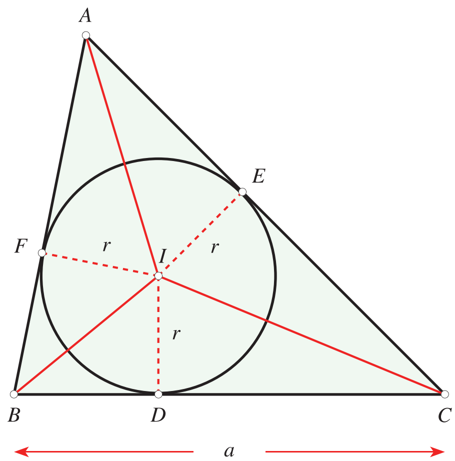
Nelsen gives a 'proof without words' but we simply use the well known addition formula for the tangent function. Since $\alpha + \beta$ and γ have a sum of 90° their tangents are reciprocals of one another:

$$\tan(\alpha + \beta) = \frac{1}{\tan\gamma}$$

But we also have:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$\therefore \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = \frac{1}{\tan\gamma}$$



The three radii of the incircle (shown dashed) are also altitudes of $\triangle IBC$, $\triangle ICA$ and $\triangle IAB$. The bases of these triangles are a, b and c, so their areas are $\frac{1}{2}ar$, $\frac{1}{2}br$ and $\frac{1}{2}cr$.

Hence $\Delta = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$. Factoring this we get: $\Delta = \frac{1}{2}(a+b+c)r = sr$.

Fig 2: Proof of the area formula $\Delta = rs$

Cross-multiplying and transposing terms, we get (1).

- *If s is the semi-perimeter of a triangle, and r is the radius of its incircle, then its area* Δ *is given by* $\Delta = rs$. The proof is given in Figure 2; it is almost a proof without words!

Now we move to Figure 3 which is the same as Figure 2 but with some extra labels. The two lengths marked x are equal ("The tangents from a point outside a circle to the circle have equal length"), as are the two lengths marked y, and the two lengths marked z; and also the two angles marked α, the two angles marked β, and the two angles marked γ.

Consider the angles marked α, β, γ:

$$\alpha = \measuredangle FAI = \measuredangle EAI,$$

$$\beta = \measuredangle DBI = \measuredangle FBI,$$

$$\gamma = \measuredangle DCI = \measuredangle ECI.$$

Since $\alpha + \beta + \gamma = 90°$, by (1) we have:

$$\tan\alpha \tan\beta + \tan\beta \tan\gamma + \tan\gamma \tan\alpha = 1.$$

But from Figure 3,

$$\tan\alpha = \frac{r}{x}, \quad \tan\beta = \frac{r}{y}, \quad \tan\gamma = \frac{r}{z}.$$

Therefore we get, by substitution,

$$(2) \qquad \frac{r^2}{xy} + \frac{r^2}{yz} + \frac{r^2}{zx} = 1, \quad \therefore \frac{r^2(x+y+z)}{xyz} = 1.$$

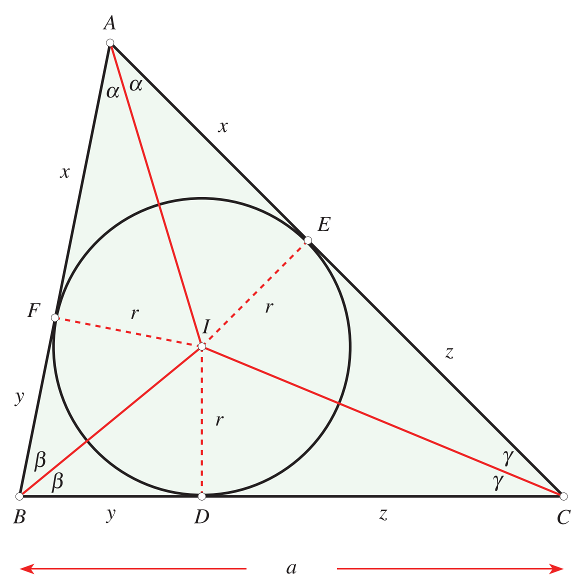


Fig 3

Now $x+y+z = s$ (the semi-perimeter); and since $y+z = a$, we have $x = s-a$. In the same way, $y = s-b$ and $z = s-c$. So result (2) may be rewritten as:

$$\frac{r^2 s}{(s-a)(s-b)(s-c)} = 1,$$

$$\text{i.e.,} \quad \frac{r^2 s^2}{s(s-a)(s-b)(s-c)} = 1.$$

Since $rs = \Delta$ this yields:

$$\Delta^2 = s(s-a)(s-b)(s-c),$$

and we have obtained Heron's formula.

Who was Heron?

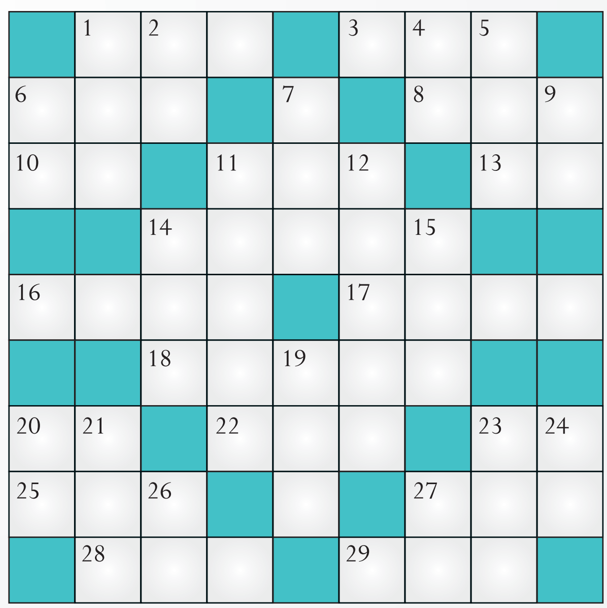
Heron lived in the first century AD, in the Roman town of Alexandria of ancient Egypt. He was a remarkably inventive person, and is credited with inventing (among other things) a wind powered machine and a coin operated vending machine — perhaps the first ever of its kind! For more information please see reference (2).

References

1. Roger B. Nelsen, *Heron's Formula via Proofs without Words*. College Mathematics Journal, September, 2001, pages 290–292
2. http://en.wikipedia.org/wiki/Hero_of_Alexandria.

number crossword-1

by D.D. Karopady



Clues Across

1: 29A minus first two digits of 14A
3: 16A minus 102
6: One less than sum of internal angles of a triangle
8: Three consecutive digits in reverse order
10: Sum of the digits of 14A reversed
11: Even numbers in a sequence
13: A number usually associated with π
14: The product of 29A and 25A
16: Sum of internal angles of an Octagon
17: The product of 6D and 28A
18: Palindrome with 4,3
20: The product of 27D and 8
22: 2 to the power of the sum of the digits of 6D
23: 5 more than 13A
25: Consecutive digits
27: 6D reduced by 1 and then multiplied by 10
28: Sum of two angles of an equilateral triangle
29: A perfect cube

Clues Down

1: 20A reduced by 2 and then multiplied by 2
2: The largest two digit number
4: Four times the first two digits of 1A
5: 20D times 10 plus 9D
6: Hypotenuse of right angle triangle with sides 8 and 15
7: 1A minus 25
9: 26D plus 10
11: First two digits are double of last two digits in reverse
12: 11D plus 18A minus 3230
14: The middle digit is the product of the first and the last digit
15: 27A times 5 then add 4
19: Two times 29A plus 26D in reverse
20: Largest two digit perfect square
21: Half of 22A written in reverse
23: 29A plus half a century
24: Difference between last 2 digits & first 2 digits of 16A
26: Four squared times two
27: A prime number

Probability taught visually

The Birthday Paradox

Simulating using MS Excel

Jonaki Ghosh

One of the fundamental concepts in statistics is that of probability. It forms an integral part of most mathematics curricula at high school level. The topic of probability can be enlivened using many interesting problems. The related experiments are, however, time consuming and impractical to conduct in the classroom. Simulation can be an effective tool for modeling such experiments. It enables students to use random number generators to generate and explore data meaningfully and, as a result, grasp important probability concepts. This section discusses a well known problem known as the Birthday Problem or the Birthday Paradox which highlights an interesting paradox in probability and lends itself to investigation. Its exploration using a spreadsheet such as MS Excel can lead to an engaging classroom activity. It highlights the fact that spreadsheets can enable students to visualize, explore and discover important concepts without necessarily getting into the rigor of mathematical derivations.

Exploring the Birthday Paradox

The birthday problem, more popularly referred to as the birthday paradox, asks the following:

How many people do you need in a group to ensure that there are at least two people who share the same birthday (birth date and month)?

The immediate response from most students is 366 (which is true). However we find some surprises here. It can be shown that in a group size of 50 we can be *almost certain* to find a birthday match (often there is more than one match), and in group sizes of 24, the chance of finding a match is around half. The argument for this can lead to an interesting classroom discussion where basic concepts of probability play an important role.

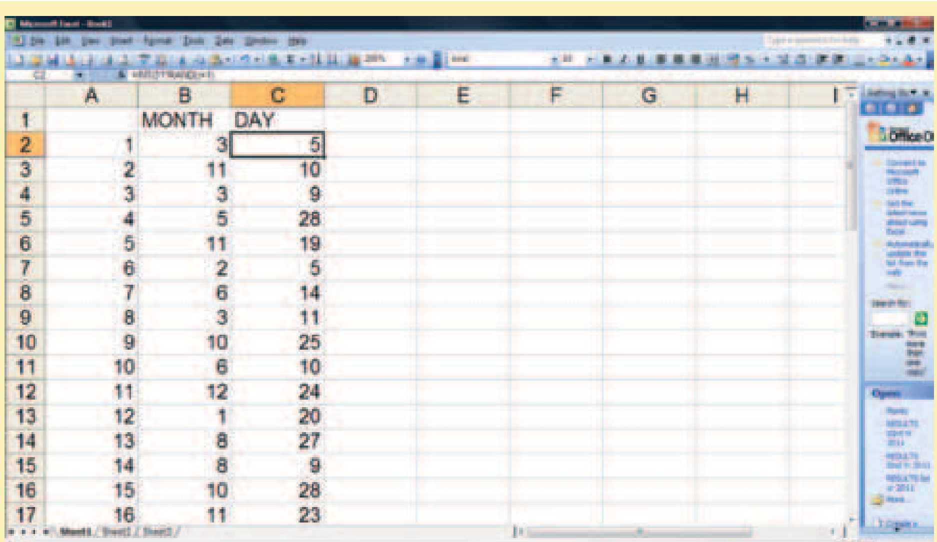
It can be an interesting, although tedious, exercise to actually verify the claim empirically by randomly collecting birthdays, randomly dividing them into groups of 50 and checking if each group has a match. Another way of conducting the experiment is to ask each student to contribute 10 birthdays of persons known to her (relatives or friends), write them on slips of paper, fold them and put them in a box. After shaking the box, each student is asked to select a slip from the box and report the date which is then marked off on a calendar. The box is circulated till a date is repeated and number of dates that were marked before finding the match is noted. After performing this experiment several times the average number of dates required to find a match is calculated. Suppose 10 sets of 24 slips each are created from the contents of the same box, then students can verify that almost invariably 5 of the 10 sets will contain a match while the other 5 will not have a match. This helps to convince them that the probability of a match among 24 randomly selected persons is around half. While the exercise is exciting it can be very time consuming.

Simulation of the Birthday Problem on Excel

Simulating the problem on Excel, on the other hand, makes it far more convenient to do the experiment. 50 birthdays can be randomly generated using the **RAND( )** and **INT( )** functions.

Step 1: The first step is to randomly generate 50 integers between 1 and 12 (inclusive) in column A to indicate the months. This may be achieved in the following manner.

- Click on cell A2 and enter 1. Then enter **= A2 + 1** in cell A3 and drag cell A3 till A51. This will create a column of numbers 1 to 50 as shown in Figure 1.
- To generate 50 integers between 1 and 12 (inclusive) for indicating the months we enter **=INT (12 * RAND( ) + 1)** in cell B2. A double click on the corner of cell B2 will fill the cells B2 to B51 with 50 randomly generated integers between 1 and 12. These represent the months of 50 birthdays (as shown in Figure 1).



	A	B	C
1		MONTH	DAY
2	1	3	5
3	2	11	10
4	3	3	9
5	4	5	28
6	5	11	19
7	6	2	5
8	7	6	14
9	8	3	11
10	9	10	25
11	10	6	10
12	11	12	24
13	12	1	20
14	13	8	27
15	14	8	9
16	15	10	28
17	16	11	23

Figure 1: Simulation of birthdays in Excel

Step 2: The next step is to generate 50 random integers between 1 and 31 (inclusive) to indicate the day of the month. This is obtained as follows

1. Enter **=INT(31*RAND( )+1)** in cell C2 and double click on the corner of cell C2.
2. 50 randomly generated integers between 1 and 31 will appear in column C. These will represent days, corresponding to the months contained in column B.

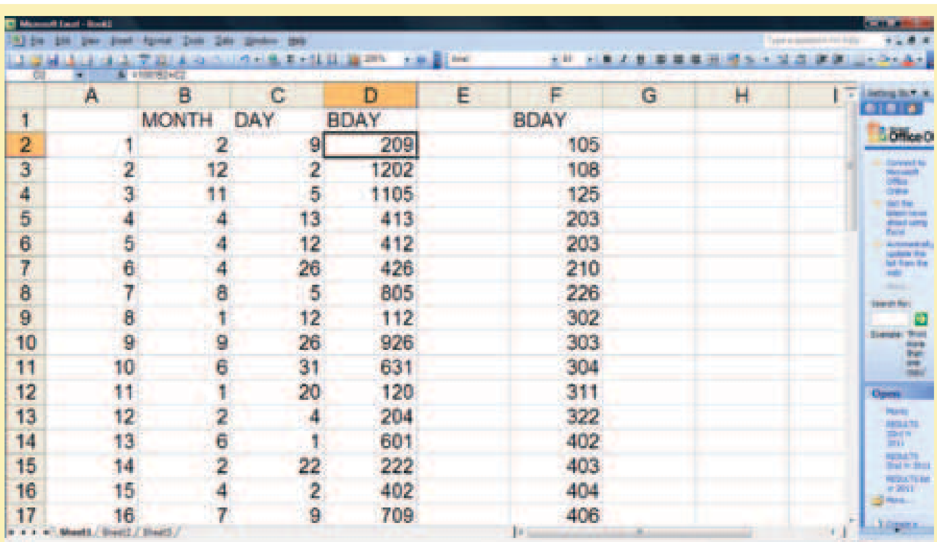
Step 3: The data in columns B and C represent 50 randomly generated birthdays. For example a 3 in cell B1 and 24 in cell C1 represent the date 24th March. We now need to browse through this list and search for a repeated date. This can be time consuming and inconvenient. In order to simplify this part of the process the dates may be converted to three or four digit numbers by entering the formula =100*B1+C1 in column D. Once this is done the first one or two digits of each number in column D will represent the month while the last two digits will represent the day. For example, the appearance of 225 in the list indicates 25th of February while 1019 indicates 19th of October. The list of numbers can then be arranged in an ascending order using the sorting feature of the spreadsheet. This will ensure that a repeated date will appear as two successive values and thus be easily identified.

To convert the dates in columns B and C to three or four digit numbers we enter **=100*B2+C2** in cell D3. Once again a double click on the corner of the cell D3 will reveal the 50 birthdays in cells D2 till D51 (see Figure 2).

