Azim Premji University

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together with Community Mathematics Centre,
Rishi Valley

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

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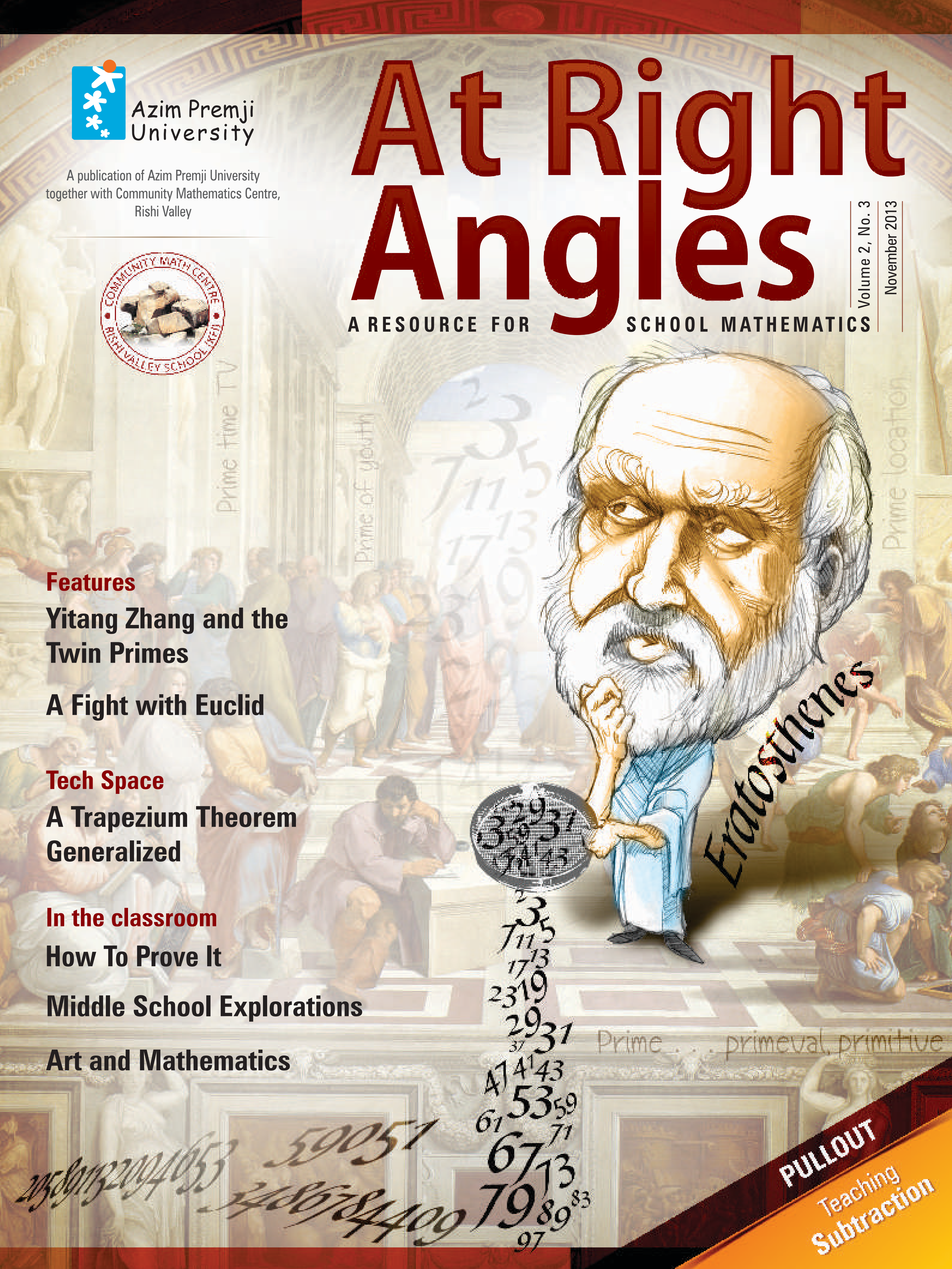

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Notes on the Cover Image

This issue has plenty of material on prime numbers, and the cover depicts, amidst a generous sprinkling of primes, the famous Greek mathematician Eratosthenes who made important contributions to Mathematics and Geography. He lived in second century BC Greece (so he was a contemporary of Archimedes, indeed, the two were good friends), and he can be said to be the person who invented the discipline of Geography as we know it. Among his many remarkable accomplishments is his measurement of the tilt of the Earth's axis to the orbital plane of the Earth (and he arrived at a very accurate estimate) and his drawing of the first map of the world, in which he drew parallels and meridians.

But what he is best remembered for now is his algorithm for generating the prime numbers - a sort of 'sieve' which is taught today to every primary school student: the idea that by sieving out every second number after 2, then every third number after 3, then every fifth number after 5, and so on, the numbers left in the end are just the prime numbers. Sieves have become an important topic in number theory, and they all derive their basic logic from the sieve that Eratosthenes proposed, those many centuries back.

From The Chief Editor's Desk . . .

Starting with this issue, the editors have decided to put a much greater emphasis on the Classroom section of the magazine. Accordingly, some new columns have been added. Punya Mishra and Gaurav Bhatnagar start the show with a fun- and pun-filled waltz through the world of ambigrams. This is part of a continuing series which will bring together themes from the world of 'Art and Mathematics'. Following this is an essay on the tangled history behind an apparently simple and innocent human creation—the calendar; A. Ramachandran describes some of the complexity hidden in this activity. Next, we have the inaugural part of a brand new series, 'How To Prove It', named in honour of Pólya's famous book, *How To Solve It*. Proof is utterly basic to mathematics, but at the school level it has acquired a fearful mystique. The approach taken in the column will mirror the one used in that book: to present —through case studies —themes, approaches and strategies connected with proof. This is followed by the inaugural piece of a 'CCE' column, on the system of Continuous and Comprehensive Evaluation which was introduced by the CBSE a few years back; it is now an integral part of much of the school system in the country, in every subject and all through upper primary and high school. It has enormous potential (for good and bad), and given its scale of usage it is important that good resources be made available to teachers and teacher-educators. This column represents an attempt to address this need, and the inaugural piece is authored by Sindhu Sreedevi, Joyita Banerjee and Sneha Titus. Next, Shiv Gaur has a piece on the rich mathematical possibilities offered by Origami, and Padmapriya Shirali gives an account of a number-theoretic exploration at the upper primary level in an evergreen topic: recurring decimals. In Problem Section too we have a new column: 'Adventures in Problem Solving', whose theme will be just what the title suggests: case studies and strategies connected with problem solving at different levels of the curriculum. In Features we have many pieces on the primes, starting with V. Tikekar's 'Infinitude of the Primes'. This is followed by a delightful 'proof by poetry' which would have gladdened Euclid's heart, and an article by Ramesh Sreekantan on twin primes. B. A. Sethuraman tells us about Fermat primes and their role in geometric constructions, and B. Sury shows how to produce the Pascal triangle starting with the reciprocals of the natural numbers. In Tech Space, Michael de Villiers writes about a generalization of a theorem about trapezoids; in Pullout, Padmapriya Shirali talks about how subtraction can be taught in a graded manner. The Review is about two books of poetic beauty on two very great individuals: S. Ramanujan and S. Chandrasekhar. N. Mukunda writes compellingly on the books written by Robert Kanigel and Kameshwar Wali. The reviews are reproduced from Journal of Genetics (Aug 1992), and we thank the editors of the Journal for permission to do so.

— Shailesh Shirali

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

At Right Angles
A Resource for School Mathematics

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Contents

Features

This section has articles dealing with mathematical content, in pure and applied mathematics. The scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

In the Classroom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The Classroom' is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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Contents contd.

Tech Space

'Tech Space' is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

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Pullout

feature

Musing on the primes

There are Infinitely many Primes – I

But how many proofs of this?

Numbers have been a subject of fascination from the most ancient times, and people keep coming up with families of numbers: integers, rational numbers, numbers, real numbers, complex numbers, prime numbers, Fermat numbers, Bernoulli numbers, ... Mathematics teacher D R Kaprekar (1905–1985) found many new families, giving them curious names like Dattatreya numbers, Demlo numbers, monkey numbers, and so on. India's great mathematician S Ramanujan who made a large number of discoveries in number theory found a new family of numbers which he called 'highly composite numbers'. Back in the Greek era, Pythagoras, steeped in mysticism, referred to numbers as sacred, lucky, evil and so on. (Sacred numbers are difficult to find these days. But 13 continues to be unlucky!) For the rest of this article, when we use the word 'number' we mean natural number or positive integer, i.e., one of the numbers 1,2,3,4,5, ...

V G Tikekar

Prime numbers

On the vast canvas of numbers there is one special category, the *prime numbers* (or just 'primes'), which have been a source of interest to mathematicians since ancient times. Not only do they display very beautiful and surprising properties, they also find unexpected application in fields like coding and cryptography.

Early in our encounter with numbers, we discover that there are infinitely many of them, meaning that their supply can never be exhausted. For, no matter how large the number that we name, we can produce a larger one by adding 1 to it.

Keywords: *Numbers, prime, composites, infinite, factorial, coprime, Euclid, contradiction, Pólya, Fermat number*

A number n exceeding 1 is said to be *prime* if it has no divisors among the set of natural numbers, other than 1 and itself. If n does have divisors other than 1 and n it is called *composite*. Note that the number 1 does not get classified by these two definitions. We call 1 a *unit*; it is neither prime nor composite. So the primes are these numbers:

$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \dots,$

and the composites are:

$4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, \dots.$

After 2, every even number is composite, so we cannot find stretches of consecutive numbers which are all prime. But we do seem to find long stretches of consecutive numbers which are all composite. For example, $14, 15, 16$ is a stretch of three such numbers, and $24, 25, 26, 27, 28$ is a stretch of five composites. Here is a stretch of seven composites: $90, 91, 92, 93, 94, 95, 96$. Can we find even longer stretches of composites? Can we, say, find a billion consecutive numbers, all composite? The surprising answer is: Yes!

Here is a simple argument showing why. Let n be any number, $n > 1$. Consider the following $n - 1$ consecutive numbers defined using the factorial function (recall that $n!$ is the product $1 \times 2 \times 3 \times \cdots \times (n-1) \times n$):

$n!+2,\ n!+3,\ n!+4,\ n!+5,\ \dots,\ n!+(n-1),\ n!+n.$

These $n - 1$ numbers are all composite; for, $n!+2$ is a multiple of 2; $n!+3$ is a multiple of 3; …; $n!+n$ is a multiple of n. (In general, $n!+k$ is a multiple of k if k lies between 2 and n.) So simply by choosing n to be extremely large, we can construct very long stretches of consecutive composite numbers. (Example: Take $n = 6$; we get the following stretch of five consecutive composites: $722, 723, 724, 725, 726$.) This establishes the claim.

How many primes are there?

There are obviously infinitely many composites (indeed, every even number after 2 is composite), but we cannot be so sure about the primes. For one thing, they start to thin out! For example, there are 168 primes between 1 and 1000; 135 primes between 1000 and 2000; 127 primes between 2000 and 3000; …and 98 primes between 20000 and 21000. The number is clearly coming down, so we may wonder whether a point will come, far down the number line, when they vanish altogether.

This question was posed by the ancient Greeks, and answered: they proved that there is no `last prime'; in short, there are infinitely many primes. The oldest proof known of this remarkable claim is found in the great text written by Euclid, *The Elements*. Since then many more proofs have been found by famous mathematicians.

Euclid's proof

It is curious that Euclid's beautiful proof is found in a book that is generally considered to be a text in geometry! But in fact there are several topics in this book which would nowadays be regarded as part of number theory.

Euclid's proof is based on the principle of ‘proof by contradiction’ . It starts by supposing that what we wish to prove is *false*, then examines what follows from this supposition—in the hope of finding something contrary. If such a contradiction is found, it shows that what was assumed at the start necessarily has to be false. In other words, the statement we wish to prove must be true. This strange-sounding strategy for proof is a corner stone for the development of modern mathematics. Let's see how Euclid carries out this strategy.

Let the primes be $p_1, p_2, p_3, \dots$; here $p_1 = 2$ is the first prime, $p_2 = 3$ is the second prime, $p_3 = 5$ is the third prime, and so on. Suppose there is a ‘last prime’ p_n. (This is precisely the supposition we hope to demolish.) We now construct the following number X by adding 1 to the product of all these primes:

$$X = p_1 p_2 p_3 \cdots p_n + 1.$$

It should be clear that X leaves remainder 1 when divided by p_1. In fact it leaves remainder 1 when divided by each of the primes $p_1, p_2, p_3, \dots, p_n$. This means, in particular, that: *X is not divisible by any of the primes $p_1, p_2, p_3, \dots, p_n$.*

What kind of number is X? It is either prime or composite. If it is the former then we have a new prime number (X itself), different from $p_1, p_2, p_3, \ldots, p_n$. If X is not prime then it has a prime divisor q different from $p_1, p_2, p_3, \ldots, p_n$. (It cannot be any of these since X is not divisible by any of these primes.) *Whichever possibility happens, we obtain a prime number different from* $p_1, p_2, p_3, \ldots, p_n$. *So* $\{p_1, p_2, p_3, \ldots, p_n\}$ *cannot be the complete set of primes*. Hence there cannot be a last prime number p_n, and the number of primes is infinite.

We can try out this argument with some actual numbers to see how it works.

- Imagine we thought that 3 is the last prime number (!); then the set of primes is $\{2, 3\}$, and $X = (2 \times 3) + 1 = 7$. It happens that 7 is prime, so we have found a new prime, contrary to our supposition that 3 is the last prime.

- Imagine we thought that 5 is the last prime number; then the set of primes is $\{2, 3, 5\}$, and $X = (2 \times 3 \times 5) + 1 = 31$. It happens that 31 is prime, so we have found a new prime, contrary to our supposition that 5 is the last prime.

- Similarly, if we imagined 13 to be the last prime number, so that the set of primes is $\{2, 3, 5, 7, 11, 13\}$, then $X = (2 \times 3 \times 5 \times 7 \times 11 \times 13) + 1 = 30031$. It happens that 30031 is composite, and its prime factorization is $30031 = 59 \times 509$. So we have found two new primes (59 and 509), contrary to our supposition that 13 is the last prime.

Notice how carefully Euclid has framed the argument. He has never claimed that X is prime, only that a new prime will be found by this means whether X is prime or composite. The proof is indeed a classic.

Variants of Euclid's proof

There are other proofs of the infinitude of primes that closely resemble Euclid's proof but are not the same (though they are clearly modelled on Euclid's proof). We sketch a few here.

(1) Instead of using the number

$$X = p_1 p_2 p_3 \cdots p_n + 1$$

we could as well work with the number $Y = p_1 p_2 p_3 \cdots p_n - 1$. We need $n > 1$, to avoid triviality. The rest of the proof is the same as earlier.

- If $n = 2$ we get $Y = (2 \times 3) - 1 = 5$ which is prime.

- If $n = 4$ we get $Y = (2 \times 3 \times 5 \times 7) - 1 = 209 = 11 \times 19$ which yields two new primes, 11 and 19.

The same reasoning works in all cases; we see that there must be a prime number other than $p_1, p_2, p_3, \ldots, p_n$.

(2) We could also use the factorial function. If K is the supposed largest prime we could work with the number defined by $Z = K! + 1$. Once again the same reasoning works and yields new primes.

(3) The following proof is due to the German mathematician Ernst Kummer (1810--1893), and it is a genuine proof by contradiction. Suppose that p_n is the last prime and that $\{p_1, p_2, p_3, \ldots, p_n\}$ is the complete set of primes. As before we construct the number $X = p_1 p_2 p_3 \cdots p_n - 1$. This number must have a prime divisor, and the divisor must be one of the primes $p_1, p_2, p_3, \ldots, p_n$, because we have supposed that these are *all* the primes that exist. Suppose that the prime divisor of X thus defined is p_k.

Now, clearly, p_k is a divisor of $X + 1$ too (since $X + 1$ is the product of the primes $p_1, p_2, \ldots, p_n$). But if p_k divides X as well as $X + 1$, then p_k must divide the difference between $X + 1$ and X, which is 1. This however is absurd: no prime number can be a divisor of 1. So we have found the desired contradiction, and the conclusion follows that there are infinitely many primes.

A presentation not based on 'proof by contradiction'

There is even a way of presenting Euclid's proof in which we do not emphasize the contradictory aspect; it would not be called a proof by contradiction. We phrase it in a positive manner by claiming that: *Given any finite set S of primes, it is possible to find a prime number that is not in S.*

The idea is exactly the same: we construct a number N which is 1 more than the product of all the numbers in S. Then N is either a prime number, or it has a prime divisor q. Either way we obtain a new prime (N or q) which does not lie in S.

It is fairly obvious that all these are 'children' of Euclid's proof.

Pólya's proof

In contrast, here is a proof which is genuinely different. It is due to the great mathematician educator George Pólya (Christian Goldbach had had exactly the same idea), and it uses the Fermat numbers F_n defined by:

$$F_n = 2^{2^n} + 1.$$

For example we have: $F_0 = 2^1 + 1 = 3$; $F_1 = 2^2 + 1 = 5$; and following these:

$$F_2 = 2^4 + 1 = 17,$$
$$F_3 = 2^8 + 1 = 257,$$
$$F_4 = 2^{16} + 1 = 65537.$$

The five numbers listed are all primes, but that should not fool us, for the very next Fermat number is not prime:

$$F_5 = 2^{32} + 1 = 4294967297 = 641 \times 6700417.$$

Pólya observed that these numbers have the following very nice property:

> *The Fermat numbers are mutually coprime:* $\gcd(F_m, F_n) = 1$ *for all* $m \neq n$.

This follows from the fact that the Fermat numbers are all odd (which is obvious), and they obey the following identity for all $n \geq 1$:

$$F_n - 2 = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1}.$$

For example, take $n = 3$; we have $255 = 3 \times 5 \times 17$. We shall leave the proof of the identity as an exercise, as also the proof of coprimeness of the Fermat numbers.

Taking the claim as proved for now, we show how it implies that there are infinitely many primes. Each Fermat number has associated with it its own set of prime divisors. These sets must all be disjoint (this is what 'coprime' implies). So for each number n we have a non-empty set of primes corresponding to n. Taking the union of these sets, we see that there must be infinitely many primes. *Note that this is not a proof by contradiction.*

Exercises

(1) Prove the relation $F_{n-2} = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1}$ for the Fermat numbers. Hint: Use the principle of induction.

(2) Show how the above relation, together with the fact that the Fermat numbers are odd, implies that these numbers are mutually coprime. Hint: Suppose some prime p divides both F_m and F_n where $m > n$. Using the above identity show that p must divide 2. But this is absurd, since p must be odd.

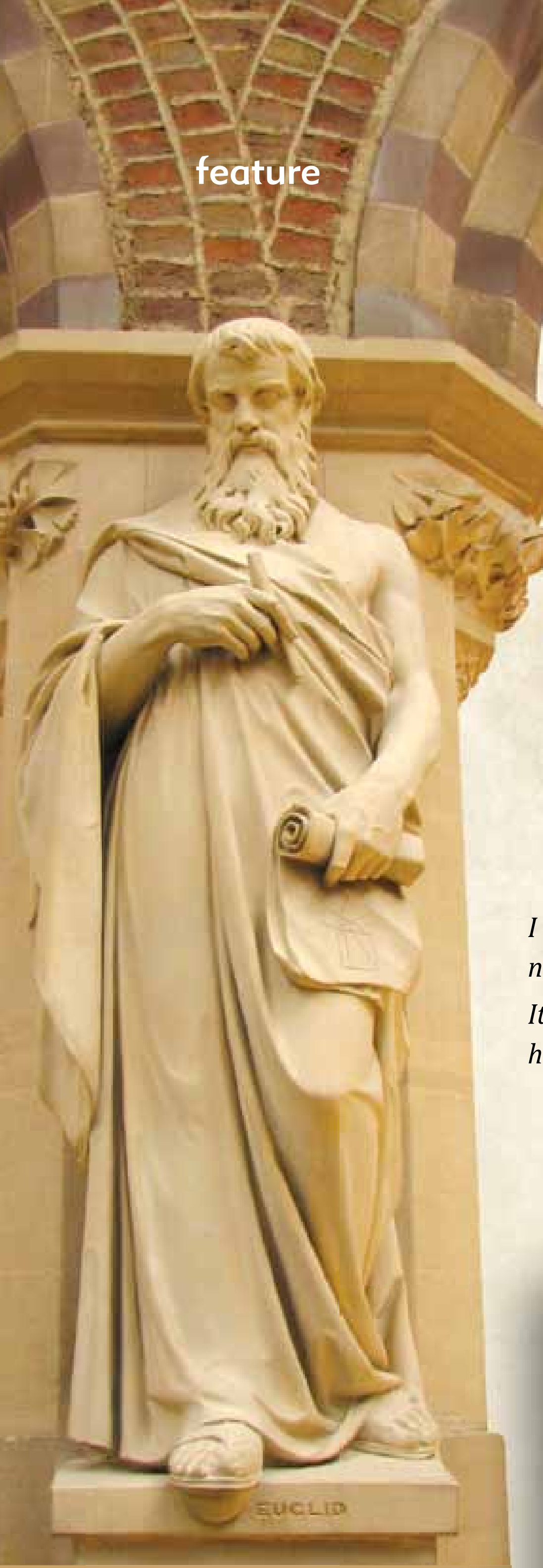
References

[1] Association of Math Teachers of India (AMTI), *The Wonder World of Kaprekar Numbers*

[2] Robert Kanigel, *The Man Who Knew Infinity: A Life of the genius Ramanujan*, Maxwell Macmillan International, 1991, p. 232

[3] Paul Hoffman, *Archimedes' Revenge*, Ballantine Books, 1988, p. 7

PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1994. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted on vgtikekar@gmail.com.



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(http://mathwithbaddrawings.com /2013/07/04/
a-fight-with-euclid/),
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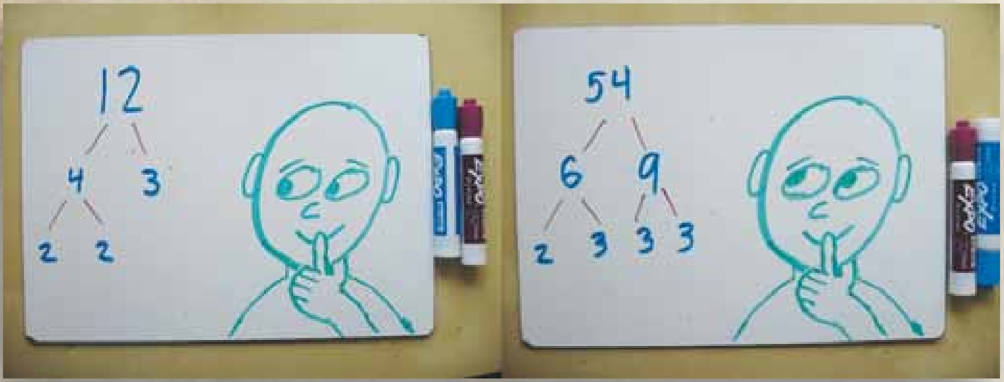
A Fight With Euclid

BEN ORLIN

I had a fight with Euclid on the nature of the primes.
It got a little heated – you know how the tension climbs.

It started out most civil, with a honeyed cup of tea;
we traded tales of scholars, like Descartes and Ptolemy.
But as the tea began to cool, our chatter did as well. We'd
had our fill of gossip. We sat silent for a spell.
That's when Euclid turned to me, and said, "Hear this, my friend:
did you know the primes go on forever, with no end?"

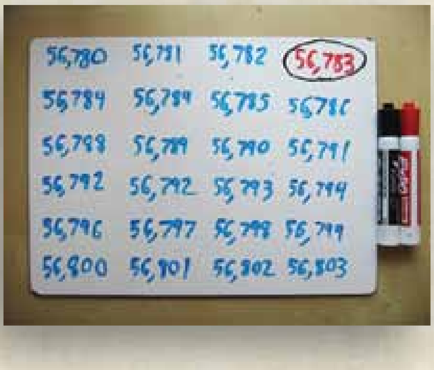
I took a napkin to my face,
to wipe the tea and shock.
At length I said, "The primes don't end?
My friend, that's crazy talk.



In general, the integers have factors we can find.
Take 12. That's 4 times 3.
Or 54. That's 6 times 9.

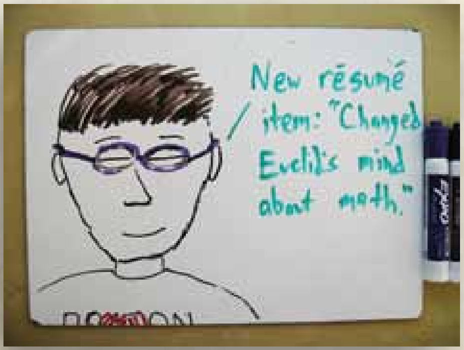
But certain numbers can't be broken down in any
way. Take 17. It has no factors.
So it's 'prime,' we say.

At first, the primes are plentiful.
There's 2, 3, 5, and 7.
There's 31 and 43.
There's 19 and 11.



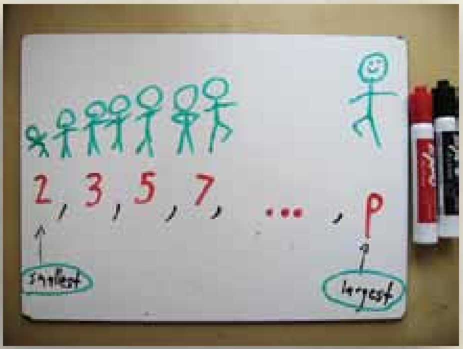
But as our sights climb higher, the primes start thinning out.
Long gaps pass without one – throwing Euclid's claim in doubt.
So when he held his ground and said, "It's true. The primes don't end."
I laughed and told him, "Euclid, boy, you've gone around the bend!"

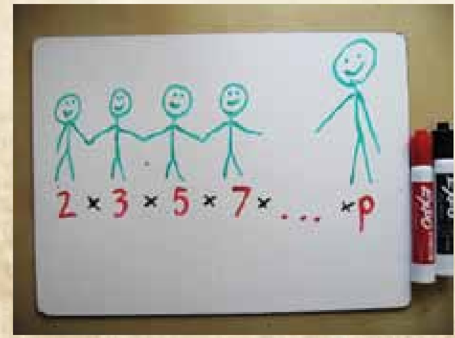
Then Euclid mused, "Suppose you're right.
Suppose they hit a max.
In that case, there's a largest prime.
Do you accept these facts?"



I nodded my acknowledgement, and felt a speck of pride that
Euclid, proud and lauded Greek, had come to see my side.

"This largest prime must have a name,"
he pressed. "Let's call it p.
The biggest prime of all the primes,"
he said. "Do you agree?"

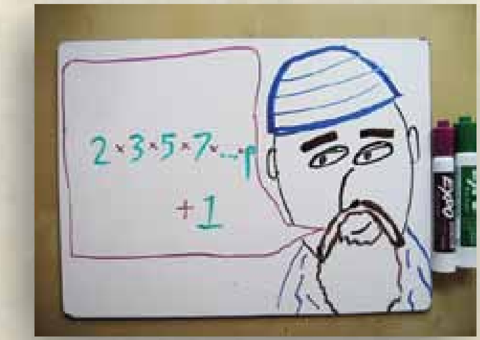


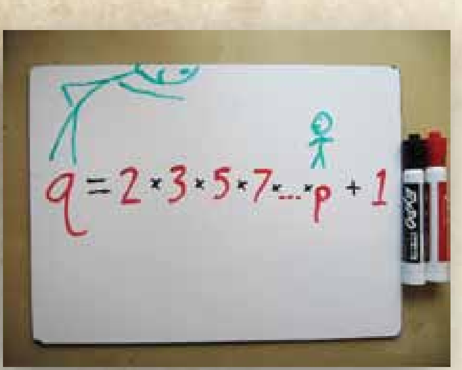


Once again I nodded, with a smile on my face, for I was
putting Euclid, mighty Euclid, in his place.

