Azim Premji University

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At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

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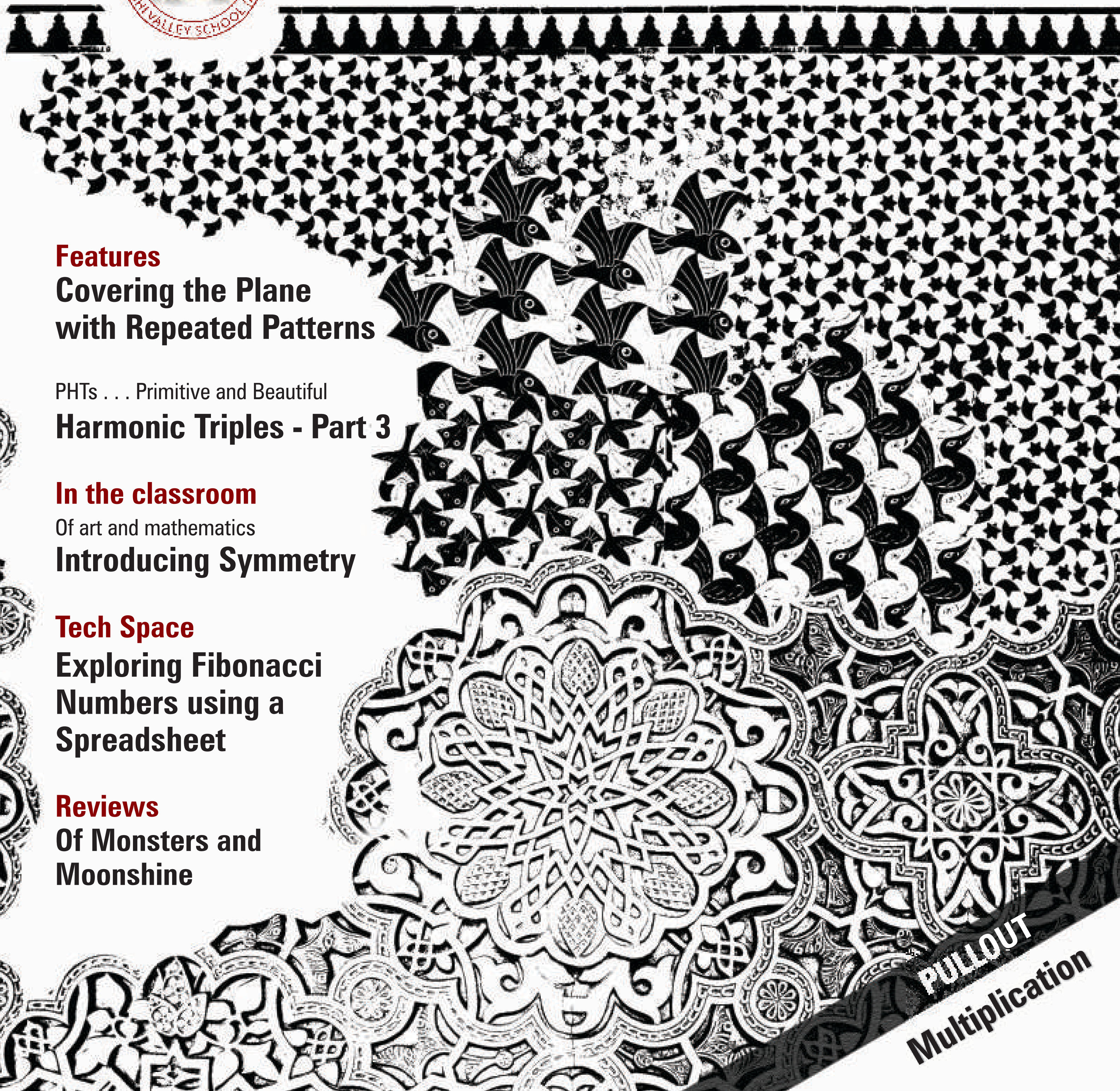

Features

Covering the Plane with Repeated Patterns

PHTs . . . Primitive and Beautiful

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In the classroom

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Exploring Fibonacci Numbers using a Spreadsheet

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Of Monsters and Moonshine

PULLOUT

Multiplication

Notes on the Cover Image

Mosaic has been around for over two millennia. Excavations have uncovered mosaic patterns from the ruins of ancient Mesopotamia, Greece and Rome. Fascination with this art form remains with us today, and it is far from a coincidence that the most striking forms of mosaic come from the Middle Eastern and Mediterranean regions; it is in these cultures that the love of mosaic goes back the longest, and it is perhaps from them that the love of mosaic was carried to distant parts of the world, including India, through trade and conquest. The tessellated patterns we find in the monuments of Granada in southern Spain deserve special mention here. This is where the famous Alhambra Palace is located, and it is to this palace that Maurits Escher came, early in his life. What he saw left a powerful impression on him, which he carried forward in his now famous paintings. This issue explores some mathematical aspects of the topic of tessellations, an art in which Escher so excelled. The sprinkling of images on this page – Alhambra and Fatehpur Sikri (Agra) and Escher's art – are testimony to the richness of this topic.

From the Chief Editor's Desk . . .

Visual riches dominate this issue. Punya Mishra and Gaurav Bhatnagar continue their series on the pattern-filled world of Ambigrams and start to formally introduce some notions of symmetry. Independently, Haneet Gandhi talks about Tessellations and the principles they derive from, and also the cultural and historical background in which they are anchored, for example, Islamic art and architecture. This article will appear in two parts. V G Tikekar continues his article on the infinitude of primes, dwelling this time on an important result in analysis first proved by Euler. The concluding part of the articles on Harmonic Triples appears next.

The Classroom section has plenty on offer. Following the Ambigrams article, we have a whimsical account by Prithwijit De, Sneha Titus and Swati Sircar of a geometric problem on dividing a triangle into two parts, and then an article by Gaurav Bhatnagar on rational approximations to irrational numbers using the device of continued fractions, and one by Padmapriya and Shailesh Shirali on the lengths of repeating portions of recurring decimals; this connects with the article on the same topic which appeared in the November 2013 issue. Following this is a brief account of a famous puzzle concerning house numbers, by A Ramachandran, and the continuation of the series begun in the last issue on the CCE system of the CBSE, by Sneha Titus, Sindhu Sreedevi and Joyita Banerjee. To close the section, we have the second part of the 'How to prove it' column, and a short cameo by Bharat Karmarkar on explorations in teaching.

In the Tech Space section, Jonaki Ghosh describes how a spreadsheet program like MS Excel can be used to explore the fascinating Fibonacci sequence, and to discover not just the Golden Ratio but also a formula for the Fibonacci numbers. Following this we have the Problem Corner, with its regular columns, and a review by Geetha Venkatraman of a book on Symmetry written by prolific math writer and TV presenter, Marcus du Sautoy. Finally, we have the next 'episode' of Padmapriya Shirali's Pullout serial — on Multiplication.

This is the right place to inform readers that we welcome letters to the Editor! Please do write to us with your comments, reflections and suggestions. Here is the e-mail ID which you can use:

AtRiA.Editor@apu.edu.in.

— Shailesh Shirali

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

At Right Angles
A Resource for School Mathematics

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Contents

Features

This section has articles dealing with mathematical content, in pure and applied mathematics. The scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

In the Classroom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The Classroom' is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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Contents contd.

Tech Space

'Tech Space' is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

Problem Corner

Reviews

Letter

Pullout

feature

Covering the Plane with Repeated Patterns - Part I

"Hey mummy, look there. It is so beautiful!"

"Yes, my dear, it is indeed. It's the epitome of Indian architecture. It has tessellations and many symmetrical patterns"

Architecture speaks of its time and place, but yearns for timelessness. Through this two part article, I will dwell on a topic that closely links mathematics with art and culture.

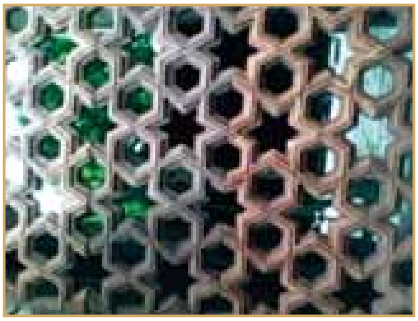
HANEET GANDHI

Recently, my daughter and I happened to take a trip to the historical places of Delhi in the famous Hop-on-Hop-off bus. In our visit to various historical places we not only admired the marvels of the civil engineering of the past, we were also deeply impressed by the complexity and beauty with which our ancestors had blended geometry and art.

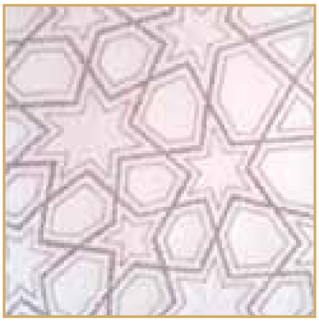
Among the interesting patterns that left us really in awe were the intricate designs we found on many pavements, ceilings and walls. We noticed that in most of these patterns a group of motifs were joined together such that they covered the entire plane. It was interesting to see how a single block of motif (s) fitted so well with each other without any gaps so as to extend in all the directions of the plane. We noticed several such designs in almost all the historical places that we had visited. I have attached some pictures that we took.

Keywords: *Pattern, tessellation, quadrilateral, triangle, architecture, tile, Mughal, Escher*

In this article I wish to share the mathematical ideas behind these perfectly fitting motifs. We will see how these complicated patterns can be broken into simpler components. This exercise can be taken up with middle grade students to encourage them to integrate mathematics with culture and art. To start with, a quick look at these beautiful images:


A partition at Qutub Minar


Delhi Metro Station, Vishwavidayala


Ceiling at Red Fort


Partition at Qutub Minar


Net partition at VC office, DU


Wall at Qutub Minar


A house in Chandni Chowk

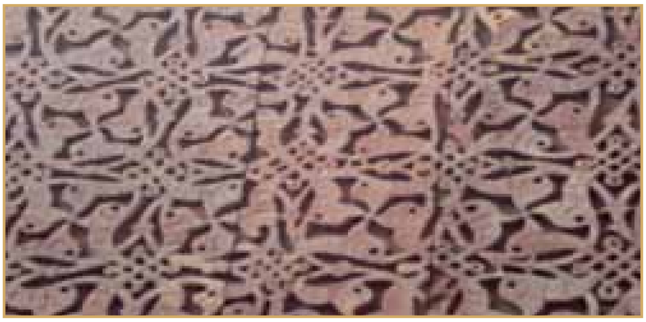

Carpet in Bangla Sahib Gurudwara

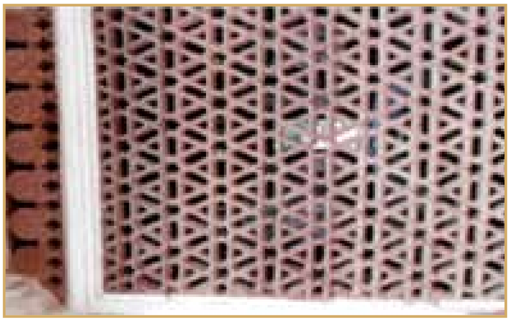

Jama Masjid

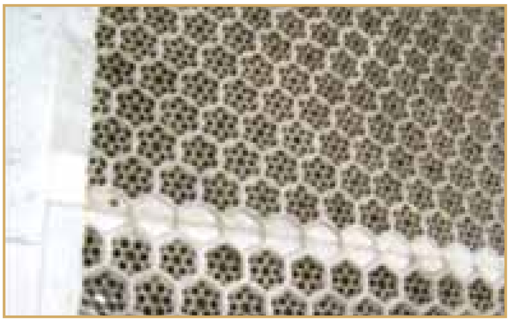

Net Partitions at Red Fort


A house in Chandni Chowk


Glimpse of Purani Dilli


Wall in Qutub Minar


Wall in Red Fort


Wall partition at Taj Mahal, Agra


Jahangir Mahal, Agra Fort

In all these pictures, you will notice that there is either one shape or a collection of shapes being repeated on a planar surface; the shapes fill the plane, with no gaps and no overlaps. The pattern can be extended in all directions of the plane. Such designs are known as ***tilings*** *or* ***tessellations*** (Greek: 'tessell' or 'tesera' which means 'tile'). The defining features of a tessellation are: (a) an infinite collection of congruent shapes, (b) the shapes fit together to fill a two-dimensional plane with no gaps and no overlaps.

The study of tessellations cuts across mathematics, art, architecture, culture and history. Tessellations can be found not only in man-made monuments but also the natural world – in the honeycombs of bees, the skins of snakes and the shells of tortoises. Today, the topic has relevance in scattered fields such as x-ray crystallography, quantum mechanics, cryptology and minimization of waste material in the cutting of metal sheets.

One of the first mathematical studies of tessellations was conducted by Johannes Kepler in 1619, emanating perhaps from his study of snowflakes. More than two centuries later, in 1891, Russian crystallographer E. S. Fedorov gave the connection between isometries and tessellations. Maurits Cornelius Escher (1898-1972) was amongst the pioneers to look into tessellating patterns in detail; he created many masterpieces of his own.

Enumerating regular and semi-regular tessellations

Before proceeding to complicated tessellating patterns, we must study the tessellations of basic shapes – so-called **regular tessellations**. To fully understand the subject matter, you will need to experiment on your own with these shapes. For this, keep at least 10 copies each of the regular polygons with sides 3, 4, 5, 6, 8, 10 and 12. Remember to keep the same edge length for all the polygons!

Start with the smallest regular polygon: an equilateral triangle. Experimentally test if these shapes fit with each other leaving no gaps and no overlaps. You will find that they do fit with each other (Figure 2a). A square tessellates as well (Figure 2b). Regular pentagons, however, do not fit with each other; they leave gaps (Figure 3a). Regular hexagons do fit with each other (Figure 2c), and the pattern extends infinitely over the plane. Regular heptagons overlap (Figure 3b) and thus do not tessellate. You may now test if the regular nonagons (9-sided), regular decagons (10-sides) or regular dodecagons (12-sided) tessellate or not.

So what makes just three regular polygons tessellate? To ensure no gaps and no overlaps, the polygons must fit with each other to make an exact 360° around each vertex. Only in the case of equilateral triangles, squares and regular hexagons can this be done, as their interior angles (60°, 90° and 120° respectively) are divisors of 360°. In other regular polygons, the interior angles are not divisors of 360°, and the shapes do not tessellate. For example, in the case of a regular pentagon, the interior angle is 108°. When we place three such pentagons together, we find a gap of 360° 324 = 36°. For regular polygons with more than six sides we find an overlap, as the sum of three interior angles exceeds 360°. So these shapes will not tessellate.

The next question is: Can we find *combinations* of regular polygons which tessellate? Will a combination of squares and equilateral triangles tessellate? How about a combination of regular pentagons and regular heptagons? Can you predict which combinations of regular polygons will tile and which will not? Table 1 shows the interior angles of regular polygons.

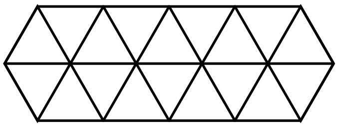


Figure 2a

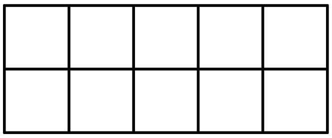


Figure 2b

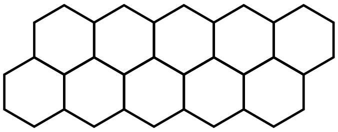


Figure 2c

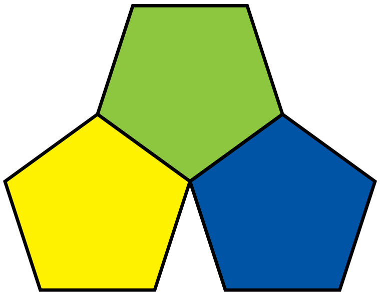


Figure 3a

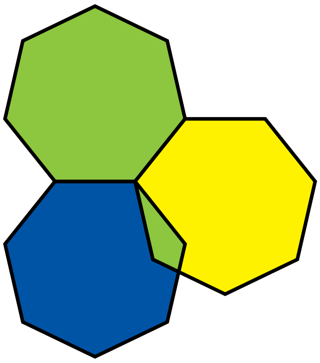


Figure 3b

Sides	3	4	5	6	8	9	10	12	18	20	n
Angle	60°	90°	108°	120°	135°	140°	144°	150°	160°	162°	$180^\circ(n-2)/n$

Table 1: Interior angles of various regular polygons

S. No.	1	2	3	4	5
Combination	(3.3.3.3.3.3)	(3.3.3.4.4)	(3.3.3.3.6)	(4.4.4.4)	(3.4.4.6)

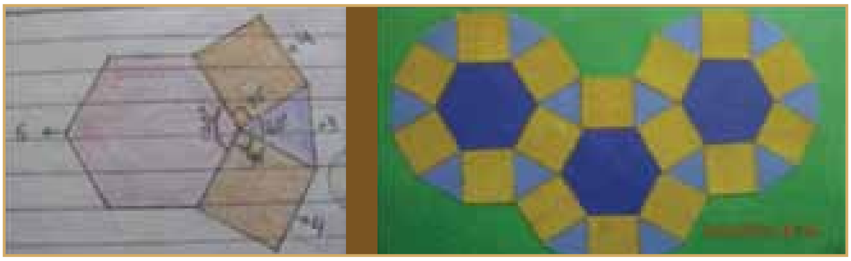
S. No.	6	7	8	9	10	11
Combination	(3.3.6.6)	(3.3.4.12)	(6.6.6)	(5.5.10)	(4.8.8)	(4.6.12)

S. No.	12	13	14	15	16	17
Combination	(4.5.20)	(3.12.12)	(3.10.15)	(3.9.18)	(3.8.24)	(3.7.42)

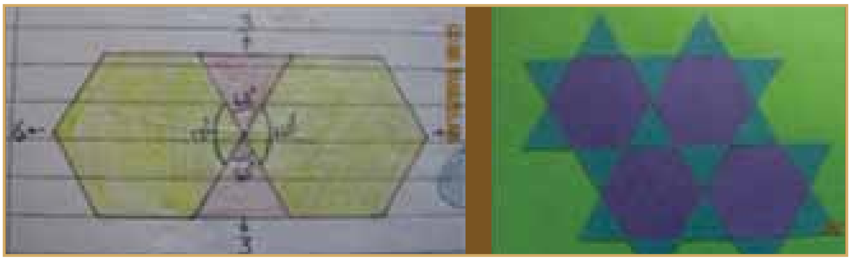
Table 2: Combination of polygons that fill 360° at a vertex

By simply adding angles, we ought to be able to predict the combinations of polygons that *may tessellate*. Three equilateral triangles and two squares may tessellate, as their interior angles add up to 360°. So also an equilateral triangle, a regular decagon and a regular 15-sided polygon; two pentagons and a decagon may tessellate. By analysing the possibilities, we find that there are 17 different combinations of regular polygons that *might* fit together at a vertex to make an angle of 360°. These possibilities are listed in Table 2.

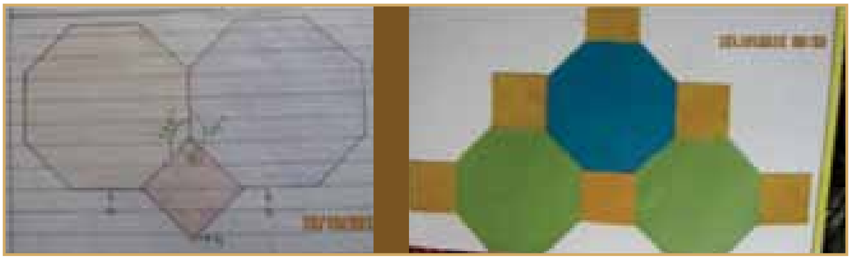
The notation in Table 2 is as follows. Consider the combination (3.7.42). This means that each vertex



(3. 4. 6. 4) combination and its tessellation



(3. 6. 3. 6) combination and its tessellation



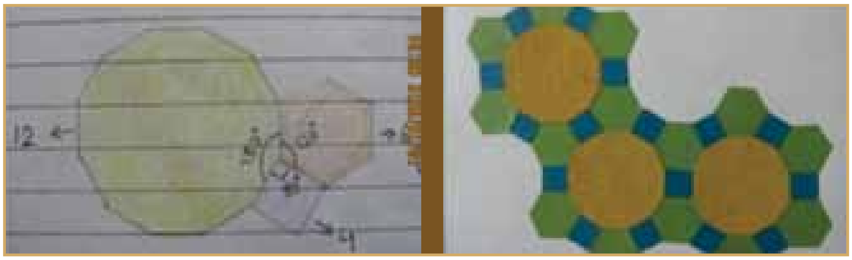
(4. 8. 8) combination and its tessellation

in the tessellation is shared by an equilateral triangle, a regular 7-sided polygon and a regular 42-sided polygon. Similarly, for the combination (3. 4. 4. 6) each vertex of the tessellation is shared by an equilateral triangle, two squares and a regular hexagon.

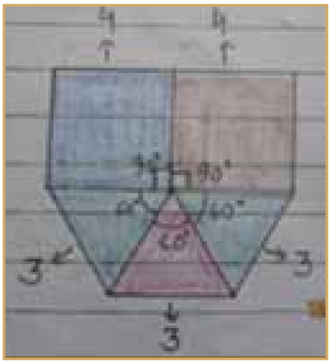
Listing the possibilities using arithmetical reasoning tells us only half of the story. Example: We have listed (3. 3. 3 .4. 4) as a possibility since $60^\circ + 60^\circ + 60^\circ + 90^\circ + 90^\circ = 360^\circ$. But this does not mean that we can actually exhibit a tessellation with this pattern. So we must ascertain *experimentally* if each of the patterns tessellates. Here are a few of the findings:

Some surprises are in store for us when we do the experimentation. For example, it may happen that a particular combination that yields 360° on summing gives rise to different spatial arrangements. Example: The interior angles of three equilateral triangles and two squares yield a complete angle ($3 \times 60^\circ + 2 \times 90^\circ = 360^\circ$), but this combination yields two possible spatial arrangements (Figure 4a and 4b).

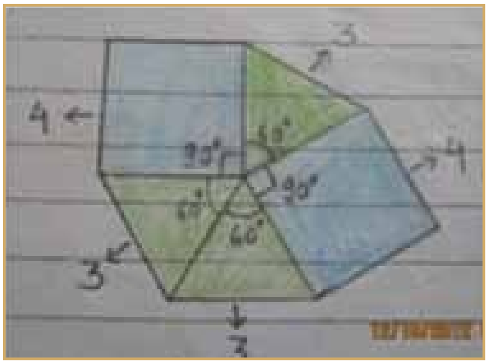
Will both arrangements yield valid tessellations, with the same spatial arrangement being maintained at all vertices? (Such a tessellation is called a **semi-regular tessellation.**)
Yes, they do!



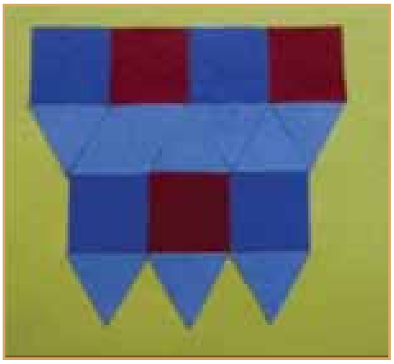
(4. 6. 12) and its tessellation



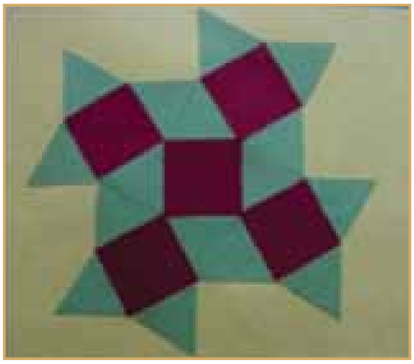
(3. 3. 3. 4. 4)
Figure 4a



(3. 4. 3. 3. 4)
Figure 4b



(3. 3. 3. 4. 4)



(3. 4. 3. 3. 4)

In other cases we find that the opposite happens! Example: Consider the case (3.10.15). It is arithmetically feasible, but by experimentation we find that it does not tessellate.

In Part-II of this article we show how to modify regular tessellations and produce attractive artwork.

(Editor's note. The mathematical note following this explains how to find all possible semi-regular tessellations, and why certain combinations like (3.10.15) do not tessellate.)

Further Reading

i. http://library.thinkquest.org/16661/index2.html#anchor-top

ii. http://euler.slu.edu/escher/index.php/Tessellations_by_Recognizable_Figures

iii. http://www.mcescher.com (biography and works of M.C. Escher)

iv. http://www.mcescher.com/Gallery/gallery-symmetry.htm
(Escher's tessellations that can be taken up for class discussions)

Sources of some pictures

i. http://www.mcescher.com/Gallery/gallery-symmetry.htm

Acknowledgements

The photographs have been taken by me and my B Ed students (batch 2012-13). I acknowledge my students' efforts in making intricate tessellations. I extend my thanks to them for allowing me to use their designs for this article.

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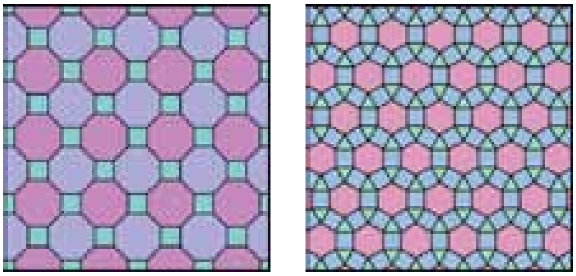
feature

Enumeration of Semi-regular Tessellations

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

In this note we show how the semi-regular tessellations can be enumerated. We describe only the approach and give a partial solution. First we recall the definition: *A semi-regular tessellation is a filling of the plane with regular polygons of two or more kinds, such that the polygons with a given number of sides are congruent copies of one another, and the pattern of placement of the polygons is the same at every vertex of the tessellation.* Figure 1 shows two semi-regular tessellations. Pattern I is made up of squares and regular octagons. Going around each vertex we meet a square and two octagons, so we associate the tuple (4,8,8) with the pattern. Pattern II is made up of equilateral triangles, squares and regular hexagons, and we associate the tuple (3,4,6,4) with the pattern.



Pattern I: (4,8,8)

Pattern II: (3,4,6,4)

Figure 1.
Source: http://en.wikipedia.org/wiki/Tiling_by_regular_polygons

Keywords: *Tessellation, semi-regular, tiling*

We now describe here how all such tuples can be enumerated, but we leave the task for you to complete. Let the tuple be $(n_1, n_2, \dots, n_k)$ where each n_i is a positive integer. Since each polygon has at least three sides, we have $n_i \geq 3$ for every i. We start by showing that $k \leq 6$. That is, *there can be no more than six polygons meeting at each vertex*.

Recall that the internal angle of a regular n-sided polygon is $180^\circ - 360^\circ/n$. Since the total angle at each vertex is 360°, it follows that

$$\left(180^\circ - \tfrac{360^\circ}{n_1}\right) + \left(180^\circ - \tfrac{360^\circ}{n_2}\right) + \cdots + \left(180^\circ - \tfrac{360^\circ}{n_k}\right) = 360^\circ. \tag{1}$$

Dividing through by 180° we get:

$$\left(1 - \frac{2}{n_1}\right) + \left(1 - \frac{2}{n_2}\right) + \cdots + \left(1 - \frac{2}{n_k}\right) = 2. \tag{2}$$

There are k bracketed terms. Opening the brackets and simplifying, we get:

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} = \frac{k}{2} - 1. \tag{3}$$

We need to find tuples $(n_1, n_2, \dots, n_k)$ of positive integers satisfying (3) and the condition that $n_i \geq 3$ for all i. This condition implies that $1/n_i \leq 1/3$ for every i, and hence that:

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} \leq \frac{k}{3}. \tag{4}$$

From (3) and (4) we deduce:

$$\frac{k}{2} - 1 \leq \frac{k}{3}, \qquad \therefore\ \frac{k}{2} - \frac{k}{3} \leq 1, \qquad \therefore\ \frac{k}{6} \leq 1, \tag{5}$$

which leads to $k \leq 6$, as claimed. On the other hand, $k \geq 3$, for we cannot have less than three polygons meeting at a vertex. So $k \in \{3, 4, 5, 6\}$. Thus, k can take just four possible values, and we can enumerate the solutions of (3) by proceeding case by case. For now we examine only the case $k = 3$, and the leave the others for you.

For convenience rename n_1, n_2, n_3 as a, b, c. There is no harm in relabeling them so that $a \leq b \leq c$. Since $k/2 - 1 = 3/2 - 1 = 1/2$, the system we have to solve is:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}, \qquad 3 \leq a \leq b \leq c. \tag{6}$$

From $a \leq b \leq c$ we get

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{a} + \frac{1}{a} + \frac{1}{a} = \frac{3}{a}. \tag{7}$$

Therefore $3/a \geq 1/2$, and $a \leq 6$. Hence $a \in \{3, 4, 5, 6\}$. We now take up each possibility in turn.

a = 3 We have: $1/b + 1/c = 1/2 - 1/3 = 1/6$. Hence $bc - 6(b + c) = 0$. Adding 36 to both sides to achieve a factorization we get $(b - 6)(c - 6) = 36$. From the factorization of 36 we infer that $(b - 6, c - 6)$ is one of the pairs $(1, 36)$, $(2, 18)$, $(4, 9)$, $(6, 6)$. Hence (b, c) is one of the following: $(7, 42)$, $(8, 24)$, $(9, 18)$, $(10, 15)$, $(12, 12)$.

a = 4 In the same way we get: $1/b + 1/c = 1/2 - 1/4 = 1/4$, hence $bc - 4(b + c) = 0$, which yields $(b - 4)(c - 4) = 16$. So $(b - 4, c - 4)$ is one of the pairs $(1, 16)$, $(2, 8)$, $(4, 4)$, implying that (b, c) is one of the following: $(5, 20)$, $(6, 12)$, $(8, 8)$.

a = 5 This time we are led to the equation $3bc - 10(b + c) = 0$. Solving this the same way (and remembering that $a \leq b$), we find that $(b, c) = (5, 10)$. (Details omitted.)

a = 6 This time we are led to the equation $bc - 3(b + c) = 0$. Solving this the same way (and remembering that $a \leq b$), we find that $(b, c) = (6, 6)$. (Details omitted.)