Step 4: The list of dates appearing in column D needs to be sorted so that a birthday match can appear as two successive numbers and therefore be easily identified. To do this we select column D, go to **Edit**, select **Copy**, click on a column away from the data (for example choose a column from column F onwards), go to **Edit**, select **Paste Special** and click on **values** and then click on **OK**. This will ensure that all the numbers of column D will now be copied in the same sequence in the new column. Now click on the new column and select the sort (in ascending order) option from the toolbar. Once the dates are sorted a match can be easily identified as shown in Figure 2.

The experiment may be run about 10 times to confirm that in each simulation of 50 birthdays (representing the birthdays of 50 randomly selected people) there is at least one match. The pitfall of this simulation process is that impossible dates (such as 431, that is, 31st April etc) may appear in a particular list. In such a case the entire list can be ignored and the simulation may be repeated.

It might be useful, however, to follow the simulation exercise with an analysis of the problem using probability theory. Begin the discussion by finding the probability of a match in group sizes of three,



	A	B	C	D	E	F
1		MONTH	DAY	BDAY		BDAY
2	1	2	9	209		105
3	2	12	2	1202		108
4	3	11	5	1105		125
5	4	4	13	413		203
6	5	4	12	412		203
7	6	4	26	426		210
8	7	8	5	805		226
9	8	1	12	112		302
10	9	9	26	926		303
11	10	6	31	631		304
12	11	1	20	120		311
13	12	2	4	204		322
14	13	6	1	601		402
15	14	2	22	222		403
16	15	4	2	402		404
17	16	7	9	709		406

Figure 2: Column D represents 50 randomly generated birthdays. The same list is sorted in column F which represents a match (in this case 203, that is, 3rd February)

four and five. Once a pattern is evident, students can easily generalize it to find the formula for the probability of a match in a group size of n persons.

The probability that in a group of three persons, all three have distinct birthdays is

$$\frac{365}{365}\times\frac{364}{365}\times\frac{363}{365}=\frac{365\times364\times363}{365^3}.$$

Thus the probability that at least two of them share a birthday is

$$1-\frac{365\times364\times363}{365^3}.$$

It needs to be emphasized here that in a group of three people there are three possible cases:

1. All three have distinct birthdays;
2. Two people have the same birthday while the third has a different birthday;
3. All three have the same birthday.

Since the three cases are mutually exclusive and exhaustive, the sum of their probabilities is 1. Thus the probability that at least two people have the same birthday includes cases (ii) and (iii) and can be obtained by subtracting the probability of (i) from 1.

The above expression can be extended to find the probability of at least one birthday match in a group of 4 persons, that is,

$$1-\frac{365\times364\times363\times362}{365^4}.$$

Similarly, in a group of 5 persons the probability of a match is

$$1-\frac{365\times364\times363\times362\times361}{365^5}$$

Extending this it can be shown that the probability of a match in a group of size n is

$$1-\frac{365\times364\times\ldots\times\left(362-(n-1)\right)}{365^n}=$$

$$1-\frac{365!}{(365-n)!\times365^n}$$

The value of the above expression approaches 1, as n approaches 50.

While generalizing the formula, students may need help in relating the last number of the product in the numerator to the group size, n. For example, the last number for $n=3$ is $365-2=363$, for $n=4$ it is $365-3=362$, for $n=5$ it is $365-4=361$; for $n=k$, it is $365-(k-1)$. Once the generalized expression is obtained the knowledge of factorials may be used to write the expression in a concise manner.

Conclusion

The topic of probability has a plethora of interesting problems which can be made accessible to high school students through spreadsheets. The experiments related to these problems may be impractical to conduct manually but simulation can be an effective modeling tool for imitating such experiments. Microsoft Excel proves to be a very handy tool for conducting the explorations and investigations in the classroom. The Birthday problem discussed in this article can be conducted with students of grades 9 and 10 without getting into the mathematical derivations. However in grades 11 and 12 the spreadsheet verification of the problems can be followed by an analysis of the underlying concepts which are rooted in probability theory.

References

1. Microsoft Excel Training & Word 2007 Tutorial, Retrieved from http://www.free-training-tutorial.com/format-cells.html
2. Birthday Problem from Wikipedia, the Free Encyclopedia, Retrieved from http://en.wikipedia.org/wiki/Birthday_problem

JONAKI GHOSH is an Assistant Professor in the Dept. of Elementary Education, Lady Sri Ram College, University of Delhi where she teaches courses related to mathematics education. She obtained her Ph.D in Applied Mathematics from Jamia Milia Islamia University, New Delhi and Masters in Mathematics from Indian Institute of Technology, Kanpur. She has also taught mathematics at the Delhi Public School R K Puram for 13 years, where she was instrumental in setting up the Mathematics Laboratory & Technology Centre. She has started a Foundation through which she regularly conducts professional development programmes for mathematics teachers. Her primary area of research interest is in use of technology in mathematics instruction. She is a member of the Indo Swedish Working Group on Mathematics Education. She regularly participates in national and international conferences. She has published articles in proceedings as well as journals and has also authored books for school students. She may be contacted at jonakibghosh@gmail.com

Student Learning and Progress

How About A Math Portfolio?

Maintaining records of math work

A not-so-common method by which the student and teacher can trace how the student's mathematical understanding develops. The article describes what a math portfolio is and how it can be used, what kind of mathematical skills it fosters, the kind of problems that can go into it and the logistical details of setting up this system.

Sneha Titus

"*What are the main goals of mathematics education in schools? Simply stated, there is one main goal — the mathematisation of the child's thought processes. In the words of David Wheeler, it is 'more useful to know how to mathematise than to know a lot of mathematics'.*" (Goals of Mathematics Education Math, NCF 2005)

A math portfolio is an excellent way to trace the mathematisation of a child's thought processes. It is a collection of memorabilia that traces a student's growing understanding of the subject and of herself as a student of the subject. The mathematics portfolio is one of the ways by which a teacher can encourage children "to learn to enjoy mathematics", "to use abstractions to perceive relationships and structure", to realize that "Mathematics (is) a part of children's life experience which they talk about" and "to engage every child in class" – all worthwhile outcomes mentioned in NCF 2005.

Wikipedia gives the literal meaning of a portfolio as “a case for carrying loose papers” and this definition works very well for a student math portfolio. Look at the skills and attitudes developed by giving a student such a portfolio to maintain - ‘a case’: *organization*; ‘for carrying’: *enthusiasm, attachment to the subject*; ‘loose papers’: *spontaneity, ability to mathematise real life situations*.

Running a math portfolio project is, however, not so much a play on words as thought given to the task assigned to the student.

The math portfolio is intended to be a collection of the student’s work and her reflections on mathematics over a period of time. While some teachers have chosen to use the portfolio to highlight the student’s best work, I feel that the portfolio is better used as a record of progress. The work put in the portfolio need not necessarily be all ‘correct’ or ‘perfect’ – rather, it is more like a work in progress giving insights (to both teacher and student) into the development of the student’s mathematical knowledge and problem solving skills. While it is maintained by the student, the teacher plays a major role in defining the student’s understanding - both of the subject as well as of the level of engagement with it. Regular appraisal and review is therefore, an important part of the project.

I have found that the math portfolio is a wonderful way to acquaint myself with students at the beginning of a course and so the first task I usually set is an essay which encourages students to share their experiences in mathematics and their attitude to it. One such essay was based on the 2001 movie ‘A Beautiful Mind’.

Arrange the logistical details

- Where will the portfolios be stored?
- Can the students take the work home?
- Are there any restrictions on quality of printing/publishing?
- Are e-portfolios an option?

I used the quote ‘There has to be a mathematical explanation for how bad that tie is’ and asked students “Do you agree with John Nash’s underlying sentiment that mathematics is all pervasive? Describe some of your encounters with mathematics in unexpected situations.” When a homesick international student spoke of being in a minority and how his emotions were related to numbers, I knew that the math portfolio had done what no counselor could do – get an adolescent boy to speak about his feelings!

Nothing encourages a student like success; therefore, in the initial stages, it is better to start with problems that the students find interesting and easy. Consequently, it is important to ensure that the problems are open to interpretation and exercise several skills and competencies. For example, a well-known problem states: “A person starts from home at 6.00 a.m. and reaches her destination after a journey in which she made several stops and travelled at varying speeds. If she stays at her destination overnight and starts the return along the very same route at the same time (i.e. 6.00 a.m.) the next morning, would she at any point on the return journey be at exactly the same spot at the same time as the previous day?”

I have posed this question to several groups of students and have always been struck by the diverse ways in which students attempt to understand and solve this problem. One student – who incidentally, has a learning disability – stunned me by restating the problem so as to have two people starting at the same time from both ends of the journey; the question, she said, thereby became: will they meet at any point? The more visual thinkers tried to solve the problem using a graph. And those of my students with a more theoretical bent of mind attempted a formal mathematical solution using continuous functions. I found that, in attempting this question, students are engaging with a mathematical question that exercises several skills, including that of making a logical mathematical argument. It can be attempted by students who process data in different ways and it is not so difficult that students get discouraged by its complexity. It can be used by the teacher to set the stage for teaching a variety of topics from plotting graphs to using numerical methods to solve equations. More importantly, using this problem as a portfolio problem gives students time to mull over the problem and the freedom to interpret and solve it by the method that makes most sense to them. Instead of mining a rich diversity of thoughtful solutions from students who are set this as a portfolio problem, such problems are often set as classroom exercises where they suffer from the 'fastest hand first' method of solution.

Cryptarithms (see Fun Problems in this issue) are perfectly suited to be portfolio problems, being appropriate for students who know basic multiplication, some divisibility rules and who are interested in exercising systematic logical reasoning. The teacher may even provide some scaffolding to the student in order to begin the solution. This may seem like giving away the solution, but as the explorations in a portfolio can be completely individualized, this should present no difficulty; the teacher can decide, based on her own understanding of the children, the degree of scaffolding which is appropriate for each individual child. Providing such support will make it a more meaningful exercise for students who simply say that they are 'bad at doing this kind of problem'; it may shed some light on how to start the problem and solve something that they thought they could not.

That being said, the point of the portfolio is that students move from the 'exercise paradigm' to 'landscapes of investigation'. The next problem is from a Japanese source and is reproduced in the book on "The Open-Ended Approach" by Jerry P. Becker & Shigeru Shimada. It was part of a presentation on open-ended questions at the NCTM Annual Meeting.

A transparent flask in the shape of a right rectangular prism is partially filled with water. When the flask is placed on a table and tilted, with one edge of its base being fixed, geometric shapes of various sizes are formed by the cuboid's faces and the surface of the water. The shapes and sizes may vary according to the degree of tilt or inclination. Try to discover as many shapes and sizes as possible and classify these shapes according to their properties. Write down all your findings.

The link http://mste.illinois.edu/users/aki/open_ended/flask_problem.html includes an interactivity which allows students to understand and visualize the problem before attempting the solution. Here, it is important that the teacher clearly states his expectations from the student. So it will be necessary to elaborate on the problem (which only asks for a classification based on the observed properties of the shapes). The teacher will first want the student to develop the skills of observation and abstraction and should therefore ask for diagrams. Then, the teacher will want the student to exercise recall of concepts taught in geometry and he should therefore ask that the diagrams be classified according to shape. After this, there should be a process by which a student can recognize the flow from one shape to another and the teacher could ask for a short paragraph describing the process. Since the dimensions of the flask are given in the link, there can be questions on the sizes of the shapes too.

In the words of Young (1992), the communication pattern in a traditional mathematics classroom is "Guess What the Teacher Thinks". This pattern tends to classify answers as 'right' or 'wrong' in absolute terms and consists of trying to avoid making mistakes and moving towards a predetermined outcome. Unfortunately, this means that the huge learning opportunities afforded by mistakes are not exploited by teachers or students. Neither can the student stumble upon discoveries that, on investigation, provide fresh insight into familiar concepts.

For example, in the activity suggested by Figure 1, the teacher should expect a thoughtful essay backed by calculations on whether the student would go for such a deal on not. For example, whether spending ₹5000 in a single month on clothes would sit well in the family budget (use of a pi chart). Whether 36 clothes bought at an average price of ₹138/- (calculation and concept of average) would ensure quality. Drawing up several deals (can this deal include a most desirable pair of jeans, if so what would the remaining 35 clothes be like, would you end up spending ₹5000 for that pair of jeans plus 35 useless clothes) and so on.

- A possible portfolio problem based on this is:
- Will you be saying "Bye Bye to Big shopping bills" if you go in for this deal?
- Do you agree with the statement in this advertisement?

Present your argument in a short essay, showing all the calculations you would base your reasoning on. Remember that there is no 'right' or 'wrong' decision, your argument should be based on your family's current monthly spend on clothes and on quantity vs. quality

Fig, 1 shows an advertisement in a bus that I travelled in recently.

Skills Tested

1. Understanding of mathematics concepts used.
2. Fluency in the language of mathematics.
3. Ability to
   i. Perform correct calculations.
   ii. Understand and evaluate answers obtained in order to take decisions
   iii. Use charts to communicate
   iv. Place this problem in the larger context of a monthly budget and evaluate its significance.
   v. Draw a logical conclusion based on calculations made.

As students gain a certain level of comfort with the portfolio project, the problems can increase in complexity and cross traditional boundaries between geometry, algebra, arithmetic and so on. It is useful for the teacher to have a collection of problems at different levels. Students could even be told that the portfolio must consist of 'at least one problem' from each level.

Tips on Building Student Portfolios

- Clarify how the portfolio system works. Explain the logistics of storage, submission for assessment and maintenance.
- Issue the portfolio calendar with dates for publishing and submission of problems.
- Explain the system of assessment- with a rubric that gives credit to recording and cognition of learning rather than just solutions to problems.
- Ensure that they understand that this is neither a design competition nor a collection of stories of success but a record of stories of their learning.
- Encourage them to record their process of problem solving (in words, diagrams, calculations), to give reasons for procedures chosen or rejected and to analyse the implications of answers obtained.
- Explain copyright issues and the dangers of plagiarism and the need to cite all sources.
- Set basic standards of neatness and organization - an index, record of dates and so on.

Along with the content, an assessment rubric that has both a problem specific as well as a general component needs to be created by the teacher. The former would assess logical reasoning and accuracy without stressing on a particular method. The general component on the other hand, would record and give credit to innovative problem solving, conjecture and structured investigation as well as clarity of presentation. Of course, students would need some benchmarks to guide them in this direction.

It is important to understand that student specific and not uniform standards of excellence should be the goal here. A component of self-assessment can also elicit student understanding of the lack of a particular skill and the teacher and student can assess whether there is improvement in that skill as more problems are added to the portfolio.

Much has been written about math portfolios but how practical is it for students who are studying the Indian curriculum to maintain portfolios? It may not be feasible for a student to spend more than an hour and a half a week on the portfolio. Since this time also must include components of work, presentation and reflection, it may not be practical to issue more than 4 portfolio problems a term. While many students and even parents may complain about additional work, particularly in the senior classes, the freedom as well as the responsibility of maintaining a portfolio has long term benefits on the student's independent study skills, problem solving skills, interest in the subject and knowledge of mathematics that may not be curtailed by the syllabus.

Set up the System

1. Portfolio calendar- dates/days on which problems are published, submission dates, etc.
2. Are the portfolios going to be shared with other students/parents, etc. If so, when?
3. How is this portfolio going to feed into the overall assessment along with tests, class quizzes, lab activities?
4. Will students be encouraged to create problems? If so, what are the guidelines to be given?
5. Will there be an end-of-portfolio activity- such as a summarizing of learning, a selection of 'good' problems or solutions and so on?