"Now, let's gather all the primes, from 2 on up to p, and
multiply them all together," Euclid said to me.

"Fine," I said, a note of worry ringing in my thoughts.
What was Euclid plotting? Had he given up or not?
"Multiply out all those primes," he said, "and when you're done,
take that final product, and simply add on 1."





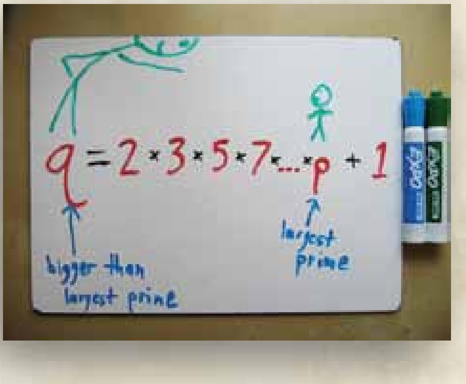
My confidence was fading. I was swiftly feeling dumber.
I could not see the reason Euclid conjured up that number.

"Let's call this number q," he said,
"this newly minted figure.
And notice that, compared with p,
our q is much, much bigger."

I could not disagree with this. His argument rang true.
After all, we'd multiplied by p to get to q.

Since q is larger than the largest prime,"
old Euclid said,
"our q cannot be prime itself.
It factors out instead."

I felt a trap was being sprung, but I could not resist.
A prime that's larger than our p could simply not exist.



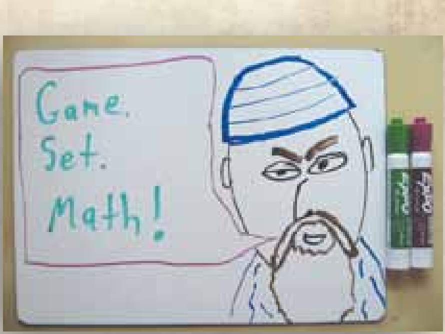
"Now, q has factors," Euclid said.
"What can those factors be?
We can't divide our q by 2.
We can't divide by 3.
We can't divide by 5 or 7.
Can't divide by p.
We can't divide by any prime," he said,
"Do you agree?"

I saw his logic, blinding now. It scorched me like the sun.
Divide our q by any prime; you'll get remainder 1.
That means it has no factors.
That means it must be prime.
But already, we've said it's not.
And if that's true, then I'm ...

"You're wrong!" he cried. "You see the flaw?"
I felt like such a dunce.
"You say q's prime,
then say it's not.
It can't be both at once!

"So take your claim that 'primes must end,' and
stick it on a shelf.
I've shown you now. That stance is flawed.
It contradicts itself!

"That only leaves one option.
And now, you see the light.
The primes must never, ever end."
I sighed. The man was right.



We poured another cup of tea,
and smoothed our ruffled shirts.
I said, "Your argument hit hard,
and I confess, it hurts.
I thought you had conceded,
but the whole charade was fake.
You only took my side
so you could show me my mistake."

Euclid sipped his teacup with a twinkle in his eye.
"The proof by contradiction," he agreed, "is rather sly.
You stand upon the sidelines.
Your opponent takes the field.
You let him play against himself,
until his flaw's revealed."

"The truth wins out then, I suppose." I glumly drained my cup.
"The truth will win out even when it seems it's given up."

"So it is," said Euclid, "and so may it always be."
And then he kindly offered up another cup of tea.

feature

Prime Time

Yitang Zhang and The Twin Primes Conjecture

Reducing the generation gap

RAMESH SREEKANTAN

In early May 2013 a lecture was announced at Harvard university, which got a lot of mathematicians (especially the analytic number theorists) cautiously excited. A person by the name of Yitang Zhang had announced a proof of a theorem which could be considered a first step towards the **Twin Primes** conjecture — long standing in the theory of numbers. The conjecture is easy to state; so easy, in fact, that it would not be surprising for anyone who spends a few moments thinking about to come up with it.

To state the conjecture we recall some facts about prime numbers. A **prime number** is a number not divisible by any number other than 1 and itself. The first few primes are 2, 3, 5, 7, 11, 13, ... With a few moments thought one might wonder: *Are there only finitely many such numbers, or does the list go on forever?* Over two thousand years ago, the Greek mathematician Euclid showed that there are infinitely many prime numbers. (Editor's note: The companion article in this issue by V G Tikekar gives several proofs of this assertion. We even have a proof in verse, by guest columnist Ben Orlin.)

Keywords: *Prime, twin primes, Polignac, bounded gaps, Brun, Brun sieve, Yitang Zhang, lim inf*

Continuing to look at the set of prime numbers, one might notice something else. There are **pairs** of primes such as 3 and 5; 5 and 7; 11 and 13; 17 and 19; 29 and 31; and so on. Once again, one might wonder: *Are there infinitely many such pairs of prime numbers? Namely, are there infinitely many numbers p such that p and p + 2 are both prime?* The statement that there are indeed infinitely many such primes is the **Twin Primes Conjecture.** The conjecture remains open as of late 2013.

Unlike many well known conjectures such as 'Fermat's Last Theorem' (which is now a theorem) or the 'Goldbach conjecture' (still a conjecture!), there is no one person who can be clearly identified as having first formulated the twin primes conjecture. It is usually attributed to Euclid. The first place it arose in print was in 1849, in the work of Alphonse de Polignac, a French mathematician.

Figure 1. Euclid of Alexandria, as depicted by Raphael; source: http://en.wikipedia.org/wiki/Euclid

The first person to make some progress towards this was the Norwegian mathematician, Viggo Brun. A well known theorem, due to Leonhard Euler, states that if one considers the sum of the reciprocals of the primes,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots=\sum_{p \text{ prime}}\frac{1}{p}$$

the sum *diverges*; it 'grows without bound'. This shows in particular that there are infinitely many prime numbers, as a sum of finitely many numbers would yield a finite number.

Brun showed that the sum of reciprocals of twin primes *converges!* So unfortunately this argument cannot be used to show that there are infinitely many twin prime pairs. But his method of proof, now called the *Brun sieve*, is an important technique in the analytic theory of numbers.

A natural generalization of the twin primes conjecture is the following question—called the **Bounded Gaps between Primes** conjecture or **Polignac** conjecture. *Given an even number k, are there infinitely many numbers p such that p and p+k are prime?* The twin prime conjecture is the case when $k = 2$. It is towards this conjecture that Yitang Zhang made his remarkable contribution. Zhang showed that this conjecture is true for some $k < 70$ million. It is the first time that such a claim has been proved. But note that we do not know any single value of k for which Polignac's conjecture is true.

The precise and rather technical statement of the theorem he proved is the following (see the box for an informal explanation of the meaning of 'liminf'; you need not feel worried at this stage if you do not quite get it).

Theorem. [6] *Let p_n denote the n^{th} prime number. Then*

$$\liminf_{n\to\infty}(p_{n+1}-p_n) < 7\times 10^6.$$

Figure 2. Yitang Zhang; source: [2]

Meaning of 'inf' and 'liminf'

The word 'min' (short for 'minimum') is familiar to most of us, e.g., we have the usage: min{2, 3, 4} = 2. However there are naturally occurring sets for which one expects to see a minimum or least element, but, contrary to expectation, they do not have such an element. For example, consider the set $R_{>0}$ of all positive real numbers. We cannot describe 0 as "the minimum element of $R_{>0}$" because 0 does not even belong to $R_{>0}$. At the same time, 0 is the only number which could be regarded as "lying at the bottom end" of $R_{>0}$. To get around this difficulty, mathematicians have come up with a concept called 'infimum' or 'inf' for short. Put briefly, the inf of a set of real numbers S is the largest number a such that no number in S is smaller than a. By this definition the inf of the set of positive real numbers is 0. Similarly, the inf of the set {1, 1/2, 1/3, 1/4, . . .} is 0.

Using this notion we define the 'limit inferior' or 'liminf' of a sequence. Given a sequence $\{x_n\}$, by its limit inferior we mean the quantity

$$\lim_{n\to\infty} \left(\inf\{x_n,\ x_{n+1},\ x_{n+2},\ \ldots\}\right).$$

As n increases, the quantity $\inf\{x_n, x_n + 1, x_n + 2, \ldots\}$ naturally increases, since we are considering the infimum over smaller and smaller sets. Therefore the sequence whose n^{th} term is

$$\inf\{x_n,\ x_{n+1},\ x_{n+2},\ \ldots\}$$

is an increasing sequence. Consequently it possesses a limit (which may be infinite). This is called the 'liminf' of the sequence $\{x_n\}$.

Zhang's theorem states that the increasing sequence

$$y_n := \inf\{p_{n+1} - p_n,\ p_{n+2} - p_{n+1},\ p_{n+3} - p_{n+2},\ \ldots\}$$

is bounded above for all n. In other words, no matter how large n is, there is a pair of consecutive prime numbers p_k and p_{k+1} with $k \geq n$ such that $p_{k+1} - p_k < 70$ million. Hence there must be infinitely many such pairs of primes.

While 70 million seems like a large number (certainly very far from 2!), experts believe that it is only a matter of time before the number is drastically reduced. In fact, in the few weeks since the result was announced, an internet based project proposed by Terrence Tao has reduced the number substantially; and as of Aug 29, the number is 4680 [3]. So in a matter of weeks the gap has been reduced by four whole orders of magnitude! I'm sure by the time this article appears it will be reduced still further.

The mathematics involved in Yitang Zhang's proof is far too technical for this article, but several expositions of his work are available online. One which is very good may be found on Terence Tao's blog [4].

According to the experts, the best bound that can be obtained by such methods is 16. Hence the original **Twin Primes** conjecture is unlikely to be resolved very soon. However, the rapid progress from 70 million to 4680 is quite remarkable.

One interesting aspect of the better bounds is that the best bounds are obtained by using what is known as the 'Weil Conjectures', which were finally proved by the mathematician Pierre Deligne in the early 1970s. They are important theorems in Algebraic Geometry and at first glance far removed from Twin Primes! This shows the universality of Mathematics: seemingly unrelated questions can turn out to be closely related.

A related question but one which, assuming the conjecture is true, is a continuing exercise in futility, is to find the largest *known* twin prime pair. The current record is the following pair of numbers which have 200700 digits each:

$$3756801695685 \bullet 2^{666669} \pm 1.$$

Unlike Fermat's last theorem (Fermat famously wrote in the margin of a book that he had a proof of this theorem, but that the margin of the book was too small to write it), the origin of the twin primes conjecture is not so romantic. However, Zhang's story is quite romantic. Zhang entered graduate school at Purdue University in January 1985 and worked with T.T.Moh, an Algebraic Geometer. According to Moh he was hard working and intelligent but chose to work on a longstanding and as yet unresolved conjecture called the 'Jacobian conjecture'. Attempting to resolve a difficult conjecture while a graduate student is not quite the most pragmatic thing to do! In the current academic world, with a difficult and competitive job market, it is risky to attempt too difficult a task as one runs the risk of failure; and at an early stage of one's career, failure could end it.

After graduating with a thesis which made some progress in the direction of the Jacobian conjecture, Zhang struggled. He did not try to get in to the regular academic career path of post-doctoral work followed by a tenure track assistant professorship; he perhaps thought he would not be able to make it. Instead, he worked at a Subway sandwich shop for some time and ended up as a lecturer at the University of New Hampshire teaching several large basic mathematics classes. It appears, though, that the difficulties he underwent did not extinguish the 'fire in his belly'. He persevered, working on hard mathematical questions, and finally — after a few unsuccessful attempts — had a breakthrough which allowed him to be the first to make progress on the Bounded Gap conjecture.

Moh [1] writes: *When I looked into his eyes, I found a disturbing soul, a burning bush, an explorer who wanted to reach the north pole, a mountaineer who determined to scale Mt. Everest, and a traveler who would brave thunders and lightnings to reach his destination.*

A lesson one can learn from his story is to never give up on your dreams, to continue pursuing what makes you happy, regardless of what the rest of the world thinks. It is often said that mathematics is a young persons game, and that one's greatest work comes before 40; but that is perhaps a myth propagated by G.H. Hardy in 'A Mathematician's Apology'. Zhang, among others, shows that great things can be done after 40.

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Some problems for you to do

(1) Let p, $p+2$ be a pair of twin primes where $p > 5$. Show that $p+1$ is a multiple of 6. (For example, consider the twins 11,13; 17,19; 29,31).

(2) The two twin pairs 3,5 and 5,7 share a prime (namely, the prime 5). Show that there is no other instance where such a sharing happens.

(3) Show that the only pair of twin primes which have a power of 2 in-between them is 3,5.

(4) Let p, $p+2$ be a pair of twin primes where $p > 5$. Show that p is necessarily of one of the following three forms: 11 mod 30, 17 mod 30, 29 mod 30.

(5) It has been shown (by P A Clement) that if $n>1$, the two integers n and $n+2$ are both prime if and only if the number $4[(n-1)!+1]+n$ is a multiple of $n(n+2)$.

(a) Verify this statement for the twins 3,5 and 5,7.

(b) Verify this statement for the 'non-twins' 7,9. (That is, show that the divisibility condition is falsified.)

(c) Find a way of verifying the stated divisibility condition for the twin primes 17,19 without having to compute 18! (which is rather a large number).

(6) Wilson's theorem states that an integer $n > 1$ is prime if and only if the number $(n-1)!+1$ is divisible by n. You could apply it to the cases $n = 2,3,4,5$ and check that it works.

Show how Wilson's theorem leads to Clement's theorem quoted above.

feature

Learning from mistakes

Fermat Numbers

A false conjecture leading to fun and fascination

Interspersed with historical and biographical details, this article has rich nuggets of information. These don't just exercise a student's understanding of exponents, they also provide solvable proofs for school students. Best of all, the article weaves random results into a coherent whole, giving direction to ideas, conjectures and proofs.

B. A. SETHURAMAN

Introduction

The French mathematician Pierre de Fermat (1601–1665) was a veritable giant of number theory whose discoveries, and especially, whose conjectures and unproven assertions, kept mathematicians hard at work for several centuries that followed. Indeed, the first two issues of this magazine both featured his work: the first issue reviewed a book ([1]) on the history of what is known as "Fermat's Last Theorem," while the second issue, in an article on the four squares theorem ([2]), describes Fermat's work on primes that are representable as sums of two squares.

Besides these two well known contributions, Fermat is known for a whole host of other theorems in mathematics. He was a lawyer by training, but his passion was mathematics. He shone in arithmetic (which in its more advanced form is what we call number theory today), but made seminal contributions in other parts of mathematics as well, and even in physics.

Keywords: *Fermat, Fermat number, Fermat prime, infinity, regular polygons, constructibility*

Great mathematicians, and Fermat was squarely in that league, are characterized by deep intuition that enables them to see mathematical truths that others are not yet able to see. But great mathematicians are also human, and occasionally, they are wrong. Fermat himself was wrong on at least one mathematical matter: the issue of whether numbers of the form $2^{2^n}+1$ are prime. These numbers are the subject of this article.

Recall first the convention when interpreting numbers written with repeated exponents: $2^{2^n}+1$ is to be interpreted as $2^{(2^n)}+1$ (and *not* $(2^2)^n+1$). Let us write F_n for the number $2^{2^n}+1$, so that $F_0=2^{2^0}+1=2^1+1=3$, $F_1=5$, $F_2=17$, etc. Fermat claimed that the numbers F_n are prime for all integers $n=1,2,\ldots$. In fact, he first claimed to have a proof, but later discovered an error in it ([3, Foreward]). However, he appeared to still believe in the truth of his claim. It is thus fair to rename his claim as a conjecture.

Indeed, F_0, F_1, and F_2 above are clearly prime. So is $F_3=2^8+1=257$ and $F_4=2^{16}+1=65537$. *But there the list is broken!* Euler, who lived approximately a century after Fermat (1707–1783) showed that F_5, a ten-digit integer, is not prime: it is divisible by 641. Thus, Fermat's conjecture on the numbers $2^{2^n}+1$ was false!

But there is another characterization of great mathematicians that is relevant here—the very objects they think about turn out to be fascinating and deep, even if these mathematicians occasionally make false assertions about them! Such is indeed the case with numbers of the form $2^{2^n}+1$, now appropriately called *Fermat Numbers.* (Numbers of the form $2^{2^n}+1$ that are prime are now referred to as *Fermat primes.*) Fermat numbers have many charming properties, and have turned out to have intriguing connections to other parts of mathematics, as well as to computer science.

Identities and the infinitude of primes

Let us start with some pretty identities that Fermat numbers satisfy. Their proofs are fun exercises for high school students, involving nothing more than simple algebra and induction.

1. $F_n=(F_{n-1}-1)^2+1$, for $n\geq 1$.
2. $F_n=F_0\times F_1\times F_2\times\cdots\times F_{n-1}+2$, for $n\geq 1$.
3. $F_n=2^{2^{n-1}}\cdot F_0\cdot F_1\cdot F_2\cdot\cdots\cdot F_{n-2}+F_{n-1}$, for $n\geq 2$.
4. $F_n=F_{n-1}^2-2(F_{n-2}-1)^2$, for $n\geq 2$.

We will prove the first one here: Note that

$$2^{2^n}=2^{2^{n-1+1}}=2^{2^{n-1}\cdot 2}=(2^{2^{n-1}})^2=(F_{n-1}-1)^2.$$

Adding one everywhere, we find $F_n=2^{2^n}+1=(F_{n-1}-1)^2+1$, as desired.

There is an immediate consequence of the second identity above: the last digit of every Fermat number (for $n\geq 2$) must be 7. This is because for $n\geq 2$, we have

$$F_n=3\cdot 5\cdot F_2\cdot\cdots\cdot F_{n-1}+2=5(3\cdot F_2\cdot\cdots\cdot F_{n-1})+2.$$

So F_n is of the form 2 plus an odd multiple of 5 and hence has last digit 7. Pretty!

The second consequence is that the Fermat numbers are pairwise relatively prime; that is, for distinct non-negative integers i and j, $\gcd(F_i,F_j)=1$. This is attributed to Christian Goldbach (who is well known for a conjecture that is as yet unproven: Every even integer greater than 2 is expressible as a sum of two primes). As noted in a companion article in this issue, *There are Infinitely many Primes*, this property leads to another proof of the infinitude of primes.

Fermat numbers and constructibility of polygons

Recall the problems of constructibility handed to us by the Greeks: using only straight-edge and compass, construct line segments of specified lengths, and angles of specified measures. It was an open problem for a very long time, for instance, (i) whether one could trisect an arbitrary angle using straight-edge or compass, (ii) whether one could "square the circle," that is, construct the side of a square whose area is that of a given circle, and (iii) whether one could "double the cube," that is, construct the side of a cube whose volume is twice that of a given cube. These problems are easy to solve once one has at one's command techniques from Field Theory (known earlier as the 'Theory of Equations'); but this theory was not known to the Greeks. We now know that the answer all three questions is: No!

A specific problem in this context was the constructibility of regular n-gons for various values of n. Thus, a regular 3-gon is an equilateral triangle, a regular 4-gon is a square, a regular 5-gon is a regular pentagon, and so on. Whether a regular n-gon can be constructed using a straight-edge and compass quickly reduces to the question of whether the angle $360^\circ/n$ can be constructed using straight-edge and compass.

This problem was investigated by the Great Master, Carl Friedrich Gauss. (Gauss ranks among the greatest mathematicians ever, when measured not just by his own productivity but by the new areas of mathematics he initiated; his results to this date are a source of joy and wonder. His influence on mathematics and indeed all sciences ranks with that of Newton.) Gauss showed that the regular 17-gon is constructible (note that 17 is F_2), and went on to show that a regular n-gon ($n \geq 3$) can be constructed if the prime factorization of n is of the form $2^k p_1 p_2 \cdots p_l$, where the p_i are distinct primes of the form $2^{2^t} + 1$; Fermat numbers again! This is an instance of how questions about objects considered by great mathematicians (in this case Fermat) can turn out to have deep significance in mathematics, far from apparent at first. Thus, the question of whether for a given k the k^{th} Fermat number $2^{2^k} + 1$ is prime turns out to be more than just a curiosity: it is vitally connected to whether an n-gon can be constructed.

The reason why a regular n-gon with the stated prime factorization of n is constructible, lies in Field Theory. For the case where n is a prime—call it p instead—the theory tells us that the regular p-gon is constructible *if and only if* $p - 1$ is a power of 2. Thus, a regular p-gon is constructible if and only if $p = 2^k + 1$ for some integer k.

Now one can see quite easily that $2^k + 1$ cannot be prime unless k is itself a power of 2. For, suppose $k = 2^l b$ for some odd integer $b > 1$. Then

$$2^k + 1 = 2^{2^l b} + 1 = (2^{2^l})^b + 1.$$

Now it is a fact (it can be proven as a high school exercise) that $x^b + 1$ is divisible by $x + 1$ if b is odd. Hence if $b > 1$, $p = (2^{2^l})^b + 1$ would have the strictly smaller divisor $2^{2^l} + 1$, contradicting the fact that p is prime. Hence, the condition from Field Theory becomes: for prime $p \geq 3$, a regular p-gon is constructible if and only if p is a Fermat prime!

The condition for the constructibility of a regular n-gon for a general n follows from the condition just described for the case where n is prime, using standard reductions also furnished by Field Theory. Indeed, the condition for a general n-gon is also an *if and only if* statement: a regular n-gon is constructible if and only if n is of the form described by Gauss. Gauss proved the 'if' part of the condition, but his proof of the 'only if' part had a gap that was filled only later ([3, Chap. 16]).

Primality of the Fermat numbers

Let us turn to the original conjecture of Fermat, that the numbers $2^{2^n} + 1$ are prime for all $n = 0, 1, \ldots$. We know, thanks to Euler, that while F_0 through F_4 are prime, F_5 is not. For what other values of n is F_n known to be prime? The answer, more than three hundred and fifty years after Fermat made his first conjecture, is: None!

That does not mean that no F_n is prime for $n \geq 5$. All it means is that no one has as yet found a prime F_n for $n \geq 6$. What has been established are many results in the opposite direction (similar to the case of F_5): the numbers F_6 through F_{32} have all been shown to be composite ([4]). Besides these, F_n is known to be composite for other sporadic values of n, such as $n = 36,\ 71,\ 99,\ 517,\ 2059,\ 6390,\ 17748$, to select just a sample ([4]).

What makes determination of the primality of F_n so difficult is that, thanks to the presence of the double exponent, the number of digits in F_n grows very rapidly as n becomes large. In fact, Exercise (2) shows that the growth in the number of digits is exponential.

On the other hand, there is a very pretty result on the possible prime factors of F_n: Euler showed that any prime that divides F_n must be of the form $k \cdot 2^{n+1} + 1$, for some positive integer k. (The proof of this itself involves another famous theorem of Fermat known as Fermat's Little Theorem: for any prime p and any integer a, the number $a^p - a$ is divisible by p.) Euler's result was further sharpened by Lucas, who showed

Figure 1. Stamp commemorating the 400th birth anniversary of Fermat; perhaps one day there will be another stamp depicting the next Fermat prime after F_4? Source for image: [8] and [9].

that the k in Euler's result must be even. Thus we have Lucas's result that any prime divisor of F_n must be of the form $l \cdot 2^{n+2}+1$ for some positive integer l. This result is the basis of certain attempts at showing F_n is prime for various n: run through all possible integers of the form $l \cdot 2^{n+2}+1$ that are less than $\sqrt{F_n}$ and check if they divide F_n. Though easy to state, the computational power required to perform these calculations, even allowing for various tricks used to speed up the process, is stupendous for large n, because the numbers F_n are so large. There are distributed searches currently taking place over the internet: various groups of people fascinated by Fermat primes collectively divide the work among themselves by looking for divisors in restricted ranges of l (see [5]). Anybody with a computer and access to the internet can join these searches: we encourage the reader to do so too!

Further readings and exercises

We have only touched on some aspects of Fermat numbers: there are many more charming features of these numbers, and many more connections with other parts of mathematics and computer science that we have not described. A wonderful reference for Fermat numbers is [3] (note the pun in its title!). Although quite advanced for a high school student, it conveys the fun and the fascination of these numbers, and students will profit by simply thumbing through the book. We also recommend the Wikipedia article ([6]) for another overview of some features of these numbers, as well as the MacTutor ([7]) biography of Fermat.

We end with some more exercises that can be tackled by high school students.

Exercises

(1) Prove Identities (3) and (4) in Section . (Hint: Use Identities (1) and (2).)

(2) Show that the number of digits in F_n is approximately $\lfloor 2^n \log_{10}(2)+1 \rfloor$ (here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x).

(3) Show that for $n \geq 1$, F_n is of the form $6k-1$ for some integer k. (Hint: You may find Identity (2) in the text helpful.)

(4) Show that no F_n $(n \geq 2)$ is a sum of two primes. (Hint: If it were, then one prime would have to be 2.)

(5) Show that every F_n is the difference of two square integers. (Hint: Show that every odd integer is the difference of two squares.)

(6) Using Euler's theorem on the possible prime factors of F_n, show that no F_n is a perfect square. (Hint: assume that F_n is a perfect square. First show that if integers a and b both leave a remainder of 1 when divided by a certain integer m, then so does the integer ab. Now combine this results with Euler's description of the possible prime factors of F_n to describe $\sqrt{F_n}$.)

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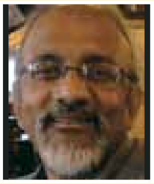

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The ubiquitous triangle

Harmonic Sequence and Pascal’s Triangle

An unexpected surprise!

We show in this note how, starting with the infinite harmonic sequence 1, 1/2, 1/3, 1/4, 1/5, 1/6, . . . , *a natural process yields the well-known Pascal triangle and, further, a curious procedure yields back the harmonic sequence. (‘Harmonic sequence’ is another name for the sequence of reciprocals of the positive integers.)*

B SURY

Start with the harmonic sequence arranged in a row of infinite length as

$$1,\quad \frac{1}{2},\quad \frac{1}{3},\quad \frac{1}{4},\quad \frac{1}{5},\quad \frac{1}{6},\quad \frac{1}{7},\quad \cdots.$$

Subtract each term from the previous term to get the sequence of first differences, with $1 - 1/2 = 1/2,\ 1/2 - 1/3 = 1/6$, and so on:

$$\frac{1}{2},\quad \frac{1}{6},\quad \frac{1}{12},\quad \frac{1}{20},\quad \frac{1}{30},\quad \frac{1}{42},\quad \cdots.$$

As we do for the Pascal triangle, write the terms of the second sequence one row below and in-between the terms of the first sequence:

$$\begin{array}{ccccccccccccc}
1 & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \frac{1}{6} & & \frac{1}{7} \\
& \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{42} &
\end{array}$$

Keywords: *Harmonic sequence, Pascal triangle*

Continue in this manner by taking successive differences to get an infinite array:

$$\begin{array}{ccccccccccccc}
1 & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \frac{1}{6} & & \frac{1}{7} \\
& \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{42} & \\
& & \frac{1}{3} & & \frac{1}{12} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{105} & & \\
& & & \frac{1}{4} & & \frac{1}{20} & & \frac{1}{60} & & \frac{1}{140} & & & \\
& & & & \cdots & & \cdots & & \cdots & & & &
\end{array}$$

Now turn the above array by 60° (clockwise) to form a triangular array:

$$\begin{array}{ccccccccc}
& & & & 1 & & & & \\
& & & \frac{1}{2} & & \frac{1}{2} & & & \\
& & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\
& \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\
\frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} \\
& \cdots & & \cdots & & \cdots & & \cdots &
\end{array}$$

Finally, divide each row by the first term in that row:

$$\begin{array}{ccccccccc}
& & & & 1 & & & & \\
& & & 1 & & 1 & & & \\
& & 1 & & \frac{1}{2} & & 1 & & \\
& 1 & & \frac{1}{3} & & \frac{1}{3} & & 1 & \\
1 & & \frac{1}{4} & & \frac{1}{6} & & \frac{1}{4} & & 1 \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

We have obtained a triangle of reciprocals of the Pascal numbers! We call this the *reciprocal Pascal triangle*.