Hence (a, b, c) is one of the following triples: $(3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$, $(3, 10, 15)$, $(3, 12, 12)$, $(4, 5, 20)$, $(4, 6, 12)$, $(4, 8, 8)$, $(5, 5, 10)$, $(6, 6, 6)$. Of these, the last is a *regular* tessellation, as there is just one type of polygon (a regular hexagon).

We now examine the other triples in the list. We shall eliminate many of them using a clever parity argument. Here is the claim: *If any one out of a, b, c is odd, then the other two numbers must be equal*. The proof may be grasped by examining any triple with an odd entry, say $(3, 10, 15)$. The numbers tell us that around each vertex there is an equilateral triangle, a regular decagon and a regular 15-sided polygon. Focusing attention on the triangle and going around its edges, we see that the decagon and 15-sided polygon must come in alternation. But this is impossible, since the triangle has an odd number of sides!

This argument extends to all triples with one odd number and two other numbers which are unequal. After eliminating the triples which do not conform to the rule, we find that only these remain: $(3, 12, 12)$, $(4, 6, 12)$ and $(4, 8, 8)$. Each of these corresponds to a genuine semi-regular tessellation.

The arguments for the cases $k = 4, 5, 6$ may be conducted along similar lines, though there are many more subtleties involved. But for now, we leave these to the reader. For further reading please refer to the following:

http://en.wikipedia.org/wiki/Tiling_by_regular_polygons

http://www.mathsisfun.com/geometry/tessellation.html

http://mathforum.org/sum95/suzanne/whattess.html

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A needle - ing problem

It was a peculiar looking board, with all sorts of odd shapes carved out of a wooden plank. I stood there, examining it for a while, trying to understand its place in an exhibition on the ‘Mathematics of Planet Earth’. Observing my confusion, one of the organisers walked up to me. “Do you see the little sticks with the red bulb at one end that those kids are holding? At the other end of it is a thin metallic strip. Do you think you can rotate it 180 degrees within each of these shapes carved out of the board? Naturally, you are not permitted to lift the rod off the board at any point in time, as our enterprising young friend is attempting here.” he said, walking a couple of steps to explain to the game to the child in question.

While patiently waiting for my turn, I thought about the task at hand. It seemed unlikely. The rod was rigid, unbending and the shapes, although they began fairly regularly with the circle, soon became strange. When my turn finally came, I picked up the rod and gradually tried to manoeuvre it this way and that. With some effort, however, I finally managed to wiggle the rod around and rotate it the desired way within each of the figures cut out. Feeling rather pleased with myself for having worked out the brain teaser, I looked up, only to find our organiser friend observing my handiwork. “Not bad!”, he exclaimed. “Do you see this triangle with the sides having an ‘outward bulge’? Say that has area A. Then this regular triangle alongside it should, intuitively, have area slightly less than A. What about the area of this ‘inward bulging’ triangle adjacent to this? That will have even smaller area!” I could see where he was going with this line of reasoning: Would this ever stop? Is there a ‘smallest’ set (in terms of area) within which you can perform this rotation? I thought about it for a while before I ventured a guess. Why don’t you try the same, and then turn to page 35 to check if our ideas match!

Musing on the primes

There are Infinitely Many Primes – II

But how many proofs of this?

In Part–I of this article we dwelt on various proofs that show the infinitude of the primes. These were mostly based on Euclid's proof — the one for which G H Hardy had such high praise. All of these start by assuming that there exists a 'last prime'. Then in a clever way they construct a number whose prime factors exceed this last prime. The one proof discussed which does not belong to this category is Pólya's; he makes use of the Fermat numbers. The first proof of a completely different nature is Euler's; he shows that the sum of the reciprocals of the primes is infinite, and hence there must exist infinitely many primes. In Part–II we dwell on this proof.

V G Tikekar

Convergent and divergent series

Say you have a set S of numbers. You want to know whether there are finitely many elements in S, or infinitely many. How may we do this? Here is a possible strategy: *Add up all the numbers in* S. If the sum if infinite, then surely S must have infinitely many elements!

Note that this strategy works only in one direction: If the sum is infinite, then S has infinitely many elements. But if the sum is finite, we cannot say anything about the size of S. This strange situation at one time in history looked impossible, and all kinds of paradoxes arose because of that, like Zeno's paradox. But it is easy

Keywords: *Prime, composite, infinite, Euler, fundamental theorem of arithmetic, divergent series*

to show that one *can* add infinitely many numbers and reach a finite sum. The simplest example of this is the following infinite decimal:

$$x = 0.1111111\ldots,$$

i.e., the recurring decimal made up of 1s. It is clearly a sum of infinitely many numbers:

$$x = 0.1 + 0.01 + 0.001 + 0.0001 + 0.00001 + 0.000001 + 0.0000001 + \cdots.$$

It is easy to show that $x = 1/9$ (to see this, work out $1/9$ in decimal form using long-division; or multiply the above relation by 10 and then subtract the original relation). So here we have a case where infinitely many positive quantities when added yield a finite number.

The above is of course a special case of the general result:

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x}, \quad \text{valid for all real } x \text{ with } |x| < 1. \tag{1}$$

Two nice special cases of this result are:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 1 \quad (\text{with } x = \tfrac{1}{2}),$$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots = \frac{1}{2} \quad (\text{with } x = \tfrac{1}{3}).$$

What about cases where the sum is infinite? An instance which is quite uninteresting is:

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + \cdots = \infty.$$

(This is (1) with $x = 1$.) We note an important point here. When we write "$\cdots = \infty$"it is not as though ∞ is a number, like 1 or 2. The phrase "$\cdots = \infty$"is merely a short form to mean that the sum in question has no bound; *by adding a sufficient number of terms, we can get the sum to exceed any given bound*.

Historically, the first result in this area which has genuine surprise value is this:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty, \quad \text{i.e.,} \sum_{1}^{\infty} \frac{1}{n} = \infty. \tag{2}$$

This result is known as 'divergence of the harmonic series'. (The numbers $1, 1 + 1/2, 1 + 1/2 + 1/3, \ldots$are called the 'harmonic numbers', and the series $1 + 1/2 + 1/3 + \cdots$ is called the 'harmonic series'.) The proof (first given by Nicolo Oresme) is a standard result in the subject called 'Analysis'. Why do we say that the result has surprise value? *Because it is counter-intuitive*. If we introduce the harmonic series to students in (say) class 10 or 11, most of them would venture to guess that the series adds up to a finite number. The numerical evidence does suggest this: the first 1000 terms yield a sum of just 7.48, and the following terms appear to not add very much. (For the case of completeness we do give a proof of divergence of the harmonic series in the Appendix.)

What Euler proved is far more counter-intuitive —he showed that *the sum of the reciprocals of the primes is divergent*:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \infty, \quad \text{i.e.,} \sum_{1}^{\infty} \frac{1}{p_n} = \infty, \tag{3}$$

where p_n is the n^{th} prime. This shows right away that there are infinitely many primes; but it proves much more. Since the corresponding sum for the powers of 2 is finite (as noted above), and the same is true for the perfect squares, that is,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots = \text{a finite number},$$

it means that in some sense the primes are more 'dense' than either the powers of 2 or the perfect squares. We now give a brief sketch of Euler's proof.

Fundamental theorem of arithmetic (FTA)

The number 60 can be written as a product of prime powers thus: $60 = 2^2 \times 3 \times 5$. Is there any other way of writing 60 as such a product? No. How about $1001 = 7 \times 11 \times 13$? Can it be written as a product of prime powers in any other way? No again. (We do not count $11 \times 7 \times 13$ as different from $7 \times 11 \times 13$.) Both these are instances of the Fundamental Theorem of Arithmetic (FTA), a crucial theorem of number theory; and Euler's proof uses the FTA in a basic way.

Here is the statement of the FTA: *Every positive integer greater than* 1 *can be expressed in precisely one way as a product of powers of prime numbers*. The FTA is often taken by students to be 'obviously' true, and the proof is omitted. But there is need for a formal proof. Interested readers should look up reference [1] (pages 3 and 21) or reference [2] (page 23). Crucial to the proof is the following property of prime numbers: *If p is a prime number, and p divides the product ab of two integers a and b, then p divides a or b or both*. Using this property and the principle of induction, a proof for the FTA may be devised.

Euler's observation

In (eq:1) substitute $x = 1/p$ where p is a prime number; we get:

$$\frac{1}{1-1/p} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \cdots. \tag{4}$$

Next put $x = 1/q$ where q is a prime number different from p; we get:

$$\frac{1}{1-1/q} = 1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} + \frac{1}{q^4} + \cdots. \tag{5}$$

Now multiply the corresponding sides of (eq:4) and (eq:5). On the left side we get:

$$\frac{1}{1-1/p} \times \frac{1}{1-1/q}.$$

On the right side we multiply together two infinite series and get another such series:

$$1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{p^2} + \frac{1}{pq} + \frac{1}{q^2} + \frac{1}{p^3} + \frac{1}{p^2 q} + \frac{1}{pq^2} + \frac{1}{q^3} + \cdots.$$

The important thing here is that on the right side we get the reciprocals of all integers of the form $p^i q^j$ where i, j are non-negative integers. *Each such integer occurs just once in the above equality*. It is here that FTA plays its role.

Example: The case $p = 2$ and $q = 5$ yields the following equality, since $1/(1-1/2) \times 1/(1-1/5) = 5/2$, and the positive integers with no prime factors other than 2 and 5 are 1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, 64, 80, 100, ...:

$$\frac{5}{2} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{10} + \frac{1}{16} + \frac{1}{20} + \frac{1}{25} + \frac{1}{32} + \frac{1}{40} + \frac{1}{50} + \frac{1}{64} + \frac{1}{80} + \frac{1}{100} + \cdots.$$

If we consider a third prime r, we get the following equality:

$$\frac{1}{1-1/p} \times \frac{1}{1-1/q} \times \frac{1}{1-1/r} = 1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{pq} + \frac{1}{pr} + \frac{1}{qr} + \frac{1}{p^2} + \cdots,$$

and now on the right side we have the reciprocals of all integers of the form $p^i q^j r^k$, i.e., all positive integers which have no prime factors other than p, q, r. So if p, q, r, ...are distinct prime numbers, then:

- The sum of the reciprocals of all positive integers divisible by no prime number other than p is $1/(1 - 1/p)$.
- The sum of the reciprocals of all positive integers divisible by no prime numbers other than p, q is $1/(1 - 1/p) \times 1/(1 - 1/q)$.
- The sum of the reciprocals of all positive integers divisible by no prime numbers other than p, q, r is $1/(1 - 1/p) \times 1/(1 - 1/q) \times 1/(1 - 1/r)$.

And so on. It was Euler who first thought along such lines, and this led him to say to himself, "Why not list the corresponding relation involving *all* the prime numbers?" Then on the right side we get the reciprocals of *all* the positive integers, each occurring just once. Euler thus wrote:

$$\frac{1}{1-1/2} \times \frac{1}{1-1/3} \times \frac{1}{1-1/5} \times \frac{1}{1-1/7} \times \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots. \tag{6}$$

This is just how Euler wrote the relation. (On the left side, 2, 3, 5, 7, ... are the primes, in sequence, and on the right side, 1, 2, 3, 4, 5, 6, ... are the positive integers, in sequence.) Today, with standards of rigour having changed over the centuries, we do not write it in quite this way, since neither side of (**6**) is a finite number! However in this article we will go with Euler and write the relation in his style.

Euler's proof

The rest of the proof now writes itself. Suppose that there are only finitely many primes $p_1, p_2, ..., p_n$; that is, there are just n primes, the largest one being p_n. Then statement (**6**) reads:

$$\frac{1}{1-1/p_1} \times \frac{1}{1-1/p_2} \times \cdots \times \frac{1}{1-1/p_n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots. \tag{7}$$

The left side of this equality is obviously a finite number, being the product of finitely many fractions. But the right side is the harmonic series, and we know that this series diverges; so the right side is infinitely large! It follows that statement (**7**) is an absurdity.

Tracing backwards we see that the contradiction arises from the supposition that the number of primes is finite. We conclude that there are infinitely many primes.

Exercises

1. Show that the sum of the reciprocals of all positive integers with no prime factors other than 2 and 3 (i.e., the integers 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, ...) is 3.
2. Find the sum of the reciprocals of all positive integers with no prime factors other than 3 and 5, i.e., the integers 1, 3, 5, 9, 15, 25, 27, 45, 75, ...

Appendix: Divergence of the harmonic series

Here is one way of proving that the sum of the reciprocals of the positive integers is infinite. Remember that to show divergence means showing that by adding sufficiently many terms, we can get the sum to exceed any bound given in advance. This is what we shall do for the harmonic series. First we group the positive integers into finite sets as follows: $\{1\}, \{2, 3\}, \{4, 5, 6, 7\}$, $\{8, 9, 10, 11, 12, 13, 14, 15\}$, ... Observe that each set starts with a power of 2 and goes up to the number just short of the next higher power of 2. We now show the sum of the reciprocals of the numbers in each set exceeds $1/2$. For example, take the set $\{4, 5, 6, 7\}$. Since 8 exceeds each number in the set, and there are 4 numbers in the set, we have:

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7} > 4\times\frac{1}{8}=\frac{1}{2}.$$

Similarly, take the set $\{8, 9, 10, 11, 12, 13, 14, 15\}$. Since 16 exceeds each number in the set, and there are 8 numbers in the set, we have:

$$\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15} > 8\times\frac{1}{16}=\frac{1}{2}.$$

In the same way we show that for each set, the sum of the reciprocals exceeds $1/2$. So if we want the sum to, say, exceed 10, it suffices if we include 20 of these sets. Since the 20^{th} set has the integers from 2^{19} till $2^{20}-1=1048575$, this means that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{1048575} > 10.$$

Hence the sum can exceed any bound given in advance. This means that the series diverges, as claimed. See reference [3] for more such proofs.

References

1 G H Hardy & E M Wright, *An Introduction to the Theory of Numbers*, Oxford (Fourth edition)

2 I Niven, H S Zuckerman & H Montgomery, *An Introduction to the Theory of Numbers*, John Wiley (Fifth edition)

3 http://en.wikipedia.org/wiki/Harmonic_series_(mathematics)

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feature

PHTs . . . Primitive and beautiful

Harmonic Triples

Part 3

Observe a relationship, then prove it – satisfying in itself.
Take this one step further and find the geometrical connect.
Excitement squared!

Shailesh Shirali

In Parts–1, 2 of this article which appeared in the March 2013 and July 2013 issues of *At Right Angles*, we introduced the notion of a *primitive harmonic triple* ('PHT') as a triple (a, b, c) of coprime positive integers satisfying the equation

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c}.$$

Examples: (3,6,2) and (6,30,5). We showed that the equation surfaces in many contexts, and we explored ways of generating PHTs. Now we explore some properties of these triples. Table 1 lists many of the triples. (To avoid duplication we have added the condition $a \le b$.) It is worth studying the list to identify features of interest.

(2,2,1),	(3,6,2),	(4,12,3),	(5,20,4),
(6,30,5),	(7,42,6),	(8,56,7),	(9,72,8),
(10,15,6),	(10,90,9),	(14,35,10),	(18,63,14),
(21,28,12),	(22,99,18),	(24,40,15),	(30,70,21),
(33,88,24),	(36,45,20),	(44,77,28),	(55,66,30),
(60,84,35),	(65,104,40),	(78,91,42),	(105,120,56).

Table 1. A list of some primitive harmonic triples (PHTs)

Keywords: *Primitive harmonic triples, perfect squares, rhombus, triangle*

Perfect squares

Among the many noticeable features of PHTs, the one that strikes the eye most is the presence of numerous perfect squares associated with each triple.

Proposition 1 *If* (a, b, c) *is a primitive harmonic triple, then* $a + b$*,* $a - c$ *and* $b - c$ *are perfect squares.*

> Example: Take the PHT $(10, 15, 6)$; we have: $10 + 15 = 5^2$, $10 - 6 = 2^2$, $15 - 6 = 3^2$. But still more can be said.

Proposition 2 *If* (a, b, c) *is a primitive harmonic triple, then* abc *is a perfect square.*

> Example: Take the PHT $(10, 15, 6)$; we have: $10 \times 15 \times 6 = 900 = 30^2$.

Four perfect squares associated with each primitive harmonic triple! Remarkable. But the claims are easier to prove than one may expect. To do so we use the complementary factor algorithm obtained in Part–2 for finding such triples.

The algorithm recalled

For readers' convenience we derive the algorithm afresh. Suppose that a, b, c are positive integers such that $a \le b$ and $1/a + 1/b = 1/c$. By clearing fractions we get:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}, \quad \therefore \ \frac{a+b}{ab} = \frac{1}{c}, \quad \therefore \ ac + bc = ab,$$

hence $ab - ac - bc = 0$. Adding c^2 to both sides we get:

$$ab - ac - bc + c^2 = c^2, \quad \therefore \ (a - c)(b - c) = c^2.$$

So $a - c$ and $b - c$ are a pair of divisors of c^2 whose product is c^2. (Thus, they are a pair of 'complementary divisors' of c^2.) So the algorithm to generate harmonic triples is:

1. Select a positive integer c.
2. Write c^2 as a product $u \times v$ of two positive integers, where $u \le v$.
3. Let $a = c + u$ and $b = c + v$.
4. Then $(a, b, c) = (c + u, c + v, c)$ is a harmonic triple in which $a \le b$.

In Part–2 of the article (July 2013 issue of *At Right Angles*) we remarked that to ensure that a, b, c are coprime (i.e., ensure that the triple is primitive), it is necessary as well as sufficient that u and v be coprime. Now we establish this claim.

Suppose that $(a, b, c) = (c + u, c + v, c)$ is *not* primitive; then there exists a number $k > 1$ which divides each number in the triple. Since k divides both $c + u$ and c, it divides u. Since k divides both $c + v$ and c, it divides v. Therefore, k divides both u and v. Hence u and v are not coprime. Taking the contrapositive of this finding, we deduce that if u and v are coprime, then the triple $(c + u, c + v, c)$ is primitive.

Next we must show the converse: if u and v are not coprime then $(c + u, c + v, c)$ is not primitive. Let p be a prime number which divides u as well as v. Then p^2 is a divisor of c^2, since $uv = c^2$. Since p is prime, it follows that p divides c. Hence p divides each number in the triple $(c + u, c + v, c)$. Consequently the triple is not primitive.

Proof of Proposition 1

Let (a, b, c) be a PHT; then there exist coprime positive integers u and v such that $uv = c^2$, $a = c + u$, $b = c + v$. Since u and v are coprime and their product is a perfect square, each of them must be a perfect square, say $u = r^2$ and $v = s^2$; this yields $a - c = r^2$ and $b - c = s^2$, showing directly that both $a - c$ and $b - c$ are perfect squares, as claimed. Now consider $a + b$. Since $a = c + r^2$ and $b = c + s^2$ and $rs = c$, we have:

$$\begin{aligned} a + b &= c + r^2 + c + \frac{c^2}{r^2} = r^2 + 2c + \frac{c^2}{r^2} \\ &= \left(r + \frac{c}{r}\right)^2 = (r + s)^2. \end{aligned}$$

So $a + b$ too is a perfect square.

Another pretty relation

We have shown that if (a, b, c) is a PHT, then

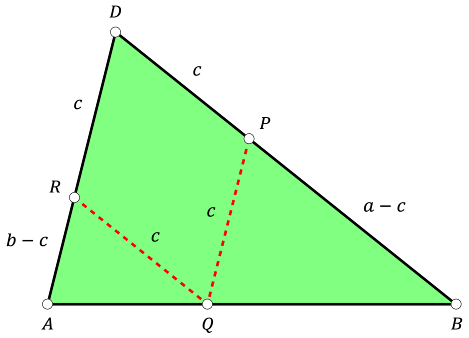
$$a + b = \left(\sqrt{a - c} + \sqrt{b - c}\right)^2.$$

Example: Take the PHT $(10, 15, 6)$; we have: $10 + 15 = 25 = \left(\sqrt{10 - 6} + \sqrt{15 - 6}\right)^2$.

Geometric interpretation of an algebraic relation

We now give a geometric interpretation to the following fact: If (a, b, c) is a harmonic triple then $(a - c)(b - c) = c^2$.

Given a $\triangle ADB$, let a rhombus $DPQR$ be inscribed in the triangle, with P on DB, Q on AB, and R on DA. Figure 1 shows the completed picture. Let a, b, c be the lengths indicated ($a = DB$, $b = DA$, $c =$ side of the rhombus). We had shown earlier that $1/a + 1/b = 1/c$.



- $DB = a$
- $DA = b$
- DP, PQ, QR, RD all have length c

Figure 1: *Rhombus inscribed in a triangle*

Now note the similarity $\triangle BPQ \sim \triangle QRA$, which follows from the relations $PQ \parallel DA$ and $RQ \parallel DB$. From this we get the following relation:

$$\frac{a - c}{c} = \frac{c}{b - c}.$$

By cross-multiplying we get the desired relation: $(a - c)(b - c) = c^2$.

More propositions

Proposition 3 *If* (a,b,c) *is a PHT then* $a+b+c$ *is coprime to* 6.

Proposition 4 *If* (a,b,c) *is a PHT then* $a+b-c$ *is coprime to* 10.

Proposition 5 *If* (a,b,c) *is a PHT then* $\gcd(a,b)=\gcd(a,c)+\gcd(b,c)$.

Example: Consider the PHT $(10,15,6)$. Observe that:

- $10+15+6=31$ is coprime to 6;
- $10+15-6=19$ is coprime to 10;
- $\gcd(10,15)=5=2+3=\gcd(10,6)+\gcd(15,6)$.

Proof of Proposition 3

- Let $a=c+r^2$, $b=c+s^2$ where r and s are coprime, and $rs=c$. Then:

  $$a+b+c=3c+r^2+s^2.$$

  Since r and s are coprime, it cannot be that both are divisible by 3. So at most one of r,s is a multiple of 3.

- Recall that if n is not a multiple of 3, then $n^2\equiv 1 \pmod 3$.

  [We do not really need to `recall' it, for it has a one-line proof. We only need to check that $1^2\equiv 1 \pmod 3$ and $2^2\equiv 1 \pmod 3$.]

- Suppose that just one of r,s is a multiple of 3; assume it is r. Then $r^2\equiv 0 \pmod 3$ and $s^2\equiv 1 \pmod 3$, therefore $r^2+s^2\equiv 1 \pmod 3$, and $a+b+c\equiv 1 \pmod 3$.
- If both r and s are non-multiples of 3, then $r^2\equiv 1 \pmod 3$ and $s^2\equiv 1 \pmod 3$, hence $r^2+s^2\equiv 2 \pmod 3$, and $a+b+c\equiv 2 \pmod 3$.
- So in both cases $a+b+c$ is a non-multiple of 3.
- Now we must show that $a+b+c$ is odd. We do so by focusing on the parity of c, i.e., its odd/even nature.
- If c is odd, then r and s are both odd, hence $a+b+c=3c+r^2+s^2$ is the sum of three odd numbers and so is odd. If c is even, then since $rs=c$ and r and s are coprime, it must be that one of $\{r,s\}$ is even and the other is odd. So $a+b+c$ is the sum of two even numbers and one odd number, and hence is odd.

  So in all cases, $a+b+c$ is odd and a non-multiple of 3, and so is coprime to 6.

Proof of Proposition 5

Since the general PHT (a,b,c) has $c=rs$, $a=c+r^2$, $b=c+s^2$ where $\gcd(r,s)=1$, we must prove the following:

$$\gcd(rs+r^2,rs+s^2)=\gcd(rs+r^2,rs)+\gcd(rs+s^2,rs).$$

But this is clear, because:

$$\begin{aligned}\gcd(rs+r^2,rs+s^2)&=\gcd(r(s+r),s(s+r))=s+r,\\ \gcd(rs+r^2,rs)&=\gcd(r(r+s),rs)=r,\\ \gcd(rs+s^2,rs)&=\gcd(s(s+r),sr)=s.\end{aligned}$$

Proofs of Propositions 2 and 4

We leave these for you. Readers must do some work, too!
There are yet more such properties that you may want to explore. For example:

Proposition 6 *If* (a, b, c) *is a PHT, then* $a + b + c$ *is not divisible by* 7.

Proposition 7 *If* (a, b, c) *is a PHT, then* $a + b - c$ *is not divisible by* 11.

See if you can spot (and prove) more such properties!

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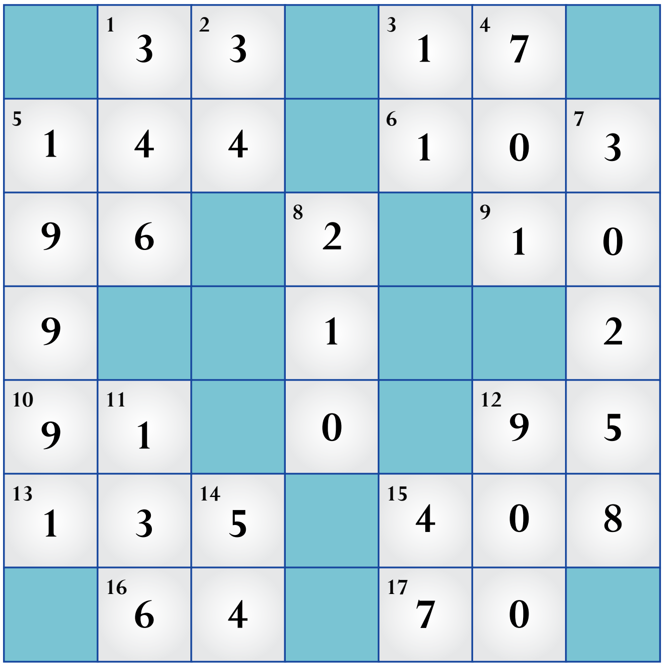
Solution for the Number Crossword in Issue-II-3 (November 2013)



■	^{1}3	^{2}3	■	^{3}1	^{4}7	■
^{5}1	4	4	■	^{6}1	0	^{7}3
9	6	■	^{8}2	■	^{9}1	0
9	■	■	1	■	■	2
^{10}9	^{11}1	■	0	■	^{12}9	5
^{13}1	3	^{14}5	■	^{15}4	0	8
■	^{16}6	4	■	^{17}7	0	■

Of Art & Math:

Introducing Symmetry

PUNYA MISHRA
GAURAV BHATNAGAR

In our November column we introduced the concept of ambigrams—the art of writing words in surprisingly symmetrical ways. Consider an ambigram of the word "Symmetry" (Figure 1).

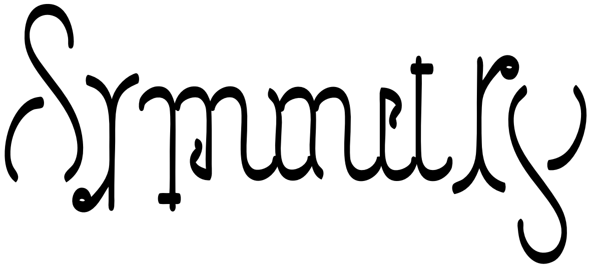


Figure 1. A symmetric ambigram for Symmetry

The design itself displays *rotational symmetry, i.e. it looks the same even when rotated by 180 degrees.* In other words, it remains *invariant* on rotation. Figure 2 shows an ambigram for "invariant" with a similar property.

Keywords: *Ambigrams, calligraphy, symmetry, perception, mapping, transformation, reflection, translation, invariance, function, inverse*

Figure 2. An ambigram for “invariant” that remains invariant on rotation

Invariance can also be seen in reflection. Figure 3 gives a design for the word “algebra” that is invariant upon reflection, but with a twist. You will notice that the left hand side is NOT the same as the right hand side and yet the word is still readable when reflected. So the invariance occurs at the level of meaning even though the design is not visually symmetric!

Figure 3. An ambigram for ‘algebra’ that remains invariant on reflection. But is it really symmetric?

In this column, we use ambigrams to demonstrate (and play with) mathematical ideas relating to symmetry and invariance.

There are two common ways one encounters symmetry in mathematics. The first is related to graphs of equations in the coordinate plane, while the other is related to symmetries of geometrical objects, arising out of the Euclidean idea of congruence. Let’s take each in turn.

Symmetries of a Graph

First let us examine the notions of symmetry related to graphs of equations and functions. All equations in x and y represent a relationship between the two variables, which can be plotted on a plane. A graph of an equation is a set of points (x, y) which satisfy the equation. For example, $x^2 + y^2 = 1$ represents the set of points at a distance 1 from the origin—i.e. it represents a circle.

Let (a, b) be a point in the first quadrant. Notice that the point $(-a, b)$ is the reflection of (a, b) in the y-axis (See Figure 4a). Thus if a curve has the property that $(-x, y)$ lies on the curve whenever (x, y) does, it is symmetric across the y-axis. Such functions are known as even functions, probably because $y = x^2, y = x^4, y = x^6, \ldots$ all have this property. Figure 4b shows the graph of $y = x^2$; this is an even function whose graph is a parabola.

Similarly, a curve is symmetrical across the origin if it has the property that $(-x, -y)$ lies on the curve whenever (x, y) does. Functions whose graph is of this kind are called odd functions, perhaps because $y = x, y = x^3, y = x^5, \ldots$ all have this property. See Figure 4c for an odd function.