References

1. http://www.ncert.nic.in/rightside/links/pdf/framework/nf2005.pdf
2. Wheeler, David, "Mathematisation Matters," 1982.
3. Polya, George, "The goals of Mathematical Education", in ComMuniCator, the magazine of the California Mathematics Council, 1969.
4. Helle Alro and Ole Skovsmose: Dialogue and Learning in Mathematics Education

Sneha Titus, a teacher of mathematics for the last twenty years has resigned from her full time teaching job in order to pursue her career goal of inculcating in students of all ages, a love of learning the logic and relevance of Mathematics. She works as a retainer at the Azim Premji Foundation and also mentors mathematics teachers from rural and city schools. Sneha uses small teaching modules incorporating current technology, relevant resources from the media as well as games, puzzles and stories which will equip and motivate both teachers and students. She may be contacted on sneha.titus@azimpremjifoundation.org

in the classroom

Visual Connect in Teaching

Paper Folding And The Theorem of Pythagoras

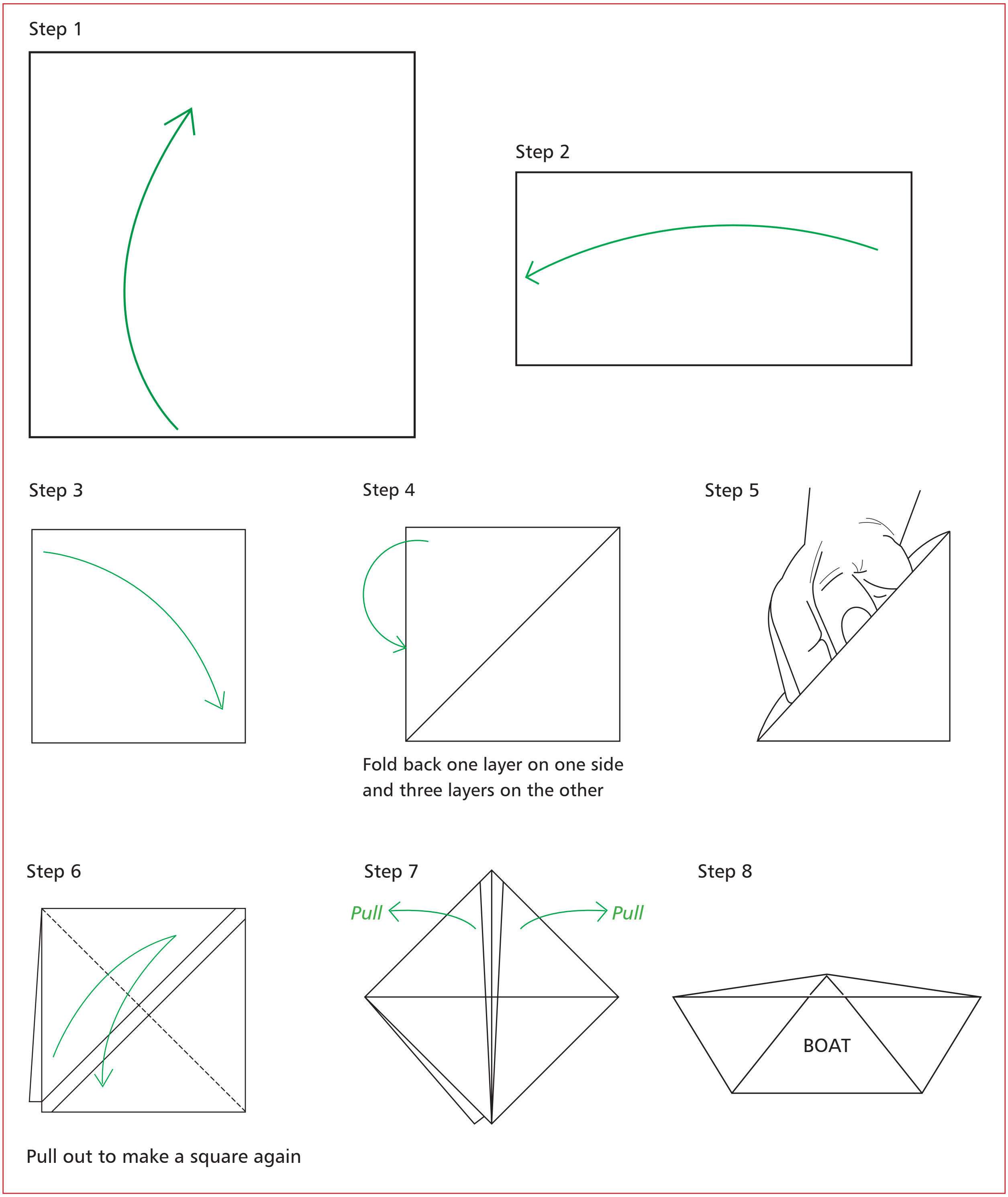
Can unfolding a paper boat reveal a proof of Pythagoras' theorem? Does making a square within a square be anything more than an exercise in geometry at best? Art and math come together in delightful mathematical exercises described in this article.

Sivasankara Sastry

Pythagoras' theorem is one of the most popular theorems in geometry. Reams of paper have been used to write different proofs of this theorem but in this article we cut and fold paper to demonstrate two different proofs.

Make a boat and prove Pythagoras' theorem

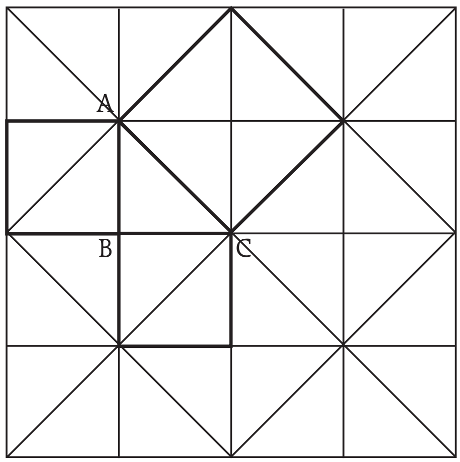
Remember how children float paper boats in running water after heavy rain? There are many types of boats that can be made by folding a single paper sheet. Here, we make the simplest and most common type of paper boat using a square sheet of paper. In case you have forgotten how to fold a boat here are the steps:-



After step 8 you will have a boat. What is its shape? If you look closely you find creases which show many right angled triangles.

Now unfold the boat. Remember we started with a plain square paper. Now look at the creases appearing in the unfolded boat. You will see a pattern which is known mathematically as a *tessellation* ("making tiles"). This particular tessellation consists of squares with creases along the diagonals which divide each square into two right angled triangles. The way we folded the paper ensures that all the squares (and therefore the triangles too) are identical in all respects.

They look like this:



Choose any right angled triangle ABC. Here angle B is a right angle. Shade the squares on sides AC, BC and AC. Look closely at these squares. All have creases along the diagonals and are divided into right-angled triangles. How do we measure their areas? *Area need not only be measured in terms of unit squares. We can also measure the area by counting the number of identical right angled triangles contained in them*.

The square upon AB has 2 right-angled triangles; so AB^2 = 2 right-angled triangles.

The square upon BC has 2 right-angled triangles; so BC^2 = 2 right-angled triangles.

The square upon AC has 4 right angled triangles; so AC^2 = 4 right-angled triangles.

Hence: $AC^2 = AB^2 + BC^2$ This is the theorem of Pythagoras applied to triangle *ABC*.

Make a square within a square and prove Pythagoras' Theorem

Take a square sheet of paper. Fold along a diagonal and make a sharp crease (Fig. 1).

Fold the bottom right corner towards the diagonal, so that the edge of the sheet lies parallel to the diagonal. Make a crease. You will have folded a right angled triangle (Fig. 2).

Fig 1

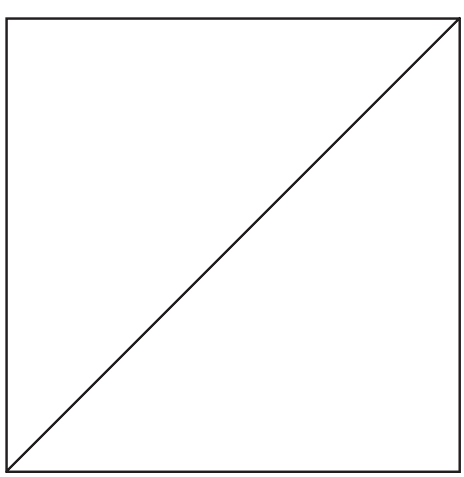
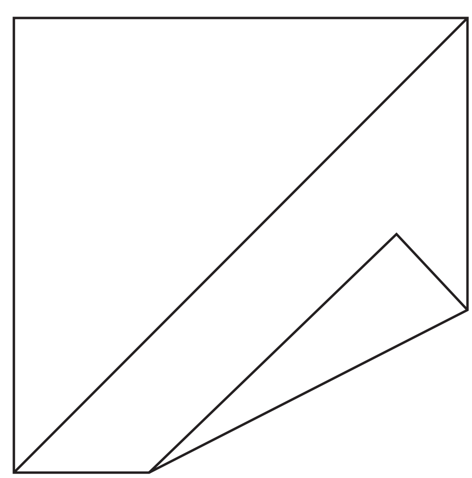


Fig 2



Now fold the next side of the square to the side of the right angle already folded (Fig. 3)and make a crease. Repeat the same with the remaining two corners.(fig. 4 and fig. 5)

Fig 3

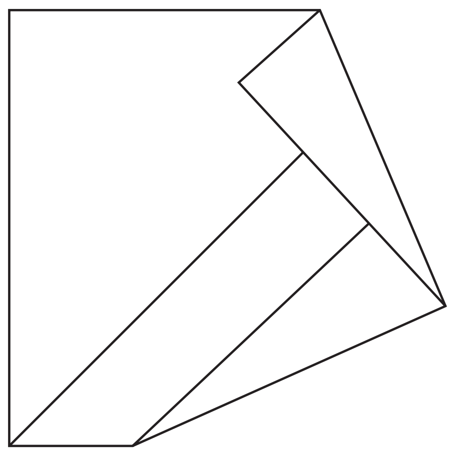
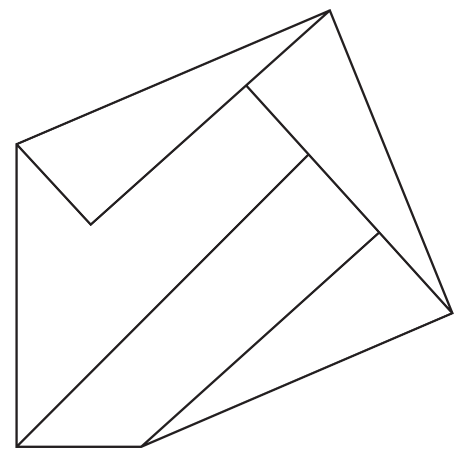


Fig 4



Now you have a square with a square hole in the middle. (Fig. 5). Crease all the sides and unfold(Fig.6). Let $AP = a$, $AQ = b$ and $PQ = c$. In triangle APQ, angle PAQ is a right angle. $PQRS$ is a square with side $PQ = c$. Hence area of $PQRS = c^2$.

Fig 5

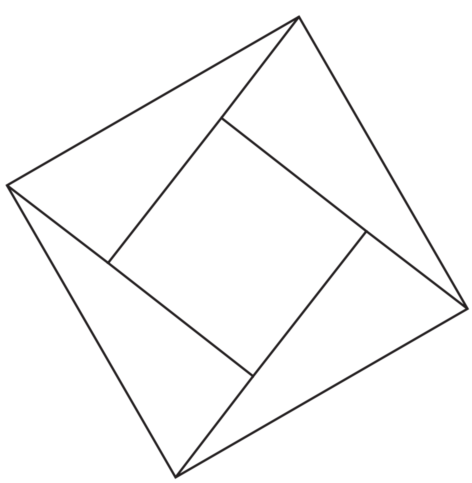
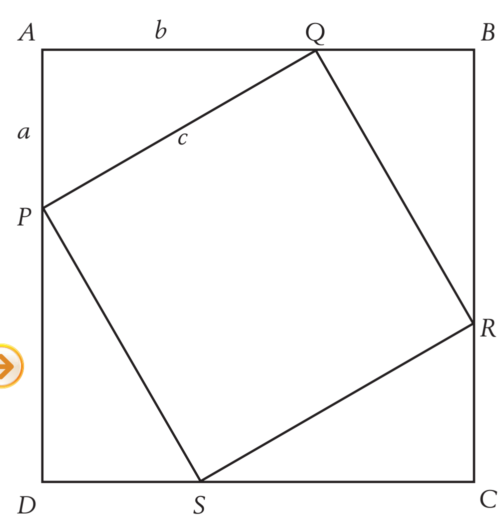
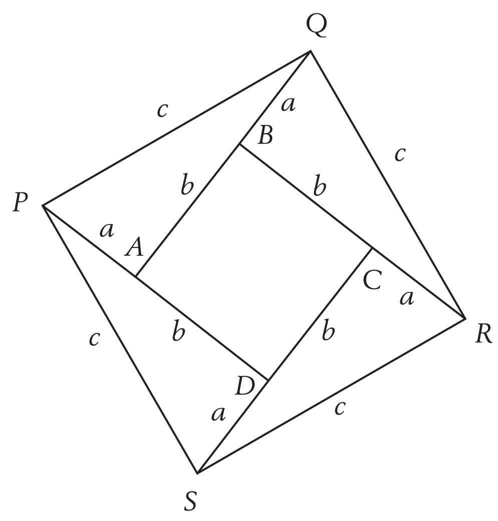


Fig 6





Fold back triangles *PAQ, BQR, RCS, SDP* inside, as before.

Now in the square *PQRS* standing on *PQ* we have identical triangles *PAQ, BQR, RCS, SDP,* and a small square *ABCD*.

Square *PQRS* = Triangle *PAQ* + Triangle *QBR* + Triangle *RCS* + Triangle *SDP* + Square *ABCD*

$= \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab +$ square $ABCD$

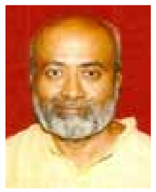
$= 4 \quad \frac{1}{2}ab + AB^2$

$= 2ab + (b - a)^2$.

Hence $c^2 = 2ab + b^2 + a^2 - 2ab$

and so $c^2 = a^2 + b^2$.

Hence: $PQ^2 = AP^2 + AQ^2$. This is the theorem of Pythagoras applied to triangle *APQ*.



SIVASANKARA SASTRY'S interests range from origami, kirigami, paleography and amateur astronomy to clay modelling, sketching and bonsai. He is also a published author having written 27 books in Kannada on science and mathematics. Mr. Sastry may be contacted a vsssastry@gmail.com

Classification of Quadrilaterals

The Four-Gon Family Tree

A Diagonal Connect

Classification is traditionally defined as the precinct of biologists. But classification has great pedagogical implications — based as it is on the properties of the objects being classified. A look at a familiar class of polygons — the quadrilaterals — and how they can be reorganized in a different way.

in the classroom

A Ramachandran

Quadrilaterals have traditionally been classified on the basis of their sides (being equal, perpendicular, parallel, ...) or angles (being equal, supplementary, ...). Here we present a classification based on certain properties of their diagonals. In this approach, certain connections among the various classes become obvious, and some types of quadrilaterals stand out in a new light.

