At Right Angles

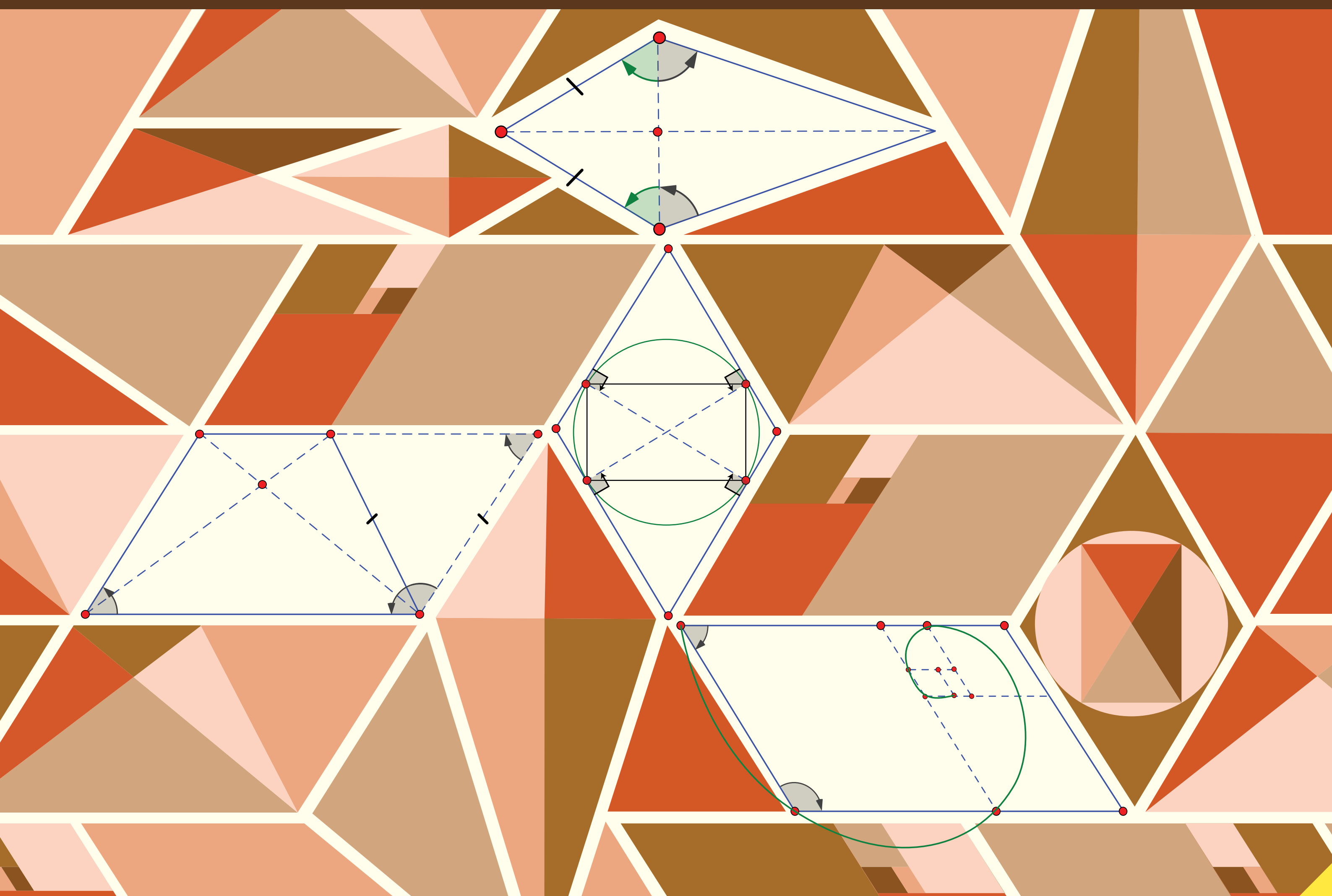
A RESOURCE FOR SCHOOL MATHEMATICS

Volume 6, No.2 August 2017

Azim Premji University

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

COMMUNITY MATH CENTRE RISHI VALLEY SCHOOL (KFI)



PULLOUT
SECRET WORLD OF LARGE NUMBERS

Michael de Villiers' two part series on Constructive Defining – Golden Quadrilaterals, has sparked off the cover images for this issue. Beauty is inherent in mathematics and the word *'Golden'* has prefixed many mathematical objects, some of which are described in the TechSpace section of this and the previous issue. But what of the **Golden Quadrilateral** which is a highway network connecting many of the major industrial, agricultural and cultural centres of India? Wikipedia says that *A quadrilateral of sorts is formed by connecting Chennai, Kolkata, Delhi and Mumbai, and hence its name*. Improving speed, connectivity and accessibility and hence impacting economic development clearly led to the appellation before the name of this network. Here is a map showing the Golden Quadrilateral, the image has been superimposed on to a GeoGebra sketch and the quadrilateral sketched out. How about investigating if this quadrilateral is golden in more ways than one?

https://en.wikipedia.org/wiki/Golden_Quadrilateral

From the Editor's Desk . . .

It is a pleasure to share with you, our readers, the collection of articles in this, the July 2017 issue. The (un)popular view of mathematics being a terrifying subject takes a completely new twist with the first article in which a mathematician takes on a terrorist threat! The hunt for answers to a mathematical problem is usually an absorbing one, at least to aficionados of the subject but Arun Vaidya's fascinating story *I M Code* makes it a matter of life and death. Following this, we have an article on another application of mathematics: *Interpolation* by Sankaran Viswanath. You will see again how mathematics is a tool for prediction, and how data can be fitted into mathematical expressions which then provide a mathematical model. From here, we move on to card tricks; yes, fun and mathematics can go together – and *At Right Angles* shows you how in Suhas Saha's *Ternary Base Magic Trick*. A quick peek behind the magic reveals patterns based on the ternary base, it's not as complex as it sounds, read on to find out. Our Features section ends with Shailesh Shirali's exposition on *Quadrilaterals with Perpendicular Diagonals*, a nice bouquet of Arithmetic, Algebra and Geometry for you.

In ClassRoom, we have the second part of the *Inequalities* series started in the March 2017 issue, again, both Algebra and Geometry are used to first prove the arithmetic mean- geometric mean inequality and then apply it in several situations to illustrate the power of this relationship and also view its implications in graphs, geometric figures, functions………the list of connections seems endless! Moshe Stupel and David Ben-Chaim appear next with their article *Three Elegant Proofs*, the name says it all, we promise it lives up to its title. CoMaC, as usual, provides an in-depth analysis of an often-asked question, now increasingly appearing even in WhatsApp forwards: *What's the next number*? Is the answer really unique as the question implies it to be? More on numbers with Swati Sircar and Sneha Titus, writing on the *Sums of Consecutive Natural Numbers*; mental mathematics becomes visual all of a sudden, and this Low Floor High Ceiling activity is sure to appeal to a variety of learning styles. Vinay Nair takes up the theme of *Divisibility by Primes* and provides some powerful tests using an 'osculator'. Students are sure to be intrigued. ClassRoom concludes with a *Proof Without Words* on a property of the Orthocentre of a triangle.

For some time now, we have been featuring articles by students and we are particularly happy when they write in with their own discoveries. So much so, that from this issue onwards, we have devoted space to Student Corner in the ClassRoom section. Featured this time are Bodhideep of class 6 and Parthiv of class 11, you are sure to be impressed with their discoveries.

Our cover this time, features Golden Quadrilaterals and the illustrations have been provided by Michael de Villiers, who continues his series on constructive defining. These beautiful quadrilaterals have been defined by investigation and are an interesting activity for students who believe that everything in mathematics is pre-defined and that *there is nothing new in mathematics to be discovered*.

Problem Corner has seen some changes over the last few issues. In a deliberate attempt to avoid a 'camp' approach to problem solving and to make this section more inclusive, we have a wide variety to interest our readers. Prithwijit De sets the ball rolling with his article on *Problem*

Posing. This is followed by Middle and Senior Problems addressed to different age groups. CoMaC presents a theorem about a triangle and a problem about a rational number; the titles are deliberately bland but these are as fun as Shailesh Shirali's *Adventures in Problem Solving*.

The Review this time will certainly have you leaping to order this book: *The Cartoon Guides to Calculus and Algebra*, a series whose name says both all and nothing. Can such a serious subject be illustrated with cartoons? With mathematical rigour? Read the Review and I'm sure you'll be convinced.

Our issue concludes with the PullOut – Padmapriya Shirali focuses on *Large Numbers* and how students can grasp this concept. I am sure that adults too will enjoy this refresher course and pick up tips on how to make this topic child- friendly and approachable.

So it's over to you now! Happy reading…..

Sneha Titus
Associate Editor

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Azim Premji University
A Publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley and Sahyadri School, Pune

Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Azim Premji University

A Publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley and Sahyadri School, Pune

Contents contd.

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

CONNECTING THE DOTS...

The art and science of interpolation and extrapolation

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

The following is an extremely common scenario in the sciences: two variables *y* and *x* are connected by a functional relation, $y = f(x)$, but *f* is unknown and our task is to find it. The only actions available to us are to perform experiments and find the values of *y* corresponding to selected values of *x*. After doing these experiments, we obtain the following *n* pairs of values of *x* and *y*:

We may plot these points on a sheet of graph paper and get something which looks like this:

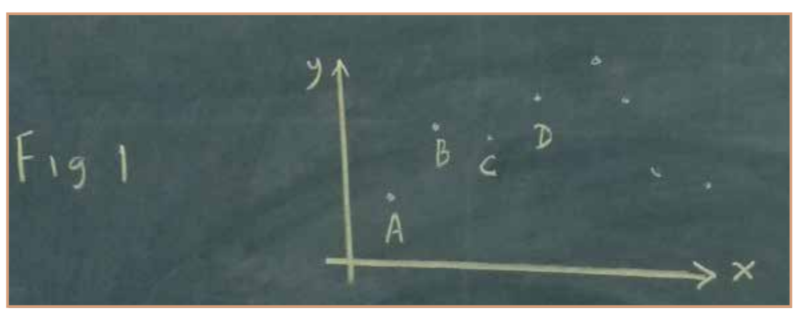


Figure 1

Armed with only these data points, can we determine the unknown function *f*? Another way of expressing this question is the following: *Can we fit a definite, unique curve to these data points?* Note that the curve must pass through all the points. (So this is not a problem of finding the "line of best fit" or the "curve of best fit").

A moment's reflection will tell us that the answer is **No**. The question is too broad to admit an answer when stated this way. Even if we were only interested in polynomial functions *f* (this is very often the case), it is still not possible for the data to yield a definite, unique answer, for there are infinitely many polynomials which will fit the given data.

Let us start with the simplest possible case: just two data points ($n = 2$). The situation now appears as shown in Figure 2:

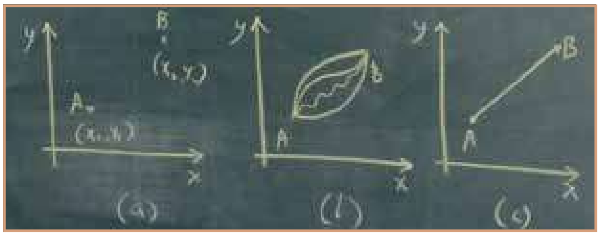


Figure 2

As Figure 2(b) suggests, many different curves may be drawn through A and B. If we are to progress, we need to impose additional conditions. The most obvious such condition is: *with the least possible degree*. In other words, we seek *the polynomial curve with the least possible degree passing through all the given points*. In the case of two points A and B, this curve is obviously the unique straight line passing through A and B, as shown in Figure 2(c). The equation of a general straight line is $y = ax + b$, with two unknown coefficients a and b. As there are also two data points, they serve to uniquely fix a and b.

In the case of three given points A, B and C (Figure 3), it may happen that the points lie in a straight line. But generally this will not be the case, and the polynomial curve with the least degree passing through the points will be a quadratic curve (upward facing or downward facing). As earlier, once we have imposed the condition of 'least possible degree' the answer becomes unique. For, the equation of a general quadratic curve is $y = ax^2 + bx + c$, with three unknown coefficients a, b and c. As there are also three data points, they serve to uniquely fix a, b and c.

In the same way, given four points A, B, C and D, the polynomial curve with least possible degree passing through these points will be a *cubic curve*. And so on.

The emphasis on the word 'least' needs to be commented upon. It is common in the sciences, when we are attempting to explain some phenomenon, that a number of competing theories are available. Other things being equal, the general principle used to choose among the theories is that of *simplicity: choose the simplest theory available*. Here the word 'simplest' could mean: *that which makes the fewest assumptions*. In our curve-finding context, it could mean: *that polynomial curve with the least possible degree*. This principle is often referred to as **Occam's razor** (see [1]). It is regarded as an extremely important principle in the philosophy of science.

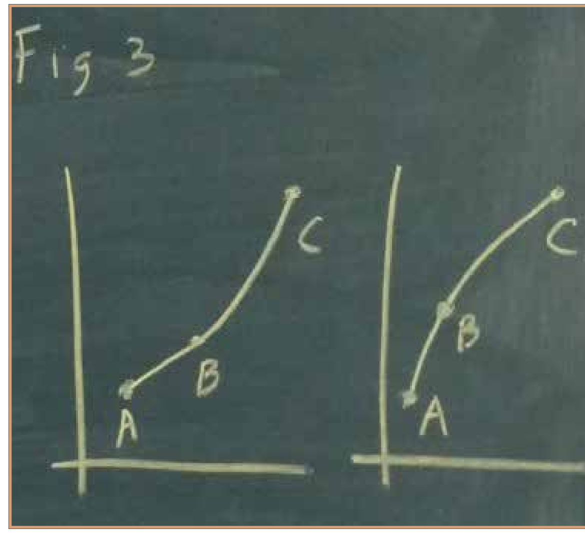


Figure 3

Note: We need to point out the difference between the two terms 'interpolation' and 'extrapolation'. Suppose we are given n data points, and we have been able to find a curve corresponding to a polynomial f of degree n which passes through all the n points. We may now want to use this knowledge to estimate the y-value corresponding to some x-value lying within the range of the given data. All we need to do now is to substitute this value of x into the function f, and we get the desired estimate. This process is known as ***interpolation***. What happens if the x-value lies outside the range of the given data? We may *assume* now that the same functional relationship holds between y and x even if such is the case. Therefore, to get the desired estimate, all we do (as earlier) is to substitute this value of x into the function f. This process is known as *extrapolation*.

Interpolation and extrapolation are vitally important elements in numerical analysis and in the application of mathematics to the sciences. The interested reader could refer to web links [2] and [3] for more details on interpolation and extrapolation respectively.

[1] Wikipedia, Occam's razor, https://en.wikipedia.org/wiki/Occam%27s_razor

[2] Wikipedia, Interpolation, https://en.wikipedia.org/wiki/Interpolation

[3] Wikipedia, Extrapolation, https://en.wikipedia.org/wiki/Extrapolation

---

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Finding the SQUARE of an INTEGER:
A SIMPLE & QUICK METHOD

We know that finding the square of a large number is a difficult job. Here we give an easy method to find the square of any number whose units digit is 9. Also we have an algorithm to find the square of any two digit number just by doing a few calculations.

The square of a two digit number $n = 10a + b$ as given by the following procedure.

Add b to the given number; then multiply by $10a$.
Add b^2 to the number obtained in Step 1.

The resulting number is the square of n.

Proof: Given number is $n = 10a + b$. Add b; we get $10a + 2b$. Multiplying by $10a$, we get $(10a)^2 + 20ab$. Lastly, add b^2; we get $(10a)^2 + 20ab + b^2$, which is the square of $10a + b$.

Illustration: $(87)^2$

Old method: $87 \times 87 = 7569$

New method:
$87 + 7 = 94$
$94 \times 80 = 7520$
$7520 + 49 = 7569$

Here's another shortcut for squaring a number n whose units digit is 9:
n^2 can be calculated by using the formula $n^2 = (n - 1)(n + 1) + 1$.
Example: $39^2 = (39 - 1)(39 + 1) + 1 = (38 \times 40) + 1 = 1520 + 1 = 1521$.

The formula is easily verified. Its advantage is that $n + 1$ is a multiple of 10.

Acknowledgement: The author is grateful to Prof B.N.Waphare for encouraging him to send this finding to *At Right Angles*.

Vikram A. Ghule

Ekalavya Shikshan Sanstha's, Sant Tukaram Vidyalaya Ghatshiras Tal. Pathardi, Dist. **Ahmednagar** 414106 (India) **Email:** vikramghule2016@gmail.com

Predicting the Future

SANKARAN VISWANATH

Said Tweedledum to Tweedledee[1]
"Now looking back, I clearly see
That life has been indeed carefree.
But what the future holds in store,
Of happiness, less or more?
I do so wish I knew before!"

Said Tweedledee: "Oh, that's easy.
You'll soon admit, I guarantee
That here's the perfect recipe:
To estimate your future state,
To guess, but still be accurate,
Just simply ex-trapolate!"

We often find ourselves needing to predict the future value of something based on past trends. For instance, climate scientists have been trying to estimate the future rise in ocean temperatures based on (among other things) temperature data of the last 100 or more years. Or maybe you are a cricket enthusiast and want to predict how many centuries your favourite batsman will score this year based on his scoring statistics for past years. We all probably have our own ways of arriving at an estimate! In this article, let us explore one approach.

[1]Tweedledum and Tweedledee appear in Lewis Carroll's *Through the Looking Glass*. Carroll, whose real name was Charles Lutwidge Dodgson, was (among other things) a mathematician who taught at Oxford. The poem above is inspired by Carroll's *literary nonsense* style which is the hallmark of the Alice books.

Keywords: Patterns, polynomials, extrapolation, interpolation

First, imagine this: you are growing a bacterial culture in the lab, and would like to understand how the population of bacteria changes over time. So, at the end of every hour you measure the bacterial population, and write down the sequence of measurements as follows: $a_1, a_2, a_3, \ldots$, where a_n denotes the population after n hours. Now, suppose you find that the first five measurements are: $1, 4, 9, 16, 25$. Can you "predict" the population after 6 hours?

You probably guessed 36, right? If asked why you think this a good estimate, you would probably say "the pattern of the previous measurements suggests that the population is varying as the square of the natural numbers, i.e., $1^2,\ 2^2,\ 3^2, \ldots$; so at the end of 6 hours, it must be $6^2 = 36$."

This illustrates the following point: to predict a future value precisely, one has to presuppose that there is some pattern or orderly mechanism by which the values are being generated. If the values are just a collection of randomly generated numbers, then future values bear no relation to past ones and precise prediction is impossible.[2]

The simplest patterns are the sequences of powers of the natural numbers; for example, the sequences: (a) $1, 2, 3, 4, \ldots$ (b) $1, 4, 9, 16, \ldots$ (c) $1, 8, 27, 64, \ldots$. As before, letting a_n denote the n^{th} term of a sequence, the sequences above are given by the explicit formulas: (a) $a_n = n$, (b) $a_n = n^2$, and (c) $a_n = n^3$. Here n ranges over the natural numbers. Similarly one can look at the sequence of d^{th} powers, $a_n = n^d$ where d is some fixed natural number.

A somewhat more general sequence is obtained by taking combinations of powers of n; for example

$$a_n = n^3 - 4n^2 + 7n + 1. \tag{1}$$

The first few terms of the sequence (obtained for $n = 1, 2, 3, 4$) are: $5, 7, 13, 29$. An expression of the above form is called a *polynomial* in n, and the highest power of n that occurs (3 in the above example) is called the *degree* of the polynomial. The most general form of a polynomial of degree $\leq d$ is:

$$c_d\, n^d + c_{d-1}\, n^{d-1} + \cdots + c_1\, n + c_0 \tag{2}$$

Here $c_0, c_1, \ldots, c_d$ are real numbers. They are called the *coefficients* of the polynomial. The degree of this polynomial is the largest value of m for which $c_m \neq 0$. For instance, the degree 3 polynomial in equation (1) has coefficients $c_0 = 1, c_1 = 7, c_2 = -4, c_3 = 1$.

Let us now formulate a specific version of our original *prediction problem* for the bacterial populations. Recall that we have a sequence of measurements $a_1, a_2, a_3, \ldots$, with a new term added to this list every hour. Fix $d \geq 1$; suppose we are given the information that the bacterial population varies as a polynomial in n of degree $\leq d$, i.e., a_n is given by a formula of the form of Equation (2).

[2] There is a more nuanced version of this: if the values are randomly generated, but from a given *probability distribution*, i.e., when the probability of taking each value is known, then one can still make predictions which hold true with some probability. We won't deal with this situation here.

Problem 1. Determine the coefficients $c_0, c_1, \ldots, c_d$ of this polynomial, using only the first few population values.

Observe that in our problem statement, we don't yet quantify the word "few"; we are allowed to use any finite number of initial values; this corresponds to the measurements that have been made until a given point of time. Knowing these, we would like to determine the coefficients of the polynomial. Once the coefficients are known, the polynomial is fully specified, and we can compute the value of a_n for any desired value of n. We are thus asking if we can just take finitely many measurements, and use those to determine what *all* future measurements will be. This process is usually termed *extrapolation*. [3]

Example 1. Suppose a_n is known to be a polynomial in n, of degree ≤ 1 (i.e., degree is 0 or 1). Given $a_1 = 3$ and $a_2 = 7$, find the formula for a_n.

To solve this, we start with the general form in Equation (2), with $d = 1$, i.e., $a_n = c_1\, n + c_0$. Here, the coefficients c_0, c_1 are unknown. Substituting $n = 1, 2$ we obtain two equations for these two unknowns:

$$\begin{aligned} c_1 + c_0 &= 3, \\ 2c_1 + c_0 &= 7. \end{aligned}$$

Solving, we obtain $c_1 = 4, c_0 = -1$; thus $a_n = 4n - 1$ is the desired formula.

This example suggests a general method; suppose d is any given natural number and we know that a_n is a polynomial of degree $\leq d$. This means a_n is given by Equation (2), but the $(d + 1)$ coefficients $c_0, c_1, \ldots, c_d$ are unknown. If we also knew the values of $a_1, a_2, \ldots, a_{d+1}$, then we could substitute $n = 1, 2, \ldots, (d + 1)$ in Equation (2) to obtain $d + 1$ linear equations in the $d + 1$ unknowns. Solving these equations would give us the values of $c_0, c_1, \ldots, c_d$. Why don't we try this in a slightly larger example, before proceeding ahead?

Exercise 1. You are given that a_n is a polynomial in n of degree ≤ 3 and that $a_1 = 1$, $a_2 = 2$, $a_3 = 9$, $a_4 = 28$. Using the above strategy, determine the coefficients c_0, c_1, c_2, c_3. You can use any method you like to solve the equations you get, but a simple elimination of variables (by subtracting successive pairs of equations) will work.

A second approach. Now, in this particular example, it turns out that another method of solution is possible. Observe that the given terms of the sequence are just one more than the cubes of the first four non-negative integers, i.e., $0^3 + 1, 1^3 + 1, 2^3 + 1, 3^3 + 1$. Thus, the first four terms are given by the simple formula: $(n - 1)^3 + 1$ for $n = 1, 2, 3, 4$.

Let us define $b_n = (n - 1)^3 + 1$; then we know the following: (i) Both a_n and b_n are polynomials in n of degree ≤ 3 (in fact, b_n has degree exactly 3 as is evident from the above formula, but we won't need this). (ii) $a_n = b_n$ for $n = 1, 2, 3, 4$.

We claim this implies that $a_n = b_n$ for all $n = 1, 2, \ldots$; in other words the formula for a_n is just $a_n = (n - 1)^3 + 1 = n^3 - 3n^2 + 3n$; you should therefore have got $c_0 = 0$, $c_1 = 3$, $c_2 = -3$, $c_3 = 1$ in Exercise 1; did you? In fact, this is more generally true:

[3] A closely related term is *interpolation*. This refers to the process of computing the value at an intermediate time that lies between two measurement times. In our case, both these come down to the same problem, that of determining the coefficients of the polynomial.

Theorem 1. *Let $d \geq 1$ and suppose a_n, b_n are both polynomials in n of degree $\leq d$. If the first $d+1$ terms of both sequences match, then the sequences are identical, i.e., $a_n = b_n$ for $1 \leq n \leq (d+1)$ implies $a_n = b_n$ for all $n \geq 1$.*

Proof. To prove Theorem 1, it is better to enlarge our perspective a little, and work with *functions* instead of sequences. A *polynomial function* or *polynomial in x* of degree $\leq d$ is an expression of the form:

$$f(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$$

where c_i are real numbers for $i = 0, 1, \ldots, d$ and the variable x can take any real value.

The degree, denoted $\deg f(x)$, is as before the largest value of m for which $c_m \neq 0$. If all c_m are zero, i.e., $f(x)$ is the zero polynomial, then its degree is not defined. If $f(x)$ and $g(x)$ are nonzero polynomials, then so is their product and $\deg(f(x)g(x)) = \deg f(x) + \deg g(x)$. For instance, $f(x) = x^2 - x$ has degree 2, $g(x) = x^3 + x^2 + x + 1$ has degree 3 and their product $f(x)g(x) = x^5 - x$ has degree 5.

Given a sequence a_n which is a polynomial in n, we can replace n by x to construct a polynomial function $f(x)$. For example, if $a_n = n^3 - 4n^2 + 7n + 1$, then $f(x) = x^3 - 4x^2 + 7x + 1$. To get the sequence back from the function, we note that $a_n = f(n)$ for $n = 1, 2, 3, \ldots$. Using this, it is now easy to see that the following theorem implies Theorem 1.

Theorem 2. *Let $d \geq 1$ and suppose $f(x), g(x)$ are polynomials of degree $\leq d$. Let $x_1, x_2, \ldots, x_{d+1}$ be any $d+1$ distinct real numbers. If $f(x_n) = g(x_n)$ for $n = 1, 2, \ldots, d+1$, then $f(x) = g(x)$ for all real x.*

Proof. To prove Theorem 2, we first set $h(x) = f(x) - g(x)$ and observe that (i) $h(x)$ has degree $\leq d$, and (ii) $h(x_n) = 0$ for $n = 1, 2, \ldots, d+1$. We now use the following important lemma:

Lemma 1. *Let $p(x)$ be a polynomial and suppose $p(a) = 0$ for some real number a. Then $p(x) = (x-a)q(x)$ for some polynomial $q(x)$.*

Proof. To prove the lemma, we use the *division algorithm* to write

$$p(x) = (x-a)q(x) + r(x),$$

where $r(x)$ is the remainder and $q(x)$ is the quotient. Here, since $x - a$ is of degree 1, the remainder $r(x)$ has degree < 1, i.e., it is just a constant (a polynomial of degree 0). Evaluating both sides at $x = a$ shows $r(x) = 0$. □

Now, back to the proof of Theorem 2.

Since $h(x_1) = 0$, the lemma implies $h(x) = (x - x_1)\, q_1(x)$ for some polynomial $q_1(x)$. Next, $h(x_2) = 0$ implies $q_1(x_2) = 0$ and by the lemma again, $q_1(x) = (x - x_2)\, q_2(x)$. Repeating this process, we obtain finally:

$$h(x) = (x - x_1)\,(x - x_2) \cdots (x - x_{d+1})\; q_{d+1}(x). \tag{3}$$

We now claim that $q_{d+1}(x)$ is the zero polynomial. If not, then the left side has degree $\leq d$, while the right side has degree $\geq d+1$. This contradiction shows that $q_{d+1}(x) = 0$ and hence $h(x) = 0$. This proves Theorem 2, and thereby Theorem 1 as well. □

Thus, a polynomial function $f(x)$ of degree $\leq d$ is uniquely determined once its values at any $d+1$ distinct points are given. So, if you can somehow produce one polynomial that takes the prescribed values at those points (as in the second approach above, where our candidate polynomial $(x-1)^3 + 1$ had the required values at $x = 1, 2, 3, 4$) then you can be sure that that is indeed the solution.

A systematic method to find the polynomial

The next natural question is: does such a polynomial always exist, and if so, can we find it systematically, without having to make informed guesses?

Problem 2. Fix $d \geq 1$. Let $x_1, x_2, \ldots, x_{d+1}$ be *distinct* real numbers, and let $y_1, y_2, \ldots, y_{d+1}$ be any (not necessarily distinct) real numbers. Does there exist a polynomial $f(x)$ of degree $\leq d$ such that $f(x_n) = y_n$ for all $n = 1, 2, \ldots, d+1$?

Let us convince ourselves that the answer is 'Yes'. [4]

For $d = 1$, the polynomial is easy to produce by following the same procedure as in Example 1. It turns out to be (as you should check!):

$$f(x) = (y_2 - y_1)\,\frac{x - x_1}{x_2 - x_1} + y_1 \tag{4}$$

Next we show that such a polynomial exists in general, by mathematical induction on d. Let $d \geq 2$; our induction hypothesis is that the desired result holds for $d - 1$. Considering only x_i and y_i for $1 \leq i \leq d$, the induction hypothesis ensures that there is a polynomial $g(x)$ of degree $\leq d - 1$ satisfying $g(x_i) = y_i$ for $i = 1, 2, \ldots, d$. The polynomial $f(x)$ that we seek should have degree $\leq d$ and satisfy $f(x_i) = y_i$ for $i = 1, 2, \ldots, d+1$. This means in particular that $h(x) = f(x) - g(x)$ has degree $\leq d$ and vanishes at $x_1, x_2, \ldots, x_d$. By repeated application of Lemma 1, we obtain:

$$h(x) = C\,(x - x_1)\,(x - x_2)\cdots(x - x_d)\,,$$

where C is a constant. We can now solve for C by plugging in $x = x_{d+1}$. Carrying out this process, we obtain finally:

$$f(x) = (y_{d+1} - g(x_{d+1}))\;\frac{(x - x_1)\,(x - x_2)\cdots(x - x_d)}{(x_{d+1} - x_1)\,(x_{d+1} - x_2)\cdots(x_{d+1} - x_d)} + g(x), \tag{5}$$

a polynomial that satisfies all the required conditions. □

The above argument not only proves the existence of such a polynomial, it also allows us to construct it step-by-step by use of Equations (4) and (5). To see this, for each $n = 1, 2, \ldots, d$, let $f_n(x)$ denote the unique polynomial of degree $\leq n$ satisfying $f_n(x_i) = y_i$ for $i = 1, 2, \ldots, n+1$. The above argument shows that (i) $f_1(x)$ is given by the right hand side of Equation (4), and (ii) for all $2 \leq n \leq d$,

$$f_n(x) = (y_{n+1} - f_{n-1}(x_{n+1}))\prod_{i=1}^{n}\frac{x - x_i}{x_{n+1} - x_i} + f_{n-1}(x)$$

This recursion relation can be used repeatedly to find $f_d(x)$.

Exercise 2. Using the above procedure, find the polynomial $f(x)$ of degree ≤ 3 such that $f(1) = 1, f(2) = 2, f(3) = 9, f(4) = 28$. Check that your answer matches what you got for Exercise 1.

The Wikipedia articles given in the references are good starting points to get more information about the subject matter of this article.

---

[4] One way to do this is along the lines of Exercise 1, i.e., write out a system of linear equations for the unknown coefficients, and argue that this system always has a solution. This can be done, but requires some facts about matrices. In particular, the famous *Vandermonde matrix* makes an appearance here, and the fact that it has nonzero *determinant* implies the result. For more, see the references at the end.

Another systematic approach to interpolation

Let us begin by rewriting Equation (4) in a more "symmetrical" form as follows:

$$f(x) = y_1 \left(\frac{x - x_2}{x_1 - x_2}\right) + y_2 \left(\frac{x - x_1}{x_2 - x_1}\right).$$

This, we recall, is the unique polynomial of degree ≤ 1 satisfying $f(x_n) = y_n$ for $n = 1, 2$. Let us define

$$p_1(x) = \frac{x - x_2}{x_1 - x_2} \text{ and } p_2(x) = \frac{x - x_1}{x_2 - x_1}.$$

We then observe $f(x) = y_1\, p_1(x) + y_2\, p_2(x)$. The key properties of these polynomials are:

(1) $p_1(x)$ and $p_2(x)$ have degree ≤ 1.
(2) $p_1(x_1) = 1,\ p_1(x_2) = 0;\ p_2(x_1) = 0,\ p_2(x_2) = 1$.

This suggests the following idea to solve Problem 2 in general. Given $x_n, y_n,\ 1 \leq n \leq d+1$ as in Problem 2, we wish to find a polynomial $f(x)$ of degree $\leq d$ such that $f(x_n) = y_n$ for $n = 1, 2, \ldots, d+1$. Let us first try to find polynomials $p_1(x), p_2(x), \ldots, p_{d+1}(x)$ satisfying the following properties:

(1) $p_n(x)$ has degree $\leq d$ for all $1 \leq n \leq d+1$.
(2) For all $1 \leq n, m \leq d+1$,

$$p_n(x_m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Suppose we can manage this, then

$$f(x) = y_1\, p_1(x) + y_2\, p_2(x) + \cdots + y_{d+1}\, p_{d+1}(x)$$

is a polynomial satisfying all the required conditions (check this!). Now, it only remains to find the $p_n(x)$. We know that $p_n(x) = 0$ for $x = x_m,\ m \neq n$. Using Lemma 1 repeatedly as before we obtain $p_n(x) = C\,(x - x_1)\,(x - x_2)\cdots\widehat{(x - x_n)}\cdots(x - x_{d+1})$, where C is a constant, and $\widehat{(x - x_n)}$ means that term is omitted from the product. We solve for C by plugging in $x = x_n$ and obtain:

$$p_n(x) = \frac{(x - x_1)\,(x - x_2)\cdots\widehat{(x - x_n)}\cdots(x - x_{d+1})}{(x_n - x_1)(x_n - x_2)\cdots\widehat{(x_n - x_n)}\cdots(x_n - x_{d+1})}.$$

Exercise 3. Redo Exercise 2 using the above method.

The polynomials obtained by the two procedures outlined above (Exercises 2 and 3) have very different forms, but check that they are equal. These are respectively called the *Newton form* and the *Lagrange form* of the *interpolating polynomial*.