Can we retrieve the harmonic sequence by some natural process? We can. Let us compute the *alternating sums* of the rows of the reciprocal Pascal triangle. Here's what we get: $1, 1-1=0, 1-1/2+1=3/2$, followed by these numbers:

$$\begin{aligned}
1-\tfrac{1}{3}+\tfrac{1}{3}-1 &= 0, \\
1-\tfrac{1}{4}+\tfrac{1}{6}-\tfrac{1}{4}+1 &= \tfrac{5}{3}, \\
1-\tfrac{1}{5}+\tfrac{1}{10}-\tfrac{1}{10}+\tfrac{1}{5}-1 &= 0, \\
1-\tfrac{1}{6}+\tfrac{1}{15}-\tfrac{1}{20}+\tfrac{1}{15}-\tfrac{1}{6}+1 &= \tfrac{7}{4},
\end{aligned}$$

and so on. Thus we get the sequence 1, 0, 3/2, 0, 5/3, 0, 7/4, 0, ...

Every second term is 0 (which is not surprising). Deleting these terms we are left with the sequence 1, 3/2, 5/3, 7/4, ... Subtracting each term from 2, we get 1, 1/2, 1/3, 1/4, ... We have got back the harmonic sequence!

Explanation. Inductively, it is easily seen that the r-th row of the array at the top of page 25

$$\frac{1}{r\binom{r}{r}},\quad \frac{1}{r\binom{r+1}{r}},\quad \frac{1}{r\binom{r+2}{r}},\quad \frac{1}{r\binom{r+3}{r}},\quad \cdots\cdots.$$

This follows from the following identity:

$$\frac{1}{r\binom{n}{r}}-\frac{1}{r\binom{n+1}{r}}=\frac{1}{(r+1)\binom{n+1}{r+1}}.$$

After the 60° turn we get array (2); its r-th row is:

$$\frac{1}{r\binom{r}{r}},\quad \frac{1}{(r-1)\binom{r}{r-1}},\quad \frac{1}{(r-2)\binom{r}{r-2}},\quad \frac{1}{(r-3)\binom{r}{r-3}},\quad \cdots,\quad \frac{1}{\binom{r}{1}}.$$

One may rewrite this row as

$$\frac{1}{r\binom{r-1}{r-1}},\quad \frac{1}{r\binom{r-1}{r-2}},\quad \frac{1}{r\binom{r-1}{r-3}},\quad \frac{1}{r\binom{r-1}{r-4}},\quad \cdots,\quad \frac{1}{r\binom{r-1}{0}}.$$

The triangle is thus

$$\begin{array}{c}
1\\[6pt]
\frac{1}{2\binom{1}{1}}\qquad\qquad \frac{1}{2\binom{1}{0}}\\[6pt]
\frac{1}{3\binom{2}{2}}\qquad\qquad \frac{1}{3\binom{2}{1}}\qquad\qquad \frac{1}{3\binom{2}{0}}\\[6pt]
\frac{1}{4\binom{3}{3}}\qquad\qquad \frac{1}{4\binom{3}{2}}\qquad\qquad \frac{1}{4\binom{3}{1}}\qquad\qquad \frac{1}{4\binom{3}{0}}\\[6pt]
\cdots\qquad\qquad \cdots\qquad\qquad \cdots\qquad\qquad \cdots\qquad\qquad \cdots\\[6pt]
\frac{1}{r\binom{r-1}{r-1}}\qquad\qquad \frac{1}{r\binom{r-1}{r-2}}\qquad\qquad \cdots\qquad\qquad \cdots\qquad\qquad \frac{1}{r\binom{r-1}{1}}\qquad\qquad \frac{1}{r\binom{r-1}{0}}
\end{array}$$

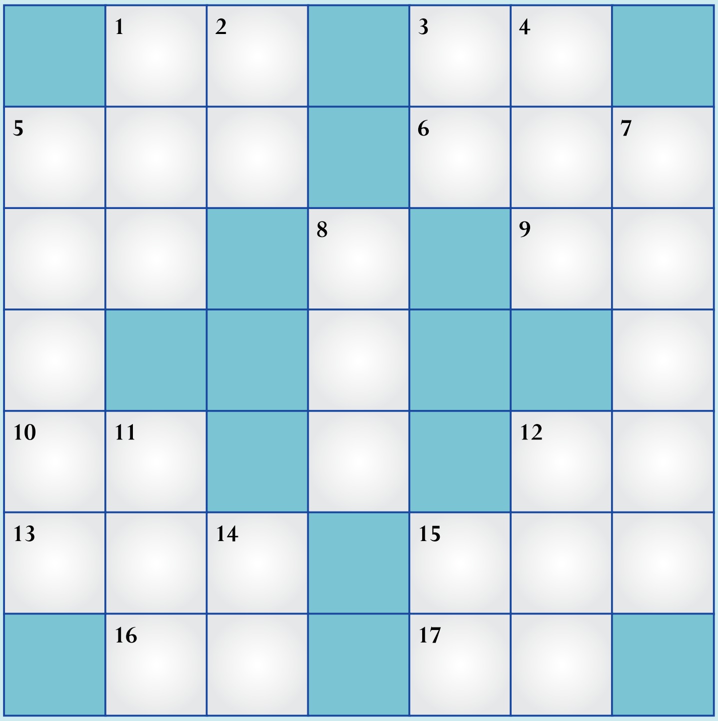
Dividing each row by the left-most entry (which is the same as the right-most entry), we get the reciprocal of the Pascal triangle, as claimed. Understanding why the alternating sums of the rows give back the harmonic numbers is more intricate, and we refer the reader to the article [1].

References

[1] B.Sury, Tianming Wang & Feng-Zhen Zhao, Identities involving reciprocals of binomial coefficients, Journal of Integer Sequences, Vol.7 (2004), Article 04.2.8.

B SURY got his Ph.D. in Mathematics from the Tata Institute of Fundamental Research in Bombay, under the supervision of M.S. Raghunathan, F.R.S. After 18 years in TIFR, he moved to the Indian Statistical Institute in Bangalore in 1999. He has been interested in expository writing at the school and college level and also, in interacting with mathematically talented students. He is the regional co-ordinator for the mathematical olympiad in Karnataka and a member of the editorial committee of the mathematics newsletter of the Ramanujan Mathematical Society, and also of the magazine Resonance. His research interests are in algebra and number theory. Mathematical limericks are an abiding interest. He may be contacted at sury@isibang.ac.in. His professional web page is www.isibang.ac.in/~sury.

number crossword-4



■	1	2	■	3	4	■
5			■	6		7
		■	8	■	9	
	■	■		■	■	
10	11	■		■	12	
13		14	■	15		
■	16		■	17		■

The design for this crossword's grid as well as several clues were given by Indira Bulhan of class 10 from Techno India Group Public School (Garia). Her interests are astronomy, guitar, dance, singing and drawing.

Down

(1) The reflex angle of 14 degrees

(2) The square of the hypotenuse of a triangle with sides 3 and 5

(3) The arithmetic mean of 2, 15 and 16

(4) 9 more than 1D x 2

(5) 9 less than 2×10^4

(7) 8! – 7! with the digits muddled up

(8) Product of the first four prime numbers

(11) 3A multiplied by the last digit of 15A

(12) Sum of the interior angles of a regular septagon

(14) 16A minus 9A

(15) 3 short of a half century

Clues Across :

(1) Product of first 3 odd numbers subtracted from product of first three even numbers

(3) Half of 2D

(5) A dozen dozens

(6) 3 more than 5 score

(9) Number of right angles in 12D

(10) The sum of the first 13 natural numbers

(12) Five times 10A written in reverse

(13) One of the exterior angles of an isosceles right angled triangle

(15) 3A multiplied by one sixth of 5A

(16) A natural number which is both a perfect square as well as a perfect cube.

(17) One third of 8D

Of Art & Math:

Introducing Ambigrams

Punya Mishra
Gaurav Bhatnagar

Mathematicians love puzzles—they love to play with numbers and shapes but often their love can turn to words and other areas that, at least on the surface, have little to do with mathematics. In this article we are going to focus on a very specific kind of artistic wordplay (and its relationship to mathematics) called *ambigrams*. The word ambigram was coined by cognitive scientist Douglas Hofstadter from 'ambi' which suggests *ambiguous* and 'gram' for *letter*. Ambigrams exploit *how* words are written and bring together the mathematics of symmetry, the elegance of typography *and* the psychology of visual perception to create surprising, artistic designs. Most of all, they are great fun!

All right, let's start with the example in Figure 1. Can you read it?

Figure 1. A 180-degree rotation ambigram for the word "Wordplay"

Keywords: *ambigrams, calligraphy, symmetry, perception, palindrome, mapping, transformation, reflection*

Rotating the page you are holding will reveal something interesting. The word stays the same! In other words, it has rotational symmetry.

Thus ambigrams are a way of writing words such that they can be read or interpreted in more than one way. Figure 2 is another one, an ambigram for the word "ambigram."

Figure 2. A 180-degree rotation ambigram for "ambigram"

Incidentally, you may have noticed something interesting in these two examples. In the "wordplay" design each letter of the first half of the word maps onto *one letter* (w to y, o to a, and so on). Some transformations are straightforward (as in the "d" becoming a "p") while others need some level of distortion to work visually (the w-y being the most obvious example). This distortion of course is constrained since whatever shape you come up with has to be readable as specific letters in two different orientations.

Now consider Figure 2, the design for the word "ambigram." There is a lot more distortion going on here. The "stroke" that emerges from the "a" becomes the third leg of the "m." More interesting is how the "m" after the "a" actually maps onto two letters ("r" and "a") when rotated. Isn't it interesting to see that what looks like *one* letter becomes *two* when rotated? On a different note, the g-b transformation is of particular interest to the authors! Can you guess why?

Given that ambigrams work because of the specific mappings of letters (either individually or in groups) to each other implies, that even one change in the letters of the word can lead to a very different design. Thus the solution for the word "ambigrams" (plural) is quite different from the solution for "ambigram" (singular). Note how in Figure 3, many of the mappings have shifted, and the natural "g-b" transformation that made so much sense in the design for "ambigram" has now shifted to a "b-a" transformation while "g" now maps onto itself.

Figure 3.The first of two ambigrams, for "ambigrams." This design reads the same when rotated 180-degrees.

Another important aspect of *why* ambigrams work can be seen in Figure 3. Notice the initial "A" and the final "S." In the case of the "A" the gap at the bottom looks exactly like what it is, a gap. On the other hand, when rotated 180 degrees, our mind imagines a connection across this gap – to make the topmost stroke of the "S." How cool is that!

Rotation is not the only way one can create ambigrams. Figure 4 is another design for the word "ambigrams" –this time as a reflection. This design has bi-lateral symmetry (a symmetry most often found in living things – such as faces, leaves and butterflies). If you place a mirror—perpendicular to the page—in the middle of the 'g', the right half of the design will reflect to become the left part of the word.

Figure 4. Another ambigram for "ambigrams" this time with bilateral symmetry

Designs such as these read the same from right to left. This is a feature of a Palindrome. A palindrome is a word or a sentence that reads the same forwards and backwards. For example, some believe that the first sentence ever spoken was:

Madam, I'm Adam

Notably the response to this palindrome was also a single word palindrome:

Eve

Even longer examples of palindromes can be created. Here is, for instance, a palindromic poem.

I think
a lot
about
very little
and care
very little
about
a lot
I think

Reverse the sequence of lines (from bottom to top) and you will have the same poem! Here the palindrome is at the level of a line of the poem. The first line is "I think" and so is the last line. Similarly, the second line from both top and bottom is "a lot". The poem is symmetric about the phrase "and care" which comes in the middle of the poem. The symmetry is very similar to the mirror symmetry mentioned earlier, though not quite the same.

Limiting ourselves to just mirror symmetry, we can find many examples of its relevance to mathematics. For example, consider an isosceles triangle, a triangle with two sides equal. It has the same symmetry as the design above.

It is possible to prove that the base angles of an isosceles triangle are equal, just by exploiting this mirror symmetry? Here is a hint:

An Isosceles Triangle
Looks in the mirror
And finds itself un-reversed.
"My base angles are equal"
it says.

Visually this can be represented as a triangle-ambigram for the word "isosceles", see Figure 5.

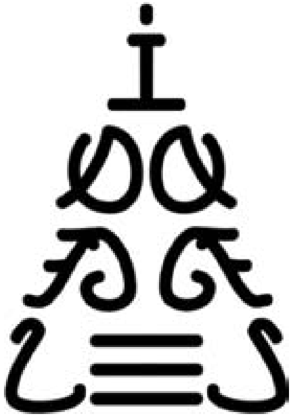


Figure 5. An isosceles triangle that reads "isosceles" when reflected in a mirror

Different types of ambigrams

Every ambigram design need not read the *same* word when rotated and/or reflected. Figure 6 is a design that reads "darpan" (the Hindi word for mirror), and "mirror" (the English word for darpan) when rotated 180 degrees.

Figure 6.The word "darpan" (hindi for mirror) becomes "mirror" on rotation by 180-degrees

So far we have seen ambigrams with a vertical line of symmetry like the designs for "ambigrams" or "isosceles" having a vertical line of symmetry. Hofstadter has called this a "wall reflection." The other is a "lake reflection" such as the example in Figure 7 – where the word "abhikalpa" (the Sanskrit word for architect) which has a horizontal line of symmetry. Mathematically speaking, a wall-reflection is a reflection across the "y-axis" while a "lake-reflection" is a reflection across the "x-axis."

Figure 7. Ambigram for "abhikalpa", an example of a lake reflection

Incidentally, the use of Hindi words in the above two designs brings up an interesting challenge. Is it possible to create an ambigram that can be read in two different languages? Here is the Sanskrit sound “Om” as traditionally written in Devanagri script. This design if rotated 90-degrees magically transforms into the letters “Om” in English!

Figure 8. The Sanskrit word “om”

Figure 9. The English “om” formed by rotating the Sanskrit “om” by 90-degrees.

Not all reflection ambigrams have to be reflected across the x- or y-axes. Consider this design (Figure 10), where the word “right” when reflected across the 45-degree axis reads “angle.” (This design was inspired by a solution first put forth by Bryce Herdt).

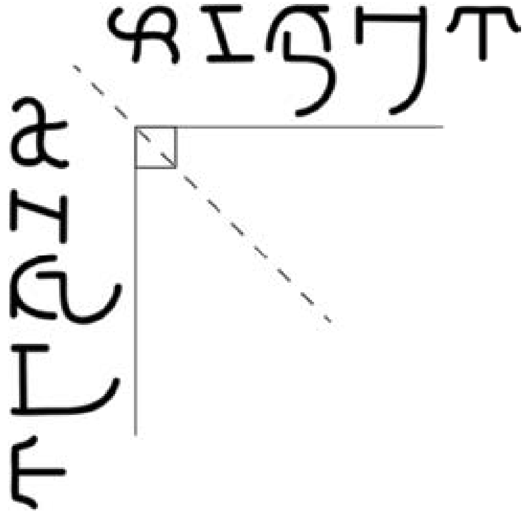


Figure 10. A special “Right angle” made specially for this special magazine

Those who are familiar with tessellations will like the next kind of designs—space-filling ambigrams. See for instance Figure 11, this design for the word “space” – where replications of the word form a network that cover a surface – in this case the surface of a sphere.

Figure 11. A space-filling ambigram for “space”

Here is an example of a rotational chain ambigram for the word “mathematics.” In chain-ambigrams a word is broken into two parts – each of which maps to itself. In Figure 12 “math” maps onto itself and the rest of the word “ematics” maps onto itself.

Figure 12. An ambigram for “mathematics”

Effective chain-ambigrams can be quite rich in meaning. Consider Figure 13. This example of a chain ambigram for “action-re-action” where the letters “-re-“ switch loyalty depending on whether you are reading the top part of the circle or the bottom.

Figure 13. Ambigram for “Action-re-action”

Given this idea of breaking words into shorter ambigramable pieces, it is easy to create such chain-reflection ambigrams as well—such as Figure 14 for the word “reflect.” This design will read the same when you hold it up against a mirror (or peer at it from the other side of the page holding it up to a light).

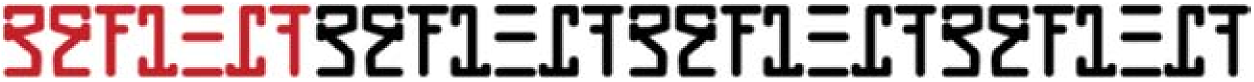


Figure 14. A chain-reflection ambigram for “reflect”

A couple of other types of ambigrams are called “figure-ground” ambigrams and “triplets.” A figure-ground ambigram is akin to a tessellation – where the space between the letters of a word can be read as another word altogether. What do you see in Figure 15? Good? Evil? Can you see both? Can you see both at the same time? A good pun-ya?

Figure 15. A Figure-Ground ambigram for “Good” and “Evil”

Mathematicians who love solid geometry will love triplets! A triplet is 3-dimensional shape designed in such a way that it casts different shadows depending on where you shine light on it. For instance the design below (Figure 16) is a shape that allows you to see the letters “A,” “B” and “C” depending on where you shine light on it.

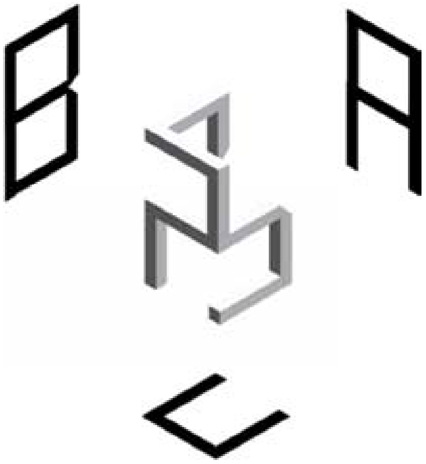


Figure 16. A triplet ambigram for “A”,“B” and“C”

Even seeing patterns in parts of a word can lead to interesting designs, such as the star-shaped design in Figure 17 for the word Astronomy. In designs like this one takes advantage of specific letters to create visually attractive designs. The designer in this case noted that the letter “R” could be rotated 60-degrees to make the letter “N.”

Figure 17. A star shaped ambigram for “astronomy”

Aesthetics, ambigrams & mathematics

Some mathematicians speak of what they do in aesthetic terms. The famous mathematician George Polya remarked: “Beauty in mathematics is seeing the truth without effort.” This mirrors

Keat's famous line "Beauty is truth, truth beauty." As Bertrand Russell said, "Mathematics, rightly viewed, possesses not only truth, but supreme beauty." Figure 18 attempts to capture this idea.

Figure 18. A design for "truth & beauty", where Beauty becomes Truth & Truth becomes Beauty.

When mathematicians speak of beauty they usually talk of theorems or proofs that are elegant, surprising, or parsimonious. They speak of "deep" theorems. Mathematical insights that are not obvious, but explained properly seem inevitable. Finally mathematicians delight in doing mathematics, which often means solving problems set by themselves or by other mathematicians.

Effective ambigram designers, in small ways, see the creating of ambigrams as sharing many of these characteristics that mathematicians speak of. The creation of ambigrams can be a highly engaging activity that can lead to seemingly inevitable and yet surprising and elegant solutions. In that sense, both mathematicans and ambigram-artists engage in what we have called "Deep Play" (DP) – a creative, open-ended engagement with ideas through manipulating abstract symbols. We must admit, however, that our teachers have often considered what we do as being more TP (Time Pass) than DP (Deep Play!). We hope we have been able to give you some of the flavor of the art and mathematics of ambigrams. In subsequent articles we will delve deeper into the mathematical aspects of these typographical designs, and use them to communicate mathematical ideas such as symmetry, paradoxes, limits, infinities and much more.

About the authors

PUNYA MISHRA, when not creating ambigrams, is professor of educational technology at Michigan State University. GAURAV BHATNAGAR, when not teaching or doing mathematics, is Senior Vice-President at Educomp Solutions Ltd.

Loving both math and art, Punya's and Gaurav's collaboration began over 30 years ago when they were students in high-school. Since then, they have individually or collectively, subjected their friends, family, classmates, and students to a never ending stream of bad jokes, puns, nonsense verse and other forms of deep play. To their eternal puzzlement, their talents have not always been appreciated by their teachers (or other authority figures). Punya's email address is punya@msu.edu and his website is at http://punyamishra.com. Gaurav's email address is bhatnagarg@gmail.com and his website is at http://gbhatnagar.com

All the ambigrams presented in this article are original designs created by Punya Mishra (unless otherwise specified). Please contact him if you need to use them in your own work.

You should note, dear reader, that Punya and Gaurav have hidden a secret message in this article. If you can work out what it is, or if you have any input, thoughts, comments, original ambigram designs to share, please drop them a note at their e-mail IDs.

The Gregorian Calendar

An absorbing story that brings together history, religion, natural science and mathematics

A Ramachandran

The calendar is a common device that regulates our day to day (even week to week and month to month) activities. The system of tracking days and assigning dates to them that is widely used now is called the Gregorian calendar, after Pope Gregory XIII, who held office from 1572 to 1585.

A calendar basically aims to integrate the cycle of earth's rotation around itself with the cycle of its revolution around the sun. The former phenomenon gives rise to the 'day' and the latter to the 'year.' Now a year is not an integral number of times as long as a day. The whole history of calendar making is the story of attempts to reconcile these two. (A calendar also tries to integrate a third phenomenon – the movement of the moon around the earth, which defines the month. This makes things a lot more complicated and so we shall not bring this in now.)

Keywords: *history, religion, natural science, Gregorian, Julian, calendar, year, leap year*

A year is 365.2422 days long, correct to 4 decimal places. The immediate response to this situation would be to approximate this figure to 365.25 days in a year. As it would be awkward to have a quarter day added to 365 days to complete a year we could just have an extra day once in 4 years – a leap year. Though the earth's revolution around the sun and our reckoning of the year would get a bit out of phase every year, the situation would get more or less restored every 4 years. This is the basis of the *Julian calendar* which served much of Europe from 325 AD to the mid-sixteenth century.

The length of the year is actually a bit less than 365.25 days. Though the difference is small, over the centuries it accumulated to a noticeable extent. By the sixteenth century the actual revolutionary movement of the earth and the reckoning of the Julian year had fallen out of phase by 10 days. At this stage one may ask, 'So what?' To answer this question we consider the observed effects of the earth's revolution. This (combined with the tilt of the earth's rotation axis) gives rise to the seasons and associated phenomena such as temperature variation, variation in length of daylight, angle of sun's rays at a place, the equinoxes and solstices. These affect natural events such as rainfall patterns, germination and flowering in plants, breeding and migration in animals, etc. Besides, some religious and cultural events are tied to these phenomena. If our calendar year is not in phase with the earth's revolution, then there will be no correlation between a natural event and its date in the calendar. For instance, the date of the spring equinox will not be a constant. In fact this observation is what precipitated action on the part of Pope Gregory.

When the Julian calendar was initiated, the spring equinox occurred on 21 March. But in the following twelve centuries it moved gradually to 11 March. (Convince yourself that when the calendar year is longer than necessary, a natural event like the spring equinox moves backward in the calendar.) The festival of Easter is linked to the spring equinox (in a rather complicated way; we shall not go into it) and if nothing was done, the day of Easter would shift widely with the passage of decades. Conversely, Christmas, celebrated on a fixed date, 25 December, would move away from its seasonal moorings (winter in the Northern hemisphere) and take place in other seasons. This situation was not pleasing to the Catholic Church; so, taking inputs from astronomers and mathematicians, Pope Gregory ordered a reform to the calendar.

Two tasks needed to be done at this stage to arrest the trend. Firstly, the average length of the year had to be adjusted to be closer to reality. Secondly, the accumulated phase difference of 10 days had to be erased.

Taking up the former issue, it is apparent that the average length of the year in the Julian calendar is longer than necessary. This could be rectified by converting some leap years to ordinary years. The suggestion that came up was that 3 leap years could be dropped in a period of four hundred years; that is, instead of 100 leap years in 400 years we would have only 97. The average length of the year would then be 365+ (97/400) = 365.2425 days. The difference between this and the actual year length (correct to 4 decimal places) of 365.2422 days is 0.0003 days which is an error of less than one part in a million.

A further suggestion was that the years that reverted from leap years to normal years would be the century years except those that were divisible by 400. So we now have a block of four hundred years as the repeating unit of the calendar. The condition for a leap year could be stated as "If it is divisible by 4 it is a leap year; if it is also divisible by 100 it isn't; but if it is also divisible by 400 then it is."

Surprisingly, this block of 400 years is an exact number of weeks. (Convince yourself that this is true.) So any two dates differing by exactly 400 years would fall on the same weekday. Indeed, 400 is the least value of k for which the following statement can be made: For any x, the calendar of year x is the same as that for the year $x + nk$ for any integer n. But as only one cycle of 400 years has transpired after the Gregorian calendar came into effect, we cannot use this formula retrospectively.

Let's move to the other issue, namely, erasing the 10 day phase difference that had built up by the middle of the sixteenth century. To restore the spring equinox to its original date of 21 March, 10 days had to be dropped from the calendar. It was decided to declare that the day following 4 October 1582 would be 15 October 1582, a really bold step to take in the face of much public apprehension. This was effected smoothly in Catholic countries but other countries initially resisted the move. Britain and the British Empire made the transition in 1752, having to drop 11 days, as the discrepancy had grown. Russia switched to the new system in 1918, having to drop 13 days. The last European nation to adopt the Gregorian calendar was Greece, in 1923. Eventually the Gregorian system was almost universally accepted. But for a time different European countries followed different calendars and this led to some confusion regarding historical dates.

Some countries decided to make the transition in a phased manner, dropping a leap year here and there, but this led to further confusion – in the interim period they were out of phase with both calendars.

When the jump from 4 October to 15 October was made in 1582, the weekly cycle was not disturbed. 4 October fell on Thursday, and 15 October was taken to be a Friday. An unintended consequence of this was that the first day of this millennium, 1 January 2001, was a Monday.

An alternative to the Gregorian system would be to drop 4 leap years in 5 centuries, which would give an average year length of 365 + (121/500) = 365.242 days. Though this would be slightly closer to the true value (it would be a negative error), a 500-year cycle does not seem to have as much appeal as a 400-year cycle. (The number four seems to be a recurring theme in calendar circles.) Also, a 500-year cycle would not be an exact number of weeks. So it looks as though the legacy of a medieval Pope will be with us for a long time to come.

A RAMACHANDRAN has had a long standing interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for over two decades, and now stays in Chennai. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

in the classroom

How To Prove It

Starting with this issue we will run a regular column on the art and science of proof, and in honour of George Pólya's book, 'How To Solve It', we have named it "How To Prove It." There is of course no single way to prove things in mathematics. But there are many general ideas and strategies that do help, and that's what this column is about.