Three parameters have been identified as determining various classes of quadrilaterals:

1. Equality or non-equality of the diagonals
2. Perpendicularity or non-perpendicularity of the diagonals
3. Manner of intersection of the diagonals. Here four situations are possible:
   a. The diagonals bisect each other.
   b. Only one diagonal is bisected by the other.
   c. Neither diagonal is bisected by the other one, but both are divided in the same ratio.
   d. Neither diagonal is bisected by the other one, and they divide each other in different ratios.

	Diagonals equal		Diagonals unequal	
	Perpendicular	Not-perpendicular	Perpendicular	Not-perpendicular
Both diagonals bisected	Square	General rectangle	General rhombus	General parallelo-gram
Diagonals divided in same ratio (not 1:1)	Isosceles trapezium with perp diagonals	Isosceles trapezium	General trapezium with perp diagonals	General trapezium
Only one diagonal bisected	Kite with equal di-agonals	Slant kite with equal diagonals	Kite	Slant kite
Diagonals divided in different ratios, neither bisected	General quadrilateral with equal and perp diagonals	General quadri-lateral with equal diagonals	General quadrilateral with perp diagonals	General quadrilat-eral

Table 1. Quadrilateral classes based on properties of the diagonals

	Diagonals equal		Diagonals unequal	
	Perpendicular	Not-perpendicular	Perpendicular	Not-perpendicular
Both diagonals bisected	Cyclic (4)	Cyclic (2)	Non-cyclic (2)	Non-cyclic (0)
Diagonals divided in same ratio (not 1:1)	Cyclic (1)	Cyclic (1)	Non-cyclic (0)	Non-cyclic (0)
Only one diagonal bisected	Non-cyclic (1)	Non-cyclic (0)	Either (1)	Either (0)
Diagonals divided in different ratios, neither bisected	Non-cyclic (0)	Non-cyclic (0)	Either (1)	Either (0)

Table 2. Cyclic/non-cyclic nature of quadrilateral & the number of reflection symmetry axes (shown in parentheses)

These parameters allow us to identify 16 classes of quadrilaterals as listed in Table 1. The meaning of the phrase ‘slant kite’ though not in common parlance should be clear.

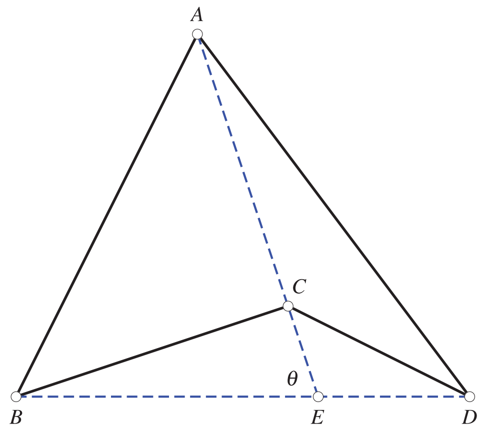
Certain relations among these classes are obvious. Members of columns 2 and 3 are obtained from the corresponding members of column 4 by imposing equality and perpendicularity of diagonals, respectively, while members of column 1 are obtained by imposing both of these conditions.

Sequentially joining the midpoints of the sides of any member of column 1 yields a square, of column 2 yields a general rhombus, column 3 a general rectangle, and column 4 a general parallelogram.

Table 2 gives the cyclic/non-cyclic nature and number of reflection symmetry axes of each type. An interesting pattern is seen in both cases.

Quadrilaterals with equal diagonals divided in the same ratio (including the ratio 1:1) are necessarily cyclic, as the products of the segments formed by mutual intersection would be equal. Quadrilaterals with unequal diagonals divided in the same ratio and those with equal diagonals divided in unequal ratios are necessarily non-cyclic, as the segment products would be unequal. Quadrilaterals with unequal diagonals divided in different ratios could be of either type.

The quadrilateral with the maximum symmetry lies at top left, while the one with least symmetry lies at bottom right. A gradation in symmetry properties is seen between these extremes.



- $[ABCD] = [\triangle ABC] + [\triangle ADC]$
- $[\triangle ABC] = [\triangle ABE] - [\triangle CBE]$
- $[\triangle ADC] = [\triangle ADE] - [\triangle CDE]$
- $[\triangle ABE] = \frac{1}{2} AE \cdot BE \cdot \sin\theta$
- $[\triangle CBE] = \frac{1}{2} CE \cdot BE \cdot \sin\theta$
- $[\triangle ADE] = \frac{1}{2} AE \cdot DE \cdot \sin\theta$
- $[\triangle CDE] = \frac{1}{2} CE \cdot DE \cdot \sin\theta$

Fig. 1 Square brackets denote area: [△ABC] denotes area of △ABC, etc

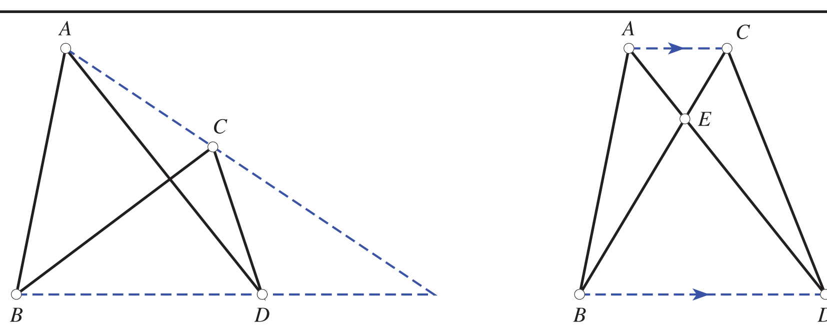
The present scheme also suggests a common formula for the areas of these figures. The area of any member of column 4 can be obtained from the formula $A = \frac{1}{2} d_1 d_2 \sin\Theta$ where d_1, d_2 are the diagonal lengths and Θ the angle between the diagonals. The formula simplifies to $A = \frac{1}{2} d_1 d_2$ for column 3, $A = \frac{1}{2} d^2 \sin\Theta$ for column 2, and $A = \frac{1}{2} d^2$ for column 1.

The general area formula $A = \frac{1}{2} d_1 d_2 \sin\Theta$ is applicable in the case of certain other special classes of quadrilaterals too.

Non-convex or re-entrant quadrilaterals are those for which one of the (non-intersecting) diagonals lies outside the figure. the applicability of the formula to these is demonstrated in Figure 1. The computation shows that

$$\begin{aligned} [ABCD] &= [\triangle ABC] + [\triangle ADC] = [\triangle ABE] - [\triangle CBE] + [\triangle ADE] - [\triangle CDE] \\ &= \frac{1}{2}\sin\Theta\,(AE \cdot BE - CE \cdot BE + AE \cdot DE - CE \cdot DE) \\ &= \frac{1}{2}\sin\Theta\,(AC \cdot BE + AC \cdot DE) = \frac{1}{2}\sin\Theta\,(AC \cdot BD) \end{aligned}$$

Reflex quadrilaterals with self-intersecting perimeters such as the ones shown in Figure 2 can be considered to have both the (non-intersecting) diagonals lying outside the figure. The applicability of the area formula to such figures is demonstrated in Figure 3. Here the area of the figure is the *difference* of the areas of the triangles seen, since in traversing the circuit *ABCDA* we go around the triangles in opposite senses. Hence in Figure 3,



(a) (b)

Fig. 2 Reflex quadrilateral: AC and BD are diagonals lying outside the reflex quadrilateral ABCDA, whose perimeter is self-intersecting

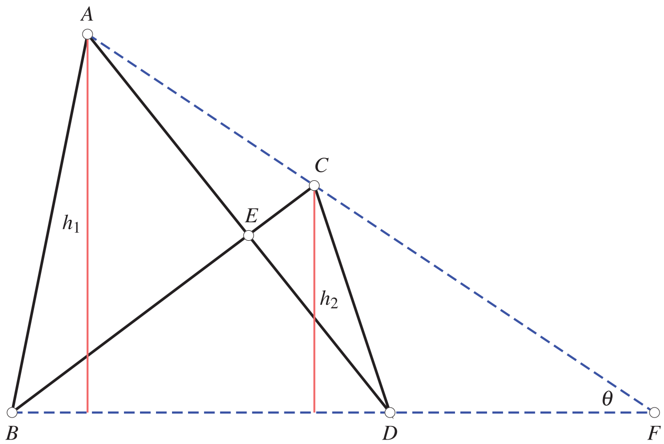


Fig. 3

$$\begin{aligned}[ABCDA] &= [\triangle ABE] - [\triangle CDE] = [\triangle ABD] - [\triangle CBD] \\ &= \tfrac{1}{2} BD \cdot h_1 - \tfrac{1}{2} BD \cdot h_2 = \tfrac{1}{2} BD \cdot (h_1 - h_2) \\ &= \tfrac{1}{2} BD \cdot (AF \sin \Theta - CF \sin \Theta) = \tfrac{1}{2} BD \sin \Theta \cdot (AF - CF) \\ &= \tfrac{1}{2} BD \cdot AC \sin \Theta.\end{aligned}$$

In the particular case when $AC \,||\, BD$, the formula implies that the area is 0. This makes sense, because, referring to Figure 2(b), the area of quadrilateral *ABCDA* is:

$$\begin{aligned}[ABCDA] &= [\triangle ABE] - [\triangle CDE] \\ &= ([\triangle ABE] + [\triangle EBD]) - ([\triangle CDE] - [\triangle EBD]) \\ &= [\triangle ABD] - [\triangle CBD] = 0.\end{aligned}$$

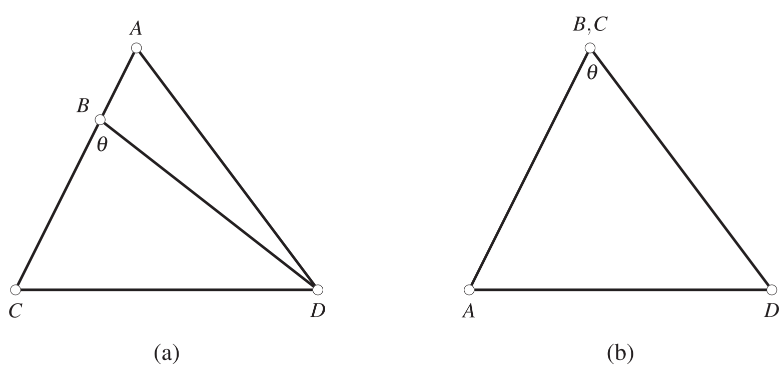


Fig. 4 Two examples of a degenerate quadrilateral ABCD; in (a) vertex B lies on diagonal AC, and in (b) vertices B and C coincide

If one or both the diagonals just touches the other one, the figure degenerates to a triangle, and the formula reverts to the area formula for a triangle, $A = \frac{1}{2} ab \sin C$ (see Figure 4). The formula $\frac{1}{2} AC \cdot BD \sin \Theta$ continues to remain valid.

To conclude, the above scheme integrates the various quadrilateral types normally encountered in classroom situations and highlights a few others. It suggests new definitions such as: *A trapezium is a figure whose diago-*

nals intersect each other in the same ratio. Non-convex and reflex quadrilaterals are also brought in as variations on the diagonal theme. The formula $A = \frac{1}{2} d_1 d_2 \sin \Theta$ is also shown to be the most generally applicable formula for quadrilaterals.

A Ramachandran has had a long standing interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He has been teaching science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at ramachandran@rishivalley.org.

a new approach to solving equations?

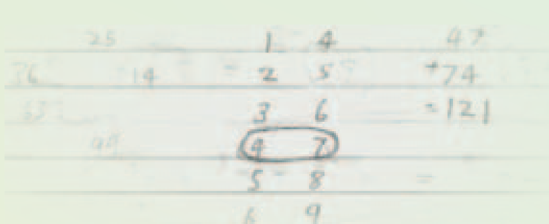
by Shibnath Chakravorty

"Algebra reverses the relative importance of the factors in ordinary language" – Alfred North Whitehead (1861-1947)

Having taught for many years one would think that one cannot be surprised by student responses anymore; one has seen it all! But I discovered that this is not so. I give below an instance of a student's out-of-the-box thinking. During a class the question posed was: *The sum of a two digit number and the number obtained by reversing the order of the digits is* 121. *Find the number, if the digits differ by* 3.

After some explaining, I wrote this pair of equations on the board: $(10x + y) + (10y + x) = 121$, $x - y = 3$. Further processing gave: $y = 7$, $x = 4$. Therefore the answer to the question is 47.

Sheehan, a student of the class who likes to think independently, worked out this problem differently. I reproduce below a copy of his working.



Courtesy: Sheehan Sista, Grade 9, Mallya Aditi International School

He had worked out all the possibilities for this occurrence. They are not many, of course. When I went around looking into their work, I was taken aback by this approach. Later I tried to impress upon him the necessity of solving problems the 'normal' way. The 'board' wanted things done in a particular way. But I must say that it was I who was impressed.

The confidence of these students is high. They are willing to tackle most problems without knowing the 'correct' way to a solution. A positive trait surely. But on the negative side, these students often block out new learning. They sometimes refuse to learn a method as they perhaps feel secure about their own capacity to tackle problems.

on running a math club

It is important to have a Math Club in one's school. It offers a forum to bring together students and teachers who share a love for mathematics.

Let it be emphasized right from the start that a Math Club is not meant only for gifted students. Nor is it a forum for coaching students for math competitions. Rather, it is a forum for 'doing' and exploring mathematics, in an atmosphere of freedom and sharing. As such it should be open to anyone who has an interest in the subject and wishes to see math ideas in action, learn about mathematicians, hear about new areas of application, and so on.

In the same way, the Club should not be restricted only to mathematics teachers. Indeed, it would be good if the convenor of the club goes out of the way to persuade interested colleagues to join the club.

We list here some ideas for running such a Club.

Convenor :
The Club should have a teacher convenor. The convenor could change after one term or one year.

Weekly meeting :
The Club should meet once a fortnight (or once a week if there is enough interest), for 1 to 1½ hours. The time slot should be chosen so that it does not get cut into by other activities.

Presentations :
Presentations can be made at the meetings. These could be of short duration (20–30 minutes); maybe one presentation per meeting, made by either a teacher or a student. The rest of the time can be spent in problem solving, done in small groups, collaboratively. Or it can be spent making some artefacts, using paper, wood or straw.

Bulletin board :
Maintain a bulletin board. It can be of a modest size, but its contents should be updated regularly. *Do not start too ambitiously and then allow the club to decay*.
The bulletin board should feature a problem corner and also a math news corner (with news from the math world).
The problem corner should have separate sections for senior and junior students. Having a single mixed problem set could lead to negative consequences.
Do not set problems which are 'straight from the textbook'. The problems should be mildly challenging. Try to set non-routine problems.

Ideas for problems :

- Cryptarithms
- Coin weighing problems
- Logic puzzles
- Problems on divisibility and numbers
- Problems on triangle and circle geometry

Ideas for presentation :

- Paper folding
- Lives of mathematicians: Ramanujan, Newton, Gauss, Euler, Riemann, Pascal, ...
- Ancient Indian mathematics – work of individuals like Aryabhata, Brahmagupta, Bhaskara I, Bhaskara II (*Lilavati*)
- Mathematics behind cryptography
- Mathematics behind GPS
- Mathematics behind a CT scan
- Mathematics of astronomy
- The story behind Fermat's Last Theorem
- Pythagorean triples and their properties
- Fibonacci numbers and their properties
- Prime numbers, perfect numbers, amicable numbers
- Ideas of infinity
- Use of math software

Sample resource :

Wolfram Math World, http://mathworld.wolfram.com/
Plus Online Magazine, http://plus.maths.org/content
NRICH, http://nrich.maths.org/public/

Note : This will be an ongoing column in this magazine, and teachers are invited to write to us and share their ideas. We will discuss a few ideas in some detail in each issue. Once our web portal is set up, some of these discussions will be continued online.