References

1. Newton Polynomial, https://en.wikipedia.org/wiki/Newton_polynomial
2. Lagrange Polynomial, https://en.wikipedia.org/wiki/Lagrange_polynomial
3. Vandermonde matrix, https://en.wikipedia.org/wiki/Vandermonde_matrix

SANKARAN VISWANATH is a faculty member at the Institute of Mathematical Sciences (IMSc), Chennai. He received his PhD in Mathematics from UC Berkeley (2004). His research interests lie in Algebraic Combinatorics and Lie Theory. He has been actively associated with mathematics outreach programmes at IMSc, and especially enjoys conveying the excitement of mathematics to school students and teachers. He may be contacted at svis@imsc.res.in.

NUMBER CROSSWORD

Solution on Page 29

D.D. Karopady & Sneha Titus

■	1	2	■	■	■	3	4	■
■	5		6	■	7			■
8		■	9			■	10	
■	11	12		■	13	14		■
■	■		■	■	■		■	■
■	15		16	■	17		18	■
19		■	20			■	21	
■	22	23		■	24	25		■
■	26		■	■	■	27		■

CLUES ACROSS

1 A prime number
3 2 more than 3 dozen
5 Product of the fourth power of 2 and 7
7 Product of a power of 2 and its reverse
8 Square root of 441
9 Perfect cube
10 A fortnight
11 Product of the 29th prime and 7
13 Multiple of 9
15 LCM of 3A and 10A
17 9 days short of a year
19 HCF of 2D and 19A is 13
20 14D divided by 7 x 37
21 2 score
22 Twin prime with 59 times 3
24 10 times 27 A
26 3.5 feet
27 20A divided by 25 and then digits reversed

CLUES DOWN

1 A hundred years hence
2 HCF of 2D and 19A is 13
6 Its negative signifies absolute zero
7 1 short of a millenium
12 Number remains a palidrome when divided by 2, 3 and 6
14 Quarter less than two centuries
15 Hundreds digit is the sum of the tens and units digit
17 Difference between middle digit and the end digits is the same

Teaching the TERNARY BASE Using a Card Trick

SUHAS SAHA

Any sufficiently advanced technology is indistinguishable from magic.

— Arthur C. Clarke, "Profiles of the Future: An Inquiry Into the Limits of the Possible"

Magic has the ability to grab the attention of adults and children alike, whether it is a traditional trick of pulling rabbits out of a hat or a more sophisticated mind-reading card trick. We started a project at our school to excite children about mathematics using card tricks. We were pleasantly surprised to see such a positive response from children across grades.

In the following article, we describe how we used the 27-card trick to introduce ternary bases to students in Grade 9.

The Magic Trick

Grab a deck of cards, remove the jokers and shuffle the deck. Select any 27 cards from the deck. Fan them out in front of your class and ask a volunteer to choose one card at random, show it to the rest of the class (turn your back or close your eyes so that it is clear you have not seen the selected card) and put the card back into the deck. This card will henceforth be called the secret card. Invite the volunteer to shuffle the deck as many times to the satisfaction of the spectators that the secret card is lost in the deck. While taking the deck back from the volunteer you proclaim that in three steps you will reveal the secret card.

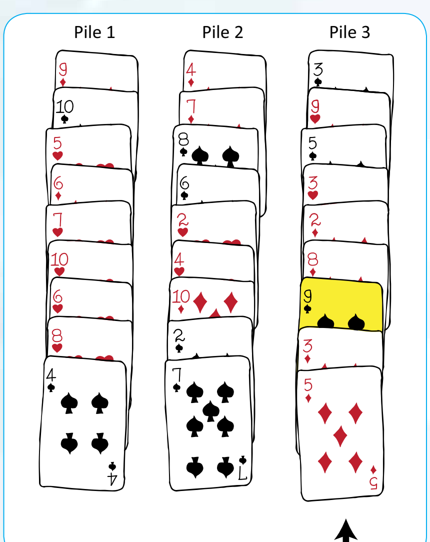


Figure 1

In our example, suppose the secret card is 9 of spades. We will use the notation 9S for 9 of spades, 9D for 9 of diamond, 9C for 9 of clubs and 9H for 9 of hearts. In Step I, deal out the cards face-up into 3 piles, each pile containing

Keywords: 27-card trick, ternary base, magic, ceiling function

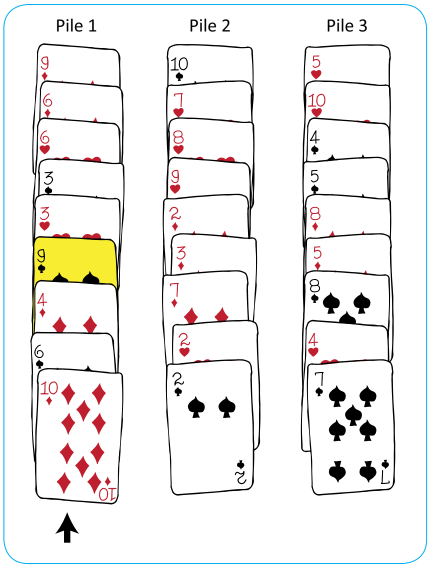


Figure 2

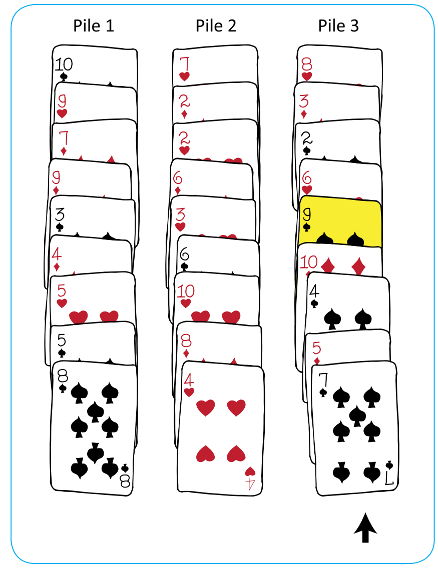


Figure 3

9 cards while the spectators keep watching. Referring to figure 1, the first card dealt is 9D, the second card is 4D, the third card is 3S, the fourth card is 10S and so on with card numbers 25, 26, 27 being the 4S, 7S and 5D respectively. Stack the cards in the first pile face-up, so that card #25 becomes the top card, and card #1 becomes the last card in the pile. Similarly, stack up the cards in the other two piles. For example, in the third pile, card #27 is the top card while card #3 is the last card. The volunteer now indicates the pile containing the secret card.

Turn the three piles over so that all the cards are face down. Pick up and stack the piles on your palm, one over the other, making sure to place the pile containing the secret card in-between the other two piles. In our example, since the secret card is in pile 3, this pile should be placed between pile 1 and 2. You may choose to place pile 2 on your palm first, follow it up with pile 3 and finish by placing pile 1 on the top. In other words, pile 2 is at the bottom, pile 3 at the middle and pile 1 is at the top.

In Steps II and III, we repeat the process of dealing the 27 cards face-up into 3 piles of 9 cards each, asking the volunteer to indicate the pile containing the secret card and placing this pile in-between the other two piles while collecting the cards. At the end of the three steps, we are ready to reveal the secret card.

Figure 2 shows the configuration of the cards when they are dealt in Step II. Since the secret card is now in pile 1, the piles should be collected and placed on the palm so that pile 1 is in between pile 2 and pile 3. Let us suppose you choose to place the piles such that pile 2 is at top, pile 1 in the middle and pile 3 at the bottom. Figure 3 shows the configuration of the cards when they are dealt out in Step III. Now the secret card is in pile 3. Therefore you must collect the piles in the order that ensures pile 3 is at the middle. Notice that at the end of Step III, the secret card will always be at position 14 from top (or bottom), which happens to be the middle card of the stack. This is clear from Figure 3.

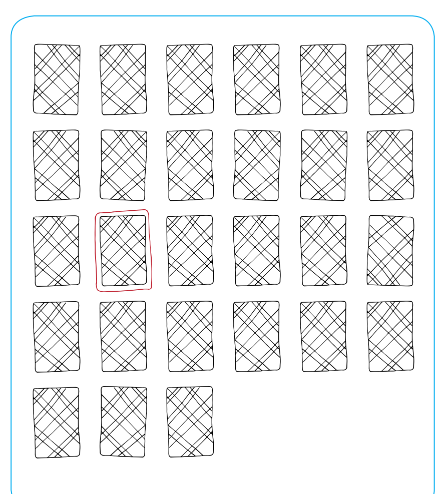


Figure 4

With the stack in your hand, all the cards face down deal out the cards (either face up or face down), placing them randomly on the table but carefully keeping track of the 14th card. After dealing out all the cards, pretend that you have lost track of the secret card, and at the moment the audience is about to acknowledge your defeat, point out the 14th card, proclaiming that this is indeed the secret card. Acknowledge and enjoy the applause from the audience!

Why does the trick work?

Why does the trick work? Denote by n the position of the secret card in the deck (counting from the top) after the volunteer has inserted it into the deck and the deck has been shuffled thoroughly. Thus $1 \leq n \leq 27$. For example, if $n = 15$, it means there are 14 cards on top of the secret card and 12 cards below it. Let us compute the new position of the secret card after Step I, i.e., after the cards have been dealt out and the cards collected back with the pile containing the secret card in-between the other two piles.

When the cards are dealt into three piles, the first, second and third cards in the deck become the first cards of piles 1, 2 and 3 respectively. Similarly, the fourth, fifth and sixth cards in the deck become the second cards in piles 1, 2 and 3 respectively, and so on. You can check for yourself that the 16th, 17th and 18th cards in the deck become the sixth cards of piles 1, 2 and 3 respectively.

From this we construct a mathematical function f which captures the position of the selected card in the deck after the 27 cards have been dealt into three piles of 9 each.

- If n is a multiple of 3, say $n = 3m$, then n goes into the third pile and assumes position m, i.e., $f(n) = m$. For example, if the initial position of the secret card is $n = 12$, then the secret card assumes position 4 in the third pile.
- Now consider the situation when n is not a multiple of 3. Suppose that $n = 20$. Note that 20 lies between 18 and 21 (these two numbers being consecutive multiples of 3 on either side of 20). With 18 cards already dealt into three piles containing 6 cards each, card #19 becomes the 7th card in pile 1, card #20 becomes the 7th card in pile 2, and card #21 becomes the 7th card in pile 3.

Note that

$$\frac{19}{3} = 6 + \frac{1}{3}, \qquad \frac{20}{3} = 6 + \frac{2}{3}, \qquad \frac{21}{3} = 7.$$

So we want f to be such that $f(19) = 7, f(20) = 7$ and $f(21) = 7$. From these observations, it is not difficult to see that

$$f(n) = \left\lceil \frac{n}{3} \right\rceil, \tag{1}$$

where $\lceil x \rceil$ is the *ceiling function*, defined thus: $\lceil x \rceil$ *is the smallest integer greater than or equal to* x. (Examples: $\lceil 1.7 \rceil = 2$, $\lceil \pi \rceil = 4$, $\lceil 8 \rceil = 8$.)

In the discussion below, we will make repeated use of the following important property of the ceiling function (try to find a proof of it for yourself):

Theorem. *For any two positive real numbers m and n,*

$$\left\lceil \frac{\lceil m \rceil}{n} \right\rceil = \left\lceil \frac{m}{n} \right\rceil.$$

Position of the selected card after Step I. We deduce the following from the considerations above: if the selected card is at position n_0 at the start, then after Step I, when a pile of 9 cards is placed over the pile containing the secret card, it is at position n_1, where

$$n_1 = 9 + \left\lceil \frac{n_0}{3} \right\rceil. \tag{2}$$

We call this operation M, with M signifying that the pile containing the secret card is placed in the 'middle position'. For example, if the secret card was at position $n_0 = 23$ initially, then after Step I the card will be at position n_1 where

$$n_1 = M(23) = 9 + \left\lceil \frac{23}{3} \right\rceil = 17.$$

At the end of Step II the card will be at position n_2 from the top, where

$$n_2 = M(M(n_0)) = 9 + \left\lceil \frac{n_1}{3} \right\rceil$$

$$= 9 + \left\lceil \frac{17}{3} \right\rceil = 9 + 6 = 15.$$

So the 23rd card will land up in the 15th position. At the end of Step III the position of the card is n_3 where

$$n_3 = M(M(M(n_0))) = 9 + \left\lceil \frac{n_2}{3} \right\rceil$$

$$= 9 + \left\lceil \frac{15}{3} \right\rceil = 14.$$

For the trick to work, it must happen that for any n_0 with $1 \leq n_0 \leq 27$,

$$M(M(M(n_0))) = 14. \qquad (3)$$

To show this, note that

$$n_3 = 9 + \left\lceil \frac{n_2}{3} \right\rceil,$$

$$n_2 = 9 + \left\lceil \frac{n_1}{3} \right\rceil = \left\lceil \frac{27 + n_1}{3} \right\rceil,$$

$$\therefore \; n_3 = 9 + \left\lceil \frac{1}{3} \left\lceil \frac{27 + n_1}{3} \right\rceil \right\rceil$$

$$= 9 + \left\lceil \frac{27 + n_1}{9} \right\rceil = 12 + \left\lceil \frac{n_1}{9} \right\rceil.$$

Next,

$$n_1 = 9 + \left\lceil \frac{n_0}{3} \right\rceil = \left\lceil \frac{27 + n_0}{3} \right\rceil,$$

$$\therefore \; \left\lceil \frac{n_1}{9} \right\rceil = \left\lceil \frac{1}{9} \left\lceil \frac{27 + n_0}{3} \right\rceil \right\rceil = \left\lceil \frac{27 + n_0}{27} \right\rceil$$

$$= 1 + \left\lceil \frac{n_0}{27} \right\rceil = 1 + 1 = 2,$$

since $\left\lceil \frac{n_0}{27} \right\rceil = 1$ (because $1 \leq n_0 \leq 27$). This leads to:

$$n_3 = M(M(M(n_0))) = 13 + \left\lceil \frac{n_0}{27} \right\rceil = 14. \qquad (4)$$

It is to be noted that 14 is the middle position in a deck of 27 cards. The trick would also have worked if we had used 21 cards and dealt the cards into 3 piles, each with 7 cards and repeated Steps I, II, III as described above. After three steps, the selected card will be at position 11, the middle position in a deck of 21 cards; see [1].

Generalization of the Trick

We now modify the above magic trick in a way that leads to the development of the ternary base. (See Box 1 for an explanation of what is meant by *ternary base*.) The question is this: *Rather than bring the selected card to position* 14 *(i.e., the middle of the pack), can we perform the steps in such a way as to bring the selected card to some other desired position?* Let's try out an example.

As earlier, fan out the 27 cards face-down and ask a volunteer to select a card, show it to the audience and put it back in the deck. All this while you turn your back so that it is clear that you have not seen the secret card. Now ask the volunteer to select a number n between 1 and 27. While n is being chosen, shuffle the deck to the satisfaction of the audience. Unlike the previous trick, where in three steps the card is positioned at the centre of the deck, now the task is to place the card at position n by repeating three times the process of distributing the cards into 3 piles of 9 cards each and picking them up in some order. Naturally, the order in which you collect the cards at each step is crucial for the trick to work.

We will illustrate this with three examples.

Example 1. Let $n = 23$. This means that at the end of Step III, we want the secret card to be at position 23.

Recall that Step III consists of laying out the cards in three piles, asking the volunteer to indicate the pile containing the secret card followed by picking up the three piles. Observe that there are three ways in which the piles can be collected:

T: Put the pile containing the secret card on **Top** of the other two piles.

M: Put the pile containing the secret card in the **Middle**, between the other two piles.

B: Put the pile containing the secret card at the **Bottom** of the other two piles.

If the secret card is to be placed in the 23rd position, we must have 22 cards on top of this card. Since each pile contains 9 cards, we collect the two piles which do not contain the secret card (a total of 18 cards) and place them over the pile containing the secret card. This is procedure **(B)**. Obviously, this is not enough to ensure that the secret card is in position 23. For example, suppose

Ternary base

Ternary base is simply 'base 3'. Most of you will be familiar with base 2, also called the **binary system**, in which the only digits used are 0 and 1. In this system, we express the positive integers as sums of distinct powers of 2. For example:

$$6 = 2^2 + 2^1 = (110)_{\text{base two}},$$

$$7 = 2^2 + 2^1 + 2^0 = (111)_{\text{base two}},$$

$$11 = 2^3 + 2^1 + 2^0 = (1011)_{\text{base two}},$$

$$19 = 2^4 + 2^1 + 2^0 = (10011)_{\text{base two}},$$

and so on. The system works because every positive integer can be expressed in a **unique** manner as a sum of distinct powers of 2; the word 'unique' signifies that there is just one way of writing the sum. Each power of 2 is either present in the sum or not present, and this leads to the digits being either 1 or 0.

In the same way, we can express every positive integer in terms of the powers of 3. However, here we cannot say 'sum of distinct powers of 3'; for example, we cannot write 6 as a sum of distinct powers of 3. But we can express every positive integer as a sum of powers of 3 provided that *we permit each power to be not used, or used once, or used twice*; further choices are not needed. For example:

$$4 = 3^1 + 3^0 = (11)_{\text{base three}},$$

$$5 = 3^1 + 2 \cdot 3^0 = (12)_{\text{base three}},$$

$$6 = 2 \cdot 3^1 = (20)_{\text{base three}},$$

$$7 = 2 \cdot 3^1 + 3^0 = (21)_{\text{base three}},$$

$$8 = 2 \cdot 3^1 + 2 \cdot 3^0 = (22)_{\text{base three}},$$

and so on. Each of the above expressions is unique.

the secret card is at position 7 in pile 1. Then if we put piles 2 and 3 over pile 1 and deal out the cards, the secret card will be at position $18 + 7 = 25$. To be at position 23, we must ensure that the secret card is at position 5 *in its own pile* after the cards are dealt out in Step III.

How do we do this? When we deal out the 27 cards at the start of Step III, the 5th positions in piles 1, 2 and 3 will be filled by the 13th, 14th and 15th cards in the deck respectively. So at the end of Step II, when we have collected the 3 piles, the secret card must be either the 13th, 14th or 15th card in the deck. Thus our problem is reduced to placing the secret card at positions 13, 14 or 15 in the deck at the end of Step II. This requires that we follow procedure **(M)** in Step II: take the pile of 9 cards containing the secret card, put one pile on top of it and another pile below it. Is this enough to ensure that the secret card is at positions 13, 14 or 15 at the end of Step II? The answer is Yes only if the secret card is in the 4th, 5th or 6th position in its own pile when the cards are dealt out at the start of Step II.

Note that when a deck of 27 cards is dealt out, the cards in positions 10 to 18 will always take positions 4, 5 or 6 in their respective piles. For example, the 11th card will be placed at position 4 in the second pile, the 15th card will be placed at position 5 in the third pile, and the 16th card will be placed at position 6 in the first pile. Therefore the problem of placing the secret card in the 4th, 5th or 6th rank in its pile at the beginning of Step

II is now further reduced to placing the secret card in any of the positions 10 to 18 in the deck at the end of Step I. Which of the operations **(T)**, **(M)**, **(B)** of collecting the cards at the end of Step I will ensure this? The answer is operation **(M)**, so that there are 9 cards on top of the pile containing the secret card, and the secret card will thus be placed in one of the positions 10–18 in the deck.

To summarize, we perform operations **(B)**, **(M)**, **(M)** in Steps III, II and I respectively to bring the secret card at position 23. Let us put some numbers on these operations, say, **(T)** $= 0$, **(M)** $= 1$, **(B)** $= 2$. The idea behind these numbers is that with operations **(T)**, **(M)** and **(B)**, we are putting 0, 1 and 2 piles respectively on top of the pile containing the secret card. Note that $23 - 1 = 22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$, which shows that the number of cards on top of the secret card, 22 in our example, can be constructed by using the operations in the sequence **(M)**, **(M)**, **(B)**. This is nothing but the ternary base representation of 22, in disguise.

Example 2. Let $n = 21$. We wish to find the operations that will place the secret card in position 21.

As $21 = 18 + 3$, the third operation has to be **(B)** so that there are 18 cards on top of the pile containing the secret card. Moreover, the secret card must be in the 3rd position *in its own pile* at the end of Step III. This means that at the beginning of Step III the secret card has to be in the 7th, 8th or 9th position in the deck. How do we arrange for this configuration? This is only possible if in Step II we collect the cards by putting two piles of 18 cards below the pile containing the secret card. That is, we apply operation **(T)** in Step II. In addition to this, when we deal out the cards in Step II, the secret card must be either the 7th, 8th or 9th card in its own pile. Note that when a deck of 27 cards is dealt out, the cards ranked 19 to 27 will take positions 7, 8 or 9 *in their own piles*. For example, the 19th card will be placed at position 7 in the first pile, the 23rd card will be placed at position 8 in the second pile, and the 27th card will be placed at position 9 in the third pile. Therefore the problem of placing the secret card in the 7th, 8th or 9th rank *in its own pile* when the cards are dealt out in Step II is reduced to placing the secret card in the any of the positions 19 to 27 in the deck at the end of Step I. Hence in Step I we should collect the cards by applying the operation **(B)**, that is, put two piles of cards on top of the pile containing the secret card.

To summarize, we perform the operations **(B)**, **(T)**, **(B)** in Steps III, II and I respectively to bring the secret card at position 21. As **(T)** $= 0$, **(M)** $= 1$, **(B)** $= 2$ and $21 - 1 = 20 = 2 \times 3^2 + 0 \times 3^1 + 2 \times 3^0$, this shows that the number of cards on top of the secret card, 20 in our example, can be constructed by using the operations in the sequence **(B)**, **(T)**, **(B)**. This is again the representation of 20 in the ternary base.

Example 3. Let $n = 16$. We wish to find the operations that will place the secret card in position 16.

As $16 = 9 + 7$, the third operation has to be **(M)** so that there are 9 cards on top of the pile containing the secret card. Moreover, the secret card must be in the 7th position *in its own pile* in Step III. Therefore when we collect the cards at end of Step II, we have to put two piles of cards on top of the pile containing the secret card, which is operation **(B)**. In addition, we have to ensure that the secret card will be in the 1st, 2nd or 3rd position in its own pile when the deck is dealt out in Step II. Similar to the argument used in the above examples, note that when a deck of 27 cards is dealt out, the cards ranked 1 to 9 will take the positions 1, 2 or 3 *in their own piles*. Hence in Step I we should collect the cards by applying operation **(T)**.

Thus we are performing the operations **(M)**, **(B)**, **(T)** in Steps III, II and I respectively to bring the secret card to position 16. As **(T)** $= 0$, **(M)** $= 1$, **(B)** $= 2$ and

$$16 - 1 = 15 = 1 \times 3^2 + 2 \times 3^1 + 0 \times 3^0, \quad (5)$$

we see that the number of cards on top of the secret card, 15 in our example, can be constructed by using the operations in the sequence **(T)**, **(B)**, **(M)**. This is the ternary base representation of 15.

Mathematical Analysis

To find out how to place the card in the chosen position, let us revisit Step I in the previous trick. Suppose the secret card is at position n_0 from the top, after the card has been put back in the deck and the deck thoroughly shuffled. This means there are $n_0 - 1$ cards above it. You deal the 27 cards into 3 piles of 9 card each, in the manner described earlier, and ask the volunteer to indicate the pile which holds the selected card. At this point, the secret card is at position $\lceil \frac{n_0}{3} \rceil$ in the pile. Recall that there are now three ways in which you can collect the piles:

T: Put the pile containing the secret card on **Top** of the other two piles.

M: Put the pile containing the secret card in the **Middle**, between the other two piles.

B: Put the pile containing the secret card at the **Bottom** of the other two piles.

If the operation is **(T)**, then the position of the secret card in the deck remains unchanged at $\lceil \frac{n_0}{3} \rceil$. If the operation is **(M)**, the new position is $9 + \lceil \frac{n_0}{3} \rceil$, as 9 cards are put over the pile containing the secret card. If the operation is **(B)**, the secret card will be at the position $18 + \lceil \frac{n_0}{3} \rceil$, as 18 cards are put over the pile containing the secret card.

Let us denote the operation by F_a so that

$$F_a(n_0) = 9a + \left\lceil \frac{n_0}{3} \right\rceil, \tag{6}$$

where $a = 0$, 1 or 2. Note that $a = 0$ when the operation is **(T)**, $a = 1$ when the operation is **(M)** and $a = 2$ when the operation is **(B)**. Hence the trick will work if, given the chosen number n, we can succeed in finding three numbers $a, b, c \in \{0, 1, 2\}$ such that for any number n_0 between 1 and 27, we have

$$F_c(F_b(F_a(n_0))) = n. \tag{7}$$

The values of a, b, c will decide whether we have to perform operations **(T)**, **(M)** or **(B)** when collecting the cards in each of the three steps.

Now let us see what exactly is $F_c(F_b(F_a(n_0)))$. We have:

$$F_a(n_0) = 9a + \left\lceil \frac{n_0}{3} \right\rceil,$$

$$\therefore\ F_b(F_a(n_0)) = 9b + \left\lceil \frac{1}{3}\left(9a + \left\lceil \frac{n_0}{3} \right\rceil\right) \right\rceil.$$

Now,

$$\frac{1}{3}\left(9a + \left\lceil \frac{n_0}{3} \right\rceil\right) = \left\lceil \frac{1}{3}\left\lceil 9a + \frac{n_0}{3} \right\rceil \right\rceil$$

$$= 3a + \left\lceil \frac{n_0}{3^2} \right\rceil,$$

implying that

$$F_b(F_a(n_0)) = 9b + 3a + \left\lceil \frac{n_0}{3^2} \right\rceil. \tag{8}$$

Using the same reasoning, it follows that

$$F_c(F_b(F_a(n_0))) = 9c + 3b + a + \left\lceil \frac{n_0}{3^3} \right\rceil$$

$$= 9c + 3b + a + 1, \tag{9}$$

since $0 < \frac{n_0}{27} \le 1$ and hence $\lceil \frac{n_0}{3^3} \rceil = 1$.

Coming back to the trick, the analysis above demonstrates that if we want the secret card to be placed at position n in the deck of 27 cards, we must find numbers $a, b, c \in \{0, 1, 2\}$ such that $n = 9c + 3b + a + 1$. A shorthand way of writing this is

$$n = cba_{\text{base three}} + 1, \tag{10}$$

where the subscript 'base three' denotes that we are expressing the number n in the ternary base. For example, if the chosen number is $n = 17$, then we have to find a, b, c such that $9c + 3b + a = 16$. If we divide 16 by 9, we get 1 as the quotient and 7 as the remainder. Therefore $c = 1$. If we divide 7 by 3, we get 2 as the quotient and 1 as the remainder. Therefore $b = 2$ and $a = 1$. Thus $16 = 121_{\text{base three}}$ and after each dealing of the cards, they should be collected using the operations **(M)**, **(B)**, **(M)** respectively. This would make the secret card land up in the 17th position. As another example, if we want the secret card to be in the position 25, then we have to express 24 in the ternary base as $24 = 2 \times 9 + 2 \times 3 + 0 \times 1$ so that $a = 0, b = 2, c = 2$. Hence the operations of gathering the cards after they have been dealt out must be in the order **(T)**, **(B)**, **(B)** respectively.

Conclusion

The 27-card trick is a fascinating way to introduce the ternary base system to students. It intrigued the students to find out how the trick works. The same trick was shown repeatedly to the students until they saw a pattern emerging. We have been successful in making the students interested in learning about the ternary base through this magic.

References

1. https://www.youtube.com/watch?v=gcgvFTfOpD8
2. Gardner, Martin. *Mathematics, Magic and Mystery*. Dover Publications 1956.

SUHAS SAHA teaches mathematics to students of Grade 9 -12 at Isha Home School, Coimbatore. Prior to Isha, he worked as an actuary in the Life Insurance sector. He did a M Sc (Integrated) in Physics from IIT Kanpur, a M Phil in Economics from Indira Gandhi Institute of Development Research, Mumbai and a Ph D in Financial Economics from the University of Minnesota, USA. He may be contacted at suhas.s@ishahomeschool.org.

SOLUTIONS
NUMBER CROSSWORD

Page 21

	(1) 2	(2) 9				(3) 3	(4) 8	
	(5) 1	1	(6) 2		(7) 9	7	6	
(8) 2	1		(9) 7	2	9		(10) 1	4
	(11) 7	(12) 6	3		(13) 9	(14) 1	8	
		0				7		
	(15) 2	6	(16) 6		(17) 3	5	(18) 6	
(19) 6	5		(20) 9	2	5		(21) 4	0
	(22) 1	(23) 8	3		(24) 7	(25) 3	0	
	(26) 4	2				(27) 7	3	

Are you interested in creating the number crossword for the next issue? Send a mail to AtRiA.editor@apu.edu.in and we can help you puzzle your fellow readers.

Features

QUADRILATERALS
with Perpendicular Diagonals

SHAILESH SHIRALI

In this article, we study a few properties possessed by any quadrilateral whose diagonals are perpendicular to each other. A four-sided figure possessing such a property is known as an ***ortho-diagonal quadrilateral***. Many special four-sided shapes with which we are familiar have this property: *squares*, *rhombuses* (where all four sides have equal length) and *kites* (where two pairs of adjacent sides have equal lengths). It may come as a surprise to the reader to find that such a simple requirement (diagonals perpendicular to each other) can lead to so many elementary and pleasing properties.

First Property: Sums of squares of opposite sides

Theorem 1. *If the diagonals of a quadrilateral are perpendicular to each other, then the two pairs of opposite sides have equal sums of squares.*

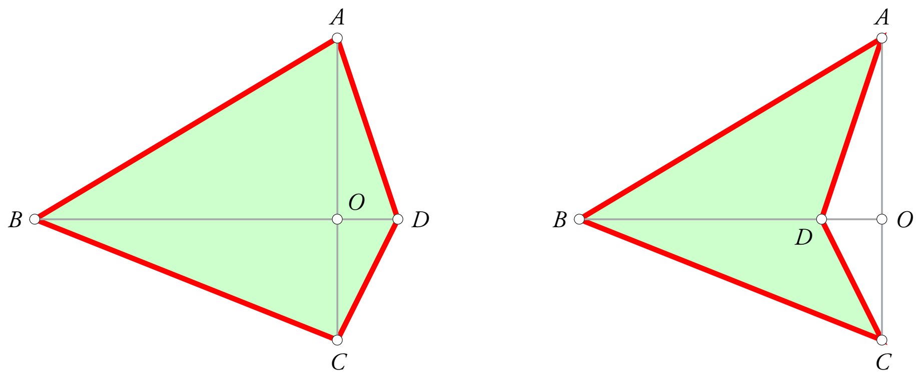
Let the quadrilateral be named $ABCD$. (See Figure 1.) Then the theorem asserts the following: *If* $AC \perp BD$, *then* $AB^2 + CD^2 = AD^2 + BC^2$. Here is a proof. Denoting by O the point where AC meets BD, we have, by the Pythagorean theorem:

$$\begin{cases} AB^2 = AO^2 + BO^2, & CD^2 = CO^2 + DO^2, \\ AD^2 = AO^2 + DO^2, & BC^2 = BO^2 + CO^2, \end{cases} \tag{1}$$

and we see that

$$AB^2+CD^2 = AO^2+BO^2+CO^2+DO^2 = AD^2+BC^2. \quad \square$$

Keywords: *Quadrilateral, perpendicular, diagonals, ortho-diagonal, cyclic quadrilateral, midpoint parallelogram, midpoint rectangle, midpoint circle, maltitude, eight-point circle, indeterminate*



(a) (b)

Figure 1

Observe that the conclusion remains true even if the quadrilateral is not convex and the diagonals meet as in Figure 1 (b). The same proof works for both figures.

What is more striking is that the proposition has a converse:

Theorem 2. *If the two pairs of opposite sides of a quadrilateral have equal sums of squares, then the diagonals of the quadrilateral are perpendicular to each other.*

That is, if $AB^2 + CD^2 = AD^2 + BC^2$, then $AC \perp BD$. We urge the reader to try proving this before reading on.

Our proof uses the generalized version of the theorem of Pythagoras: *In $\triangle ABC$, the quantity a^2 is less than, equal to, or greater than $b^2 + c^2$, in accordance with whether $\measuredangle A$ is less than, equal to, or greater than* $90°$.

We wish to show that if $AB^2 + CD^2 = AD^2 + BC^2$, then $\measuredangle AOB = 90°$. Our approach will be to show that $\measuredangle AOB$ cannot be either acute or obtuse; this leaves only one possibility, the one we want. (Euclid was fond of this approach. Several proofs in THE ELEMENTS are presented in this style. Sherlock Holmes too was fond of this principle! Holmes enthusiasts will remember his unforgettable sentence, "How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?") To start with, suppose that $\measuredangle AOB$ is acute. Then $\measuredangle COD$ too is acute, and $\measuredangle BOC$ and $\measuredangle DOA$ are obtuse. By the generalized version of the PT, we have:

$$\begin{cases} AB^2 < AO^2 + BO^2, & CD^2 < CO^2 + DO^2, \\ AD^2 > AO^2 + DO^2, & BC^2 > BO^2 + CO^2. \end{cases} \tag{2}$$

By addition we get the following double inequality:

$$AB^2 + CD^2 < AO^2 + BO^2 + CO^2 + DO^2 < AD^2 + BC^2, \tag{3}$$

implying that $AB^2 + CD^2 < AD^2 + BC^2$. This contradicts what we were told at the start: that $AB^2 + CD^2 = AD^2 + BC^2$. *So our supposition that $\measuredangle AOB$ is acute must be wrong; $\measuredangle AOB$ cannot be acute.*

If we suppose that $\measuredangle AOB$ is obtuse, we get $AB^2 + CD^2 > AD^2 + BC^2$. This too contradicts what we were told at the start and must be discarded. *$\measuredangle AOB$ cannot be obtuse.*

The only possibility left is that $\measuredangle AOB$ is a right angle; i.e., that AC and BD are perpendicular to each other. Which is the conclusion we were after. □

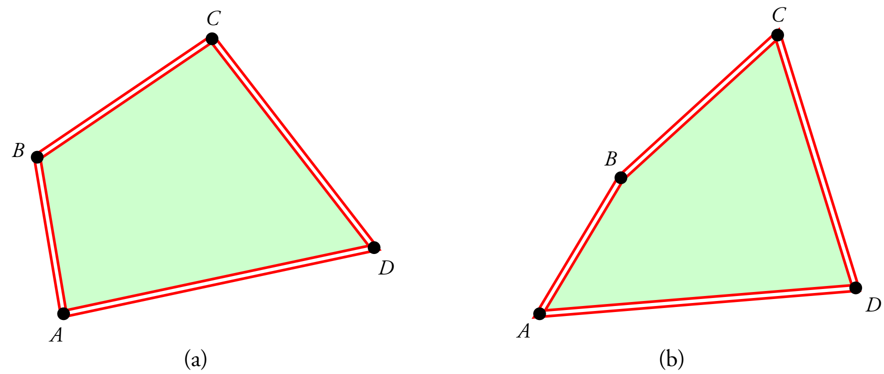


Figure 2. A pair of jointed quadrilaterals with the same side lengths

A proof using vectors. Students of classes 11 and 12 may be interested in seeing that there is a vector proof of Theorems 1 and 2; both theorems are proved at the same time.