SHAILESH A SHIRALI

Formal proof is one of the striking features of mathematics. You do not find this feature in any of the sciences. What you do meet in the sciences would be more accurately described as 'verification'. You may for example perform an experiment in the laboratory to verify the formula $t = 2\pi\sqrt{l/g}$ for the time period of oscillation of a pendulum. What do you do? You set up the apparatus and take a lot of readings, then draw a graph or two and check how close are your results to the prediction. At the end you say, 'The formula has been verified to be true within experimental error' or something like that. This is done routinely in the sciences. It is important to see that *this is not the same as proof in mathematics.*

In a proof what you are attempting to do is to build a logical bridge from one set of statements (or suppositions) to another statement, using intermediate steps that are small and of a kind which no one would dispute. The jump from the initial statement to the final one may seem large, but when broken down to a sequence of small steps it does not appear so. The logic used in mathematics is

Keywords: *Polya, formal proof, number patterns, algebra, pattern, sequence*

actually no different from that used in ordinary life (though it may seem different, especially when expressed using symbols and formal mathematical language); indeed, daily life is the source of all logical methods. You could say, in fact, that much of mathematical logic is plain and simple 'kitchen logic'!

It is believed by many that at the school level proof is encountered mainly in the realm of geometry; and that geometry is the only platform available for teaching proof. Both these statements are false. *Proof lies at the heart of mathematics, in every single branch*. At the school level, one resource that is heavily underutilized with regard to the teaching of proof is *Number Patterns and Algebra*. In this column we shall demonstrate many principles of proof using themes from number theory (which at this level is mainly applied algebra). Of course, we shall consider themes from geometry too.

It is equally a fallacy to imagine that proof can be introduced only when students are in their upper primary classes or in high school. Formal written proof, yes; symbolic proof, yes; but informal and clearly articulated, verbalized reasoning can and should be introduced much earlier — indeed, in the lower primary years. We shall elaborate on this theme in subsequent columns.

An example from algebra

In the first 'episode' of this serial we study an example from number theory:

Show that the square of any odd number leaves remainder 1 when divided by 8.

We experiment with some numbers to get a sense of the task: $1^2 = 0 \times 8 + 1$, $3^2 = 9 = 1 \times 8 + 1$, $5^2 = 25 = 3 \times 8 + 1$, $7^2 = 49 = 6 \times 8 + 1$, $9^2 = 81 = 10 \times 8 + 1$, $11^2 = 121 = 15 \times 8 + 1$, $13^2 = 169 = 21 \times 8 + 1$, ... We see that the claim has worked for the odd squares from 1^2 till 13^2. Is this enough evidence to conclude that the pattern will always be true?

Not quite! As we said earlier, empirical evidence is suggestive of the truth of a proposition — but that's all. In number theory there are numerous instances of statements which fail despite the evidence in their favour being very strong. A well known example of this is Euler's prime-generating function $n^2 + n + 41$, which yields prime values for 40 consecutive values of n (namely, $n = 0, 1, 2, 3, \ldots, 39$; we get the primes 41, 43, 47, . . . , 1447, 1523, 1601), and just as we are beginning to be certain that the expression will always yield a prime, the formula disappoints us: the pattern breaks, with $n = 40$ yielding a composite number. (It is easy to check that $n = 40$ does yield a composite number, for $40^2 + 40 + 41$ is clearly a multiple of 41. Indeed, it equals 41^2.)

So if we want actual proof then we have to produce something that will stand up in the 'mathematical court' before the toughest lawyer, who will be looking for ways to dash your arguments to bits. Here are some approaches which should satisfy such a lawyer.

First proof. What is an odd number? Clearly, one that leaves remainder 1 when it is divided by 2. This means that an odd number A is of the form $2 \times$ an integer $+ 1$, i.e., $A = 2n + 1$ where n is a positive integer. Let us see what happens when we square this expression:

$$A^2 = (2n + 1)^2 = 4n^2 + 4n + 1.$$

We see readily that A^2 is of the form $4 \times$ (some integer)+1. That is, A^2 leaves remainder 1 when divided by 4. While this comes close, it is not good enough: we need division by 8, not by 4. What do we do now?

Let's look more closely. We see that $A^2 = 4n\,(n + 1) + 1$. If only we can show that $n\,(n + 1)$ is an even number, then our task will be done, for the number $4n\,(n + 1)$ will then be twice a multiple of 4, and therefore a multiple of 8.

But $n\,(n + 1)$ is even; for, it is the product of two consecutive numbers, of which one clearly must be even. So our job is done!

Second proof. This approach may appear a bit strange at first but is perfectly valid. The idea comes from the fact that the problem has to do with division by 8, so it seems natural to check if there is some underlying pattern which repeats each time n increases by 8. So we consider the expression: $(n + 8)^2 - n^2$. We have:

$(n + 8)^2 - n^2 = (n^2 + 16n + 64) - n^2 = 16n + 64 = 8 (2n + 8)$.

We see clearly that the last quantity is a multiple of 8. *So when n increases by* 8, *the remainder in the division $n^2 \div 8$ stays unchanged.*

It follows that if the given statement is true for the odd squares 1^2, 3^2, 5^2 and 7^2, then it will necessarily be true for 9^2, 11^2, 13^2 and 15^2; and therefore it will necessarily be true for 17^2, 19^2, 21^2 and 23^2; and so on, indefinitely. But the statement is indeed true for 1^2, 3^2, 5^2 and 7^2, as is easily checked. Therefore it is true for the square of every odd number!

Remark. This proof can be hugely improved once we notice that we do not need to consider integers separated by a gap of 8. In fact, since we are studying the squares only of odd numbers, a gap of 2 is good enough! For, if we consider any two consecutive odd numbers, say $2n - 1$ and $2n + 1$, the difference between their squares is

$(2n + 1)^2 - (2n - 1)^2 = (2n - 1 + 2n + 1) \times 2 = 4n \times 2 = 8n$,

which is a multiple of 8. So if the hypothesis is true for the first odd square (namely: 1^2), which it clearly is, then it will be true for every subsequent odd square. Hence proved!

Third proof. Just for variety we give a third proof. It is based on the fact that the sum of the first n odd numbers is n^2. For example, $1 + 3 = 4 = 2^2$ and $1 + 3 + 5 = 9 = 3^2$. So to show that $(2n - 1)^2$ is 1 more than a multiple of 8, we must show that the sum of the first $2n - 1$ odd numbers is 1 more than a multiple of 8.

Now we observe the following simple pattern in the sequence of odd numbers: the sums 3 + 5, 7 + 9, 11 + 13, 15 + 17, . . . are all *multiples of* 8. It is easy to see why this must be so; for, 3 + 5 = 8, and in advancing from 3 + 5 to 7 + 9 we increase the sum by 4 + 4 = 8. Likewise, in advancing from 7 + 9 to 11 + 13 we increase the sum by 4 + 4 = 8. As the sums increase by 8 each time, and we start off at a multiple of 8, the sum will always be a multiple of 8.

The statement now proves itself; for, in the sum of the first $2n - 1$ odd numbers, we can pair the last two odd numbers, then the two odd numbers just before that pair, and so on, down to {3, 5}. The sum of each pair is a multiple of 8, and the remaining number, 1, ensures that the sum is 1 more than amultiple of 8. The following depicts a typical situation:

$9^2 = 1 + \underbrace{3 + 5} + \underbrace{7 + 9} + \underbrace{11 + 13} + \underbrace{15 + 17}$.

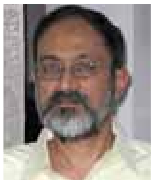
Closing remarks. We quote Professor Gila Hanna, from [1]:

The recognition that proofs can convey new mathematical techniques effectively, and thus should be treated as important bearers of mathematical knowledge, is a fertile point of view that mathematics educators seem to have overlooked to a large extent. Adopting this approach to proof in the classroom does not challenge in any way the accepted "Euclidean" definition of a mathematical proof (as a finite sequence of formulae in a given system, where each formula of the sequence is either an axiom of the system or is derived from preceding formulae by rules of inference of the system), nor does it challenge the teaching of proof as a Euclidean derivation. It is rather an acknowledgement that the teaching of proof has the potential to further students' mathematical knowledge in other ways. It offers an opportunity to make new connections between the process of proving and mathematical techniques, and also gives us an additional reason for keeping proof in the mathematics curriculum.

References

[1] Gila Hanna, *Proof can teach you new methods*, http://www.unige.ch/math/EnsMath/ Rome2008/WG1/Papers/ HANNA.pdf

[2] David Reid, *Understanding proof and transforming teaching*, http://www.pmena.org/2011/ presentations/ PMENA_2011_Reid.pdf



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A poem on the prime number theorem

The prime numbers are mysterious because they have the two `opposing' properties: there are arbitrarily large gaps in between them and they satisfy no simple formula, while simultaneously their distribution is regular in the sense of the famous prime number theorem. This theorem can be informally stated as saying that the probability of a number n being prime is 1/log(n). This can be poetically worded as:

Numbers in their prime --
for no reason or rhyme,
show up at a rhythm
with probability 1/logarithm.
If this is a law they knew,
they also break quite a few
but then, that is not a crime!

-- B Sury

Teacher's Diary on Classroom Assessment

Sindhu Sreedevi
Joyita Banerjee
Sneha Titus

The moment we hear the word 'assessment' it makes us conscious of being put under a scanner, and gives us a feeling that we will be compared with our peers. It highlights our weaknesses instead of our strengths. A child in the school environment goes through similar feelings on hearing about assessment. If the academic calendar of a school is observed, the most anxious moments for the teacher and children are during examinations. This defeats the purpose of assessment. There are various purposes for doing an assessment and its benefits are manifold. It enables the teacher to identify whether students have any learning gaps and then modify his/her instructional strategies. Though as a teacher I wanted to use assessment to improve overall learning, it is taking me time to learn how to assess without inducing stress and fear. With CCE becoming mandatory, I resolved not to reduce the activity to mere tallying and book-keeping but to deeply integrate assessment with everyday classroom activities and to use it to plan my next steps. There is enough body of research evidences across the world to show that continuous assessment leads to drastic improvement in students' learning levels.

Keywords: *formative assessment, CCE, mathematical skills, abstraction, diagnostic, mensuration*

In short, assessment is not a program which is merely done ‘to’ the child; rather, as a teacher we need to consider it as a process we are doing for the child to facilitate his/her learning (National Council of Teachers of Mathematics). To put this into practice, I see the classroom practices in three stages. At the beginning of a concept, I assess the students’ understanding of previous concepts. While teaching a concept I use various assessment techniques to check the progress of the students. I also believe that it is very important to provide appropriate and timely feedback to the child while assessing him/her on a continuous basis. This helps the child to identify his/her key strengths and possible areas of improvement. This does not mean that term-end or year-end assessments are of no significance. It is also important to conduct assessment at the end of a chapter, term or year. It provides evidence of achievement to parents, students themselves and to school authorities. Hence assessment must be looked at in a comprehensive manner. While framing the questions I need to focus on all areas which will equip the child to apply the knowledge in different situations and also develop problem solving skills and mathematical communication.

Looking at the NCERT syllabus (std. VI – VIII) at the beginning of the year, I decided that as a mathematics teacher of std. VIII, I would identify an overarching skill which I wanted my students to develop in preparation for Std. IX mathematics. I zoomed in on the readiness to move from concrete to abstract as I recognized that the lack of this skill caused many students to abandon mathematics at the higher level. The sub-skills were also extracted from the syllabus, and these were the ability to:

- Use and understand mathematical language including symbols
- Generalize from specific results
- Apply logical thinking
- Appreciate the notion of proof

My first task on starting each unit was to design an entry level test to measure the extent of the student’s knowledge and skills in a particular topic. The responses would lead me to determine whether he/she needed a review in the topic or he/she was ready for greater challenges. This test would assess the student’s mastery of the content standards that are building blocks for the next topic. It would enable me to place the student at an appropriate starting point. It would help me find answers for three questions:

(1) Is the student equipped with the pre-requisite knowledge required for acquiring the concept?

(2) How ready is the student to move from concrete to abstract?

(3) Does the student have prior knowledge about content to be covered in the coming year?

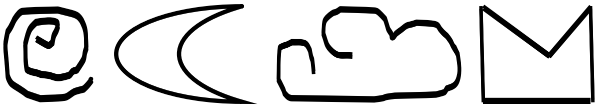
For this I needed to closely examine the particular topic and create a test that would help me diagnose all of the above. I decided on grouping questions into four categories. For example for the test for the topic ‘Mensuration’ in grade 8, I proceeded as follows:

Category 1	Concept of area and perimeter
Category 2	Derivation of simple formulae for rectangles using logical steps and then generalizing it to shapes like triangles and parallelograms and combinations of these
Category 3	Application in daily life example, with problem solving
Category 4	Establishing the relationship between area and perimeter

After this classification I designed a few questions for each category and used them to analyze the student’s responses.

Category 1:

(1) For which of the following given shapes can we find the area and perimeter? Justify your answer.



(2) Help Radha to plan 'Project Lawn'. The dimensions of the rectangular plot of land are 5 m and 10 m. If she wants to buy grass seedlings for the lawn, what would we have to find out and what will the unit be for it?

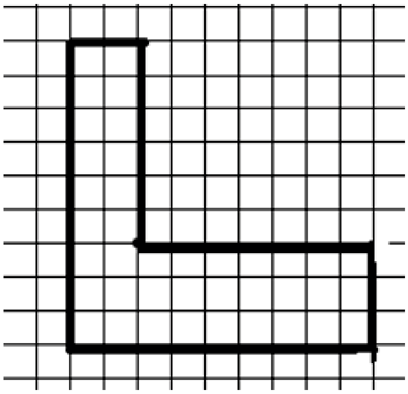
a. Area in meters

b. Area in square meters.

c. Perimeter in meters

d. Perimeter in square meters

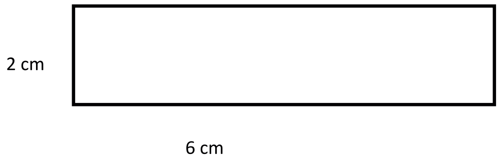
(3) a) Find the area and perimeter of the figure shown below. Each square in the grid has a side of unit length.



b) If Ram has a similar L-shaped flower bed, in his garden then how will you find the area of the flower bed?

Category 2:

(1) Choose the correct calculation to find the area of the given picture .



a. 6 *cm* × 2 *cm* × 6 *cm*

b. 6 *cm* × 2 *cm*

c. 6 *cm* + 2 *cm*

d. 6 *cm* + 2 *cm* + 6 *cm* + 2 *cm*

(2) To find the area of a triangle which of the following formula will you consider?

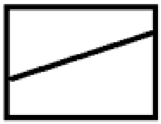
e. $\frac{1}{2}$ × *length* × *breadth*

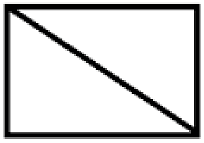
f. *Length* × *height*

g. $\frac{1}{2}$ × *length* × *height*

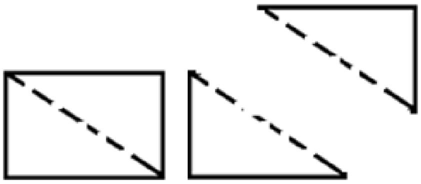
h. $\frac{1}{2}$ × *base* × *height*

(3) a) How will you find the area of each of these figures?

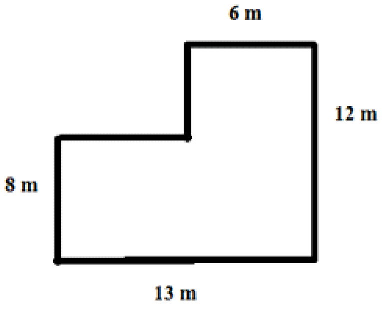




b) If the rectangular shape given below is cut along the broken line, as shown and the parts are separated, then will the sum of the perimeters of the parts be same as the perimeter of the whole rectangle?

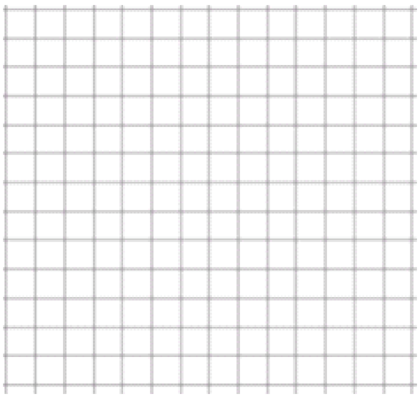


(4)

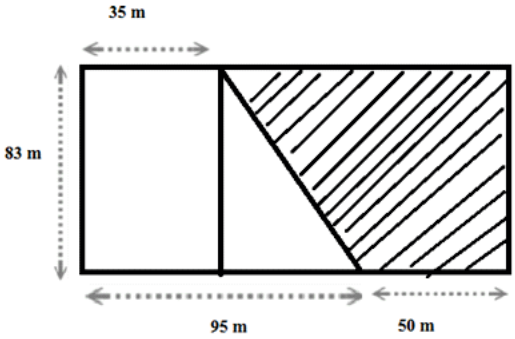


What is the perimeter and area of the above figure?

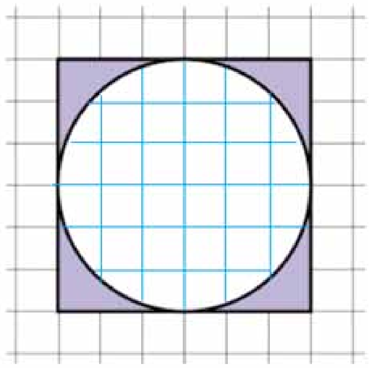
(5) The area of each square in the grid is 1 square unit. Draw a right angled triangle with area 10 square units.



(6) Find the area of the shaded portion of the following figure:



(7) In the figure given below, what is the area of the shaded portion? Each square of the grid is of area 1 cm^2.

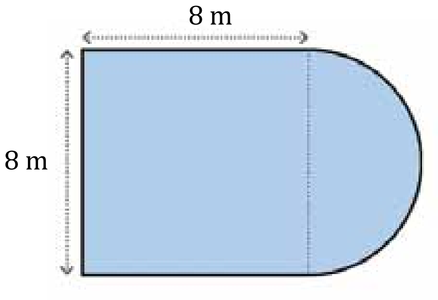


Category 3:

(1) A park is circular in shape. It has a jogging track all around it with width 3.5 m. If the inner radius of the park is 7 m, find the cost of cementing the path at the rate of Rs 15 per square meter.

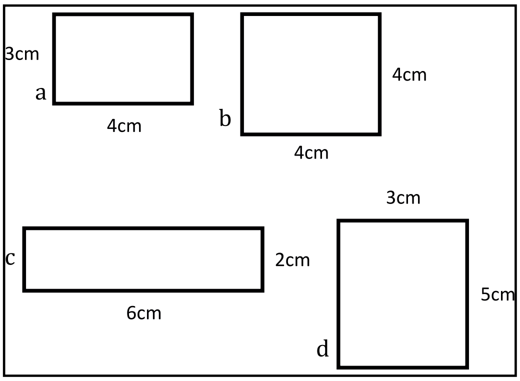
(2) Rama's study room is rectangular in shape with dimension 10 m x 8 m. She wants to tile her floor with tiles of 24 cm by 24 cm. How many such tiles will be required by her to tile the floor?

(3) A swimming pool is in the shape of a combined square and a half circle as shown in the figure below. The half circle is the shallow part of the pool where children are allowed and the square is the deep part where only adults are allowed. What fraction of the area of the pool are the children allowed to swim in?



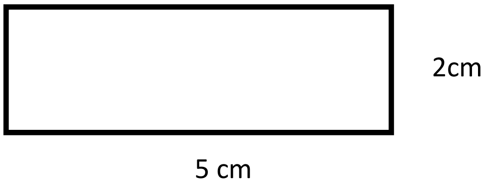
Category 4:

(1) Which of the following figures has area 12 square centimeter and perimeter 16 cm?



(2) A wire is bent into a square shape of length 5 cm. If the same wire is bent to form a triangle, what will be the perimeter and the area of the triangle formed?

(3) The perimeter of this rectangle is 14 cm, and its area is 10 square cm.



a. Draw a diagram of a rectangle with the same perimeter, but a larger area. Write down the area of your rectangle.

b. Draw a diagram of a rectangle with the same perimeter, but a smaller area. Write down the area of your rectangle.

(4) The perimeter of a rectangle is 22 cm, and its area is 24 square cm. Is it possible to draw a rectangle with the same area but a larger perimeter?

(5) a) Draw a circle of diameter 5 cm using a compass. What is the area of a square with the same perimeter as the circumference of the circle?

b) Draw two different shapes where the numerical value of the perimeter of one is the same as the numerical value of the area of the other.

Closing comment:

The children who were not successful in categories 2, 3 and 4 give me indicators for designing a bridge course for the content of grade VII. For children who did not cover category 1, I must design a remedial class for the concept of area and perimeter. For children able to solve category 4, I can think of more challenging strategies for assessment.

In preparation to develop the overarching skill and the sub-skills identified by me, I would pay particular attention to the student's interpretation of the diagrams and the word problems as well as the ability to move from the specific to the general by the comfort level with using mensuration formulae. The few open ended questions used would help me test their logical skills. Calling on students to defend their reasoning for such questions would help them develop an idea of the reason for proof.

References

[1] NCERT text books

[2] http://www.mathmammoth.com/

[3] National Council of Teachers of Mathematics (NCTM). Principles and Standards for School Mathematics.

The CCE column is the product of the Azim Premji University Resource Centre. The team members who are working on it are SINDHU SREEDEVI, JOYITA BANERJEE and SNEHA TITUS.
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Sneha Titus is Associate Editor of At Right Angles and Assistant Professor and Mathematics resource person at the University Resource Centre.

An 'Origamics' Activity: X-lines

ORIGAMICS: Activities based on exploration, conjecture and proof
by Kazuo Haga

Shiv Gaur

Dr. Kazuo Haga is a retired professor of biology at the University of Tsukuba, Japan. During his career as a biology professor, while waiting for his experiments to progress, he used to while away the time doing paper-folding and noting his mathematical findings through these paper-folding sessions.

He devised a set of activities and classified them under the name 'Origamics' (coined by him) as the end product was different from Origami. Unlike Origami, his exercises don't produce paper models but rather they lead to the study of the effects of the folding and seek patterns.

Haga's Origamic activities require students to explore simple, geometric properties found when we fold paper in prescribed ways.

The aim of these activities is to give students easy-to-explore paper-folding puzzles so that they can experience a micro-version of the three stages of mathematical research: exploration, conjecture and proof.

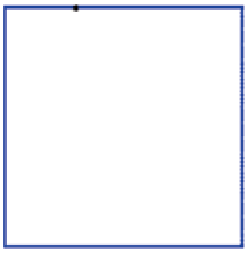
Here we look at one such activity from the chapter "X-Lines with lots of Surprises".

Keywords: *Kazuo Haga, origamics, paper folding, exploration, conjecture, proof, dynamic geometry, Geogebra*

Observe the following procedure:

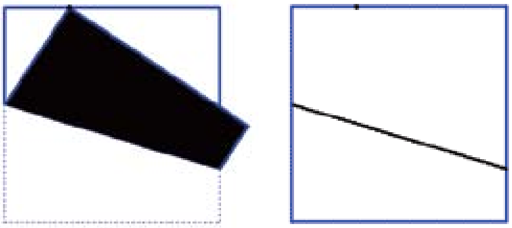
Step 01:

Take an arbitrary point on the upper edge of a square sheet of paper.



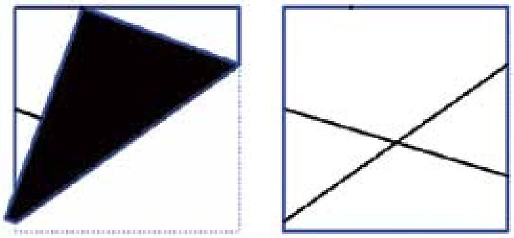
Step 02:

Place the lower left vertex onto the arbitrary point and unfold. We obtain one creased line.



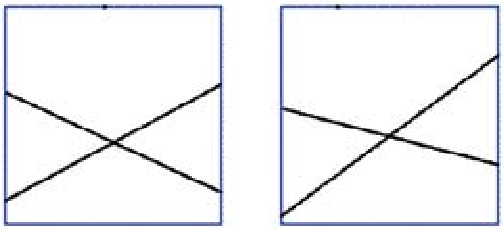
Step 03:

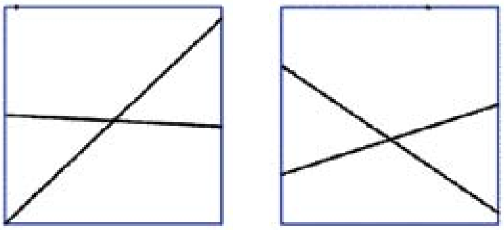
Place the lower right vertex onto the same point and unfold. We obtain two X-shaped creases. We shall call the pair of creases obtained as X-creases.



Repeat this procedure on different pieces of paper with different starting points.

Different X-creases will be obtained by different starting points, and therefore the position of the point of intersection may vary.

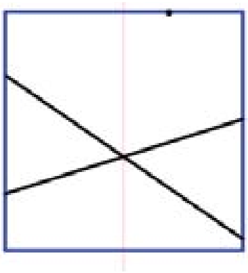




Now take one piece of paper. Make a vertical book fold to obtain the vertical midline of the square. Do likewise with your other X-creases.

What do you observe from your various X-creases?

It seems that regardless of the starting point, the intersection falls on the midline!



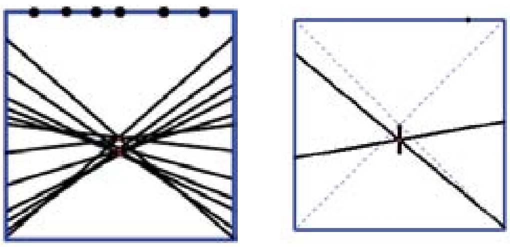
We now state our first observation:

The points of intersection of the X-creases fall on the vertical midline.

Pile up the pieces of paper which you used to make X-creases, and hold the pile up to the light.

What's your observation?

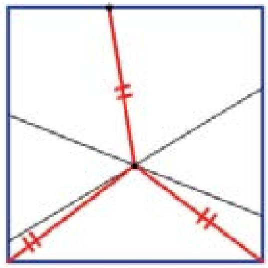
You see that the points of intersection seem to vary up and down along the midline, although within a small range.



We state our second observation:

The points of intersection of the X-creases lie along the midline and lie below the centre of the square within a certain range.

Next select a starting point and the corresponding X-creases. Draw a line from the intersection point to the starting point. Also draw lines from the intersection point to the lower vertices of the square. Then fold along an X-crease and hold the paper to the light. It comes out that two of the spokes coincide. Repeat with the other X-crease. It appears that the third spoke also has the same length!



We state our third observation:

The distances from the point of intersection to the starting point and to each of the lower vertices are equal.

We'll leave it as an exercise for the reader to prove the first and the third observation. Proofs will be provided in the next issue of At Right Angles.

Please note you can use any Dynamic Geometry Software such as Geogebra to simulate the above mentioned paper folding exercise.

Reference

[1] ORIGAMICS: Mathematical Explorations through Paper Folding, Kazuo Haga (World Scientific Publishing Co. Pvt. Ltd)

A B.Ed. and MBA degree holder, SHIV GAUR worked in the corporate sector for 5 years and then took up teaching at the Sahyadri School (KFI). He has been teaching Math for 13 years, and is currently teaching the IGCSE and IB Math curriculum at The Gandhi Memorial International School, Jakarta. He is deeply interested in the use of technology (Dynamic Geometry Software, Computer Algebra System) for teaching Math. His article "Origami and Mathematics" was published in the book "Ideas for the Classroom" in 2007 by East West Books (Madras) Pvt. Ltd. He was an invited guest speaker at IIT Bombay for TIME 2009 and TIME Primary 2012. Shiv is an amateur magician and a modular origami enthusiast. He may be contacted at shivgaur@gmail.com.