Ramanujan's Number

by C⊗MαC

They say that each number has its own special property, unique and peculiar to it. It is not always easy to find such a property; but sometimes, by luck, or by hard work, one stumbles upon it.

Mathematicians such as John E Littlewood and Godfrey H Hardy who worked closely with the great mathematician Srinivasa Ramanujan (1887–1920) would say of him that he seemed to know the positive integers as friends. He would know the individual peculiarities of each number!

Here is a well known property that Ramanujan noted about the number 1729: he said to Hardy one day:

"It is the smallest positive integer that can be written as the sum of two positive cubes in more than one way".

'Cubes' are numbers like 1, 8, 27, 64, 125, 216,…. Ramanujan obviously had in mind the following identity: $1729 = 10^3 + 9^3 = 12^3 + 1^3$.

The identity is easy to check. But how could he know that 1729 is the *smallest* positive integer that can be written in two such ways? Only if he had looked at lots and lots of integers before that ….

Ramanujan is said to have made this observation to Hardy who happened to be visiting him while he was recovering in a sanatorium in England, in the year 1918; on entering Ramanujan's room, Hardy apparently said (perhaps just to start a conversation), "I came in a taxi whose number was 1729. I could not see anything interesting about that number" — thereby inviting the response quoted above.[1]

We shall call a number with such a property a *Ramanujan number*. Thus, 1729 is the least Ramanujan number.

Here is our challenge to you: *Find the next Ramanujan number after* 1729.

Probably you will need to use a computer to make a systematic search for such numbers.

And some extensions …

1. *Which is the smallest positive integer that can be written as the sum of two squares in more than one way?*
2. *Which is the smallest positive integer that can be written as the sum of two triangular numbers in more than one way?* (The 'triangular numbers' are the numbers 1, 1+2, 1+2+3, 1+2+3+4, …; i.e., the numbers 1, 3, 6, 10, 15, 21, ...)
3. *Which is the smallest positive integer that can be written as the sum of two fourth powers in more than one way?* (You will certainly need to use a computer to solve this.)

We will discuss the answers in the next issue, and at the same time make some general comments about such problems.

---

1 Editor's note: Ramanujan had gone to England four years earlier, in 1914, at the invitation and insistence of Hardy. We shall have more to say about Ramanujan in future issues of this magazine.

Book Review:
Three centuries of brain racking discovery

$$x^n + y^n = z^n \, ?$$

Fermat's Enigma – The Epic Quest to Solve The World's Greatest Mathematical Problem

by Simon Singh

Reviewed by Tanuj Shah

To talk about a book on mathematics as 'entertaining' or a 'page-turner' may look out of place; but that is exactly how one would describe Simon Singh's book, *Fermat's Enigma*. The book starts in a dramatic manner: "This was the most important mathematics lecture of the century". Singh is writing about a lecture to be delivered by Andrew Wiles on 23 June, 1993; he was going to sketch a proof of Fermat's last theorem in this lecture. It was known as the 'last' theorem because it was the only remaining 'theorem' stated by the 17th century mathematician Pierre de Fermat which had neither been proved nor disproved, despite close attention given to it over the course of three and a half centuries by some of the greatest mathematicians. (Technically it ought to have been called a 'conjecture' as no proof had been found as yet). One can imagine an atmosphere of tension and excitement in the lecture hall at the prospect of the theorem finally being proved.

What Singh manages to do in the book is weave a story with several strands into a colourful tapestry. The story navigates between biographical, historical and mathematical topics in a fluid and intriguing manner. It captures the spirit that drives and inspires mathematicians to take on intellectual challenges. The protagonist is Andrew Wiles, who as a ten year old dreamed of solving one of the most enduring problems of mathematics – that of finding a proof of Fermat's last theorem, or FLT as it is called – and ultimately went on to solve it.

The FLT states that for the equation $x^n + y^n = z^n$ there are no solutions in positive integers when n is an integer greater than 2.

Singh starts by looking at the origins of the equation in ancient Greece in what we call the Theorem of Pythagoras.[1] This is the case $n = 2$ of the equation, that is, $x^2 + y^2 = z^2$. There is a short biography of Pythagoras, describing how he starts the 'Pythagorean Brotherhood' dedicated to discovering the meaning and purpose of life. He believed that numbers held a special key to unlocking the secrets of the universe. The Brotherhood was fascinated by notions such as *perfect numbers*, i.e., numbers whose proper divisors add up to the number itself (for example, 6). Their world of numbers consisted of the counting numbers and rational numbers, which are ratios of counting numbers. They found a surprising number of connections between these and nature, including the ratios responsible for harmony in music. However their strong belief in the importance of rational numbers proved to undermine further progress by the Brotherhood in the field of Mathematics. There is an apocryphal story where one of the disciples proved that $\sqrt{2}$ is irrational; the Brotherhood felt that this threatened their worldview, which was based on rational numbers, and supposedly the disciple was put to death. However Pythagoras can be credited with laying the foundation of modern mathematics by introducing the notion of proof: starting with a statement that is self evident (an axiom) and arriving at a conclusion through a step-by-step logical argument. The theorem that goes by his name is true for all right angled triangles and it is not necessary to test it on all right angled triangles, as it rests on logic that cannot be refuted. (One of the few hundreds of proofs that exist is given in the appendix of the book). Teaching the Pythagorean Theorem by giving the historical back ground would surely broaden students' horizons and deepen their interest in the topic.

Singh makes a detour at this point, making a distinction between *mathematical proof* and *scientific proof*. The demands made by mathematical proof are absolute; it has to be true for all cases whereas a scientific theory is only a model or an approximation to the truth. This is one reason why the Pythagorean Theorem remains accepted 2500 years after it was first proved, while many scientific theories have been supplanted over the years. These are ideas that a teacher can incorporate while teaching the theorem. Further along, different types of proofs like *proof by contradiction* and *proof by induction* are explained in a lucid manner with examples given in the appendix, which an average 14 or 15 year old would easily be able to follow.

Chapter 2 and 3 focuses on some prominent mathematicians who tried to tackle the problem, starting with Fermat, the person who posed the problem. Fermat was a civil servant – indeed, a judge – who devoted all his leisure time to the study of Mathematics. He was a very private man and hardly met any other mathematician; the only one with whom he collaborated with was Blaise Pascal, on formulating the laws of probability. With Father Marin Mersenne he would share his findings, and Mersenne in turn would pass on the news to other mathematicians. Fermat also had a hand in developing calculus; he was one of the first mathematicians to develop a way of finding tangents to curves. However, the reason he has become a household name is for his 'last' theorem, which he jotted in the margins of the *Arithmetica*, adding: "I have a truly marvelous demonstration of this proposition which this margin is too narrow to contain." This statement spurred a large number of mathematicians to try and prove it, while others contested the claims.

Following this, the author gives a brief biography of Leonhard Euler, one of the greatest mathematicians of the 18th Century. There is enough material here that can be shared with students to give them a glimpse of Euler, and they will certainly enjoy tackling the problem of the Königsberg bridges. There is also a small diversion into the structure of numbers to introduce the idea of imaginary or complex numbers. Euler, who made the first breakthrough on the problem by giving a proof of the case when $n = 3$, had to make use of imaginary numbers. Teachers can test students' understanding of exponents by asking what other cases of the theorem have been proved once you prove it for $n = 3$. (The theorem in this case states that the equation $x^3 + y^3 = z^3$ has no solutions in positive integers.)

The next mathematician to make a breakthrough on this problem was Sophie Germain. The author engagingly brings through the difficulties women had to undergo to establish themselves in this field, which was regarded as a domain for men only. The story of how her father tried to dissuade her from pursuing mathematics by taking away the candles will bring a tear to most. The determination with which she continued her studies including taking up an identity of a man will inspire girls; even now, mathematics tends to be thought of as a subject for boys. It will definitely help in puncturing some stereotypes that people hold.

The second part of the book focuses on the discoveries made in the 20th Century that finally helped in cracking the theorem. Though one may not understand the mathematics behind 'modular forms' or 'elliptic curves', Singh provides good analogies to help the reader keep up with the story without making any excessive technical demands. How two Japanese mathematicians linked the above two areas of mathematics with the Taniyama-Shimura conjecture which led to new approaches in tackling Fermat's last theorem is described in a riveting manner, with a touching account of the tragedy that befell one of them. All along Singh keeps the story moving by giving details of Wiles' career and his attempts at solving the problem, which finally culminates in his lecture in Cambridge in June 1993. However, this is not the end of the story; at the beginning there had been a hint that there was more to come by saying "While a general mood of euphoria filled the Newton Institute, everybody realised that the proof had to be rigorously checked by a team of independent referees. However, as Wiles enjoyed the moment, nobody could have predicted the controversy that would evolve in the months ahead."

Simon Singh has managed to show that mathematicians are people with a great passion to discover the highest truth, and he has certainly succeeded in portraying mathematics as a subject of beauty. The book will inspire teachers and students alike, and is recommended to all classes of readers.

Reference

[1] Editor's note: The article by J Shashidhar, elsewhere in this issue, gives more information on the history of the Pythagorean Theorem.

TANUJ SHAH teaches Mathematics in Rishi Valley School. He has a deep passion for making mathematics accessible and interesting for all and has developed hands-on self learning modules for the Junior School. Tanuj Shah did his teacher training at Nottingham University and taught in various Schools in England before joining Rishi Valley School. He may be contacted at tanuj@rishivalley.org

Review of a YouTube clip

http://www.youtube.com/watch?v=z6lL83wl31E&feature=youtu.be

Among many origami videos on YouTube, you will find this proof of Pythagoras' theorem. I was sent the link by an ex-student and thought that the video merited a review for the many teaching and learning opportunities it afforded.

Reviewed by Sneha Titus

The video starts slowly and clearly. Instructions are easy to follow because the demonstration backed by the excellent audio track is very student friendly, though the paper folding happens rather fast.

As the activity proceeded I noticed a visible increase in the pace of the narration. I managed to keep up for a while, but towards the end my energies flagged, I gave up and had to press 'pause' till I figured out what was happening.

And therein lay my learning opportunity! Stimulated by the excellent trigger afforded by the video, I stayed on task, deconstructed the instructions and arrived at the proof.

Which brought me to the second learning opportunity: the excellent questions offered by the video. A lot of 'whys' encourage students not to accept shapes unquestioningly, give reasons for congruence of triangles, understand the significance of starting with a square sheet of paper, and so on. Using the properties of the geometric shapes that have been recognized, the student can understand relationships between the areas of these shapes. The video stimulates the thinking but does not over-explain.

A word on the abstraction: Undoubtedly, a viewer friendly aspect of the video is the use of descriptive terms for the sides of the right angled triangle. 'Little leg' and 'Big leg' intimidate young viewers much less than the standard 'a', 'b' and 'c' which mean so much to math teachers and so little to students. Of course, these symbols do make their way in at the end but only when all the tough work is done and the student can cope with the finer points of the proof.

A great classroom exercise to introduce or reinforce the theorem; the lesson can be conducted with planned pauses for questioning, clarifying, summarizing and predicting and will provide several learning opportunities for students.

Preamble

Problems, The Life Blood of Mathematics

problem corner

Many mathematicians take great pleasure in problem solving, and 'Problem Corner' is where we share interesting problems of mathematics with one other: talk about experiences connected with memorable problems, show the interconnectedness of problems, and so on.

It has been said that "problems are the lifeblood of mathematics." This short, pithy sentence contains within it a great truth, and it needs to be understood.

What is a problem? It refers to a task or situation where you do not know what to do; you have no way already worked out to deal with the situation, no 'formula'; you have to discover the way afresh, by thinking on the spot.

In this sense, a problem is not the kind of exercise you meet at the end of a chapter. On completing a chapter on quadratic equations one may be assigned a list of ten or twenty quadratic equations to solve. But these are 'drill exercises' — they must not be called 'problems'. *A problem is essentially non-routine.* You have to throw yourself at it in order to solve it.

In the history of mathematics it has happened time and again that problems posed by mathematicians — to themselves or to others — would lie unsolved for a long time. Perhaps the most famous instance of this is that of Fermat's Last Theorem ('FLT'), whose origin lies in a remark casually inserted by Fermat in the margin of a mathematics book he was reading; the eventual solution to the problem came after a gap of three and a half centuries! (See the Review of FERMAT'S ENIGMA elsewhere in this issue for more about this story.) And each time this happens, in the struggle between mathematician and problem, the winner invariably is mathematics itself; for in the encounter are born fresh concepts and ideas, fresh ways of organizing and looking at old ideas, fresh notation. In the case of FLT, number theory developed enormously as a result of this encounter, and a whole new field was born, now called *Algebraic number theory*.

The problem corner is a very important component of this magazine. It comes in three parts: Fun Problems, Problems for the Middle School, and Problems for the Senior School. For each part, the solutions to problems posed will appear in the next issue.

To encourage the novice problem solver, we start each section with a few solved problems which convey an idea of the techniques used to understand and simplify problems, and the ways used to approach them.

We hope that *you* will tackle the problems and send in your solutions. We may choose your solution to be the 'official' solution! 'Visual proofs' are particularly welcome — proofs which use a minimum of words.

Another instance where this happened was in the struggle to solve polynomial equations. Quadratic equations (i.e., equations of degree 2, like $x^2+3x+2=0$) were mastered a long time back, perhaps as early as the seventh century (though there was no concept of negative numbers back then); cubic equations (degree 3) were solved by several mathematicians independently over the twelfth to fifteenth centuries; and biquadratic equations (degree 4; also called 'quartic equations') were solved soon after. Naturally, attention then turned to the quintic equation (degree 5). Here researchers hit what seemed to be a wall; no matter what approach was tried they could not cross this barrier. Eventually the matter was resolved but not in the way that everyone expected; it was shown by a young Frenchman named Evaristé Galois that in a certain sense the problem was not solvable at all! In the process was born one of the gems of higher algebra, now called *Galois theory*.

It is not difficult to see why a struggle of this kind will bring up something new. Take any real problem, tackle it, struggle with it and do not give up, no matter what happens; and examine at the end how much you have learnt in the process. What you find may surprise you … It is remarkable that this happens even in those instances where you do not get the solution. But for that, it is essential not to 'give up' ….

In the problem section of Crux Mathematicorum, which is one of the best known problem journals, there occur these memorable words: *No problem is ever closed*, and the editor adds that solutions sent in late will still be considered for publication, provided they yield some new insight or some new understanding of the problem. We are happy to adopt a similar motto for our three problem sections.

Submissions to the Problem Corner

The Problem Corner invites readers to send in proposals for problems and solutions to problems posed. Here are some guidelines for the submission of such entries.

(1) Send your problem proposals and solutions by e-mail, typeset as a Word file (with mathematical text typeset using the equation editor) or as a LaTeX file, with each problem or solution started on a fresh page. Please use the following ID: **AtRiA.editor@apu.edu.in**

(2) Please include your name and contact details in full (mobile number, e-mail ID and postal address) on the solution sheet/problem sheet.

(3) If your problem proposal is based on a problem published elsewhere, then please indicate the source (be it a book, journal or website; in the last case please give the complete URL of the website).

Notation used in the problem sets

For convenience we list some notation and terms which occur in many of the problems.

Coprime
Two integers which share no common factor exceeding 1 are said to be coprime.

Example: 9 and 10 are coprime, but not 9 and 12. Pairs of consecutive integers are always coprime.

Pythagorean triple ('PT' for short)
A triple (a, b, c) of positive integers such that $a^2+b^2=c^2$.