A graph can also be symmetric across the x-axis. Here $(x, -y)$ lies on the curve whenever (x, y) does. The graph of the equation $x = y^2$ (another parabola) is an example of such a graph. Can a (real) function be symmetric across the x-axis?

Figure 5 shows a chain ambigram for “parabola”. Compare the shape of this ambigram with the graph in Figure 4b. The chain extends indefinitely, just like the graph of the underlying equation!

Figure 5. A parabolic chain ambigram for “parabola”

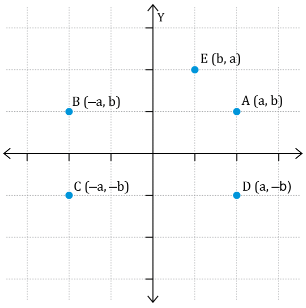


Figure 4a. The point *A (a, b)* and some symmetric points

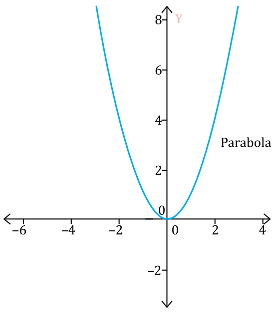


Figure 4b. An even function: $y = x^2$

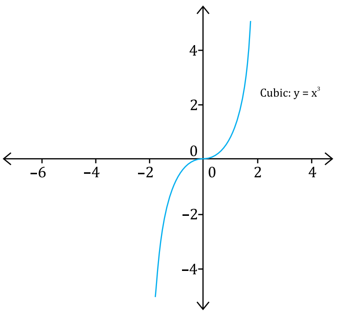


Figure 4c. An odd function: $y = x^3$

The ambigram for “axis of symmetry” (Figure 6) is symmetric across the y-axis. You can see a red y as a part of the x in the middle. So this ambigram displays symmetry across the y axis. At the same time it is symmetric across the letter x!

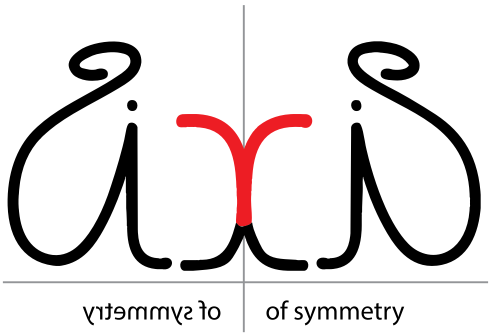


Figure 6. The axis of symmetry: Is it the y-axis or the x?

Another possibility is to interchange the x and y in an equation. Suppose the original curve is C_1 and the one with x and y interchanged is C_2. Thus if (x, y) is a point on C_1, then (y, x) lies on C_2. By looking at Figure 4a, convince yourself that the point (b, a) is the reflection of (a, b) in the line $y = x$, the straight line passing through the origin, and inclined at an angle of 45° to the positive side of the x-axis. Thus the curve C_2 is obtained from C_1 by reflection across the line $y = x$. If C_1 and C_2 (as above) are both graphs of functions, then they are called inverse functions. An example of such a pair: the *exp* (exponential $y = e^x$) and *log* (logarithmic $y = \ln x$) functions (Figure 7).

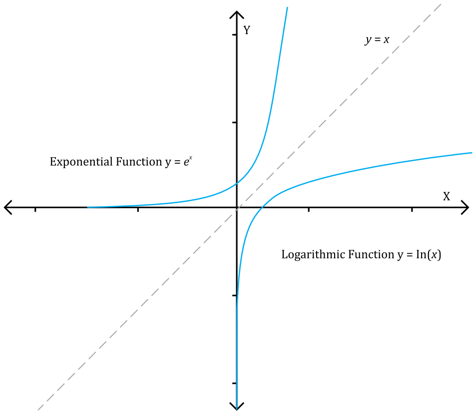


Figure 7. Inverse functions are symmetric across the line $y{=}x$.

Figure 8 is a remarkable design that where *exp* becomes *log* when reflected in the line $y = x$.

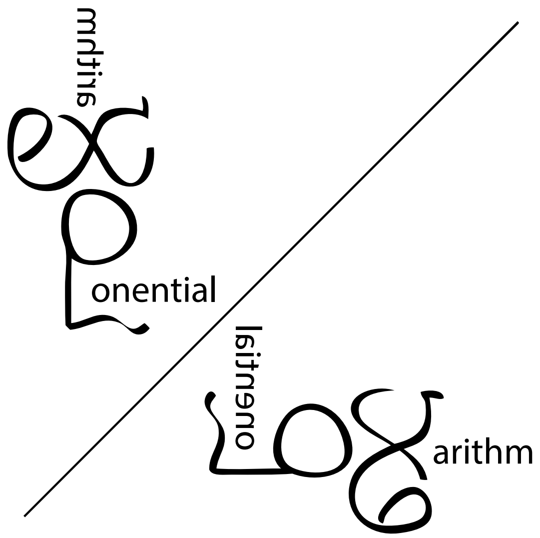


Figure 8: *Exp* becomes *Log* when reflected across the diagonal line!

A great example of an inverse function is the hyperbola $y = 1 / x$, defined for all non-zero real numbers x (Figure 9). Its inverse is obtained by interchanging x with y. But $x = 1 / y$ can be written $y = 1 / x$. So it is its own inverse, and thus symmetric across the line $y = x$.

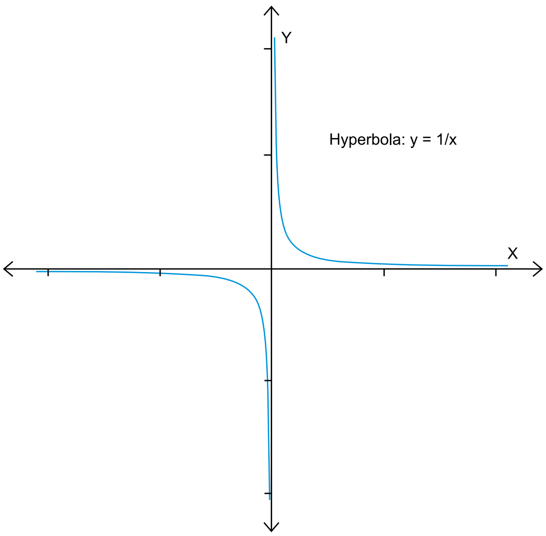


Figure 9. The symmetrical graph of the hyperbola. It is its own inverse. And it’s odd, too!

The ambigram for “inverse” in Figure 10 is inspired by the hyperbola.

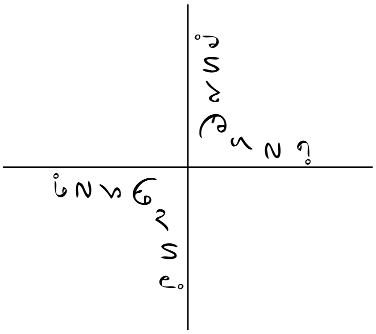


Figure 10. An ambigram for the word “inverse” shaped like a hyperbola.

It is symmetric across the origin and across the line joining the two S's.

Seeking congruence

Another type of symmetry consideration arises from the notion of congruence in plane geometry. Two objects are considered to be congruent if one object can be superposed on the other through rotation, reflection and/or translation.

Figure 11 shows an ambigram of the word “rotate”.

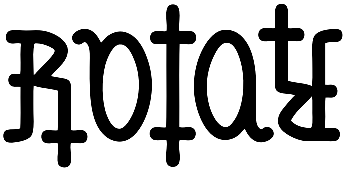


Figure 11. An ambigram for “rotate”. What happens when you rotate it through 180°?

This leads to the question: if we can rotate “rotate” can we reflect “reflect”? Figure 12 is an ambigram for “reflect” that is symmetric around the vertical line in the middle.

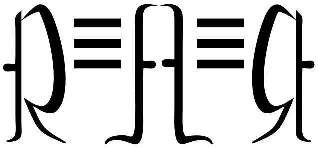


Figure 12. An ambigram for “reflect.” What happens when you hold it to a mirror?

Finally, the third operation is translation. An example of this symmetry is shown by the chain ambigram for sine in Figure 13.

Figure 13. A sine wave ambigram. It displays translation symmetry.

The sine function satisfies many symmetry properties. Perhaps the most important of them is that it is periodic, i.e., if you shift (in other words, translate) the functions by 2π, then you get the same function back. In addition, it is an odd function, and the ambigram is both periodic and odd.

One can of course combine these transformations. This is best understood by looking at the symmetries of an equilateral triangle (see Figure 14) involving both rotation and reflection.

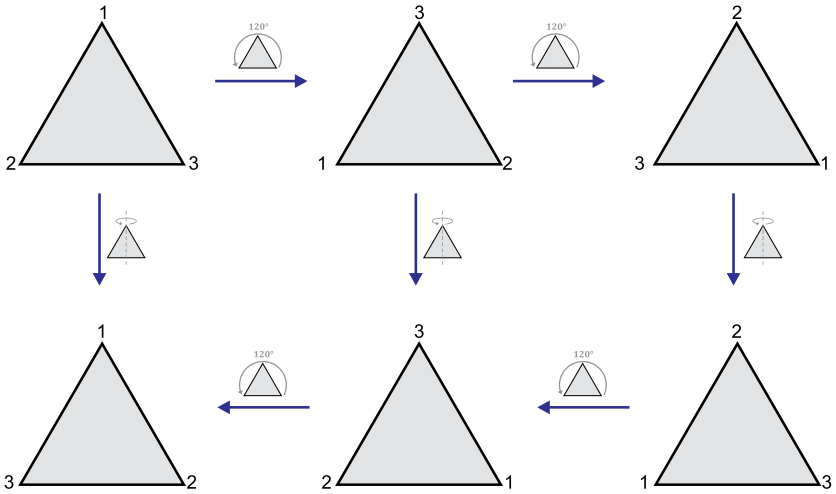


Figure 14. The 6 symmetries of an equilateral triangle.

The 6 symmetries of the equilateral triangle are all found from two fundamental operations: (a) rotation by 120°; and (b) reflection across the line passing through the vertex 1, and At Right Angles to the base of the triangle. Figure 15 shows ambigrams for the word triangle and pentagon. Do they display all the symmetries of the equilateral triangle and regular pentagon (respectively)?

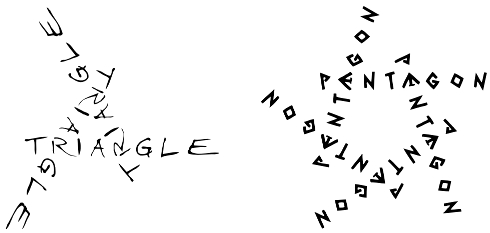


Figure 15: Ambigrams for "triangle" and "pentagon" showing rotational symmetries.

In conclusion

Clearly we have just scratched the surface of the power of symmetry as an idea in mathematics. The philosopher Aristotle once observed that, "the mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful." We agree with Aristotle, but perhaps we would have said "arts" instead of "sciences." In our next article, we will continue to use ambigrams to explore more beautiful mathematical ideas.

Our last article had a secret message. The first letter of each paragraph read: "Martin Gardner lives on in the games we play". This is our homage to Martin Gardner whose writings inspired us when we were growing up. This article has a different puzzle (see Figure 16).

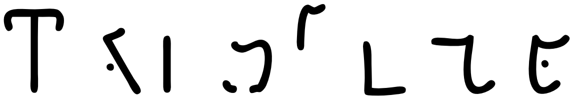
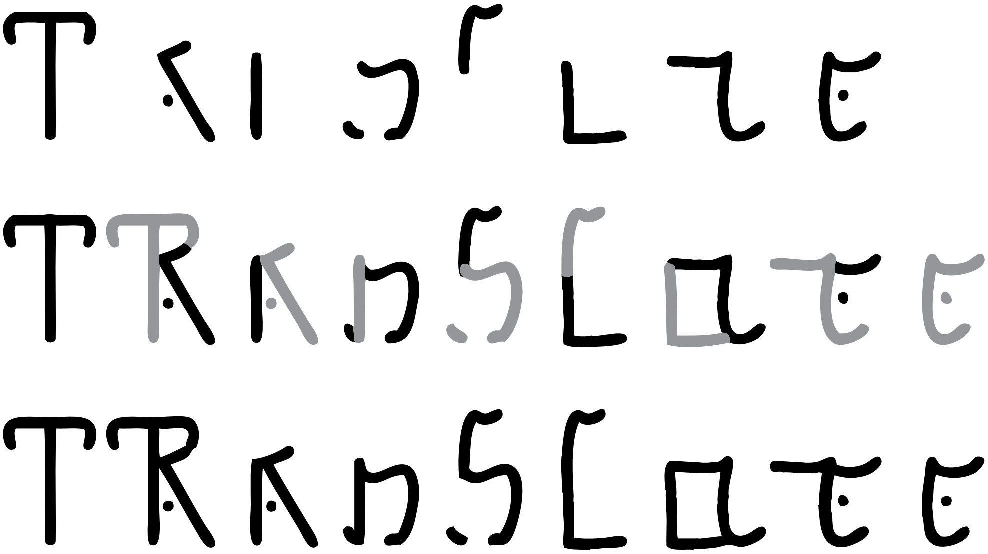


Figure 16. Can you translate this hieroglyphic code? The answer appears below.

ANSWER TO THE PUZZLE

We asked you to "translate" the code. Once you translate (i.e. move) the shapes and align them, you get the answer—the word "Translate." An example of translation symmetry!



About the authors

PUNYA MISHRA, when not creating ambigrams, is professor of educational technology at Michigan State University. GAURAV BHATNAGAR, when not teaching or doing mathematics, is Senior Vice-President at Educomp Solutions Ltd.

Loving both math and art, Punya's and Gaurav's collaboration began over 30 years ago when they were students in high-school. Since then, they have individually or collectively, subjected their friends, family, classmates, and students to a never ending stream of bad jokes, puns, nonsense verse and other forms of deep play. To their eternal puzzlement, their talents have not always been appreciated by their teachers (or other authority figures). Punya's email address is punya@msu.edu and his website is at http://punyamishra.com. Gaurav's email address is bhatnagarg@gmail.com and his website is at http://gbhatnagar.com

All the ambigrams presented in this article are original designs created by Punya Mishra (unless otherwise specified). Please contact him if you need to use them in your own work.

You, dear reader, are invited to share your thoughts, comments, math poems, or original ambigrams at the addresses above.

A RIVER PUZZLE FROM MASTER PUZZLIST SAM LOYD

Two ferry boats ply the same route between ports on opposite sides of a river. They set out simultaneously from opposite ports, but one is faster than the other, so they meet at a point 720 yards from the nearest shore. When each boat reaches its destination, it waits 10 minutes to change passengers, then begins its return trip. Now the boats meet at a point 400 yards from the other shore. How wide is the river?

Comment from Sam Loyd: "The problem shows how the average person, who follows the cut-and-dried rules of mathematics, will be puzzled by a simple problem that requires only a slight knowledge of elementary arithmetic. It can be explained to a child, yet I hazard the opinion that ninety-nine out of every hundred of our shrewdest businessmen would fail to solve it in a week. So much for learning mathematics by rule instead of common sense which teaches the reason why!"

See ***page* 53** for the solution.

Source: http://www.futilitycloset.com/category/puzzles/

A Fair Division

PRITHWIJIT DE, SNEHA TITUS
AND SWATI SIRCAR

Wills and inheritances have traditionally been a cause of much discord in families. Which is why, when Mr. Jagirdar drew up his will, he invited his friend, the mathematician Mr. Z to meet him at a popular coffee shop and sought his help! You see, the problem was that he wanted to bequeath his triangular plot of land (shown in Fig. 1), bordered on three sides by roads, to his sons, and he wanted to divide the property so that both sons received equal land area *and* equal access to the road. Mr. Z heard out Mr. Jagirdar's problem and looked at the Google map plot of the land shown in figure 1.

Figure 1: Source: http://www.obj.ca/Real-Estate/Construction/2011-11-08/article-2799423/From-homes-to-chips-to-churches/1

Keywords: *Area, perimeter, division, quadratic, discriminant, nature of roots*

Then he looked down at the mouth watering cake that Mr. Jagirdar had ordered for him. Following his gaze, Mr. Jagirdar, exclaimed: *If my sons had to divide the triangle so that each of them got equal cake as well as equal lengths of that delicious chocolate border, then I could have my cake and eat it too!*


Figure 2

But...he added gloomily, If I cut along the median starting from the top to the midpoint of the longest side and give the left portion to my older son, then he is sure to protest that this is not a fair share since the younger fellow has more of that chocolate line (a.k.a. greater access to the road at the right), no matter that the areas are equal.

And if I measure out the chocolate line and cut in such a way that they get equal amounts then they will get unequal areas. I would hate to cause fights simply because I did not make a fair division!

Mr. Z nodded and in true mathematical style, he restated the problem succinctly: *Given a triangle, you want to know if there exists a line which bisects the area as well as the perimeter of the triangle.* Mr. Jagirdar looked abashed at this – was the problem which had troubled him for so long so very easy to state? He had to agree that it was!

Well, said Mr. Z, *if your plot was circular, square or even rectangular, your solution would have been easy.* Yes, said Mr. Jagirdar, *I remember enough about high school geometry to see that for any of these shapes, any line through the centre of the plot would divide it along my specifications.*

Mr. Z smiled and said, Yes; *even if the land had been in the shape of an equilateral or isosceles triangle, your problem would be simpler. Your sons would accept your decision as calmly as when you divided a grilled sandwich for them by cutting it along the median.* Mr. Jagirdar frowned, the conversation was getting too technical for him but Mr. Z had already sketched an equilateral triangle and an isosceles triangle and was briskly explaining that the median to the base of these would also be the perpendicular bisector and voilà, the problem would be solved. Figure 3 shows Mr. Z's sketch.

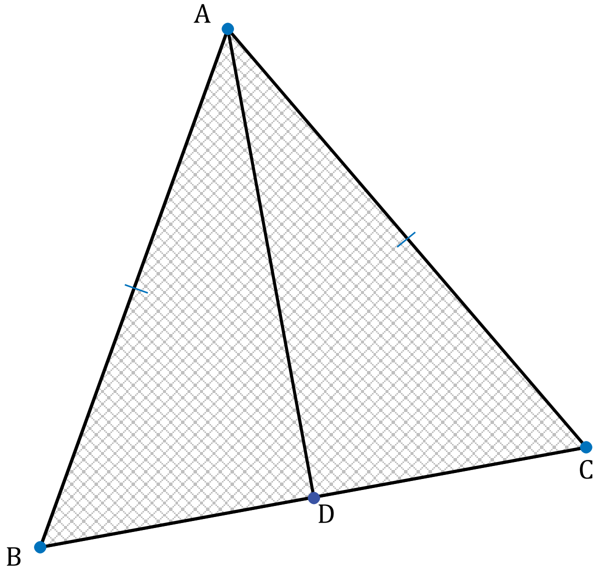


Figure 3

Suppose the given triangle ABC is isosceles and AB = AC. Let D be the midpoint of BC. From fig. 3, we see that AB + BD = AC + CD.

Also, area of triangle *ABD* = ½ *BD* x *AD* = ½ *DC* x *AD* = area of triangle *ADC*

Mr. Jagirdar gloomily stabbed at the cake with his knife. *This scalene triangle makes life difficult! Relax*, said Mr. Z authoritatively. *It is possible, and I will tell you how to make the cut XY so that even the most careful measurements will leave your sons equally satisfied. And I will provide the justification for the division if they want to prove that you have been fair.*

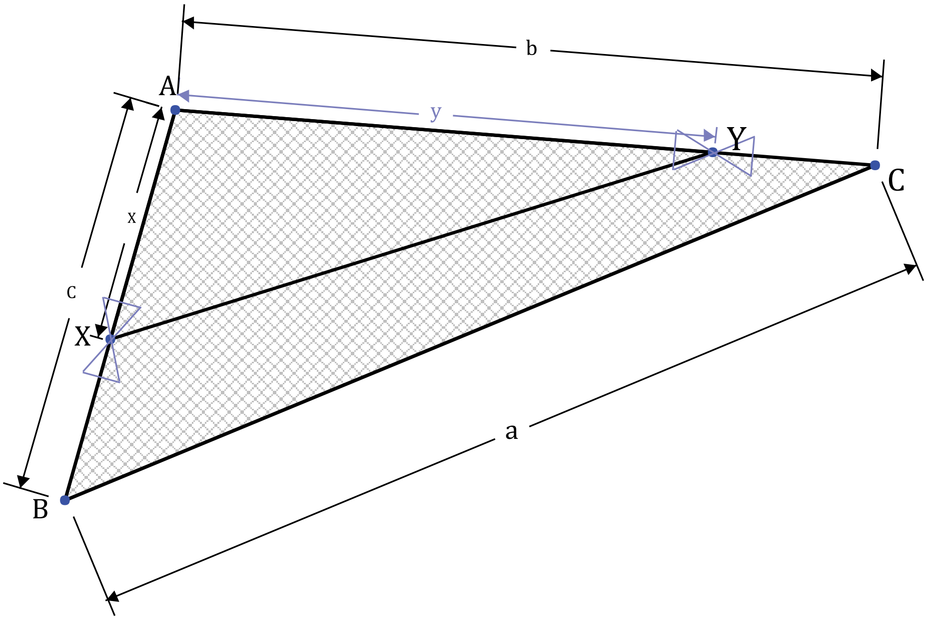


Figure 4

I must see this, said Mr. Jagirdar eagerly. And this is the proof that Mr. Z provided him with:

Let BC = a, CA = b, AB = c and $2s = a + b + c$. Let line L cut the sides AB and AC at X and Y, respectively. Suppose AX = x and AY = y. (See Fig. 4.)

Stage 1: Prove that Area of ΔAXY = ½ Area of ΔABC if and only if $2xy = bc$.

Area of ΔAXY = ½ xy sin A

Area of ΔABC = ½ bc sin A

Substitution in the given expression gives $2xy = bc$. The converse is proved similarly.

Stage 2: Show that XY bisects both the area and the perimeter of the triangle if and only if the simultaneous equations $x + y = s$, $2xy = bc$ have real solutions.

If line XY bisects the area, then $2xy = bc$; and if XY bisects the perimeter of the triangle, then perimeter of triangle AXY = perimeter of quadrilateral XYCB. Substituting the given lengths we get $x + y = s$. So if XY bisects the perimeter and the area, then this set of equations has real solutions. The converse is proved similarly.

Stage 3: Using these simultaneous equations arrive at the condition that the roots are real if $s^2 - 2bc \geq 0$.

Arrive at the quadratic $2x^2 - 2sx + bc = 0$ and find its discriminant.

Stage 4: Prove that $s^2 - 2bc \geq 0$ is equivalent to $(s - b)^2 + (s - c)^2 - (s - a)^2 \geq 0$

Expand $(s - b)^2 + (s - c)^2 - (s - a)^2$. This becomes

$$s^2 + b^2 - 2bs + s^2 + c^2 - 2cs - s^2 - a^2 + 2as$$

$$= s^2 - 2s\,(b + c - a) + (b^2 + c^2 - a^2)$$

$$= s^2 - 2s\,(b + c - a) + ((b + c)^2 - a^2 - 2bc)$$

$$= s^2 - 2s(b + c - a) + (b + c + a)\,(b + c - a) - 2bc$$

On substituting $2s = a + b + c$ we get $s^2 - 2bc$.

Stage 5: Prove that $(s - b)^2 + (s - c)^2 - (s - a)^2 \geq 0$ in turn is equivalent to

$$(s - b)^2 + (a - c)\,b \geq 0.$$

Expand $(s - b)^2 + (s - c)^2 - (s - a)^2$ using the difference of squares formula for the last two terms. The expression becomes

$(s - b)^2 + (s - c - s + a)(s - c + s - a)$. Now substitute $2s = a + b + c$. The expression now reduces to $(s - b)^2 + (a - c)\, b$ which is positive since BC is the longest side

Stage 6: Explain why this expression ensures that the quadratic has real roots.

The discriminant being non-negative ensures that the quadratic has real roots

Stage 7: Prove that the farmer can be successful in his endeavor if he takes

$$x = \frac{s \mp \sqrt{(s^2 - 2bc)}}{2}$$

and $$y = \frac{s \mp \sqrt{(s^2 - 2bc)}}{2}$$

Find the roots of the quadratic equation using the formula.

Awesome said Mr. Jagirdar. *I remember memorizing the conditions for the nature of the roots of a quadratic! I never thought that I would rely them on them to bring peace among my sons!*

That's the problem, said Mr. Z cryptically.

Teacher Note:

Introduction: This problem may seem intimidating at first sight. But it is based on simple principles learnt at school and can be solved by students if they are given a little scaffolding (see blue text for suggested scaffolding and green text for answers which students may be able to provide).

Prior Knowledge:

1. Basic trigonometry

2. Formula for area of a triangle in terms of 2 sides and the included angle.

3. Concept of median of a triangle.

4. The discriminant and the nature of the roots of a quadratic.

Suggestion: Discuss with the students, the seeming contradiction between the diagram which seems to suggest that $x \neq y$ and the result itself which seems to indicate the opposite.

Explain that the positive square root for x coincides with the negative square root for y and vice-versa.

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A needle-ing problem (Part 2)

It seemed reasonable to expect that there will be a set with least area. After all, moving this rod around, even if the rod was assumed to be needle thin, would require quite a bit of 'wiggle-room'! I put forward my thoughts to our friend but was quite surprised by his response. "What you say *intuitively* seems to be correct. However, intuition is no substitute for a proof. And if you attempt to find a proof, then you shall not succeed, for your claim is false! No matter how small an area you give me (positive, of course), I can design a set of still smaller area in which a rod of unit length can be rotated through 180 degrees!" He seemed to be enjoying my look of disbelief for he had an excited glint in his eye. "This question is called *Kakeya's needle problem*. Feel free to go home and read about it at length, it really is quite fascinating!"

And read about it, I did. Apparently the problem was first posed by a Japanese mathematician S. Kakeya in 1917, and this surprising theorem was proved by the Russian mathematician Abram S. Besicovitch in 1919. The result has enthralled mathematicians ever since. While the proof itself uses very sophisticated mathematics such as harmonic analysis, the statement of the problem, along with other related questions, requires very little technical background, and this only adds to its allure! I encourage you all to look this problem up on the internet and explore it in greater detail.

International Centre for Theoretical Sciences of the Tata Institute of Fundamental Research (ICTS-TIFR) is a partner institution in Mathematics Of Planet Earth (MPE2013), a year-long worldwide program designed to showcase ways in which the mathematical sciences can be useful in understanding and tackling our world's problems. If you wish to know more about the exhibition itself (organised in Bangalore in December 2013 by Tata Institute of Fundamental Research, in collaboration with Srishti School of Art, Design and Technology), please visit http://mpe2013.org/

A Puzzle Leading to Some Interesting Mathematics

A. Ramachandran

A well-known puzzle that can be posed to even young children is as follows:

> *There is a row of houses numbered sequentially from 1. A resident of one of the houses notices one day that the sum of the door numbers to his left is the same as the sum of the door numbers to his right. How many houses are there in the row and in which of these does the speaker live?*

Matters can be made simpler by stating that the number of houses is less than 10. The solution is then not difficult to get by trial and error; see if you can find it.

The next level of challenge is to ask for another solution to the problem. (The numbers involved are still less than 50.) Now trial and error may not be a good option. One could use the well-known formula for the sum of the first n natural numbers

$$S_n = \frac{n(n+1)}{2}$$

to proceed. Let n be the number of houses in the row and m the door number of the house occupied by the speaker. Then the sum of the numbers from 1 to $m-1$ is equal to the sum of the numbers from $m+1$ to n. That is,

$$\frac{m(m-1)}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}, \qquad (1)$$

which leads to

$$m^2 = \frac{n(n+1)}{2}. \qquad (2)$$

The right hand side of the above equation represents a *triangular number*, while the left hand side represents a square number. So we look for numbers that are both square and triangular.

Lists of such numbers are available. The lowest such (apart from 1 itself) is 36: it is the 6^{th} square number and the 8^{th} triangular number. This corresponds to $m = 6$ and $n = 8$, and the door number sums involved are $1 + 2 + 3 + 4 + 5 = 15$ and $7 + 8 = 15$. The next larger values of m and n are, respectively, 35 and 49:

$$1 + 2 + 3 + \cdots + 33 + 34 = 595,$$
$$36 + 37 + 38 + \cdots + 48 + 49 = 595.$$

Though it is not obvious, there are infinitely many pairs (m, n) of whole numbers which satisfy (2). The table below displays some of these pairs:

m	1	6	35	204	1189	6930	...
n	1	8	49	288	1681	9800	...

(3)

As m and n get larger, the ratio n/m gets gradually closer to $\sqrt{2}$. For example:

$$\frac{49}{35} = 1.4, \quad \frac{288}{204} \approx 1.41, \quad \frac{1681}{1189} \approx 1.414, \quad \ldots .$$

We see a slow approach to $\sqrt{2}$. It is not difficult to see why this must be so: we rewrite the equation $m^2 = n(n+1)/2$ as $n^2 + n = 2m^2$, from which we get:

$$\frac{n^2}{2m^2} = 1 - \frac{n}{2m^2},$$
$$\therefore \frac{n^2}{2m^2} = 1 - \frac{n}{n(n+1)} = 1 - \frac{1}{n+1}.$$

As n gets large, $1/(n + 1)$ gets close to 0, which means that n^2/m^2 gets close to 2.

In (3) we see many striking patterns in both the m-row and the n-row. Example: Write the numbers in the n-row as follows:

$$1^2,\ 8 = 3^2 - 1,\ 7^2,$$
$$288 = 17^2 - 1,\ 1681 = 41^2,\ \ldots . \quad (4)$$

We see that there is an underlying sequence of numbers:

$$1, \quad 3, \quad 7, \quad 17, \quad 41, \quad \ldots, \qquad (5)$$

and we see that this sequence itself has an underlying pattern! Each successive number can be formed using the two previous numbers (much like the rule for the Fibonacci numbers): if a, b, c are three successive numbers in (5), then

$$c = 2b + a. \qquad (6)$$

Example: $41 = (2 \times 17) + 7$. If this pattern is valid, then we ought to be able to guess more numbers in the sequence! After 41 we should have $(2 \times 41) + 17 = 99$, so the next n-value after 1681 should be $99^2 - 1 = 98 \times 100 = 9800$, and it really is so.