Math Club

Exploration of Recurring Decimals

PADMAPRIYA SHIRALI

We first expose children to the topic of converting fractions to decimal numbers in class 5 or 6. At that point children notice that some fractions terminate and some do not, and they come across terms like *terminating decimals* and *recurring decimals*. They are also shown the usage of the bar or dot notation. Generally most textbooks do not proceed beyond this point. Later (class 8 or 9) they are taught how to rationalize numbers. The activity I describe here is one which I have tried with class 8 children. It proved to be an interesting investigation into the patterns in recurring decimals leading to generalization and looking at the reverse process initially through a trial and error approach followed by arriving at the procedure for rationalization.

I first posed a question to the children, "Will all fractions give rise to either terminating decimals or recurring decimals of some periodicity"? They were not too certain. Some confidently said yes. I asked in return "Can you prove why they should either terminate or recur with some periodicity?" They were not yet exposed to formal proof. So though they knew the answer intuitively, they found it

Keywords: *recurring, terminating, decimals, fractions, pattern, repetend*

difficult to articulate it. So I asked them further questions: “What remainders can you get when you divide a number by 5?” They responded, “0, 1, 2, 3, 4”. I asked the same question for other divisors, and within a short while they saw that if the divisor is n, then there are just n possible remainders (including zero), and once the same remainder appears again, the quotient pattern repeats itself from that point.

I then set them the task of finding the decimal expansions for all unit fractions (i.e., fractions of the type $1/n$ where n is a positive integer) with denominators from 2 till 100. The class had twenty students, and each one computed the value of five such fractions within an hour. In the case of fractions whose decimals terminated, they had to write the complete answer. In the case of fractions whose decimals recurred, they had to stop at the point where the digits began to recur for the second time. However I asked them to omit fractions for which repetition had not happened by the tenth decimal place. (Later, I provided the computer generated result.) As they did this, some began to notice some interesting patterns in their answers.

We then collated all the fractions on a chart classifying them into the following groups. Fractions which terminated were grouped together; then fractions with period 1 (i.e., where the repeating portion or *repetend* has just one digit) were grouped together; then came fractions with period 2, period 3, period 4, etc. Here are the summarized results.

Fractions with terminating decimals. The unit fractions with terminating decimals were:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{8}, \frac{1}{10}, \frac{1}{16}, \frac{1}{20}, \frac{1}{25}, \frac{1}{32}, \frac{1}{40}, \frac{1}{50}, \frac{1}{64}, \frac{1}{80}, \frac{1}{100}.$$

Notice the denominators:

$$2,\ 4,\ 5,\ 8,\ 10,\ 16,\ 25,\ 32,\ 40,\ 50,\ 64,\ 80,\ 100.$$

Children noticed that the list contains all the powers of 2 (i.e., 2, 4, 8, 16, 32, 64) and another set 5, 10, 20, 25, 40, 50, 80 which could be rewritten as $5, 5 \times 2, 5 \times 4, 5 \times 8, 25 \times 2, 5 \times 16, 5^2 \times 4$. After some discussion they generalized the result by stating that the denominators are of the form 2^n, or 2×5^n, or 5×2^n, with n belonging to the set of positive integers N. Each denominator listed above is of one of these forms.

Generalizing further, we may say that whenever the denominator has the form $2^a \times 5^b$ where a and b are non-negative integers, the decimal expansion terminates.

Fractions where the repetend has one digit. Here are the fractions for which the repetend has just one digit:

$$\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}, \frac{1}{18}, \frac{1}{24}, \frac{1}{30}, \frac{1}{36}, \frac{1}{45}, \frac{1}{48}, \frac{1}{60}, \frac{1}{72}, \frac{1}{75}, \frac{1}{90}, \frac{1}{96}.$$

Children quickly noticed that the denominators are not consecutive and have gaps emerging after the initial set of numbers. They factorized the denominators as $3 \times 1, 3 \times 2, 3 \times 3, 3 \times 4, 3 \times 5, 3 \times 6, 3 \times 8, 3 \times 10, 3\times12, 3 \times 15, 3 \times 16, 3 \times 20, 3 \times 24, 3 \times 25, 3 \times 30, 3 \times 32$.

These numbers could now be sorted as a set consisting of 3 times powers of 2 ($3 \times 1, 3 \times 2, 3 \times 4, 3 \times 8, 3 \times 16, 3 \times 32$), another set consisting of 3 times multiples of 2 and 5 ($3 \times 5, 3 \times 10, 3 \times 15, 3 \times 20, 3 \times 25, 3 \times 30$) and a third set consisting of 3^2 times powers of 2 (3×3, 3×6 or $3 \times 3 \times 2$, 3×12 or $3^2 \times 2^2$, 3×24 which is $3^2 \times 2^3$); more generally, fractions in which the denominator has the form $3 \times 2^n, 3^2 \times 2^n, 3 \times 5^n$. Generalizing, we may say that all fractions where the denominator has the form $3 \times 2^a \times 5^b$ and $3^2 \times 2^a \times 5^b$ give rise to decimal numbers with period 1. Each denominator listed above is of one of these forms.

Fractions where the repetend has two digits. Fractions which resulted in a decimal with two repeating digits (i.e., period 2) were:

$$\frac{1}{11}, \frac{1}{22}, \frac{1}{33}, \frac{1}{44}, \frac{1}{55}, \frac{1}{66}, \frac{1}{88}, \frac{1}{99}.$$

When we first saw the list we concluded that the denominators were all the multiples of 11, but then we noticed that 77 is not in this list. This initially came as a surprise. We could see why it was not so only later when we studied fractions with period 6.

Fractions where the repetend has three digits. Fractions which resulted in a decimal with three repeating digits (i.e., period 3) were:

$$\frac{1}{27}, \frac{1}{37}, \frac{1}{54}, \frac{1}{74}.$$

We noted that the denominators were multiples of 27 and 37; however, 81 was 'missing'.

Fractions where the repetend has four digits. Fractions which resulted in a decimal with four repeating digits (i.e., period 4) were . . . : none! A question which naturally crossed our minds was: Is this true only for the first 100 unit fractions, or will this always be the case? And can we prove it either way?

Fractions where the repetend has five digits. Fractions which resulted in a decimal with five repeating digits (i.e., period 5) were:

$$\frac{1}{41}, \frac{1}{82}.$$

This was the third time we noticed a multiple of the first denominator appearing in the list, and it provoked us to look for an explanation for this.

Fractions where the repetend has six digits. Fractions which resulted in a decimal with six repeating digits (i.e., period 6) were many in number:

$$\frac{1}{7}, \frac{1}{13}, \frac{1}{14}, \frac{1}{21}, \frac{1}{26}, \frac{1}{28}, \frac{1}{35}, \frac{1}{39}, \frac{1}{42}, \frac{1}{52}, \frac{1}{56}, \frac{1}{63}, \frac{1}{65}, \frac{1}{70}, \frac{1}{77}, \frac{1}{78}, \frac{1}{84}, \frac{1}{91}.$$

Multiples of 7 and 13 could be seen in the denominators, but it was interesting to see that 49 or 7×7 does not appear in the list. We asked ourselves why this should be so.

Fractions where the repetend has seven digits. There were no such fractions.

Fractions where the repetend has eight digits. There was just one fraction which resulted in a decimal with eight repeating digits (i.e., period 8): $\frac{1}{73}$.

Fractions where the repetend has nine digits. There was just one fraction which resulted in a decimal with nine repeating digits (i.e., period 9): $\frac{1}{81}$.

Fractions where the repetend has ten digits. There were no such fractions.

Finding the fraction matching a given recurring decimal

Following the above exercise I wanted them to look at the process in reverse. I posed the question “Given a decimal number how do we convert it into a fraction?” Terminating decimals do not pose any difficulty as they can be written as whole numbers divided by suitable powers of 10 and then reduced (e.g., 0.034 = 34 / 1000 = 17 / 500). So the question got narrowed down to: what does one do with recurring decimals?

We used a trial-and-error approach. We took the recurring decimal .027027027. . .. What fraction will give this decimal? The children first looked at 1 / 2 (= .5) and 1 / 4 (= .25) and realized that the denominator had to be bigger than 4. Then they tried with 1 / 10 (= .1) and 1 / 20 (= .05) and realized that the denominator had to be bigger than 20 as well. They now tried with 1/30 (= .0333. . .) and 1 / 40 (= .025) and realized that the fraction lies between 1 / 30 and 1 / 40. Then they tried 1 / 35 (= .02814. . .) and further narrowed down the range to 1 / 35 to 1 / 40. Soon they obtained the result that 1 / 37 = .027027. . ..

Now they had to be introduced to a more systematic procedure for ‘rationalization’. We looked at the result obtained by multiplying a decimal number with powers of 10. We followed this by looking at various recurring decimals to determine by what power of 10 they had to be multiplied in order to move the recurring part to the whole number position. (For example, .027027. . . needs to be multiplied by 1000 to change it to 27.027027. . ., whose repeating portion is identical to that of the original number). Then I asked them what operation could be performed using this and the original number that would result in the elimination of the recurring part. Soon they saw that subtraction of the original number from the new one would yield the answer 27. Now I introduced them to the procedure of treating the original number as x and subtracting x from $1000x$ to give $999x$ which must equal 27. From this we deduce that $x = 27 / 999$, which when reduced yields 1 / 37.

This whole exercise took us three classes (some calculations were set as homework). We found many interesting patterns; for example, 1 / 81 = 0.012345679 012345679 . . ., and we see that 8 is missing from the repetend. It also raised many questions, not all of which we could answer. For example, why did we not find any fractions which give rise to decimals with period 4, 7 or 10 in the unit fractions from 1 / 2 to 1 / 100? Does this continue to hold if we extend our range? And is there a way by which we can prove this conclusively?

Maybe some day we will find answers to some of these questions! [Editor’s note. We shall take up some of these questions in the next issue of *At Right Angles*.]

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A Trapezium Theorem Generalized

> *"... the deductive method starting from seemingly dogmatic axioms provides a shortcut for covering large territory. But the constructive Socratic method that proceeds from the particular to the general and eschews dogmatic compulsion leads the way more surely to independent productive thinking."* – Richard Courant (1964, p. 43).

MICHAEL DE VILLIERS

A nice investigation with dynamic geometry for students at high school is the so-called *Midpoint Trapezium theorem*, which appears in many popular geometry textbooks such as Serra (2008, pp. 276-277). It can be stated as follows: 'Given any trapezium *ABCD* with *AD // BC*, and if *E* and *F* are the respective midpoints of *AB* and *CD*, then *EF* is parallel to the other two sides *AD* and *BC*, and equal to half of their sum.'

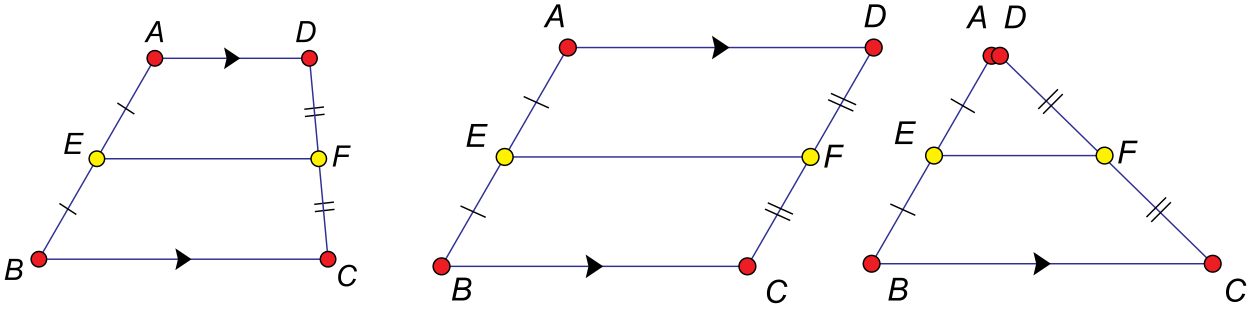

Figure 1

Keywords: *Trapezium (trapezoid), half-turn, parallelogram, triangle inequality, translation, midpoint triangle theorem.*

With the aid of dynamic geometry, children can easily demonstrate this as indicated in Figure 1; and by dragging D they can show that the theorem applies to a parallelogram as a special case as well as to a triangle in the degenerate case, giving us the Midpoint Triangle theorem as another special case.

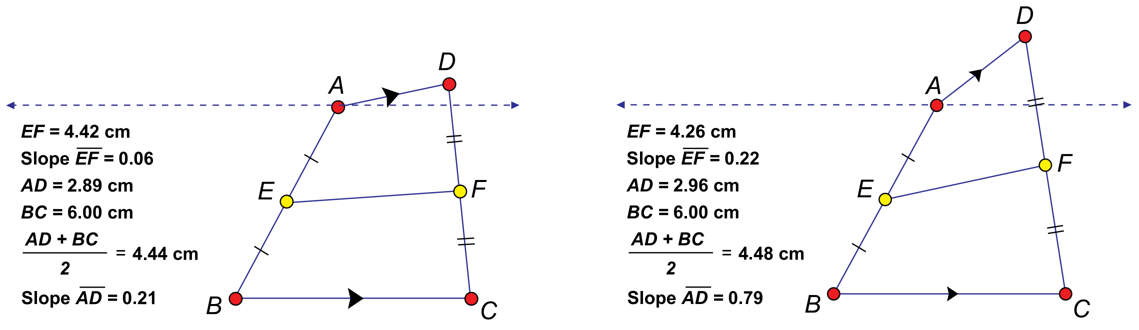

Figure 2

Conjecturing

It seems natural to ask: What happens if $ABCD$ is not a trapezium? Instead of having to construct a new quadrilateral, and redo all the measurements and calculations, a useful feature of *Sketchpad*, not present in most other dynamic geometry programmes, is that one can simply select the command "Split point D from the line" through A parallel to BC, to obtain Figure 2 for a general quadrilateral $ABCD$.

Apart from EF obviously being no longer parallel to the other two sides, it is apparent and easily checked by repeated experimentation that EF is less than or equal to half the sum of AD and BC, with equality just when $AD // BC$. So by asking a simple question and then exploring it with dynamic geometry, we've obtained a nice, new conjecture (which as far as I've been able to ascertain so far seems to be new, or at the very least, not well-known).

Before going on, readers are now invited to first explore and convince themselves of the truth of this conjecture by using a dynamic, interactive sketch online, by dragging any of the vertices of $ABCD$ at this URL: http://dynamicmathematicslearning.com/trapezium-theorem-generalized.html

Explaining

But **why** is the result true? Note that I'm NOT asking whether it is true, as one can easily obtain sufficient conviction through experimentation to answer that question. What is lacking in this case is *insight* and *understanding*: in other words, not a proof to verify it, but a proof to *logically explain it* (compare Hersh, 1993; De Villiers, 1997).

As pointed out by Pólya (1946) and others, a useful strategy in problem solving is often to look at the special case first, as that may give insight into why the general result is true (and thus help one to construct a deductive argument). So how can we explain (prove) the original trapezium result stated at the beginning?

A simple transformation approach might come to mind as follows. Give the trapezium $ABCD$ a half-turn around the midpoint F as shown in Figure 3. From the properties of a half-turn it immediately follows that the formed quadrilateral $ABA'B'$ is a parallelogram, for which the result is intuitively and visually obvious. To formally prove this and understand why that is the case for the parallelogram, note that A and B can both be translated by the same vector AB' to respectively map onto B' and A'; but the same translation by vector AB' maps midpoint E onto midpoint E' since a translation preserves distance, direction and segment-length; hence EF is parallel to the other two sides and $2EF = (AD + DB') \Rightarrow EF = (AD + BC)/2$.

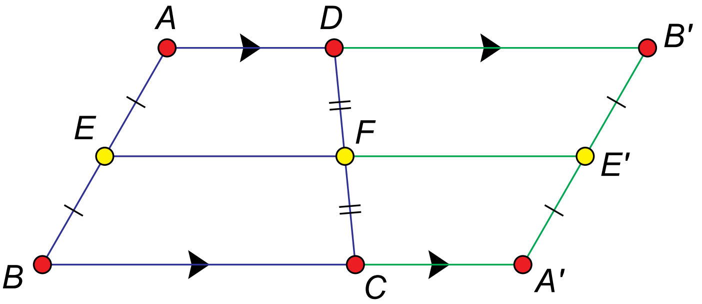

Figure 3

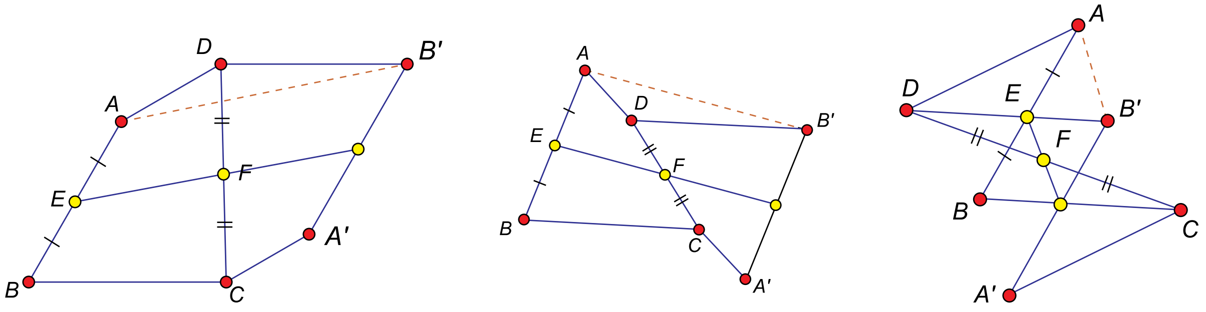

Figure 4

But what happens when $ABCD$ is not a trapezium, but a general convex, concave or crossed quadrilateral? As shown in Figure 4, a half-turn around F produces a parallelo-hexagon (a hexagon with opposite sides equal and parallel), and therefore as before $ABA'B'$ is still a parallelogram. Therefore, $EF = AB' / 2$. But from the triangle inequality $AB' < AD + DB' = AD + BC$, and this concludes the proof.

So the general inequality $EF \leq (AD + BC) / 2$ is merely a straightforward consequence of the triangle inequality, and nicely explains why the result is true. The traditional Euclidean proof for the Midpoint Trapezium theorem relies on the midpoint triangle theorem by drawing a diagonal AC with its midpoint G as shown in the first sketch in Figure 5 (compare Kay, 1994: 220; Alexander & Koeberlein, 2007: 209). But as shown for the convex case when $ABCD$ is not a trapezium in Figure 5, the general inequality again easily follows from the triangle inequality. It is left to the reader to fill in the details and to check the concave and crossed cases as well.

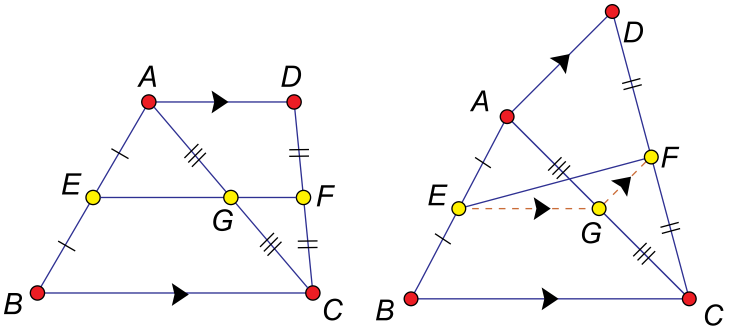

Figure 5

Conclusion

This extension of the Midpoint Trapezium theorem can be used to engage students using dynamic geometry in observing and making an interesting further conjecture, well within their means to logically explain (prove) with the triangle inequality theorem.

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Adventures in Problem Solving

A Tale of Two Formulas

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Starting with this issue, we will run a new column in the Problem Section, titled 'Adventures in Problem Solving'; it replaces the 'Fun Problems' column we have had till now. In each column we will study a few problems that typify some theme.

Among the most familiar and humble formulas we meet in high school algebra are the 'completing the square' formula and the 'difference of two squares' factorization. In this note we showcase some unexpected and pleasing uses of these formulas.

Completing the square

Given the expression $a^2 + 2ab$ we can 'complete the square' by adding the missing last term b^2; we get $a^2 + 2ab + b^2 = (a + b)^2$. In Figure 1 we depict this method visually, a depiction we feel should be made better known to students, for its impact as well as its historical connection.

Similarly, if we are given the expression $a^2 + b^2$ we can complete the square by adding the missing middle term which is $2ab$; once again we get $(a + b)^2$. This way of completing the square is much less well known and probably not used to its potential.

Example: Given the expression $n^2 + 10n$, by adding $(10/2)^2 = 25$ to it we get the perfect square: $n^2 + 10n + 25 = (n + 5)^2$; this is what has been depicted in Figure 1. Or, given the expression $n^2 + 25$ we can add $10n$ to it and obtain the same perfect square.

Here is an example showing how 'completing the square' is used at the higher secondary level to solve an integral. Consider the indefinite integral of $1/(x^2 + 10x + 29)$; we have:

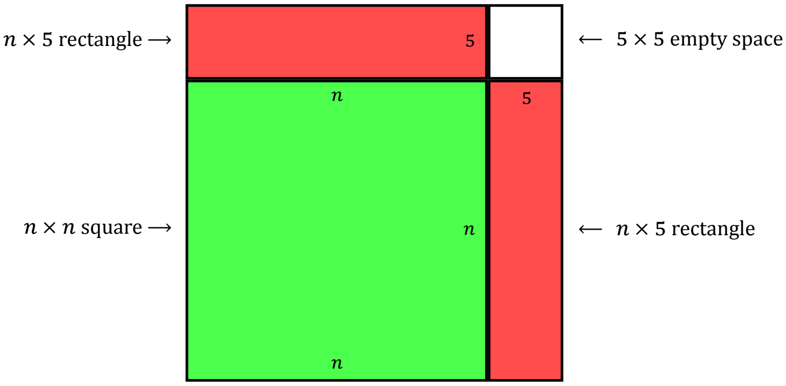
$$\int \frac{dx}{x^2+10x+29} = \int \frac{dx}{(x^2+10x+25)+4}$$

$$= \int \frac{dx}{(x+5)^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x+5}{2}\right) + \text{constant.}$$

Difference of two squares

This refers to the identity best remembered in the following form: $a^2 - b^2 = (a - b)(a + b)$. Examples of factorization using the $a^2 - b^2$ formula are (no doubt) familiar to the reader. Here is one involving numbers: $899 = 30^2 - 1 = 29 \times 31$.

There is a lovely geometric interpretation of the act of completing the square which goes all the way back to al Khwarizmi. We illustrate it with the expression $n^2 + 10n$. Here n^2 corresponds to the area of a $n \times n$ square, and $10n$ corresponds to the total area of **two** $n \times 5$ rectangles. The three shapes can be fitted together as shown below.



Note the empty space in the upper right-hand corner. Its dimensions are clearly 5×5. By filling this space, we 'complete the square' . This is the geometric equivalent of adding 5^2 to $n^2 + 10n$ to get $(n+5)^2$.

Figure 1. *A visual way of 'completing the square'*

And here is an example of its usage in factorizing quadratics. Given the second-degree expression $n^2 + 10n + 16$ we complete the square associated with $n^2 + 10n$, i.e., we add 5^2 to it and get $(n+5)^2$, and then proceed as follows:

$$n^2 + 10n + 16 = (n+5)^2 - 9 = (n+5)^2 - 3^2 = (n+5+3)(n+5-3) = (n+8)(n+2).$$

In this note we study a few problems that show the two formulas working in unison. We hope to convince you that these simple and unassuming formulas have great power, and we should never underestimate them.

Problems

Problem 1

Find all integers $n > 0$ such that $n^2 + 12n$ is a perfect square.

Solution

Let us first 'complete the square' with the given expression, $n^2 + 12n$. By adding $(12/2)^2 = 36$ we get: $n^2 + 12n + 36 = (n+6)^2$.

Suppose that $n^2 + 12n$ is a perfect square. Let it be denoted by x^2. From what was noted above, $n^2 + 12n + 36 = (n+6)^2$ too is a perfect square. Let it be denoted by y^2. Here x and y are two positive integers, with $y > x$, and their squares are connected by the relation $y^2 - x^2 = 36$. *So 36 has been expressed as a difference of two squares.*

In what ways can 36 be written as a difference of two squares? We find all the ways by calling upon our second friend, the difference of two squares formula! Since a difference of two squares is also a product of two terms, let us first list all the ways of writing 36 as a product of two distinct positive integers. There are several ways: 1×36, 2×18, 3×12, 4×9.

Since $y^2 - x^2 = 36$ we get $(y-x)(y+x) = 36$; thus, $y - x$, $y + x$ are two positive integers whose product is 36. Hence it must be that $(y-x, y+x)$ is one of the pairs $(1, 36)$, $(2, 18)$, $(3, 12)$, $(4, 9)$. This information enables us to set up pairs of equations in x and y, which can then readily be solved. The results are shown below (though we have left the working out of the solutions to you):

- If $(y-x, y+x) = (1, 36)$, we get $y = 18\tfrac{1}{2}$, $x = 17\tfrac{1}{2}$, with $36 = \left(18\tfrac{1}{2}\right)^2 - \left(17\tfrac{1}{2}\right)^2$.
- If $(y-x, y+x) = (2, 18)$, we get $y = 10$, $x = 8$, with $36 = 10^2 - 8^2$.
- If $(y-x, y+x) = (3, 12)$, we get $y = 7\tfrac{1}{2}$, $x = 4\tfrac{1}{2}$, with $36 = \left(7\tfrac{1}{2}\right)^2 - \left(4\tfrac{1}{2}\right)^2$.
- If $(y-x, y+x) = (4, 9)$, we get $y = 6\tfrac{1}{2}$, $x = 2\tfrac{1}{2}$, with $36 = \left(6\tfrac{1}{2}\right)^2 - \left(2\tfrac{1}{2}\right)^2$.

In just one case are both x and y are integers. So only that one possibility works out, and we get: $x = 8$, $y = 10$.

Therefore we get: $(n+6)^2 = 10^2$, hence $n + 6 = \pm 10$. The negative sign yields $n = -16$, which we do not accept, while the positive sign yields $n = 10 - 6 = 4$. *Hence there is precisely one positive integer n for which $n^2 + 12n$ is a perfect square; namely, $n = 4$, for which $n^2 + 12n = 64 = 8^2$.*

Problem 2

Factorize the fourth degree polynomial $x^4 + x^2 + 1$.

Solution

Since the exponents of the powers of x occurring in the problem are 4 and 2, the given polynomial may be viewed as a quadratic in x^2. We may now attempt to complete the square in two ways, by suppressing either the x^2 or the 1:

$$x^4 + x^2 = \left(x^4 + x^2 + \frac{1}{4}\right) - \frac{1}{4} = \left(x^2 + \frac{1}{2}\right)^2 - \frac{1}{4},$$

and

$$x^4 + 1 = \left(x^4 + 2x^2 + 1\right) - 2x^2 = \left(x^2 + 1\right)^2 - 2x^2.$$

The first possibility yields:

$$x^4 + x^2 + 1 = \left(x^2 + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

This is a 'sum of two squares' and does not immediately yield results, so we do not pursue it. The other possibility yields:

$$\begin{aligned} x^4 + x^2 + 1 &= \left(x^2 + 1\right)^2 - 2x^2 + x^2 \\ &= \left(x^2 + 1\right)^2 - x^2 \\ &= \left(x^2 + 1 - x\right) \cdot \left(x^2 + 1 + x\right). \end{aligned}$$

Rearranging terms, we get: $x^4 + x^2 + 1 = \left(x^2 - x + 1\right) \cdot \left(x^2 + x + 1\right)$.