Primitive Pythagorean triple ('PPT' for short)
A triple (a, b, c) of coprime, positive integers such that $a^2+b^2=c^2$. Thus a PPT is a PT with an additional condition — that of coprimeness.

Example : The triples (3, 4, 5) and (5, 12, 13). The set of PPTs is a subset of the set of PTs.

Arithmetic Progression (AP for short)
Numbers $a_1, a_2, a_3, a_4, \cdots$ are said to be in AP if $a_2-a_1=a_3-a_2=a_4-a_3=\cdots$ The number $d=a_2-a_1$ is called the **common difference** of the AP.

Example : The numbers 3, 5, 7, 9 form a four term AP with common difference 2, and 10, 13, 16 form a three term AP with common difference 3.

Fun Problems

Problems for the Middle School

Problems for the Senior School

problem corner

Fun Problems

1. A Problem from the 'Kangaroo' Math Competition

Here is a charming and memorable problem I came across the other day, adapted from a similar problem posed in the 'Kangaroo' Math Competition of the USA.

In a particular month of some year, there are three Mondays which have even dates. On which day of the week does the 15th of that month fall?

At first sight this look baffling. But as one looks more closely, a solution emerges. Please try it out before reading any further!

Let x denote the date of the first Monday of that month. Clearly, x is one of the numbers 1, 2, 3, 4, 5, 6, 7. The Mondays of that month have the following dates:

$$x,\ x+7,\ x+14,\ x+21,\ x+28\,(?),$$

with a possible question mark against $x+28$; for that particular day may fall in the next month (this will depend on which month it is, and on the size of x). For example, if $x = 4$, then $x+28$ is not a valid date, whichever month it is; and if $x = 2$ and the month is February, then too $x+28$ is not a valid date.

Now the numbers $x,\ x+14,\ x+28$ are either all odd or all even; and $x+7,\ x+21$ are either both odd or both even.

Since we are told that the month has *three* Mondays on even dates, it is the first possibility which must apply. The Mondays of that month thus come on dates $x,\ x+14,\ x+28$.

From this we deduce two things: (i) x is even; (ii) $x+28$ is a valid date for that month.

From (ii) we deduce that x is either 1, 2 or 3. Combining this with (i), we get $x = 2$.

So the 2nd of the month falls on Monday, as does 16th. Hence the 15th falls on Sunday.

2. Problems For Solution - What is a cryptarithm?

Two of the fun problems in this issue deal with *cryptarithms*, so we first explain what they are and how they are to be approached.

A cryptarithm is a disguised arithmetic problem, in which digits have been replaced by letters — each digit being mapped to a different letter. (This implies that two different letters cannot represent the same digit.) The problem, of course, is to 'decode' the mapping — i.e., find which digit stands for which letter.

For example, consider the following 'long multiplication' cryptarithm:

$$\begin{array}{cc} A & B \\ & 4 \\ \hline C & A \\ \hline \end{array}$$

To solve this we argue as follows. Since 4 times the two-digit number AB yields another two-digit number, the tens digit of AB cannot exceed 2. So $A = 0$, 1 or 2. But since A is the tens digit of AB, we cannot have $A = 0$ (else, AB would be a one-digit number); hence $A = 1$ or 2.

Again, since 4 times any number is an even number, the units digit of CA is even; i.e., A is even.

Combining this what we got earlier, we find that $A = 2$.

Now we ask: What can B be, so that the units digit of $4 \quad B$ has units digit 2? Clearly it must be 3 or 8 (because $4 \quad 3 = 12$ and $4 \quad 8 = 32$).

But if $B = 8$ then $AB = 28$, and $4 \quad 28 = 112$ is a three-digit number; too large.

So $B = 3$, and the answer is: $23 \quad 4 = 92$. Thus: $A = 2$, $B = 3$, $C = 9$.

Cryptarithms do not always come this easy! But they generally yield to persistence and a long, careful examination of the underlying arithmetic. And, of course, there is no harm in doing a bit of 'trial and error'! Once one solves a cryptarithm there is a great feeling of satisfaction, and one finds that one has learnt some useful mathematics in the process.

Sometimes we come across a cryptarithm with more than one solution. People who design cryptarithms consider this to be a 'design flaw'. *They maintain that a really well designed cryptarithm has a unique solution, and it should be possible to find it using 'pure' arithmetical reasoning, possibly with a small component of trial.*

Problem I-1-F.1

Solve this cryptarithm:

$$ABCD \quad 4 = DCBA.$$

Thus, $ABCD$ is a four digit number whose digits come in reverse order when the number is multiplied by 4.

Problem I-1-F.2

Solve this cryptarithm: $(TWO)^2 = THREE$.

Problem I-1-F.3

The numbers 1, 2, 3, . . . , 99, 100, 101, . . . , 999998, 999999 are written in a line. In this enormously long string of numbers, what is the total number of 1s?

Middle School

Problem Editor : R ATHMARAMAN

1. An Unusual Multiple of 15

The problem we take up for discussion is the following; it was asked in the American Invitational Mathematics Examination (AIME), which is one of the examinations taken by students aspiring to do the national level olympiad of USA:

Find the least positive integer n such that every digit of 15*n is either* 0 *or* 8*.*

Such questions look a bit baffling at first sight, but if one looks carefully, then some facts emerge. Let us try to solve to this problem.

A number whose digits are all 0s and 8s is clearly divisible by 8; hence $15n$ is divisible by 8. But $15 = 3 \quad 5$ and thus has no factors in common with 8 (we say that 8 and 15 are 'coprime'); hence n itself is divisible by 8. Let $m = n/8$. Then m has the property that every digit of $15m$ is 0 or 1. So the problem to solve is:

> *Find the least positive integer m such that every digit of* 15*m is* 0 *or* 1*.*

Once we get this least m, we multiply by 8 to get the required n. So let us look for the least such m.

Since $15m$ is a multiple of 3, the test for divisibility by 3 must apply. Thus, the sum of the digits of $15m$ must be a multiple of 3. Since 0 does not contribute to the sum of the digits it follows that *the number of* 1*s must be a mutiple of* 3. That is, there must be three 1s, or six 1s, or nine 1s, and so on. The least positive multiple of 3 which has no digit outside the set. (0, 1) is clearly the number 111. Since we want the number to be divisible by 5 as well, we simply append a 0 at the end; we get 1110. Hence: 1110 *is the least multiple of* 15 *which has the stated property*.

Since $1110 = 15 \quad 74$, it follows that $m = 74$. Hence the required value of n is $8 \quad 74 = 592$. (Please check for yourself that $592 \quad 15 = 8880$.)

---

2. Problems for Solution

Problem I-1-M.1

Find: (a) Four examples of right triangles in which the lengths of the longer leg and the hypotenuse are consecutive natural numbers.

(b) Two examples of right triangles in which the lengths of the legs are consecutive natural numbers.

(c) Two differently shaped rectangles having integer sides and a diagonal of length 25.

(d) Two PPTs free from prime numbers.

Problem I-1-M.2

Let a right triangle have legs a and b and hypotenuse c, where a, b, c are integers. Is it possible that among the numbers a, b, c:

1. All three are even?
2. Exactly two of them are even?
3. Exactly one of them is even?
4. None of them is even?

Either give an example for each or prove why the statement is false.

Problem I-1-M.3

Let a right triangle have legs a and b and hypotenuse c, where a, b, c are integers. Is it possible that among the numbers a, b, c:

1. All three are multiples of 3?
2. Exactly two of them are multiples of 3?
3. Exactly one of them is a multiple of 3?
4. None of them is a multiple of 3?

Either give an example for each or prove why the statement is false.

Problem I-1-M.4

How many PPTs are there in which one of the numbers in the PPT is 60?

Problem I-1-M.5

Take any two fractions whose product is 2. Add 2 to each fraction.

Multiply each of them by the LCM of the denominators of the fractions. You now get two natural numbers. Show that they are the legs of an integer sided right triangle. (Example: Take the fractions $\frac{5}{2}$ and $\frac{4}{5}$; their product is 2. Adding 2 to each we get $\frac{9}{2}$ and $\frac{14}{5}$. Multiplying by 10 which is the LCM of 2 and 5, we get 45 and 28. Now observe that $45^2 + 28^2 = 53^2$.)

Problem I-1-M.6

The medians of a right triangle drawn from the vertices of the acute angles have lengths 5 and $\sqrt{14}$. What is the length of the hypotenuse?

Problem I-1-M.7

Let *ABCD* be a square of side 1. Let *P* and *Q* be the midpoints of sides *AB* and *BC* respectively. Join *PC*, *PD* and *DQ*. Let *PC* and *DQ* meet at *R*. What type of triangle is $\triangle PRD$? What are the lengths of the sides of this triangle?

Problem I-1-M.8

Find all right triangles with integer sides such that their perimeter and area are *numerically* equal.

Problem I-1-M.9

If *a* and *b* are the legs of a right triangle, show that

$$\sqrt{a^2 + b^2} < a + b \leq \sqrt{2(a^2 + b^2)}$$

Hint. A suitable diagram with a right triangle inscribed in a square may reveal the answer.

Acknowledgement

Mr. Athmaraman wishes to ackowledge the generous help received from Shri Sadagopan Rajesh in preparing this problem set.

Senior School

Problem Editors : PRITHWIJIT DE & SHAILESH SHIRALI

1. A problem in number theory

We start this column with a discussion of the following problem which is adapted from one asked in the first Canadian Mathematical Olympiad (1969):

Find all integer solutions of the equation $a^2 + b^2 = 8c + 6$.

At first sight it looks rather daunting, doesn't it? — a single equation with *three* unknowns, and we are asked to find *all* its integer solutions! But as we shall see, it isn't as bad as it looks.

Note the expression on the right side: 8c + 6, eight times some integer plus six. That means it leaves remainder 6 when divided by 8. Hmmm ...; so we want pairs of integers such that their sum of squares leaves remainder 6 when divided by 8. Put this way, it invites us to first examine what kinds of remainders are left when squared numbers are divided by 8. We build the following table. We have used a shortform in the table: 'Rem' means 'remainder', so 'Rem $(n^2 \div 8)$' means 'the remainder when n^2 is divided by 8'.

n	1	2	3	4	5	6	7	8	9	10	...
n^2	1	4	9	...	...	...	...	...	...	...	...
Rem $(n^2 \div 8)$	1	4	1	...	...	...	...	...	...	...	...

Please complete the table on your own and study the data. What do you see?

Here are some striking patterns we see (and there may be more such patterns):

1. *Every odd square leaves remainder* 1 *when divided by* 8.

2. *The even squares leave remainders* 0 *and* 4 *when divided by* 8*, in alternation:* 4, 0, 4, 0,.... (To say that 'the remainder is 0' means that there is no remainder, i.e., the square is divisible by 8.)

Some thought will convince us that these patterns are 'real'; *they stay all through the sequence of squares and so are genuine properties of the squares*.

For example, consider pattern (1). Every even number can be represented as $2n$ and every odd number as $2n + 1$, for some integer n. The quoted property concerns odd squares.

Hence we have:

$$\begin{aligned}(2n+1)^2 &= 4n^2+4n+1\\ &= 4n(n+1)+1\\ &= (4 \quad \text{an even number})+1\\ &= (\text{a multiple of } 8)+1.\end{aligned}$$

We see that every odd square leaves remainder 1 under division by 8.

In the same way, we find that the even squares leave remainders 0 and 4 under division by 8. Please prove this on your own. (You may want to figure out which squares leave remainder 0, and which squares leave remainder 4.)

With these findings let us look again at the expression a^2+b^2, which is a sum of two squares. We have just seen that under division by 8, the only remainders possible are 0, 1 or 4. So the possible remainders when a^2+b^2 is divided by 8 are the following:

$$0+0, \quad 0+1, \quad 1+1, \quad 4+0, \quad 4+1, \quad 4+4.$$

Hence, the possible remainders are 0, 1, 2, 4 *and* 5*.*

Some numbers are missing in this list. We see that *a sum of two squared numbers cannot leave remainder* 3 *under division by* 8*; nor can it leave remainder* 6*; nor remainder* 7. Note in particular that 'remainder 6' is not possible.

So we have found our answer: *The equation* $a^2+b^2=8c+6$ *has* ***no*** *integer solutions!*

In fact our analysis has shown us rather more: There are no integer solutions to *any* of the following three equations:

$$a^2+b^2=8c+3, \quad a^2+b^2=8c+6, \quad a^2+b^2=8c+7.$$

The reasoning we have used in solving this problem is typical of such solutions. We call it 'number theoretic reasoning'.

Another example of number theoretic reasoning

Here is another problem from number theory, of a kind often encountered. It has clearly been composed keeping in mind the year when India became independent.

Find all possible square values taken by the expression $n^2+19n+47$ *as n takes on all integer values.*

Let $n^2+19n+47=m^2$. We need to find the possible values of m.

We shall now use the humble and time honoured technique of 'completing the square'. However to avoid fractions we first multiply by 4; this is acceptable: if $n^2+19n+47$ is a square number, then so is $4\,(n^2+19n+47)$. Here is what we get:

$$4(n^2+19n+47) = 4n^2+76n+188$$
$$= (4n^2+76n+19^2)+(188-19^2)$$
$$=(2n+19)^2-173.$$

It so happens that 173 is a prime number. This will play a part in the subsequent analysis!

Now we transpose terms and factorize:

$$(2n+19)^2-173 = (2m)^2,$$

$$\therefore (2n+19-2m)\cdot(2n+19+2m) = 173.$$

As 173 is prime, it can be written as a product of two integers in the following four ways (where we have permitted the use of negative integers):

$$-1 \quad -173 = 1 \quad 173 = -173 \quad -1 = 173 \quad 1.$$

Hence the pair $(2n+19-2m, 2n+19+2m)$ must be one of the following:

$$(-1,-173),\ (1,173), (-173,-1),\ (173,1).$$

By addition we get $4n+38 = \pm 174$, i.e., $n = 39$ or $n = -53$.

So there are precisely two values of n for which $n^2 + 19n + 47$ *is a perfect square, namely:* $n = 39$ *and* $n = -53$.

Next, by subtraction we get $4m = \pm 172$, i.e., $m = \pm 43$.

Hence, $n^2+19n+47$ *takes precisely one square value,* namely: 43^2 or 1849.

(Query: Was there something noteworthy happening in India in 1849?)

---

2. Problems for solution

Problem I-1-S.1

Let (a, b, c) be a PPT.

1. Show that of the numbers a and b, one is odd and the other is even.
2. Show that the even number in $\{a, b\}$ is a multiple of 4.

Problem I-1-S.2

Let (a, b, c) be a PPT. Show that abc is a multiple of 60.

Problem I-1-S.3

Show that any right-angled triangle with integer sides is similar to one in the Cartesian plane whose hypotenuse is on the x-axis and whose three vertices have integer coordinates. (Source: Problem of the Week column, Purdue University.)

Problem I-1-S.4

Let a, b and c be the sides of a right-angled triangle. Let θ be the smallest angle of this triangle. Show that if $1/a$, $1/b$ and $1/c$ too are the sides of a right-angled triangle, then

$$\sin\theta = \tfrac{1}{2}(\sqrt{5}-1)$$

(Source: B Math entrance examination of the Indian Statistical Institute.)