Given a quadrilateral $ABCD$, we denote the vectors **AB**, **BC**, **CD** by **a**, **b**, **c** respectively; then $\mathbf{AD} = \mathbf{a} + \mathbf{b} + \mathbf{c}$. We now have:

$$AB^2 = \mathbf{a} \cdot \mathbf{a}, \quad BC^2 = \mathbf{b} \cdot \mathbf{b}, \quad CD^2 = \mathbf{c} \cdot \mathbf{c}, \tag{4}$$

$$\begin{aligned} AD^2 &= (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{c} + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}). \end{aligned} \tag{5}$$

Hence:

$$\begin{aligned} AD^2 + BC^2 - AB^2 - CD^2 &= 2(\mathbf{b} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}) \\ &= 2(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} + \mathbf{c}) = 2\mathbf{AC} \cdot \mathbf{BD}. \end{aligned} \tag{6}$$

Hence $AD^2 + BC^2 - AB^2 - CD^2 = 0$ if and only if $AC \perp BD$. □

Second Property: Rigidity

Theorem 2 has a pretty consequence related to the notion of *rigidity* of a polygon.

We know that a triangle is rigid: given three lengths which satisfy the triangle inequality (i.e., the largest length is less than the sum of the other two lengths), there is just one triangle having those lengths for its sides. So its shape cannot change. If we make a triangle using rods for sides, joined together at their ends using nuts and bolts, the structure is **rigid** and **stable**; it will not lose shape when subjected to pressure.

But a quadrilateral made this way is not rigid; if at all one can make a quadrilateral using four given lengths as its sides (this requires that the largest length is less than the sum of the other three lengths), one can 'push' it inwards or 'pull' it outwards and so deform its shape. There are thus infinitely many distinct quadrilaterals that share the same side lengths. Figure 2 illustrates this property.

Now let a 'jointed quadrilateral' be formed in this manner using four given rods. Then, as just noted, we can deform its shape by applying pressure at the ends of a diagonal, inward or outward. Suppose it happens that in some position the diagonals of the quadrilateral are perpendicular to each other. *Then this property is never lost, no matter how we deform the quadrilateral. The diagonals always remain perpendicular to each other.*

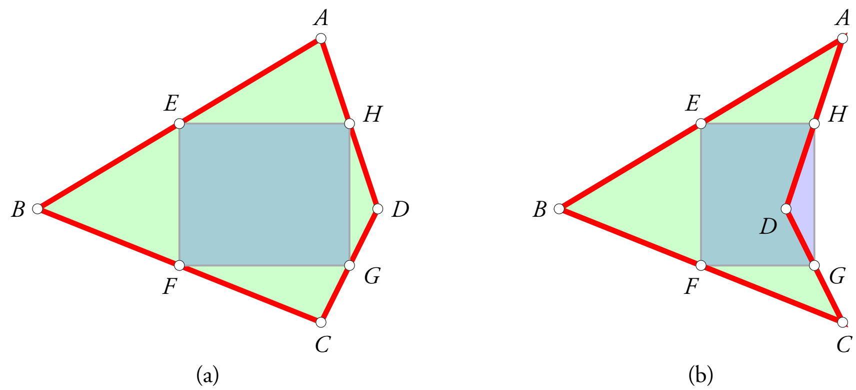


Figure 3

Third Property: Midpoint rectangle
This property will not come as a surprise to students in classes 9 and 10, who will be familiar with the midpoint theorem for triangles. It is illustrated in Figure 3.

Given a quadrilateral $ABCD$, let the midpoints of its sides AB, BC, CD, DA be E, F, G, H respectively. Then, by the Midpoint Theorem, segments EF and HG are parallel to diagonal AC, and segments FG and EH are parallel to diagonal BD. Hence the quadrilateral $EFGH$ is a parallelogram; it is called the **midpoint parallelogram** of $ABCD$. (This result by itself is known as **Varignon's Theorem**. It is true for any quadrilateral.) Hence, if the diagonals of $ABCD$ are perpendicular to each other, all the angles of $EFGH$ are right angles. This yields Theorem 3.

Theorem 3. *If the diagonals of the quadrilateral are perpendicular to each other, then its midpoint parallelogram is a rectangle.*

As earlier, the result remains true if the quadrilateral is non-convex, as in Figure 3 (b). The midpoint parallelogram may now be called the **midpoint rectangle**.

The converse of Theorem 3 is also true, namely:

Theorem 4. *If the midpoint parallelogram of a quadrilateral is a rectangle, then the diagonals of the quadrilateral are perpendicular to each other.*

We omit the proof, which is quite easy.

Fourth Property: Midpoint Circle
A rectangle is a special case of a cyclic quadrilateral. Hence, for any quadrilateral whose diagonals are perpendicular to each other, there exists a circle which passes through the midpoints of its four sides. Now a line which intersects a circle must do so again at a second point (possibly coincident with the first point of intersection, which would be a case of tangency). So the midpoint circle must intersect each of the four sidelines again, giving rise to four special points; see Figure 4. What are these points?

Figure 4 depicts the situation. The midpoint of the sides are E, F, G, H respectively, and circle ($EFGH$) intersects the four sides at points I, J, K, L. (It so happens that in this particular figure, L has coincided with H. However, this will obviously not happen in general.) Since the line segments EG and FH (which are diagonals of the rectangle $EFGH$) are diameters of the circle, it follows that I is the foot of the perpendicular from G to line AB; J is the foot of the perpendicular from H to line BC; K is the foot of the perpendicular from E to line CD; and L is the foot of the perpendicular from F to line DA. (This follows from the theorem of Thales: "the angle in a semicircle is a right angle.") So we have now identified what

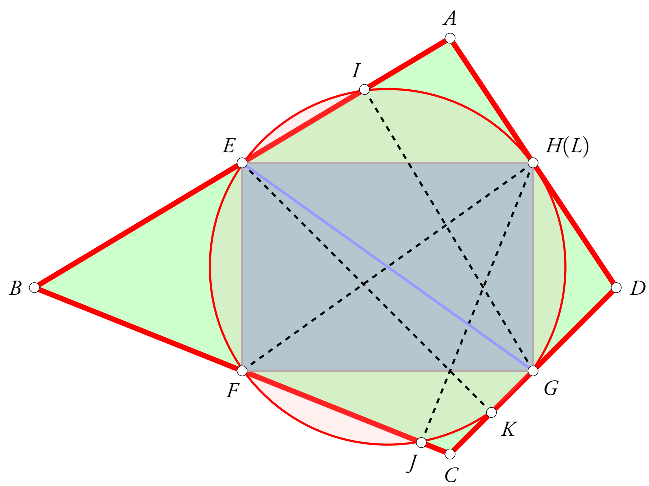


Figure 4

the four new points are: they are the feet of the perpendiculars drawn from the midpoints of the sides to the opposite sides. The circle passing through these eight points is referred to as the **eight-point circle** of the quadrilateral (in analogy with the much better-known nine point circle of a triangle).

These four perpendiculars we have drawn (from the midpoints of the sides to the opposite sides) are called the **maltitudes** of the quadrilateral. Clearly, this construction can be done for any quadrilateral. The theorem enunciated above can now be written in a stronger form:

Theorem 5. *If the diagonals of the quadrilateral are perpendicular to each other, then the midpoints of the four sides and the feet of the four maltitudes lie on a single circle.*

The converse of this statement is also true:

Theorem 6. *If the midpoints of the four sides and the feet of the four maltitudes lie on a single circle, then the diagonals of the quadrilateral are perpendicular to each other.*

Fifth Property: Brahmagupta's theorem

In this section, we study an interesting property of a cyclic quadrilateral whose diagonals are perpendicular to each other; thus, we have imposed an additional property on the quadrilateral, namely, that it is *cyclic*. The property in question was first pointed out by the Indian mathematician Brahmagupta (seventh century AD).

Figure 5 depicts the property. The cyclic quadrilateral in question is $ABCD$; its diagonals AC and BD are perpendicular to each other. The midpoints of its sides are E, F, G, H, and $EFGH$ is a rectangle. The circle through E, F, G, H intersects the four sides again at points I, J, K, L respectively. If we draw the segments EK, FL, GI, HJ respectively, we find that they all pass through the point of intersection of AC and BD. So we have six different segments passing through a common point. Not only that, but we also have $EK \perp CD$, $FL \perp DA$, $GI \perp AB$ and $HJ \perp BC$! Indeed a beautiful result. We state the result formally as a theorem (see [3]):

Theorem 7 (Brahmagupta)**.** *If a quadrilateral is both cyclic and ortho-diagonal, then the perpendicular to a side from the point of intersection of the diagonals bisects the opposite side.*

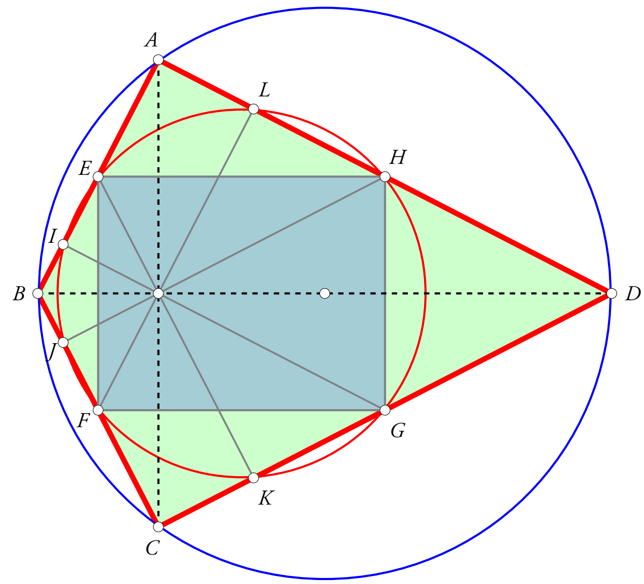


Figure 5

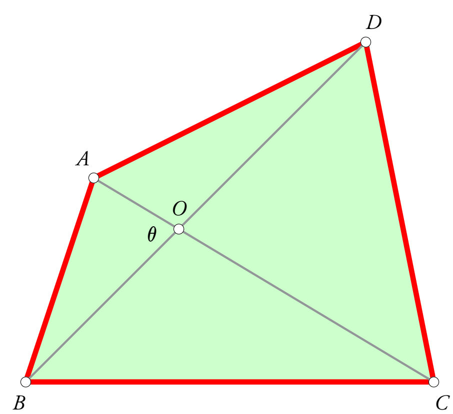


Figure 6

Sixth Property: Area

We conclude this article with a discussion concerning the **area** of a quadrilateral. We examine the following question: *If we know the four sides of a quadrilateral and the angle between its two diagonals, can we find the area of the quadrilateral?* We shall show that the answer in general is 'Yes.' But it is not an unqualified Yes!—there is an unexpected twist in the tale.

Let the quadrilateral be named $ABCD$, let the sides AB, BC, CD, DA have lengths a, b, c, d respectively, let the point of intersection of the diagonals AC and BD be O, and let it be specified that the angle between the diagonals has measure θ. We need to find a formula for the area k in terms of a, b, c, d, θ. Using the sine formula for the area of a triangle, it is easy to show that the area of the quadrilateral is

$$k = \frac{1}{2} AC \times BD \times \sin\theta. \tag{7}$$

We had shown in Section I that $AD^2 + BC^2 - AB^2 - CD^2 = 2\mathbf{AC} \cdot \mathbf{BD}$, i.e.,

$$\left|AD^2 + BC^2 - AB^2 - CD^2\right| = 2\,AC \times BD \times \left|\cos\theta\right|. \tag{8}$$

(We have used absolute value signs here as the signs do not matter any longer at this stage; we are only interested in the absolute magnitudes.) Hence:

$$AC \times BD = \frac{|AD^2 + BC^2 - AB^2 - CD^2|}{2\ |\cos\theta|}. \tag{9}$$

This yields the desired formula for the area of the quadrilateral:

$$k = \frac{|AD^2 + BC^2 - AB^2 - CD^2| \times |\tan\theta|}{4}. \tag{10}$$

At this point, we uncover something quite fascinating. Suppose the quadrilateral in question is of the type studied here; i.e., its diagonals are perpendicular to each other. Then in the above formula, we encounter an indeterminate form! For, in the numerator, when $\theta = 90^\circ$, we see the product $0 \times \infty$. Examining the situation geometrically, we realise that this is not just a numerical quirk; the area really cannot be determined!

The comments made in Section II should make this clear; we pointed out that a quadrilateral is not rigid; but that as it changes shape, the property that its diagonals are perpendicular to each other is invariant. If the property is true in one particular configuration, it always remains true. We see directly in this situation that the area can assume a whole continuum of values, implying that it is indeterminate.

The case of a kite. Figure 7 displays a particular case of this which is easy to grasp: when adjacent pairs of sides have equal length (i.e., the figure is a kite).

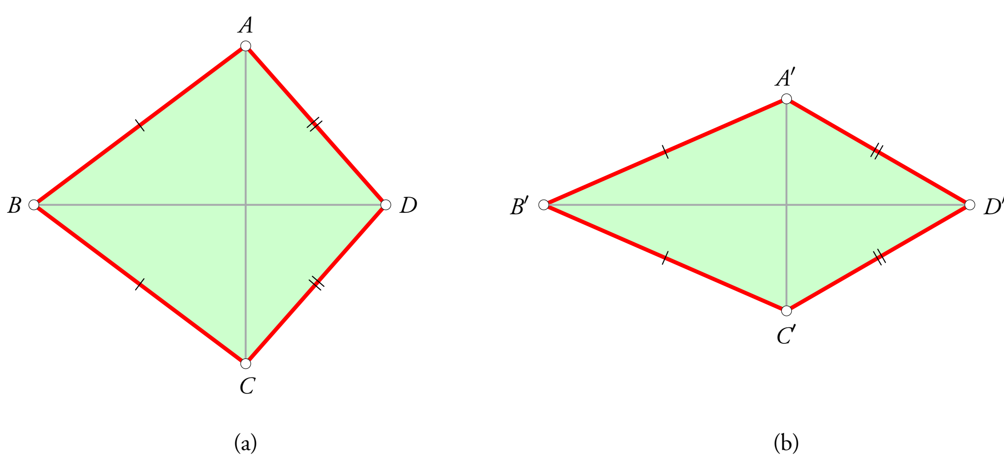


Figure 7

Figures $ABCD$ and $A'B'C'D'$ (which are kites, with $AD = CD = A'D' = C'D'$ and $AB = BC = A'B' = B'C'$) have unequal area. By further squashing the kite along the vertical diagonal, we can make its area as small as we wish. The limiting case in this direction would be that of a **degenerate quadrilateral**, i.e., one with zero area.

The case of a rhombus. As noted in [2], the phenomenon described above may be visualized still more easily in the case of a rhombus. Let the rhombus have side a, and let its angles be α and $180^\circ - \alpha$ where $0^\circ < \alpha \le 90^\circ$ (see Figure 8). Then the area of the rhombus is equal to $a^2 \sin\alpha$.

Since α can assume any value between 0° and 90°, it follows that the area of the rhombus can assume any value between 0 and a^2.

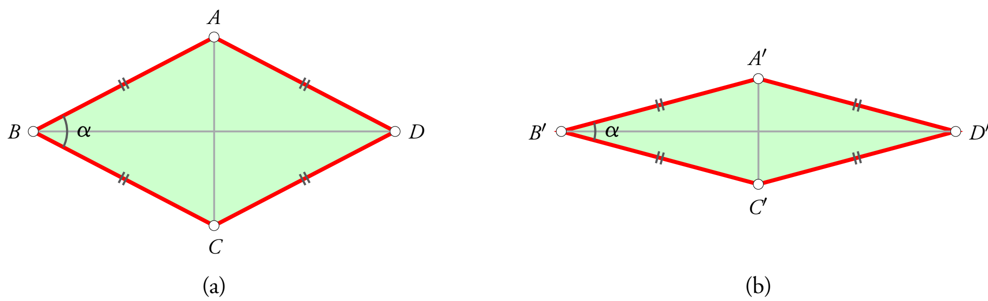


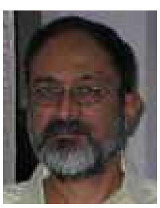
Figure 8

References

1. Martin Josefsson, *Characterizations of Orthodiagonal Quadrilaterals*. http://forumgeom.fau.edu/FG2012volume12/FG201202.pdf

2. Wikipedia, *Orthodiagonal quadrilateral*. https://en.wikipedia.org/wiki/Orthodiagonal_quadrilateral

3. Wikipedia, *Brahmagupta theorem*. https://en.wikipedia.org/wiki/Brahmagupta_theorem



SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as an editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

ClassRoom

INEQUALITIES in Algebra and Geometry Part 2

SHAILESH SHIRALI

This article is the second in the 'Inequalities' series. We prove a very important inequality, the Arithmetic Mean-Geometric Mean inequality ('AM-GM inequality') which has a vast number of applications and generalizations. We prove it using algebra as well as geometry.

The AM-GM inequality for two numbers

Let a and b be non-negative numbers. Their arithmetic mean m is the quantity $(a+b)/2$, and their geometric mean g is the quantity $\sqrt{ab}$. For example, if the numbers are 2 and 8, then the arithmetic mean is $(2+8)/2 = 5$ and the geometric mean is $\sqrt{2 \times 8} = 4$. If the numbers are 1 and 25, then the arithmetic mean is 13 and the geometric mean is 5. Observe that in both these instances, the arithmetic mean (AM) exceeds the geometric mean (GM). We shall show that this invariably happens; that is, it is invariably the case that $m \geq g$.

Theorem 1 (AM-GM inequality for two numbers)**.** *For any two non-negative numbers a and b, it is always the case that*

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Moreover, the equality sign holds if and only if $a = b$.

Proof. The most straightforward approach is to rewrite the inequality in various equivalent forms as shown below:

$$\frac{a+b}{2} \geq \sqrt{ab} \quad \text{for all } a, b \geq 0$$

$$\Longleftrightarrow \quad a + b \geq 2\sqrt{ab} \quad \text{for all } a, b \geq 0$$

$$\Longleftrightarrow \quad a - 2\sqrt{ab} + b \geq 0 \quad \text{for all } a, b \geq 0.$$

The inequality in the last line is clearly true since

$$a - 2\sqrt{ab} + b = \left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0.$$

Keywords: AM-GM inequality, harmonic mean, root-mean-square, area, perimeter, maximise, minimise

This proves the AM-GM inequality for two numbers. Observe how the inequality has reduced to the well-known fact that a squared number is non-negative.

Equality holds if and only if $\sqrt{a} - \sqrt{b} = 0$, i.e., if and only if $a = b$, as claimed. □

Alternative formulation of the same idea. The proof presented above can be recast in a different manner as follows. Let $m = (a+b)/2$ be the AM of a and b, and let $g = \sqrt{ab}$ be their GM. Then a and b are equidistant from m, and we can write

$$a = m + c, \quad b = m - c$$

for some number c which could be either positive or negative or zero (note that $c = 0$ corresponds to the case when $a = b$). This yields:

$$ab = (m+c)(m-c) = m^2 - c^2 \leq m^2, \text{since } c^2 \geq 0.$$

Since $g^2 = ab$, we get:

$$g^2 \leq m^2, \quad \text{i.e.,} \quad g \leq m,$$

since $g \geq 0$, $m \geq 0$. The equality sign will hold if and only if $c = 0$, i.e., if and only if $a = b$. It follows that the GM is less than or equal to the AM, with equality if and only if the two numbers are identical. □

Isoperimetric property of the square. In Part-1 of this article (November 2016 issue of *At Right Angles*) we proved the isoperimetric property of the square, namely: *Among all rectangles sharing the same perimeter, the square has the largest area. Among all rectangles sharing the same area, the square has the least perimeter.* Let us now show how this property follows from the AM-GM inequality.

Consider a rectangle with sides a and b. Its area is ab, and its perimeter is $2(a+b)$. Hence the side of the square with equal perimeter is $2(a+b)/4 = (a+b)/2$. Therefore the area of the square whose perimeter is the same as that of the given rectangle is equal to $(a+b)^2/4$. Now we have, by the AM-GM inequality:

$$\sqrt{ab} \leq \frac{a+b}{2}, \quad \therefore \ ab \leq \frac{(a+b)^2}{4},$$

with equality if and only if $a = b$. This proves that among all rectangles sharing the same perimeter, the square has the largest area. The proof of the second assertion ("among all rectangles sharing the same area, the square has the least perimeter") follows in exactly the same way. (Please do fill in the details.) □

Here is a nice example of a result which follows from the AM-GM inequality:

Proposition 1. *The sum of a positive number and its reciprocal cannot be less than* 2. *Moreover, the only positive number for which the sum of the number and reciprocal equals* 2 *is* 1.

Stated in symbols: if x is any positive number, then

$$x + \frac{1}{x} \geq 2,$$

with equality if and only if $x = 1$. For proof, we apply the AM-GM inequality to the two positive numbers x and $1/x$. Their geometric mean is $\sqrt{x \cdot 1/x} = 1$, hence their arithmetic mean cannot be less than 1. In other words:

$$\frac{x + 1/x}{2} \geq 1, \quad \therefore \ x + \frac{1}{x} \geq 2.$$

Moreover, equality holds if and only if $x = 1/x$, i.e., $x = 1$. (The other solution of the equation $x = 1/x$ is $x = -1$, but this does not need to be considered since x is supposed to be positive.) □

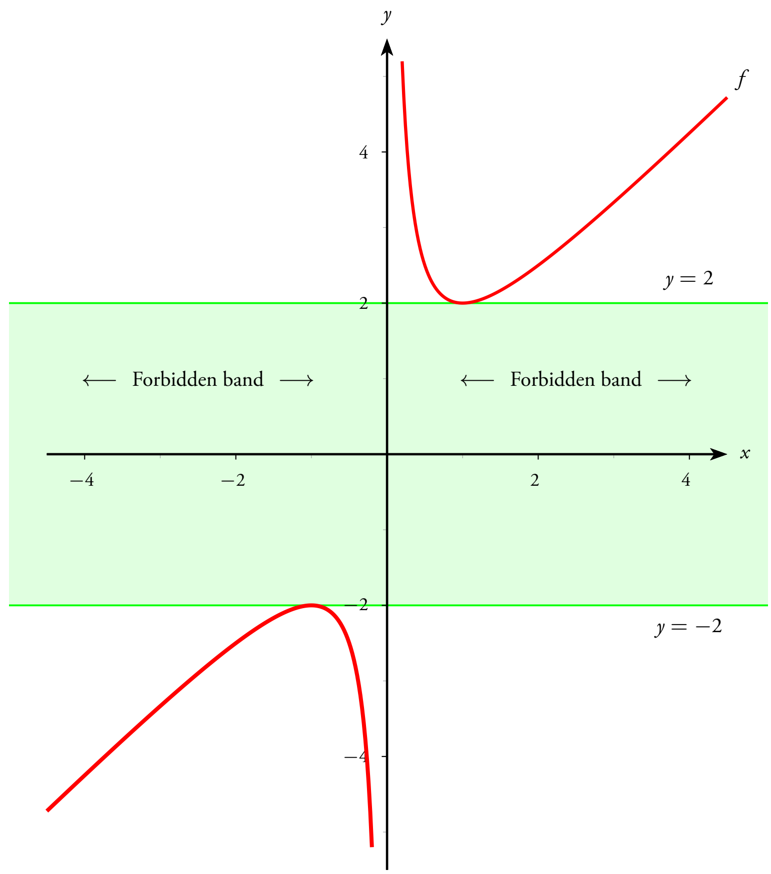


Figure 1. Graph of $f(x) = x + 1/x$

An extension, and a graphical representation. Proposition 1 can be extended in the following manner: *The sum of a nonzero real number and its reciprocal cannot lie between* -2 *and* 2. That is, if $x \neq 0$ is any real number, then it is **not** possible that

$$-2 < x + \frac{1}{x} < 2.$$

The inequality shows itself in striking form when we draw the graph of the function $f(x) = x + 1/x$ over the real numbers (see Figure 1). There is a "forbidden band" between the lines $y = -2$ and $y = 2$, within which the graph of the function never enters.

The graph itself is a hyperbola with centre $(0, 0)$ and asymptotes $x = 0$ and $y = x$.

Applications of the AM-GM inequality for two variables

We now demonstrate the utility and versatility of the two-variable AM-GM inequality.

Proposition 2. *Let a, b be positive real numbers. Then:*

$$(a + b) \cdot \left(\frac{1}{a} + \frac{1}{b}\right) \geq 4,$$

with equality if and only if $a = b$.

Proof. Multiplying out, we get:

$$\begin{aligned}(a+b)\cdot\left(\frac{1}{a}+\frac{1}{b}\right) &= 1+1+\frac{a}{b}+\frac{b}{a}\\ &= 2+\frac{a}{b}+\frac{b}{a}.\end{aligned}$$

Invoking Proposition 1, we have: $\frac{a}{b}+\frac{b}{a}\geq 2$, with equality if and only if $\frac{a}{b}=1$, i.e., if and only if $a=b$. The stated inequality thus follows. □

Proposition 3. *Let a, b, c be positive real numbers. Then:*

$$(a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9,$$

with equality if and only if $a=b=c$.

Proof. Multiplying out, we get:

$$\begin{aligned}(a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) &= 3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)\\ &\geq 3+2+2+2=9,\end{aligned}$$

with equality if and only if $\frac{a}{b}=1$ and $\frac{b}{c}=1$ and $\frac{c}{a}=1$, i.e., if and only if $a=b=c$.

Generalisation. The obvious generalisation of Propositions 2 and 3 is the following, which we state without proof:

Proposition 4. *Let $a_1, a_2, \ldots, a_n$ be n positive real numbers. Then:*

$$(a_1+a_2+\cdots+a_n)\cdot\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\geq n^2,$$

with equality if and only if $a_1=a_2=\cdots=a_n$.

Proposition 5. *Let a, b, c be positive real numbers such that $abc=1$. Then:*

$$(1+a)(1+b)(1+c)\geq 8,$$

with equality if and only if $a=b=c=1$.

Proof. The AM-GM inequality applied to the pairs $\{1,a\}$, $\{1,b\}$ and $\{1,c\}$ yields:

$$\frac{1+a}{2}\geq\sqrt{a},\quad \frac{1+b}{2}\geq\sqrt{b},\quad \frac{1+c}{2}\geq\sqrt{c}.$$

Hence we have:

$$1+a\geq 2\sqrt{a},\quad 1+b\geq 2\sqrt{b},\quad 1+c\geq 2\sqrt{c}.$$

Multiplication yields:

$$(1+a)(1+b)(1+c)\geq 8\sqrt{abc}.$$

Since $abc=1$, the desired result follows. Equality holds if and only if $a=b=c=1$.

This proposition can be extended quite easily. For example, if a, b, c, d are positive real numbers such that $abcd=1$, then $(1+a)(1+b)(1+c)(1+d)\geq 16$. Equality holds if and only if $a=b=c=d=1$. □

Proposition 6. *Let a, b, c be real numbers. Then:*

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Equality holds if and only if $a = b = c$.

Proof. This is Problem 5 from the set given in Part-1 of this article (November 2016 issue).

Let us apply the AM-GM inequality to the pairs $\{a^2, b^2\}$, $\{b^2, c^2\}$ and $\{c^2, a^2\}$. We get:

$$a^2 + b^2 \geq 2ab,$$
$$b^2 + c^2 \geq 2bc,$$
$$c^2 + a^2 \geq 2ca.$$

Adding the three inequalities we get $2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca$. On dividing by 2, we get the desired inequality.

For equality to hold, we must have equality in each of the three inequalities listed above. This obviously requires that $a = b = c$. Hence the claim. □

Remark. As noted in the November 2016 issue, in the article on Napoleon's theorem, the following assertion is true in the domain of real numbers: *If $a^2 + b^2 + c^2 = ab + bc + ca$, then $a = b = c$.* But in the domain of complex numbers, a totally different conclusion holds, namely: *If $a^2 + b^2 + c^2 = ab + bc + ca$, then the points corresponding to a, b, c are the vertices of an equilateral triangle.*

Proposition 7. *Let a, b, c be positive real numbers. Then:*

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

Equality holds if and only if $a = b = c$.

Proof. The method used to prove Proposition 5 (above) works here as well. We leave the details for you to fill in.

Geometric proof of the AM-GM inequality

We close this article by offering a geometric proof of the AM-GM inequality. It proves more than just the AM-GM; several inequalities get proved simultaneously simply by exhibiting a certain diagram. The geometric facts used are the following:

- *From a point P outside a circle ω are drawn a tangent PT to ω and a line which intersects ω at points Q and R. We now have the equality: $PT^2 = PQ \cdot PR$.* (This follows from the **intersecting chords theorem** which we have used in previous articles of this magazine. Here is its general statement: *In any circle ω, if AB and CD are two chords which intersect at a point P which may lie either inside or outside ω, then $PA \cdot PB = PC \cdot PD$.*)
- *In any triangle, the largest side is the one opposite the largest angle.* This implies in particular that in a right-angled triangle the largest side is the hypotenuse.

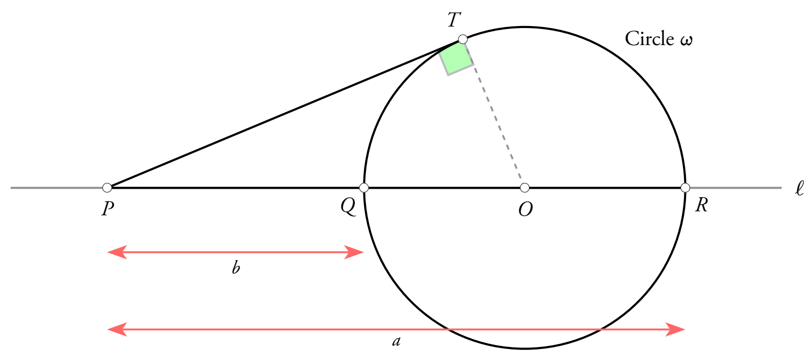


Figure 2

Given two positive numbers a, b $(a > b)$, we construct the above diagram (Figure 2).

On a line ℓ we mark three points P, Q, R (in that order) such that $PQ = b$ and $PR = a$. Next, we draw a circle ω on QR as diameter; let its centre be O. We also draw a tangent PT to the circle from P. Note that the radius of the circle is $(a - b)/2$, and also that $PO = (a + b)/2$ and $PT \perp TO$. The length of PT can be computed in two different ways:

- By using the intersecting chords theorem:

$$PT^2 = PQ \cdot PR = ab, \quad \therefore \; PT = \sqrt{ab};$$

- By using the Pythagoras theorem:

$$PT^2 = PO^2 - TO^2 = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab, \quad \therefore \; PT = \sqrt{ab}.$$

Either way we see that $PT = \sqrt{ab}$.

Now in $\triangle PTO$, which is right-angled at vertex T, the hypotenuse is PO; this is therefore the longest side of the triangle. Hence we have: $PT \leq PO$, i.e.,

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Therefore the geometric mean of a and b cannot exceed the arithmetic mean of a and b.

For equality to hold, $\triangle PTO$ must be *degenerate*, with side TO shrinking to zero length. In other words, we must have $a = b$ for the geometric mean to be equal to the arithmetic mean. □

Extensions. A small modification of the diagram yields a substantial generalisation of the AM-GM inequality.

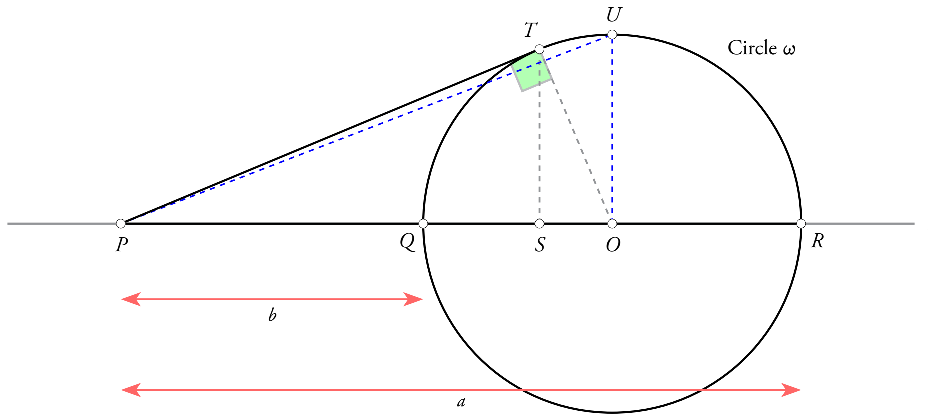


Figure 3

Figure 3 is the same as Figure 2 but with some extra line segments drawn. From T draw a perpendicular TS to ℓ. Also draw a perpendicular to ℓ at O; let it intersect the circle at U. We now have the following string of inequalities:

$$PS \le PT \le PO \le PU,$$

with equality if and only if $a = b$. We already have expressions for the lengths of PT and PO. Let us now do the same for PS and PU. We have:

$$\frac{PS}{PT} = \cos\measuredangle TPS = \frac{PT}{PO}, \quad \therefore\ PS = \frac{PT^2}{PO},$$

i.e.,

$$PS = \frac{ab}{(a+b)/2} = \frac{2ab}{a+b}.$$

We also have:

$$PU^2 = PO^2 + OU^2 = \frac{(a+b)^2}{4} + \frac{(a-b)^2}{4}, \quad \therefore\ PU = \sqrt{\frac{a^2+b^2}{2}}.$$

The quantities

$$\frac{2ab}{a+b}, \qquad \sqrt{\frac{a^2+b^2}{2}}$$

are known respectively as the **harmonic mean** (HM) and the **root mean square** (RMS) of a and b. So we have established that:

$$\text{HM}(a,b) \le \text{GM}(a,b) \le \text{AM}(a,b) \le \text{RMS}(a,b),$$

with equality if and only if $a = b$. Is it not beautiful that we have managed to get four inequalities from a single diagram?