Problem 3

Show without actually dividing out that 9901 *is a divisor of* 100010001.

Solution

Since the given claim can be checked by actual division, this cannot be termed a difficult problem! The challenge is to find a solution which is 'pretty and sweet'.

Note that 100010001 may be written as $10^8 + 10^4 + 1$. This means that it is of the form $x^4 + x^2 + 1$ for $x = 10^2$. So we can call upon the result of the preceding problem! We get: $100010001 = \left(10^4 - 10^2 + 1\right) \times \left(10^4 + 10^2 + 1\right) = 9901 \times 10101$.
We see that 9901 is a divisor of the given number. (It turns out that 9901 is prime. The other divisor can itself be factored as $10^4 + 10^2 + 1 = \left(10^2 - 10 + 1\right) \times \left(10^2 + 10 + 1\right) = 91 \times 111$, which then further factorizes, giving $10101 = 3 \times 7 \times 13 \times 37$. So the given number factorizes as $100010001 = 3 \times 7 \times 13 \times 37 \times 9901$.)

Problem 4

Find all integers $n > 0$ such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square.

Solution

This looks extremely daunting, but we shall tame it using the very same weapons. Our first challenge will be to find an expression which is identically a perfect square and is very close to the given expression, $n^4 - 4n^3 + 22n^2 - 36n + 18$. Since this has degree 4, the desired expression will have to be the square of a quadratic expression. Therefore it must have the form $(n^2 + an + b)^2$ where a and b are coefficients to be found.

Since the coefficient of n^3 in the given expression is -4, we must have $a = -2$. This is the 'completing the square' rule used yet again! So $n^2 + an + b = n^2 - 2n + b$.

What value should b have so that $(n^2 - 2n + b)^2$ is as close to the given expression as possible? By squaring we get:

$$(n^2 - 2n + b)^2 = n^4 - 4n^3 + (2b + 4)n^2 - 4bn + b^2.$$

So for the coefficients of n^2 to 'match' we must have $2b + 4 = 22$, i.e., $b = 9$. But for this value of b we see that the coefficients of n agree as well; only the constant terms differ. Indeed we have:

$$(n^2 - 2n + 9)^2 = n^4 - 4n^3 + 22n^2 - 36n + 81.$$

Now suppose that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square; let it be equal to x^2 where x is a positive integer. From what we found above, $n^4 - 4n^3 + 22n^2 - 36n + 81$ too is a perfect square (it is so for any integer n); let it be equal to y^2, where y is a positive integer.

Looking closely at the two expressions we see that $y^2 - x^2 = 63$. *So 63 has been written as a difference of two squares*. Ah! Now we are on familiar ground.

In what ways can 63 be so written? From the three different factorizations of 63 (as 1 × 63, 3 × 21, 7 × 9) we get all the possible ways by setting up three different sets of equations and solving them:
$63 = 32^2 - 31^2 = 12^2 - 9^2 = 8^2 - 1^2$.

Therefore it follows that $n^2 - 2n + 9$ is one of the numbers 32, 12, 8. Hence to find n we must solve these three separate quadratic equations:

$$n^2 - 2n + 9 = 32,$$
$$n^2 - 2n + 9 = 12,$$
$$n^2 - 2n + 9 = 8.$$

The first of these gives $n = 1 \pm 2\sqrt{6}$ which is not an integer at all (or even rational). The second one yields $n = 3$ or -1, while the third one yields $n = 1$.

Since n must be a positive integer we see that $n = 1$ or 3, and we have fully answered the question. The values that the given expression takes at these values of n are 1^2 and 9^2.

Some problems for you to solve, using these ideas

Problem #4 is interesting in that it can be solved in two different ways. Problem #6 is challenging, but do try it out.

1. Factorize these numbers: (i) 3599 (ii) 8099 (iii) 4087.
2. Factorize the polynomial $x^4 + 4$.
3. Find all integers n such that $n^2 + 10n + 20$ is a perfect square.
4. Find all integers n such that $n^2 + n$ is a perfect square.
5. Find all integers n such that $n^2 + n + 1$ is a perfect square.
6. Find all integers n such that $n^4 + n^3 + n^2 + n + 1$ is a perfect square.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at comm.math.centre@gmail.com or shailesh.shirali@gmail.com

Problems for the Middle School

Problem Editor : R. ATHMARAMAN

Problems for Solution

Problem II-3-M.1

Find the value of the following in as simple a way as possible (and without using a calculator!):

$$\frac{(2013^2 - 2019) \times (2013^2 + 4023) \times 2014}{2010 \times 2012 \times 2015 \times 2016}.$$

Problem II-3-M.2

It is easy to find a pair of perfect squares that differ by 2013; for example, $47^2 - 14^2 = 2013$. Now that the new year (2014) is close upon us, we ask: Can you find a pair of perfect squares that differ by exactly 2014?

Problem II-3-M.3

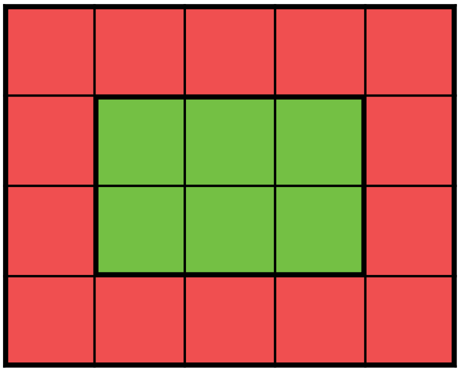
From a two-digit number *n* we subtract the number obtained by reversing its digits. The answer turns out to be a perfect cube. What could be the value of *n*?

Problem II-3-M.4

To a certain two-digit number *m* we add the number obtained by reversing its digits. The answer turns out to be a perfect square. What could be the value of *m*?

Problem II-3-M.5

The rectangle shown has been divided into equal squares. The squares along the perimeter are shaded red, while the rest of the squares are shaded green. You will notice that the number of red squares is greater than the number of green shares. What should be the dimensions of the rectangle if the number of red squares exactly equals the number of green squares?



Solutions of Problems in Issue-II-2

Solution to problem II-2-M.1

Find all integers $n > 0$ *such that* $n^4 - 4n^3 + 22n^2 - 36n + 18$ *is a perfect square.*

This problem has been solved in full in the newly started 'Adventures' column (page 59 – 60).

Solution to problem II-2-M.2

A railway line is divided into 10 *sections by stations* $A, B, C, D, E, F, G, H, I, J, K$. *The distance from* A *to* K *is* 56 *km. A trip along any two successive sections never exceeds* 12 *km. A trip along any three successive sections is at least* 17 *km. What is the distance between* B *and* G? (See Figure 1.)

We have: $AD + DG + GJ + JK = 56$. But $AD \geq 17$, $DG \geq 17$, $GJ \geq 17$. Hence $JK \leq 5$.

Again, $HK \geq 17$ and $JK \leq 5$ (just found); hence $HJ \geq 12$. On the other hand, $HJ \leq 12$ (given condition). Hence $HJ = 12$.

Next, since $HK \geq 17$, $HJ = 12$, $JK \leq 5$ we must have $JK = 5$.

In just the same way we get $AB = 5$ and $BD = 12$. Also:
$DH = AK - AB - BD - HJ - JK =$
$56 - 5 - 12 - 12 - 5 = 22$.

Again, $GJ \geq 17$ and $HJ = 12$, hence $GH \geq 5$.

Further, $DG \geq 17$, and $DH = DG + GH = 22$.

Hence $DG = 17$ and $GH = 5$.

Finally we get:

$$BG = BD + DG = 12 + 17 = 29.$$

This is the required answer.

Solution to problem II-2-M.3

In the right $\triangle ABC$ *with* BC *as hypotenuse,* $AB = x$ *and* $AC = y$ *where* x *and* y *are positive integers. Squares* $APQB$, $BRSC$ *and* $CTUA$ *are drawn externally on sides* AB, BC *and* CA, *respectively. When* QR, ST *and* UP *are joined, a convex hexagon* $PQRSTU$ *is formed. Let* k *be its area. Prove that* $k \neq 2013$. (See Figure 2.)

Using the sine formula for area of a triangle ("half the product of two sides and the sine of the included angle") and arguing as shown in the figure (in the itemized list), we see that $\triangle ABC$, $\triangle QBR$, $\triangle PAU$ and $\triangle CTS$ have equal area. Hence the total area of the four triangles is $4 \times \frac{1}{2}xy = 2xy$. The areas of the squares are x^2, y^2 and $x^2 + y^2$ (Pythagoras!), so the area of hexagon $PQRSTU$ is $2\left(x^2 + y^2 + xy\right)$ which clearly is even. Hence the area cannot be equal to 2013.

Solution to problem II-2-M.4

The numbers $1, 2, 3, \ldots, n$ *are arranged in a line in such a way that each number is either strictly bigger than all the numbers to its left, or strictly smaller than all the numbers to its left. In how many ways can this be done?*

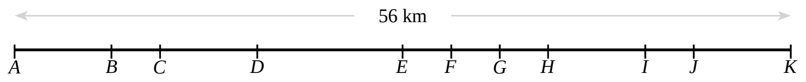
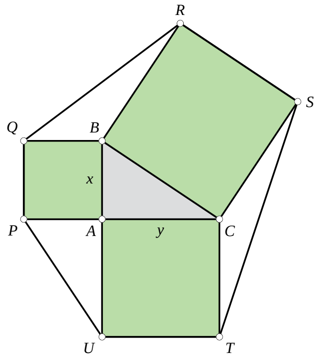


Figure 1.



- ∡ABC + ∡QBR = 180^0, hence sin∡ABC = sin∡QBR
- Hence ΔABC and ΔQBR have equal area
- Similarly, ΔABC and ΔPAU have equal area
- Similarly, ΔABC and ΔCTS have equal area

Figure 2.

Let the required number be denoted by $f(n)$. Trivially, $f(1) = 1$. For $n = 2$ both permutations of $(1, 2)$ satisfy the stated condition, so $f(2) = 2$.

For $n = 3$ the permutations of $(1, 2, 3)$ which satisfy the stated condition are $(1, 2, 3)$, $(2, 3, 1)$, $(2, 1, 3)$ and $(3, 2, 1)$; so $f(3) = 4$.

For the general case, with n numbers, let us focus on the *last* number. Since it must be either less than all the other $n - 1$ numbers, or greater than all of them, the last number can only be 1 or n. Whichever it is, the $n - 1$ numbers to its left satisfy exactly the same conditions as the given problem; if the last number is n the numbers are $1, 2, 3, \ldots, n - 1$, and if the last number is 1 the numbers are $2, 3, 4, \ldots, n$. In the latter case, by subtracting 1 from each number we get a permutation of the numbers $1, 2, 3, \ldots, n - 1$ for which the same conditions are satisfied.

From this it follows that for the problem with n numbers there are $f(n - 1)$ permutations in which the last number is n, and an equal number in which the last number is 1. Hence $f(n) = 2f(n - 1)$. Thus the sequence of f-values is a doubling sequence, and since $f(2) = 2 = 2^1$ it follows that $f(n) = 2^{n-1}$.

Solution to problem II-2-M.5

If a, b, c are three real numbers with $1/a + 1/b + 1/c = 1/(a + b + c)$; prove: $1/a^{2n+1} + 1/b^{2n+1} + 1/c^{2n+1} = 1/\left(a^{2n+1} + b^{2n+1} + c^{2n+1}\right)$ for any positive integer n.

The given condition yields, on simplification:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a + b + c},$$

$$\therefore (a + b + c)(bc + ca + ab) - abc = 0.$$

The expression $(a + b + c)(bc + ca + ab) - abc$ vanishes if we put $b + c = 0$, or if we put $c + a = 0$, or $a + b = 0$. Invoking the factor theorem, we arrive at this surprising and nice factorization:

$$(a+b+c)(bc+ca+ab)-abc = (a+b)(b+c)(c+a).$$

So the given condition implies that one of $a + b$, $b + c$, $c + a$ is 0. Suppose that $b + c = 0$. Then $b = -c$, so the quantities $1/a^{2n+1} + 1/b^{2n+1} + 1/c^{2n+1}$ and $1/\left(a^{2n+1} + b^{2n+1} + c^{2n+1}\right)$ are both equal to $1/a^{2n+1}$. This is so because $2n + 1$ is an odd integer. Similarly if $a + b = 0$ or $c + a = 0$. Hence proved.

Problems for the Senior School

Problem Editors : Prithwijit De & Shailesh Shirali

Problems for Solution

Problem II-3-S.1

Let P be a polynomial such that $P(x) = P(0) + P(1)x + P(2)x^2$ and $P(-1) = 1$. Find the value of $P(3)$.

Problem II-3-S.2

In $\triangle ABC$, the midpoint of BC is D; the foot of the perpendicular from A to BC is E; and the foot of the perpendicular from D to AC is F. Given that $BE = 5$, $EC = 9$, and the area of $\triangle ABC$ is 84, compute the length of EF.

Problem II-3-S.3

In how many ways can the integers $-8, -7, -6, -5, \ldots, 4, 5, 6, 7, 8$ be arranged in a line such that reading from left to right the absolute values of the numbers do not decrease?

Problem II-3-S.4

Two ships sail on the sea with constant speeds and fixed directions. It is known that at 9:00 am the distance between them was 20 miles; at 9:35 am, 15 miles; and at 9:55 am, 13 miles. What was the least distance between the ships, and at what time was it achieved? [IMO Short list, 1968]

Solutions of Problems in Issue-II-2

Solution to problem II-2-S.1

A circle has two parallel chords of length x that are x units apart. If the part of the circle included between the chords has area $2 + \pi$, find the value of x.

Let AB and CD be the two parallel chords of length x units which are x units apart. Observe that $ABCD$ is a square with area x^2 square units. Since the diagonal of the square coincides with a diameter of the circle, the radius R of the circle is given by $2R = x\sqrt{2}$. Thus the area of the circle is $\frac{1}{2}\pi x^2$. The four arcs into which the circumference of the circle is divided by the vertices A, B, C, D of the square are congruent and so are the four segments of the circle (created by the four sides of the square). If each segment has area k square units then

$$\frac{\pi}{2}x^2 = x^2 + 4k,$$

and we also have $x^2 + 2k = 2 + \pi$ (given data). From these two equations we get $x^2 = 4$ and therefore $x = 2$ (since $x > 0$).

Solution to problem II-2-S.2

The prime numbers p and q are such that $p+q$ and $p+7q$ are both perfect squares. Determine the value of p.

Let $p+q=x^2$ and $p+7q=y^2$. Then $6q=y^2-x^2=(y-x)(y+x)$. Thus $(y-x)(y+x)$ is an even number. As $y-x$ and $y+x$ are of the same parity (this is short for saying that they are both odd or both even) and their product is even, both $y-x$ and $y+x$ must be even. Therefore their product is a multiple of 4, i.e., 4 divides $6q$. But then q must be even, so $q=2$. Therefore $(y-x)(y+x)=6q=12$. Assuming x and y to be positive we see that $y-x=2$ and $y+x=6$, which gives $(x,y)=(2,4)$. Thus $p=x^2-2=2$.

Solution to problem II-2-S.3

Determine the value of the infinite series

$$\frac{1}{3^2+1}+\frac{1}{4^2+2}+\frac{1}{5^2+3}+\frac{1}{6^2+4}+\cdots.$$

For $r\geq 1$ let t_r be the r^{th} term of the series. Then:

$$\begin{aligned}t_r=\frac{1}{(r+2)^2+r}&=\frac{1}{(r+1)(r+4)}\\&=\frac{1}{3}\left(\frac{1}{r+1}-\frac{1}{r+4}\right)\\&=\frac{1}{3}\left(\left(\frac{1}{r+1}-\frac{1}{r+2}\right)\right.\\&\quad+\left(\frac{1}{r+2}-\frac{1}{r+3}\right)\\&\quad\left.+\left(\frac{1}{r+3}-\frac{1}{r+4}\right)\right).\end{aligned}$$

Hence if the sum of the series is S, then:

$$\begin{aligned}3S&=\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\cdots\right)\\&\quad+\left(\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}+\cdots\right)\\&\quad+\left(\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}+\frac{1}{6}-\frac{1}{7}+\cdots\right)\\&=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12},\qquad\therefore S=\frac{13}{36}.\end{aligned}$$

Remark Properly speaking we should show first that the given series has a finite sum; to use the technical and accepted term, it 'converges' . But we shall leave this step to you.

Solution to problem II-2-S.4

In trapezium $ABCD$, the sides AD and BC are parallel to each other; $AB=6$, $BC=7$, $CD=8$, $AD=17$. Sides AB and CD are extended to meet at E. Determine the magnitude of $\angle AED$.

Choose a point F on AD such that $BCDF$ is a parallelogram. Then $BF=CD=8$ and $AF=AD-FD=AD-BC=17-7=10$. Thus in triangle ABF, $AB=6$, $BF=8$ and $AF=10$. Hence $\angle ABF=90^\circ$. But $\angle AED=\angle ABF$. Therefore $\angle AED=90^\circ$.

Book Reviews

N MUKUNDA

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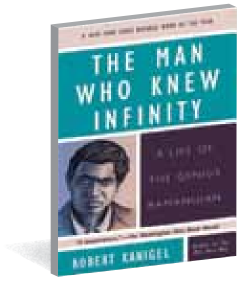
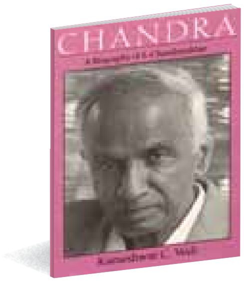


The Man Who Knew Infinity – A Life of the Genius Ramanujan, by ROBERT KANIGEL; Charles Scribner's New York, 1991; Rupa & Co, Calcutta 1991; 438 pp; Price: Rs. 195.00.



Chandra – A Biography of S. Chandrasekhar, by KAMESHWAR C. WALI; The University of Chicago, 1990; Viking Penguin India, 1991; 341 pp; Price: Rs. 250.00.

Keywords: *Ramanujan, S. Chandrashekhar, Kanigel, Wali, Hardy, mathematician, physicist, sociology, beliefs*

It is an unusual and irresistible opportunity to be invited by the Editor to jointly review these two recent biographies. Srinivasa Ramanujan, who was born in 1887 and died in 1920, is the most luminous Indian mathematician in centuries. He came upon the scene as suddenly as a meteor and left as suddenly. Subrahmanyan Chandrasekhar – Chandra – is the finest and most accomplished theoretical physicist from India, who has set standards of achievement and style. To write this review is therefore a privilege. It also provides the occasion to look upon the Indian scientific scene, past and present, and to reflect upon several social and psychological issues that are raised by these books.

Both biographies are written in an easy and non-technical style for a wide readership.

The references to mathematics in the one, and to physics in the other, are handled with a light touch. Kanigel's book is definitely journalistic in style. It is very much concerned with social conditions that prevailed in India and in England during Ramanujan's life time. Thanks to the somewhat greater distance in time, from about the last decade of the last century to about the end of the first World War, it is easier to speak of Ramanujan's family and social conditions quite objectively. Overall it reads like a fairy tale. In contrast – and quite understandably – Wali is able to paint a more personal and intimate portrait of Chandra, thanks to direct contact with his subject lasting many years. In this review I will look first at Kanigel's book, then at Wali's, and finally turn to some points of comparison and reflection.

It is appropriate to begin by recalling how Kanigel came to write his book. During 1987, India and the world celebrated Ramanujan's birth centenary. Many seminars and conferences were held – mainly in India and the U.S.A.– to look back upon his life, assess his work, and trace its impact on mathematics over the decades. It was around this time that Kanigel first heard of Ramanujan, when he was approached with the proposition that he write a biography of Ramanujan. The result is this first full-scale life story of the mathematician. It is quite astonishing that in such a short time Kanigel has been able to research and put together such a detailed, balanced and absorbing account of Ramanujan's life and work. This in spite of the cultural gap he faced in the process. The title chosen by Kanigel is also wonderfully evocative.

I will not describe here in any detail the events of Ramanujan's life. What is more interesting is to see how Kanigel has treated his material, in the process recalling only in broad outline some facts from Ramanujan's life. And then to say something about the several important issues the author raises.

Kanigel's narrative breaks naturally into four parts. It begins with the period of Ramanujan's childhood and early education; exposure to the book by Carr entitled Synopsis of Elementary Results in Pure and Applied Mathematics; his experiences and disappointments at college; difficulties in his attempts to secure recognition, understanding, support and employment; till the departure for England in March 1914 by which time many around him had sensed his extraordinary talents and come together to help him. Next Kanigel turns to a brief life sketch of the English mathematician G. H. Hardy, aptly called the "discoverer" of Ramanujan, covering his social background, education, personality and achievements. The third part is in a way the happiest part – the first two years of Ramanujan's stay in Cambridge – the fulfilment of all of Hardy's hopes, and the flowering of the collaboration between the two. Last comes the tragic part – tragic in a manner distinct from Ramanujan's early years – the onset of World War I; its impact on Ramanujan; his illness, breakdown and hospitalisation in the midst of recognition of various kinds; his return to India in 1919; and death in April 1920 while still at the height of his creative powers. The conclusion of the narrative takes up later events – Hardy's and others' assessments of Ramanujan's work; publication of the Collected Works; the finding of the "Lost Notebooks" and the rediscovery or rebirth of Ramanujan in recent times, thanks to the devoted efforts of the U.S. mathematicians George Andrews, Richard Askey and Bruce Berndt. Within this broad framework each chapter covers a short period of a few years, building up in rich detail the atmosphere and events of that period.

Kanigel's account of Ramanujan's early years pays great attention to South Indian life and customs characteristic of the time and the social group to which he belonged. There are many references to the river Cauvery and its bounties, which remind one of passages from Hesse's Siddhartha; descriptions of the countryside; and occupations of ordinary people. Family details – a dominating mother completely overshadowing a weak and ineffectual father, death of many of Ramanujan's siblings at very young ages – are recalled. Ramanujan is outstanding in school, and by age thirteen discovers for himself Euler's infinite series expressions for exponential and trigonometric functions. His extreme sensitivity to perceived humiliations also surfaces early – indeed, incidents and behaviour attributable to this aspect of his personality occur often later in the book. The exposure in 1903 to Carr's book becomes a turning point in Ramanujan's life.

One realises, however, that it both stimulated him and limited his growth. Being a logically arranged compendium of some 5000 formulae stated without proof, Ramanujan was led to find proofs for them, extend them and find new results, but record everything in the same style completely omitting derivations. This style, which became his hallmark, was later to cause difficulties. On the other hand, since the mathematics underlying the book stopped at around 1850, Ramanujan was denied access to all later European advances. One cannot help feeling that there is a bittersweet quality to the entire episode. Carr's book became Ramanujan's entry point to mathematics, but also left him "cabined, cribbed, confined". ,

Total absorption in mathematics leads to neglect of other subjects, failure in college, and loss of scholarship. The celebrated "Notebooks" also start in this period. After an arranged marriage to Janaki in 1909, the need for employment and steady income becomes acute. But all through this troubled period, Ramanujan's self-confidence and belief in his powers remain supreme. Kanigel describes Ramanujan's travels up and down South India, meeting so many people with letters of introduction, seeking recognition and support. Fortune finally smiles when Ramachandra Rao, Collector of Nellore and Secretary of the Indian Mathematical Society, recognises his genius and begins supporting him so that he may devote all his energies to mathematics. Soon a job at the Madras Port Trust also materialises. In this part of the narrative, Kanigel gives an account of the origin of the Indian Mathematical Society set up in 1906. And Ramanujan's earliest papers start appearing in the Society's Journal. By about 1912, there are many around Ramanujan wanting to help, including the Englishmen Francis Spring, Gilbert Walker, and the Madras University Registrar Francis Dewsbury. But if one had to pick just two, they would have to be Ramachandra Rao and Narayana Iyer, Ramanujan's superior at the Port Trust. Recognition had come, but not comprehension of his work, and with it the realisation that contact with the West was absolutely essential.

At this point Kanigel turns to England and a life sketch of Hardy. In a biography of Ramanujan this has an essential place, and what Kanigel has presented is a special attraction of this book. In social aspects, educational systems, and care and opportunities for the gifted, South India and England are total contrasts. Kanigel highlights the Public School and University systems with their centuries-old traditions, and speaks of both their strengths and weaknesses objectively. We are given an insightful account of Hardy's social and family milieu. Both parents were school teachers, and though they came from modest backgrounds they bred in Hardy and his sister the desire always to excel. He inherited a softness from the father and a sternness from the mother, and grew up into a very private and reserved person. The description of intellectual life in Cambridge; Hardy's crucial role as the leading British mathematician of the era bringing back an appreciation of the Continental values of purity and rigour; and his extraordinary lecturing and writing skills, all make fine reading. One also sees the "other side" of the much vaunted Tripos examinations, and what they had been reduced to by this time.

Readers of this Journal would be interested in the account Kanigel gives of the discovery of the celebrated Hardy-Weinberg Law of Population Genetics. There are also several historical aspects worth recalling here. The problem concerns the question whether a dominant trait in a given population would proliferate and completely wipe out a recessive one as one progresses from generation to generation. R. C. Punnett, who was editor of this journal from 1910 to 1946, sometimes alone and sometimes with Bateson or Haldane, mentioned to Hardy that there was an argument to this effect, due to one Mr.Udny Yule. However, by a rather elementary mathematical analysis, Hardy was able to show that this would not happen – the ratio of dominant to recessive genes would quickly stabilise and then stay constant from generation to generation. Hardy communicated his results in a letter to the Editor of Science in July 1908, but did not pursue the subject any further. In the same year, independent of Hardy, the German physician Wilhelm Weinberg arrived at the same law, but published it in a comparatively less well known journal. Much later, Curt Stern in a note in Science in February 1943 recalled the entire episode and in particular emphasized that Weinberg had followed up his original work in several directions, which Hardy, had not. It was Stern who suggested that as a matter of justice one should attach the names of both discoverers to the law. This incident also illustrates well the fact that the significance of a mathematical result need have no relation at all to the complexity of the mathematics involved!

Perhaps the high point, the most gripping part of the book is the story of the day in January 1913 when Hardy received Ramanujan's first letter. Kanigel reconstructs the events in great detail, tracing hour by hour the growing impact on Hardy, until late at night Hardy and Littlewood realised they were looking at the work of a mathematician "of altogether exceptional originality and power". Hardy responds appreciatively but stresses the importance of supplying proofs for Ramanujan's numerous claimed results. Thanks to Hardy's prestige and influence, after much negotiation, Madras University rises to the occasion and awards Ramanujan a two-year scholarship to pursue his researches. Hardy's efforts to bring Ramanujan to Cambridge, however, face much opposition – from Ramanujan himself, more so from his mother. Hardy "solves" the problem by sending his younger colleague E. H. Neville to Madras to speak to Ramanujan on the spot and persuade him to go. The problem is only resolved by a family trip to the Namagiri temple, planned by Narayana Iyer, to seek divine guidance from the family deity – in a manner Hardy would never have understood, and involving psychological aspects Kanigel analyses sympathetically.

The period of adjustment and attendant strains on Ramanujan, upon reaching England in early 1914, are overtaken by happiness in work. Hardy is soon reassured that he had taken the right initiatives. The meeting of two persons with such vastly different backgrounds, training and gifts, their collaboration, the delicacy and care exercised by Hardy in making up for Ramanujan's ignorance of modern advances while protecting his brilliantly intuitive mind – all these make fascinating reading. In time Hardy would declare: "I have never met his equal, and I can compare him only with Euler or Jacobi".