Let's continue the pattern. In (5), the number following 99 should be $(2 \times 99) + 70 = 239$, so in (3) the number following 1681 should be $239^2 = 57121$.

Can we now anticipate which number will come after 6930 in the m-row (the top row) of (3)? These numbers too have a simple pattern: if a, b, c are any three consecutive numbers in the row, then

$$c = 6b - a. \qquad (7)$$

Example: $35 = (6 \times 6) - 1$. Therefore after 6930 we should get:

$$(6 \times 6930) - 1189 = 40391.$$

So we expect the following equality to be true:

$$1+2+3+\cdots+40390 = 40392+40393+\cdots+57121.$$

Indeed, it is true: both sides are equal to 815696245. (Please check! Use the fact that the sum of the first n positive integers is $n(n + 1)/2$.)

The reader is invited to find more patterns in (3), and maybe supply some proofs.
(Yes, they *are* needed ....)

A RAMACHANDRAN has had a long standing interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for over two decades, and now stays in Chennai. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

Explorations in Teaching

Bharat Karmarkar

In this short note we describe some incidents in mathematics teaching— as they actually occurred in the classroom.

Finding the sine of 15 degrees

It is class IX. The teacher is teaching trigonometric ratios of 30°, 45°, 60°. Suddenly an idea pops up in his mind. He draws Figure 1 and asks the students, "Can anybody calculate sin 15° from this figure?" This topic is not in the IX-th standard text book. But a few students accept this challenge, a few more silently appreciate their efforts to solve the problem, and the rest wait for the period to end! Two days later some students find a nice solution.

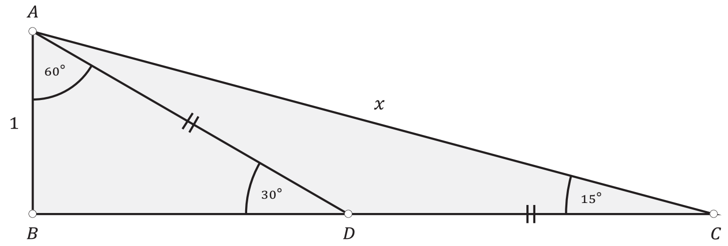


Figure 1.

They write: Let $AB = 1$; then $AD = 2$ (since $\sin 30^\circ = 1/2 = AB/AD$), and $CD = 2$ as well because $\measuredangle DCA = \measuredangle DAC$. Also, $BD = \sqrt{3}$, so $BC = 2 + \sqrt{3}$. Let $AC = x$. Using the Pythagorean theorem,

$$x^2 = AB^2 + BC^2 = 1^2 + \left(2+\sqrt{3}\right)^2 = 1 + \left(7+4\sqrt{3}\right) = 8+4\sqrt{3}.$$

Keywords: *Exploration, trigonometry, sine, surd, square root, ratio*

Hence:

$$\sin^2 15^\circ = \frac{1}{x^2} = \frac{1}{8+4\sqrt{3}}.$$

In taking the square root, some simplification occurs, since

$$4+2\sqrt{3} = \left(1+\sqrt{3}\right)^2.$$

Hence:

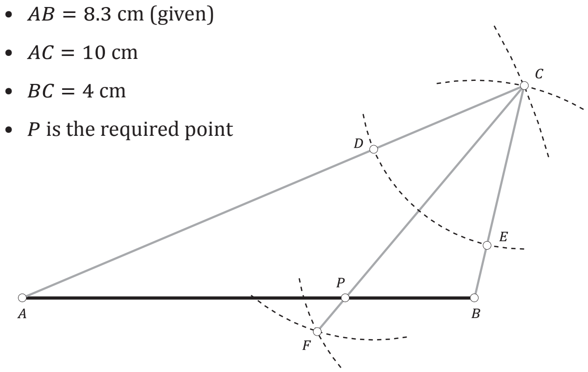
$$\sin 15^\circ = \frac{1}{\sqrt{2\left(4+2\sqrt{3}\right)}} = \frac{1}{\sqrt{2}\left(1+\sqrt{3}\right)}.$$

This can be written this in different ways, e.g.:

$$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Dividing a line segment in a given ratio

It is class X. The teacher is teaching the chapter on similarity and angle bisectors. A question is posed: "Draw a line segment AB of length 8.3 cm. Using compass and ruler, locate a point P on it such that $AP : PB = 5 : 2$." Most students use a method based on parallels, but Figure 2 shows an imaginative solution, based on a different idea; it is presented without any words. Can you explain why it works? Try it!



- $AB = 8.3$ cm (given)
- $AC = 10$ cm
- $BC = 4$ cm
- P is the required point

Figure 2.

BHARAT KARMARKAR is a freelance educator. He believes that learning any subject is simply a tool to learn better learning habits and a better aptitude; what a learner really carries forward after schooling is learning skills rather than content knowledge. His learning club, located in Pune, is based on this vision. He may be contacted at learningclubpune@gmail.com.

in the classroom

Exploration of Recurring Decimals: Some Explanations

SHAILESH SHIRALI
PADMAPRIYA SHIRALI

In the article on 'Recurring Decimals' published in the November 2013 issue of *At Right Angles*, many questions remained unanswered in the end. They had emerged as empirical observations during the course of the exploration. We study these observations closely here, examine their validity and explain them using simple principles of divisibility.

(A) ***Is it true that all fractions give rise to either terminating decimals or to recurring decimals of some periodicity?***

Yes! Let the fraction be m/n where m, n are positive integers with no common factors. Let the decimal form of m/n be computed. If the decimal terminates, well and good; so we examine what happens in the case of non-termination. When the digits of m (the dividend) 'run out', what do we do? — we simply tack on zeros to the units 'end' of the number and continue the division. At each stage we get some non-zero remainder which is one of the numbers $1, 2, 3, \ldots, n-1$. So, after at most n divisions, we necessarily get a remainder that we had already got earlier.

Keywords: *Recurring decimal, terminating decimal, repetend, prime factorization*

From this point onwards, the string of digits we obtain from the division will be identical to what had been obtained earlier; in other words, the decimal will recur from this point onwards.

An example will make this clear. Consider the fraction 10/13. The long division working (of $10 \div 13$) yields the display shown in Table 1. Observe that in the last row, the remainder is the same as in the first row. It follows that the row following this one will be identical to the second row, the row after that will be identical to the third row, and so on. Note the sequence of quotients: $0, 7, 6, 9, 2, 3, 0, \ldots$. Therefore, $10/13 = 0.076923\ 076923\ \ldots = 0.\overline{076923}$.

Quotient	Remainder	Updated dividend
0	10	100
7	9	90
6	12	120
9	3	30
2	4	40
3	1	10
0	10	100

TABLE 1. Steps in the division $10 \div 13$

(B) ***Is it true that the decimal expansion of*** $\mathbf{1/n}$ ***terminates precisely when*** $\mathbf{n = 2^a \times 5^b}$ ***where*** $\mathbf{a}$ ***and*** $\mathbf{b}$ ***are non-negative integers?*** (The values of n for which the decimal expansion of $1/n$ terminates were found to be the following: 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, 64, 80, 100, ... All these numbers have the stated form.)

Yes, again! For, suppose that the decimal expansion of $1/n$ terminates; say $1/n = 0.a_1a_2 \ldots a_k$ where $a_1, a_2, \ldots, a_k$ are decimal digits. Let A be the k-digit number $\overline{a_1a_2 \ldots a_k}$. Then, clearly,

$$\frac{1}{n} = \frac{A}{10^k}, \qquad \therefore A = \frac{10^k}{n}. \tag{1}$$

Relation (eq:1) tells us that n is a divisor of 10^k. Since the prime factors of 10 are 2 and 5, we deduce that n cannot have any prime factors other than 2 and 5. Hence $n = 2^a \times 5^b$ for some non-negative integers a and b.

A numerical instance will make this clear. Take $n = 32$; we get: $1/32 = 0.03125$, so $A = 3125$ and $k = 5$. Relation (eq:1) reduces to:

$$\frac{1}{32} = \frac{3125}{100000}, \qquad \therefore 3125 = \frac{100000}{32}.$$

The converse statement too is true: if $n = 2^a \times 5^b$ for some non-negative integers a and b, then the decimal expansion of $1/n$ terminates. For, if $a \geq b$ then we have:

$$\frac{1}{2^a \times 5^b} = \frac{5^{a-b}}{2^a \times 5^a} = \frac{5^{a-b}}{10^a}.$$

This yields a terminating decimal with a digits after the decimal point. Example: Consider $n = 2^4 \times 5^1 = 80$:

$$\frac{1}{80} = \frac{1}{2^4 \times 5^1} = \frac{5^3}{2^4 \times 5^4} = \frac{125}{10000} = 0.0125.$$

Similarly, if $b \geq a$ we have:

$$\frac{1}{2^a \times 5^b} = \frac{2^{b-a}}{2^b \times 5^b} = \frac{2^{b-a}}{10^b}.$$

Now we get a terminating decimal with b digits after the decimal point. Example: Consider $n = 2^1 \times 5^3 = 250$:

$$\frac{1}{250} = \frac{1}{2^1 \times 5^3} = \frac{2^2}{2^3 \times 5^3} = \frac{4}{1000} = 0.004.$$

(C) ***Is it true that the values of*** $\mathbf{n}$ ***for which the repetend of*** $\mathbf{1/n}$ ***is a single-digit number are given by the formulas*** $\mathbf{n = 3 \times 2^a \times 5^b}$ ***and*** $\mathbf{n = 3^2 \times 2^a \times 5^b}$***?*** (The values of n we found for which the repetend of $1/n$ is a single-digit number were: 3, 6, 9, 12, 15, 18, 24, 30, 36, 45, 48, 60, 72, 75, 90, 96, ... All these fit the given formula.) We shall show that the answer again is Yes. What does it mean for the repetend to have just one digit? Suppose that the repetend is the single digit d. Then the decimal expansion of $1/n$ must be of the form

$$0.a_1a_2a_3 \ldots a_k dddd \ldots$$

where $a_1, a_2, a_3, \ldots, a_k$ are digits. Let A be the k-digit number $\overline{a_1a_2a_3 \ldots a_k}$. From the relation

$$\frac{1}{n} = 0.a_1a_2a_3 \ldots a_k dddd \ldots,$$

we get the following two relations, by multiplication by 10^k and then by 10 (these multiplications shift the decimal point, first by k steps and then by 1 more step):

$$\frac{10^k}{n} = \overline{a_1a_2a_3 \ldots a_k}.dddd \ldots = A.dddd \ldots, \tag{2}$$

$$\frac{10^{k+1}}{n} = \overline{a_1 a_2 a_3 \dots a_k d}.dddd \dots = (10A + d).dddd \dots. \tag{3}$$

Here the term $10A + d$ is the number $\overline{a_1 a_2 a_3 \dots a_k d}$ which equals $\overline{a_1 a_2 a_3 \dots a_k 0} + d$, i.e., $10 \times \overline{a_1 a_2 a_3 \dots a_k} + d = 10A + d$.

If we do "(eq:3) minus (eq:2)", the portion after the decimal point ($.dddd \dots$) gets wiped out by the subtraction, and we get:

$$\frac{10^{k+1} - 10^k}{n} = 10A + d - A = 9A + d. \tag{4}$$

From this relation we deduce that n is a divisor of the number $10^{k+1} - 10^k = 10^k \times 9$. Since the prime divisors of $10^k \times 9$ are 2, 5, 3, it follows that n is the product of a divisor of a power of 10 and a divisor of 9 (which can only be 3 or 9; if the divisor were 1, then the decimal would terminate). Hence $n = 3 \times 2^a \times 5^b$ or $n = 3^2 \times 2^a \times 5^b$. This conclusion matches the observed finding.

Example: Working with actual numbers will make the argument clear. Take $n = 18$. Then we have $1/n = 0.05555 \dots$, so $d = 5$ (this is the repetend). Also, $k = 1$ (there is one digit before the repeating portion). So we multiply both sides first by $10^1 = 10$ and then by $10^2 = 100$. We now get:

$$\frac{10}{18} = 0.55555 \dots,$$

$$\frac{100}{18} = 5.55555 \dots.$$

Subtracting, we get:

$$\frac{100}{18} - \frac{10}{18} = 5, \quad \text{i.e.,} \quad \frac{90}{18} = 5.$$

So 18 is a divisor of $90 = 10 \times 9$. Note that $18 = 2 \times 9$. This will illustrate what we mean when we say that "n is the product of a divisor of a power of 10 and a divisor of 9". (Here the divisor of 10 is 2, and the divisor of 9 is 9 itself.)

(D) ***For which values of* n *does the decimal expansion of* 1/n *have a two-digit repetend?***

To answer this we argue exactly the same way as we did earlier. We see that the decimal expansion must be of the form

$$0.a_1 a_2 a_3 \dots a_k d_1 d_2 d_1 d_2 \dots \tag{5}$$

where $a_1, a_2, a_3, \dots, a_k, d_1, d_2$ are digits, and the repetend is the two-digit number $D = \overline{d_1 d_2}$. Let A be the k-digit number $\overline{a_1 a_2 a_3 \dots a_k}$. From the relation

$$\frac{1}{n} = 0.a_1 a_2 a_3 \dots a_k d_1 d_2 d_1 d_2 \dots,$$

we get the following two relations, by multiplication by 10^k and then by 10^2:

$$\frac{10^k}{n} = A.d_1 d_2 d_1 d_2 \dots, \tag{6}$$

$$\frac{10^{k+2}}{n} = (100A + D).d_1 d_2 d_1 d_2 \dots. \tag{7}$$

Subtraction, (eq:7) minus (eq:6), now yields:

$$\frac{10^{k+2} - 10^k}{n} = 100A + D - A = 99A + D. \tag{8}$$

From this we deduce that n is a divisor of the number $10^{k+2} - 10^k = 10^k \times 99$. Hence n is the product of a divisor of a power of 10 and a divisor of 99. The latter divisor (of 99) cannot be a divisor of 9, because in that case we would have a repetend consisting of only one digit. So the divisor of 99 must be one of the following: 11, 33, 99. These numbers when multiplied by divisors of powers of 10 yield the n's we want. So n must be one of the following: 11, 22, 33, 44, 55, 66, 88, 99, 110, 132, ... This conclusion matches the observed finding exactly.

(E) ***For which values of* n *does the decimal expansion of* 1/n *have a three-digit repetend?***

As the strategy by now will be familiar, we skip the initial steps. The conclusion now is: the decimal expansion of $1/n$ will have a three-digit repetend precisely when n is a divisor of the number $10^k \times 999$ (for some k) but not a divisor or either $10^k \times 99$ or $10^k \times 9$. Since the prime factorization of 999 is $999 = 3^3 \times 37$, the divisor of 999 contained in n must be one of the following: 27, 37, 111, 333, 999. These numbers when multiplied by divisors of powers of 10 yield the n's we want. So n must be one of the following: 27, 37, 54, 74, 108, 148, 135, 175, ... Yet again, this conclusion matches the observed finding exactly.

(F) ***For which values of* n *does the decimal expansion of* 1/n *have a four-digit repetend?***

While answering this, we must explain the observed fact that there is no n in the range $1 \leq n \leq 100$ for which the repetend is a four-digit number.

Following the same steps, we see that the decimal expansion of $1/n$ will have a four-digit repetend precisely when n is a divisor of the number $10^k \times 9999$ (for some non-negative integer k) but not a divisor or either $10^k \times 999$ or $10^k \times 99$ or $10^k \times 9$. Since the prime factorization of 9999 is $9999 = 3^2 \times 11 \times 101$, the divisor of 9999 contained in n must be one of the following: 101, 303, 909, 1111, 3333, 9999. These numbers when multiplied by divisors of powers of 10 yield the n's we want. Observe that none of the n's we have listed is a two-digit number (the least n in the list is 101, a three-digit number). This explains the observed finding: that we did not find even one value of n in the range $1 \leq n \leq 100$ for which the repetend is a four-digit number. (*Remark*. If we had persisted for just one number more, i.e., till 101, we would have found an instance of a four-digit repetend!)

(G) *For which values of* n *does the decimal expansion of* 1/n *have a five-digit repetend?*

To answer this, we need to examine the factorization of 99999. More specifically, we need the divisors of 99999 which are not divisors of 9999, 999, 99 or 9. Since the prime factorization of 99999 is $3^2 \times 41 \times 271$, it follows that n necessarily has one of the following numbers as a divisor: 41, $3 \times 41 = 123$, $3^2 \times 41 = 369$, 271, $3 \times 271 = 813$, $3^2 \times 271 = 2439$, 41×271, $3 \times 41 \times 271$, $3^2 \times 41 \times 271$. These numbers when multiplied by divisors of powers of 10 yield the n's for which the repetend is a five-digit number. We get these values of n: 41, 82, 123, 164, 205, ..., in agreement with our finding.

(H) *For which values of* n *does the decimal expansion of* 1/n *have a six-digit repetend?*

To answer this, we need to examine the factorization of 999999. More specifically, we need the divisors of 999999 which are not divisors of 99999, 9999, 999, 99 or 9. Since $999999 = 3^3 \times 7 \times 11 \times 13 \times 37$, the new primes which have entered the picture are 7 and 13, and so the divisors we want are: 7, 13, 21, 39, 63, 77, 91, 117, ... These numbers when multiplied by divisors of powers of 10 give us the n's we want. We leave the remaining cases for you to tackle; when the repetend has seven, eight, nine or ten digits. The same approach may be followed in each case. You will need the following prime factorizations: $9999999 = 3^2 \times 239 \times 4649$, $99999999 = 3^2 \times 11 \times 73 \times 101 \times 137$, $999999999 = 3^4 \times 37 \times 333667$ and $9999999999 = 3^2 \times 11 \times 41 \times 271 \times 9091$. Good luck in your endeavour!

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Teacher's Diary on Classroom Assessment – II

SINDHU SREEDEVI &
SNEHA TITUS

In the last entry in my teacher's diary on classroom assessment, I had reiterated the importance of and the need for assessment-both formative and summative. I had decided to start the year with the topic of mensuration and had included the sample diagnostic test for the students I would be meeting that year in std. VIII.

After administering this test, I was able to gain good insights about my students' understanding of this topic. The performance of the class encouraged me to divide the group into five groups in such a way that students who had solved particular questions could be part of particular groups. Each group was assigned specific questions from the test and were required to present their solutions to the rest of the class. This engaged the students in collaborative learning within groups. Each group then made a small presentation to the rest of the class and I also tried to ensure that students, who had not solved some of the questions during the test, did so later. Finally, each student worked individually on, and submitted their corrections.

Keywords: *formative assessment, CCE, mathematical skills, abstraction, mensuration, area, trapezium, GeoGebra*

In token form at least, there is now a paradigm shift in the role of a child in the classroom from that of a passive listener to that of an active participant. The position paper *Teaching of Mathematics* (NCF 2005) says *"The higher aim (of teaching mathematics) is to develop the child's resources to think and reason mathematically, to pursue assumptions to their logical conclusion and to handle abstraction. It includes a way of doing things, and the ability and the attitude to formulate and solve problems"*. To this end, my role as a teacher is to facilitate learning in the classroom using various pedagogical techniques. Consequently, assessment takes on an even more important role. Observing students' work, keeping anecdotal records and timely feedback are key components of this process. Hands on activities such as paper folding, writing graphing stories, working with dynamic geometry software, etc., home assignments and classroom dialogues can be a source of evidence for both teacher and student alike. Assessment needs to provide answers for two questions:

- How is the student evolving as a learner?
- What can I do to facilitate that learning?

As I planned my work for the year, based on the CCE Manuals for Teachers brought out by CBSE in 2009, I realised that continuous and comprehensive evaluation would be daunting in terms of work load.

In the scholastic areas, subject teachers (Hindi, third language, English, Social Science, Science, Science and Mathematics) are required to award marks for various tests conducted as part of Formative and Summative Assessment for each student studying in classes VI- X. In each year, the formative tests have 40% weightage and summative tests carry 60% weightage. In my school, there were four formative assessments each year and within each formative assessment, written tests (called pen and paper tests) get 50% weightage and other activities get 50% weightage.

Under CCE, for all the subjects taken together, a student is required to work on 70 projects and appear for 40 examinations within a period of 8 to 9 working months (this includes 4 quiz /oral tests in mathematics). Each teacher is required to complete the following proforma (see Figure 1) subject wise for each formative assessment containing the details of marks scored by each student. Besides conducting four formative assessment tests, the teachers are required to conduct two summative tests as well.

I realised that unless I modified the assessment criteria, I would not be able to spend sufficient time on reading students' note books and looking carefully at the projects and on assessing their projects and activities. The sheer variety of modes of assessment would leave no time for remedial teaching based on my formative assessment.

Class :									
Subject : Maths									
S.No.	*Name of Student*	*Pen-Paper Test*	*Lab Activities*	*Project Work*	*Assignments*	*Home Work & Class Work*	*MCQ/ Quiz/ Oral*	*Total*	100 Marks are to be reduced to 10%
		50	*10*	*10*	*10*	*10*	*10*	*100*	

Figure 1 : Formative test proforma for Mathematics

With the encouragement of a supportive principal, I re-worked the model for formative assessment and proposed the following format

FA1: Pen - pencil 40% + Lab Activity 50% + Homework 10%

FA2: Pen - pencil 40% + MCQ/Quiz/Oral 50% + Homework 10%

FA3: Pen - pencil 40% + Individual Project Work 50% + Homework 10%

FA4: Pen - pencil 40% + Group Work & Presentation 50% + Homework 10%

I tried to plan my assessment in such a way that my over arching goal of students moving from the concrete to the abstract would be assessed in each session. I also kept in mind the following sub-skills while proceeding with the assessment.

i. Use and understand mathematical language including symbols
ii. Generalize from specific results
iii. Apply logical thinking
iv. Appreciate the notion of proof

Since remedial teaching was core to the success of formative assessment, I had to allocate time for this after each test. For FA 3 & 4, I decided to assign projects to the class in groups of 10. These would be designed such that the individual project work in FA3 would be consolidated and put together in FA4. Through this, co-scholastic areas such as collaborative work as well as artistic and presentation skills could be assessed. Students would also have an opportunity to improve on their individual projects based on the teacher's feedback which is the true meaning of formative assessment.

Along with the annual plan I also structured my classroom assessment carefully. I used questioning as a method to keep track of students' knowledge and understanding of the concepts.

The next section of the article describes a lab activity which was conducted by me to enable the students to 'discover' the formula of the area of a trapezium. Students already knew the definition of a trapezium. They were also familiar with the use of GeoGebra, an open source dynamic geometry software for exploring geometrical shapes.

LAB ACTIVITY TO 'DISCOVER' THE FORMULA FOR THE AREA OF A TRAPEZIUM

Task 1: Draw a trapezium ABCD with parallel sides AD and BC.

Task 2: Find the perpendicular distance between the parallel sides.

Task 3: Divide the trapezium into 2 right angled triangles and a rectangle. (See Figs. 2 and 3 for the GeoGebra sketches)

Task 4: Find the area of each of these polygons and hence find the area of the trapezium. Show your calculations. *(This is important as GeoGebra can give the area of each polygon if required)*

Task 5: Repeat the above calculation for a trapezium for which the parallel sides are of lengths *a* & *b* with *h* being the perpendicular distance between them.

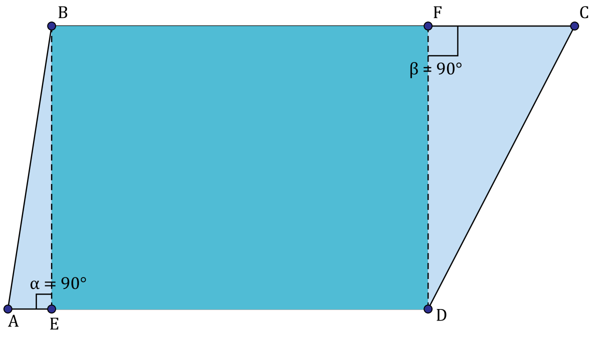

Figure 2

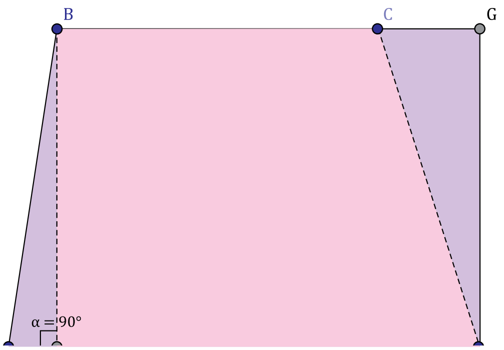

Figure 3

After this we had a group discussion on the formula for the area of the trapezium and arrived at the result *Area* = ½ $h(a + b)$

This exercise was followed by a home assignment in which students were required to solve problems based on the area of a trapezium. The object was to reinforce their understanding of the formula. Some of the problems in the assignment were as follows:

1-3. Find the area of the following trapezia.

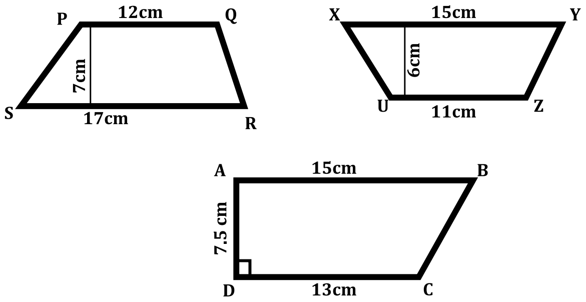

Figure 4

4. Find the side EF of the trapezium EFGH.

Area of EFGH = 152 square cm.

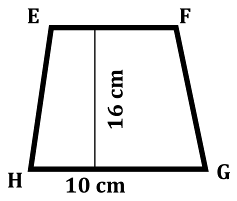

Figure 5

5. Mohan wants to buy a trapezium shaped field. Its side along the river is parallel to and twice the side along the road. If the area of this field is 10500 m^2 and the perpendicular distance between the two parallel sides is 100 m, find the length of the side along the river.

The next day, I made notes on students responses based on the lab activity worksheets with Geogebra and the homework. Finally I displayed a copy of the answer key using the overhead projector and we discussed each answer in detail. My strategy was driven by Wiggins (1993) *"an authentic education makes self-assessment central"*. Students need to reflect on their understanding and modify or adjust, based on their performance and feedback is an important tool for self-assessment. The class ended with the students completing the corrections for homework.

The same concept may be assessed in different ways and through different activities. Students may be given the following paper cutting activity either to reinforce understanding, evaluate progress or as remedial teaching after assessment. I have found that re-teaching a concept does not mean a re-explanation. A different route often helps students grasp better.

1. Cut the trapezium EFGH on the dotted lines. If EH = x and FG = y and the distance between the parallel sides is z, prove by summing the areas of the polygons formed that *Area* = ½ $z\ (x + y)$.

   Can the same formula be obtained by cutting the trapezium into a parallelogram and a triangle?

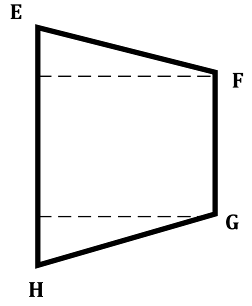

Figure 6

2. Where should point O be located along CB so that if ABCD is cut into two parts along the dashed line, the parts may be joined to form a triangle? If AB = l and CD = b and the height of this trapezium is h prove using the triangle so formed that the area of both the triangle and the trapezium is
   ½ $h\ (l + b)$

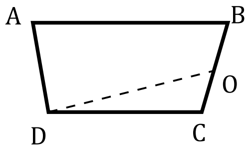

Figure 7

3. Fold the rectangle PQRS along the dotted lines PA and SB so as to make a trapezium. If PQ = l, PS = b, QA = BR = x, find a formula for the area of trapezium PABS in terms of l, b and x. What happens if BR = y where y is not equal to x?

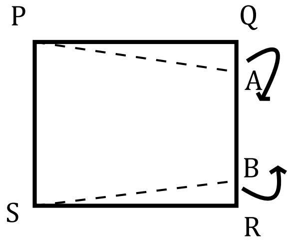

Figure. 8

I see this as a good opportunity for peer assessment for which the parameters will be given to the students as:

- Identification of polygons after folding or cutting.
- Use of mensuration formulae learnt earlier.
- Deriving the proposed formula and giving justification for the steps taken

Peer assessment enables students to openly discuss different problem solving strategies as well as justify specific strategies used. Once children assess their peers they tend to assess themselves on the same parameters. For the formative assessment, I now had the lab activity as well as the homework. In both these, I had ensured that the skills that I had identified were being developed in the students. The paper-pencil test that followed assessed the students' understanding of the formulae for the area of all quadrilaterals. After this, I did a final round of remedial teaching and corrections and was able to start the next unit.

Prior to this exercise, I had only assessed students' understanding through anecdotal evidence. However the use of a combination of assessment methods such as lab activities and paper folding activities I obtained a deeper insight into the students' understanding of concepts.

We tend to use or interpret words such as assessment, evaluation and examination interchangeably everywhere, especially in the context of education and particularly towards the year end or term end or end of the unit. But assessment is not the end of anything - rather, it is the beginning of better learning for the student as well as the teacher. And evaluation need not always be based on quantitative assessment rather, qualitative assessment can direct richer learning for both student and teacher.

The CCE column is the product of the Azim Premji University Resource Centre. The team members who are working on it are SINDHU SREEDEVI, JOYITA BANERJEE and SNEHA TITUS.