Remark. You may wonder why the HM is a mean, and why the RMS is a mean. In other words, what is 'mean' about the HM and the RMS? Why do we call them 'means'? The underlying logic is revealed when we state the commonalities between the HM, GM and RMS. Each one makes use of a particular function-inverse function pair, (f, f^{-1}). In each case, given the positive numbers a, b, we first apply f to the numbers, thereby getting the f-numbers $f(a), f(b)$ respectively. Then we compute the AM of these two numbers; i.e., we compute the number

$$\frac{f(a)+f(b)}{2}.$$

Lastly, we apply the inverse function f^{-1} to this AM, i.e., we compute the number

$$f^{-1}\left(\frac{f(a)+f(b)}{2}\right).$$

We call this number the f-mean of a and b. Observe how this prescription applies to the HM, GM and RMS:

HM: Let $f(x) = 1/x$; i.e., f maps each number to its reciprocal. (Here, the inverse function of f is f itself; i.e., the function is its own inverse.) So from a, b we get the numbers $1/a, 1/b$; then we get the AM of these numbers, i.e.,

$$\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{a+b}{2ab}.$$

Lastly we apply f to this number; we get

$$f^{-1}\left(\frac{a+b}{2ab}\right) = \frac{2ab}{a+b}.$$

We have obtained the harmonic mean of a and b.

GM: Let $f(x) = \log_2 x$; i.e., f maps each number to its logarithm to base 2. (In this case, the inverse function of f is $f^{-1}(x) = 2^x$.) So from a, b we get the numbers $\log_2 a, \log_2 b$; then we get the AM of these numbers, i.e.,

$$\frac{1}{2}\left(\log_2 a + \log_2 b\right) = \frac{1}{2}\log_2(ab) = \log_2 \sqrt{ab}.$$

Lastly we apply f^{-1} to this number; we get

$$2^{\log_2 \sqrt{ab}} = \sqrt{ab}.$$

We have obtained the geometric mean of a and b.

RMS: Let $f(x) = x^2$; i.e., f maps each number to its square. (In this case, the inverse function of f is $f^{-1}(x) = \sqrt{x}$.) So from a, b we get the numbers a^2, b^2; then we get the AM of these numbers, i.e.,

$$\frac{a^2+b^2}{2}.$$

Lastly we apply f^{-1} to this number; we get

$$\sqrt{\frac{a^2+b^2}{2}}.$$

We have obtained the root mean square of a and b.

Problems for you to solve

We close by offering a small list of problems for you to tackle. Most of them are based on the AM-GM inequality.

(1) Let a, b, c be positive real numbers. Show that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3.$$

(2) Let a, b, c be positive real numbers. Show that:

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 3,$$

with equality if and only if $a = b = c$.

(3) Let a, b, c be positive real numbers. Show that:

$$\left(a^2 b + b^2 c + c^2 a\right) \cdot \left(ab^2 + bc^2 + ca^2\right) \geq 9a^2 b^2 c^2,$$

with equality if and only if $a = b = c$.

(4) Let a, b, c be positive real numbers. Show that:

$$a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a,$$

with equality if and only if $a = b = c$.

Solutions to problems from November 2016 issue

(1) (a) *Which is larger,* $3^{1/3}$ *or* $4^{1/4}$*?*

Raising both numbers to the 12th power, we get the numbers $3^4 = 81$ and $4^3 = 64$ respectively. Since $81 > 64$, it follows that $3^{1/3} > 4^{1/4}$.

(b) *Which is larger,* $4^{1/4}$ *or* $5^{1/5}$*?*

Since $4^5 = 1024$ and $5^4 = 625$ and $1024 > 625$, we conclude that $4^{1/4} > 5^{1/5}$.

(2) (a) *Which is larger,* $2^{1/3}$ *or* $3^{1/4}$ *?*

Since $2^4 < 3^3$, we conclude that $2^{1/3} < 3^{1/4}$.

(b) *Which is larger,* $3^{1/4}$ *or* $4^{1/5}$ *?*

Since $3^5 < 4^4$, we conclude that $3^{1/4} < 4^{1/5}$.

(3) *Which is larger:* $1.1 + \dfrac{1}{1.1}$ *or* $1.01 + \dfrac{1}{1.01}$ *?*

Let $x = 1.1$ and $y = 1.01$; then $x > y > 1$. We have now:

$$\left(x + \frac{1}{x}\right) - \left(y + \frac{1}{y}\right) = x - y + \frac{y - x}{xy} = \frac{(x - y)(xy - 1)}{xy} > 0,$$

since $xy > 1$. Hence $1.1 + \dfrac{1}{1.1} > 1.01 + \dfrac{1}{1.01}$.

(4) *If a, b are non-negative real numbers with constant sum s, what are the least and greatest values taken by $a^2 + b^2$? Express the answers in terms of s.*

We give a graphical solution. The set of pairs (a, b) of non-negative real numbers with constant sum s corresponds to a line segment PQ (Figure 4). The set of pairs (a, b) for which $a^2 + b^2$ has some fixed value k^2 corresponds to a circle centred at the origin O, with radius k. This means that we need to identify the smallest and the largest circles centred at the origin which have some contact with segment PQ. These are clearly the two circles shown. The smaller one touches PQ at its midpoint M, and the larger one passes through both P and Q.

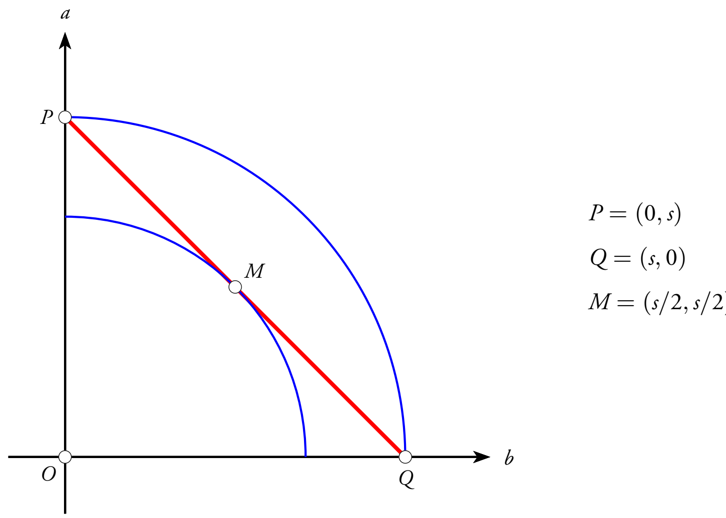
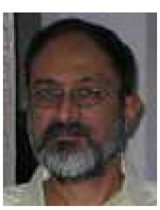


Figure 4

Therefore, given that $a + b = s$ and $a \geq 0$, $b \geq 0$, the least possible value of $a^2 + b^2$ is $s^2/2$ and the greatest possible value is s^2.



SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as an editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

ClassRoom

THREE ELEGANT PROOFS of mathematical properties

MOSHE STUPEL and AVI SIEGLER

Elegant proofs of mathematical statements present the beauty of mathematics and enhance our learning pleasure. This is particularly so for 'proofs without words' which significantly improve the visual proof capability. In this article, we present three largely visual proofs which carry a great deal of elegance. In a strict sense they are not pure proofs without words because of the mathematical expressions and formulas that appear in them. Nevertheless, they carry a lot of appeal.

Many important statements have been made about the value of aesthetics and beauty in mathematics. Various studies in mathematics education have dwelt on their significance in the teaching of the subject. Here are a few relevant quotes:

- "*Mathematics is one of the greatest cultural and intellectual achievements of human-kind and citizens should develop an appreciation and understanding of that achievement, including its aesthetics and even recreational aspects*"[1].
- "*The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics*"[2].
- "*Although it seems to us obvious that the aesthetics is relevant in mathematics education, the aesthetics also seems to be elusive when attempting to purposefully incorporate it in the mathematical experience*"[3].
- "*Mathematical beauty is the feature of the mathematical enterprise that gives mathematics a unique standing among the sciences*"[4].
- "*Mathematicians who successfully solve problems say that the experience of having done so contributes to an appreciation for the power and beauty of mathematics*"[5].

Among mathematicians and mathematics educators, many believe that aesthetics should be an integral part of the mathematics class.

In this article, we shall feature three elegant proofs which highlight the beauty of mathematics.

Keywords: Proof, proofs without words, pictures, areas, inscribed circles, right-angled triangles, hypotenuse, similar triangles

Products of four consecutive natural numbers

The product of four consecutive natural numbers plus 1 *is the square of a natural number.*

$$\underbrace{n\overbrace{(n+1)(n+2)}^{n^2+3n+2}(n+3)}_{n^2+3n}+1=(n^2+3n)(n^2+3n+2)+1=$$

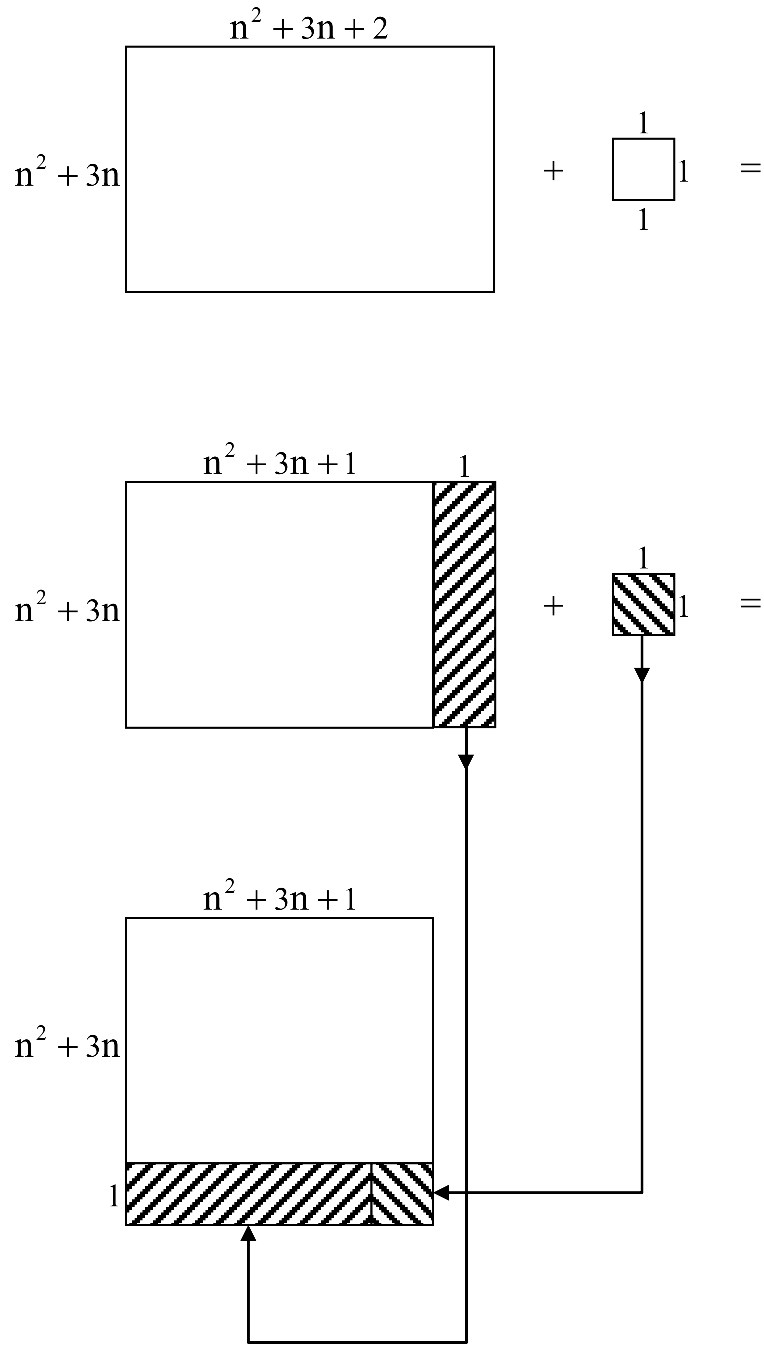


Figure 1

The mathematics of the above derivation is the following:

$$n(n+1)(n+2)(n+3)+1=(n^2+3n+1)^2.$$

Note 1: The proof is based on the following property: *The product of two natural numbers differing by* 2 *is* 1 *less than the square of the number in-between.* (For example, $5\times 7=35=6^2-1$.)

When four natural numbers are multiplied, the difference between the product of the two extreme numbers and the product of the two middle numbers is 2; the property noted above applies, and so the product of the products plus 1 is the square of a natural number. (For example, consider the four consecutive numbers 4, 5, 6, 7. The product of the numbers at the extremes is 28, and the product of the two middle numbers is 30. Observe that the difference between the two products is 2.)

Note 2: The above identity elegantly generalises to the case of an arithmetic progression: if a is the first term of the AP, and d is the common difference, then:

$$a(a+d)(a+2d)(a+3d)+d^4=(a^2+3ad+d^2)^2.$$

That is, *the product of four consecutive terms of the progression plus the fourth power of the common difference is a perfect square.*

Radii of three inscribed circles of a right angled triangle

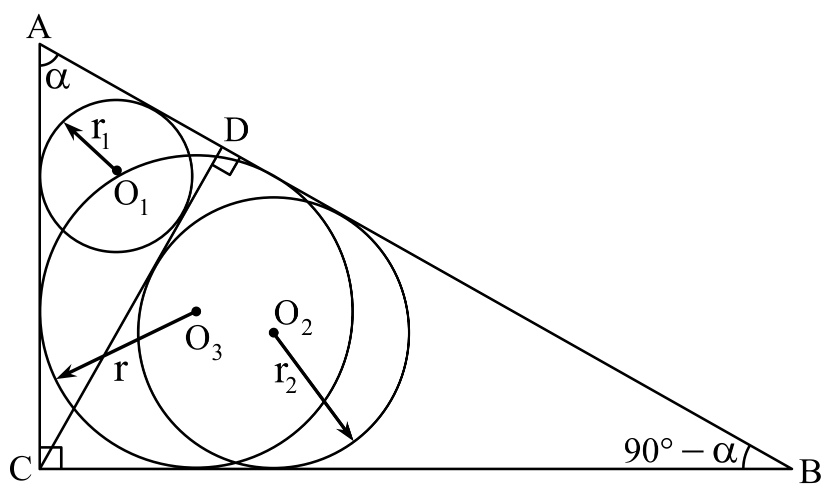


Figure 2

Let $[X]$ denote area of a figure X. Then:

$$\triangle ABC\sim\triangle ACD\sim\triangle CBD,$$

$$\therefore\ \frac{[\triangle ACD]}{[\triangle ABC]}=\frac{r_1^2}{r^2},\quad \frac{[\triangle CBD]}{[\triangle ABC]}=\frac{r_2^2}{r^2}.$$

Since $[\triangle ACD]+[\triangle CBD]=[\triangle ABC]$, we get

$$r_1^2+r_2^2=r^2.$$

Corollary. If $AC=BC$, then

$$r_1=r_2=\frac{r}{\sqrt{2}}.$$

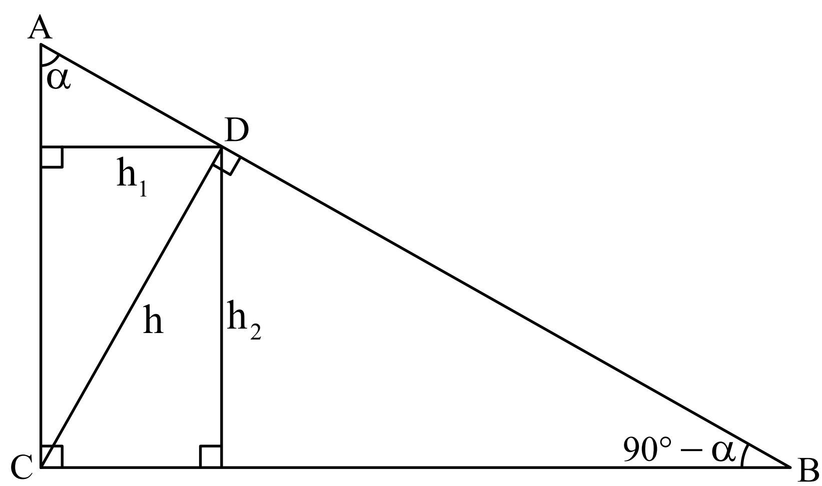
Notes.

(1) In a similar way, if R denotes radius of the circumscribed circle of a triangle; h denotes the corresponding height (altitude) of the triangle; l denotes the length of the bisector of the corresponding angle; and m denotes the length of the corresponding median of the triangle, then:

$$R_1^2 + R_2^2 = R^2,$$
$$h_1^2 + h_2^2 = h^2,$$
$$l_1^2 + l_2^2 = l^2,$$
$$m_1^2 + m_2^2 = m^2.$$

Note: For a pair of similar triangles, the proportional relationship that holds between the lengths of corresponding sides also holds between the lengths of any two corresponding segments in the two triangles; for example, the corresponding heights, the corresponding angle bisectors, the corresponding medians; and so on.

(2) The statement that $h_1^2 + h_2^2 = h^2$ is actually another way of stating the Pythagorean theorem. So this yields another proof of that famous theorem.



More relationships in a right angled triangle

Let point D lie on the hypotenuse of a right-angled triangle ABC. Then:

$$AB^2 \cdot DC^2 + AC^2 \cdot BD^2 = AD^2 \cdot BC^2$$

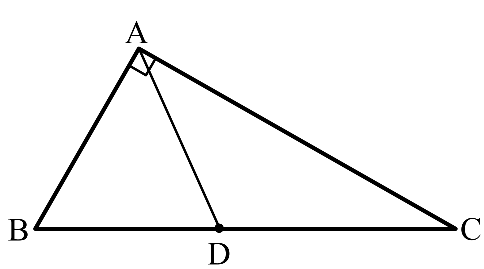


Figure 3.1

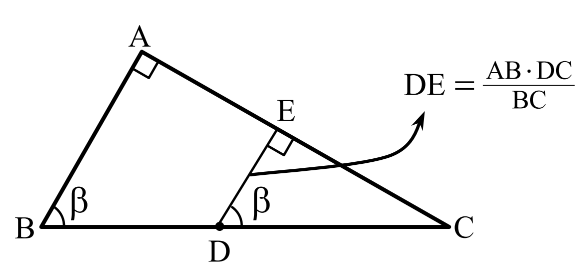
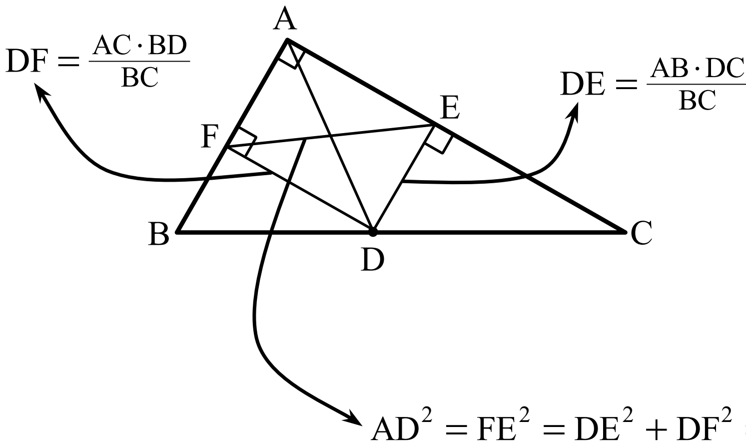


Figure 3.2

This is derived from the similarities of the triangles *ABC* and *EDC* (as shown in Figure 3.2). In the same manner, triangles *ABC* and *FBD* are similar (see Figure 3.3)



$$AD^2 = FE^2 = DE^2 + DF^2 = \frac{AB^2 \cdot DC^2 + AC^2 \cdot BD^2}{BC^2}$$

$$\Downarrow$$

$$AB^2 \cdot DC^2 + AC^2 \cdot BD^2 = AD^2 \cdot BC^2$$

Figure 3.3

If *DF* is parallel to side *AC*, then *AFDE* is a rectangle and hence its diagonals are equal (as shown in Fig. 3.3).

Note: If *D* is the midpoint of *BC* then, using the above, the theorem 'The median to the hypotenuse of a right triangle equals half the hypotenuse' can be easily proved.

References

1. National Council of Teachers of Mathematics (NCTM), (2000, p. 4). *Principles and standards for school mathematics*. Reston, VA: NCTM.
2. Hardy, G. H. (1940, p. 40). *A mathematician's apology*. Cambridge, UK: Cambridge University Press.
3. Gadanidis, G., & Hoodland, C. (2003). "The aesthetics in mathematics as story." *Canadian Journal of Science, Mathematics and Technology Education*, 3(4), 487–498.
4. Rota, G.C. (1997). "The phenomenology of mathematical beauty." *Syntbase*, 111(2), 171–182.
5. National Council of Teachers of Mathematics (NCTM). (1989, p. 79). *Curriculum and Evaluation standards for school mathematics*. Reston, VA: NCTM.
6. Sinclair, N., & Crespo, S. (2006). "What makes a good problem? An aesthetics lens." In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 129–136. Prague.

Classroom usage

These three proofs can be used in various ways in the classroom.

1. Giving just the pictures and asking students to find relationships in each of them.
2. Giving the students the pictures and relationships and asking them to explain 'why' the relationships exist; i.e., why they are true.
3. Asking them to prove the new results given at the end of each proof.
4. Asking them to suggest new relationships (example: trigonometric ratios in the case of proof II when the triangle is right-angled and isosceles).

MOSHE STUPEL is Professor, Head of Mathematics Department, a researcher, and a pre-service teacher educator in two academic colleges: the Shaanan Academic College and the Gordon Academic College. In the past, he was a high school principal. He has published and presented 40 papers on mathematics and mathematics education. Recently, his research is focused on various methods of problem solving and on variance and invariance property in dynamic geometrical environments (DGEs) in mathematics. He may be contacted at stupel@bezeqint.net.

DR. AVI SIEGLER has spent nearly four decades in mathematics education in secondary schools. In recent years, he engaged in the framework of Colleges of Teacher Education as a senior lecturer at the Shaanan College, Haifa, Israel. He has published many articles in the field of geometric constructions and on special features that exist in different geometric shapes.

ClassRoom

What's the NEXT NUMBER?

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

"What's the next number?" is an extremely popular item in so-called mental ability tests which feature in vast numbers of entrance tests in this country and elsewhere. Thus, one may be required to replace the question mark by the most suitable number in each of the sequences shown below:

(i) 8, 7, 16, 5, 32, 3, 64, 1, 128, (?)
(ii) 16, 33, 65, 131, (?), 523,
(iii) 5, 2, 17, 4, (?), 6, 47, 8, 65

These questions have been taken from the National Talent Search (First Level) & National Means-Cum-Merit Scholarship Examination, 2012.

Typically in such questions, the sequence has been generated by the paper-setter according to some pattern, and the student is expected to spot the pattern and then to find the unknown number using that pattern. Such questions make sense, given the fact that patterns are so central to mathematics as well as science, which means that the ability to spot patterns is of great value, in numerous ways. (For example, it is highly valued in a field like cryptography. Some of you may recall seeing, in the film *A Beautiful Mind*, the character played by Russell Crowe (John Nash) displaying an uncanny pattern-spotting ability.)

However, there is an interesting twist to this tale. The underlying question is this: given the initial (say) five terms of a sequence, can we say with any degree of certainty what the next term must be? Let's say we have found a nice pattern in the given initial portion; can we be sure that the sequence has been generated with just that pattern in mind?

Keywords: sequence, pattern, mental ability test, prediction, ambiguity

For example, suppose the first five terms of a sequence $\{f(n)\}_{n \geq 1}$ are

$$1,\ 2,\ 3,\ 4,\ 5$$

and we are asked to guess the sixth term. This looks like a very simple sequence. It *appears* that $f(n) = n$ for all n; if so, it implies that $f(6) = 6$. But is this the only solution possible? Or can it happen that there are multiple patterns possible which match the initial portion of the sequence (i.e., the given terms)? If multiple patterns are possible, then there is no logically justifiable way of predicting the next term. We shall show that this is the case. Indeed, *we shall show that the sixth term can be any number whatsoever!*

Here is a simple way of showing this. Let k be any non-zero number. Consider the function f given by the following expression:

$$f(n) = n + k(n-1)(n-2)(n-3)(n-4)(n-5).$$

The expression $k(n-1)(n-2)(n-3)(n-4)(n-5)$ takes the value 0 when n assumes any of the values $1, 2, 3, 4, 5$. This is true regardless of the value of k. Consequently, for $n = 1, 2, 3, 4, 5$, we have $f(n) = n$. But for $n \neq 1, 2, 3, 4, 5$ we have $f(n) \neq n$, since $k(n-1)(n-2)(n-3)(n-4)(n-5) \neq 0$. The discrepancy between $f(n)$ and n for $n \geq 6$ can be made arbitrarily large by choosing k appropriately. For example, if we take $k = 1$, we get:

$$f(6) = 126, \quad f(7) = 727, \quad f(8) = 2528, \quad \ldots;$$

and if we take $k = 2$, we get:

$$f(6) = 246, \quad f(7) = 1447, \quad f(8) = 5048, \quad \ldots.$$

These values may be compared with the predictions $f(6) = 6, f(7) = 7, f(8) = 8$ which we get if we assume that $f(n) = n$.

Or we may let the function f take the following form:

$$f(n) = n + g(n)(n-1)(n-2)(n-3)(n-4)(n-5),$$

where g is an arbitrary function. It should be clear that by tweaking the expression appropriately, we can arrange for the sixth term to be any number whatsoever.

What this tells us is that if we are given some initial terms of a sequence, there is no logical way of predicting the next term. Indeed, *the next term can be any number whatsoever*. This is so, no matter how striking the pattern which appears to govern the initial terms.

However, there is another way in which the problem can be posed. We can ask: *Given the first few terms of a sequence, what is* ***most likely*** *to be the next term?* Or: *Given the first few terms of a sequence, what is* ***most likely*** *to be the generating formula of the sequence?* It is meaningful to use words like "most likely" only if we assume that the sequence has been generated by some simple rule or pattern. Note that we are asking a conditional question now; *we are imposing a condition of simplicity on the given situation*; we are assuming that the maker of the sequence is a simple person, not inclined to act in a devious manner! With this proviso, it is reasonable to claim that if the first five numbers in a sequence are $1, 2, 3, 4, 5$, then the most likely next number is 6, and the most likely generating formula is: the n^{th} term is equal to n.

The same point of view can arise in another way. It often happens in mathematics and the sciences that it is the simplest function that fits the given data which turns out to be the most satisfactory one. (Not always, but often enough for us to wonder at it.) Or if not the simplest, then a function that is "reasonably simple." More often than not, it happens that nature opts for something simple and elegant. Many great stories can be told around this theme if one dips into the history of science.

The best-known such story is perhaps the one concerning the structure of the solar system. We narrate it very briefly here. To early man, it would have seemed self-evident that we lie at the centre of the universe, with all the heavenly bodies circling around us in geometrically perfect orbits. (Our everyday experience and observation certainly support this view.) During the Greek era, this became formalised as the ***geocentric model***. Now the strength of any model lies in its predictive ability and its ability to account for newly observed phenomena. (Indeed, this is the very purpose of having a model.) In the case of the geocentric model, observers noticed soon enough that there were discrepancies between what this simple model suggests and what is actually seen. To account for this, the model was modified by introducing *epicycles*. Over the centuries, more discrepancies began to be observed, and the response was to introduce more epicycles: more adjustments. This process iteratively continued, until finally an extremely complicated model was obtained: epicycles upon epicycles upon epicycles! And then all of a sudden, late in the 16th century, a new theory emerged—the ***heliocentric theory***. In contrast to the epicycles, it was a very much simpler model, and it explained the observed phenomena beautifully. This model has, of course, survived to the present day.

This account has been extremely brief; perhaps much too brief! We will say more in a future article on this theme, and also showcase more such episodes from the history of science. Hang on for those stories!

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Low Floor High Ceiling Tasks

SUMS OF CONSECUTIVE NATURAL NUMBERS

SWATI SIRCAR & SNEHA TITUS

We continue our Low Floor High Ceiling series in which an activity is chosen – it starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that students are pushed to their limits as they attempt their work. There is enough work for all, but as the level gets higher, fewer students are able to complete the tasks. The point however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

As we developed this series, we realised that most of our activities began with an investigation. Mathematical investigation refers to the sustained exploration of a mathematical situation. It distinguishes itself from problem solving because it is open-ended.

In mathematical investigations, students are expected to pose their own problems after initial exploration of the mathematical situation. The exploration of the situation, the formulation of problems and their solution give opportunity for the development of independent mathematical thinking, and in engaging in mathematical processes such as organizing and recording data, pattern searching, conjecturing, inferring, justifying and explaining conjectures and generalizations. It is these thinking processes which enable an individual to learn more mathematics, apply mathematics in other disciplines and in everyday situations and to solve mathematical (and non-mathematical) problems. Teaching anchored on mathematical investigation allows for students to learn about mathematics, especially the nature of mathematical activity and thinking. It also makes them realize that learning mathematics involves intuition, systematic exploration, conjecturing, reasoning, etc., and that it is not about memorizing and following existing procedures.

Keywords: numbers, consecutive, sum, pattern, digits

While we have developed questions based on the investigation we carried out, we would urge you to get your students to engage in their own investigations. The following questions may help:

- What did I observe?
- What did I know?
- What did I discover?
- What was challenging?
- Can I check this in another way?
- How many solutions?
- What happens if I change ……….?
- What else did/can I learn from this?

Here are our questions for this investigation on the Sum of Consecutive Natural Numbers. As usual, they go from Low Floor to High Ceiling:

1. Investigate the numbers from 1 to 50 to pick out numbers which can be written as the sums of a series of consecutive natural numbers. For example, 3 = 1 + 2; 12 = 3 + 4 + 5, and so on.

2. Are there numbers which can be written as sums of two or more consecutive natural numbers in more than one way, for example, can the same number be written as the sum of two or three consecutive natural numbers?

3. Find a pattern for numbers which:
   i. Can always be written as a sum of two consecutive numbers
   ii. Can always be written as a sum of three consecutive numbers

4. If we add (2n + 1) consecutive natural numbers and the sum is N, investigate factors of N.

5. If we add (2n + 2) consecutive natural numbers and the sum is N, investigate factors of N.

6. Based on your investigations, can you generalise about the kinds of numbers which can be written as a sum of two or more consecutive natural numbers?

7. Given such a number, investigate in how many ways it can be written as a sum of two or more consecutive natural numbers.

8. Which numbers cannot be written as the sum of two or more consecutive natural numbers?

Additional question: Predict the number of terms in any SCNN given N and its odd factor 2n + 1.

Sums of Consecutive Natural Numbers (SCNN)
This is a great example of how visual representation can give insight into proofs.

The number 18 can be written as the sum of two or more consecutive natural numbers (SCNN) in two ways:

$$3 + 4 + 5 + 6 \text{ and } 5 + 6 + 7$$

SCNN can be illustrated by columns of tiles as shown in Figure 1. Investigate the numbers from 1 to 50 to determine which numbers can or cannot be written as SCNN

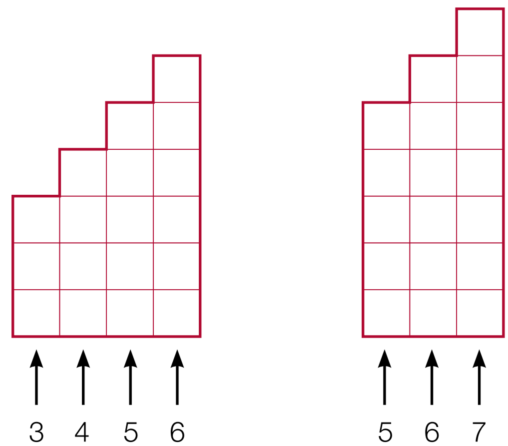


Figure 1 from ref. 1

i. When we add two consecutive numbers, we get an odd number.
n + (n+1) = 2n+1. The converse statement: any odd number ≥ 3 can be written as the sum of two consecutive natural numbers.

ii. When we add three consecutive natural numbers, we get a multiple of 3.

$$(n - 1) + n + (n + 1) = 3n$$

The converse statement: any multiple of 3 which is ≥ 6 can be written as a sum of three consecutive natural numbers. Putting (i) and (ii) together, we see that an odd multiple of 3 can be written both as the sum of two consecutive natural numbers and as the sum of three consecutive natural numbers.

More interestingly, the left hand side of the algebraic representation of (n-1) + n + (n + 1) = 3n gives us an intriguing line of thought to pursue. Note that the -1 in (n – 1) is compensated for by the +1 in (n + 1).

Now, this can be easily extended if there are 5, 7, 9 ……..in fact, for any odd number of numbers.

Algebraically, suppose there are an odd number, (say 2n + 1), of natural numbers which are added, then this can be written as (m – n) + … + (m – 2) + (m – 1) + m + (m + 1) + (m + 2) + … + (m + n); then every number that is added to the right of m is subtracted to the left of m, so that the figure can be rearranged as in Figure 2, to get a m × (2n + 1) rectangle.

Figure 2 illustrates this for n = 3 and m = 6.