To this happy period and soon after belongs their finest joint work on the theory of partitions. Each with his unique gifts, together they achieved what neither could have done on his own; yet as Kanigel says, Ramanujan was the irreplaceable component in the collaboration! But soon the war commences, devastating life in Cambridge and for Ramanujan too. Privations and lack of proper nourishment build up till Ramanujan's health gives way in early 1916. Here Kanigel finds fault with Hardy for having been blind to Ramanujan's emotional needs, being concerned only with progress in work, and also pushing him unduly hard. There may be truth in this, but it is no easy matter to take sides. Hardy did not pay attention to, never bridged, the cultural gap – each was being true to his own nature. Those aspects of Hindu life which were crucial as sources of sustenance to Ramanujan, as Kanigel says, made little sense to the very British Hardy. Yet through Hardy's efforts Ramanujan won the formal recognition he wanted and richly deserved – Fellowships of the London Mathematical Society, Cambridge Philosophical Society, Royal Society and Trinity College.

Kanigel brings out the pathos of the situation – years of illness and hospitalization, the wasting of a luminous intellect, the effects of loneliness, cut away from his roots, and lack of support from the family, all coupled with the war. The return to India leads to more treatment and diagnoses and movement from place to place, all to no avail. All the while Ramanujan keeps working on his mathematics, while surrounded by family tensions and strife. He finally succumbed on April 26, 1920, aged just over 32.

Of further developments, Kanigel writes about Hardy's later career at Oxford and then back at Cambridge; his lectures at the Harvard Tercentenary in 1936; and the contributions of Watson, Wilson, Askey, Andrews, Berndt and others to the "rediscovery" of Ramanujan. Kanigel also refers to the impact of Ramanujan on contemporary figures like the physicist Freeman Dyson and the mathematician Atle Selberg. There is a good discussion on the roots of creativity contrasting Hardy's and Hadamard's attitudes. While Hadamard acknowledges the roles of unconscious mental activity, intuitive thinking and flashes of inspiration in the creative process, and while to some extent Hardy sympathised, in the end Hardy seemed to believe that there was nothing inexplicable, and indeed was reluctant to speak on the issue.

Kanigel's sympathy for and involvement with Ramanujan shine through the book. With pain we recognise the tensions between his mother and his wife, that hurt Ramanujan so badly. His life and fortunes were often so delicately balanced that they have a razor's edge quality. Maybe Kanigel sometimes gives excessive detail, yet it is good to see what an outsider finds most interesting. In the end, Neville's assessment comes back to us: "Srinivasa Ramanujan was a mathematician so great that his name transcends jealousies, the one superlatively great mathematician whom India has produced in the last thousand years". And seeing how our institutions failed him, we are moved to say that Ramanujan turned out to be the man who knew too much about infinity for the society into which he was born.

Now let me turn to Wali's biography of Chandra. Here the personalities, the times, the social conditions, the fields, and even the backgrounds of the two biographers are all totally different. Yet, as we shall see later on, there are several points of contact which any perceptive reader will surely notice. Chandra has already attained the status of a legend. When the Nobel Prize in Physics for 1983 came to him, at the age of seventy three, it was in a way a recognition of a lifetime of extraordinary achievement in theoretical astrophysics in particular, and theoretical and mathematical physics in general. Just as in mathematics Ramanujan could only be compared to Euler and Jacobi, so here, on the occasion of the award of the Dannie Heinemann Prize in 1974, the citation of the American Physical Society compared Chandra to Lord Rayleigh and Henri Poincaré for the range and depth of his scholarship. It was sometime around 1970 that Wali first thought of writing an account of Chandra's life. There were many obviously unique and unusual aspects to it – initial training in India; a six-year stay in Cambridge in the early 30's immediately following the Golden Age of Theoretical Physics; contacts with so many leading personalities of physics and astrophysics over such a long period of time; and then from 1937 onwards a member of the faculty of the University of Chicago. Wali felt convinced, and rightly so, that a record of so rich a life and so much accomplishment ought to be made. This conviction grew, with a sense of urgency, after Chandra's heart attack and by-pass surgery in 1975. It was in 1977 that Wali formally obtained permission from Chandra to proceed with his plans, and the result is this delightfully written, absorbing and splendid book. Wali is acutely conscious of the difficulties inherent in being biographer to a living person, but he has handled the situation with sensitivity. This is a book written at a personal level, touching only lightly upon technical matters. And the author has capped his effort by a beautiful device – he provides at the end an extended conversation with Chandra, covering both a wide variety of issues and a great span of time.

It is necessary to recapitulate here in barest outline the course of Chandra's career, so that later comments can be understood in the proper perspective. Chandra was born in October 1910 into a family which had, within a span of two generations, come to value highly academic attainment and scholarship.

Indeed his uncle, C. V. Raman, was later to win in 1930 the Nobel Prize for Physics, for the effect named after him. By the time Chandra completed his undergraduate studies in Madras in 1930, he had already begun independent researches in theoretical astrophysics and published several papers. He was also noticed by and known to the leading Indian physicists at that time. All his formative years in India were spent during the days of British rule; and Wali succeeds in capturing the moods, aspirations and, values of educated Indians of those days extremely well. As Chandra recalls, those were inspiring times for the young in India, thanks to figures like Rabindranath Tagore and C. V. Raman, Ramanujan, S. N. Bose and M. N. Saha, and men like Gandhi and Nehru leading the Independence Movement.

With the aid of a Government scholarship, Chandra went to Cambridge in 1930, planning to work with R. H. Fowler on problems of theoretical astrophysics. Already before leaving India, he had a most fortunate chance to meet the German physicist Arnold Sommerfeld, from whom he learnt of the most recent advances in quantum mechanics and statistics. Based on this and Fowler's earlier work, during the sea voyage to England, Chandra was able to work out the startling consequences of applying the relativistic quantum statistics and degeneracy formula for electrons in the late stages of evolution of massive stars. This led to the famous mass limit named after him, the Chandrasekhar limit of 1.4 solar masses; below and above this limit the evolution of stars follows dramatically different courses. During his stay in Cambridge, leading to the Ph.D. in 1933, and then as a Fellow of Trinity College, he branched into many areas of astrophysics, and steadily built up a world-wide reputation. Among those who influenced him deeply in this period, and who had intimate contact with him, were Fowler, Paul Dirac, Arthur Eddington and Edward Milne. (In passing it is interesting to note that generally Chandra found no sympathy, among any of these luminaries, for India's freedom). There were also extended visits to Bohr's Institute in Copenhagen and to Gottingen, which greatly increased his circle of contacts.

In particular, the relationship with Eddington is a very peculiar one, carrying all sorts of overtones. During Chandra's continuing researches into the processes of stellar evolution, he had the clear feeling all along that he was being supported and encouraged by Eddington. But the occasion of his final presentation of his results at a January 1935 meeting of the Royal Astronomical Society turned out to be a shattering experience – in a totally unexpected manner, Eddington publicly disagreed with and ridiculed Chandra's results and humiliated him.

Such treatment at the hands of the most distinguished astrophysicist of that time deeply influenced Chandra's attitude to research and manner of working as well. Instead of carrying on a public controversy with Eddington, which would not have led him anywhere, he decided to complete his researches in that area, write them up in a book, and then move on to other things. This he has made into the pattern of his life, devoting his attention in turn to stellar structure, stellar dynamics, radiative transfer, hydrodynamic and hydromagnetic stability, ellipsoidal figures of equilibrium, the general theory of relativity, and the mathematical theory of black holes. To quote Goldberger, during each period "he has produced an infinite series of papers followed by an infinitely thick book on the subject".

After his marriage to Lalitha in 1936, Chandra moved in 1937 to the Yerkes Observatory of the University of Chicago. He has remained with this University ever since, becoming Professor in 1944, and Morton D. Hull Distinguished Service Professor in 1952.

Wali's success in recounting Chandra's story lies in combining the elements of involvement and objectivity to just the right extent. It is this that enables him to trace the development of Chandra's personality, and describe his triumphs and troubles, in so moving and eloquent a manner.

Wali's own background as an Indian-born physicist settled in the U.S.A. makes it possible for him to bring out aspects of Chandra's personality, relationships with people, views on science and scientists in India, in a rather special way. In a sense this is comparable to the American Kanigel, rather than an Indian, writing on Ramanujan from the perspective of a total outsider. There are so many things one learns about Chandra and about many others with whom he interacted, which are sobering, and are worth recall and comment.

It appears that in the early stages of his career Chandra definitely felt that he had begun to work in astrophysics more or less by chance. There was always a strong desire to change over to "mainstream physics", which then meant atomic and nuclear science, and not merely remain in the periphery. Indeed many times he tried to make this change, but for various reasons it never worked out. This was also the period of a feeling of inadequacy and self-doubt. Chandra keenly felt that even being among the most distinguished Indian scientists of the time, such as Raman and Bose and Saha, was nowhere near being in the stimulating environment of Cambridge, in the company of people like Fowler, Eddington and Dirac.

This strong desire to move into the mainstream of physics, and the events following the incident with Eddington, form very interesting material for an analysis into the psychology and sociology of science. In turn Chandra sought help and vindication from such leading figures as Bohr, Dirac, Pauli and Rosenfeld. But while everyone of them agreed privately that Chandra was in the right and Eddington had erred, not one wished to say so publicly. Apparently all these leaders in "mainstream physics" had concluded that Eddington was not to be taken seriously, that he was past his prime. But within the "peripheral field" of astrophysics Eddington's reputation and standing were enormous, and there was no resolution available to Chandra. The whole episode held back the development of the subject, of neutron stars and black holes, by almost half a century. One is reminded of Huxley's well-known remark that "a man of science past sixty does more harm than good", and is led to believe that sometimes this can happen at a younger age!

Chandra's attitude towards his father is in the Indian tradition – deep respect, a sense of obedience and duty, yet the desire to be left free and alone to follow one's own path. He was also extremely anxious to be, and to appear to be, totally independent of his uncle Raman. This was the advice given to him by his mother (to whom he was very close), and also by his father, at various times. On many occasions we read of Chandra's fears about not being left alone to continue his work, were he to return to India. And his descriptions of events and relationships among many of the leading Indian physicists of those days – the bickering and sniping – the desire of each to be treated and to appear as a prima donna – makes one understand the causes of his fears. While all this makes sad reading, one may hope that the situation is somewhat better today. But here one must remember Chandra's account of a visit to the TIFR in Bombay in 1961 – all he heard was constant criticism of its founder and director Homi Bhabha, not a single good word! In spite of all this, comparing life in India to that in the U.S.A., he would write of the latter: "Life here, in spite of its wholesome climate for my intellectual work, has the quality of distilled water, and I feel curiously desiccated".

All these tales retold make one ponder over such questions as: what should a country such as India provide for its most gifted, and in turn what may it expect from them?

Originally Chandra had gone to Cambridge with a Government scholarship which stipulated that he return and serve the Government for a certain term. Over the years there were many opportunities, many occasions to return; and on every occasion Chandra was under great pressure from his father to do so. But each time something intervened – either the award of the Trinity Fellowship, or the wish to avoid competing with a dear friend, or a three-year position at Yerkes. And in the case of the invitation from Raman to join the Indian Institute of Science, his father advised Chandra by cable to "keep off his orbit". All this against the constant background of fear of return to an unhelpful environment.

In other countries and at other times, we read of remarkable individuals who showed great sensitivity to the needs of scientists and to the growth of science at the level where it really matters. Thus Constance Reid, in her biography of David Hilbert, speaks of Friedrich Althoff who "was no bureaucrat but an administrator who had been academically trained. His great goal was to build up mathematics in Germany". Similarly in the Italy of the early 30's we have "Professor Corbina's boys" led by Enrico Fermi revitalizing Italian physics. In India, sad to say, since the days of Asutosh Mukherjee in Calcutta in the early part of this century, there have been precious few visionaries of this kind.

In his own long career at the University of Chicago, Chandra was subjected to subtle and not so subtle acts of discrimination on many occasions. These are openly and honestly documented by Wali. But while it is true that Chandra did not allow such events and treatment to distract him from his work, one suspects that the trust he enjoyed in the eyes of the President of the University, Robert M. Hutchins, was a source of much strength to him. What could be more revealing than Hutchins' 1971 remark: "I have always been proud that I had a part in bringing you to the University of Chicago".

The years from 1952 to 1971 spent by Chandra as editor of the Astrophysical Journal merit an entire chapter in Wali's book. One is astonished and humbled by this account of total dedication, integrity and self-sacrifice that characterised Chandra's stewardship of the Journal over this long period. Starting from an in-house publication of the University of Chicago, it grew in his hands into the world's leading journal in the field.

This kind of total dedication and involvement characterise many other aspects of his life and work. His deep aesthetic sense and feel for language are well known among physicists, and they deserve being known more widely. Indeed he deliberately studied the literary styles of writers like Virginia Woolf and T. S. Eliot, in order to fashion a distinctive style of his own. Sometime ago the theoretical physicist Victor Weisskopf lamented: "It is regrettable that among scientists the presentation of ideas is not as highly valued as the creation of ideas. This is in stark contrast to music, where the performer is a partner equal to the composer". In recent times, few have done more than Chandra to redress the balance. As Weisskopf says of him: " ....His deep education, his humanistic kind of approach to these problems, his knowledge of world literature, and in particular English literature, are outstanding. I mean you'd hardly find (another) physicist or astronomer who is so deeply civilized"; and "Good English style is a lost art in physics, but he has it and this wonderful feeling for the essential, and a feeling for beauty".

With the passage of time, Chandra has evolved a very personal and detached attitude to achievement and recognition in science. Thus he reveals a deep humility when he says: "One's place in science, as posterity will duly assign, depends very largely on one's continuous exertion, at the edge of one's ability;... I think one could say that a certain modesty towards understanding nature is a precondition to the continued pursuit of science". Chandra has had a great deal to do directly with preserving Ramanujan's legacy in recent times. This is a most absorbing part of the conversations at the end of Wali's book, and supplements Kanigel's account in important respects. Just as Hardy became the discoverer of Ramanujan, Chandra regards his "discovery" of Ramanujan's passport photo as one of his most important ones!

Returning finally to what we may learn from Ramanujan and Chandra – in Chandra's own words, Ramanujan was a giant and rare fluctuation, an event one could hardly be prepared for, though a stronger academic environment would surely have helped. There are no rules to be made in advance to care for such genius. And the only reasonable answer to the question in what sense Ramanujan belonged to India is in Gibran's words from The Prophet: "Your children are not your children. They are the sons and daughters of Life's longing for itself. They come through you but not from you, and though they are with you yet they belong not to you. You may give them your love but not your thoughts. For they have their own thoughts. You may house their bodies but not their souls, for their souls dwell in the house of tomorrow, which you cannot visit, not even in your dreams”.

As for Chandra himself, as we recalled, there were several times when he might have returned to India, but it was not to be. In his concluding conversations Chandra admits that if he had come back, and had done only half as much for Indian science as he has done for science in the U.S.A., the net gain would have been greater. One can only agree with this assessment. Still, while it may be hard indeed to change the sociology of science, the appearance of these two books at this time, the unbelievable story of Ramanujan, and the example of standards and dedication set by Chandra, should surely inspire many. If so, then varying somewhat Chandra’s favourite passage from Virginia Woolf, we may say that that will be our consolation and their triumph.

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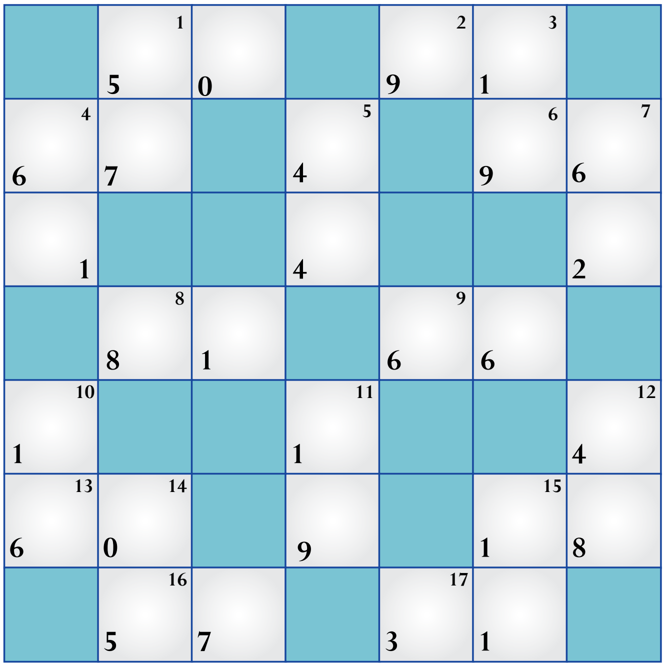
Solution to number crossword 3

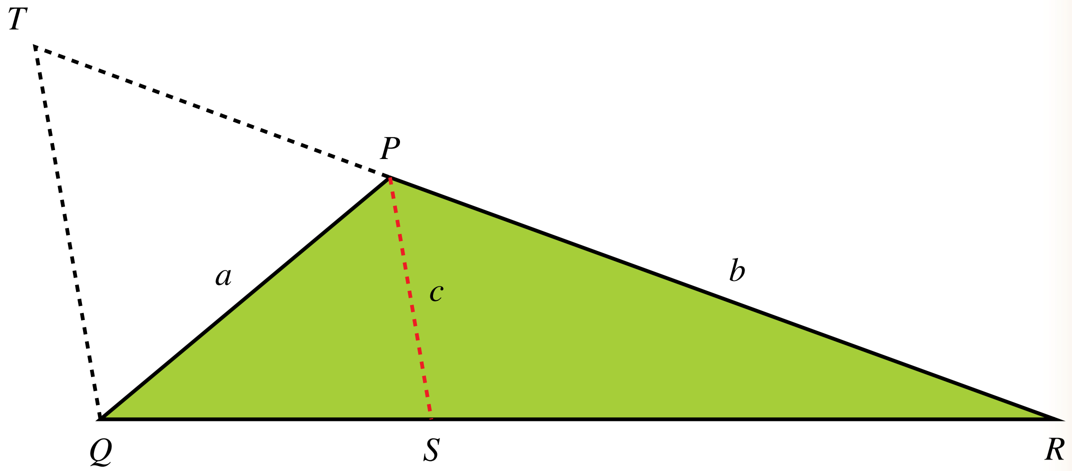


■	1 5	0	■	2 9	3 1	■
4 6	7	■	5 4	■	6 9	7 6
1	■	■	4	■	■	2
■	8 8	1	■	9 6	6	■
10 1	■	■	11 1	■	■	12 4
13 6	14 0	■	9	■	15 1	8
■	16 5	7	■	17 3	1	■

GEOMETRIC SOLUTION TO THE 120 DEGREE PROBLEM

A. RAMACHANDRAN

In the March 2013 issue of *AtRiA,* the following result had been stated in the article on ‘Harmonic Triples’: Let ΔPQR have $\measuredangle P = 120^\circ$. Let PS be the bisector of $\measuredangle QPR$, and let $PQ = a$, $PR = b$, $PS = c$; then $1/a + 1/b = 1/c$. It had been proved using trigonometry, and the question was asked: Is there a proof using ‘pure geometry’? We give just such a proof here.



Extend RP to T so that $PT = PQ$. Since $\measuredangle QPT = 60^\circ$ it follows that ΔPQT is equilateral, and hence (since $\measuredangle TQP = 60^\circ = \measuredangle QPS$) that $TQ \parallel PS$. Hence we have:

$$\frac{TQ}{PS} = \frac{TR}{PR} = \frac{TP+PR}{PR}, \qquad \therefore \frac{a}{c} = \frac{a+b}{b}, \qquad \therefore \frac{1}{c} = \frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b}.$$

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Keywords: *Harmonic triples, pure geometry, 120 degree problem, proof*

The Closing Bracket . . .

Many pieces in this issue are about the prime numbers, a topic that has fascinated human beings for over two thousand years, and continues to fascinate and please. It is remarkable that in a topic that can be introduced to a child in the primary school there are so many mysteries. In recent decades the primes have been put to uses that the ancients would have been amazed to hear — cryptography and the like. But the charm of the primes transcends their usage. It is reassuring to know this, living as we are in times when a use must be found for everything, that use quite often being perfectly diabolical in nature. It is one of the great tragedies of our time that side-by-side with so many discoveries being of great beauty being made, the world is steadily becoming a more insecure place to live in, with ever new ways being found to spread disorder and violence, and to exploit others. On the one hand we read of results like Yitang Zhang's theorem on primes separated by small gaps, Gauss's discovery of the role that certain primes play in geometric constructions, and the Prime Number Theorem, in which you find the primes and logarithms secretly holding hands under the table; on the other hand you read of rape on a scale that boggles the mind, of the destruction of the natural environment on an equally horrifying scale, and of war and ever spreading forces of darkness. How do these two worlds coexist? The question remains a compelling mystery. And what are our education systems doing about it? It is surely a matter of immense relevance to address this challenge, and to nurture minds that can look at these matters with equal insight and feeling. Recently we read of the exciting news that Voyager 1 has left the bounds of the Solar System, and in a discussion on the Scientific American website I came across the following evocative and charming poem. I thought I would close by sharing it with you.

> *Oh! I have slipped the surly bonds of earth,*
> *And danced the skies on laughter-silvered wings;*
> *Sunward I've climbed, and joined the tumbling mirth*
> *Of sun-split clouds, –and done a hundred things*
> *You have not dreamed of –Wheeled and soared and swung*
> *High in the sunlit silence. Hov'ring there*
> *I've chased the shouting wind along, and flung*
> *My eager craft through footless halls of air...*
> *Up, up the long, delirious, burning blue*
> *I've topped the wind-swept heights with easy grace*
> *Where never lark or even eagle flew –*
> *And, while with silent lifting mind I've trod*
> *The high untrespassed sanctity of space,*
> *Put out my hand, and touched the face of God*
>
> John Gillespie Magee, Jr (1922–1941)

— Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.

2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.

3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.

4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.

5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.

6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.

7. Provide a compact list of references, with short recommendations.

8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.

9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.

10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.

11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).

12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.

13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.

14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.

15. Submit articles in MS Word format or in LaTeX.

Azim Premji University

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together with Community Mathematics Centre,
Rishi Valley

TEACHING SUBTRACTION

PADMAPRIYA SHIRALI

A PRACTICAL APPROACH

At Right Angles
A Resource for School Mathematics

TEACHING SUBTRACTION

Most assessments conducted across the country indicate that the first stumbling block for many children is the subtraction operation, followed by division. On looking more closely we see that the difficulties often arise in subtraction contexts involving double digit or larger numbers. This difficulty is largely caused by three factors: (i) improper understanding of place values (ii) lack of understanding of the rationale behind the formal subtraction procedure, (iii) not seeing the connection between addition facts and subtraction facts.

This article follows the sequence of place values and addition operations and is closely linked to the ideas introduced in the preceding articles of this series. I proceed on the assumption that the reader is acquainted with the earlier ideas and activities talked about.

When do we first introduce children to subtraction? We use the concept of subtraction to introduce ‘zero’ by removing one object after another: say 10 balls onwards till we reach zero balls. In my childhood it was taught as a nursery rhyme: “Ten green bottles hanging on the wall, if 1 green bottle were to accidentally fall, 9 green bottles hanging on the wall, etc”.

ACTIVITY ONE

Counting Backwards From 10 to 1 in Steps of 1

Let children show 10 initially with their fingers and say "10 are open". Close one finger, say "9 open", then close another, say "8 open" and so on, all the way down to "1 open"; then close all and say "zero fingers". Make sure that there is a correspondence between what they say and what they show. Some children have a tendency to count mechanically and not see the correspondence. This naturally leads to inaccuracy in counting and improper development of the number sense.

The ability to recite numbers backwards from 10 to 1 comes a little after mastering forward counting. The challenge can be raised to reciting numbers from 20 to 1, or 50 to 30, or 83 to 65, etc.

If the child has difficulty in reciting backwards one can allow the child to use a number line or number chart or tens and units material as an aid. Providing some visual support is necessary till the child internalises the pattern and observes the transition points. Teachers can draw the number line 1 to 20 on the floor and have children walk back from 10 to 1 in steps of 1 and 2, saying the numbers aloud.

While conducting this activity a teacher will surely notice that the stumbling blocks are the transition points (60, 59; 50, 49; etc). Teachers can use the number chart to make children observe these points.

Reciting numbers backwards is of value at various points in Classes 1, 2, 3, 4; it reinforces the child's understanding of the way numbers are sequenced and their place values. It helps the child to handle numbers in a flexible manner which is a prerequisite for doing mental arithmetic.

One can choose the right challenge from the extensions given for different age groups (classes 2 to 4)

- Extension 1: Counting backwards from 10 to 0 or 20 to 0 in steps of 2.
- Extension 2: Counting backwards from 100 to 0 in steps of 10.
- Extension 3: Counting backwards from 100 to 0 in steps of 5.
- Extension 4: Counting backwards from 100 to 0 in steps of 20.
- Extension 5: Counting backwards from 300 to 0 in steps of 25.
- Extension 6: Counting backwards from 600 to 0 in steps of 75. (Much more challenging.)

ACTIVITY TWO

The Three Subtraction Contexts

In order to familiarise the child with the three subtraction situations, the teacher needs to use the three different contexts (take away, comparison, inverse addition) over a period of time, in a gradual way. Problems posed in one form can be posed in another form so that children begin to see that they lead to the same operation, and are thus able to deal with it in a flexible manner.

Subtraction as take away: Initial introduction to subtraction is through the situation of 'Take away' or 'Remove' from a pile: "Remove 3 seeds from 7 seeds".

How does the child obtain the answer? Usually he counts out 7 seeds, removes 3, again by counting, and then counts the leftover and says "4". When it is given as a problem in the text with an illustration he crosses out the number of seeds to be removed and counts the rest. Counting of the leftover objects begins from 1 again (this changes to forward counting when the child begins to use the number line), so the child counts 1, 2, 3, 4 and gives the answer as "7 take away 3 is 4". Various contexts are used to demonstrate the take away situation. Also the child in his daily life often shares what he has with his siblings or friends, so they have already internalised these subtraction contexts which the teacher can fall back on.

Subtraction as comparison: Comparison situations arise both at home and in school. Teacher needs to make use of these comparison situations consciously to show that they give rise to subtractions.

Children should be thoroughly familiar with the usage of language: "How many more?", "How much less?", "What is the difference?" etc.

Subtraction as the inverse of addition: Often we solve subtraction problems by converting them into addition problems.

Ex 1. How many more to be added to 8 to make 12?

Ex 2. A shopkeeper when he has to give change. When given a Rs. 100 note for a purchase of Rs. 67, he would first return Rs. 3 and then Rs. 30, turning it into an addition problem: $67 + 3 + 30 = 100$.

ACTIVITY THREE

Backward and Forward Counting

However at some point the child will need to start visualising the numbers to do subtractions without actual objects and without resorting to counting. An intermediate step to help the child to visualise the number sequence is to use bead chains, number line, 10 frames and number charts to learn backward counting and forward counting. We will discuss here the usage of bead chains and number line to give practice in forward and backward counting.