Problem I-1-S.5

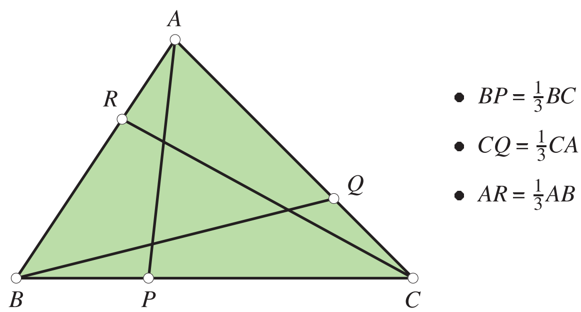
Find all Pythagorean triples (a, b, c) in which:
(i) one of a, b, c equals 2011; (ii) one of a, b, c equals 2012.

Problem I-1-S.6

Find all PPTs (a, b, c) in which a, b, c are in *geometric progression*; or show that no such PPT exists.

Problem I-1-S.7

In any triangle, show that the sum of the squares of the medians equals $\frac{3}{4}$ of the sum of the squares of the sides.



Problem I-1-S.8

The figure shows a $\triangle ABC$ in which P, Q, R are points of trisection of the sides, with $BP = \frac{1}{3}BC$, $CQ = \frac{1}{3}CA$, $AR = \frac{1}{3}AB$. Show that the fraction

$$\frac{AP^2 + BQ^2 + CR^2}{BC^2 + CA^2 + AB^2}$$

has the same value for every triangle. What is the value of the constant?

The Closing Bracket …

At a function held in December 2011 at the Institute of Mathematical Sciences in Chennai, Prime Minister Shri Manmohan Singh declared 2012 to be 'National Mathematics Year' and 22 December (birth date of the great mathematician Srinivasa Ramanujan) to be 'National Mathematics Day'. He inaugurated a series of year-long celebrations to mark the 125th year of Ramanujan's birth, and said in his speech, "*It is a matter of concern that for a country of our size, the number of competent mathematicians that we have is badly inadequate … There is a general perception in our society that the pursuit of mathematics does not lead to attractive career possibilities. This perception must change. [It] may have been valid some years ago, but today there are many new career opportunities available [in] mathematics. …*" He urged the mathematical community to find ways and means to address the shortage of top quality mathematicians and reach out to the public.

These remarks bring home the need to ponder the state of mathematics education in our country: why, for vast numbers of students, mathematics remains a subject of dread, a subject that causes one to 'switch off' at an early age. What are we doing to make this happen?

In contrast we have the extraordinary story of Ramanujan, who was completely in love with mathematics, to a degree that seems scarcely imaginable, and whose life may be described as a passionate celebration of mathematics.

It seems natural to ask what I — as a mathematics teacher — can do to address the situation in the country, and to ask, "Can I bring about a love for mathematics in my students? Can I help children explore this beautiful garden and show them that it is a world in which great enjoyment is possible, even if one is not highly talented at it? "

The answer surely is: Yes. And I do not think it is so very difficult to do. But two things are required at least — a love for one's subject, and a love of sharing with human beings. If these are there, then ways can be found and techniques developed that will bridge most barriers. If as a teacher I have a love of exploration, a love of inquiry, a love of playing with numbers, then surely I will be able to communicate it to children. It seems to me that before I ask for techniques of instruction, I must ask if I have that kind of feeling for the subject and for sharing it with children.

What are the factors which for so many children bring about a fear-filled and alienating relationship with mathematics? It is obvious that a huge contributory factor is a hostile learning environment, in which early contact with fear and comparison as instruments of learning serve to chip away at one's childhood. If there is one area where techniques need to be found, it is this: to find ways of assessment, of feedback and communication, which do away with these traditional and intrinsically violent instruments.

Mathematics education may be in a state of crisis, but this is true of education as a whole, and in a far more serious sense. The world today is in a very grave situation: divisive forces are tearing us apart, and our greed is destroying the earth. We seem to be blind to the fact that our way of life is not sustainable. In what way can we teachers help in bringing some sanity to the world around us, through our teaching and our contact with children? In what way can we convey the beauty of exploration and sharing so that it extends beyond the boundaries of the classroom and spills over into life? In what way can we convey a love for what we are doing, a love which is not bound to the classroom? Let us keep these questions in the foreground, so that they thoroughly permeate our work and everything we do.

— Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

Call for Articles!!!

At Right Angles welcomes articles from math teachers, educators, practitioners, parents and students. If you have always been on the lookout for a platform to express your mathematical thoughts, then don't hesitate to get in touch with us.

Suggested Topics and Themes

Articles involving all aspects of mathematics are welcome. An article could feature: a new look at some topic; an interesting problem; an interesting piece of mathematics; a connection between topics or across subjects; a historical perspective, giving the background of a topic or some individuals; problem solving in general; teaching strategies; an interesting classroom experience; a project done by a student; an aspect of classroom pedagogy; a discussion on why students find certain topics difficult; a discussion on misconceptions in mathematics; a discussion on why mathematics among all subjects provokes so much fear; an applet written to illustrate a theme in mathematics; an application of mathematics in science, medicine or engineering; an algorithm based on a mathematical idea; etc.

Also welcome are short pieces featuring: reviews of books or math software or a YouTube clip about some theme in mathematics; proofs without words; mathematical paradoxes; 'false proofs'; poetry, cartoons or photographs with a mathematical theme; anecdotes about a mathematician; 'math from the movies'.

Articles may be sent to :
AtRiA.editor@apu.edu.in

Please refer to specific editorial policies and guidelines below.

Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to 'At Right Angles'.

While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

Two familiar suspension bridges

The Ram Jhula and the Lakshman Jhula in Rishikesh, Uttarakhand (India), spanning the River Ganga
(Source: http://www.snapshotsofindia.com/Rishikesh/Rishikesh.html)

A well built suspension bridge is a magnificent sight, a marvel of architecture and engineering; a 'symphony in steel'. There are many bridges of this kind across the world, and the most famous is surely the Golden Gate Bridge in the bay area of San Francisco (USA); it is more than 1 km long. Still longer is the Akashi Kaikyo Bridge in Japan. Two examples which will probably be familiar to most Indian readers are the Ram Jhula and the Lakshman Jhula in Rishikesh (Uttarakhand), both spanning the River Ganga. The beauty of such a bridge may hide the mathematics that lies 'underneath' and which is just as beautiful — the fact that the two immensely strong curving cables which hold the bridge with vertical rods have a parabolic shape.

After the straight line (the simplest geometric object possible), we have the conic sections which are second degree curves: the circle, parabola, ellipse and hyperbola (and one degenerate case — a 'pair of straight lines'). Each of these is richly endowed with geometric properties; and the parabola, richly so. Engineers have found ingenious ways of putting these properties to use. Shown here are two more 'incarnations' of the parabola - the curve seen in a fountain, and a solar reflector. (In the latter case, the shape of the mirror is a paraboloid, which is obtained by rotating a parabola about its central axis.)

A water fountain, and a solar reflector
(Sources: http://mathforum.org/ and http://www.tinytechindia.com/)

FRACTIONS

A PAPER FOLDING APPROACH

PADMAPRIYA SHIRALI

A PAPER FOLDING APPROACH

NOTE TO THE TEACHER

These activities comprise a possible approach that can be used at the Upper Primary level to **revisit** the concept of fraction, and deepen their understanding of the various rules used in the arithmetic of fractions.

The activities suggested must be repeated with many examples before patterns in results are observed, and an attempt can be made towards generalization.

At Right Angles
A Resource for School Mathematics

Azim Premji University
A publication of Azim Premji Foundation and Rishi Valley School

FRACTIONS

A PAPER FOLDING APPROACH

SOME CAUTIONS THAT THE TEACHER NEEDS TO OBSERVE

1. **Allow children to experiment with paper freely even if there is some wastage.** They need to figure out things for themselves. Step in only when they seem to be stuck.
2. **While holding any activity it is important to do a consolidation at the end in written form** – the teacher writing on the board, and the children recording in the notebook. If possible, all the materials created should be preserved in some form, maybe pasted in the notebook, or drawn.
3. Often at the initial stage itself teachers refer to to $\frac{1}{2}$ as '1 by 2' instead of 'half'; $\frac{1}{3}$ as '1 by 3' instead of 'one third'; $\frac{1}{4}$ as '1 by 4' instead of 'one fourth'; and so on. **It is best that in the early stages teachers say 'one half', 'one third', 'one fourth', 'two thirds', 'three quarters', 'seven eighths' and so on.** This is important at the initial stage, till the children gain a proper understanding of fractions.

ACTIVITY ONE

Purpose:
To consolidate the idea of fraction and its relationship to the whole.

Materials required:
Six paper strips of equal size - either cut from ordinary ruled notebook paper or from A-4 paper or plain colour paper rolls (available in gift shops), crayons or coloured pencils, scissors

Step 1: Each strip can be considered as 1 whole.

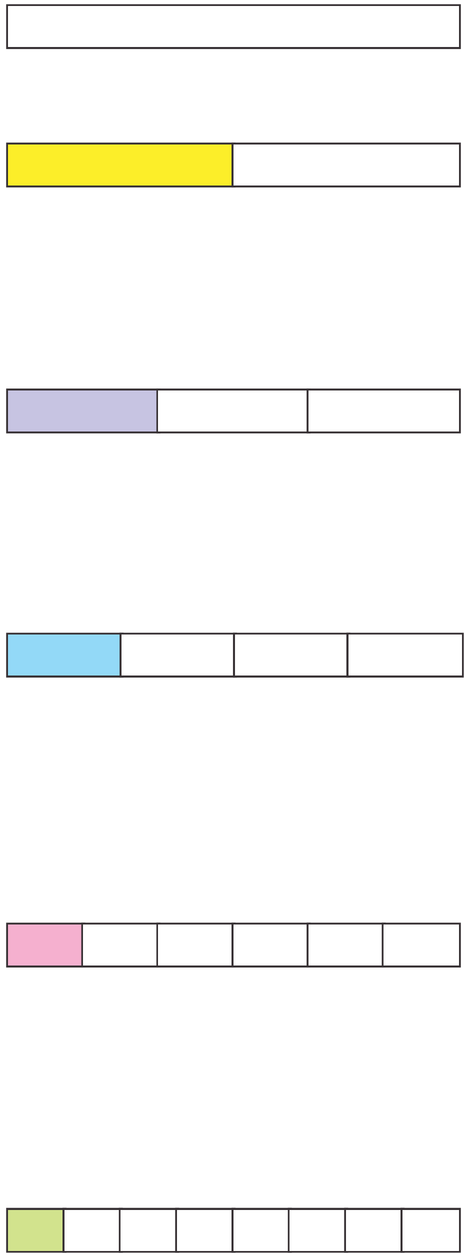
Step 2: Fold the second strip into 2 equal parts. Colour one part to show half. Show how $\frac{1}{2}$ is written. Point out: the denominator represents the total number of parts in the whole, and the numerator is the number of parts we have taken or coloured. Point out that 2 halves make 1 whole.

Step 3: Fold the third strip into 3 equal parts. Colour the first part. (Children will need guidance in folding a paper into 3 equal parts. They should not crease it till they can see 3 equal parts.) Point out that each part is called a third and that 3 thirds make 1 whole. Show that $\frac{1}{3}$ and $\frac{1}{3}$ make $\frac{2}{3}$, and that $\frac{2}{3}$ and $\frac{1}{3}$ make $\frac{3}{3}$. You can also discuss: "What is $1-\frac{1}{3}$?", "What is $1-\frac{2}{3}$?" At each step, let them record the answer in fraction form.

Step 4: Fold the fourth strip into 4 equal parts. Colour the first part. Point out that each part is called a fourth, that 4 fourths make 1 whole, that one-fourth and one-fourth makes two-fourths, and so on. Discuss: "What is $1-\frac{1}{4}$?", "What is $1-\frac{2}{4}$?", "What is $1-\frac{3}{4}$?"

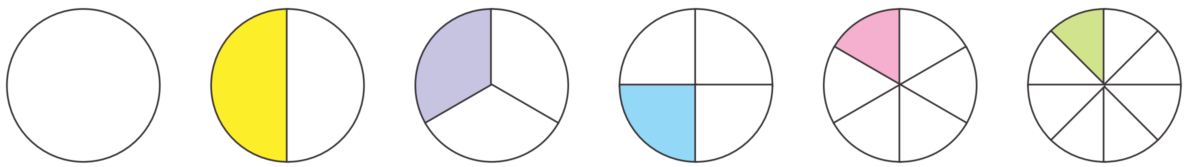
Step 5: Fold the fifth strip into 6 equal parts. (Children will again need guidance in folding a paper into 6 equal parts. Discuss with them till someone points out that they will have to first make 3 equal parts and then halve them, or first make 2 equal parts and then make 3 parts of each one.) Colour the first part. Point out that each part is called a sixth. Ask them: "How many sixths make 1 whole?" Show that that one-sixth and one-sixth make two-sixths, and so on. Discuss: "What is $1-\frac{1}{6}$?", "What is $1-\frac{2}{6}$?", "What is $1-\frac{3}{6}$?", etc.

Step 6: Fold the sixth strip into 8 equal parts. Colour the first part. Ask: "What is each part called?" and "How many eighths make 1 whole?" Show that one-eighth and one-eighth make two-eighths, and so on. Discuss: "What is $1-\frac{1}{8}$?", "What is $1-\frac{2}{8}$?", "What is $1-\frac{3}{8}$?" etc.



Step 7: Extend the activity by asking: "How many ninths make a whole?", "How many twelfths make a whole?", and so on.

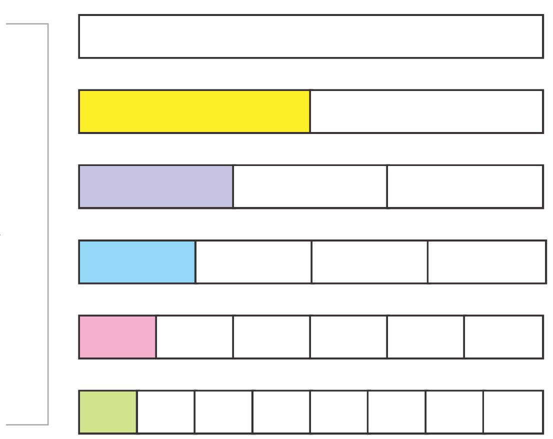
VARIATION: The same activity can be done using paper plates (6 plates are needed). They can cut the parts and arrange them one over the other to understand ordering. To divide a paper plate into three equal parts, children will need to use a protractor. If they do not have the necessary skills to measure 120 degrees, they can be given a cutout for outlining. (Some children may know how to use a compass to divide a circle into six equal parts.)



ACTIVITY TWO

Purpose:
To understand the relationship between unit fractions.

Materials required:
Use the paper strips prepared for activity 1.



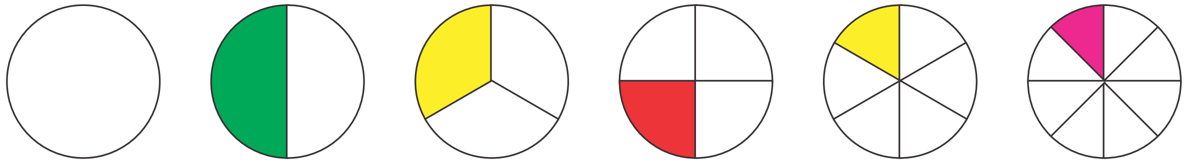
Step 1: Ask the children to arrange the strips one below the other.

Step 2: Observe what they notice. As the denominator increases, the part size decreases.

Step 3: Children record the result using standard mathematical symbols, either in a record book for pasting paper strips, or a notebook: 1/2>1/3>1/4....

Step 4: Extend the activity by asking them further questions: "What is less than $\frac{1}{9}$ but greater than $\frac{1}{11}$?"