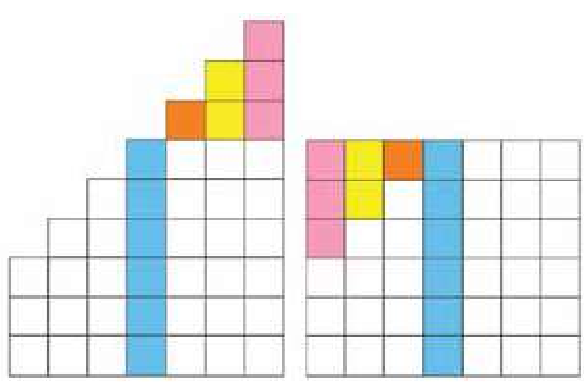


Figure 2

Apart from the neatness of this rearrangement, notice that the sum of $2n + 1$ consecutive natural numbers is divisible by the odd number $2n + 1$. For example, when we add 5 consecutive numbers say, $8 + 9 + 10 + 11 + 12$, the sum is divisible by 5.

What if we add an even number of consecutive natural numbers?

Algebraically, suppose there are an even number, (say $2n + 2$), of natural numbers which are added, then this can be written as

$$(m - n) + \ldots + (m - 2) + (m - 1) + m + (m + 1) + (m + 2) + \ldots + (m + n) + (m + n + 1)$$

Now, pairing the terms from the ends, then the next two, ... finally the middle two, we get:

$$\begin{aligned} (m - n) + (m + n + 1) &= 2m + 1 \\ (m - n + 1) + (m + n) &= 2m + 1 \\ &\vdots \\ (m - 1) + (m + 2) &= 2m + 1 \\ m + (m + 1) &= 2m + 1 \end{aligned}$$

and there are $(n + 1)$ such pairs, so that

$$(m - n) + \ldots + (m - 2) + (m - 1) + m + (m + 1) + (m + 2) + \ldots + (m + n) + (m + n + 1) = (n + 1) \times (2m + 1)$$

Figure 3 illustrates this for $n = 2$ and $m = 6$.

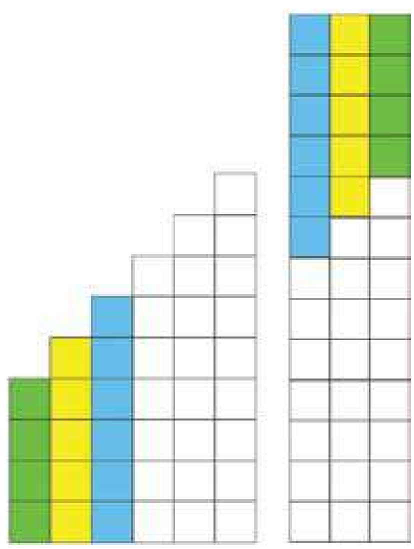


Figure 3

Notice that when $2n + 2$ consecutive natural numbers are added, the sum is divisible by $(n + 1)$. Not just that, the sum is divisible by $2m + 1$, an odd number. For example, when we add 8 consecutive numbers say, $4 + 5 + 6 + 7 + 8 + 9 + 10 + 11$, the sum is divisible by 4 and by 15 (which is $2 \times 7 + 1$). The rearranged rectangular array has a height of 15 since $4 + 11 = 5 + 10 = 6 + 9 = 7 + 8 = 15$. In other words the height of the column is $(m - n) + (m + n + 1) = 2m + 1$, an odd number.

Summing up these two findings we see that any sum of two or more consecutive natural numbers always has an odd factor.

Now, we come to a very interesting question: If a number N has an odd factor, then can it be written as a sum of two or more consecutive natural numbers?

Find any odd factor 2n + 1 of the given number N and create an array of N = m × (2n + 1) tiles. If we can rearrange the tiles to form steps then N can be written as a sum of two or more consecutive natural numbers. Further exploration with numbers yields that this can be done in mainly two ways:

a. When m > n:

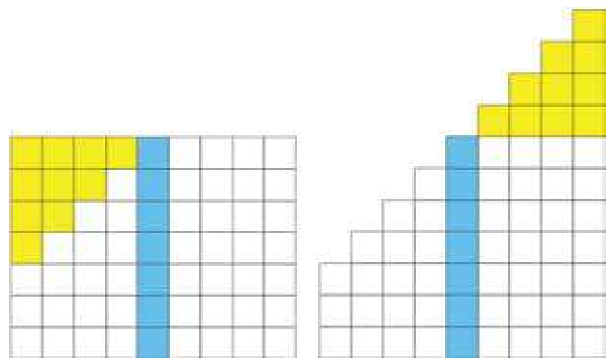


Figure 4

From the (2n + 1) columns, take the 1st n columns (of length m each) and cut off steps n, (n – 1) … 2, 1. Rotate these steps by 180° and place them over the last n columns. This generates the sum of 2n + 1 consecutive natural numbers m – n, m – n + 1, … m + n.

Figure 4 depicts the case m = 7 and n = 4.

b. When m < n:

It should be clear why we can't use the above method.

Cut off n, (n – 1) … (n – (m – 1)) steps from the left end of the array. Rotate by 180° and place them below the remaining rows to get the sum of (n – m + 1) + … + n + (n + 1) + ... + (2n + 1 – (n – m + 1)), i.e., (n – m + 1) + … + n + (n + 1) + ... + (n + m).

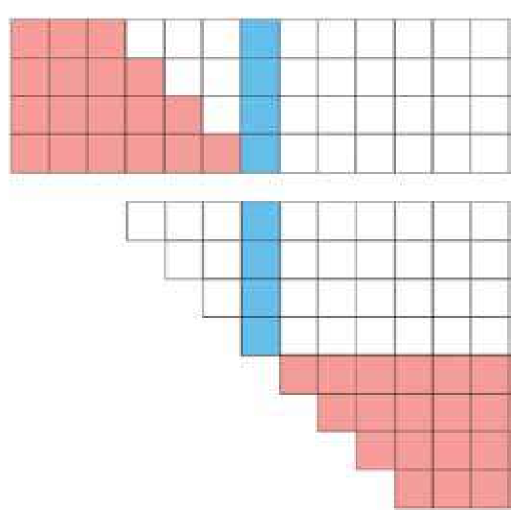


Figure 5

We leave it for the reader to explore what happens when m = n and, in particular, what kind of number N turns out to be in that case.

Given such a number N which has at least one odd factor, in how many ways can it be written as a SCNN?

If $N = 2^a \times {p_1}^{b_1} \times \ldots \times {p_k}^{b_k}$ where $p_1, \ldots, p_k$ are odd primes, then there will be $(b_1 + 1), \ldots (b_k + 1) - 1$ odd factors > 1, i.e., $(b_1 + 1) \ldots (b_k + 1) - 1$ possible SCNN

E.g. $N = 360 = 2^3 \times 3^2 \times 5$ has $(2 + 1)(1 + 1) - 1 = 5$ odd factors, viz. 3, 5, 9, 15, 45

$2n + 1 = 3 \Rightarrow m = 360 \div 3 = 120 \Rightarrow N = 119 + 120 + 121$

$2n + 1 = 5 \Rightarrow m = 360 \div 5 = 72 \Rightarrow N = 70 + 71 + 72 + 73 + 74$

$2n + 1 = 9 \Rightarrow m = 360 \div 9 = 40 \Rightarrow N = 36 + 37 + 38 + 39 + 40 + 41 + 42 + 43 + 44$

$2n + 1 = 15 \Rightarrow m = 360 \div 15 = 24 \Rightarrow N = 17 + \ldots + 24 + \ldots + 31$

$2n + 1 = 45 \Rightarrow m = 360 \div 45 = 8 \Rightarrow N = 15 + \ldots + 22 + 23 + \ldots + 30$

It will be a good idea to try different numbers and observe that that there is a unique SCNN for each odd factor.

The reader is advised to predict the number of terms in any SCNN, given N and its odd factor 2n + 1.

Claim: One can explore and see that if the number of terms is divisible by 2^n, then the SCNN will be divisible by 2^{n-1}. From the above, we see that any number which has an odd factor, can be written as a sum of two or more consecutive natural numbers. So which numbers cannot be written as the sum of two or more consecutive natural numbers?

Any number without even one odd factor must be a power of 2. So, the only numbers which cannot be written as the sum of two or more consecutive natural numbers are numbers of the form $2^n\ \forall\ n \in \mathbb{N}$.

We invite responses from our readers for the additional question: Predict the number of terms in any SCNN, given N and its odd factor 2n + 1.

Reference:

1. http://highered.mheducation.com/sites/0072533072/student_view0/math_investigations.html
2. https://us.corwin.com/sites/default/files/upm-binaries/7047_benson_ch_1.pdf
3. http://math4teaching.com/2010/03/09/what-is-mathematical-investigation/

SWATI SIRCAR is Senior Lecturer and Resource Person at the School of Continuing Education and University Resource Centre, Azim Premji University. Math is the second love of her life (first being drawing). She has a B.Stat-M.Stat from Indian Statistical Institute and a MS in math from University of Washington, Seattle. She has been doing mathematics with children and teachers for more than 5 years and is deeply interested in anything hands on, origami in particular. She may be contacted at swati.sircar@apu.edu.in.

SNEHA TITUS works as Asst. Professor in the School of Continuing Education and University Resource Centre, Azim Premji University. Sharing the beauty, logic and relevance of mathematics is her passion. Sneha mentors mathematics teachers from rural and city schools and conducts workshops in which she focusses on skill development through problem solving as well as pedagogical strategies used in teaching mathematics. She may be contacted on sneha.titus@azimpremjifoundation.org.

Divisibility by PRIMES

VINAY NAIR

In school we generally study divisibility by divisors from 2 to 12 (except 7 in some syllabi). In the case of composite divisors beyond 12, all we need to do is express the divisor as a product of coprime factors and then check divisibility by each of those factors. For example, take the case of 20; since 20 = 4 × 5 (note that 4 and 5 are coprime), it follows that a number is divisible by 20 if and only if it is divisible by 4 as well as 5. It is crucial that the factors are coprime. For example, though 10 × 2 = 20, since 10 and 2 are *not* coprime, it cannot be asserted that if a number is divisible by both 10 and 2, then it will be divisible by 20 as well. You should be able to find a counterexample to this statement.

Divisibility tests by primes such as 7, 13, 17 and 19 are not generally discussed in the school curriculum. However, in *Vedic Mathematics* (also known by the name "High Speed Mathematics"; see Box 1), techniques for testing divisibility by such primes are discussed, but without giving any proofs. In this article, proofs of these techniques are discussed.

Divisibility test for numbers ending with 9

The author of Vedic Mathematics gives divisibility tests for the divisors 19, 29, 39, ... (These are all numbers ending with 9.) These tests all have the same form. We add 1 (according to the sutra '*Ekadhikena Purvena*' which means: "by one more than the previous") to 19, 29, 39, ... Doing so, we get 20, 30, 40, ... The **osculators** for 19, 29, 39 are then 2, 3, 4, respectively (the digits left in 20, 30, 40 after ignoring the 0). The manner in which the osculator is to be used is explained below.

Keywords: Divisibility, divisor, osculator, Vedic mathematics

Notation. Any positive integer N can be written in the form $10a + b$, where b is the units place digit (so $b \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$) and a is an integer. We refer to a as 'the rest' of the number (after deleting the units digit). For example, if $N = 2356$, then $b = 6$ and $a = 235$. If d is the divisor, we denote the osculator for d by k.

Example 1. We check if 114 is divisible by 19. We follow these steps:

Step 0: This is the "initialisation step." The divisor is $d = 19$; its osculator is $k = 2$. The initial values of N, b, a are $N = 114$ (the given number), $b = 4$ (units digit), $a = 11$ ('rest of the number'). The values of N, b, a will get updated in subsequent steps, as shown below.

Step 1: Compute c using the formula c = units place digit of $N \times$ the osculator for 19, i.e., $c = bk$. We get $c = 4 \times 2 = 8$.

Step 2: Add the number c obtained in Step 1 to a, i.e., to the rest of N after ignoring the digit b considered for the operation in Step 1; so we do $a + c = 11 + 8 = 19$. *This is the updated value of N. We now update the values of a and b, using the updated value of N.*

Step 3: Check mentally if the updated value of N is divisible by 19. If yes, then we conclude that the original number is divisible by 19. If we are not sure, then repeat Steps 1–2–3 till we get a number which we know mentally is divisible (or not divisible) by 19. Please note that we must update the values of N, a, b each time we go through the cycle.

Example 2. We check whether 2356 is divisible by 19.

Step 0: Initialisation: $d = 19$; $k = 2$; $N = 2356$ (the given number), $b = 6$ (units digit), $a = 235$ ('rest of the number').

Step 1: Compute $c = bk = 6 \times 2 = 12$.

Step 2: Compute $a + c = 235 + 12 = 247$. This is now the updated value of N. So the updated values are: $N = 247$, $b = 7$, $a = 24$.

Step 3: Is 247 divisible by 19? As we are not sure, we repeat Steps 1–2.

Step 1: Compute $c = bk = 7 \times 2 = 14$.

Step 2: Compute $a + c = 24 + 14 = 38$.

About Vedic Mathematics

Vedic Mathematics is a way of doing calculations using 16 sutras (aphorisms) in Sanskrit discovered by Swami Bharati Krishna Tirtha, a saint in the Shankaracharya order, during 1911–1918 CE. For tests of divisibility, the founder uses something called *osculators*.

Note from the editor. Readers could refer to the following Wikipedia reference [1]. Its opening paragraph is the following:

> Vedic Mathematics is a book written by the Indian Hindu priest Bharati Krishna Tirthaji and first published in 1965. It contains a list of mental calculation techniques claimed to be based on the Vedas. The mental calculation system mentioned in the book is also known by the same name or as "Vedic Maths". Its characterization as "Vedic" mathematics has been criticized by academics, who have also opposed its inclusion in the Indian school curriculum.

References

i. Wikipedia, "Vedic Mathematics (book)", https://en.wikipedia.org/wiki/Vedic_Mathematics_(book)

Step 3: Is 38 divisible by 19? Yes. Hence 2356 is divisible by 19.

Example 3. We check whether 1234 is divisible by 19.

Step 0: Initialisation: $d = 19$; $k = 2$; $N = 1234$ (the given number), $b = 4$ (units digit), $a = 123$ ('rest of the number').

Step 1: Compute $c = bk = 4 \times 2 = 8$.

Step 2: Compute $a + c = 123 + 8 = 131$. Now $N = 131$, $b = 1$, $a = 13$.

Step 3: Is 131 divisible by 19? As we may not be sure not sure of this, we repeat Steps 1–2–3.

Step 1: Compute $c = bk = 1 \times 2 = 2$.

Step 2: Compute $a + c = 13 + 2 = 15$.

Step 3: Is 15 divisible by 19? No. Hence 1234 is not divisible by 19.

A doubt. One may not know when to stop the process. Steps 1–2–3 have to be continued till we come across a number which is small enough that we know directly whether it is or is not a multiple of 19. Consider the number 1121. After the first iteration, we get 114. If one knows that 114 is divisible by 19, then the process can be stopped here. If one does not know it, then the steps can be continued from 114 as in Example 1.

The process for 29, 39, 49, 59, . . . is the same; the osculators are 3, 4, 5, 6, . . . respectively.

Rationale behind the process

Consider the number 114. When we are doing $4 \times 2 = 8$ and adding it to 11, getting $11 + 8 = 19$, what we are 'actually' doing is to compute 110 + 80. But now we have:

$$110 + 80 = 114 - 4 + 80 = 114 + 76 = 190.$$

In effect, therefore, we have added 76 to the original number. The significant point here is that 76 is a multiple of 19. Since both 190 and 76 are multiples of 19, the original number 114 too must be a multiple of 19.

Instead of 114, think of a number ending with the digit 1; e.g., 171. When we multiply the units digit with the osculator 2 and add the product to the rest of the number, we are actually adding 20 and subtracting 1 (as we ignore the units digit 1). So we are actually adding $20 - 1 = 19$ to the number; and 19 is a multiple of 19. Similarly, when the units digit is another digit, say 5, the calculation is now $5 \times 2 = 10$. This means we are adding $100 - 5 = 95$, which again is a multiple of 19.

Similarly for the tests of divisibility by 29, 39, 49, ... Essentially the same logic works in each case.

Finding osculators for other prime numbers

Once the osculators are known, the process of checking divisibility remains the same. We only need to know how to find the osculator.

Consider a prime number like 7 which does not end with the digit 9. In such cases, we consider a multiple of 7 that ends with 9. The smallest such multiple is 49. We use the sutra *Ekadhikena purvena* and add 1 to it to get 50. Ignore the 0 and consider the remaining part of 50 as the osculator; we get 5. Hence 5 is the osculator for checking divisibility by 7.

If we look carefully, the process is simple. By taking the osculator as 5, we are really checking if the number is divisible by 49. Since the number is divisible by 49, it is definitely divisible by 7.

In the same way, the osculators for 13, 17, 23, . . . can be obtained; they are 4, 12, 7, respectively.

Divisibility test for numbers ending with 1

Let's take the case of divisibility by numbers like 21, 31, 41, ... Here, instead of adding 1 to the numbers (as in the case of divisibility by numbers ending with the digit 9), we subtract 1; this is the instruction given in the sutra *Ekanyunena Purvena* ("by one less than the previous"). So the numbers to be considered for divisibility by 21, 31, 41 are 20, 30, 40, respectively. As earlier, we consider the digits other than 0 to be the osculator, i.e., for 20, 30 and 40, the osculators are 2, 3 and 4, respectively. After this we follow the same process as in the case for divisibility by numbers ending with 9; but now we ***subtract*** rather than add. An example will make this clear.

Example 4. We check whether 441 is divisible by 21.

Step 0: Initialisation: $d = 21$; $k = 2$; $N = 441$ (the given number), $b = 1$ (units digit), $a = 44$ ('rest of the number').

Step 1: Compute $c = bk = 1 \times 2 = 2$.

Step 2: Compute $a - c = 44 - 2 = 42$. Now $N = 42$, $b = 2$, $a = 4$.

Step 3: Is 42 divisible by 21? Yes. Hence 441 is divisible by 21.

Rationale. The rationale is the same as earlier. When we multiply the units digit 1 by 2 and subtract from 44, we are actually doing $440 - 20 = 441 - 1 - 20 = 441 - 21$.

So we have subtracted 21 from the number. Since $441 - 21 = 420$, and 420 is a multiple of 21, the original number 441 is a multiple of 21.

If the units digit of a number happens to be 3, and if we multiply by 2 and subtract from the rest of the number, the actual process happening is $3 \times 2 = 6$. When the units digit 3 is ignored, we are subtracting 3 from the original number. When 6 is subtracted from the rest of the number, because of decimal place value system, we are actually subtracting 60. Since 3 is ignored, we are subtracting 3 as well, resulting in a total subtraction of 63. Since 63 is a multiple of 21, if the final number is a multiple of 21, the original number itself must be a multiple of 21. Likewise for any other units digit.

Remark. To find the osculator for 7, we can use the osculator of 49, as seen above. But we can also use the osculator of 21, since 21 is a multiple of 7. Thus there will be two osculators for every prime number depending on whether we use multiplication and addition or multiplication and subtraction. Given below is a table of some divisors and their two osculators.

Divisor	7	13	17	23	27	37
Multiple ending with 9	49	39	119	69	189	259
Corresponding osculator	5	4	12	7	19	26
Multiple ending with 1	21	91	51	161	81	111
Corresponding osculator	2	9	5	16	8	11

A study of the table reveals that *the sum of the two osculators of a divisor is the divisor itself*. For example, the osculators for 13 are 4 and 9, and $4 + 9 = 13$. One has the liberty to choose whichever seems more convenient. For example, in the case of 17, if we choose the osculator 12, the calculation is cumbersome. Instead, the osculator 5 makes it easier.

With this approach in Vedic Mathematics, tests of divisibility by any number can be devised.

VINAY NAIR is a co-founder of *Raising a Mathematician Foundation*. He conducts various online and offline programs in different parts of India on exploratory learning in Mathematics and ancient Indian Mathematics. He aspires to create a research mentality in the minds of school children. He can be reached at vinay@sovm.org.

Student Corner – Featuring articles written by students.

A Property of Primitive PYTHAGOREAN TRIPLES

BODHIDEEP JOARDAR

A *primitive Pythagorean triple*, or *PPT* for short, is a triple (a, b, c) of coprime positive integers satisfying the relation $a^2 + b^2 = c^2$. Some well-known PPTs are: $(3, 4, 5)$, $(5, 12, 13)$ and $(8, 15, 17)$. See Box 1 for some basic facts about PPTs.

A note on PPTs

There have been several articles in past issues of *At Right Angles* exploring PPTs and ways of generating them. Here are some features about PPTs which you need in this article (we invite you to provide proofs): If (a, b, c) is a PPT, then:

(i) c is odd;

(ii) one out of a, b is odd and the other one is even;

(iii) the even number in $\{a, b\}$ is a multiple of 4.

We agree to list the numbers in the PPT so that a is the odd number and b is the even number.

The following property is worth noting: *b is a multiple of 4.* To see why, write $b^2 = c^2 - a^2$. Note that a and c are odd, and recall that any odd square is of the form $1 \pmod 8$. This implies that b^2 is a multiple of 8 and hence that b is a multiple of 4. (If b were even but not a multiple of 4, then b^2 would be a multiple of 4 but not a multiple of 8.)

This article focuses on one particular family of PPTs, those having $b = c - 1$. For this family we have:

$$a^2 + (c-1)^2 = c^2, \quad \therefore \quad a^2 = 2c - 1,$$

so:

$$c = \frac{a^2+1}{2}, \quad b = \frac{a^2-1}{2}.$$

Keywords: Pythagorean triple, primitive, divisibility, modulus, proof

This note describes a feature of PPTs (a, b, c) in which $b = c - 1$. Here are some PPTs with this feature:

$$(3, 4, 5), \quad (5, 12, 13),$$
$$(7, 24, 25), \quad (9, 40, 41),$$
$$(11, 60, 61), \quad (13, 84, 85),$$
$$(15, 112, 113), \quad (21, 220, 221),$$
$$(33, 544, 545), \quad (35, 612, 613),$$
$$(39, 760, 761), \quad \dots .$$

Here is the property I discovered:

If (a, b, c) is a PPT with $b = c - 1$, then $a^b + b^a$ is divisible by c.

For example:

- For the PPT $(3, 4, 5)$: $3^4 + 4^3 = 145 = 5 \times 29$;
- For the PPT $(5, 12, 13)$: $5^{12} + 12^5 = 244389457 = 13 \times 18799189$.

But in the other PPTs such as $(15, 8, 17)$, $(21, 20, 29)$, $(33, 56, 65)$, $(35, 12, 37)$, $(39, 80, 89)$, etc., where $b \neq c - 1$, this property is not to be seen. Why should the property belong to just this type of PPT?

I will prove the following: if (a, b, c) is a PPT with $b = c - 1$, then $a^b + b^a$ is divisible by c.

Proof. Since $b = c - 1$ we have (see Box 1):

$$c = \frac{a^2 + 1}{2}, \quad b = \frac{a^2 - 1}{2}.$$

From $b = c - 1$ we get $b \equiv -1 \pmod{c}$, therefore

$$b^a \equiv (-1)^a \pmod{c} \equiv -1 \pmod{c},$$

since a is odd. Next, from $a^2 = 2c - 1$ we get $a^2 \equiv -1 \pmod{c}$. Raising both sides to power $b/2$ (remember that b is an even number), we get

$$a^b \equiv (-1)^{b/2} \pmod{c} \equiv 1 \pmod{c},$$

since, as per Box 1, b is a multiple of 4 (which implies that $b/2$ is an even number). Hence

$$a^b + b^a \equiv 1 - 1 \equiv 0 \pmod{c}.$$

In other words, $a^b + b^a$ is divisible by c. □

BODHIDEEP JOARDAR (born 2005) is a voracious reader of all kinds of mathematical literature. He is a student of South Point High School in Calcutta who is interested in number theory, Euclidean geometry, higher algebra, foundations of calculus and infinite series. He feels inspired by the history of mathematics and by the lives of mathematicians. His other interests are in physics, astronomy, painting, and the German language. He may be contacted at ch_kakoli@yahoo.com.

Student Corner – Featuring articles written by students.

An investigation with TRIANGLES & CIRCLES

PARTHIV DEDASANIYA

Often concepts as simple as triangles and circles can give rise to very interesting problems. One such problem was given to us by our Math Teacher in class when we were learning about incircles and incentres. When given the problem, it did not occur to me that the problem had to be solved using incentres. Instead, I thought of the problem in a completely different method and was able to solve it.

Problem. Given an arbitrary triangle *ABC*, the problem is to draw three circles with their centres at the three vertices of the triangle, in such a way that each circle touches the other two circles (i.e., is tangent to the other two circles), as in Figure 1.

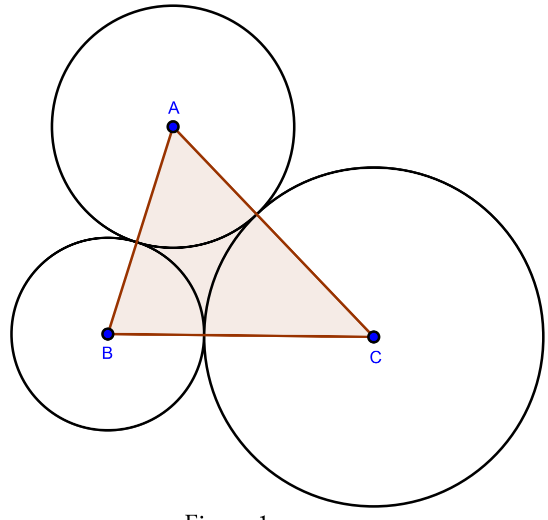


Figure 1

Keywords: Triangle, incircle, incentre

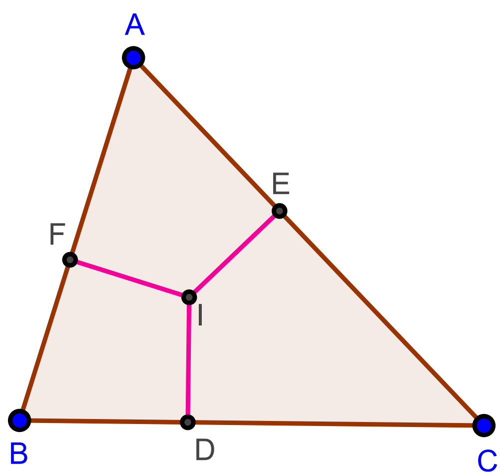


Figure 2

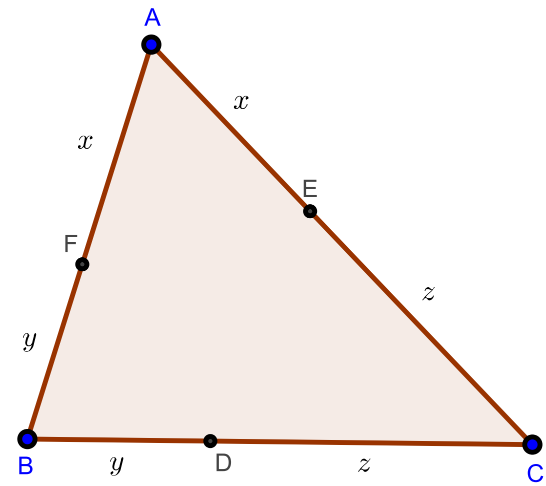


Figure 4

Solution. Here is how the problem was *expected* to be solved. First locate the incentre I of the triangle (this is the point where the internal bisectors of the three angles of the triangle meet). Next, drop perpendiculars ID, IE and IF from I to the sides of the triangle. (See Figure 2; the angle bisectors have not been shown.)

Lastly, draw circles: centred at A and passing through E and F; centred at B and passing through F and D; and centred at C and passing through D and E. These circles are the ones we seek. (See Figure 3.)

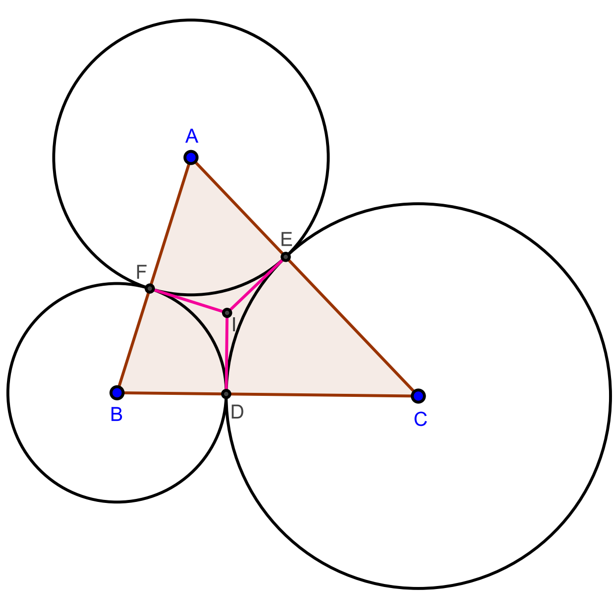


Figure 3

Explanation. As the incentre lies on the bisectors of the three angles of the triangle, it is equidistant from the three sides of the triangle; that is, the segments ID, IE, IF have equal length (Figure 2). From this it is easy to deduce, using triangle congruence, that $BD = BF$, $AF = AE$, $CE = CD$. (Draw the segments IA, IB, IC to see why.) Hence the circles can be drawn as shown in Figure 3.

My solution. I reasoned out a solution in the following way. Let the lengths of the sides BC, CA, AB be a, b, c, respectively, and let the radii of the circles centred at A, B, C be x, y, z respectively. The picture appears as shown in Figure 4. The problem is to find x, y, z respectively.

In the discussion below, I shall assume that side AB is less than side AC in length. This means that $x + y < x + z$, i.e., $y < z$. Mark off a length AU on side AC such that $AU = AB$ (see Figure 5.1). This is possible since $AB < AC$. The length of CU is $z - y$.

Next, locate a point V on side BC such that $CV = CU$ (see Figure 5.2), and locate the midpoint D of segment BV. Then BD is the radius of one of the desired circles, the one centred at B.

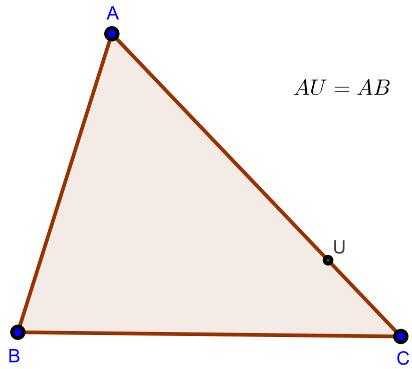


Figure 5.1

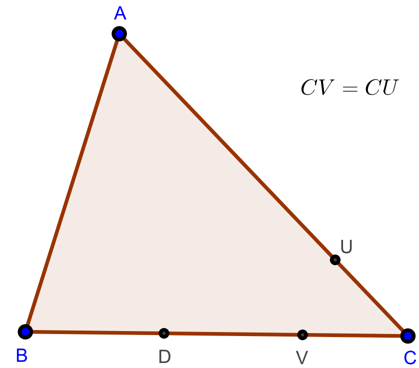


Figure 5.2

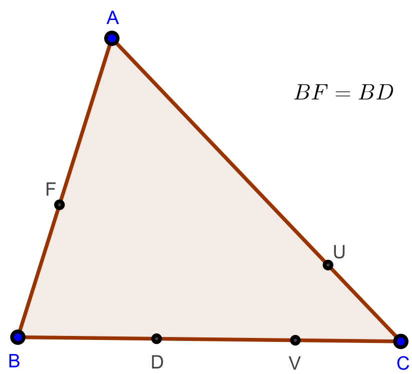


Figure 6.1

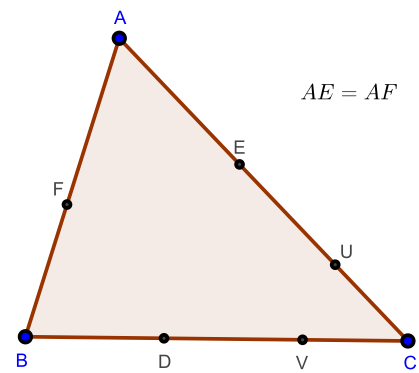
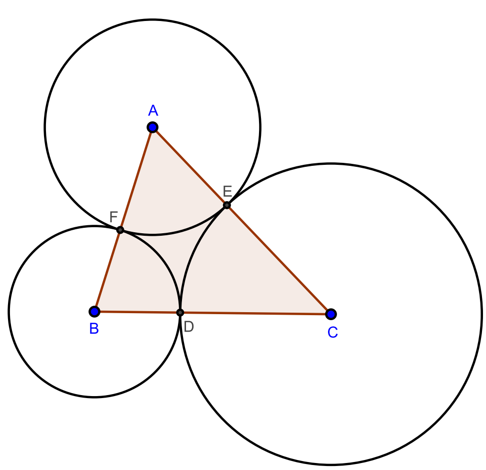


Figure 6.2

Follow this by locating a point F on side AB such that $BD = BF$ (Figure 6.1), and then by locating a point E on side AC such that $AE = AF$ (Figure 6.2). These constructions provide the points D,E,F. Once these three points have been located, the circles can be drawn as earlier (Figure 7).

Explanation. The reason this procedure works is this. Remember that $BC = a,\ CA = b,\ AB = c$. In Figures 5.1, 5.2, 6.1 and 6.2, $AU = c$, hence $CU = b - c = CV$, therefore $BV = a - (b - c) = a - b + c$. This implies that $BD = (a - b + c)/2 = s - b$, where s is the semi-perimeter of the triangle, $s = (a + b + c)/2$. From this we get:

$$AF = c - \frac{a-b+c}{2} = \frac{-a+b+c}{2} = s - a;$$



so $AE = s - a$ as well. Next, notice that

$$CE = b - \frac{-a+b+c}{2} = \frac{a+b-c}{2} = s - c,$$

and also

$$CD = a - \frac{a-b+c}{2} = \frac{a+b-c}{2} = s - c;$$

so $CE = CD$. Therefore we have:

$$BD = BF,\ AF = AE,\ CE = CD,$$

which means that the circle centred at B, with radius BD, passes through F; the circle centred at A, with radius AF, passes through E; and the circle centred at C, with radius CE, passes through D. These three circles are therefore identical with the three circles drawn in the earlier construction (Figure 3). Hence the circles can be drawn as described.