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How to discover 22/7 and other rational approximations to π

Gaurav Bhatnagar

Introduction

In math and physics books, we often find the words "Take $\pi = 22/7$". You and I, dear reader, know that this cannot be right. We know perfectly well that π is an irrational number, and so cannot *equal* 22/7; for 22/7 is clearly a ratio of integers and therefore a rational number. So the best we can say is $\pi \approx 22/7$, that is, π is *approximately* 22/7.

In this article, I will explain a method to find such rational approximations for π and other irrational numbers. The key idea here is to use a calculator (or a spreadsheet program) to find what is called a *continued fraction* for an irrational number.

Keywords: *Pi, approximation, rational, continued fraction, square root, spreadsheet, Excel*

The continued fraction for π

It is easy to see that:

$$\begin{aligned}\pi = 3.14159265\ldots &= 3 + 0.14159265\ldots\\ &= 3 + \frac{1}{1/0.14159265\ldots}\end{aligned}$$

We now compute $1/0.14159\ldots$ in the denominator, using a calculator, and obtain:

$$\pi = 3 + \frac{1}{7.06251331\ldots}$$

Repeating these steps, we obtain:

$$\begin{aligned}\pi &= 3 + \frac{1}{7 + 0.06251331\ldots} = 3 + \cfrac{1}{7 + \cfrac{1}{1/0.06251331\ldots}}\\[2ex] &= 3 + \cfrac{1}{7 + \cfrac{1}{15.99659440\ldots}} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1/0.99659440\ldots}}}\\[2ex] &= \cdots\\[1ex] &= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}.\end{aligned}$$

This process can be continued indefinitely, and we obtain what is called a *continued fraction* representation of π. To get approximations of π, chop off the continued fraction suitably to get:

$$\pi \approx 3 + \frac{1}{7} = \frac{22}{7};$$

$$\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106};$$

$$\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113}.$$

The approximations $22/7$ and $355/113$ are quite popular, and have been known for thousands of years.

So now you know how to discover $22/7$ and other rational approximations to π. Let us try the same thing with another familiar irrational number, namely $\sqrt{2}$.

The square root of 2

Unlike π, the continued fraction of $\sqrt{2}$ has a beautiful pattern. Here are the calculations:

$$\begin{aligned}\sqrt{2} &= 1.414213562\ldots = 1 + 0.414213562\ldots\\ &= 1 + \frac{1}{1/0.414213562\ldots} = 1 + \frac{1}{2.414213562\ldots}\\ &= 1 + \frac{1}{2 + \frac{1}{1/.414213562\ldots}} = 1 + \frac{1}{2 + \frac{1}{2.414213562\ldots}}\\ &= \cdots\end{aligned}$$

Notice that the 2.414 … has occurred earlier. So you would expect that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

This is an infinite continued fraction representation of $\sqrt{2}$, and is surely much nicer than its decimal expansion 1.414213562 … .

One can *prove* that the pattern repeats, and also avoid the use of a calculator, by noticing the following equality which happens to be exact:

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}.$$

Replace the $\sqrt{2}$ on the RHS by the expression

$$1 + \frac{1}{1 + \sqrt{2}}$$

and see if you can tell why the pattern repeats!

I leave it to you to chop off the terms of the continued fraction and find nice rational approximations for $\sqrt{2}$. The first few approximations are: $3/2, 7/5, 17/12, 41/29$ and $99/70$.

You may find it interesting to find the continued fractions for $\sqrt{3}$ and $\sqrt{5}$ in the same way. The patterns are every bit as nice as those in the continued fraction for $\sqrt{2}$. In fact, you are well equipped to find out rational approximations of a host o irrational numbers, such as e, π^2, e^{π}. Why don't you try some experiments of your own?

Some experiments on the simplest infinite continued fraction

You should also try your hand at the simplest of all continued fractions:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}.$$

Chop off the continued fractions and calculate a few fractions. The fractions are closely related to the famous *Fibonacci sequence* 1, 1, 2, 3, 5, 8, 13, … .

Can you figure out the number to which these approximations converge? Here's a hint: It is a number that stars prominently in the movie *The da Vinci Code*. Another hint: It is the only positive number that is one more than its reciprocal.

Notes

- Of course, the simplest way to approximate π by a rational number would be to take the first few digits of its decimal expansion. For example, 3.14 is a perfectly legal rational approximation of π. And 3.14159 is an even better approximation. But surely, 22/7 is much prettier!

- It is said that Archimedes found the approximation 22/7 of π by laboriously approximating a circle with a 96-sided polygon. Google the phrase "approximations to pi" to find the story of approximations of π from ancient times to Ramanujan; and to a record computation of trillions of digits of π.

- One can prove that the continued fractions of numbers such as $\sqrt{2}$, $\sqrt{3}$ etc. are nice, in the sense that the numbers repeat periodically, in much the same way as decimal expansions of some rational numbers.

- We have not considered whether an infinite object such as

  $$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}$$

  has a meaning. After all, in the finite amount of time accorded to us in this lifetime, we cannot hope to perform all the calculations that are indicated by the '… '.

  The key idea is to consider a sequence of fractions x_n that are obtained by chopping off the continued fraction, and to show this sequence *converges*.

  One way to convince yourself of this is to use a computer and calculate a number of fractions by suitably chopping the infinite continued fractions. Now find their decimal expansions, and see if they seem to be stabilizing. That is to say, if you take the difference $x_n - x_{n-1}$ of successive fractions, does the difference come close to 0 as n becomes large? This is not enough to prove the sequence converges, but it does give evidence of convergence.

  If you study the values of $x_n - \sqrt{2}$ for increasing n, you may discover another interesting feature of these approximations. But I won't tell you what it is. Find out for yourself!

- You can use this approach to approximate complicated rational numbers too. For example,

  $$\frac{985}{304} \approx 3 + \frac{1}{4} = \frac{13}{4}.$$

- There are much better-looking continued fractions for π than the one given in this article. One of them, found by Lambert in 1770, is

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cdots}}}}.$$

- I could have *proved* that $\sqrt{2}$ is irrational, but then I would have been obliged to prove that π is irrational too, which is quite tough. The best such proofs involve continued fractions. Had $\sqrt{2}$ been a rational number, its continued fraction would have been finite. This is because when the process of finding continued fractions is applied on a rational number of the form p/q, it will stop after a finite number of steps. If you don't believe me, try the process on a few fractions, and discover just why it stops after a few steps.

  A nice book which contains such a proof is *Mathematics: A very short introduction* by T. Gowers, Oxford University Press (2002).

- If you liked what you saw here, then you will definitely enjoy the classic: *Continued Fractions* by C. D. Olds, Random House (1963).

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SOLUTION TO THE 'RIVER PUZZLE' GIVEN BY SAM LOYD HIMSELF

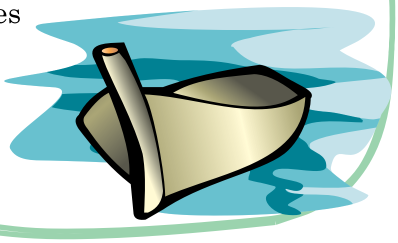
See ***page* 30** for a statement of the problem.

At their first meeting, 720 yards from one shore, the boats together have travelled a distance equal to the width of the river. When each reaches its destination, the combined distance is twice the width of the river. At their second meeting, 400 yards from the other shore, the combined distance is three times the river's width. So at the second meeting each boat has gone three times as far as it had at the first meeting.

At the first meeting one boat had travelled 720 yards, so at the second meeting it has gone three times this distance, or 2,160 yards. At this point it has covered 400 yards since doubling back, so the river is 2,160 - 400 = 1,760 yards, or one mile, wide.

The amount of time each ship consumed at the landing does not affect the problem.

Comment from the Editor. This beautiful solution tells us that Sam Loyd is called the Master for very good reason!



How To Prove It

This continues the 'Proof' column begun in the last issue. In this 'episode' too we study some problems from number theory; more specifically, from patterns generated by sums of consecutive numbers.

SHAILESH SHIRALI

Sums of consecutive numbers

Few of us would be impressed by the relation

$$1 + 2 = 3,$$

but when we set it alongside the following:

$$4 + 5 + 6 = 7 + 8,$$

our eyebrows go up a bit. And they climb up very much further if we list the following:

$$9 + 10 + 11 + 12 = 13 + 14 + 15.$$

At this point the mathematician in us will surely demand the clear statement of some general relation, and its proof as well. Let us respond to this challenge.

Note the sequence of first numbers in these relations: 1, 4, 9, ... It is clear that the first number in the n^{th} relation is n^2. Noting the *number* of numbers on the left side and the right side in the relations (2, 3, 4, ...on the left side, and 1, 2, 3, ...on the right side), it appears that we are claiming the following:

> *For each positive integer* n*, the sum of* $n + 1$ *consecutive numbers starting with* n^2 *is equal to the sum of the next* n *consecutive numbers.*

Keywords: *Sequence, consecutive number, generalization, triangular number, pictorial*

For $n = 4$ the claim is that $16 + 17 + 18 + 19 + 20$ is equal to $21 + 22 + 23 + 24$, and this is true (each sum is 90). How do we check whether this claim is true for every n?

Let's look more closely at the statements. In the statement relating $9 + 10 + 11 + 12$ to $13 + 14 + 15$, note that on the left side the first number in the list is $9 = 3^2$ and the last number is $12 = 3^2 + 3$. In the statement relating $16 + 17 + 18 + 19 + 20$ to $21 + 22 + 23 + 24$, note that on the left side the first number in the list is $16 = 4^2$ and the last number is $20 = 4^2 + 4$. The pattern is clear: in the n^{th} statement, on the left side the first number in the list is n^2, and the last number is $n^2 + n$. Also, there are $n + 1$ numbers. Using the well-known rule for the sum of the terms of an arithmetic progression ("half the sum of the first term and the last term, times the number of terms"), we see that the sum of the $(n + 1)$ numbers on the left side is

$$\frac{n^2 + (n^2 + n)}{2} \times (n + 1) = \frac{n(n + 1)(2n + 1)}{2}.$$

How about the sum on the right side? The first number in the list is clearly the number following the last number on the left side, and therefore it is $n^2 + n + 1$; and the last number is the one preceding the first number of the next such relation, i.e., the predecessor of $(n + 1)^2$; hence it is $(n + 1)^2 - 1 = n^2 + 2n$. As there are n numbers, their sum is

$$\begin{aligned}\frac{(n^2 + n + 1) + (n^2 + 2n)}{2} \times n &= \frac{n(2n^2 + 3n + 1)}{2} \\ &= \frac{n(n + 1)(2n + 1)}{2}.\end{aligned}$$

We have obtained the same expression as earlier, so the two sums are equal. Hence proved.

A more informal approach

Here is a more informal way of arguing. Consider the two sets $\{9, 10, 11, 12\}$ and $\{13, 14, 15\}$. If we divide the last number (12) of the first set into three equal parts of 4 each ($12 \div 3 = 4$), and add one part to each of the other numbers in the set, we get, from 9, 10, 11, the numbers $9 + 4 = 13$, $10 + 4 = 14$, $11 + 4 = 15$. So it will naturally be the case that $9 + 10 + 11 + 12$ is equal to $13 + 14 + 15$.

Similarly, if we take the two sets $\{16, 17, 18, 19, 20\}$ and $\{21, 22, 23, 24\}$, divide the last number in the first set into four equal parts of 5 each and add one part to each of the other numbers in the set, we get, from $\{16, 17, 18, 19\}$ the set $\{21, 22, 23, 24\}$. So it will naturally be the case that $16 + 17 + 18 + 19 + 20$ is equal to $21 + 22 + 23 + 24$.

In the general case we have the two sets

$$A = \{n^2,\ n^2 + 1,\ n^2 + 2,\ \ldots,\ n^2 + n\},$$
$$B = \{n^2 + n + 1,\ n^2 + n + 2,\ \ldots,\ n^2 + 2n\}.$$

We take away the largest number $(n^2 + n)$ from A, divide it into n parts of $n + 1$ each, and add this amount $(n + 1)$ to each of the remaining numbers; we get the set A' given by:

$$A' = \{n^2 + n + 1,\ n^2 + n + 2,\ \ldots,\ n^2 + 2n\},$$

which is exactly the set B. It follows that the sum of the numbers in A is the same as the sum of the numbers in B.

Triangular number identities

Triangular numbers offer a rich environment for exploration of number patterns and identities. Bring a group of youngsters to this fertile ground, and you will soon have a few discoveries on your hands, including some you may not have seen earlier.

The triangular numbers ("T-numbers" for short) are the numbers 1, $1 + 2 = 3$, $1 + 2 + 3 = 6$, $1 + 2 + 3 + 4 = 10$, …; thus, they are the partial sums of the sequence of natural numbers. The n^{th} such number is denoted by T_n ($n = 1, 2, 3, \ldots$):

$$T_n = 1 + 2 + 3 + 4 + \cdots + (n - 1) + n,$$

or in summation notation: $T_n = \sum_{k=1}^{n} k$. Here are the first ten T-numbers:

$T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$,
$T_5 = 15$, $T_6 = 21$, $T_7 = 28$,
$T_8 = 36$, $T_9 = 45$, $T_{10} = 55$.

It is well known that

$$T_n = \frac{n(n + 1)}{2},$$

and there are several ways of proving this relation alone. Here are some pretty relations that children quickly discover for themselves:

1 *The sum of any two consecutive T-numbers is a perfect square.*

For example, $T_1 + T_2 = 4 = 2^2$ and $T_6 + T_7 = 49 = 7^2$.

2 *If* x *is a T-number, then* $8x + 1$ *is a perfect square. Conversely, if* $8x + 1$ *is an odd perfect square, then* x *is a T-number. Otherwise expressed: if* x *is an odd perfect square, then* $(x - 1)/8$ *is a T-number.*

For example, 3 is a T-number, and $8 \times 3 + 1 = 25 = 5^2$ is a perfect square. Similarly, 81 is an odd perfect square, and $(81 - 1)/8 = 10$ is a T-number.

3 *If* x *is a T-number, then so is* $9x + 1$.

For example, take the T-numbers 3, 10 and 36. Multiplying them by 9 and adding 1 we get the numbers 28, 91 and 325. It remains to check that each of these is a T-number; indeed, they are: $28 = T_7$, $91 = T_{13}$ and $325 = T_{25}$.

The next challenge is to get the children to find *proofs* of these various relations. We now take up this theme.

"The sum of two consecutive T-numbers is a perfect square.":

Many different approaches are possible. Perhaps the most direct way is the one based on 'pure algebra'. We know that $T_n = \frac{1}{2}n(n + 1)$. So:

$$T_{n-1} + T_n = \frac{1}{2}(n-1)n + \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1+n+1)$$

$$= \frac{1}{2}n \times 2n = n^2.$$

Hence the sum of the $(n-1)^{th}$ and n^{th} triangular numbers is the n^{th} perfect square.

Other approaches: But it is fun to seek other ways. Here is a way which draws on the definition of T_n as the sum of the first n positive integers together with the well-known and often-used fact that the sum of the first n odd positive integers equals n^2. These two facts acting in concert with another simple fact —that each odd number is the sum of two consecutive numbers (e.g., $5 = 2 + 3$) —yield a nice proof; all we need to do is re-bracket the numbers and add them in a slightly different order. We illustrate the idea for $n = 3$:

$$\begin{aligned} 3^2 &= 1 + 3 + 5 = (0 + 1) + (1 + 2) + (2 + 3) \\ &= (0 + 1 + 2) + (1 + 2 + 3) \\ &= T_2 + T_3. \end{aligned}$$

Similarly, consider $n = 5$:

$$\begin{aligned} 5^2 &= 1 + 3 + 5 + 7 + 9 \\ &= (0 + 1) + (1 + 2) + (2 + 3) + (3 + 4) + (4 + 5) \\ &= (0 + 1 + 2 + 3 + 4) + (1 + 2 + 3 + 4 + 5) \\ &= T_4 + T_5. \end{aligned}$$

Without having to elaborate on the details, it should be clear that such a re-arrangement of summands will always work. But for those who are keen on seeing how the idea can be expressed symbolically, here is how we do it:

$$\begin{aligned} n^2 &= \sum_{k=1}^{n}(2k - 1) \;=\; \sum_{k=1}^{n}\big((k-1) + k\big) \\ &= \sum_{k=1}^{n}(k - 1) + \sum_{k=1}^{n} k \\ &= \sum_{k=1}^{n-1} k + \sum_{k=1}^{n} k \\ &= T_{n-1} + T_n. \end{aligned}$$

A pictorial way: There's even a way of expressing the relation using pictures! We regard the numbers T_{n-1} and T_n as representing the areas of two staircase-shaped polygons as depicted in Figure 1, which show the polygons for $n = 6$. Observe how neatly they fit together to form a 6×6 square.

Though the construction has been shown only for the specific case $n = 6$, it is not hard to see that the same idea will work for any n.

"8 times a T-number plus 1 is a perfect square."

Here is an algebraic proof. We know that $T_n = \frac{1}{2}n(n + 1)$. So if x is a T-number then $x = \frac{1}{2}n(n + 1)$ for some positive integer n.

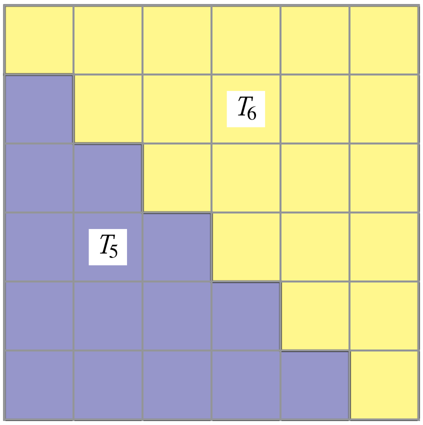


Figure 1. Illustrating "why" $T_5 + T_6 = 6^2$

But in this case we have:

$$\begin{aligned} 8x + 1 = 8 \times \frac{n(n+1)}{2} + 1 &= 4n(n+1) + 1 \\ &= 4n^2 + 4n + 1 \\ &= (2n+1)^2. \end{aligned}$$

Proof of the converse: Suppose that $8x + 1$ is an odd perfect square. Then $8x + 1 = (2n+1)^2$ for some integer n. Hence:

$$x = \frac{(2n+1)^2 - 1}{8} = \frac{4n^2 + 4n}{8} = \frac{n(n+1)}{2} = T_n.$$

This property provides a simple way of checking whether a given number x is a T-number. (Example: The number 3003 is a T-number, because $8 \times 3003 + 1 = 24025 = 155^2$.)

A pictorial way: As earlier there is an elegant way of depicting the relation "If x is a T-number, then $8x + 1$ is a perfect square". We know that if x is a T-number, then $x = \frac{1}{2}n(n+1)$ for some integer $n \geq 0$. Otherwise put, if x is a T-number, then $2x = n(n+1)$ for some integer $n \geq 0$. Hence we may associate with $2x$ a rectangle of dimensions $n \times (n+1)$. Note that the length of this rectangle exceeds the breadth by 1 unit. (This fact has a bearing on the outcome as we shall see shortly.) Four such rectangles may be fitted together as shown in Figure 2, in which we have taken $n = 3$.

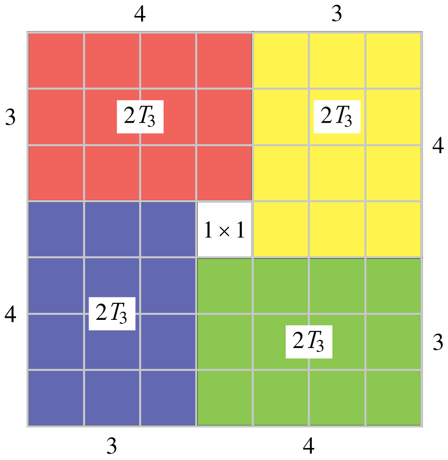


Figure 2. Illustrating 'why' $8T_3 + 1$ is a perfect square

The four rectangles neatly enclose a square measuring 1×1 in the centre, since $4 - 3 = 1$. We see from this that $8T_3 + 1$ is a perfect square, indeed, $8T_3 + 1 = (4+3)^2$. In general we have: $8T_n + 1 = \big((n+1) + n\big)^2$.

The third property

It remains to show this: *If x is a T-number, then so is $9x + 1$.* But we shall leave the proof to you. (*Hint.* Use the property: "n is a T-number if and only if $8n + 1$ is a perfect square".) Indeed we shall challenge you with the following problem:

> *The pair of integers $a = 9$ and $b = 1$ has the property that if x is a T-number, then so is $ax + b$. Find all such pairs of integers.*

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Exploring Fibonacci Numbers using a spreadsheet

In this article we are going to explore a very interesting sequence of numbers known as the Fibonacci sequence. The exploration of the sequence can lead to an absorbing classroom activity for students at the middle school and secondary school level. Students can explore many patterns within the sequence using a spreadsheet like MS Excel and the observations can lead to an enriching discussion in the classroom.

JONAKI GHOSH

The Fibonacci Puzzle

The Fibonacci puzzle was posed by Leonardo of Pisa, also known as Fibonacci (hence the name of the sequence). The puzzle describes the growth of an idealized rabbit population. A newly born pair of rabbits comprising a male and a female rabbit is put in the field and are able to mate at the age of one month. At the end of the second month the female rabbit produces a pair of rabbits (again a male and a female). Rabbits never die and every mating pair always produces a new pair every month from the second month on. How many pairs will there be in one year?

Let us try to find out the number of pairs of rabbits at the end of every month starting from the first month. To begin with there is one pair of rabbits. At the end of the first month, they mate, but there is still only 1 pair. At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the

***Keywords**: sequence, Fibonacci, golden ratio, investigation, Excel*

field. At the end of the third month, the female of the original pair produces a second pair, making 3 pairs altogether in the field. Remember that the second pair which was born at the end of the second month is only able to mate at the end of the third month. At the end of the fourth month, the female of the original pair has produced yet another new pair and the female born at the end of the second month produces her first pair, making 5 pairs in all. This process continues. To obtain the number of pairs at the end of any given month, say n, we need to add the number of pairs at the end of month $n-1$ and the number of pairs at the end of month $n-2$.

Hence the Fibonacci sequence can be written in the form of the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

The first two terms of the sequence are 1 and 1. So if $F_1 = F_2 = 1$, the recurrence relation can also be written as

$$F_{n+2} = F_{n+1} + F_n .$$

Sometimes F_0 is taken to be 0. Here are the first fifteen Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610.

Can you guess how large the 50th Fibonacci number will be? One can continue the process of adding pairs of consecutive terms to find the next term, but after a while it can get quite cumbersome. Let us take the help of a spreadsheet to generate the Fibonacci sequence and find its 50th term.

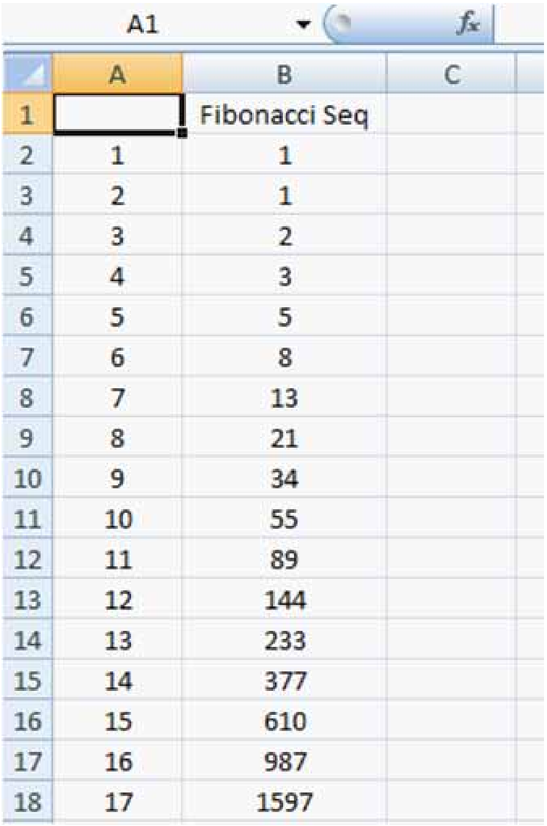
Generating the Fibonacci sequence on Excel: The following steps will help you obtain the sequence on Excel.

Step 1: The first step is to create a column of integers from 1 to 50 (in column A). This may be achieved in the following manner.

- Click on cell A2 and enter 1. Then enter = **A2 + 1** in cell A3 and drag cell A3 till A51. This will create a column of numbers 1 to 50.

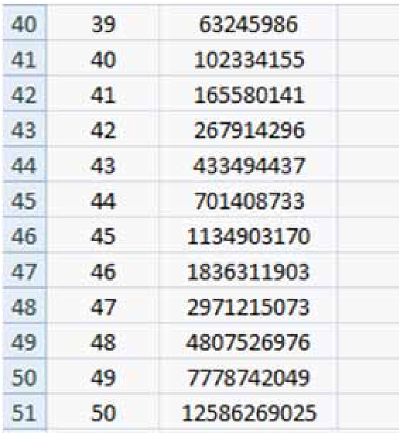
Step 2: Enter 1 in cells B2 and B3. In cell B4, enter = **B3 + B2** and press enter. A double click on the corner of cell B4 will generate the Fibonacci sequence as shown in Figure 1. As you scroll down the sequence in column B, you will notice that the column width is too small to accommodate the numbers. For example the 40th Fibonacci number appears as 1.02E + 08. This means that Excel has approximated the number and the number is close to 1.02×10^8. To get the actual terms of the sequence beyond the 40th term, the column width needs to be increased. This can be done by taking the cursor to the end of the column (were the columns are named as A, B, C etc) and dragging it to the required width.

Note that the 50th Fibonacci number is 12586269025. Clearly, this is a very fast growing sequence. See Figure 2.



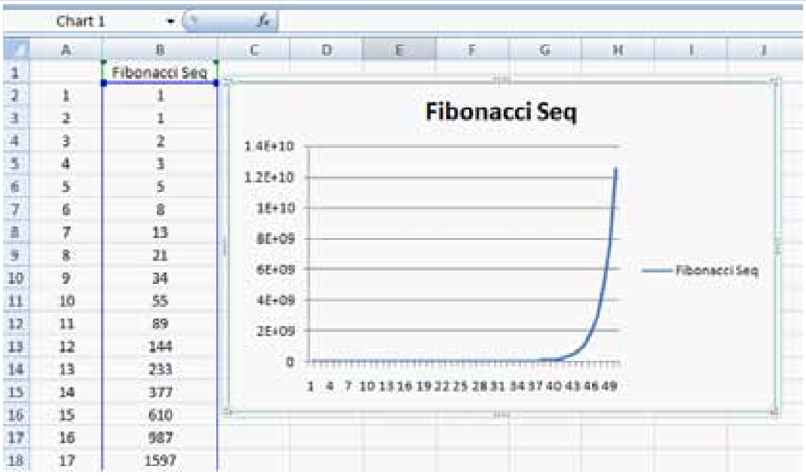
	A	B	C
1		Fibonacci Seq	
2	1	1	
3	2	1	
4	3	2	
5	4	3	
6	5	5	
7	6	8	
8	7	13	
9	8	21	
10	9	34	
11	10	55	
12	11	89	
13	12	144	
14	13	233	
15	14	377	
16	15	610	
17	16	987	
18	17	1597	

Figure 1: Generating the Fibonacci sequence on Excel.



40	39	63245986
41	40	102334155
42	41	165580141
43	42	267914296
44	43	433494437
45	44	701408733
46	45	1134903170
47	46	1836311903
48	47	2971215073
49	48	4807526976
50	49	7778742049
51	50	12586269025

Figure 2: A part of the Excel sheet showing the 39th to 50th terms of the Fibonacci sequence.



	A	B
1		Fibonacci Seq
2	1	1
3	2	1
4	3	2
5	4	3
6	5	5
7	6	8
8	7	13
9	8	21
10	9	34
11	10	55
12	11	89
13	12	144
14	13	233
15	14	377
16	15	610
17	16	987
18	17	1597

Figure 3: Graphing the Fibonacci sequence on Excel.

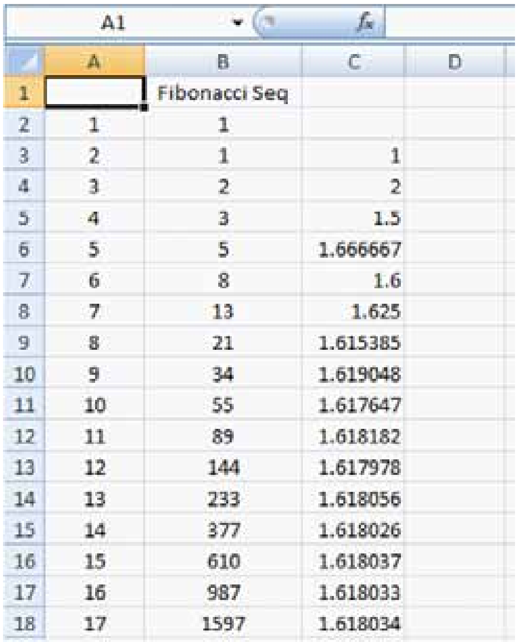
Step 3: We can also draw the graph of the sequence. For this, select column B by clicking on B, go to **Insert** on the toolbar, select **chart** and then **line**. See Figure 3.

As you already know,the first 15 Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610. Let us see what happens when we take the ratios of consecutive terms $\frac{F_{n+1}}{F_n}$ of the Fibonacci sequence. They are:

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \frac{233}{144}, \frac{377}{233}, \frac{610}{377}, \ldots\ldots,$$

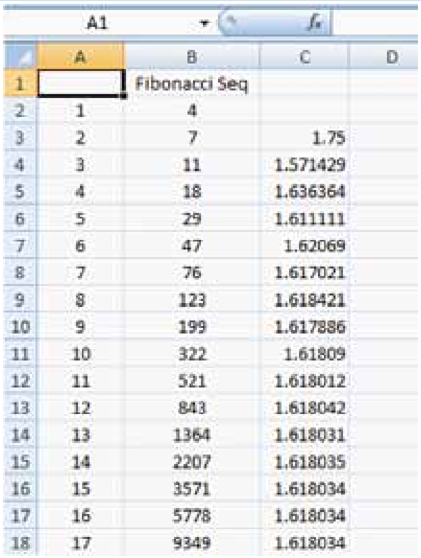
Let us compute the ratios on Excel as follows. In cell C3 of column C, enter = B3 / B2 and double click on the corner of the cell. You will observe that after a certain number of terms the ratios become steady at 1.618034.