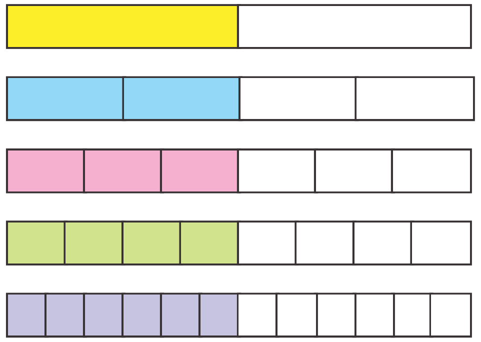
VARIATION: The same activity can be done using paper plates (6 plates are needed)



ACTIVITY THREE

Purpose:
To understand equivalent fractions.

Materials required:
Five paper strips of equal size; crayons or coloured pencils; scissors



Step 1: Fold the first strip into 2 equal parts. Colour one part.

Step 2: Fold the second strip into 4 equal parts. Colour the first 2 parts.

Step 3: Fold the third strip into 6 equal parts. Colour the first 3 parts.

Step 4: Fold the fourth strip into 8 equal parts. Colour the first 4 parts.

Step 5: Fold the fifth strip into 12 equal parts. Colour the first 6 parts.

Step 6: Arrange the strips one under the other.

Step 7: Pose questions to point out the equalities: 1/2=2/4=3/6=.... Tell them that these are called equivalent fractions.

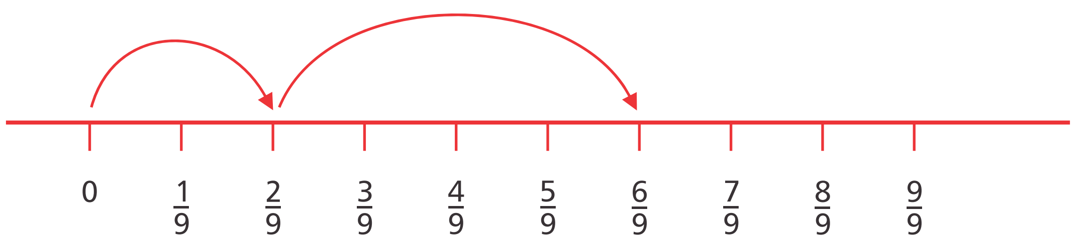
Step 8: Draw out the rule from the students for obtaining equivalent fractions for a given fraction by asking them: "How are the numerators related?" and "How are the denominators related?" Ask them to record the rule in the notebook.

Step 9: Now help them apply the rule for $\frac{1}{3}$ and get the first few equivalent fractions of $\frac{1}{3}$. Discuss: "Into how many parts will you divide the strip?"

Step 10: Extend the activity by asking the children to build equivalent fractions of $\frac{1}{4}$. Children will need guidance to do this till they understand it thoroughly.

ACTIVITY FOUR

Purpose:
To show addition and subtraction of like fractions. Additions and subtractions of like fractions can be easily shown using either a strip folded into the required parts, or on a number line. For example, for $\frac{2}{9}+\frac{4}{9}=\frac{6}{9}$:



ACTIVITY FIVE

Purpose:
To show addition and subtraction of unlike fractions.

Materials required:
8 equal paper strips

Problem:
To show $\frac{1}{2}+\frac{1}{3}$

Step 1: Take the first strip, fold into 2 equal parts and shade $\frac{1}{2}$.

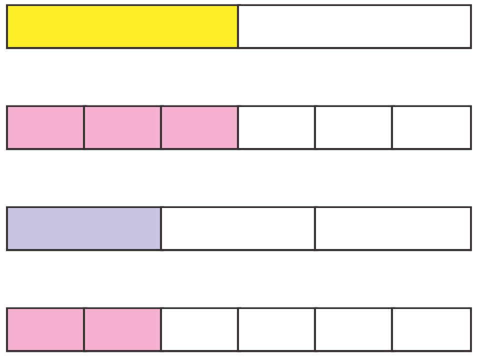
Step 2: Take the second strip, fold into 3 equal parts and shade $\frac{1}{3}$.

Step 3: Point out: The '$\frac{1}{2}$' and '$\frac{1}{3}$' are of two different sizes and cannot be counted together. Discuss their understanding of like and unlike fractions.

Step 4: Ask: "Is there a way of turning them into like fractions without changing their value?" Give a hint about equivalent fractions.

Step 5: Point out: By halving $\frac{1}{2}$ they will get fourths, and halving $\frac{1}{3}$ they will get sixths; but they are still of different sizes.

It is possible that some child will now come up with the solution that $\frac{1}{2}$ can be made into sixths (by folding into three equal parts).



Step 6: Take the first strip with $\frac{1}{2}$ shaded and fold into 3 equal parts to get sixths. Children will now see that the earlier $\frac{1}{2}$ now equals $\frac{3}{6}$.

Step 7: Take the second strip with $\frac{1}{3}$ shaded and fold it into 2 equal parts to get sixths. Children will now see that the earlier $\frac{1}{3}$ now equals $\frac{2}{6}$.

Step 8: Now point out to them that $\frac{1}{2}+\frac{1}{3}$ equals $\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$.

Step 9: Extend this activity to other addition problems which lend themselves to paper folding; for example, $\frac{1}{4}+\frac{1}{8}$.

Step 10: Let children record the results. Then help them find the rule using the LCM of the denominators in finding the equivalent fractions to be added. (We can subtract fractions the same way).

ACTIVITY SIX

Purpose:
To show conversion between a mixed fraction and an improper fraction.

Materials required:
Paper strips or square papers

Problem:
To show $1\frac{1}{2}=\frac{3}{2}$.

Step 1: Take 2 strips or square papers.

Step 2: Fold one strip into half.

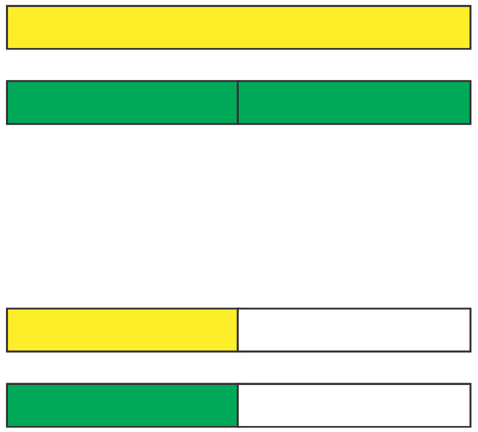
Step 3: Shade 1 whole paper and a half.

Step 4: Pose the question "How much is shaded?" Children respond: 1 and $\frac{1}{2}$. (Or 1 whole and $\frac{1}{2}$.)

Step 5: "Fold the fully shaded whole to make 2 halves. Now how many halves are shaded?" Children respond: 3 halves.

Step 6: Reinforce the fact that 1 whole is the same as 2 halves and therefore $1\frac{1}{2}$ is the same as 3 halves.

Step 7: Extend the activity to conversion of other mixed fractions to improper fractions; e.g., $1\frac{1}{4}=\frac{5}{4}$ and $1\frac{1}{3}=\frac{4}{3}$. Reinforce their understanding that a whole is equal to $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, and so on ($\frac{2}{2}$= 2 halves = 1 whole, $\frac{3}{3}$= 3 thirds = 1 whole, $\frac{4}{4}$= 4 quarters = 1 whole).



ACTIVITY SEVEN

Purpose:
To demonstrate multiplication of fractions.

Materials required:
Square sheets of paper

Problem:
To find $\frac{1}{4}$ of $\frac{1}{2}$.

Step 1: Fold a square sheet of paper vertically in half and shade $\frac{1}{2}$ of it using vertical lines.

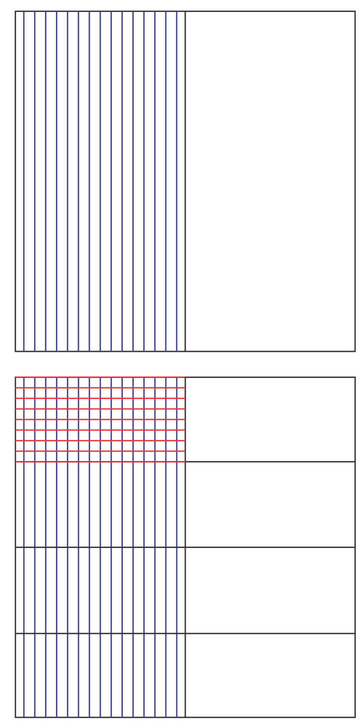
Step 2: Fold the same sheet horizontally in 4 parts. Shade $\frac{1}{4}$ of the $\frac{1}{2}$ using horizontal lines.

Step 3: Open the sheet to show the relationship of the part to the whole. Get the children to notice that one-fourth of one-half equals one-eighth.

Step 4: Extend this activity to demonstrate other multiplications of proper fractions. Ask them to record the results.

Step 5: Now help the children find the rule for multiplication of fractions by asking: How are the first two numerators related to the numerator of the product? How are the first two denominators related to the denominator of the product?

Step 6: Let children record the results in the figure.
For example: $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$, $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.



ACTIVITY EIGHT

Purpose:
To demonstrate division of a whole number by a fraction.

Materials required:
Equal paper strips; use the strips prepared for Activity 1.

Problem: $1 \div \frac{1}{2}$; $1 \div \frac{1}{3}$; $1 \div \frac{1}{4}$.

Step 1: Place the strips one under the other.

Step 2: Revisit the question "How many halves make 1 whole" Children say '2'. Emphasize: 2 halves make 1 whole.

Step 3: Point out: The question "How many halves make 1 whole?" is the same as asking "What is 1 divided by $\frac{1}{2}$?" (You can refer to their understanding of whole number division. "How many 2s in 8?" is the same as: "What is 8 divided by 2?")

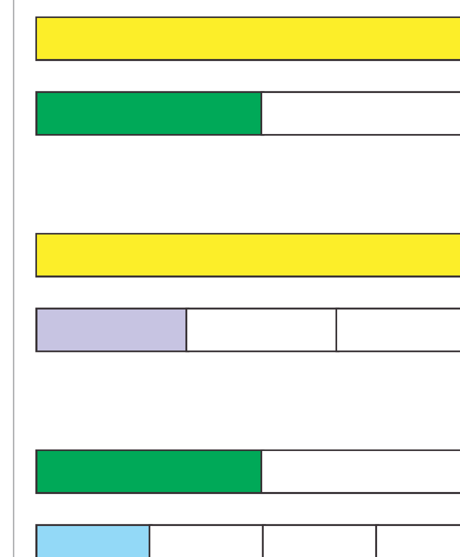
Step 4: Reiterate: "As two halves make 1 whole, therefore 1 divided by $\frac{1}{2}$ equals 2.

Step 5: Now ask: "What is 1 divided by $\frac{1}{3}$?" "What is 1 divided by $\frac{1}{4}$?" "What is 1 divided by $\frac{1}{5}$?" Establish the division facts.

Step 6: Get children to observe the pattern ($1 \div \frac{1}{2} = 2$, $1 \div \frac{1}{3} = 3$, $1 \div \frac{1}{4} = 4$, etc), and deduce the rule for getting the answer. There are 2 halves in one ($2 \times \frac{1}{2} = 1$), there are 3 thirds in one ($3 \times \frac{1}{3} = 1$), there are 4 fourths in one ($4 \times \frac{1}{4} = 1$), etc. Show them the relationship between division and multiplication.

Step 7: Now repeat these questions using 2 wholes, 3 wholes, and so on ($2 \div \frac{1}{2} = 4$, $3 \div \frac{1}{2} = 6$, $4 \div \frac{1}{2} = 8$ and so on). Explain that 1 whole = 2 halves, 2 wholes = 4 halves, and 3 wholes = 6 halves.

Step 8: Now explain the idea of reciprocal. Reciprocal of $\frac{1}{2}$ is 2, reciprocal of $\frac{1}{3}$ is 3, reciprocal of $\frac{1}{4}$ is 4, etc.



ACTIVITY NINE

Purpose:
To demonstrate division of a fraction by another fraction.

Materials required:
Equal paper strips; use the strips prepared for Activity 1.

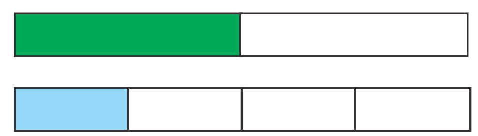
Problem: $\frac{1}{2} \div \frac{1}{4}$.

Step 1: Fold a sheet of paper in half, i.e., to get $\frac{1}{2}$.

Step 2: Fold another sheet in half and then again in half to get $\frac{1}{4}$.

Step 3: Place the strips one under the other. Ask: "How many one-fourths make $\frac{1}{2}$?" Children say: 2.

Step 4: Point out: 2 one-fourths make $\frac{1}{2}$. Emphasize the 'one-fourth' part as they should not think of the 2 as '2 wholes'.
So: $\frac{1}{2} \div \frac{1}{4} = 2$. Let them record it as 'there are 2 one-fourths in half.'



Step 5: Extend the activity to demonstrate various other fraction divisions: $\frac{1}{2} \div \frac{1}{8} = 4$, $\frac{1}{4} \div \frac{1}{8} = 2$, $\frac{1}{3} \div \frac{1}{6} = 2$ and so on. Point out: there are 4 one-eighths in half; 2 one-eighths in one-fourth; 2 one-sixths in one-third.

Step 6: Get children to observe the pattern and deduce the rule for division of fractions using reciprocal. Let them write the above in the following form: $\frac{1}{2} \div \frac{1}{8} = \frac{1}{2} \times 8 = 4$.

GAME

Game 1 : Memory

Purpose : Practice in equivalent fractions

Number of players : 2 to 4

Materials: Make a set of cards like the ones shown. Shuffle them and lay them face down. Each player picks two cards in turns. If they are equivalent fractions he retains them, else he puts them back in the same place. Players must remember where the equivalent pairs are and 'capture' as many as possible. The game finishes when all the cards have been captured. The aim is to capture as many pairs as possible.

$\frac{1}{3}$	$\frac{2}{6}$	$\frac{3}{9}$	$\frac{4}{12}$	$\frac{5}{15}$	$\frac{6}{18}$
$\frac{1}{4}$	$\frac{2}{8}$	$\frac{3}{12}$	$\frac{4}{16}$	$\frac{5}{20}$	$\frac{6}{24}$
$\frac{2}{3}$	$\frac{4}{6}$	$\frac{6}{9}$	$\frac{8}{12}$	$\frac{10}{15}$	$\frac{12}{18}$
$\frac{3}{4}$	$\frac{6}{8}$	$\frac{9}{12}$	$\frac{12}{16}$	$\frac{15}{20}$	$\frac{18}{24}$

GAME

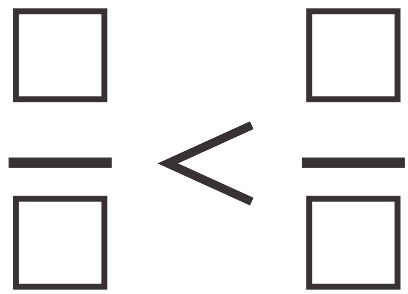
Game 2

Purpose: Practice in comparison of fractions

Number of players: 4

Materials: Number cards from 1 to 10 (2 sets), plain sheets of paper.

Each player makes a drawing like this on his sheet:



Shuffle the number cards and place them upside down. Each player takes one number card at a time and places it in one of the boxes. Once it is put in the box it cannot be changed. If a player manages to get the relation right he gets a point.

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre, in Rishi Valley School (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at **padmapriya.shirali@gmail.com**