PARTHIV DEDASANIYA is currently in grade 12 (IB) of Kodaikanal International School. He has a passion for solving mathematical puzzles and problems. He has won medals in mathematical Olympiads, including a distinction in the Waterloo test. He enjoys tutoring junior students who face difficulties in mathematics. He hopes to study robotics or nano-technology in college. Parthiv may be contacted at parthiv3215@gmail.com.

ClassRoom

Property associated with the Orthocentre of a Triangle

A PROOF WITHOUT WORDS

MOSHE STUPEL
Shaanan College, Haifa Israel
stupel@bezeqint.net

VICTOR OXMAN
Western Galilee College, Acre Israel
victor.oxman@gmail.com

Theorem. In Figure 1, $HF = FK$.

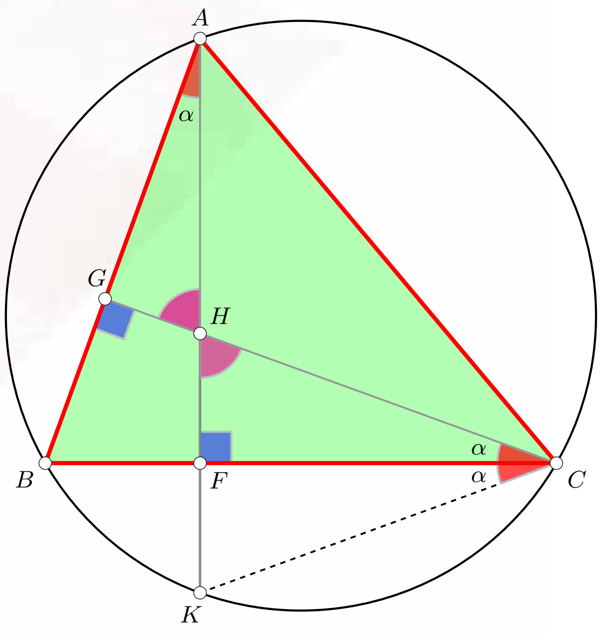


Figure 1

Proof. Please examine the figure.

Keywords: PWW (proof without words), orthocentre

An example of constructive defining: From a GOLDEN RECTANGLE to GOLDEN QUADRILATERALS and Beyond Part 2

MICHAEL DE VILLIERS

This article continues the investigation started by the author in the March 2017 issue of At Right Angles, available at: http://teachersofindia.org/en/ebook/golden-rectangle-golden-quadrilaterals-and-beyond-1 The focus of the paper is on constructively defining various golden quadrilaterals analogously to the famous golden rectangle so that they exhibit some aspects of the golden ratio phi. Constructive defining refers to the defining of new objects by modifying or extending known definitions or properties of existing objects. In the first part of the paper in De Villiers (2017), different possible definitions were proposed for the golden rectangle, golden rhombus and golden parallelogram, and they were compared in terms of their properties as well as 'visual appeal'.

In this part of the paper, we shall first look at possible definitions for a golden isosceles trapezium as well as a golden kite, and later, at a possible definition for a golden hexagon.

Constructively Defining a 'Golden Isosceles Trapezium'

How can we constructively define a 'golden' isosceles trapezium? Again, there are several possible options. It seems natural though, to first consider constructing a golden isosceles trapezium *ABCD* in two different ways from a golden parallelogram (*ABXD* in the 1st case, and *AXCD* in

Keywords: Golden ratio, golden isosceles trapezium, golden kite, golden hexagon, golden triangle, golden rectangle, golden parallelogram, golden rhombus, constructive defining

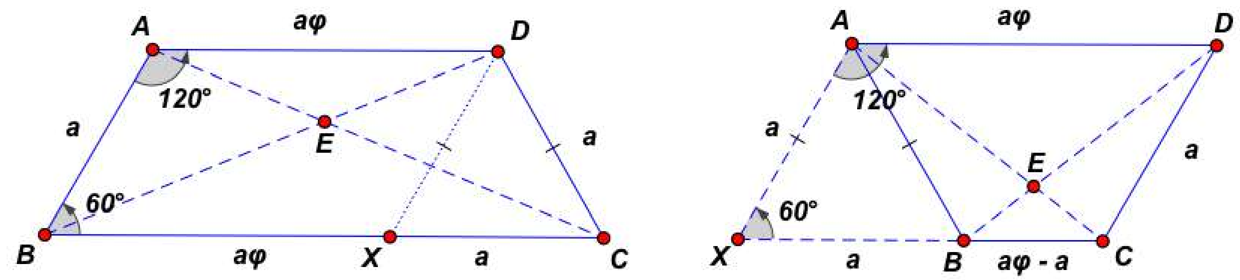


Figure 8. Constructing a golden isosceles trapezium in two ways

the 2nd case) with an acute angle of 60° as shown in Figure 8. In the first construction shown, this amounts to defining a golden isosceles trapezium as an isosceles trapezium *ABCD* with *AD // BC*, angle *ABC* = 60°, and the (shorter parallel) side *AD* and 'leg' *AB* in the golden ratio phi.

From the first construction, it follows that triangle *DXC* is equilateral, and therefore *XC* = *a*. Hence, *BC/AD* = (phi + 1)/phi, which is well known to also equal phi[1] . This result together with the similarity of isosceles triangles *AED* and *CEB*, further implies that *CE/EA* = *BE/ED* = phi. In other words, not only are the parallel sides in the golden ratio, but the diagonals also divide each other in the golden ratio. Quite nice!

In the second case, however, *AD/BC* = phi/(phi – 1) = (phi + 1) = phi squared. Also note in the second case, in contrast to the first, it is the longer parallel side *AD* that is in the golden ratio to the 'leg' *AB*, and the 'leg' *AB* is in the golden ratio with the shorter side *BC*. So the sides of this golden isosceles trapezium form a geometric progression from the shortest to the longest side, which is quite nice too!

Subdividing the golden isosceles trapezium in the first case in Figure 8, like the golden parallelogram in Figure 5, by respectively constructing a rhombus or two equilateral triangles at the ends, clearly does not produce an isosceles trapezium similar to the original. In this case the parallel sides (longest/shortest) of the obtained isosceles trapezium are also in the ratio (phi + 1), and is therefore in the shape of the second type in Figure 8. The rhombus formed by the midpoints of the sides of the first golden isosceles trapezium is also not any of the previously defined 'golden' rhombi.

With reference to the first construction, we could define the golden isosceles trapezium without any reference to the 60° angle as an isosceles trapezium *ABCD* with *AD // BC*, and *AD/AB* = phi = *BC/AD* as shown in Figure 9.

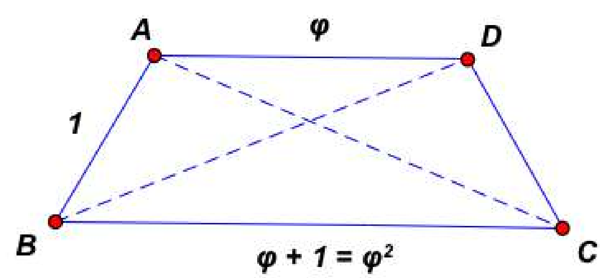


Figure 9. Alternative definition for first golden isosceles trapezium

However, this is clearly not as convenient a definition, as such a choice of definition requires again the use of the cosine formula to show that it implies that angle *ABC* = 60° (left to the reader to verify). As seen earlier, stating one of the angles and an appropriate golden ratio of sides or diagonals in the definition, substantially simplifies the deductive structure. This illustrates the important educational point that, generally, we choose our mathematical definitions for convenience and one of the criteria

[1] Keep in mind that phi is defined as the solution to the quadratic equation phi^2 – phi – 1 = 0. From this, it follows that phi = (phi + 1)/phi, phi = 1/(phi -1), phi/(phi +1) = phi + 1, or phi^2 = phi + 1.

for 'convenience' is the ease by which the other properties can be derived from it.

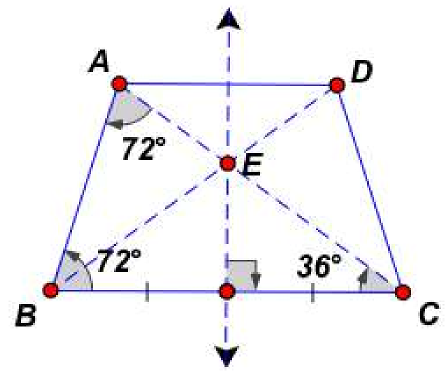


Figure 10. Third golden isosceles trapezium

Another completely different way to define and conceptualize a golden isosceles trapezium is to again use a golden triangle. As shown in Figure 10, by reflecting a golden triangle *ABC* in the perpendicular bisector of one of its 'legs' *BC*, produces a 'golden isosceles trapezium' where the ratio *BC/AB* is phi, and the acute 'base' angle is 72°. Moreover, since angle *BAD* = 108° and angle *ADB* = 36°, it follows that angle *ABD* is also 36°. Hence, *AD* = *AB* (= *DC*), and therefore the two parallel sides are also in the golden ratio, and as with the preceding case, the diagonals therefore also divide each other into the golden ratio. Of interest also, is to note that the diagonals *AC* and *DB* each respectively bisect the 'base' angles[2] at *C* and *B*.

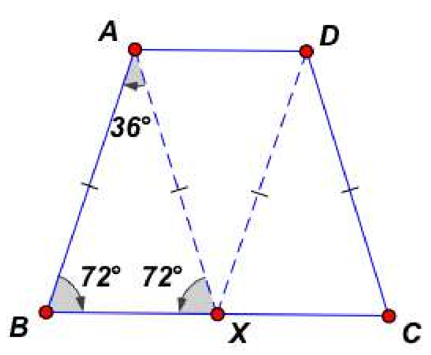


Figure 11. A fourth golden isosceles trapezium

A fourth way to define and conceptualize a golden isosceles trapezium could be to start again with a golden triangle *ABX*, but this time to translate it with the vector *BX* along its 'base' to produce a golden isosceles trapezium *ABCD* as shown in Figure 11. In this case, since the figure is made up of 3 congruent golden triangles, it follows that *AB/AD* = phi, and *BC* = 2*AD* (and therefore its diagonals also divide each other in the ratio 2 to 1).

Though one could maybe argue that the first case of a golden isosceles trapezium in Figure 8 is too 'broad' and the one in Figure 11 is too 'tall' to be visually appealing, there is little visually different between the one in Figure 10 and the second case in Figure 8. However, all four cases or types have interesting mathematical properties, and deserve to be known.

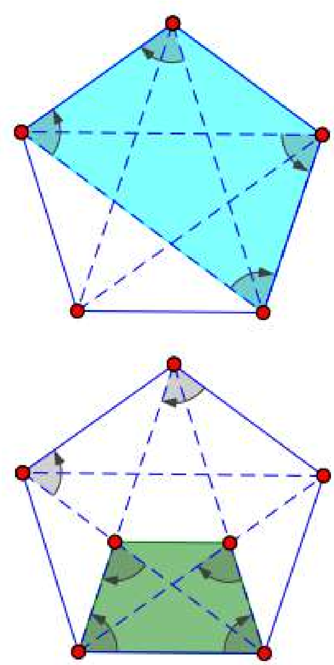


Figure 12. Golden isosceles trapezia of type 3

One more argument towards perhaps slightly favoring the golden isosceles trapezium, defined and constructed in Figure 10, might be that it appears in both the regular convex pentagon as well as the regular star pentagon as illustrated in Figure 12.

[2] In De Villiers (2009, p. 154-155; 207) a general isosceles trapezium with three adjacent sides equal is called a trilateral trapezium, and the property that a pair of adjacent, congruent angles are bisected by the diagonals is also mentioned. Also see: http://dynamicmathematicslearning.com/quad-tree-new-web.html

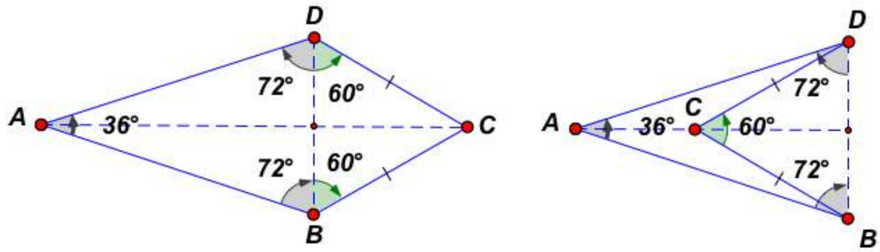


Figure 13. First case of golden kite

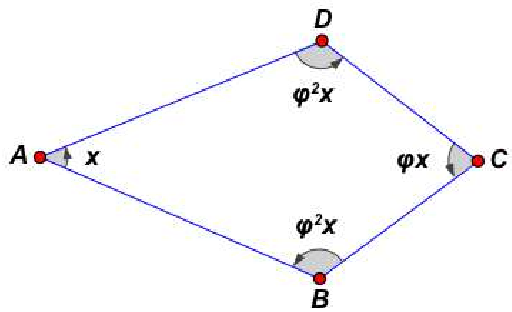


Figure 14. Second case of golden kite

Constructively Defining a 'Golden Kite'

Again there are several possible ways in which to constructively define the concept of a 'golden kite'. An easy way of constructing (and defining) one might be to again start with a golden triangle and construct an equilateral triangle on its base as shown in Figure 13. Since *AB*/*BD* = phi, it follows immediately that since *BD* = *BC* by construction, *AB* to *BC* is also in the golden ratio. Notice that the same construction applies to the concave case, but is probably not as 'visually pleasing' as the convex case.

Another way might be again to define the pairs of angles in the golden kite to be in the golden ratio as shown in Figure 14. Determining x from this geometric progression, rounded off to two decimals, gives:

$$x = \frac{360°}{1+\phi+2\phi^2} = 45.84°$$

Of special interest is that the angles at *B* and *D* work out to be precisely equal to 120°. This golden kite looks a little 'fatter' than the preceding convex one, and is therefore perhaps a little more visually pleasing. This observation, of course, also relates to the ratio of the diagonals, which in the first case is 2.40 (rounded off to 2 decimals) while in the case in Figure 14, it is 1.84 (rounded off to 2 decimals), and hence the latter is closer to the golden ratio phi.

To define a golden kite that is hopefully even more visually appealing than the previous two, I next thought of defining a 'golden kite' as shown in Figure 15, namely, as a (convex[3]) kite with both its sides and diagonals in the golden ratio.

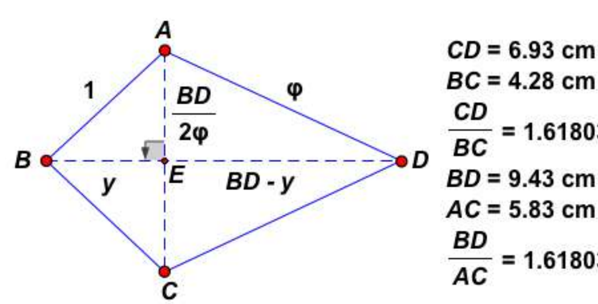


Figure 15. Third case: Golden kite with sides and diagonals in the golden ratio

[3] For the sake of brevity we shall disregard the concave case here.

Though one can drag a dynamically constructed kite in dynamic geometry with sides constructed in the golden ratio so that its diagonals are approximately also in the golden ratio, making an accurate construction required the calculation of one of the angles. At first I again tried to use the cosine rule, since it had proved effective in the case of one golden parallelogram as well as one isosceles trapezium case, but with no success. Eventually switching strategies, and assuming $AB = 1$, applying the theorem of Pythagoras to the right triangles ABE and ADE gave the following:

$$y^2 + \frac{BD^2}{4\phi^2} = 1$$

$$\frac{BD^2}{4\phi^2} + (BD - y)^2 = \phi^2.$$

Solving for y in the first equation and substituting into the second one gave the following equation in terms of BD:

$$BD^2 - 2BD\sqrt{1 - \frac{BD^2}{4\phi^2}} - \phi + 1 = 0.$$

This is a complex function involving both a quadratic function as well as a square root function of BD. To solve this equation, the easiest way as shown in Figure 16 was to use my dynamic geometry software (*Sketchpad*) to quickly graph the function and find the solution for $x = BD = 2.20$ (rounded off to 2 decimals). From there one could easily use Pythagoras to determine BE, and use the trigonometric ratios to find all the angles, giving, for example, angle $BAD = 112.28°$. So as expected, this golden kite is slightly 'fatter' and more evenly proportionate than the previous two cases. One could therefore argue that it might be visually more pleasing also.

In addition, the midpoint rectangle of the third golden kite in Figure 15, since its diagonals are in the golden ratio, is a golden rectangle.

On that note, jumping back to the previous section, this reminded me that a fifth way in which we could define a golden isosceles trapezoid might be to define it as an isosceles trapezium with its mid-segments KM and LN in the golden ratio as shown in Figure 17, since its midpoint rhombus would then be a golden rhombus (with diagonals in golden ratio). However, in general, such an isosceles trapezium is dynamic and can change shape, and we need to add a further property to fix its shape. For example, in the 1st case shown in Figure 17 we could impose the condition that BC/AD = phi, or as in the 2nd case, we can have $AB = AD = DC$ (so the base angles at B and C would respectively be bisected by the diagonals DB and AC). As can be seen, it is very difficult to visually distinguish between these two

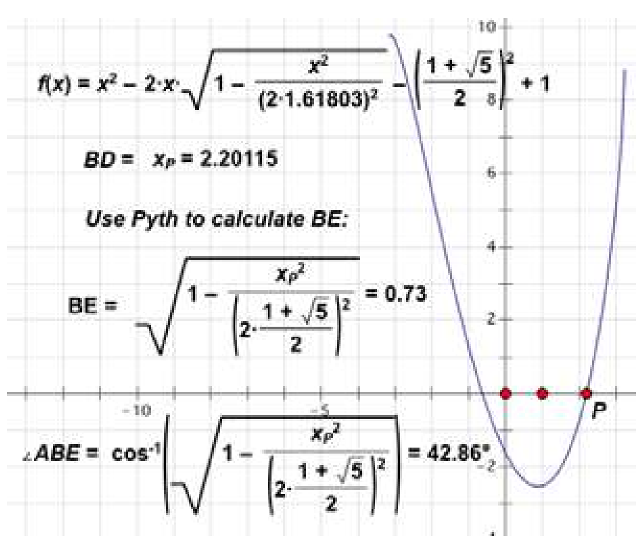


Figure 16. Solving for BD by graphing

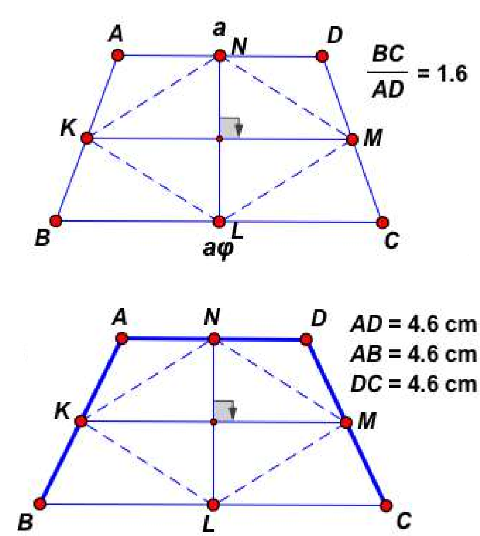


Figure 17. Fifth case: Golden isosceles trapezia via midsegments in golden ratio

cases since the angles only differ by a few degrees (as can be easily verified by calculation by the reader). Also note that for the construction in Figure 17, as we've already seen earlier, *AD* to *AB* will be in the golden ratio, if and only if, isosceles trapezium *ABCD* is a golden rectangle.

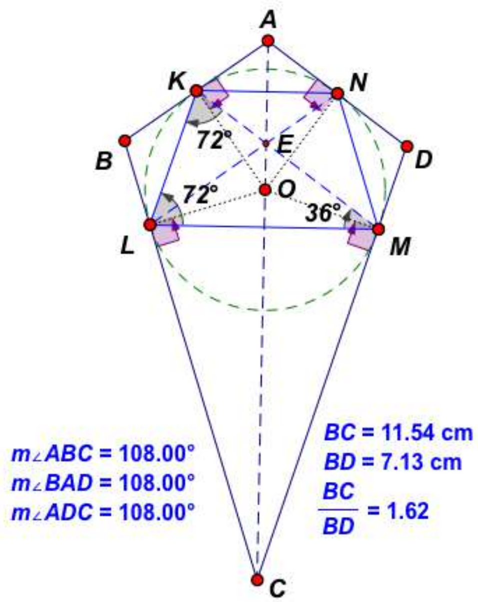


Figure 18. Constructing golden kite tangent to circumcircle of KLMN

Since all isosceles trapezia are cyclic (and all kites are circumscribed), another way to conceptualize and constructively define a 'golden kite' would be to also construct the 'dual' of each of the golden isosceles trapezia already discussed. For example, consider the golden isosceles trapezium *KLMN* defined in Figure 10, and its circumcircle as shown in Figure 18. As was the case for the golden rectangle, we can now similarly construct perpendiculars to the radii at each of the vertices to produce a corresponding dual 'golden kite' *ABCD*. It is now left to the reader to verify that *CBD* is a golden triangle (hence *BC/BD* = phi) and angle *ABC* = angle *BAD* = angle *ADC* = 108°. In addition *ABCD* has the dual property (to the angle bisection of two angles by diagonals in *KLMN*) of *K* and *N* being respective midpoints of *AB* and *AD* . The reader may also wish to verify that *AC/BD* = 1.90 (rounded off to 2 decimals), and since it is further from the golden ratio, explains the elongated, thinner shape in comparison with the golden kites in Figures 14 and 15.

Last, but not least, one can also choose to define the famous Penrose kite and dart as 'golden kites', which are illustrated in Figure 19. As can be seen, they can be obtained from a rhombus with angles of 72° and 108° by dividing the long diagonal of the rhombus in the ratio of phi so that the 'symmetrical' diagonal of the Penrose kite is in the ratio phi to the 'symmetrical' diagonal of the dart. It is left to the reader to verify that from this construction it follows that both the Penrose kite and dart have their sides in the ratio of phi. Moreover, the Penrose kites and darts can be used to tile the plane non-periodically, and the ratio of the number of kites to darts tends towards phi as the number of tiles increase (Darvas, 2007: 204). Of additional interest, is that the 'fat' rhombus formed by the Penrose kite and dart as shown in Figure 19, also non-periodically tiles with the 'thin' rhombus given earlier by the second golden rhombus in Figure 7, and the ratio of the number of 'fat' rhombi to 'thin' rhombi similarly tends towards phi as the number of tiles increase (Darvas, 2007: 202). The interested reader will find various websites on the Internet giving examples of Penrose tiles of kites and darts as well as of the mentioned rhombi.

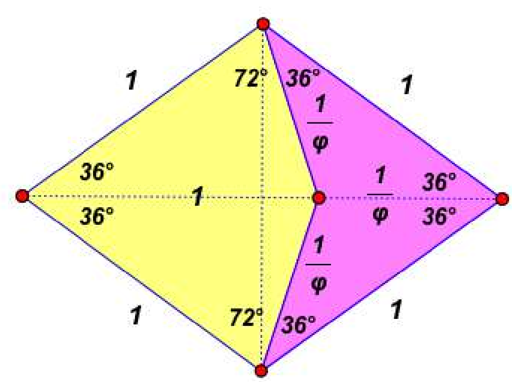


Figure 19. Penrose kite and dart

[4] In De Villiers (2009, p. 154-155; 207) , a general kite with three adjacent angles equal is called a triangular kite, and the property that a pair of adjacent, congruent sides are bisected by the tangent points of the incircle is also mentioned. The Penrose kite in Figure 19 is also an example of a triangular kite. Also see: http://dynamicmathematicslearning.com/quad-tree-new-web.html

Constructively Defining Other 'Golden Quadrilaterals'

This investigation has already become longer than I'd initially anticipated, and it is time to finish it off before I start boring the reader. Moreover, my main objective of showing constructive defining in action has hopefully been achieved by now.

However, I'd like to point out that there are several other types of quadrilaterals for which one can similarly explore ways to define 'golden quadrilaterals', e.g., cyclic quadrilaterals, circumscribed quadrilaterals, trapeziums[5], bi-centric quadrilaterals, orthodiagonal quadrilaterals, equidiagonal quadrilaterals, etc.

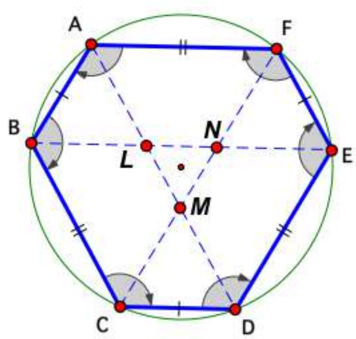


Figure 20. A golden hexagon with adjacent sides in golden ratio

Constructively Defining a 'Golden Cyclic Hexagon'

Before closing, I'd like to briefly tease the reader with considering defining hexagonal 'golden' analogues for at least some of the golden quadrilaterals discussed here. For example, the analogous equivalent of a rectangle is an equi-angled, cyclic hexagon[6] as pointed out in De Villiers (2011; 2016). Hence, one possible way to construct a hexagonal analogue for the golden rectangle is to impose the condition on an equi-angled, cyclic hexagon that all the pairs of adjacent sides as shown in Figure 19 are in the golden ratio; i.e., a 'golden (cyclic) hexagon'. It is left to the reader to verify that if *FA/AB* = phi, then *AL/LM* = phi[7], etc. In other words, the main diagonals divide each other into the golden ratio.

The observant reader would also note that *ABEF*, *ABCD* and *CDEF*, are all three golden trapezia of the type constructed and defined in the first case in Figure 8. Moreover, *ALNF*, *ABCF*, etc., are golden trapezia of the second type constructed and defined in Figure 8.

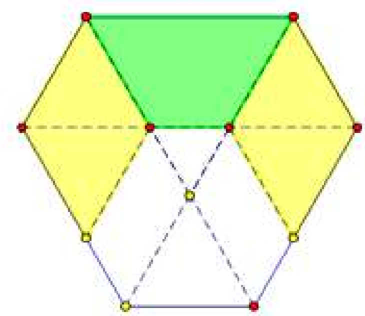


Figure 21. Cutting off two rhombi and a golden trapezium

By cutting off two rhombi and a golden isosceles trapezium as shown in Figure 21, we also obtain a similar golden cyclic hexagon. Lastly, it is also left to the reader to consider, define and investigate an analogous dual of a golden cyclic hexagon.

Concluding Remarks

Though most of the mathematical results discussed here are not novel, it is hoped that this little investigation has to some extent shown the productive process of constructive defining by illustrating how new mathematical objects can be defined and constructed from familiar definitions of known objects. In the process, several different possibilities may be explored and

[5] Olive (undated), for example, constructively defines two different, interesting types of golden trapezoids/trapeziums.

[6] This type of hexagon is also called a semi-regular angle-hexagon in the referenced papers.

[7] It was with surprised interest that in October 2016, I came upon Odom's construction at: http://demonstrations.wolfram.com/HexagonsAndTheGoldenRatio/, which is the converse of this result. With reference to the figure, Odom's construction involves extending the sides of the equilateral triangle LMN to construct three equilateral triangles ABL, CDM and EFN. If the extension is proportional to the golden ratio, then the outer vertices of these three triangles determine a (cyclic, equi-angled) hexagon with adjacent sides in the golden ratio.

compared in terms of the number of properties, ease of construction or of proof, and, in this particular case in relation to the golden ratio, perhaps also of visual appeal. Moreover, it was shown how some definitions of the same object might be more convenient than others in terms of the deductive derivation of other properties not contained in the definition.

The process of constructive defining also generally applies to the definition and exploration of different axiom systems in pure, mathematical research where quite often existing axiom systems are used as starting blocks which are then modified, adapted, generalized, etc., to create and explore new mathematical theories. So this little episode encapsulates at an elementary level some of the main research methodologies used by research mathematicians. In that sense, this investigation has hopefully also contributed a little bit to demystifying where definitions come from, and that they don't just pop out of the air into a mathematician's mind or suddenly magically appear in print in a school textbook.

In a classroom context, if a teacher were to ask students to suggest various possible definitions for golden quadrilaterals or golden hexagons of different types, it is likely that they would propose several of the examples discussed here, and perhaps even a few not explored here. Involving students in an activity like this would not only more realistically simulate actual mathematical research, but also provide students with a more personal sense of ownership over the mathematical content instead of being seen as something that is only the privilege of some select mathematically endowed individuals.

Reference:

1. Darvas, G. (2001). *Symmetry*. Basel: Birkhäuser Verlag.
2. De Villiers, M. (2009). *Some Adventures in Euclidean Geometry*. Lulu Press.
3. De Villiers, M. (2011). Equi-angled cyclic and equilateral circumscribed polygons. The Mathematical Gazette, 95(532), March, pp. 102-106. Accessed 16 October 2016 at: http://dynamicmathematiclearning.com/equi-anglecyclicpoly.pdf
4. De Villiers, M. (2016). Enrichment for the Gifted: Generalizing some Geometrical Theorems & Objects. *Learning and Teaching Mathematics*, December 2016. Accessed 16 October 2016 at: http://dynamicmathematicslearning.com/ICME13-TSG4-generalization.pdf
5. De Villiers, M. (2017). An example of constructive defining: From a golden rectangle to golden quadrilaterals & beyond: Part 1. *At Right Angles*. Volume 6, No. 1, (March), pp. 64-69.
6. Olive, J. (Undated). Construction and Investigation of Golden Trapezoids. University of Georgia, Athens; USA: Classroom Notes. Accessed 23 October 2016 at: http://math.coe.uga.edu/olive/EMAT8990FYDS08/GoldenTrapezoids.doc

MICHAEL DE VILLIERS has worked as researcher, mathematics and science teacher at institutions across the world. He retired from the University of KwaZulu-Natal at the end of January 2016. He has subsequently been appointed Professor Extraordinaire in Mathematics Education at the University of Stellenbosch. He was editor of *Pythagoras*, the research journal of the Association of Mathematics Education of South Africa, and was vice-chair of the SA Mathematics Olympiad from 1997 -2015, but is still involved. His main research interests are Geometry, Proof, Applications and Modeling, Problem Solving, and the History of Mathematics. His home page is http://dynamicmathematics-learning.com/homepage4.html. He maintains a web page for dynamic geometry sketches at http://dynamic-mathematicslearning.com/JavaGSPLinks.htm. He may be contacted on profmd@mweb.co.za.

ON Problem Posing

PRITHWIJIT DE

Problem-posing and *problem-solving* are central to mathematics. As a student one solves a plethora of problems of varying levels of difficulty to learn the applications of theories taught in the mathematics curriculum. But rarely is one shown how problems are made. The importance of problem-posing is not emphasized as a part of learning mathematics. In this article, we show how new problems may be created from simple mathematical statements at the secondary school level.

We begin with a simple problem.

Problem. Let a, b, c be three positive real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}. \tag{1}$$

This is known as *Nesbit's inequality*.

Proof. There are several proofs of this statement. One of them uses the arithmetic mean-harmonic mean (AM-HM) inequality (see Box 1). If we call the algebraic expression on the left hand side P, then by adding 1 to each term we get:

$$\begin{aligned} P+3 &= \left(1+\frac{a}{b+c}\right)+\left(1+\frac{b}{c+a}\right)+\left(1+\frac{c}{a+b}\right) \\ &= (a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right). \end{aligned}$$

Next, by the AM-HM inequality applied to the three numbers $(b+c)/2$, $(c+a)/2$ and $(a+b)/2$, we have

$$\frac{\frac{b+c}{2}+\frac{c+a}{2}+\frac{a+b}{2}}{3}\geq\frac{3}{2\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)},$$

$$\therefore\ \frac{a+b+c}{3}\geq\frac{3}{2\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)},$$

$$\therefore\ (a+b+c)\cdot\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq\frac{9}{2}.$$

Keywords: problem-posing, problem-solving, Nesbit's inequality, arithmetic mean-harmonic mean inequality

Thus $P + 3 \geq \dfrac{9}{2}$ and the desired result follows. It is easy to see that equality holds if and only if $a = b = c$.

The AM-GM-HM inequality

The AM-GM inequality states this: Given any collection of positive numbers, their arithmetic mean is never less than their geometric mean. Moreover, the two means are equal in precisely one situation: the given numbers are all identically equal.

In symbols: Let $a_1, a_2, \ldots, a_n$ be n positive numbers. Their arithmetic mean (AM) and their geometric mean (GM) are defined to be the following:

$$\text{AM} = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

$$\text{GM} = (a_1 a_2 \cdots a_n)^{1/n}.$$

Then we have:

$$\text{AM} \geq \text{GM}.$$

Equality holds in this relation if and only if $a_1 = a_2 = \cdots = a_n$.

The AM-GM inequality may be strengthened to include the harmonic mean (HM). The harmonic mean of n positive numbers $a_1, a_2, \ldots, a_n$ is defined to be:

$$\text{HM} = \frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n}.$$

The AM-GM-HM inequality states the following:

$$\text{AM} \geq \text{GM} \geq \text{HM}.$$

Moreover, the equality sign holds if and only if $a_1 = a_2 = \cdots = a_n$.

Numerical example. Consider the four numbers $8, 9, 16, 18$. Then:

$$\text{AM} = \frac{8 + 9 + 16 + 18}{4} = \frac{51}{4} = 12.75,$$

$$\text{GM} = (8 \times 9 \times 16 \times 18)^{1/4} = 12,$$

$$\text{HM} = \frac{4}{1/8 + 1/9 + 1/16 + 1/18} = \frac{192}{17} \approx 11.29.$$

Observe that AM $>$ GM $>$ HM.

The AM-GM-HM inequality is a tremendously useful inequality. It comes of use in a vast number of situations.

More often than not one leaves a problem as soon as a solution is found and does not care to see if there is more to it than meets the eye. But we will explore this problem and create more problems from it by asking more questions and answering them as they come.