A natural question now arises whether this value (that is, 1.618034) will remain the same if we change the initial values of the Fibonacci sequence (which are 1 and 1). Let us try to investigate by taking different starting values. Change the values in cells B2 and B3 in the Excel sheet to 4 and 7. Observe that the ratios of successive terms still approach 1.618034 (shown on Figure 5). You may investigate by taking different starting values. You will observe that the ratios of successive terms still approach 1.618034. A graph of the ratios of successive terms (obtained by selecting column C) reveals this behavior of the sequence. See Figures 5 and 6. This is indeed an interesting observation but we need to find a mathematical explanation for it.



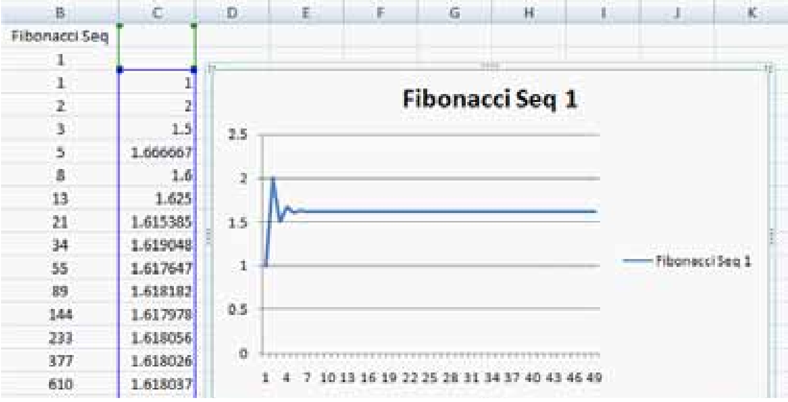
	A	B	C	D
1		Fibonacci Seq		
2	1	1		
3	2	1	1	
4	3	2	2	
5	4	3	1.5	
6	5	5	1.666667	
7	6	8	1.6	
8	7	13	1.625	
9	8	21	1.615385	
10	9	34	1.619048	
11	10	55	1.617647	
12	11	89	1.618182	
13	12	144	1.617978	
14	13	233	1.618056	
15	14	377	1.618026	
16	15	610	1.618037	
17	16	987	1.618033	
18	17	1597	1.618034	

Figure 4: The ratios of consecutive terms of the Fibonacci sequence become steady at 1.618034 (in column C)



	A	B	C	D
1		Fibonacci Seq		
2	1	4		
3	2	7	1.75	
4	3	11	1.571429	
5	4	18	1.636364	
6	5	29	1.611111	
7	6	47	1.62069	
8	7	76	1.617021	
9	8	123	1.618421	
10	9	199	1.617886	
11	10	322	1.61809	
12	11	521	1.618012	
13	12	843	1.618042	
14	13	1364	1.618031	
15	14	2207	1.618035	
16	15	3571	1.618034	
17	16	5778	1.618034	
18	17	9349	1.618034	

Figure 5: The terms of the sequence with initial values 4 and 7.



B	C
Fibonacci Seq	
1	
1	1
2	2
3	1.5
5	1.666667
8	1.6
13	1.625
21	1.615385
34	1.619048
55	1.617647
89	1.618182
144	1.617978
233	1.618056
377	1.618026
610	1.618037

Figure 6: A graph of the ratios of consecutive terms of the Fibonacci sequence becomes steady at 1.618034.

We already know that the Fibonacci sequence is given by the recurrence relation,

$$F_{n+1} = F_n + F_{n-1} \tag{1}$$

Also from the sequence of ratios we observe that after some value of n, we have:

$$\frac{F_{n+1}}{F_n} = \frac{F_n}{F_{n-1}} \tag{2}$$

Let this ratio be equal to x. We would like to show that $x = 1.618$ up to three places of decimals.

Using (1) and (2) we get

$$\frac{F_n + F_{n-1}}{F_n} = \frac{F_n}{F_{n-1}} \tag{3}$$

$$or \;\; 1 + \frac{F_{n-1}}{F_n} = \frac{F_n}{F_{n-1}} \tag{4}$$

This implies that $1 + \frac{1}{x} = x$. Simplifying this we get the quadratic equation $x^2 - x - 1 = 0$ whose roots are

$$\frac{1 \pm \sqrt{5}}{2}.$$

We ignore the value $\frac{1-\sqrt{5}}{2}$ as x cannot be negative.

Thus $x = \frac{1+\sqrt{5}}{2}$ (which is approximately equal to 1.618 up to three decimal places).

This is also known as the *Golden Ratio*.

The Fibonacci sequence $F_{n+2} = F_{n+1} + F_n$ with $F_1 = F_2 = 1$ is a recursive sequence. We have also seen from our Excel exploration that the terms of the sequence grow large very quickly. But wouldn't it be nice if we had a formula for the n^{th} Fibonacci number? After all, if we wanted the 100th or the 1000th Fibonacci number, it would be far more convenient to have a formula in which we could insert the value of n (for the 100th term, $n = 100$) and obtain the required number.

The general term of the Fibonacci sequence: To find the required formula, we need to get acquainted with a simple but useful tool. Given a sequence of numbers $x_1, x_2, x_3, \ldots$ we define the *left shift operator L as follows:*

$$L(x_n) = x_{n+1},$$

where x_n is the n^{th} element of the sequence $x_1, x_2, x_3, \ldots$. If L is made to operate on this sequence, we will obtain:

$$L(x_1), L(x_2), L(x_3), \ldots.$$

This is just: $x_2, x_3, x_4, \ldots$. i.e., the original sequence shifted to the left by one term (x_1 has been deleted).

Let us see how the left shift operator can be used on the Fibonacci sequence.

We know that $$F_{n+1} = L(F_n). \qquad (5)$$

Also, $$F_{n+2} = L(F_{n+1}) = L(L(F_n)) = L^2(F_n). \qquad (6)$$

Now, $F_{n+2} = F_{n+1} + F_n$ yields $F_{n+2} - F_{n+1} - F_n = 0$. Rewriting this in terms of L and using (6) we get

$$L^2(F_n) - L(F_n) - F_n = 0.$$

'Factoring out' F_n (treating it as though it is a number) we get $(L^2 - L - 1)F_n = 0$. (7)

This is an interesting property of the left shift operator L which allows us to treat it like a variable and to be able to manipulate it algebraically. In fact we can solve the factored version of the above equation separately.

We already know that the roots of the algebraic equation

$$x^2 - x - 1 = 0 \text{ are } \frac{1 \pm \sqrt{5}}{2}. \text{ Let } \theta = \frac{1+\sqrt{5}}{2} \text{ and let } \varphi = \frac{1-\sqrt{5}}{2}.$$

Hence to solve $(L^2 - L - 1)F_n = 0$ we solve its factored form $((L - \theta)(L - \varphi))F_n = 0$.

In other words, we need to solve $(L - \theta)F_n = 0$ and $(L - \varphi)F_n = 0$ separately and combine the results linearly.

Now $(L - \theta)F_n = 0$ implies $L(F_n) = \theta F_n = F_{n+1}$.

Further, $F_{n+1} = \theta F_n$ implies that $F_n = A\theta^n$ for some constant A. This is an example of a geometric sequence.

Similarly, solving $(L - \varphi)F_n = 0$ we get $F_n = B\varphi^n$ for some constant B.

Finally we obtain the general solution of $(L^2 - L - 1)F_n = 0$ as

$$F_n = A\theta^n + B\varphi^n \qquad (8)$$

This relation will help us find the formula for the n^{th} Fibonacci number.

Substituting $n = 0$ in (8) and recalling that $F_0 = 0$, we get $A + B = 0$ or $B = -A$.

Substituting $n = 1$ in (8) and recalling that $F_1 = 1$, we get

$$1 = A\theta^1 + B\varphi^1 = A(\theta - \varphi) = A\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = \sqrt{5}\,A.$$

This implies that $A = 1/\sqrt{5}$ and $B = -A = -1/\sqrt{5}$.

Plugging these values of A and B in (8), we have the much desired formula

$$Fn = \frac{1}{\sqrt{5}}\left(\theta^n - \phi^n\right) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Simplifying we get, $F_n = \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}}$.

Let us verify if this works by substituting $n = 2$.

$$F_2 = \frac{(1+\sqrt{5})^2-(1-\sqrt{5})^2}{2^2\sqrt{5}} = \frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1,$$

which is true!

Conclusion

Fibonacci numbers appear in nature. They appear in the arrangement of leaves in certain plants, on the patterns of the florets of a flower, the bracts of a pinecone, and on the scales of a pineapple. There are numerous examples of Fibonacci numbers and the golden ratio in nature which students can explore in the form of a project. MS Excel can prove handy in exploring the patterns in the sequence, and this activity can be done with students at the middle school level. However, finding the general term of the Fibonacci sequence can be easily illustrated to secondary school students.

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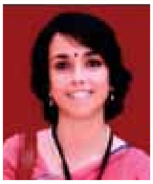
2. Numbers in Nature from Wikipedia, the free Encyclopedia, Retrieved from http://en.wikipedia.org/wiki/Fibonacci_number



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Adventures in Problem Solving

Squares between Squares

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

I. A 'Trivial' Observation

It is often the case — more often than one may expect— that in the successful solution of a problem, the seed of the solution is a very simple observation; so simple that we call it 'trivial', and we doubt its potential to play a significant role in the solution of any problem. We showcase such an instance here, starting with the following:

Observation 1.
Between two consecutive integers there does not lie another integer.

Stated this way the observation looks too trivial to be of use to anyone. But even simple observations have consequences! And these may well be of a less trivial nature than the original one. They in turn may give rise to other consequences, and so on, and these may progressively get less trivial. What we get in the end result can be significant and far from trivial!

For example, here is a consequence of the Observation 1, which we get by squaring the quantities:

Observation 2.
Let n be an integer. Between the perfect squares n^2 and $(n+1)^2$ there does not lie another perfect square.

From this we deduce the following, which looks more impressive than Observation 1:

Observation 3.
Let n and X be integers such that $n^2 \le X \le n^2 + 2n + 1$. Suppose that X is a perfect square. Then either $X = n^2$ or $X = (n+1)^2$.

II. Applications

We make use of observations 2 and 3 made above in solving two problems posed in the 'Adventures' column of the November 2013 issue of *AtRiA*.

Problem 5, 'Adventures', November 2013
Find all integers n such that $n^2 + n + 1$ is a perfect square.

Solution Suppose that $n > 0$. Then $n^2 < n^2 + n + 1 < n^2 + 2n + 1$, so $n^2 + n + 1$ lies between the two consecutive squares n^2 and $(n+1)^2$ and thus cannot be a square. So there can be no solution with $n > 0$.
Next suppose that $n < 0$; then $n \le -1$, or $n + 1 \le 0$.

Hence $n^2 + 2n + 1 < n^2 + n + 1 \leq n^2$. So the only way for $n^2 + n + 1$ to be a square is to let $n + 1 = 0$, i.e., $n = -1$.
Since $n = 0$ also yields a square value, the possible n-values are $-1, 0$. Both these n-values yield $n^2 + n + 1 = 1^2$.

Problem 4, 'Adventures', November 2013 may be solved in the same way.

Problem 6, 'Adventures', November 2013
Find all integers n such that $n^4 + n^3 + n^2 + n + 1$ *is a perfect square.*

This is a substantially more complicated problem of the same genre, but it too can be solved the same way. The first challenge is to box the given quantity between two perfect squares. To do this we shall use the 'completing the square' reasoning yet again.

Recall that to add a term to $(x^2 + ax)$ so as to get a square, we *halve* the coefficient of x and use that to create the term: $x^2 + ax + (\frac{1}{2}a)^2 = \left(x + \frac{1}{2}a\right)^2$.

The same logic applied to $n^4 + n^3 + n^2 + n + 1$ suggests that the expression to be studied has to be $\left(n^2 + \frac{1}{2}n\right)^2$. For, when we expand this expression, we get both the n^4 and n^3 terms with the correct coefficients:

$$\left(n^2 + \frac{n}{2}\right)^2 = n^4 + n^3 + \frac{n^2}{4}.$$

Note that we have not got the right coefficient for n^2 (we have got $\frac{1}{4}$ instead of 1), and we haven't got the other two terms at all.

The fractions ($\frac{1}{2}$ and $\frac{1}{4}$) make things a bit awkward, so why don't we multiply everything by 4? That way, squares remain squares (for, if X is a square, then so is $4X$; this is crucial), and at the same time we get rid of the fractions. So we ask: *For which integers n is the quantity* $A = 4\left(n^4 + n^3 + n^2 + n + 1\right)$ *a square?*

So let us now box A between two consecutive squares. Since $2 \times \left(n^2 + \frac{1}{2}n\right) = 2n^2 + n$, the candidate squares are $B = \left(2n^2 + n\right)^2$ and $C = \left(2n^2 + n + 1\right)^2$. We now have:

$$\left(2n^2 + n\right)^2 = 4n^4 + 4n^3 + n^2,$$

$$\therefore\ A - B = 3n^2 + 4n + 4.$$

We shall show that this quantity is *always positive*. There are various ways of seeing this. The simplest is to compute the discriminant of the quadratic expression $3n^2 + 4n + 4$ using the well-known "$b^2 - 4ac$" formula. We get: $4^2 - 4 \times 3 \times 4 = 16 - 48$ which is negative. As the discriminant is negative, the expression $3n^2 + 4n + 4$ never changes sign. And since it is positive for $n = 0$, it is positive for all n.

(Another way to see that $A - B > 0$ is to write the expression for $A - B$ as:

$$A - B = 3n^2 + 4n + 4$$

$$= n^2 + n^2 + \left(n^2 + 4n + 4\right)$$

$$= n^2 + n^2 + (n + 2)^2.$$

Please complete this line of reasoning on your own.)

We see that $B < A$, strictly. Now let us look at $C - A$:

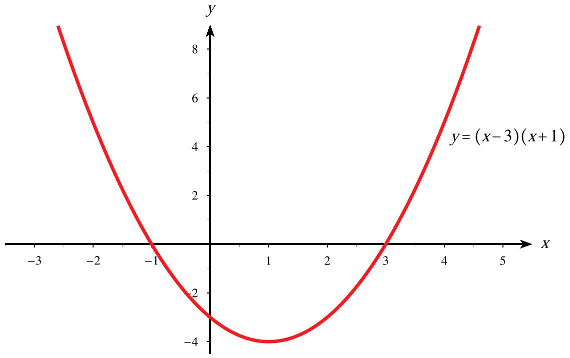
$$\left(2n^2 + n + 1\right)^2 = 4n^4 + 4n^3 + 5n^2 + 2n + 1,$$

$$\therefore\ C - A = n^2 - 2n - 3.$$

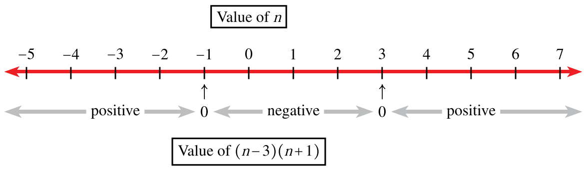
Conveniently for us, the form $n^2 - 2n - 3$ factorizes:

$$n^2 - 2n - 3 = (n-3)(n+1), \quad \therefore C - A = (n-3)(n+1).$$

The quadratic expression $(x - 3)(x + 1)$ gives rise to the following graph:



We see that the expression (n -3) (n +1) has the following sign profile:



Hence $C - A$ is positive if $n < -1$ or if $n > 3$.

This has important consequences:

Proposition 1. If either $n < -1$ or $n > 3$ then

$$(2n^2 + n)^2 < 4\,(n^4 + n^3 + n^2 + n + 1) < (2n^2 + n + 1)^2 .$$

So if the integer n is less than -1 *or greater than* 3, *the quantity* $4\left(n^4 + n^3 + n^2 + n + 1\right)$ *lies strictly between two consecutive squares and is thus not a perfect square. Therefore the quantity* $n^4 + n^3 + n^2 + n + 1$ *is not a square if* $n < -1$ *or* $n > 3$.

Proposition 2. If $-1 \le n \le 3$ then

$$(2n^2 + n)^2 < 4\,(n^4 + n^3 + n^2 + n + 1) \le (2n^2 + n + 1)^2 .$$

Equality holds in the inequality on the right side precisely when $n \in \{-1, 3\}$. *Therefore, the quantity* $4\left(n^4 + n^3 + n^2 + n + 1\right)$ *is a perfect square precisely when* $n \in \{-1, 3\}$.

We have fully solved the problem! The answer is: *The quantity* $n^4 + n^3 + n^2 + n + 1$ *is a perfect square precisely when* $n \in \{-1, 3\}$.
All this from the 'trivial' observation that between two consecutive integers there does not lie another integer! We have come a long way indeed. We invite you to apply these ideas to the following problem:

Find all integers n such that $n^4 + 2n^3 + 3n^2 + 4n + 5$ *is a perfect square.*

Remark 1. Here is a problem which closely resembles the one solved in the preceding pages: *Find all integers n such that* $n^3 + n^2 + n + 1$ *is a perfect square.*

But the resemblance is deceptive. Though the problem concerns a polynomial of lower-degree, it is significantly more difficult than the one we solved. Perhaps the main reason for this is that its degree is odd (3 rather than 4), so there is no easy way of boxing it between two squares. We shall not attempt to solve the problem here.

Remark 2. We mention here a second consequence of Observation 1 (but in another form); it is based on Problem 11121 from the well-known journal *American Mathematical Monthly*. For any positive integer n, we consider the set S_n of integers that lie strictly between n^2 and $(n+1)^2$. For example, $S_1 = \{2, 3\}$, $S_2 = \{5, 6, 7, 8\}$ and $S_3 = \{10, 11, 12, 13, 14, 15\}$. You can check that S_n has $2n$ elements. We now ask the following question:

Can we find two distinct numbers in S_n *whose product is a perfect square?*

The answer is clearly 'No' for $n = 1$ and $n = 2$ (please check). It turns out that the answer is 'No' for every n. The proof is elegant and instructive.

Proof We adopt the well-known approach of 'proof by contradiction'. Suppose that for some $n \ge 1$ there exist integers $a, b \in S_n$, $a < b$, such that ab is a square. Now any positive integer can be written as the product of a perfect square and a square-free integer (i.e., an integer not divisible by any perfect square larger than 1), simply by factoring out the largest possible perfect square from it. (For example, $20 = 2^2 \times 5$; $150 = 5^2 \times 6$.) Accordingly, we write $a = uv^2$ and $b = UV^2$ where u and U are square-free. Since ab is a perfect square, so is uU; but this implies that u and U are the same number! (Do you see why? It is because each prime factor in u and U occurs just once in each number.)

So we have $a = uv^2$ and $b = uV^2$ for some positive integers u, v, V with $v < V$. It must be that $u > 1$, else we get the chain $n^2 < v^2 < V^2 < (n+1)^2$, which is clearly absurd. Now from the relation $n^2 < uv^2 < uV^2 < (n+1)^2$ we get, by taking square roots:

$$\frac{n}{\sqrt{u}} < v < V < \frac{n+1}{\sqrt{u}}.$$

The difference between the numbers at the ends of this chain of inequalities is

$$\frac{n+1}{\sqrt{u}} - \frac{n}{\sqrt{u}} = \frac{1}{\sqrt{u}} < 1,$$

i.e., the difference is strictly less than 1. On the other hand, $V - v \ge 1$: the difference between the middle numbers is not less than 1. These relations contradict each other! Hence such a situation cannot happen. That is, integers a, b with the stated property do not exist.

III. Solutions to Problems 1–5 from the November 2013 Issue

1. (a) $3599 = 3600 - 1 = 60^2 - 1 = (60 - 1) \times (60 + 1) = 59 \times 61$

   (b) $8099 = 8100 - 1 = 90^2 - 1 = (90 - 1) \times (90 + 1) = 89 \times 91 = 89 \times 7 \times 13$

   (c) $4087 = 4096 - 9 = 64^2 - 3^2 = (64 - 3) \times (64 + 3) = 61 \times 67$

2. $x^4 + 4 = \left(x^4 + 4x^2 + 4\right) - 4x^2 = \left(x^2 + 2\right)^2 - (2x)^2 = \left(x^2 - 2x + 2\right) \cdot \left(x^2 + 2x + 2\right)$

3. *Find all integers n such that* $n^2 + 10n + 20$ *is a perfect square.*

   Let $n^2 + 10n + 20 = x^2$; then $n^2 + 10n + 25 = x^2 + 5$, so $(n + 5)^2 - x^2 = 5$, and:

   $$(n + 5 - x) \cdot (n + 5 + x) = 5.$$

   The factorizations of 5 are: $1 \times 5, 5 \times 1, (-1) \times (-5), (-5) \times (-1)$. Hence:

   $$(n + 5 - x, n + 5 + x) \in \{(1, 5),\ (5, 1),\ (-1, -5)\ (-5, -1)\}.$$

   It follows by addition that $2(n + 5) = 6$ or -6, so $2n = -4$ or -16, and therefore $n = -2$ or -8. Both these values of n correspond to the same value of x^2, namely: $x^2 = 4$.

4. *Find all integers n such that* $n^2 + n$ *is a perfect square.*

   If $n^2 + n$ is a square, so is $4(n^2 + n) = 4n^2 + 4n$. But $4n^2 + 4n + 1 = (2n + 1)^2$ is a square as well. So we have two squares differing by 1, which happens only with 0 and 1.

   (To see why the only squares differing by 1 are 0 and 1, let x, y be integers such that $x^2 - y^2 = 1$. Then $(x + y)(x - y) = 1$, hence $(x + y, x - y) = (1, 1)$ or $(-1, -1)$. The first possibility lead to $(x, y) = (1, 0)$, and the second one leads to $(x, y) = (-1, 0)$. Hence the two squares are $1^2 = 1$ and $0^2 = 0$.)

   Hence $n^2 + n = 0$, which yields $n = -1, 0$. So the answer is that the only square value taken by $n^2 + n$ is 0, which it takes when $n \in \{-1, 0\}$.

   Another approach. Note that $n^2 + n = n(n + 1)$. Since n and $n + 1$ are consecutive integers, they are coprime. Hence if their product is a square, both are squares or both are negatives of squares. Either way we get a pair of consecutive squares differing by 1. The only such squares are 0, 1. Hence it must be that one of $n, n + 1$ is 0. The two ways in which this can happen yield the solutions obtained above ($n = 0, -1$).

5. *Find all integers n such that* $n^2 + n + 1$ *is a perfect square.*

   This was solved above but we give another solution here.

   If $n^2 + n + 1 = x^2$, then $4n^2 + 4n + 4 = (2x)^2$, so $(2n + 1)^2 + 3 = (2x)^2$, which yields $(2x)^2 - (2n + 1)^2 = 3$. Hence $(2x - 2n - 1)(2x + 2n + 1) = 3$. Since $3 = 3 \times 1, 1 \times 3, (-3) \times (-1), (-1) \times (-3)$, it follows that $4n + 2 \in \{-2, 2\}$. This yields $n \in \{-1, 0\}$. So there are two integers n for which $n^2 + n + 1$ is a perfect square, -1 and 0. For both these n-values, $n^2 + n + 1$ takes the same value, which is $1 = 1^2$.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at comm.math.centre@gmail.com or shailesh.shirali@gmail.com

Problems for the Middle School

Problem Editor : R. ATHMARAMAN

Problems for Solution

Problem III-1-M.1
Show that the following number is a perfect square for every positive integer n:

$$\underbrace{111111\ldots111111}_{2n\text{ digits}} - \underbrace{222\ldots222}_{n\text{ digits}}.$$

For example, $11 - 2 = 9$ and $1111 - 22 = 1089$ are perfect squares.

Problem III-1-M.2
On a digital clock, the display reads 6 : 38. What will the clock display twenty-eight digit changes later?

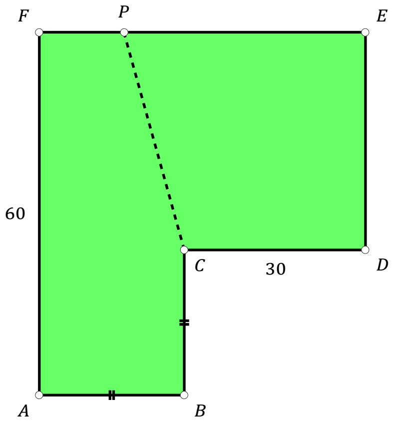
Problem III-1-M.3
The figure shows a hall $ABCDEF$ with right angles at its corners. Its area is 2520 sq units, and $AB = BC$, $CD = 30$ units, $AF = 60$ units. A point P is located on EF such that line CP divides the hall into two parts with equal area. Find the length EP.

Problem III-1-M.4
In a circle with radius 4 units, a rectangle and an equilateral triangle are inscribed. If their areas are equal, find the dimensions of the rectangle.

Problem III-1-M.5
Find the value of the following (no calculators!):

$$\left\lfloor \frac{2014^3}{2012 \times 2013} \right\rfloor - \left\lfloor \frac{2012^3}{2013 \times 2014} \right\rfloor.$$

Here the symbol $\lfloor\ \rfloor$ has the following meaning: if x is any real number, $\lfloor x \rfloor$ is the largest integer not greater than x. For example, $\lfloor 3.2 \rfloor = 3$, and $\lfloor -1.7 \rfloor = -2$. It is called the "greatest integer function".



Solutions of Problems in Issue - II - 3

Solution to problem II-3-M.1

Find the value of the following (no calculators!):

$$\frac{(2013^2 - 2019) \times (2013^2 + 4023) \times 2014}{2010 \times 2012 \times 2015 \times 2016}.$$

Let $a = 2013$; then $2019 = a + 6$, $4023 = 2a - 3$, etc., so the given expression equals:

$$\frac{\left(a^2 - a - 6\right)\left(a^2 + 2a - 3\right)(a+1)}{(a-3)(a-1)(a+2)(a+3)}$$
$$= \frac{(a-3)(a+2)(a+3)(a-1)(a+1)}{(a-3)(a-1)(a+2)(a+3)} = a + 1.$$

So the expression simplifies to 2014.

Solution to problem II-3-M.2

Can you find a pair of perfect squares that differ by 2014?

The answer is **No.** For suppose that $a^2 - b^2 = 2014$ where a, b are integers. Then $(a-b)(a+b) = 2014$. Since 2014 is even, at least one of the quantities $a - b$, $a + b$ is an even number. But $a - b$ and $a + b$ have the same parity (they are both odd or both even), so if one of them is even, then so is the other one. This means that the product $(a-b)(a+b)$ is a multiple of 4. However, 2014 is not a multiple of 4. Hence the given representation is not possible.

Solution to problem II-3-M.3

From a two-digit number n we subtract the number obtained by reversing its digits. The answer is a perfect cube. What could n be?

Let $n = 10a + b$ where a, b are digits. On subtracting its reversal, $10b + a$, we get the number $x = 9(a-b) = 3^2(a-b)$. For x to be a cube, $a - b$ would have to be 3 times a cube (this would lead to $x = 3^3 \times$ a cube). Since a and b are digits, the absolute value of $a - b$ cannot exceed 9. Hence if $a - b$ is 3 times a cube, it must be that $a - b = 3$ or -3. Therefore n is one of the following: 14, 41, 25, 52, 36, 63, 47, 74, 58, 85, 69, 96. For each of these, $x = \pm 27 = (\pm 3)^3$.

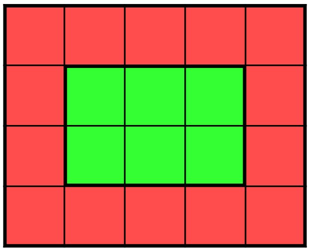
Solution to problem II-3-M.4

To a certain two-digit number m we add the number obtained by reversing its digits. The answer is a perfect square. What could m be?

Let $m = 10a + b$ where a, b are digits. On adding its reversal, $10b + a$, we get the number $y = 11(a+b)$. For y to be a square, $a + b$ would have to be 11 times a square (this would lead to $y = 11^2 \times$ a square). Since a and b are digits, the sum $a + b$ cannot exceed 18. Hence if $a + b$ is 11 times a square, it must be that $a + b = 11$. Therefore m is one of the following: 29, 92, 38, 83, 47, 74, 56, 65. For each of these, $y = 121 = 11^2$.

Solution to problem II-3-M.5

The rectangle shown has been divided into equal squares. The squares along the perimeter are shaded red; the rest of the squares are shaded green. Note that the number of red squares is greater than the number of green squares. What should be the dimensions of the rectangle if the number of red squares equals the number of green squares?