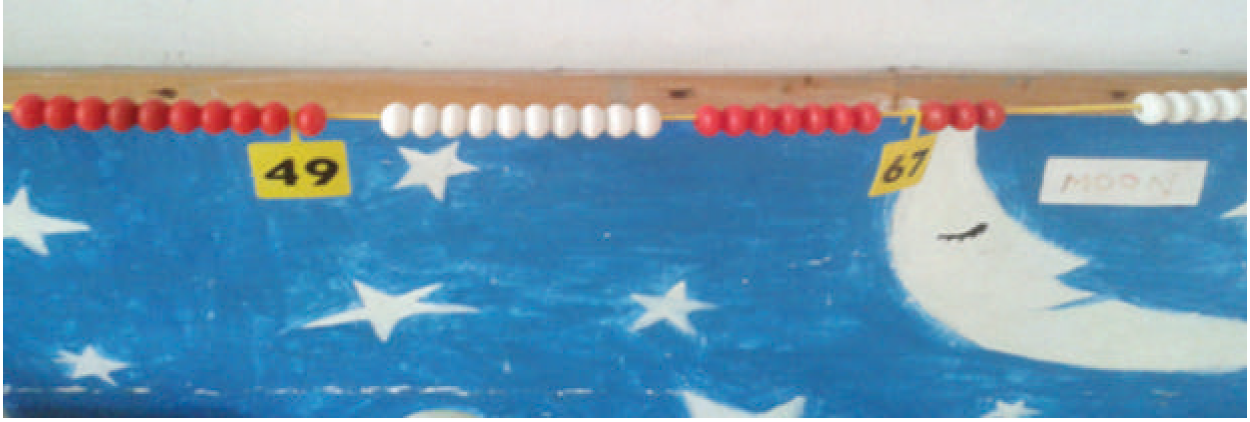
Backward counting: Backward counting is often used when the number to be taken away is small like 2 or 3. If a child has counted 17 beads and wants to remove 2, he may count backwards (perhaps using his fingers to keep track of the number to be subtracted): 17, 16 and arrive at 15 as the answer. But it is of use only in situations where the number to be subtracted is small.

Human number line: If you have a class of 30 students let each child be given a number card in order. Teacher gives a subtraction problem, say 27-3, the children upto the 27th child hold up their cards, starting from 27 three children now put down their cards.

Forward counting: Forward counting has much larger usage and application and needs to be taught in a graded manner using visuals. Bead chains with sets of ten beads in different colours can be used effectively as the change in colour denotes the break at 10, 20 etc.

Ex. 13 – 8 can be worked out as "Separate the first 8 beads from 13". Counting forward from the rest: 9, 10, 11, 12, 13, which becomes 5. (Let the children open one finger at a time while counting forward.)

Depicting 49 to 67 on a bead chain:



Counting to the nearest 10: 'Counting to the nearest ten' and using the understanding of place value is another method used as an extension of forward counting (Classes 3, 4). Example: 17 – 8 may be worked out as 8 +2 (to reach the nearest ten) and seven (as 17 is 10 + 7).

$8 + 2 + 7 = 17$, $2 + 7 = 9$, $17 - 8 = 9$

ACTIVITY FOUR | Relationship of Addition and Subtraction

Appreciation of the relationship between addition and subtraction facts can help children to use their knowledge of addition facts in solving subtraction problems.

However teachers must take care not to teach it using formal language but help children to assimilate this and use it in problem solving.

a) Show that every subtraction fact gives rise to another subtraction fact (5 – 3 = 2, 5 – 2 = 3).

This can be demonstrated using fingers. Hold up all the fingers. What If I close two fingers? Say aloud, "10 – 2 = 8". What If I close 8? "10 – 8 = 2".

Get the children to try other combinations. Each time, record both the results on the board.

10 – 2 = 8, 10 – 8= 2

10 – 3 = 7, 10 – 7 = 3

10 – 6 = 4, 10 – 4 = 6

10 – 1 = 9, 10 – 9 = 1

Let the children observe the pattern and give the related subtraction fact for a given subtraction fact.

b) Show the relationship between the subtraction fact and the addition fact (5– 3= 2, 2+3 = 5).

Often there are exercises in textbooks which show the related addition fact like this:

7–2 = 5, 5+ 2 = 7.

But this by itself cannot enhance a child's understanding of the relationship between these two facts unless the teacher points out explicitly by using materials or pictures: "Here I have 7 seeds, if I remove 3 seeds there are 4 seeds left. But if I put back the 3 seeds, I again have 7 (i.e., 4+ 3) seeds".

7 – 4 = 3, 4 + 3 = 7

7 – 2 = 5, 5 + 2 = 7

7 – 1 = 6, 6 +1 =7

The fact that the quantity removed and the quantity left sum to what we had started with needs to be internalised by the child by experiencing it.

c) Show that every addition fact gives rise to two subtraction facts.

Teacher can ask, "What do I get when I add 2 and 6?" 8. "What do I get when I take away 6 from 8?" 2. "What do I get when I take away the other number (2) from 8?" 6.

6 + 2 = 8, 8 – 6 = 2, 8 – 2 = 6

4 + 5 = 9, 9 – 5 = 4, 9 – 4 = 5

Several such examples can be done and this can be recorded on the board with pictures.

The child will see that every addition statement can be written as two subtraction statements.

One often finds children trying to subtract by counting on fingers or counting finger segments even in classes 3 and 4. Teachers must help children use number complements and addition facts to arrive at quick answers, and also commit subtraction facts to memory.

In the previous issue we discussed approaches to facilitate the learning of number complements and addition facts.

ACTIVITY *FIVE*

Subtraction Facts

Subtraction facts: (18 – 9 to 10 – 9, 17 – 8 to 9 – 8, 16 – 7 to 8 – 7, etc, down to 9 – 1 to 2 – 1)

In the previous issue we discussed how to break down the goal of learning addition facts into manageable sub–goals. In a similar manner the learning of subtraction facts can also be broken down into sub–goals as listed below. In order to achieve each goal there is a need to give targeted practice.

- Subtraction of 1: This is fairly simple for the child as he sees it as 'stepping back by 1 step on the number line' or 'counting backwards by 1': 18 – 1, 17 – 1, 16 – 1,.... 2 – 1.
- Subtraction of 2: This can be arrived at by counting backwards or visualising taking 2 steps back on a number line.
- Subtraction of 0: Ex. 7 – 0. Children need to clearly understand that subtracting zero means nothing is being removed. At the outset it may not make sense to them why zero should be written at all and it may baffle them.
- Subtraction ofthe number itself: Ex. 8 – 8. This too is quite clear to the child: that a number subtracted from itself gives 0.
- Subtraction where the first number is twice the second one: 18 – 9, 16 – 8, 14 – 7, 12 – 6, 10 – 5, 8 – 4, 6 – 3.
- Subtraction of 10: Ex. 18 – 10, 12 – 10. Children use their understanding of place value to give the answer. Since 18 is 10 + 8, removing 10 gives 8.
- Subtraction of numbers close to ten: Here, 10 is first subtracted from the number and then 1 is added to compensate for the 1 that was removed. So 17 – 9 is done in two stages: first 17 – 10, which gives 7, and then we add 1 and get 8, which is the answer.
- Subtraction where the first number is 1 more than twice the second one: 17 – 8 can be worked out as: 16 – 8 = 8, 17 is 1 more than 16, so 17 – 8 is 9.
- Subtraction where the first number is 1 less than twice the second one: 9 – 5, 11 – 6, 13 – 7, 15 – 8. Here 11 – 6 can be worked out using knowledge of doubles: 12 – 6 = 6, 11 is 1 less than 12, so the result is 1 less than 6, i.e., 5.

Subtraction by pausing at 10

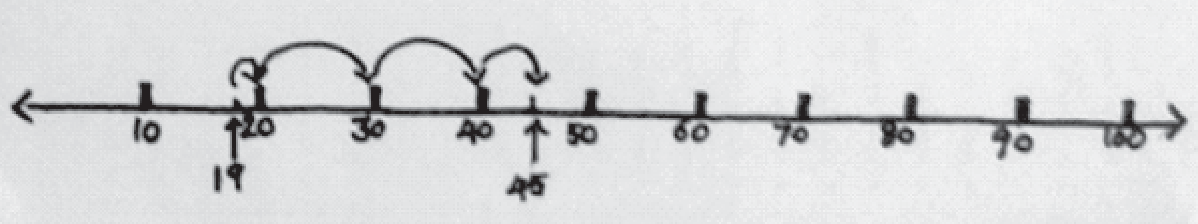
- Consider the subtraction14 – 9. We do this as follows: 14 – 9 is 1 (i.e., 9 to 10) + 4 (10 to 14), so 14 – 9 = 5.
- Similarly: 17 – 9 is 1 (9 to 10) + 7 (10 to 17), so 17 – 9 = 8.
- Subtraction of 8, pausing at 10 (using the fact that 8 is 2 less than 10): Ex. 11 – 8, 12 – 8.
- Example: 14 – 8 is 2 (i.e., 8 to 10) + 4 (10 to 14), so 14 – 8 = 6
- Subtraction of 7 can also be done by pausing at 10: 12 – 7 = 3 (7 to 10) + 2 (10 to 12) = 5

ACTIVITY *SIX* | Usage of Number Line (Upto 100) for Subtraction

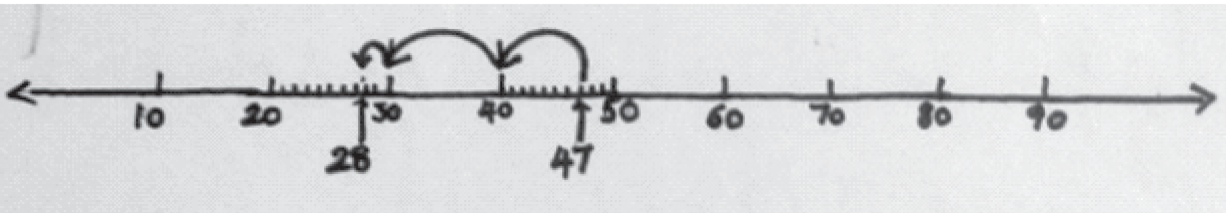
As shown in the picture it is good to get every child to prepare a foldable sturdy number line to use as an aid while solving subtraction problems.

It is good to also draw the number line on the class room wall, running along the space underneath the black board. It is less complex than a 'hundreds' chart, as movement on a number line is "forwards and backwards"; a forward move by 1 step denotes increase by 1, while a backward movement by one step denotes decrease by 1. Number line strengthens both forward and backward counting.

Forward counting: Ex. 45 – 19:



Backward counting: Ex. 47 -28:

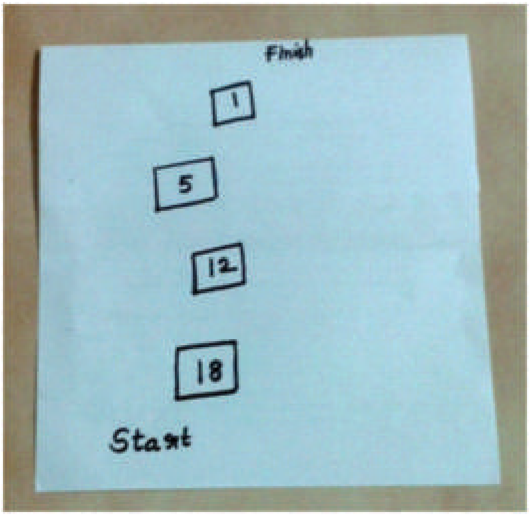


You can ask questions which involve related subtraction facts. "From 40 if you take 8 steps backwards where do you reach?" and: "On the number line, if you are on 40 and need to reach 32 how many steps back do you go?"

GAME 1

Hopscotch

Objective: To practice subtraction facts (mental arithmetic)



This outdoor game is well known. We can modify it to suit our needs. Children can make a drawing on the ground as shown in the picture, and 3 or 4 children can use one such plan. Numbers can be varied so that children practice different subtraction facts.

'Start' and 'finish' can be fixed. The child has to hop from the starting square to another square by giving the subtraction fact. He first says 18 – 6 =12 and hops to '12'. If he makes a mistake, he can be challenged by others and he loses his turn. If he gets it right he proceeds to the next by giving again the next subtraction fact, 12 –7 = 5. This continues till he reaches the last square,'1'. Then he says 1 –1 = 0, and lands outside on both his feet!

We can also play this as an indoor game by drawing such a plan on stiff paper. Differently coloured counters can be used by each child. As the child moves his counter across the squares, he has to give the correct subtraction fact.

ACTIVITY SEVEN | Using a Hundreds Chart

A hundreds chart also acts as an excellent visual aid in solving subtraction problems. While tracing paths on the chart, children will notice patterns in the subtraction process and in the organisation of numbers, and it caters both to their kinesthetic and visual abilities. Every child should have such a hundreds chart for use as an aid while solving subtraction problems.

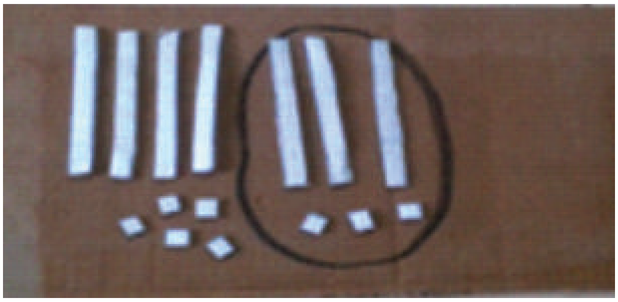
- However, a hundreds chart is quite different from a number line. The teacher must take time to point out the way numbers are organised on a hundreds chart. If we move horizontally (left to right) from one square to another, the number gets incremented by 1. When we move vertically downwards, the number gets incremented by 10. When we move diagonally across, it gives rise to another pattern. The teacher can pose questions to help children notice what patterns the different movements yield. Later, the teacher can use these patterns for solving problems, posing them in a graded manner.
- Subtracting 10 from any multiple of 10; ex. 40 – 10, 20 – 10, 30 – 10. The child realizes that on the hundreds chart a vertical move by 1 unit upwards results in subtraction by 10 and reduces the number in the tens place by 1.
- Subtracting 10 from any number; ex. 45 – 10, 28 – 10, 33 – 10. The child notices that subtraction by 10 does not alter the number in the units place.
- Subtracting multiple tens, ex. 50 – 20.
- Subtracting bigger numbers can be broken down into steps and can be done in two ways. Ex: subtract 22 from 45:
    - The child places his finger first on 45, moves vertically upwards 2 steps (to remove 20) and reaches 25 and then 2 squares to the left (to remove 2) to reach 23.
    - Or child locates his finger first on 45, moves 2 squares to the left (to subtract 2) and reaches 43, and then moves vertically up by 2 squares (to remove 20) and reaches 23.
- Subtracting numbers close to multiples of 10, ex. 45 – 19.
- The child moves his finger from 45 to 25 (remove 20) and then moves right by 1 square (add 1 to compensate the extra 1 removed) to reach 26.

BUILDING CAPACITY FOR EXTENSION

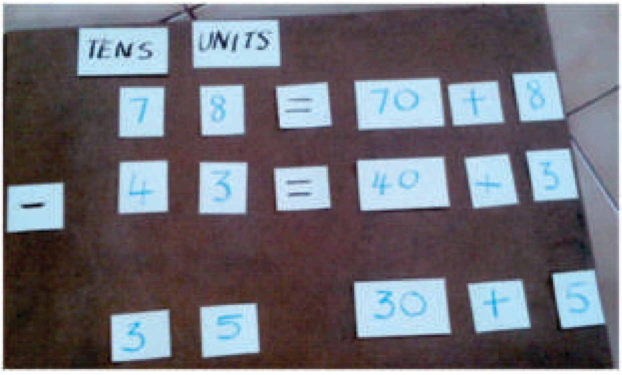
A fundamental principle involved in teaching mathematics is to derive new facts from known ones. In teaching subtraction facts we do this by relating it to addition facts which the child has already learnt. We bring in his understanding of place values. We also use his intuitive understanding of associativity. It is important to give problems which help him to see that a subtraction fact will continue to give the same answer in whatever situation it occurs.

Ex. 7 – 4 = 3. What is 17 – 4? What is 47 – 4? What is 70 – 40?

ACTIVITY EIGHT | Using a Place Value Kit



78 is shown by using tens and units materials. This is written down in the place value table along with expanded notation and the number to be removed is recorded underneath. As mentioned, the teacher needs to constantly bring it to the child's attention that units are being subtracted from units and tens are being subtracted from tens.



It should be read as: '8 units minus 3 units gives 5 units'; '7 tens minus 4 tens gives 3 tens'.

To reinforce this a few problems can be written initially in expanded form.

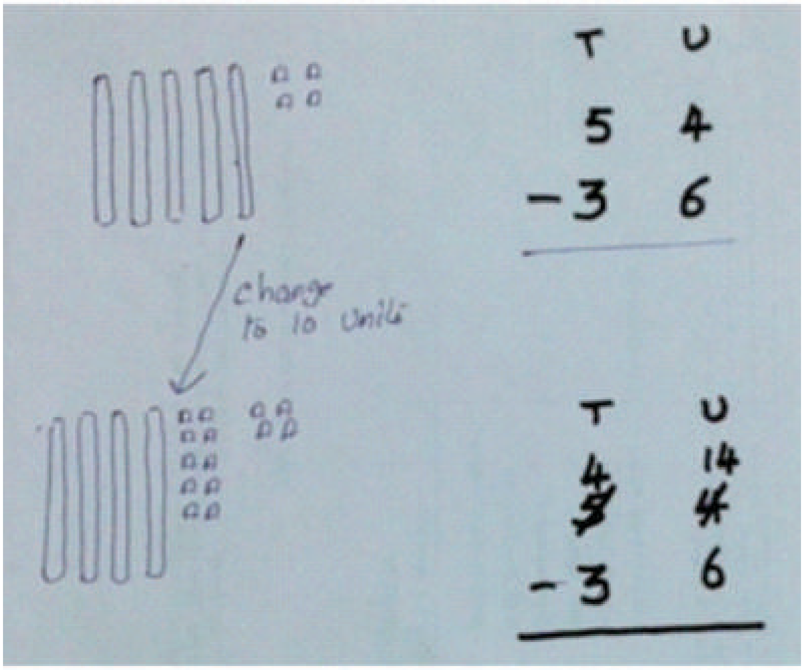
Problem exercises should initially contain visuals. Also let children use materials (tens and ones) till they gain confidence and drop usage of aids on their own.

ACTIVITY NINE

Subtraction of Tens and Units with Exchange

Materials required: Place value kit.

Subtractions like 54 – 36 require exchanging a ten into units and recording the result appropriately. Let the child pick up tens and units corresponding to 54 and place them on the place value card. The number should also be recorded using tens and units as headers. The child now has to remove 36 from this collection. He writes the number to be removed in the second row of the place value card. Then, realising that there are not an adequate number of units for removal, he exchanges a ten for 10 units. When this exchange is done the teacher needs to help the child do corresponding recording by striking out the 5 tens and writing 4 on top in the tens place, and striking out the 4 units and writing 14 on top in the units place (some teachers may follow the convention of writing 10 on top in the units place and not strike out the number in the units place).



Extension: Subtraction of tens and units (zero in the units place) with exchange

Ex. 50 – 36

This can be performed in the same way as the earlier one, stressing on the importance of the need for exchange of a ten for units.

Many children often make errors in subtraction from zero. A child who has had plenty of opportunity to handle materials while learning subtraction is unlikely to make such mistakes.

GAME 2

Race to Zero !

Objective: Practice of subtraction involving exchanging, aids in conceptual understanding of subtraction procedure.

Materials required: Kit containing flats, longs and units (Hundreds, tens, units material) and two dice.

Number of players: 4 players, 1 Banker and 1 shopkeeper

All the materials are initially kept with the banker. He then issues to each player a flat (hundred square). One child throws 2 dice and totals the number he has got (say 9). He now has to pay the shopkeeper that amount. Since he has only a hundred he goes to the banker to exchange 10 tens, and if needed he exchanges 1 of the tens for 10 units. He records the transaction in the note book as 100 – 9 = 91. Now he pays that amount (9) to the shopkeeper.

This same process is followed by the other three children in succession. Each records his transaction in his notebook.

After the first round it is now again the turn of the first child. As the game continues situations which require exchange and do not require exchange both will occur deepening the child's understanding of place value and the relationship of hundred to tens and ten to units.

Towards the end it is possible that the number left with the child is small, say 6, but when he throws the dice he may get a larger number. He will then skip his turn till he gets the required number.

Whoever reaches zero first is the winner.

(If the game seems to take too long, one can have a different goal post, say 50.)

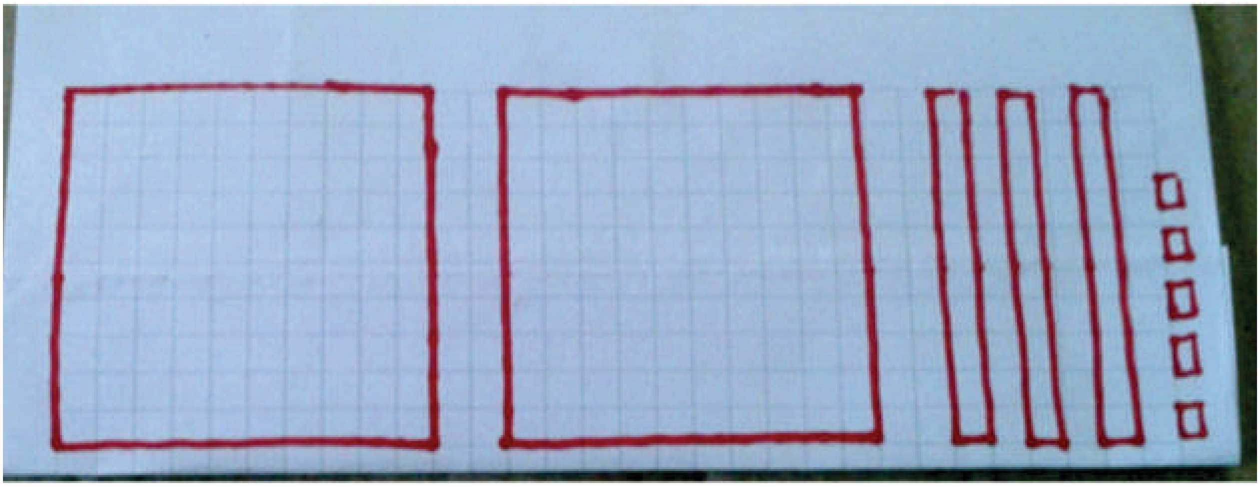
ACTIVITY **TEN**

Subtraction of Hundreds, Tens and Units with Exchange

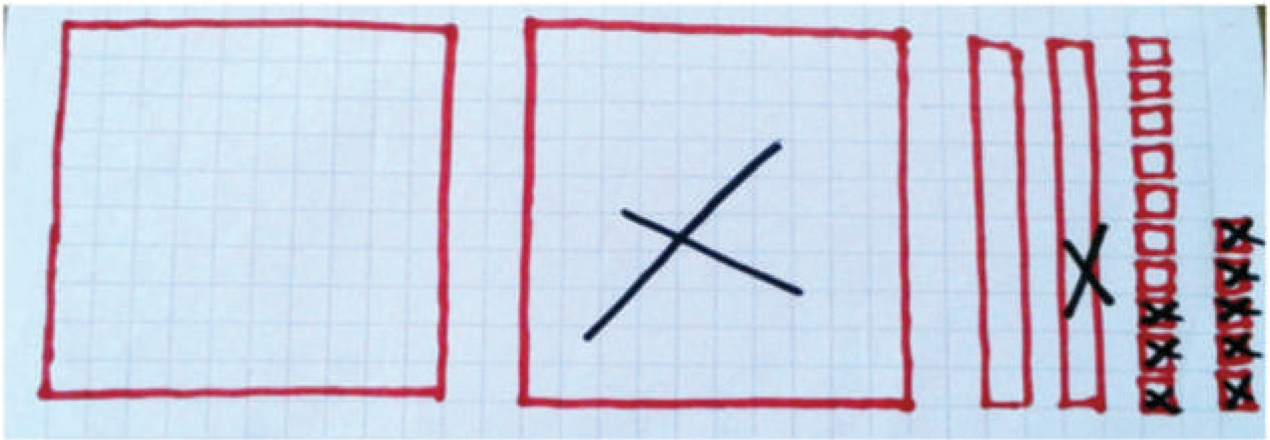
Materials required: Place value Kit.

Subtraction problems need to be taught in a graded manner: initially teach problems which require exchange of ten to units, eg., 235 – 118, and then teach problems which require exchange of hundred to tens, Ex., 342 – 161 followed by problems which require exchange of hundred to tens as well as ten to units, Ex. 245 -168.

Depiction of 235



Backward counting: Ex. 47 - 28:

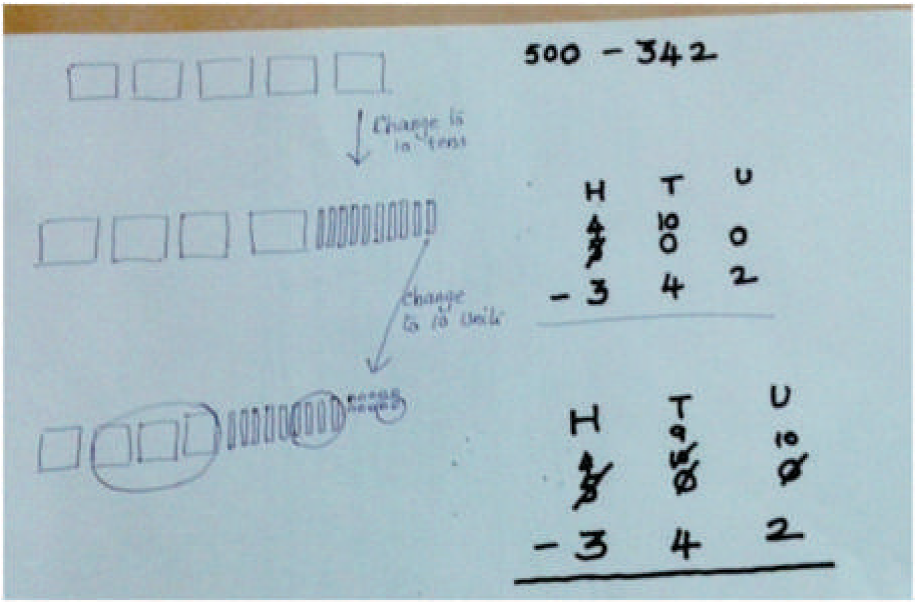


ACTIVITY ELEVEN | Subtraction Problems with Zero in Tens and Units Place

Problems involving zero could initially be with zero only in the units place, followed by problems with zero in the tens place and problems with zero both in tens and units place.

Problems with zero in both tens and units are often not understood clearly by children because they require two sequential exchanges. Unless it is demonstrated repeatedly and performed as well by the child with the help of materials, it will not be understood and the child may just pick up the procedure mechanically or may continue to make mistakes.

Ex. 500 – 342:



The teacher first takes 5 hundred square flats and poses the question, “How do I remove 342 from this?” The problem is written on the board in the usual vertical column way with place values written on top. Simultaneously there can be a visual drawn to depict the 5 flats. “I need to remove 2 units which I do not have. So I first exchange 1 hundred for 10 tens. This is how I record it here.” Teacher alters the drawing by striking out 1 flat and drawing 10 tens, as well as the written form by striking out the 5 in the hundreds place and writing 4 instead and writing 10 in the tens place.

Caution: Many books and teachers tend to skip the crucial step here and write directly 9 in tens place and 10 in the units place. This step should not be skipped as it is not obvious to the child how the digit in the tens place becomes a nine.

Now the teacher shows the need for further exchange of 1 ten for 10 units and simultaneously shows it in the drawing by striking out 1 ten and drawing 10 units, and in the written form by striking out the 10 tens, rewriting 9 in the tens place and 10 in the units place.

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at **padmapriya.shirali@gmail.com**

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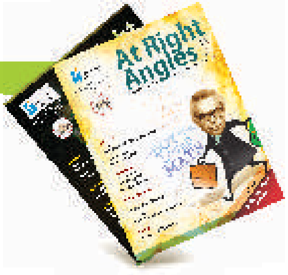

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