It is quite natural to ask whether P is *smaller* than a particular number under the given conditions. To be more precise, we rephrase this as:

Question. Does there exist a positive real number k such that $P < k$ for all positive real numbers a, b, c? Here as earlier,

$$P = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

Analysis. How does one tackle such a question? One way is to try with small values of k and see if the condition is satisfied. We want a *friendly* value of k greater than $\frac{3}{2}$. Let us try $k = 2$. Now we need to figure out whether $P < 2$ for all positive real numbers a, b and c. The easiest way to settle this is to look at any one term of the expression, say, for instance, the first term $\frac{a}{b+c}$, on the left hand side, and figure out if it can be made large compared to 2. Indeed it can be made equal to 2 by choosing $b = c = 1$ and $a = 6$. Note that this is not the only way. There are numerous choices of a, b and c for which $\frac{a}{b+c} > 2$. Thus we have infinitely many possible choices of a, b and c for which $P > 2$. There is nothing so special about the number 2. If we replace 2 by any positive real number k, then also $\frac{a}{b+c} > k$ for infinitely many positive real numbers a, b and c (one choice is $a = 2k + 2$, $b = c = 1$; it yields $\frac{a}{b+c} = k + 1$). Thus given any positive real number k, we can choose a, b, and c in such a way that $P > k$. This leads us to conclude that there does not exist any positive real number k such that $P < k$ for all positive a, b, c.

Now one may ask under what additional conditions on a, b and c will there exist a positive real number k such that $P < k$ for all positive real numbers a, b and c? What if a, b and c are restricted to assume values over some finite interval, say $(0, 1]$? That is, $0 < a \le 1, 0 < b \le 1, 0 < c \le 1$. Even in this case, P can be made larger than any given positive real number k. Because the fraction $\frac{a}{b+c}$ is unaltered if a, b, and c are replaced by ta, tb, and tc where t is a positive real number such that $ta \le 1$, $tb \le 1$ and $tc \le 1$. But if we demand that a, b, and c satisfy the following:

$$b + c > a, \qquad c + a > b, \qquad a + b > c, \tag{2}$$

then indeed we have $P < 3$. In other words if a, b, and c are the side-lengths of a triangle, then we can find a positive real number $k\,(= 3)$ such that $P < k$. But we can do better. We can make P smaller than 2. How? To see this assume that $c = \max(a, b, c)$. Then observe that

$$\frac{a}{b+c} \le \frac{a}{a+b}, \qquad \frac{b}{c+a} \le \frac{b}{a+b}, \qquad \frac{a}{b+c} < 1, \tag{3}$$

which leads to

$$P = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2. \tag{4}$$

Can P ever equal 2? Perhaps the reader may like to ponder over this.

There is another way of proving $P < 2$ by using a very elementary fact about fractions. If x and y are positive real numbers such that $x < y$ then for any positive real number t,

$$\frac{x}{y} < \frac{x+t}{y+t}. \tag{5}$$

The proof is obvious; just cross-multiply and rearrange terms. By virtue of this and the triangle inequality we have

$$\frac{a}{b+c} < \frac{a+a}{a+b+c} = \frac{2a}{a+b+c}. \tag{6}$$

Thus

$$P < \frac{2(a+b+c)}{a+b+c} = 2. \tag{7}$$

Generalization

The next step is to see if we can generalize the results obtained above, to more than 3 variables. The new problem before us is the following.

Problem. Let $n \geq 4$ be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers. Let

$$Q = \frac{a_1}{a_2 + a_3 + \cdots + a_n} + \frac{a_2}{a_1 + a_3 + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots a_{n-1}}. \tag{8}$$

What is the minimum value of Q?

Note the similarity in form between P and Q. For every term in both P and Q, the numerator and the denominator add up to the same quantity ($a + b + c$ for P and $a_1 + a_2 + \cdots + a_n$ for Q). This suggests using the same approach for Q as we did for P. Let $s = a_1 + a_2 + \cdots + a_n$. Then

$$Q + n = s\left(\frac{1}{s - a_1} + \frac{1}{s - a_2} + \cdots + \frac{1}{s - a_n}\right), \tag{9}$$

and by appealing to the AM-HM inequality we obtain:

$$\frac{\frac{1}{s - a_1} + \frac{1}{s - a_2} + \cdots + \frac{1}{s - a_n}}{n} \geq \frac{n}{ns - s} = \frac{n}{(n-1)s}. \tag{10}$$

Therefore:

$$Q + n \geq \frac{n^2}{n-1}, \tag{11}$$

which yields $Q \geq \frac{n}{n-1}$. We readily observe that $Q = \frac{n}{n-1}$ if $a_1 = a_2 = \cdots = a_n$. Therefore the minimum value of Q is $\frac{n}{n-1}$.

As in the case of P, the maximum value of Q does not exist unless some constraints are placed on $a_1, a_2, \ldots, a_n$. Let us mimic the 3-variable case and demand that

$$s - a_i > a_i \tag{12}$$

for $i = 1, 2, \ldots, n$. Then the conclusion that $Q < n$ is immediate. But what is amazing and perhaps less obvious is that even in this case we can show that $Q < 2$. We use the result on fractions stated earlier. Thus:

$$\frac{a_i}{s - a_i} < \frac{a_i + a_i}{s - a_i + a_i} = \frac{2a_i}{s} \tag{13}$$

and therefore

$$Q < \frac{2(a_1 + a_2 + \cdots + a_n)}{s} = 2. \tag{14}$$

Once again the reader may try to probe if at all Q ever attains the value 2.

Thus we see that starting with a very simple and known result in algebra we could come up with different problems either by way of changing the assumptions or through simple generalizations. Heuristics too played a role in ascertaining whether an algebraic expression would admit an upper bound.

It is not possible to teach a student how to solve each and every problem or how to pose a new one, but perhaps it is possible to plant in him or her the seeds of an inquiry-based approach towards problem-solving or problem-posing.

PRITHWIJIT DE is a member of the Mathematical Olympiad Cell at Homi Bhabha Centre for Science Education (HBCSE), TIFR. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. His other interests include puzzles, cricket, reading and music. He may be contacted at de.prithwijit@gmail.com.

Problem Corner

Problems for the MIDDLE SCHOOL

Problem Editor: SNEHA TITUS

Our focus this time is a step forward from parity (our theme for the last issue). In this issue, our problems are all based on multiples and factors; I'm sure you'll enjoy verifying and proving these number facts, and playing these games which improve your understanding of factorization and the divisibility rules.

The following basic rule will help you:

If the prime factorization of a number N is given by

$N = p_1^{b_1} \times \ldots \times p_k^{b_k}$, then the number has $(b_1 + 1)(b_2 + 1)(b_3 + 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b_k + 1)$ factors.

For example, $75 = 3 \times 5^2$ has $(1 + 1)(2 + 1) = 6$ factors, viz., $1, 3, 5, 15, 25, 75$.

This is pretty easy to reason out: the factors can have no 3s or one 3, no 5s or one 5 or two 5s. So there are 2 ways in which 3 can be a factor and 3 ways in which 5 can be a factor and so $3 \times 2 = 6$ possible factors.

Problem VI-2-M.1 **http://nrich.maths.org/4989**

A certain number has exactly eight factors, including 1 *and itself. Two of its factors are* 21 *and* 35. *What is the number?*

Keywords: Multiples, factors, divisibility, factorization, primes, place value, number of factors

Problem VI-2-M.2 From http://nrich.maths.org/480

How can I find a number with exactly 14 *factors? How can I find the smallest such number?*

How can I find a number with exactly 18 *factors? How can I find the smallest such number?*

Which numbers have an odd number of factors?

Extension: *What is the smallest number with exactly* 100 *factors?*

Which number less than 1000 *has the most factors?*

Problem VI-2-M.3 From http://nrich.maths.org/524

Choose any 3 *digits and make a* 6 *digit number by repeating the* 3 *digits in the same order (e.g.* 594594*).*

Whatever digits you choose, the number will always be divisible by 7 *and by* 11 *and by* 13*, without a remainder.*

Can you explain why?

Problem VI-2-M.4

Faded documents from Adventures in Problem Solving by Shailesh Shirali

In this type of problem, some digits have 'faded away' and restoration is based on the digits that remain legible. It is usual to place an 'x' in place of the unreadable digits.

The Lonely '8'. A division problem with a solitary '8' . . .. see Figure 1. Can you find the unreadable digits?

				x	x	8	x	x
x	x	x	x	x	x	x	x	x
		x	x	x				
					x	x		
					x	x		
						x	x	x
						x	x	x
								0

Figure 1

Solutions

Problem VI-2-M.1 http://nrich.maths.org/4989

A certain number has exactly eight factors including 1 *and itself. Two of its factors are* 21 *and* 35*. What is the number?*

We use the given property, i.e., if the prime factorization of a number N is given by

$N = p_1^{b_1} \times \ldots \times p_k^{b_k}$, then the number has $(b_1 + 1)(b_2 + 1)(b_3 + 1) \ldots\ldots\ldots\ldots (b_k + 1)$ factors.

If the number has 8 factors then

$(b_1 + 1)(b_2 + 1)(b_3 + 1) \ldots\ldots\ldots\ldots (b_k + 1) = 8 = 1 \times 8 = 2 \times 2 \times 2 = 4 \times 2.$

We know that this unknown number (call it N) has at least 3 distinct primes factors because 21 and 35 are both its factors and $21 = 3 \times 7$ and $35 = 5 \times 7$; so 3, 5 and 7 are definitely factors of the number. So the number of factors is given by $2 \times 2 \times 2$. (This also shows that the number has exactly three distinct prime factors.)

$(b_1+1)(b_2+1)(b_3+1) = 2 \times 2 \times 2$ implies that $b_1 = b_2 = b_3 = 1$

So $N = 3^1 \times 5^1 \times 7^1 = 105$

Teacher's Note:
This problem is easy enough to encourage the novice problem solver but at the same time it includes some delicate problem solving skills. Note how the options $8 = 1 \times 8 = 4 \times 2$ are rejected and how the terms 'at least' and 'exactly' are arrived at logically. A question which may need to be discussed is why $8 = 1 \times 2 \times 2 \times 2$ is not considered. It would also be interesting to ask students if this amount of data, i.e., number of factors and any two factors is sufficient to find N in all cases!

Problem VI-2-M.2 From http://nrich.maths.org/480

(a) *How can I find a number with exactly* 14 *factors? How can I find the smallest such number?*

(b) *How can I find a number with exactly* 18 *factors? How can I find the smallest such number?*

(c) *Which numbers have an odd number of factors?*

(d) ***Extension:*** *What is the smallest number with exactly* 100 *factors?*

(e) *Which number less than* 1000 *has the most factors?*

(a) If the number has 14 factors then
$(b_1+1)(b_2+1)(b_3+1)\ldots\ldots\ldots\ldots(b_k+1) = 14 = 1 \times 14 = 2 \times 7$
So $b_1 = 13$ <u>or</u> $b_1 = 1$ and $b_2 = 6$
But this opens up an infinite number of possibilities, because the number could be $2^{13} = 8192$ or $3^1 \times 5^6 = 46875$, $3^6 \times 5^1 = 3645$; by varying the primes and their powers we could get many, many numbers which have 14 factors. The smallest such number is $2^6 \times 3^1 = 192$, whose 14 factors are $1, 2, 4, 8, 16, 32, 64, 3, 6, 12, 24, 48, 96$ and 192.

(b) If the number has 18 factors then $(b_1+1)(b_2+1)(b_3+1)\ldots\ldots\ldots\ldots(b_k+1) = 18$
$18 = 1 \times 18 = 3 \times 6 = 2 \times 9 = 2 \times 3 \times 3$
So $b_1 = 17$ <u>or</u> $b_1 = 2$ and $b_2 = 5$ <u>or</u> $b_1 = 1$ and $b_2 = 8$ <u>or</u> $b_1 = 1$ and $b_2 = 2$ and $b_3 = 2$
But this opens up an infinite number of possibilities, because the number could be $3^{17} = 129140163$ or $3^8 \times 5^1 = 32805$, or $2^2 \times 5^5 = 12500$, and so on. The smallest such number is $2^2 \times 3^2 \times 5 = 180$.

(c) If a number has an odd number of factors, then
$(b_1+1), (b_2+1), (b_3+1)\ldots\ldots\ldots\ldots(b_k+1)$ are all odd which means that $b_1, b_2, b_3 \ldots\ldots\ldots\ldots b_k$ are all even and this means that $N = p_1^{b_1} \times \ldots \times p_k^{b_k}$ is a perfect square. This is an enormously useful fact which is used in many puzzles; you could find one such on http://teachersofindia.org/en/article/atria-dumble-door-rescue

(d) $100 = 2 \times 2 \times 5 \times 5$
so $2^4 \times 3^4 \times 5 \times 7$ is the smallest number with 100 divisors.

(e) The last question is quite challenging but armed with all the discoveries of the previous sub- parts, we could reason that since $2 \times 3 \times 5 \times 7 = 210$ (which is less than 1000) and 210×11 is greater than 1000, the number less than 1000 with the most factors should have only 2, 3, 5 and 7 as the prime factors.

Now we need to increase the powers of these primes to the maximum possible without letting the product exceed 1000. By trial and error, we arrive at the conclusion that $3 \times 5 \times 7 \times 8$ is less than 1000 and has 64 factors. This seems to be the number with the highest number of factors.

Teacher's Note:
A step up from the previous problem and slightly more difficult. Logical reasoning is practised and each student can get a different, but correct, answer. It is also a good idea for them to list the factors of the numbers that they get as it helps them to understand the basic property that they are using. This also gives them some practice in using their understanding of exponents. Also, the difference between the different sub-parts of the problem clarifies their understanding of viable options in each case. The additional criterion of the smallest number helps them to practise comparison of numbers. And finally, trial and error is a good mathematical strategy which students must learn to use to validate their thinking.

Problem VI-2-M.3 From http://nrich.maths.org/524
Choose any 3 digits and make a 6 digit number by repeating the 3 digits in the same order (e.g. 523523).

Whatever digits you choose, the number will always be divisible by 7 and by 11 and by 13, without a remainder.

Can you explain why?

A lovely problem which simply uses place value to arrive at the conclusion that if a number is created in this way, then one of its factors is 1001, which is $7 \times 11 \times 13$ and which is why such a number will definitely have these three prime factors.

For example
$$\begin{aligned}523523 &= 5 \times 10^5 + 2 \times 10^4 + 3 \times 10^3 + 5 \times 10^2 + 2 \times 10 + 3\\ &= 5 \times 10^2(10^3 + 1) + 2 \times 10(10^3 + 1) + 3(10^3 + 1)\\ &= 523(10^3 + 1)\end{aligned}$$

Teacher's Note:
Students can create variations of such problems to come up with interesting patterns; this is also a way for them to practise their divisibility rules while verifying the divisibility of the numbers that they have created.

Problem VI-2-M.4 Lonely 8
Figure 1 can be rewritten as

				c	d	8	e	f
a	b	p	q	r	s	t	u	v
		x	x	x				
					x	x		
					x	x		
						x	x	x
						x	x	x
								0

It must be understood that the number ab is a two digit number with a in the tens place and b in the units place, i.e., it is not the product of a and b. This notation will hold through the problem. Remember that x represents anything that is unknown.

Using our knowledge of long division, we can do more

				c	0	8	0	f
a	b	p	q	r	s	t	u	v
		p	q	r				
					s	t		
					x	x		
						x	u	v
						x	u	v
								0

As you can see, the number of unknowns has gone down significantly. We also know that the number st has 2 digits and that 8 times ab is again a 2 digit number which differs from the 2 digit number st by a single digit number.

Since the only two digit numbers which when multiplied by 8 give a 2 digit number are 10, 11 and 12, we get three possible options for ab. But ab when multiplied by c (a single digit number) gives a three digit number pqr. So ab can only be 12 and then $c = 9$ and $pqr = 108$

				9	0	8	0	f
1	2	1	0	8	s	t	u	v
		1	0	8				
					s	t		
					9	6		
						g	u	v
						g	u	v
								0

We have reduced the number of unknowns even further!

Also, st must be 97, 98 or 99.

st	g	gu (should be $<$ 12)	guv (should be a multiple of 12)	$pqrstuv$
97	1	10	108	1089708
	1	11		
98	2	20	204	1089804
	2	21	216	1089816
	2	22	228	1089828
99	3	30	300	1089900
99	3	31	312	1089912
	3	32	324	1089924
	3	33	336	1089936
	3	34	348	1089948
	3	35		

Out of all the options for *pqrstuv*, only the first option gives a quotient which has zeroes in the tens and thousands place of the quotient. So our final answer is

				9	0	8	0	9
1	2	1	0	8	9	7	0	8
		1	0	8				
					9	7		
					9	6		
						1	0	8
						1	0	8
								0

Clearly, a case of systematic reasoning and a clear understanding of the process of long division winning the day!

Teacher's Note: Getting the students to create their own faded document problems will be a tremendous learning experience for them.

Pedagogy: Problem solving strategies that have been imparted with this problem set are:

1. Systematic reasoning.
2. Listing of possibilities and elimination on a case by case basis.
3. Trial and error.
4. Learning to understand and apply given formulae.
5. Learning to use symbols appropriately.
6. Problem posing.

Problem Corner

Problems for the SENIOR SCHOOL

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem VI-2-S.1

A mathematics teacher wrote the quadratic $x^2 + 10x + 20$ on the board. Then each student either increased by 1 or decreased by 1 either the constant or the linear coefficient. Finally $x^2 + 20x + 10$ appeared. Did a quadratic expression with integer zeros necessarily appear on the board in the process? [From *Polynomials* by Ed Barbeau]

Problem VI-2-S.2

Let $p(t)$ be a monic quadratic polynomial. (The word '**monic**' indicates that the leading coefficient is 1. For example, $x^2 + 10x + 100$ is a monic quadratic; but $3x^2 + 10x + 100$ is not, as its leading coefficient is 3.) Show that, for any integer n, there exists an integer k such that $p(n)p(n + 1) = p(k)$. [From *Polynomials* by Ed Barbeau]

Problem VI-2-S.3

Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers. [From Round 1, British Mathematical Olympiad, 2011]

Problem VI-2-S.4

Find all integers n for which $n^2 + 20n + 11$ is a perfect square. [From Round 1, British Mathematical Olympiad, 2011]

Problem VI-2-S.5

Find all integers x, y and z such that $x^2 + y^2 + z^2 = 2(yz + 1)$ and $x + y + z = 4018$.

[From Round 1, British Mathematical Olympiad, 2009]

Keywords: Integer, cube, perfect square, perfect cube, progression, cyclic quadrilateral, inscribed square

Solutions to Problems in Issue-VI-1 (March 2017)

Solution to problem VI-1-S.1

Let ABCD be a cyclic quadrilateral in which AC is perpendicular to BD. Let X be the point of intersection of the diagonals. Let P be the midpoint of BC. Prove that PX is perpendicular to AD.

Solution. In triangle BXC, $\measuredangle BXC = 90^\circ$ and P is the midpoint of the hypotenuse BC. Therefore $XP = BP = CP$. Let Q be the point of intersection of PX and AD. Now

$$\measuredangle ADX = \measuredangle ADB = \measuredangle ACB = \measuredangle XCP = \measuredangle PXC = \measuredangle AXQ,$$

and

$$\measuredangle DAX = \measuredangle DAC = \measuredangle DBC = \measuredangle XBP = \measuredangle BXP = \measuredangle DXQ.$$

Therefore in triangle ADX, $\measuredangle ADX = \measuredangle AXQ$ and $\measuredangle DAX = \measuredangle DXQ$. Thus $\measuredangle AQX = \measuredangle DQX = 90^\circ$.

Solution to problem VI-1-S.2

Let a, b, c be three distinct non-zero real numbers. If a, b, c are in arithmetic progression and b, c, a are in geometric progression, prove that c, a, b are in harmonic progression and find the ratio a : b : c.

Solution. The given conditions imply

$$2b = a + c, \quad c^2 = ab.$$

Therefore

$$\frac{2bc}{b+c} = \frac{(a+c)c}{b+c} = \frac{ac+ab}{b+c} = a,$$

so c, a, b are in harmonic progression. Eliminating a we get

$$2b^2 - bc - c^2 = (b-c)(2b+c) = 0.$$

Thus $b = -\dfrac{c}{2}$ and $a = 2b - c = -2c$. Therefore $a : b : c = -2 : -\dfrac{1}{2} : 1 = 4 : 1 : -2$.

Solution to problem VI-1-S.3

Prove that the following quantity is not an integer:

$$\frac{2}{3} + \frac{4}{5} + \frac{6}{7} + \cdots + \frac{2016}{2017}.$$

Solution. Let $S = \dfrac{2}{3} + \dfrac{4}{5} + \dfrac{6}{7} + \cdots + \dfrac{2016}{2017}$. If S is an integer then so is

$$T = 1008 - S = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2017}.$$

Let $M = 3 \cdot 5 \cdot 7 \cdots 2015$. If T is an integer so is the quantity

$$MT = \frac{M}{3} + \frac{M}{5} + \frac{M}{7} + \cdots + \frac{M}{2017}.$$

Each of the terms except the last term is an integer. Therefore $\frac{M}{2017}$ is an integer. But as 2017 is prime, it does not have any factor in common with M and hence does not divide M. Thus we arrive at a contradiction. Hence S cannot be an integer.

Solution to problem VI-1-S.4
Prove that the product of six consecutive positive integers cannot be a perfect cube.

Solution. Because $6! = 720$ is not a perfect cube, we only need to consider products $t = n(n+1)(n+2)(n+3)(n+4)(n+5)$ with $n \geq 2$. Now

$$t = a^3 + 10a^2 + 24a$$

where $a = n(n+5)$. Because $a \geq 14$, we have:

$$\begin{aligned}(a+3)^3 &= a^3 + 9a^2 + 27a + 27 \\ &= t - (a-9)(a+3) - 3a < t < a^3 + 12a^2 + 48a + 64 \\ &= (a+4)^3.\end{aligned}$$

Hence t cannot be a perfect cube.

Solution to problem VI-1-S.5
A square with side a is inscribed in a circle. Find the side of the square inscribed in one of the segments thus obtained.

Solution. The radius of the circle is $\dfrac{a}{\sqrt{2}}$. By symmetry it is clear that the sides of the square inscribed in the segment are parallel to the sides of the bigger square. Let x be the side length of the inscribed square. Then:

$$\left(\frac{a}{2} + x\right)^2 + \left(\frac{x}{2}\right)^2 = \frac{a^2}{2}.$$

This reduces to

$$5x^2 + 4ax - a^2 = 0.$$

Thus $x = \dfrac{a}{5}$.

Theorem concerning a RIGHT TRIANGLE

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

The following elegant geometric result concerning a triangle is based on a problem that appeared in the Regional Mathematics Olympiad (RMO) of 2016.

Let ABC be a scalene triangle, and let D be the midpoint of BC. Draw median AD. Through D draw a line perpendicular to AD and let it meet the extended sides AB, AC at points K, L, respectively. Then points B, C, K, L lie on a circle if and only if angle BAC is a right angle. (See Figure 1.)

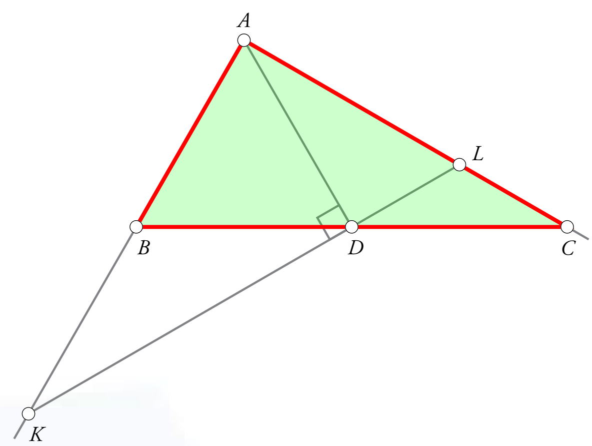


Figure 1

The implication in one direction is easy (if the triangle is right-angled, then the four points are concyclic); but the reverse implication seems more challenging. We shall give a geometric solution for the forward implication, followed by an algebraic solution in which both the implications are established at the same time.

Geometric proof that the four points are concyclic. We are given the fact that $\measuredangle BAC$ is a right angle, and we must prove that points B, K, L, C are concyclic.

Keywords: Cosine rule, intersecting chords theorem, crossed chords theorem, Apollonius, power of a point

The most obvious approach towards proving that four given points are concyclic is the angle-chasing route: prove that some two angles are equal. In the present instance, it suffices to prove that $\measuredangle AKD = \measuredangle ACD$, or that $\measuredangle ALD = \measuredangle ABD$. (These two statements are clearly equivalent to each other.) See Figure 2; we must prove that $x = y$.

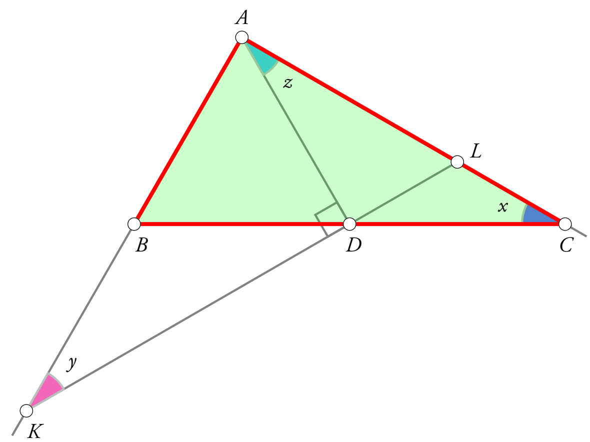


Figure 2

The desired equality follows when we notice that $x = z$ and $y = z$. To see why $y = z$, observe that both y and z are complementary to $\measuredangle ALK$ (and this follows because $\measuredangle LAK$ and $\measuredangle ADL$ are right angles). To see why $x = z$, note that since $\triangle ABC$ is right-angled at A, its circumcentre lies at the midpoint of the hypotenuse BC. This means that D is equidistant from vertices A, B, C. Hence $DA = DC$, which implies that $x = z$.

Thus the forward implication has been proved: if $\measuredangle BAC$ is a right angle, then points B, K, L, C are concyclic.

For the reverse implication, a geometric approach seems rather elusive; we opt for an algebraic approach. We shall need the following results:

(i) the cosine rule: in $\triangle ABC$, we have: $c^2 = a^2 + b^2 - 2ab \cos C$, etc;

(ii) the Intersecting Chords theorem, also called the Crossed Chords theorem, and the related notion of 'power of a point': given a circle with centre O and radius r, if two of its chords EF and GH intersect at a point P (which may lie inside or outside the circle), then we have the equality $PE \cdot PF = PO^2 - r^2 = PG \cdot PH$. (Note that the distances here are *signed*; so if PE and PF point in opposite directions, then $PE \cdot PF \leq 0$.) We need the **converse** of this theorem (which is also true): *if coplanar points E, F, G, H are placed such that the equality $PE \cdot PF = PG \cdot PH$ is true, where P is the point of intersection of lines EF and GH, then the points E, F, G, H are concyclic.*

(iii) the theorem of Apollonius which tells us that $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

We reason as follows:

$$\begin{aligned} \text{Points } B, C, K, L \text{ concylic} &\iff AB \cdot AK = AC \cdot AL \\ &\iff c \cdot \frac{AD}{\cos \measuredangle KAD} = b \cdot \frac{AD}{\cos \measuredangle LAD} \\ &\iff \frac{c}{b} = \frac{\cos \measuredangle BAD}{\cos \measuredangle CAD}. \end{aligned}$$

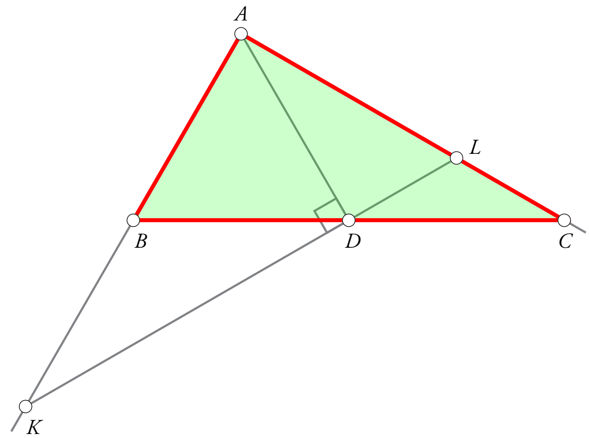


Figure 3

Next we have, by the cosine rule:

$$\cos\measuredangle BAD = \frac{AB^2 + AD^2 - BD^2}{2\,AB\cdot AD}, \qquad \cos\measuredangle CAD = \frac{AC^2 + AD^2 - CD^2}{2\,AC\cdot AD};$$

hence:

$$\frac{\cos\measuredangle BAD}{\cos\measuredangle CAD} = \frac{AB^2 + AD^2 - BD^2}{AC^2 + AD^2 - CD^2}\cdot\frac{AC}{AB} = \frac{c^2 + AD^2 - a^2/4}{b^2 + AD^2 - a^2/4}\cdot\frac{b}{c}.$$

The theorem of Apollonius implies that:

$$AD^2 = \frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4}.$$

Substituting this into the previous expression we get:

$$\frac{\cos\measuredangle BAD}{\cos\measuredangle CAD} = \frac{3c^2 + b^2 - a^2}{3b^2 + c^2 - a^2}\cdot\frac{b}{c}.$$

Hence we have:

$$\text{Points } B, C, K, L \text{ concylic} \iff \frac{c}{b} = \frac{3c^2 + b^2 - a^2}{3b^2 + c^2 - a^2}\cdot\frac{b}{c}.$$

That is:

$$\text{Points } B, C, K, L \text{ concylic} \iff c^2\left(3b^2 + c^2 - a^2\right) = b^2\left(3c^2 + b^2 - a^2\right).$$

Next we have:

$$\begin{aligned} c^2\left(3b^2 + c^2 - a^2\right) - b^2\left(3c^2 + b^2 - a^2\right) &= a^2\left(b^2 - c^2\right) - \left(b^4 - c^4\right) \\ &= \left(a^2 - b^2 - c^2\right)\left(b^2 - c^2\right). \end{aligned}$$

Hence:

$$\text{Points } B, C, K, L \text{ concylic} \iff \left(a^2 - b^2 - c^2\right)\left(b^2 - c^2\right) = 0.$$

Since $b^2 - c^2 \neq 0$ (we have specifically been told that the triangle is scalene), we deduce finally that:

$$\text{Points } B, C, K, L \text{ concylic} \iff a^2 - b^2 - c^2 = 0 \iff \measuredangle BAC = 90^\circ. \qquad \square$$

Some of you may like to take up the challenge of finding a purely geometric proof for the reverse implication.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Problem Corner

Problem Concerning RATIONAL NUMBERS

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

Take any positive rational number and add to it, its reciprocal. For example, starting with 2, we get the sum $2 + 1/2 = 5/2$; starting with 3, we get the sum $3 + 1/3 = 10/3$.

Can we get the sum to be an integer? Clearly we can, in one simple way: by starting with 1, we get the sum $1 + 1/1 = 2$. Observe that the answer is an integer. Is there any other choice of starting number which will make the sum an integer? This prompts the following problem.

Problem: *Find all positive rational numbers with the property that the sum of the number and its reciprocal is an integer.*

Try guessing the answer before reading any further!

We offer two solutions. Some properties of positive integers that we take for granted are the following.

(1) Let p be a prime number and let n be any integer; then: if p divides n^2, then p divides n. Expressed in contrapositive form: *If p does not divide n, then p does not divide n^2.*

(2) *A rational number whose square is an integer is itself an integer.* That is, if x is a rational number such that x^2 is an integer, then x is an integer. Expressed in contrapositive form: *If n is not the square of an integer, then $\sqrt{n}$ is not a rational number.*

Solution I. Let x be a positive rational number such that $x + 1/x = n$ is an integer. We argue as follows.

- Since $x + 1/x \geq 2$ for all $x > 0$, we have $n \geq 2$.

Keywords: Rational, reciprocal, integer, prime, coprime, divisibility, quadratic, discriminant

- The relation $x + 1/x = n$ yields the following quadratic equation:

$$x^2 - nx + 1 = 0.$$

Solving it, we get:

$$x = \frac{n \pm \sqrt{n^2 - 4}}{2}.$$

If $n = 2$, the square root term vanishes and we get $x = 1$. This confirms what we already know: that $1 + 1/1$ is an integer.

- If $n \geq 3$, then $2n - 1 \geq 5$, hence $n^2 - (n-1)^2 \geq 5$, which implies that

$$(n-1)^2 < n^2 - 4 < n^2.$$

Therefore $n^2 - 4$ lies strictly between two consecutive perfect squares and cannot be a perfect square itself. It follows that the quantity $\sqrt{n^2 - 4}$ is not an integer. Hence $\sqrt{n^2 - 4}$ is irrational.

- This implies that if $n \geq 3$, then x is an irrational quantity, which is contrary to the given information (namely, that x is a rational number).
- This contradiction shows that if x is rational, then $x + 1/x$ cannot assume any integer value other than 2.
- It follows that 1 is the only positive rational number that has the stated property. □

Solution II. This is a simpler solution. Let the positive rational number r/s be such that the sum of this number and its reciprocal is an integer n. Here we assume that r and s are positive integers that share no factor exceeding 1; i.e., they are coprime. (There is no loss of generality in assuming that r and s are coprime; we would write any fraction in this form.) Let p be a prime divisor of r; then p cannot be a divisor of s. We now have:

$$\frac{r}{s} + \frac{s}{r} = n,$$

$$\therefore\ r^2 + s^2 = nrs,$$

$$\therefore\ s^2 = nrs - r^2.$$

Now a contradiction arises when we check the third equality in terms of divisibility by p; for, p divides r, hence p divides nrs as well as r^2, and so p divides $nrs - r^2$; but p does not divide s^2. These statements clearly contradict each other.

The contradiction shows that there cannot be any such prime divisor p. But this means that $r = 1$, this being the only positive integer with no prime divisor. Swapping the roles of r and s in the above argument, we deduce that $s = 1$ as well. Hence $r = s = 1$, and $r/s = 1$. Thus the only such rational number is 1. □

Questions for further explorations

The main result having been proved, we need to open out the question and ask related questions. For example, we could ask:

Question 1: *With x rational, we cannot get $x + 1/x$ to assume an integer value. But how close can we get to an integer? How close can we get to 3, or 4, or 5 (or any other integer)? What meaningful question can be asked in this regard?*

For example, is the following a reasonable question to ask?

> *Do rational numbers x exist which satisfy the following inequality:*
>
> $$\left| x + \frac{1}{x} - 3 \right| < 10^{-50}\ ?$$

A possibly more interesting question to explore is the following.