Let the dimensions of the rectangle be $a \times b$, with $a \geq b$. The inner rectangle (shaded green) has dimensions $(a-2) \times (b-2)$. For the areas of the red and green regions to be the same, the area of the green region must be half the area of the large rectangle, so we must have:

$$ab = 2(a-2)(b-2),$$
$$\therefore\ ab = 2(ab - 2a - 2b + 4),$$
$$\therefore\ ab - 4a - 4b + 8 = 0.$$

We must therefore find pairs (a, b) of positive integers that satisfy the equation $ab - 4a - 4b + 8 = 0$. The way we do this is based on factorization. (It is a fairly standard procedure.) It draws on the observation that $ab - 4a - 4b + 8$ is `almost' equal to the product $(a-4)(b-4)$, but not quite: we get 16 in place of 8. This prompts the following:

$$ab - 4a - 4b + 8 = 0,$$
$$\therefore\ ab - 4a - 4b + 16 = 8,$$
$$\therefore\ (a-4)(b-4) = 8.$$

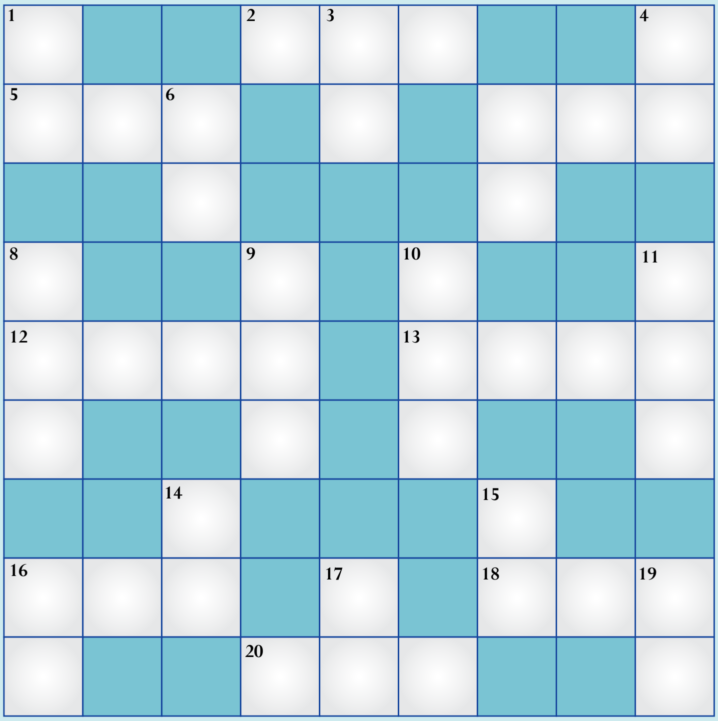
Hence $(a-4, b-4)$ are a pair of positive integers whose product is 8. In what ways can 8 be expressed as a product of two integers? The only ways are: 8×1 and 4×2. Hence $(a-4, b-4) = (8, 1)$ or $(4, 2)$, and therefore, $(a, b) = (12, 5)$ or $(8, 6)$. So the dimensions of the large rectangle are either 12×5 or 8×6. Observe that these satisfy the stated conditions:

$12 \times 5 = 60, \quad (12-2) \times (5-2) = 30,$

30 = half of 60, $8 \times 6 = 48$, $(8-2) \times (6-2) = 24$,

24 = half of 48.

number crossword–4



Clues Across :

1. Half of 16 A
3. The first digit is followed by its successor and then by its predecessor
5. The middle digit is the sum of the end digits
6. Area of a square of side 74
9. Digits in arithmetic progression
10. The square root of 417316
12. Two complete rotations and two degrees
13. Two centuries, two decades and two years

Clues Down :

1. 16A divided by 4D
3. One less than a positive multiple of 10
4. A dozen more than 19D
6. 14 D written in reverse
7. Twice the difference between 15D and 17D
8. Two and a half times 20A
9. A score of unlucky numbers
10. 3 D times the cube of 3
11. 9 times the second 3 digit prime.
14. A perfect square between 30 and 40
15. Square root of 12 A written in reverse
16. One day short of 10 weeks
17. 2 score and 2
19. One tenth of 9D

Problems for the Senior School

Problem Editors: Prithwijit De & Shailesh Shirali

Problems for Solution

Problem III-1-S.1
Let $f(x) = ax^2 + bx + c$, where a, b, c are positive integers. Show that there exists an integer m such that $f(m)$ is a composite number.

Problem III-1-S.2
Show that the arithmetic progression $1, 5, 9, 13, 17, 21, 25, 29, \ldots$ contains infinitely many prime numbers.

Problem III-1-S.3
In $\triangle ABC$, the midpoint of AB is D, and E is the point of trisection of BC closer to C. Given that $\measuredangle ADC = \measuredangle BAE$, determine the magnitude of $\measuredangle BAC$.

Problem III-1-S.4
We know that a median of a triangle bisects it into two triangles of equal area. We also know that the medians of a triangle are concurrent. Given a $\triangle ABC$, does there necessarily exist a point D on side BC such that $\triangle ABD$ and $\triangle ACD$ have equal perimeter?

If such a point exists, then we can similarly obtain points E and F on AC and AB, respectively such that BE and CF bisect the perimeter of ABC. Are the lines AD, BE, CF concurrent?

Problem III-1-S.5
Let $A = 5^{2013}$ and $B = 4^{2013}$. Is $4^A + 5^B$ a prime number? Justify your answer.

Solutions of Problems in Issue-II-3

Solution to problem II-3-S.1
Let P be a polynomial such that $P(x) = P(0) + P(1)x + P(2)x^2$ and $P(-1) = 1$. Find $P(3)$.

Putting $x = 1$ yields $P(0) + P(2) = 0$. Putting $x = -1$, we get:

$$P(-1) = P(0) - P(1) + P(2),$$

hence $P(1) = -1$. Putting $x = 2$, we get:

$$P(2) = P(0) - 2 - 4P(0) = -2 - 3P(0) = -2 + 3P(2),$$

hence $P(2) = 1$ and $P(0) = -1$. Therefore $P(x) = -1 - x + x^2$ and $P(3) = 5$.

Solution to problem II-3-S.2
In $\triangle ABC$, the midpoint of BC is D; the foot of the perpendicular from A to BC is E; the foot of the perpendicular from D to AC is F; $BE = 5$, $EC = 9$; area of $\triangle ABC$ is 84. Find EF.

There are two cases: (i) E lies between B and C. (ii) E lies to the left of B on the line BC. Possibility (i) is depicted in Figure 1.

(i) We have $[ABC] = 84 = 1/2 \times 14 \times AE$, so $AE = 12$, $ED = EC - DC = 9 - 7 = 2$. Now assign coordinates: $E = (0, 0)$, $A = (0, 12)$, $B = (-5, 0)$, $C = (9, 0)$, $D = (2, 0)$.

The slope of AC is $-4/3$, and the slope of DF is $3/4$. The equations of AC and DF are $x/9 + y/12 = 1$ and $y = (3/4)(x - 2)$, respectively. Solving these two equations for x, y, we get $F = (162/25, 84/25)$. Hence $ED = \sqrt{(162^2 + 84^2)}/5 = 6\sqrt{37}/5$.

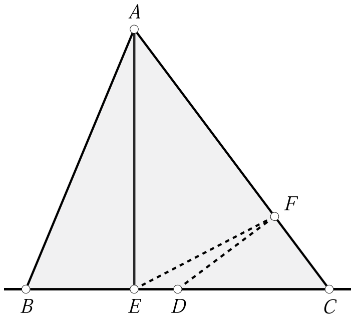


Figure 1.

(ii) Following the same steps we get $E = (0, 0)$, $B = (5, 0)$, $C = (9, 0)$, $D = (7, 0)$. The equations of AC and DF are now $x/9 + y/42 = 1$ and $y = (3/14)(x - 7)$. Solving these we get $F = (1827/205, 84/205)$, so $EF = 21\sqrt{7585}/205$.

Solution to problem II-3-S.3
In how many ways can the numbers $-8, -7, -6, \ldots, 6, 7, 8$ *be arranged in a line so that the absolute values of the numbers do not decrease from left to right?*

If the stated property is to be satisfied, then for each $a \in \{1, 2, \ldots, 8\}$, the numbers $a, -a$ must occur together. Further, as the absolute values of the entries must not decrease, the numbers $1, 2, \ldots, 8$ must appear in this order, reading from left to right. If these two conditions are met, then the stated property will hold good. Having fixed such an arrangement, we observe that for each a, the number $-a$ must occur either immediately to the left or immediately to the right of a. Thus there are just two ways to insert $-a$ for any a. Hence the total number of legitimate arrangements is $2^8 = 256$.

Solution to problem II-3-S.4
Two ships sail with constant speed and direction. It is known that at 9:00 am the distance between them was 20 *miles; at 9:35 am,* 15 *miles; and at 9:55 am,* 13 *miles. What was the least distance between the ships, and at what time was it achieved? [IMO Short list, 1968]*

As the ships are sailing with constant speed and direction, the second ship is sailing at a constant speed and direction with reference to the first ship. Let A be the constant position of the first ship in this frame, and let ℓ be the path of the second ship in relation to the first one. Let points B_1, B_2, B_3, and B on ℓ be positions of the second ship with respect to the first ship at 9:00, 9:35, 9:55, and the moment the two ships are closest to each other. Then we have the following equations: $AB_1 = 20, AB_2 = 15, AB_3 = 13$; $B_1B_2 : B_2B_3 = 7 : 4, AB_i^2 = AB^2 + BB_i^2$.

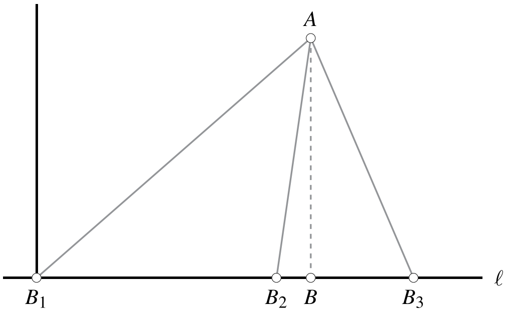


Figure 2.

We now adopt coordinates (see Figure 2). Let ℓ be taken to be the x-axis, with $B_1 = (0, 0)$, $B_2 = (7c, 0)$, $B_3 = (11c, 0)$ where $c > 0$; it is assumed that the second ship is moving along the x-axis in the positive direction; let $A = (a, b)$. Then we have:

$$a^2 + b^2 = 20^2,$$
$$(a - 7c)^2 + b^2 = 15^2,$$
$$(a - 11c)^2 + b^2 = 13^2.$$

These yield, by subtraction:

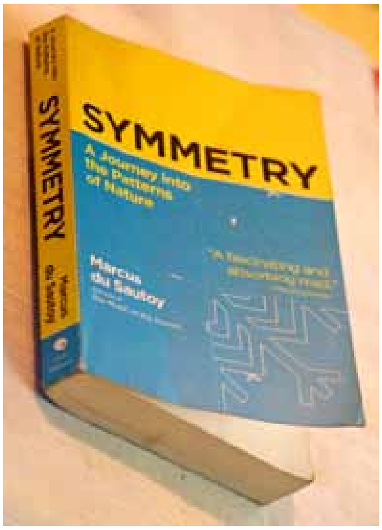
$$14ac - 49c^2 = 20^2 - 15^2 = 175,$$
$$22ac - 121c^2 = 20^2 - 13^2 = 231.$$

Treating these as a pair of simultaneous equations in ac and c^2 we get $ac = 16$, $c^2 = 1$, hence $c = 1$, $a = 16$. This in turn yields $b = \pm 12$. Hence the closest distance of A from ℓ is 12 miles, and the time at which this takes place is 16×5 minutes after 9:00, i.e., at 10:20 am.

Of Monsters and Moonshine

A review of 'Symmetry' by Marcus Du Sautoy

Geetha Venkataraman



Symmetry is a topic that resonates with audiences of varied backgrounds and levels of mathematical knowledge. One can ask a layperson to explain what symmetry means to them and there would invariably be a fairly accurate response from an informal or non-mathematical point of view. However, symmetry has deep roots in mathematics and in some sense pervades most areas of study in mathematics. The mathematical study of symmetry though has its primary residence in an area of abstract algebra called 'group theory'. What is remarkable about Marcus Du Sautoy's book on symmetry is that the mathematics underlying the study of symmetry is explained without recourse to technical mathematical language. While the word 'group' makes its debut on page 9 of the book, it is only much later that its mathematical context is explained. By then the reader has had sufficient foundation laid to absorb the mathematical context.

Keywords: *symmetry, patterns, Marcus du Sautoy, reflection, rotation, tiling, group, permutation*

As a mathematician there was a special joy in realising that it was possible to talk about areas of research in a language that would find consonance with the interested reader. The book begins with notions of symmetry that are commonplace or intuitive notions. Through the course of the book the reader is taken on a journey that explores the connections of symmetry with nature, evolution, psychology, music and even mathematics at the research level. There is also a conscious attempt to illustrate mathematical ideas with concrete examples from everyday life.

The book starts with the author on the shore of the Red Sea contemplating the fact that he has turned 40. The number 40 is important in a mathematician's life. The Nobel Prize equivalent in Mathematics is the Field's medal. In a way it is tougher to get a Field's medal than a Nobel Prize as at most four are awarded every four years and only to mathematicians who have done outstanding work and have not yet attained the age of 40.

The chapters in the book traverse the months of a calendar year beginning with the first chapter titled *August: Endings and Beginnings* and finishing with *July: Reflections.* Thus the book intertwines a year of Marcus Du Sautoy's life, his forays into searching for symmetrical objects that are part of his research, his encounters as a mathematician with 'symmetry seekers'; a term used for the mathematicians trying to classify and quantify 'indivisible collections of symmetry', with the story being told of the main protagonist, namely, symmetry.

Since a year of Marcus's life is interlinked with the story of symmetry we learn about how the author got interested in mathematics at the age of 12 because of a schoolteacher who encouraged him. This might indeed be the case for many a mathematician. A book the author was encouraged to read, as a schoolboy was *The Language of Mathematics* by Frank Land. If one thinks about it with some care, mathematics as a language is particularly efficient in expressing precisely and concisely the statements that one wishes to make. The problem though is that it is not always an easy language to master.

There are several intersecting strands that are covered in the book. One strand represents the usual story that one expects while learning about symmetry: reflections and rotations of regular geometric figures like the square, equilateral triangle to those of the five platonic solids, symmetries of infinite figures like wall-paper patterns and tilings. The chapter *October: The Palace of Symmetry,* discusses the search by the author and his son Tomer for the 17 wallpaper patterns or tessellations that exist in the Alhambra Palace in Granada, Spain, built around 1300 by Spain's Muslim rulers. Another strand brings to fore the life histories and works of the mathematicians of the Renaissance period leading to those from the early 19th century who were responsible for creating the mathematical language of group theory to analyse symmetries. These stories are the fodder for Chapters 5-8 from *December: connections* to *March: indivisible shapes.*

April: Sounding Symmetry, as the title hints at, discusses the links between western classical music and symmetry. It also points out the opposing philosophy, between when musicians use symmetrical object as a basis for creating their music but keep them secret from the audience, and the task undertaken by a mathematician of laying bare all the facts logically about the objects of study.

The strand where the book goes beyond the expected is when it moves into the difficult territory of describing one of the mammoth tasks that occupied the symmetry seekers for a large part of the previous century, namely, 'the classification of simple finite groups'. This history is explained entirely in terms of 'indivisible symmetry groups' with many anecdotes and mathematical experiences thrown in. While this thread is woven into several chapters, the last three are primarily devoted to this twentieth century tale.

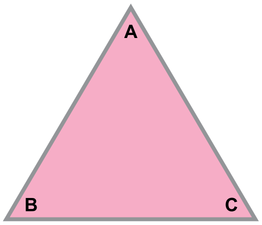
There are strands that are entirely missing, though. The book is deficient in telling the story of symmetry of non-western cultures. The Asian experience with symmetry finds no place. Indeed there is hardly any mention of the role played by symmetry in ancient or even medieval India. It is a lacuna that is not even acknowledged in passing by the author. Euro-centrism is the lens used.

Let us think of symmetry as 'a magic trick move' that keeps an object looking exactly like it did and in the exact same position as it occupied to start with. For example, as explained in the book, if we take a 50 pence coin (which is shaped like a regular heptagon or seven-sided figure) and draw its outline on a piece of paper, then a symmetry of the coin is any move or action that can be performed on the coin which brings the coin back into the outline drawn. In other words if someone had closed their eyes while the symmetry was being performed on the coin then they would assume that nothing had taken place. If we forget the markings on the coin and use this definition of symmetry then it is not too difficult to see that a regular heptagon has 14 symmetries in all.

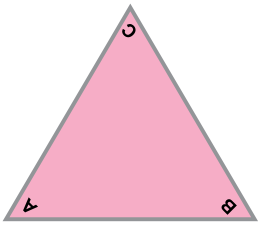
An easier example to work with is the equilateral triangle. It has six symmetries. Three are mirror symmetries or reflections, each about a line joining a vertex to the mid-point of the opposite side. These three lines of reflection meet at a point that is like the 'centre' of the equilateral triangle. The other three symmetries of the equilateral triangle are rotations through 0°, 120° and 240° respectively, say in the counter-clockwise direction, about an axis passing through the centre and perpendicular to the plane of the triangle. For example, if the three vertices of an equilateral triangle are marked A, B, C in the counter-clockwise direction then the 120° counter-clockwise rotation will take A to B, B to C and C back to A. The figures show the effect on the vertices after a reflection or rotation symmetry has been performed.

A six-pointed star with no reflection symmetries (see figure) also has exactly six symmetries; these are rotations through 0°, 60°, 120°, 180°, 240° and 300°. Here too, we can keep track of the symmetries by assuming a marking of the vertices and noting the effect of the respective symmetries on the vertices.

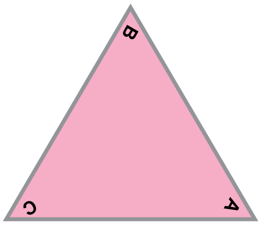
But is the collection of six symmetries of an equilateral triangle the same as the collection of six symmetries of a six-pointed star with no mirror symmetries? The answer is No: the two collections of symmetries are not the same when we consider them as 'groups'. Here is one way to see this: for the six-pointed star, if we take any two symmetries, it does not matter in which order we apply them; the final result is the same. But in the case of the equilateral triangle, if we apply a reflection followed by (say) a 120° rotation, we get a different symmetry than when we first apply a 120° rotation followed by the reflection. If we let the symbol M denote the reflection and R the rotation through 120°, then in the language of mathematics we write: $M * R \neq R * M$. (The symbol $M * R$ denotes that R is applied first and then M.)



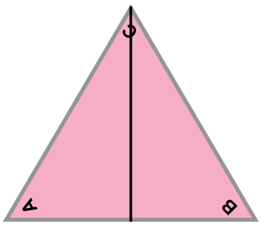
0 degree



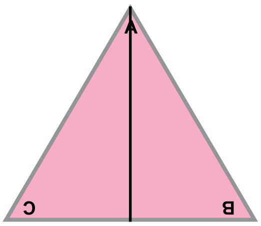
120 degree



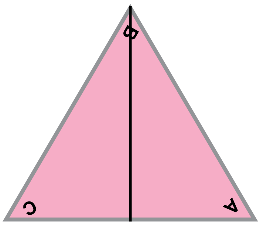
240 degree



R_C

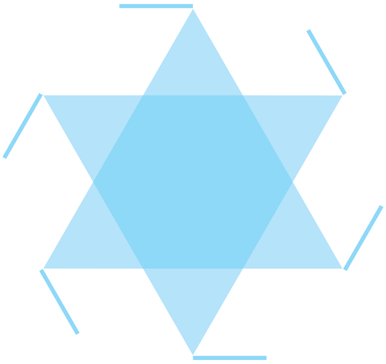


R_A



R_B

The point to note is that for any given object, one symmetry followed by another one leads to yet another symmetry of the figure. For example, the six symmetries of the equilateral triangle form such a closed system. In other words, if R and S are symmetries of the object, then so is $R * S$. It can be checked that if R, S and T are symmetries of the same object, then $R * (S * T) = (R * S) * T$.



This is called the 'associative property of $*$'. Every figure has the 'do-nothing' or 0° rotation. If we denote it by E, then we have $R * E = R = E * R$ for all symmetries R of the given object. The symmetry E is called the 'identity'. Also, for every symmetry R of the figure there is a reverse move which we denote by R'; this has the property that $R * R' = E = R' * R$. The symmetry R' is called the 'inverse' of R. For example, the inverse of the 120° rotation of the equilateral triangle is the 240° rotation.

Such a closed system as described above is a *group*. The collection of all symmetries of any object is always a group, usually referred to as its *group of symmetries*. The young French mathematician Évariste Galois discovered this in 1829 at the age of 17. He came upon the idea while investigating something that, on the surface, seemed far removed from symmetry of objects; namely, his investigations into solutions of polynomial equations of the fifth degree or higher. While Galois did not quite see the shapes hiding in the solutions of these equations, he was able to associate with each equation a 'group of permutations' of its solutions. He showed by analysing these groups that a certain property of the group would decide whether one could express the solutions of the equation in terms of the constants occurring in the equation, rather like we can for a quadratic equation (and indeed for a cubic and a quartic).

The discerning reader may recognise that integers under addition also form a group. The prime numbers play a very important role amongst natural numbers. We learn at school that any counting number greater than one can be written as a product of primes. The primes are the counting numbers with exactly two factors, 1 and the number itself. Thus, primes are like the building blocks for the natural numbers. In much the same way, with the development of group theory in the 19th century, mathematicians realised that every symmetrical object could be built from smaller objects whose collection of symmetries were indivisible. For example, the 15 rotations of a regular 15-sided figure can be built from the 5 rotations of a pentagon and 3 rotations of an equilateral triangle. The two groups of 5 rotations of a pentagon and 3 rotations of an equilateral triangle each constitute an indivisible collection (group) of symmetries, in the sense that they cannot be built up from objects with a strictly smaller collection of symmetries. Such groups are also called 'simple groups'.

In the second-half of the twentieth century a prodigious task was undertaken to create a 'periodic table' of simple groups. There were a large number of group theorists working in all corners of the globe in a concentrated effort to find and classify all finite simple groups. The commander in chief of this project was the American mathematician, Daniel Gorenstein. The thrill of the chase, the discoveries of very large finite simple groups, christened 'baby monster' and 'monster', the story of how mathematicians involved in this esoteric quest managed to more-or-less complete the Herculean task is a story of moonshine and monsters and is narrated very well, with personal anecdotes involving the author and colourful descriptions of the mathematicians involved. As an example the following is the author's description of the Cambridge mathematician Simon Norton who was a contributor to the quest: "I could see what looked like a tramp, with wild black hair sprouting out all over his head, trousers frayed at the turn-ups, wearing a shirt full of holes. He was surrounded by plastic bags which seemed to contain his worldly possessions. He looked like a scarecrow."

The research questions, which the author is engaged in answering, are also discussed. Some of these are: How many objects are there with a prime power number of symmetries? How does one build symmetrical objects living in 'higher' non-visible dimensions from say the group of rotational symmetries of an equilateral triangle?

Marcus du Sautoy, as the blurb at the back of the book tells us, is the 'Charles Simoyni Professor for the Public Understanding of Science' at the University of Oxford and with this book he more than satisfies the job description. At the heart of his writing is the genuine wish to communicate the joy of doing and thinking mathematics.
The style and substance of his writing conveys this. The book is laid out with a structure that interweaves a year in the life of the author with his quest to demystify symmetry and at the same time make the case for symmetry as an all pervasive ideal that the human brain seems to be attracted to and finds hard to disengage from. The book comes highly recommended by this reviewer.

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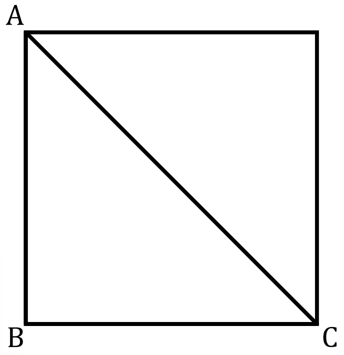
Letter to the Editor

In "An 'Origamics' Activity: X-lines" (AtRiA, November 2013) , the following result had been stated: A point is taken on the top edge of a square sheet of paper, and the bottom corners are folded to the point, generating two straight lines. The points of intersection of the X-creases fall on the vertical midline. The reader was asked to prove this observation. Here is a proof.

Let A be the point on the top edge, and let B, C be the two bottom corners. The fold produced by bringing B to A is the perpendicular bisector of segment AB, and the fold produced by bringing C to A is the perpendicular bisector of AC. So AB and AC are the perpendicular bisectors of sides AB and AC of triangle ΔABC, while the perpendicular bisector of side BC is the vertical midline! These three lines concur, so the point of intersection of the X-lines will lie on the vertical midline. So the point of intersection is nothing but O, the circumcentre of ΔABC.

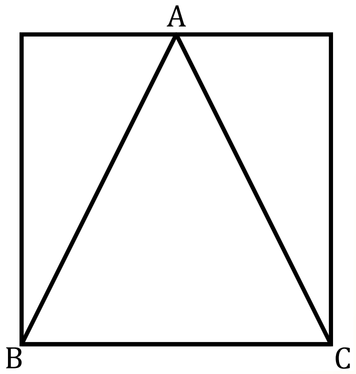
This proves the third observation: "The distances from the point of intersection to the starting point and to each of the lower vertices are equal" (each distance is the circumradius of ΔABC).

Now take the second observation. The points of intersection of the X-creases lie along the midline and **lie below the centre of the square within a certain range** (emphasis added). Let us look at O as the point of intersection of the perpendicular bisectors of AB and BC (the vertical midline).The perpendicular bisector of AB intersects AB at its midpoint, which lies on the horizontal midline, i.e., the same level as the centre of the square. The slope of AB is positive, implying that the slope of its perpendicular bisector is negative. O lies on this line and to the right of the midpoint of AB. So O lies below the midpoint of AB, i.e., below the horizontal midline, hence below the centre of the square.



O does touch the centre of the square when A is at either of the top corners. When A is at the top left corner, we get a right triangle ABC with $\angle B = 90^\circ$ and O is the midpoint of the hypotenuse AC. AC is a diagonal of the square, so its midpoint O is the centre of the square.

O reaches the other extreme, i.e., the lowest point, when A is at the middle of the top edge, i.e., the midpoint. Then AB has the lowest possible slope, hence its perpendicular bisector dips most steeply and cuts the vertical midline at the lowest possible point. Then O is at a distance 3/8 BC from BC. [This can be calculated considering $\angle BOC = 2\angle BAC$ and $r + r\cos\angle BAC$ = height of ΔABC = BC (since the paper is square), where r is the circum radius of ΔABC.]



As A moves from the top left corner to the midpoint of the top edge, O moves from the centre of the square to the above height. So far $AB < AC$, and we end with $AB = AC$. Then O reverses its trip back to the centre of the square as A continues from the midpoint of the top edge to the top right corner. Here $AB > AC$; they have switched roles, and everything now repeats in the reverse order.

Note: We don't need to restrict ourselves to a square sheet for this activity. It works equally well for a rectangular sheet. The lower extreme of the range would need to be recalculated given the ratio of height of the paper (or ΔABC) to BC, and in case BC becomes longer than twice the height of ΔABC, something interesting but inconvenient happens. It is left to the curious reader to explore this!

Contributed by

Swati Sircar

The Closing Bracket . . .

What is the single most important thing that a high school mathematics teacher can hope to convey to his or her students? It is an interesting exercise to dwell on this question. I'm sure that different answers will be forthcoming, but I would like to focus on the matter of authority.

I came across this quote from Prof Michael de Villiers' site, http://frink.machighway.com/~dynamicm/newsletter.html: ". . . that's really what the beauty of mathematics is, because you can't simply call on authority *and* say 'so-and-so said it, therefore it must be'. You need to argue logically and coherently, using all the key things in mathematics to make your case. There is no authority except a valid mathematical proof . . . ." The quote is from mathematician Sizwe Mabizela of Rhodes University. Here's a similar quote from the legendary Richard Feynman: "Doubting . . .was a reaction I learned from my father: Have no respect whatsoever for authority; forget who said it and instead look what he starts with, where he ends up, and ask yourself, 'Is it reasonable?"

I wonder where mathematics teachers stand in relation to what has been said in these extracts. Do we attempt to convey such an attitude to our students? And do we also tell them about the need for this maxim to spill across into life and not be confined just to mathematics or science? If we do not, it would be a disservice on our part. But the reality in this matter is that very few among us have found the clarity to carry the principles of mathematics beyond the boundaries of the subject.

The following quote from Albert Einstein asks a similar question in a more hardhitting and forceful way: "Why does this magnificent applied science which saves work and makes life easier bring us so little happiness? The simple answer runs: Because we have not yet learned to make sensible use of it." Einstein was not talking about mathematics, but his words apply even if quoted out of context. Something for us mathematics teachers to ponder over .

— Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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TEACHING
MULTIPLICATION

PADMAPRIYA SHIRALI

A VISUAL
APPROACH

At
Right
Angles
A Resource for School Mathematics

TEACHING MULTIPLICATION

Here are some questions which arise while teaching Multiplication: Should children memorise the multiplication tables? What is an easy and convenient way of modeling multiplication? Is it enough if one only teaches the procedure of multiplication? Perhaps answers to these questions can be found if we reflect on the importance we give to construction of knowledge. If we see that children must understand how facts are derived, how procedures are derived and how concepts can be visualized, then our approach will be dictated by that understanding.

Keywords: *Multiplication, manipulatives, pattern, cycle, symmetry, commutative, Cartesian product*

We start with two 'warm up activities' before introducing multiplication (Activity 1 and Activity 2).

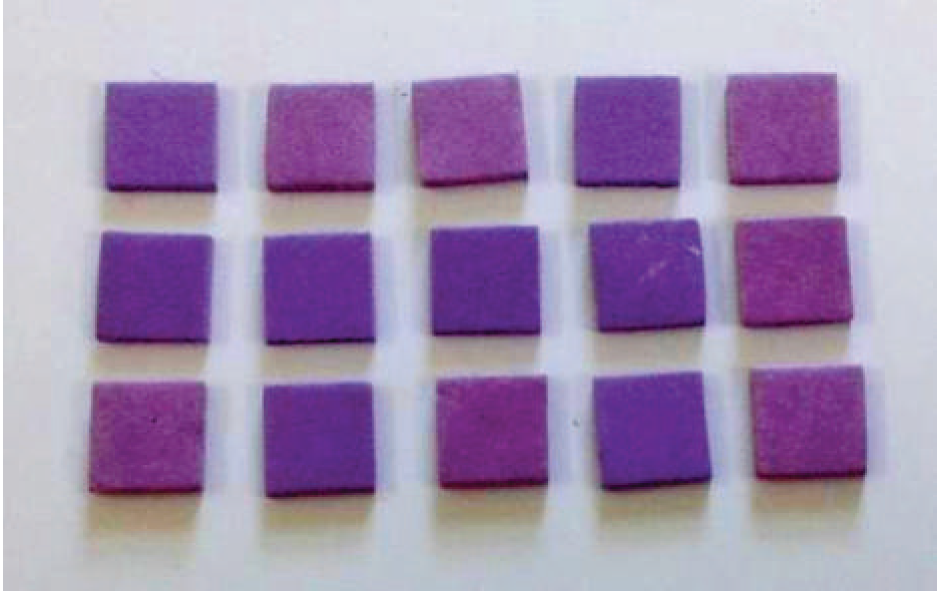
ACTIVITY **ONE**

Making equal groups and internalising multiplication contexts

Materials required: Square pieces, straws and rubber bands, coloured buttons. Peg board and pegs or a graph board and seeds

One group of children can work with straws and make bundles of straws with the same number of straws in each bundle. Another group can arrange square pieces in rows with the same number of pieces in each row. Yet another group can line up seeds on a graph board or square ruled sheets. (Seeds can also be placed in paper plates or bowls.) By rotation, all groups should work with different materials. Different children learn in different ways. We need to expose them to multiple ways of looking at things. Also, working with different materials and different arrangements will help children become familiar with different contexts in which multiplication arises. Further, it is important that children of this age group are exposed to tactile learning. This will aid in visualizing problems and strengthen their conceptual understanding. Doing these activities will also help children who learn through a kinesthetic approach.

The purpose of this activity is to focus on re-arranging objects into equal groups and distinguishing between the two numbers (the number of groups made, and the number in each group) arising from the situation. It is not necessary at this point to talk of the total number. Questions will centre around the following: 'How many groups?', 'How many in each group?'



ACTIVITY ***TWO***

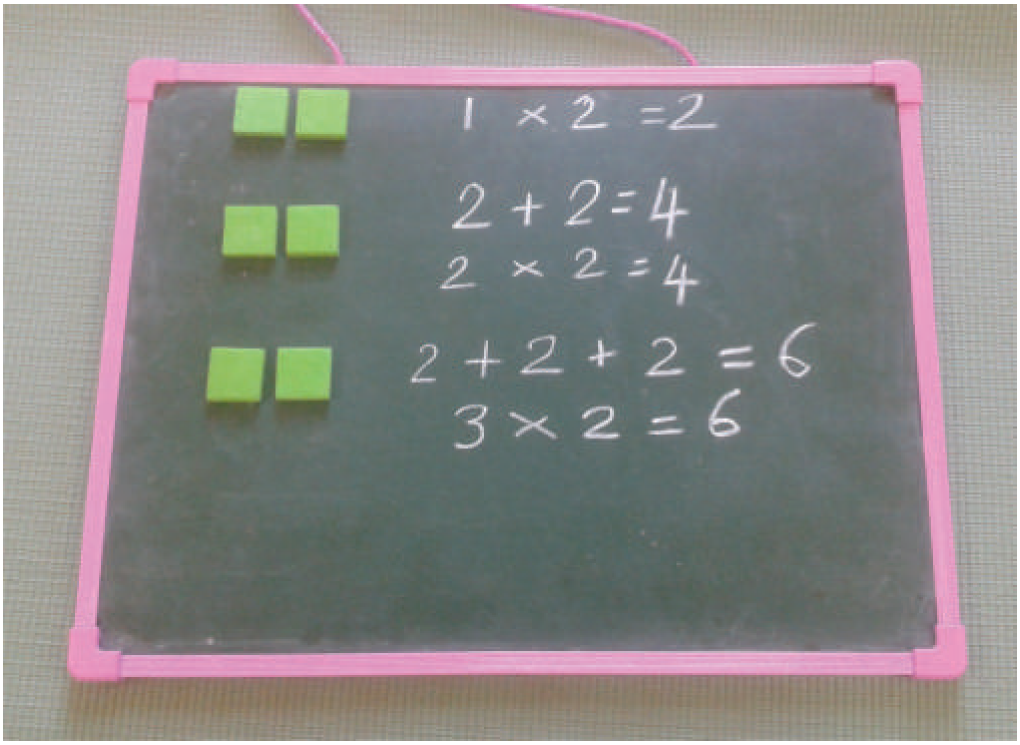
Skip counting in steps of 2, 5, 4 and 10

Materials required: String of beads, number line and number chart

With or without the aid of a number line, children can do skip counting using skips of 2, 5, 4 and 10. They could also try skip counting with other numbers if they are at ease with them. They can do forward counting as well as backward counting.

Here the questions will centre around the following: 'In what steps are we counting?' Say 2. 'How many steps of 2 did we count to reach 10?' Answer: 5 repetitions of 2 have brought us to 10.

It is fun to do this as a hopping activity on a number line drawn on the ground. Children can explore whether they can reach 12 by hopping in steps of 2 or 3, or steps of any other number.



ACTIVITY ***THREE***

Introduction to multiplication table 2 through repeated addition

Materials required: Seeds or square pieces

While introducing any multiplication table it is important to construct the table gradually in front of the children, articulating each step clearly.

Arrange 2 squares in a row and say: "This is 1 group of 2 squares." (One two is two, this is written as $1 \times 2 = 2$) Now place 2 squares under them, saying: "This is 2 groups of 2 squares" (two twos are four, this is written as $2 \times 2 = 4$). Now build the third row of 2 more squares (three twos are six, $3 \times 2 = 6$) and so on till ten twos are twenty, $10 \times 2 = 20$.

I prefer to teach multiplication tables as $1 \times 2 = 2$, $2 \times 2 = 4$, $3 \times 2 = 6$, $4 \times 2 = 8$, etc. (changing the first number and keeping the second number constant).It is the group number which increases each time while the group size remains constant. This corresponds to the way we speak about a multiplicative situation: 3 rows of 10 chairs, 4 classes of 20 students, five 2 kg packets of salt, etc.

However, if one prefers to teach the tables as $2 \times 1 = 2$, $2 \times 2 = 4$, $2 \times 3 = 6$, etc., then while arranging the squares in successive rows, one will have to say: 'two occurring once is 2, $2 \times 1 = 2$', 'two repeated twice is four, $2 \times 2 = 4$', 'two repeated thrice is six, $2 \times 3 = 6$', and so on.

Whichever approach one takes, one needs to proceed gradually, stating the number that is repeated and the number of times it is repeated.

Also, let children record their activities as drawings (as shown in the picture for Activity Two). It is important that they record the result both as a repeated addition and as a multiplication fact, in both forms till they internalize the relationship between repeated addition and multiplication.

Usage of 'into': For some reason, while reciting multiplication tables, the usage of the word 'into' has crept into our language ("2 into 4 equals 8"), but this is not appropriate. In fact, when one asks,"How many times does 2 go into 4?" it actually means *division* (4 divided by 2), and the answer is 2. We need to change this practice and read multiplication facts as "3 times 2 equals 6", "4 times 8 equals 32", etc.

Multiplication tables for 5, 4 and 3 (I prefer to teach the tables for 2 and 5 first) can be introduced in a similar manner. It is good to pause at this point and spend time consolidating these facts before we go on to further multiplication tables.

ACTIVITY *FOUR*

Patterns in multiplication tables of 2, 3, 4 and 5 as an aid in committing the tables to memory

Materials required: Multiplication Tables chart with bold numbers, number chart (1 to 100)

Discuss with the children the patterns seen in the multiplication table of 5. They can first look at the numeral in the units place and observe that 5 and 0 repeat in a cycle of 2. They will also notice that in the tens place, each number appears twice.

Next they can work on the pattern in the multiplication table of 2. They will see that numerals 2, 4, 6, 8, 0 repeat with a cycle of 5. But in the tens place, the pattern does not establish itself unless they build the table further. This is a good point to show an extended multiplication table (which we normally do not attempt).

Now they can work on the pattern in the multiplication table of 4. They will see that the numerals 4, 8, 2, 6, 0 repeat in the units place with a cycle of 5. What about the pattern in the tens place? Do we need to extend the table to notice a pattern? Is there any relationship between the sequence of digits in the units place of 4 table and the sequence of digits in the units place of 2 table?

Finally they look for patterns in the multiplication table of 3. The patterns can be found more easily if we group the digits of the units place in groups of three and place them in rows under one another:

3 6 9

2 5 8

1 4 7

Children will see that the digits of the first, second and third columns decrease by 1 each time.

Should Multiplication Tables Be Memorised?

First: Children should have plenty of exposure to the concept of multiplication and internalize it.

Second: Children should be able to build or construct any multiplication table with understanding.

Third: Usage of aural memory or visual memory in learning and memorising the multiplication tables of 2 to 10 is very useful in mental arithmetic and saves a lot of time.

Multiplication tables have also been set to tunes and are available in the market and on the internet as songs. It will help children who are musically inclined.

Many teachers either skip or rush through the first two steps in a cursory way and get children to memorise tables. This will not lead to an understanding of the concept and makes the child helpless whenever his memory fails. The capacity to build a table is enabling and empowering to the child.

Also, there is no need for panic if some children take more time to memorise. We want children to think in mathematical ways and not merely learn by rote. It is therefore advisable that we give a lot of attention to the proper understanding of this concept.

ACTIVITY *FIVE*

Constructing tables from 6 to 10

Materials required: Broom sticks or cardboard strips or plastic tongue cleaners

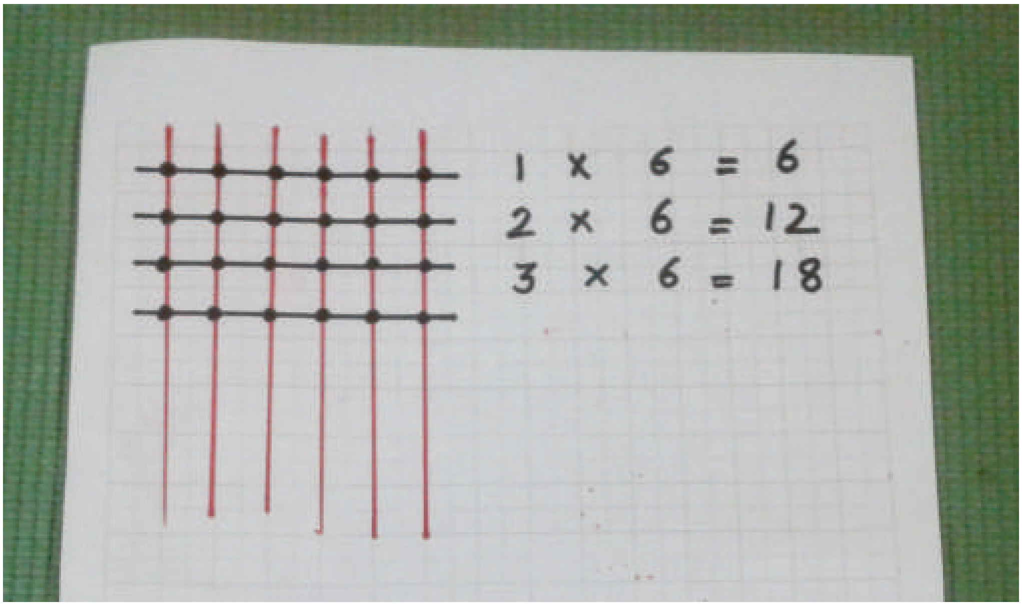
Multiplication facts for 6, 7, 8 and 9 can be taught using any of the following methods.

- Repeated addition, using seeds or buttons
- Arranging square pieces in array form (rows and columns)
- Counting the joints of intersecting lines

The third approach has the advantage of being less cumbersome than the first two methods when one is constructing a multiplication table for a larger number. It is also easier for children to make rough sketches of it in their notebooks.

Arrange 6 strips parallel to one another vertically. Lay one strip horizontally across them and point out the joints where they intersect and say 1×6=6. Lay one more strip horizontally across the vertical lines, point out the joints where they intersect and say 2 × 6 = 12. Lay one more strip horizontally across the vertical lines, point out the joints where they intersect and say 3 × 6 = 18, and so on.

Once children have understood the process by which they have created the multiplication table for 6, they will be able to do the same for 7, 8 and 9 on their own and work out the multiplication facts.



ACTIVITY SIX

Noting the patterns in multiplication tables of 6, 7, 8, 9 and 10 as an aid in committing the tables to memory

Materials required: Multiplication Tables chart with bold numbers,
Number chart (1 to 100)

The pattern for table 10 is obvious.

Discuss with children the patterns they notice in the multiplication table of 9. It has many patterns and there is a lot that children will be able to discover on their own if the teacher poses some leading questions.

Finger pattern showing 9 × 4 = 36

Finger pattern showing 9 × 5 = 45

They can first look at the numeral in the units place and see that it goes down from 9 to 0. At the same time, the tens place increases from 1 to 9. The digits of the number always add up to 9. The units digits have a cycle of 10 before they repeat. The table can be demonstrated using the fingers of both hands in a simple fashion by progressively raising the first finger, followed by the second, etc, and reading tens from the left side of the raised finger and units from the right side of the raised finger, as shown in the figure.

They can now look for patterns in the multiplication table of 8. They will see that 8, 6, 4, 2, 0 repeat in the units place with a cycle of 5. But in the tens place, the pattern does not establish itself unless they extend the table further. Is there a relationship between the sequence of the digits in the units place of the 4 table and the sequence of digits in the units place of the 8 table?

Now they can work on the pattern in the multiplication table of 6. They will see that numerals 6, 2, 8, 4, 0 repeat in the units place with a cycle of 5. What about the pattern in the tens place? Will we need to extend the table to notice a pattern?

Finally they can look for patterns in the multiplication table of 7. The patterns can be easily found if we group the digits in the units place in threes and place them in rows, one below the other (like we did in the case of multiplication by 3):

7 4 1

8 5 2

9 6 3

The digits of the first column, second column and third column are seen to increase by 1 at each stage.

ACTIVITY SEVEN

Creating visual patterns using multiples of 2, 3, 4, 5, 6, 7, 8, 9

Materials required: Square grid paper, 8 sheets per child

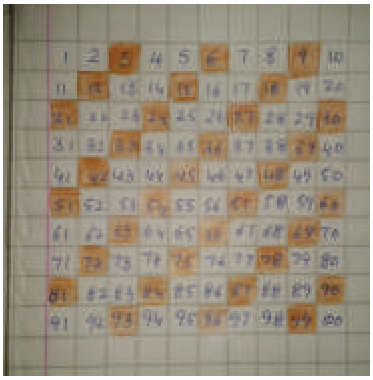
Ask the children to write the numbers from 1 to 100 with a pencil in a 10 by 10 square grid.

Let them colour the multiples of 2 in their grids and note the pattern that emerges.

Let them write 1 to 100 again on another 10 by 10 square grid and this time colour all the multiples of 3. This creates a diagonal pattern.

They can repeat this exercise for other numbers 4 to 9 on different square grids.

Discuss the patterns that emerge.



ACTIVITY EIGHT

Discovering commutativity, associativity and distributive property of multiplication

Square grid paper, cardboard strips

COMMUTATIVITY

While we want children to discover these three properties of multiplication, we can avoid mentioning the names to young children and demonstrate only the property.

Let children make 5 groups of 3 seeds. Ask them what number this gives. Record the answer 5 x 3 = 15.

Let them now show 3 groups of 5 seeds. Ask them what number this gives. Record the answer 3 x 5 =15.

Ask them: "Is 3 groups of 5 each the same as 5 groups of 3 each?" What is common? It is the answer which is common.

Let children colour a row of 6 squares. Let them make 3 such rows. They can now record what they have coloured.

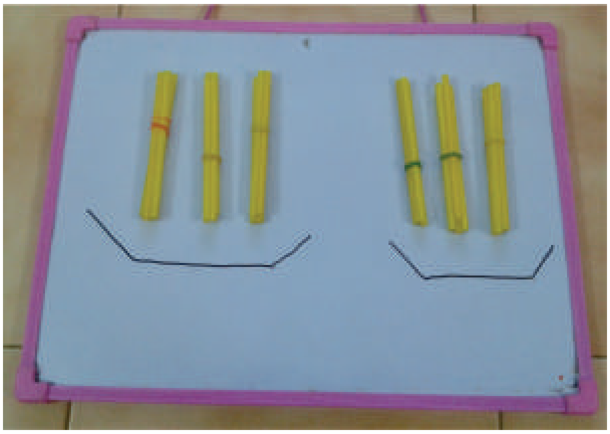
3 rows of 6 squares equals 18, i.e., $3 \times 6 = 18$.

Now ask them to turn the drawing through a right angle to make it vertical. Ask them to describe the number of rows that they see now.

They see 6 rows of 3 squares. So $6 \times 3 = 18$.

Now point out that 3×6 gives the same result as 6×3.

ASSOCIATIVITY



Ask children to bundle 4 straws together using a rubber band. Let them make 6 such bundles. Place them equally in two plates (i.e., 3 bundles in each plate). Now let us count the total number of straws.

There are 2 plates, 3 bundles in each plate, and 4 straws in each bundle.

The total number of straws can be calculated as the number of bundles times number of straws in each bundle, i.e., (2 × 3) × 4, or as the number of plates times the number of straws in one plate, i.e., 2 × (3 × 4). So: (2 × 3) × 4 = 2 × (3 × 4).

DISTRIBUTIVITY

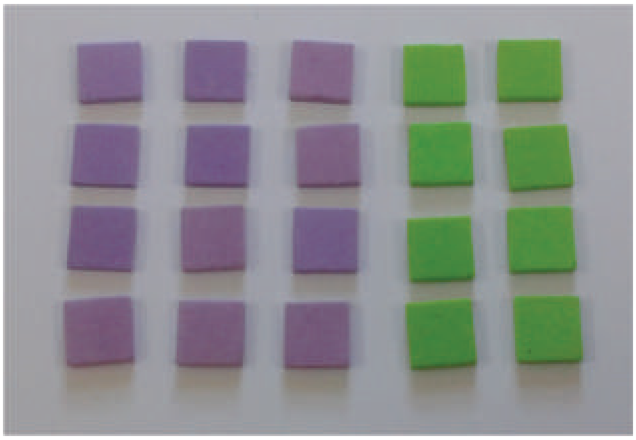
Let children colour the squares as shown. Let them state the multiplication fact for the purple squares (4 rows of 3 squares each, 4 × 3 = 12) and green squares (4 rows of 2 squares each, 4 × 2 = 8), separately.

Next, let them state the multiplication fact for the whole region: 4 rows of 5 squares each, 4 × 5 = 20.

Hence:

(4 × 3) + (4 × 2) = 4 × (3 + 2) = 4 × 5 = 20.

Several examples of each type need to be shown using various contexts and numbers for the three laws to be understood.



ACTIVITY NINE

Modelling word problems and writing stories for given multiplication facts.

Materials required: Square grid paper, plain paper, dot paper, seeds

Use all the different multiplication contexts

Equal groups: 4 bowls, 5 apples in each bowl. How many apples?

Rate: Every child needs 2 pencils. How many pencils for 24 children?

Arrays: 4 plants in each row; 3 rows. How many plants?

Scale factor: A boy has 4 books; his brother has 3 times as many. How many books does the brother have?

Cartesian product: A boy has 3 T-shirts (red, yellow, white) and 2 shorts (black, blue). What are the different ways in which he can pair them?

Many children have difficulties in interpreting word problems. This difficulty continues to persist in higher classes when they encounter word problems in linear equations or applications of percentages. Teaching modelling techniques for word problems is neglected both by text books and many teachers. Problems should be introduced in a contextual situation while we teach, and we also need to expose children to different modelling techniques.

1. A gardener plants 9 plants in each bed. There are 4 beds in the garden. How many plants?

   Let children in the initial stages model it using seeds, or use a dot paper to depict the seeds and beds.

2. A kitchen wall is covered with tiles. If there are 8 tiles in each row and the mason needs to make 7 such rows, how many tiles does he need?

   Children can work with square pieces or use square grid paper to model this.

3. Trees are planted at 5 metres distance from the start to the end of a street. If 8 trees are planted on the street, how long is the street?

   Children can use a number line to depict the situation and work out the answer.

For a scale factor they can depict it as a graph.

For a Cartesian product problem they can make a tree diagram or a network.

Writing Stories for multiplication facts

Ask children to write a story for a multiplication fact like 6 × 5 = 30. It will reveal their understanding or bring out misconceptions. I have always found this exercise very revealing; it gave me a chance to remedy my teaching. The contexts they choose will give us feedback on the kind of examples that we have used.

ACTIVITY TEN

Multiplication by 1 and 0

Sticks or straws

How does one explain to a child that $n \times 1 = n$ and $n \times 0 = 0$?

Providing a convincing explanation is not easy. If one gives repetition as an explanation, then one will be forced to say 2 occurring twice becomes four (2 times 2 = 4) and two occurring once is two (1 times 2 = 2). How about zero times 2?

One way is to use a flow technique in the reverse direction, by using sticks and counting the joints.

We start with 5×2, shown using sticks, and progressively remove one stick at a time, counting the joints each time: 5×2 (10 joints), 4×2 (8 joints), 3×2 (6 joints), 2×2 (4 joints), 1×2 (2 joints), 0×2 (0 joints). It is important that we provide consistent explanations which follow patterns and are logical.

CONCEPTUAL UNDERSTANDING AND PROCEDURAL UNDERSTANDING

Often teachers face a dilemma with regard to concepts and procedures. They are not sure whether they need to focus on one or the other. There are some (definitely a large number in India!) who think that procedures are more important, as they help in solving problems.Many techniques and shortcuts are taught. Some feel that in this day of calculators and computers, procedures are taken care of by gadgets, so they need to give importance only to conceptual understanding. However, procedures are a result of our historical research of methods, and if the logic behind the procedure is gone into by the teachers, along with the students, it addresses both the understanding of concepts and an appreciation of procedures. So we do not need to put procedures and concepts in an either-or situation.

Procedures are compatible with teaching of concepts. Students need to understand the conceptual system of multiplication in which procedures are fully integrated. Deriving procedures and finding generalizations can be rewarding at all levels and build the mathematical muscle of the brain.

ACTIVITY ELEVEN

Multiplication of a double digit number by a single digit number

Materials required: Place value Kit.

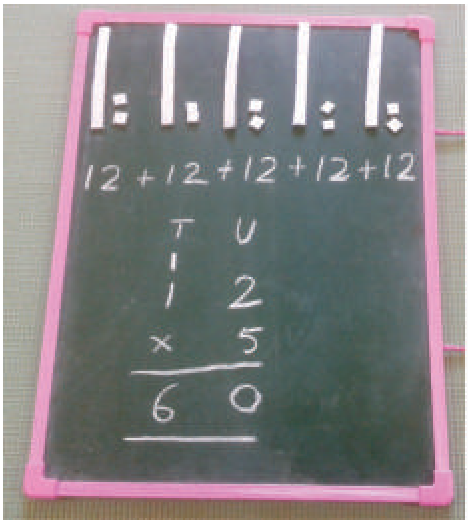
Introduction to multiplication of a double digit number by a single digit number is best done through place value material. One needs to constantly emphasize the place value aspect in all operations, as it determines the procedural knowledge needed for solving a problem. For example, in computing 32 x 8,when we multiply the digit in the units place (2) with 8 and write 1 over the number (3) in the tens place, that 1 stands for ‘1 ten’. In the next stage, when we multiply 3 by 8, we are actually multiplying 3 tens (30) by 8. The meaning of all this comes through only when we use place value materials and emphasize the place values coming into operation in each step.

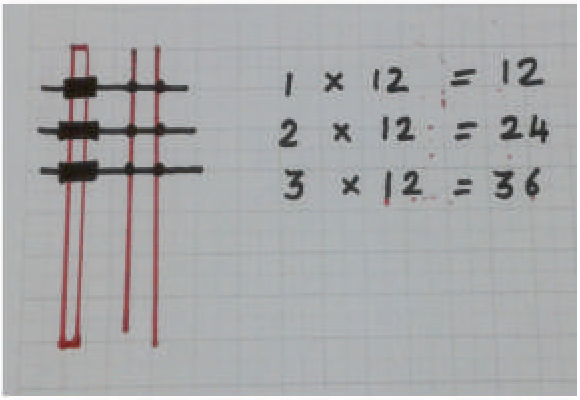
Initially, use examples which do not need exchange of units to tens:12 × 4 or 13 × 3, etc, using place value materials.

In the second stage, demonstrate 12 × 5 again with place value materials. Relate it to the procedure by showing the in-between steps for better understanding.

By focusing on the place value show that $12 \times 5 = (10 + 2) \times 5 = 10 \times 5 + 2 \times 5$.

It can also be shown by using thicker strips to indicate tens and thinner strips for units and counting the corresponding joints as tens and units, as shown in the picture.





Extension: Multiplication of a three digit number by a single digit number can be demonstrated using hundreds, tens, units material and should again be taught without any exchanges initially, followed by exchanges in tens place and later exchanges in hundreds place.

It can also be shown through expanded form: $324 \times 7 = (300 \times 7) + (20 \times 7) + (4 \times 7)$.

ACTIVITY **TWELVE**

Multiplication by 10 and multiples of 10

Multiplication by 10 comes easily to children.

Multiplication by multiples of ten (20, 30, 40) involves usage of associativity and needs to be gone into carefully. When we multiply by 20, we do it in two steps. We treat 20 as 2 × 10 and first multiply the number by 2, and then by 10. Teachers sometimes use language incorrectly here by saying,"Multiply by 2 and add zero to the answer". It is better to say "Place a zero next to the answer".

ACTIVITY **THIRTEEN**

Using multiplication facts to get new facts

Materials required: Place value Kit.

Encourage children to do mental arithmetic and find efficient ways of doing multiplication.

As children become conversant with the three laws, they will be able to use them in simplifying multiplications. For example:

To do $4 \times 8 \times 25$, they may first multiply 4×25 to get 100, and then multiply by 8 to get 800.

To do 7×35, they may first do $7 \times 30 = 210$ and then $7 \times 5 = 35$, and then add 210 and 35 to get 245.

They may use a 'halving and doubling' technique. To do 16×4, they may multiply 8 (half of 16) with 8 (double of 4).

They may use 'rounding and subtraction'. To do 28×5, they may do: $(30 - 2) \times 5 = 30 \times 5 - 2 \times 5 = 150 - 10 = 140$.

ACTIVITY FOURTEEN

Multiplication of a double digit number by a double digit number

Children face a lot of difficulty in understanding the procedure of double digit multiplication and many errors happen in this area. A chief cause of this problem is focusing on procedures mechanically and not paying sufficient attention to the logic of the procedure and the concept.

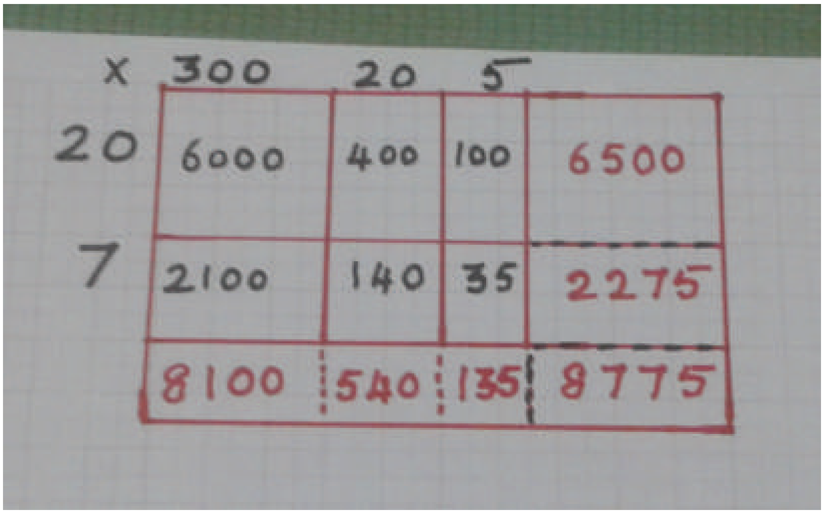
Multiplication by double digit involves the distributive law, place value and the usage of zero as a place holder. When we multiply 24×32, one needs to show it initially by writing it in full expanded form as $24 \times 30 + 24 \times 2$.

24×30 in turn can be seen as $24 \times 3 \times 10$ which is 72 tens (720), and $24 \times 2 = 48$. Repeatedly one needs to draw the child's attention to the place value of the digits with which we are multiplying.

While multiplying with the number in the tens place, one must start by placing a zero in the units place. Leaving it blank, or using some other symbol like a star or a cross, does not make sense, nor does it aid in understanding what is happening.

Multiplication of a three digit number by a two digit number can be shown through expanded form and partial products.

Ex.325×27:



x	300	20	5	
20	6000	400	100	6500
7	2100	140	35	2275
	8100	540	135	8775

Extension: Similarly, when we multiply a number by a three digit number, we must point out that multiplying by a digit in the hundreds place will result in zeroes in the units and tens places in the product.

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at **padmapriya.shirali@gmail.com**

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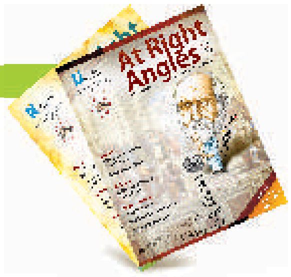

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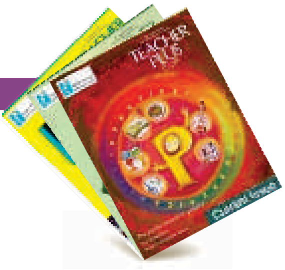

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