Question 2: *We have already shown that if x is a rational number, then $x + 1/x$ cannot assume an integer value. But what kind of values can it assume? Given an arbitrary rational number a/b, how do we check whether this number lies within the range of the function $f(x) = x + 1/x$?*

It goes without saying that we need a test which can be executed rapidly.

We invite the reader to explore both these questions. Please send in your responses!

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Finding the area of a **circle** given the circumference: A **simple** and **quick** method

We know that if, instead of the radius of a circle, the circumference is given, to find the area A enclosed by the circle, we usually find the radius first and then use the formula $A = \pi r^2$. But the following formula gives a quick and easy method to find the area of a circle without finding its radius. Note that we assume that $\pi = \frac{22}{7}$.

Claim: Let C be the circumference of a circle. Then the area A of the circle is given by

$$A = \left(\frac{C}{4}\right)^2 + 33\left(\frac{C}{44}\right)^2.$$

$$\text{Consider RHS} = \left(\frac{C}{4}\right)^2 + 33\left(\frac{C}{44}\right)^2 = \left(\frac{2\pi r}{4}\right)^2 + 33\left(\frac{2\pi r}{44}\right)^2 \qquad (\text{because } C = 2\pi r)$$

$$= \left(\frac{\pi r}{2}\right)^2 + 33\left(\frac{\pi r}{22}\right)^2 = \frac{\pi^2 r^2}{4} + \frac{33\,\pi^2 r^2}{22\times 22}$$

$$= \frac{\pi^2 r^2}{4} + \frac{3\,\pi^2 r^2}{44} = \frac{11\pi^2 r^2 + 3\,\pi^2 r^2}{44} = \frac{14\pi^2 r^2}{44} = \frac{7\pi^2 r^2}{22} = \pi r^2 = \text{Area of circle.}$$

Now, we claim the following:

Let C be the circumference of a circle and S be the side of a square such that $C = 4S$. Then the difference between the area of the circle and the area of the square is $33\left(\frac{C}{44}\right)^2$.

Can you prove this using the above formula?

Acknowledgement: The author is grateful to Professor B.N. Waphare for encouraging him to send this formula to *At Right Angles.*

Vikram A. Ghule

Ekalavya Shikshan Sanstha's, Sant Tukaram Vidyalaya Ghatshiras Tal. Pathardi, Dist. **Ahmednagar** 414106 (India) **Email:** vikramghule2016@gmail.com

ADVENTURES IN PROBLEM SOLVING
A Math Olympiad Problem … and a few cousins

SHAILESH SHIRALI

In this edition of *Adventures in Problem Solving*, we study in detail a problem which first appeared in the 1987 USA Mathematics Olympiad. However, we adopt a different strategy this time. Faced with the given problem which looks quite challenging, we tweak it in different ways and obtain related problems which are simpler than the original one. Solving this collection of problems turns out to be a fun activity and demonstrates yet one more time the importance and utility of quadratic functions and quadratic equations.

As usual, we state the problems first, so that you have an opportunity to tackle them before seeing the solutions.

The problem is to find all integer-valued solutions of the following equation:

$$\left(a^2+b\right)\left(a+b^2\right)=(a-b)^3.$$

We shall study not just this equation but others obtained by tweaking it. As noted above, we shall find that knowledge of quadratic equations and quadratic functions comes to our aid repeatedly. Those who have read earlier issues of this magazine will know that quadratic equations and quadratic functions have been studied many times in these pages. What is striking is that they invariably enter the scene in a completely natural manner, without fanfare or announcement. The manner in which this happens is worthy of close study. (For more such instances, please see [1].)

We start by studying three simpler variants of the USAMO problem (1–3 below). They are superficially similar to the original problem. In each case, our interest lies only in solutions where a and b are both nonzero (this may be captured in a compact way by the statement $ab \neq 0$). Problem 4 is the same as the one stated above. Please try solving all of them; you will find that one of them is particularly easy, but we won't tell you which one!

Problem 1: Find all nonzero integer-valued solutions of the following equation:

$$\left(a^2+b\right)\left(a+b^2\right)=(a+b)^3. \tag{1}$$

Problem 2: Find all nonzero integer-valued solutions of the following equation:

$$(a+b)\left(a^2+b^2\right)=(a+b)^3. \tag{2}$$

Problem 3: Find all nonzero integer-valued solutions of the following equation:

$$(a+b)\left(a^2+b^2\right)=a^3+kb^3. \tag{3}$$

Here k is an integer-valued parameter; the question here is, for which k do solutions exist?

Problem 4: Find all nonzero integer-valued solutions of the following equation:

$$\left(a^2+b\right)\left(a+b^2\right)=(a-b)^3. \tag{4}$$

Solutions

Solution to Problem 1. The pairs $(a,0)$ and $(0,b)$ work for any integer values of a and b; however, we have explicitly excluded such solutions from consideration.

The natural thing to do is to simplify the expressions involved and see where it leads us. We have:

$$\begin{aligned}\left(a^2+b\right)\left(a+b^2\right)-(a+b)^3&=ab+a^2b^2-3ab^2-3a^2b\\&=ab\left(ab-3a-3b+1\right).\end{aligned}$$

As we have announced in advance that we will not be looking at solutions in which either of the variables is 0, we may assume that ab is nonzero. Hence equation 1 implies that

$$ab-3a-3b+1=0. \tag{5}$$

We have encountered this kind of equation on several occasions. They are solved using a standard artifice, using factorisation. We only need to observe that

$$(a-3)(b-3)=ab-3a-3b+9,$$

which is almost the same as $ab-3a-3b+1$; only the constant term is different. We therefore proceed as follows.

$$\begin{aligned}ab-3a-3b+1&=0,\\ \therefore\ ab-3a-3b+9&=8,\\ \therefore\ (a-3)(b-3)&=8.\end{aligned}$$

Therefore, $a-3$, $b-3$ are a pair of complementary factors of 8 (i.e., their product is 8). Here we permit the presence of negative factors. It therefore follows that

$$(a-3,b-3)\in\{(-8,-1),\ (-4,-2),\ (-2,-4),\ (-1,-8),\ (1,8),\ (2,4),\ (4,2),\ (8,1)\},$$

giving:

$$(a,b)\in\{(-5,2),\ (-1,1),\ (1,-1),\ (2,-5),\ (4,11),\ (5,7),\ (7,5),\ (11,4)\}. \tag{6}$$

These are thus the desired solutions; there are eight such integer pairs.

Solution to Problem 2. Once again, we begin by simplifying the expressions involved. We have:

$$(a+b)\left(a^2+b^2\right)-(a+b)^3=-2ab(a+b). \tag{7}$$

Hence the nonzero solutions to equation 2 all have $a+b=0$, i.e., they are all of the form $(t,-t)$ for some integer t. So equation 2 has infinitely many nonzero integer solutions.

This should not come as a surprise, for the equation is *homogeneous*; if we expand out the terms in the equation, then every term has degree 3. This implies that if a, b solves the equation, then so does ka, kb for every integer k.

Problem 2 has turned out to be rather too simple!

Solution to Problem 3. This problem turns out to be quite interesting. Note that as in Problem 2, the equation is homogeneous; every term has degree 3.

On opening out the brackets and simplifying, we obtain:

$$\begin{aligned}(a+b)\left(a^2+b^2\right)&=a^3+kb^3,\\ \therefore\ \ a^3+a^2b+ab^2+b^3&=a^3+kb^3,\\ \therefore\ \ (k-1)b^3-ab^2-a^2b&=0.\end{aligned}$$

Since we have assumed that $b\neq 0$, we get:

$$(k-1)b^2-ab-a^2=0. \tag{8}$$

We regard this as a quadratic equation with b as the unknown. If the equation is to have integer solutions, then its discriminant D must be a perfect square. Hence the quantity

$$a^2+4(k-1)a^2=a^2\,(4k-3)$$

must be a perfect square. For this to happen, $4k-3$ must itself be a perfect square. Since $4k-3$ is an odd number, we must have

$$4k-3=(2n+1)^2$$

for some integer n; i.e., $4k-3=4n^2+4n+1$ for some integer n. Hence

$$k=n^2+n+1 \tag{9}$$

for some integer n. It follows that equation 3 has nonzero integer solutions if and only if k is of the form n^2+n+1 for some integer n, i.e., iff $k\in\{1,3,7,13,21,31,43,\ldots\}$. When k is of this form, equation 8 yields:

$$\begin{aligned}&\left(n^2+n\right)b^2-ab-a^2=0,\\ \therefore\ \ & b=\frac{a\pm\sqrt{a^2\left(4n^2+4n+1\right)}}{2\left(n^2+n\right)}=\frac{a(1\pm(2n+1))}{2n(n+1)},\\ \therefore\ \ & b=\frac{a}{n},\quad b=-\frac{a}{n+1}.\end{aligned}$$

Hence the solutions to equation 3, with k having the form n^2+n+1, are the following:

$$(a,b)=(nt,t),\quad (a,b)=(-(n+1)t,t)\,, \tag{10}$$

where t is an arbitrary integer. Thus equation 3 too has infinitely many nonzero integer solutions; indeed, two families of them.

Solution to Problem 4. Now we are ready to take up the equation posed at the start:

$$\left(a^2+b\right)\left(a+b^2\right)=(a-b)^3.$$

Our interest as earlier will be in the nonzero integer solutions. On opening the brackets and simplifying, we get:

$$\left(a^2+b\right)\left(a+b^2\right)-(a-b)^3=b\left(a^2b+3a^2-3ab+a+2b^2\right).$$

Since $b\neq 0$, the given equation yields:

$$a^2b+3a^2-3ab+a+2b^2=0. \qquad (11)$$

We observe the following about the polynomial $a^2b+3a^2-3ab+a+2b^2$:

- it is quadratic in a individually;
- it is quadratic in b individually;
- it is a cubic expression when a, b are both treated as variables (by virtue of the term a^2b).

To make progress, we exploit the fact that the polynomial is quadratic in each of a and b individually; but which one should we use? It turns out that if we regard the expression as a quadratic in b, less work is involved; the expressions obtained are more manageable. Here is how the analysis proceeds. We write the equation $a^2b+3a^2-3ab+a+2b^2=0$ as

$$2b^2+a(a-3)b+a(3a+1)=0. \qquad (12)$$

For this quadratic equation to have integer solutions, the discriminant must be a perfect square. Hence the quantity D given by

$$D=a^2(a-3)^2-8a(3a+1) \qquad (13)$$

must be a perfect square. We now need to identify the integer values of a for which D is a perfect square.

We first simplify this expression; we get:

$$\begin{aligned}D&=a^2(a-3)^2-8a(3a+1)\\&=a\left(a^3-6a^2-15a-8\right),\end{aligned}$$

on simplifying. Progress is achieved when we notice that $-1-6+15-8=0$, which tells us that $a+1$ is a factor of $a^3-6a^2-15a-8$. (Remember the factor theorem!) We have:

$$a^3-6a^2-15a-8=(a+1)\left(a^2-7a-8\right).$$

Next, we notice that $1+7-8=0$, which tells us that $a+1$ is a factor of a^2-7a-8:

$$a^2-7a-8=(a+1)(a-8).$$

Hence $a+1$ is a repeated factor (i.e., it occurs twice), and we have:

$$a\left(a^3-6a^2-15a-8\right)=a(a+1)^2(a-8). \qquad (14)$$

It follows that D is a perfect square precisely when $a(a-8)$ is a perfect square. We must therefore identify all integers a for which $a(a-8)$ is a perfect square. This question brings us back to familiar ground; we have tackled such questions earlier — many times! We only need the tried and trusted completing-the-square procedure and the difference-of-two-squares formula. Let $a(a-8)=c^2$. Then we have:

$$\begin{aligned} a^2-8a &= c^2, \\ \therefore\ a^2-8a+16 &= c^2+16, \\ \therefore\ (a-4)^2-c^2 &= 16, \\ \therefore\ (a-4-c)(a-4+c) &= 16. \end{aligned}$$

The ordered integer pairs (u,v) whose product is 16 are the following:

$$\begin{array}{ccccc} (-16,-1), & (-8,-2), & (-4,-4), & (-2,-8), & (-1,-16), \\ (1,16), & (2,8), & (4,4), & (8,2), & (16,1). \end{array}$$

We can equate the pair $(a-4-c, a-4+c)$ with each of these pairs and solve for a and c; and from a we then solve for b. The pairs $(\pm 16, \pm 1)$ and $(\pm 1, \pm 16)$ yield fractional values of a, and so need not be considered. (This happens because 1 and 16 have opposite parity.) That leaves six pairs. Solving for c and a and then for b, we get the following:

$(a-4-c, a-4+c)$	c	a	b
$(-8,-2)$	3	-1	-1
$(-4,-4)$	0	0	0
$(-2,-8)$	-3	-1	-1
$(2,8)$	3	9	$-21,-6$
$(4,4)$	0	8	-10
$(8,2)$	-3	9	$-21,-6$

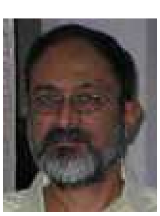
It follows that the nonzero integer pairs which satisfy equation 4 are the following:

$$(a,b)=(-1,-1),\ (9,-21),\ (9,-6),\ (8,-10). \tag{15}$$

Exercise. Write the equation that we had obtained, $a^2b+3a^2-3ab+a+2b^2=0$, as a quadratic in a and check that this approach leads to the same solutions listed above.

References

1. Chris Budd and Chris Sangwin, "101 uses of a quadratic equation" from +plus Magazine, https://plus.maths.org/content/101-uses-quadratic-equation



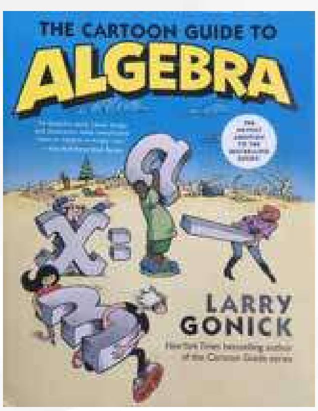
SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as an editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

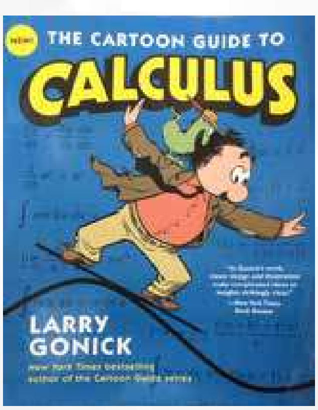
Review

Transcending the unbearable heaviness in the Teaching of Mathematics !

A Review of THE CARTOON GUIDE TO ALGEBRA and THE CARTOON GUIDE TO CALCULUS *By Larry Gonick*

SHASHIDHAR JAGADEESHAN





I don't know what your experience of learning mathematics was—but for me, till I went to graduate school and except for a few courses as an undergraduate, it was a very heavy affair. First of all, there was an overwhelming sense of being weighed down that was associated with 'knowledge'. There was so much to learn, so much to remember and so much to be tested on! There was no sense of lightness associated with learning, no sense of play or joy in discovering and understanding. I remember once, wandering around a huge library with this feeling, and coming upon a lovely poem by Justin Richardson (Punch, 1952), which gave me immense relief. It goes like this:

For years a secret shame destroyed my peace-

I'd not read Eliot, Auden or MacNeice.

But then I had a thought that brought me hope-

Neither had Chaucer, Shakespeare, Milton, Pope.

While this sense of gravitas can be associated with any area of knowledge, in mathematics it can get further compounded. This is especially so while learning courses like calculus when it is taught without any graphs and graphics, and one is expected

Keywords: Illustrations, graphics, calculus, algebra

to learn a huge bunch of formulae, and solve innumerable integrals, with no clue why you are learning what you are learning. A few years ago, a student of mine bitterly complained about how he was taught an undergraduate course in calculus and the teacher did not draw a single graph in the entire course!

I feel our relationship to mathematical knowledge and how we teach it has to change. We need to bring in a certain lightness (this is not the opposite of rigour), humour and sense of play in our teaching and also use graphical illustrations to make concepts, formulae and equations come to life!

The *Cartoon Guides* to algebra and calculus do just that. I will share some common features before going into the specifics of each book. Larry Gonick is a well known cartoonist, who studied and taught mathematics in Harvard. He has authored several cartoon guides in a wide variety of topics from history to physics. Both books referred to above are actually very rigorous introductions to the subject in hand. There is no sloppy hand-waving involved. Concepts are introduced very carefully, several examples and illustrations given from real life, and there are exercises at the end of each chapter.

What then is the difference between these cartoon guides and regular text books on these subjects? Given that it is a cartoon book the main tool for exposition and explanation is graphics. Every concept, in fact every page, is illustrated using graphics. These (see Figure 1) could be illustrations or pictorial proofs (for example, to show $a^2 - b^2 = (a + b)(a - b)$, to illustrate completing the square, the concept of a function and its inverse, the chain rule, and so on).

There is a great emphasis on using mathematical graphs to explain and illustrate concepts. Paraphrasing a famous dictum, I believe a graph is worth a thousand equations! Graphs give students an intuitive understanding of how functions work, why some results make sense and also serve as models for conjecturing results (the mean value theorem, for example). It is a real tragedy that teachers do not use them more in their classrooms.

Of course the most important feature is humour. There is a running narrative where cartoon characters (human and non-human) interact with each other while learning the various concepts. So we have, in the Algebra book, Al-Khwarizmi and Bhaskara making an appearance and in the calculus book, Newton and Leibnitz. This is refreshing from the history of mathematics point of view. It would have been really nice if the contribution from the Kerala school was mentioned in the calculus book and Omar Khayyam in the algebra book. The conversations between the various characters are a constant humorous banter, often using ‘puns’ to evoke laughter. Newton and Leibnitz, as is to be expected, are at constant loggerheads with each other (see Figure 2)!

While Gonick does have his witty and clever moments, I must confess, I found many of the jokes to be of the ‘poor joke’ or ‘sick joke’ variety! So you might well ask, then what is the point? I think humour (poor or rich) has a way of keeping interactions with students light, and also bringing a playful aspect to learning a concept. Sometimes a bad joke gets associated with a concept and may serve as an aid to memory! Here are examples of jokes that I crack in my classes, often to the sound of my students groaning. For example, while teaching complex numbers: a negative number goes to see a psychiatrist. The psychiatrist asks the negative number: “Why are you always so negative?” To which the negative number replies “That is because I am the product of two complex entities!” Another example of a joke (suitable while teaching infinite geometric series), which is actually quite deep, goes like this. An infinite number of mathematicians go to a bar. The first one orders half a mug of beer, the second one-fourth, the third one-eighth, and so on. The bartender gives them a mug full of beer and says “Know your limit guys!”

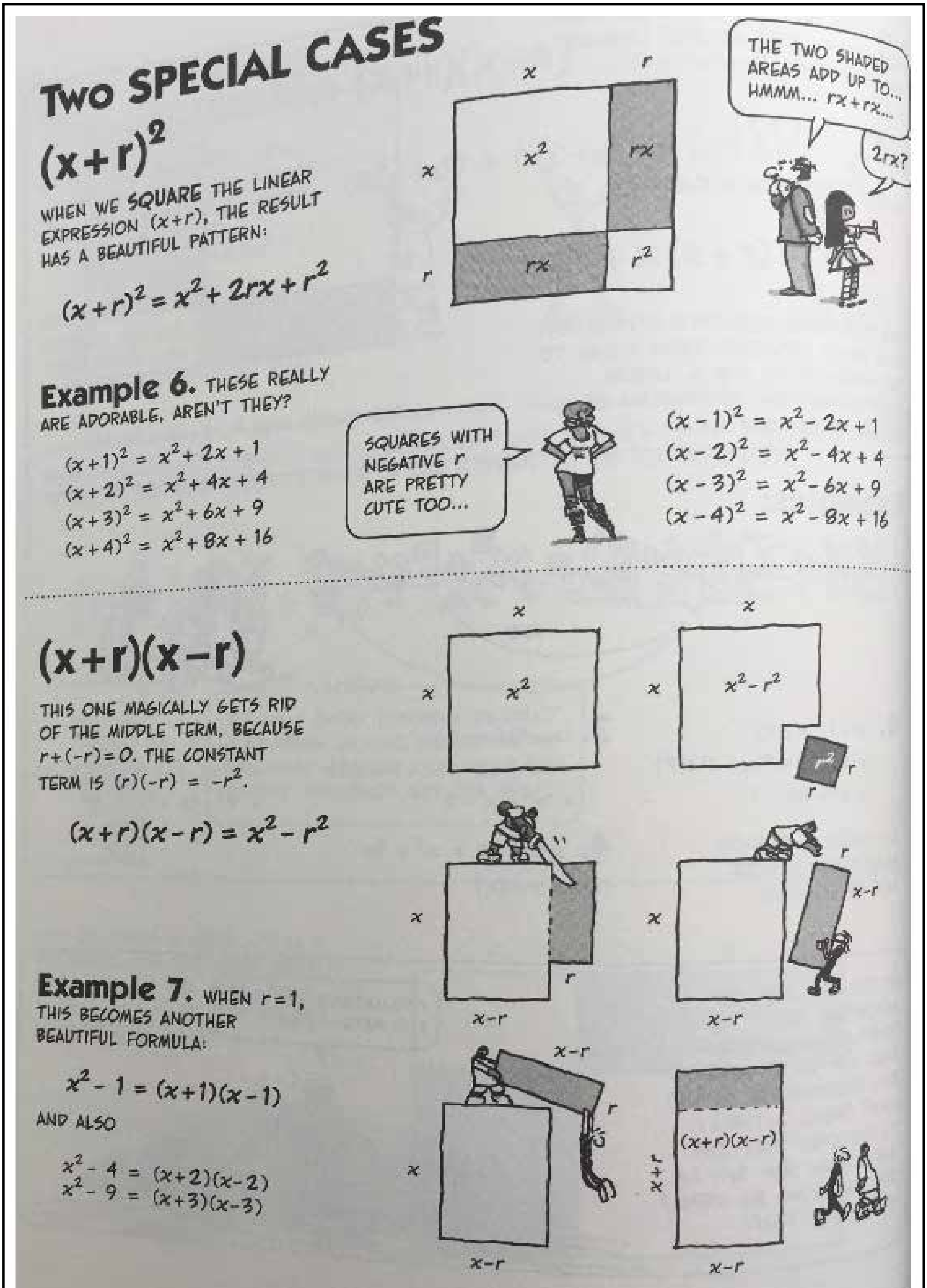

TWO SPECIAL CASES
(x+r)2
WHEN WE SQUARE THE LINEAR EXPRESSION (x+r), THE RESULT HAS A BEAUTIFUL PATTERN:
(x+r)2 = x2 + 2rx + r2
x
r
x2
rx
rx
r2
THE TWO SHADED AREAS ADD UP TO... HMMM... rx + rx...
2rx?
Example 6. THESE REALLY ARE ADORABLE, AREN'T THEY?
(x+1)2 = x2 + 2x + 1
(x+2)2 = x2 + 4x + 4
(x+3)2 = x2 + 6x + 9
(x+4)2 = x2 + 8x + 16
SQUARES WITH NEGATIVE r ARE PRETTY CUTE TOO...
(x−1)2 = x2 − 2x + 1
(x−2)2 = x2 − 4x + 4
(x−3)2 = x2 − 6x + 9
(x−4)2 = x2 − 8x + 16
(x+r)(x−r)
THIS ONE MAGICALLY GETS RID OF THE MIDDLE TERM, BECAUSE r+(−r)=0. THE CONSTANT TERM IS (r)(−r) = −r2.
(x+r)(x−r) = x2 − r2
x2
x2 − r2
r2
x−r
(x+r)(x−r)
x+r
Example 7. WHEN r=1, THIS BECOMES ANOTHER BEAUTIFUL FORMULA:
x2 − 1 = (x+1)(x−1)
AND ALSO
x2 − 4 = (x+2)(x−2)
x2 − 9 = (x+3)(x−3)

Figure 1

Figure 2

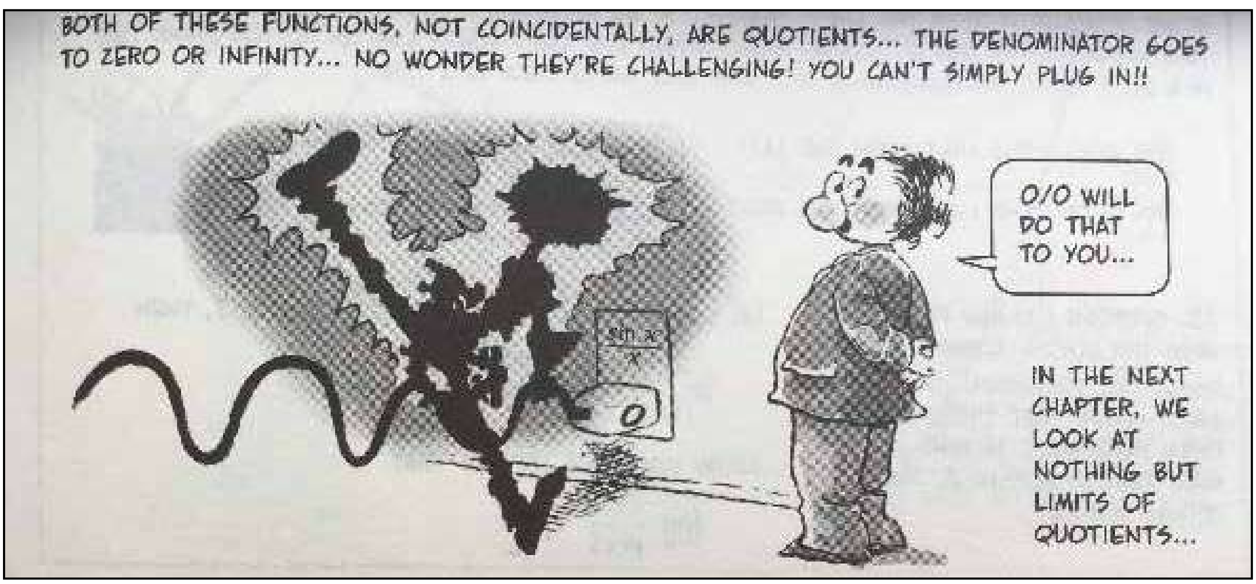


Figure 3

The Cartoon Guide to Algebra

The Cartoon Guide To Algebra has 17 chapters and starts off with very basic arithmetic, working its way up to solving quadratic equations. Along the way it covers standard topics in elementary algebra and coordinate geometry.

We are introduced to the unknown variable via arithmetic. We are then taught basic operations in mathematics, including how to deal with negative quantities, fractions, the number line, ratio and proportion, introduction to expressions, solving equations, introduction to coordinate geometry, linear and simultaneous equations and finally, quadratic equations.

There are many takeaways for a mathematics teacher. The books offer many ideas to introduce and teach concepts. For example, the author uses two 'models' to deal with negative numbers and operations with negative numbers. One is to treat real numbers as directed lengths and then have rules to show how these lengths combine with each other. The other is to treat negative numbers as debts. I liked how he illustrates the relationship between the slopes of perpendicular lines, his introduction of $a^{-n} = 1/a^n$ and his idea of checking roots of quadratic equations.

I was not too happy with the way 'zero' was treated. I feel the author could have spent more time discussing the concept as both a place holder and a number in its own right, along with its properties. The author does, however, illustrate (quite graphically!) why dividing by zero is not such a good idea (see Figure 3)!

While most of the chapters cover standard material, I found two chapters quite unusual. One is to do with rates, where Gonick introduces the 'All-purpose rate equation'

$$U = U_0 + r_U(t - t_0),$$

where U is any given quantity at time t, U_0 is the initial amount, r_U is the rate of change of U over time, and t_0 is initial time. He explains how a graph of such an equation would be a line with y-intercept U_0 and slope r_U. He then applies this equation in a variety of situations, ranging from speed and velocity to water flowing in and out of a tank.

The other unusual chapter is about weighted averages. We encounter a real-life example of computing the average monthly electricity bill, where the rate of consumption varies month by month. This chapter has been written with a great deal of humour!

The Cartoon Guide to Calculus

The Cartoon Guide To Calculus has 14 chapters and covers basic differential and integral calculus. Gonick begins by explaining the idea of instantaneous velocity and has the clever idea of introducing a so-called 'velocimeter' which is just like a speedometer, except that it assigns a negative sign to the speed of a car in reverse. Using the velocimeter he is able to introduce the notion that instantaneous velocity is closely approximated by the change in distance by change in time.
He then moves on to introducing the idea of a 'function,' via an animal which has an input and output (with predictable humour associated with such a model!). He has a nice idea of drawing two parallel lines (see Figure 4), where the input is marked on one line and the output on the other, and the function animal stands between the two, pointing with its hand to the input and its tail to the output. He then moves on to introducing us to all the standard functions, each of whose properties are very well illustrated using graphs.

Gonick spends a lot of time developing the notion of limits, and teachers will definitely benefit from using his ideas to illustrate this rather difficult concept. He then introduces the derivative using the idea of rate of change. His emphasis here is on showing that calculus is mainly concerned with understanding change and he uses speed, velocity and acceleration as his main tools to illustrate this idea. The book also covers standard applications of differentiation.

One thing that Gonick does not spend a lot of time on (except here and there) is to show how a tangent line is locally the best approximation to a curve, and consequently how calculus is extremely useful in curve sketching. I guess he had to make a choice between what to keep and what to leave out.

As in *The Cartoon Guide to Algebra*, most of the material is standard fare, and experienced teachers will recognize many techniques that one has picked up over the years to explain concepts. However, once again, I would like to emphasise that there is a lot one can learn on how to explain and illustrate difficult topics (like the Fundamental Theorem of Calculus). What I found unusual in *The Cartoon Guide to Calculus* is his treatment of the 'chain rule.' He introduces two new ideas. We need some notation here to explain these ideas.

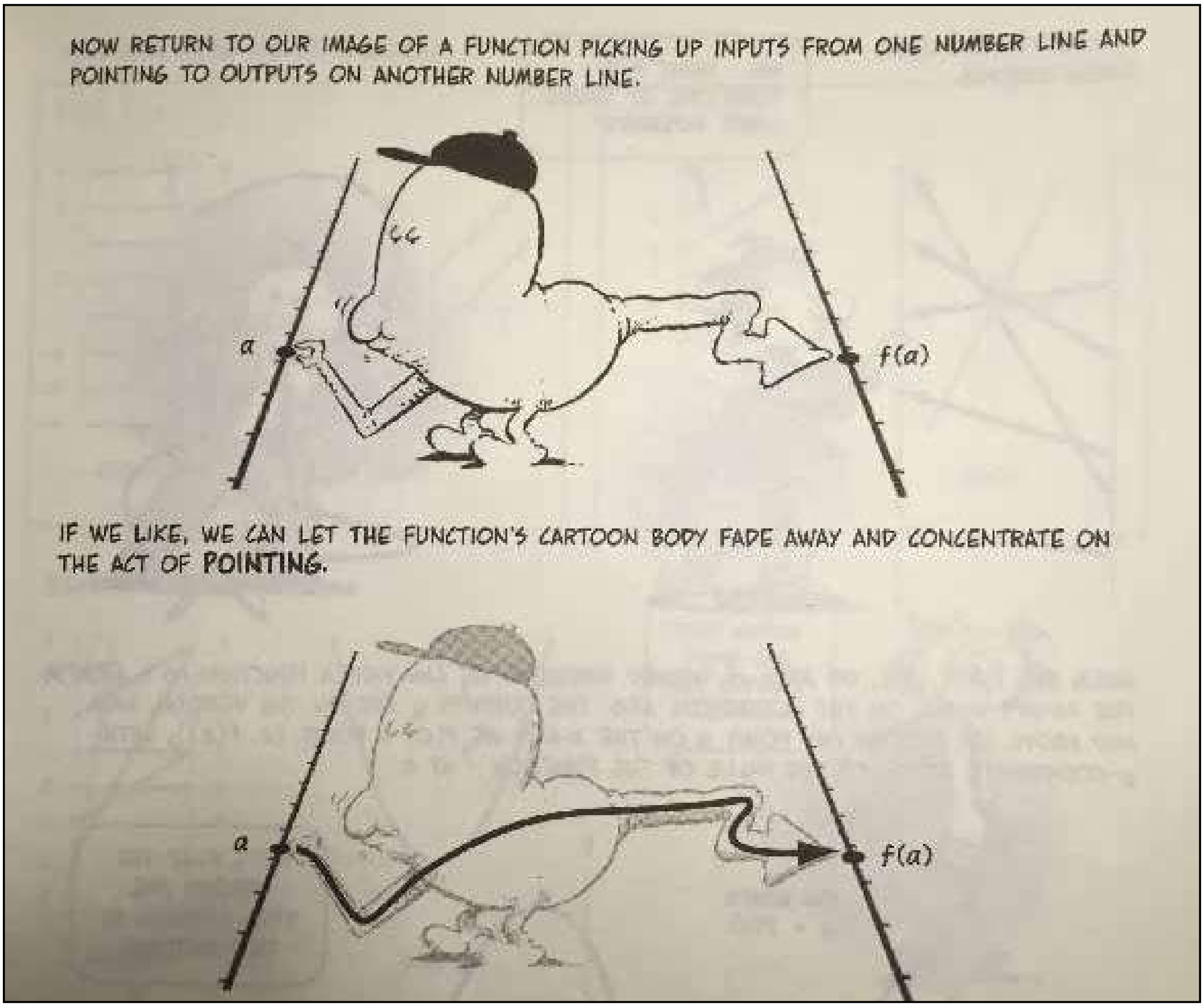


Figure 4

Let $f(x)$ be a continuous function; its derivative is $f'(x)$. As usual h is a non-zero real number and $\Delta f = f(x+h) - f(x)$. Gonick, in his irreverent manner, defines a concept called the 'Flea' to be a mathematical quantity that satisfies

$$\lim_{h\to 0} \frac{\text{Flea}}{h} = 0.$$

He shows how $\frac{\Delta f}{h}$ is the scaling factor, which when multiplied by h gives Δf. He goes on to derive what he calls the Fundamental equation of calculus,

$$\Delta f = hf'(x) + \text{Flea},$$

from this equation and the definition of Flea; we can then replace the scaling factor $\frac{\Delta f}{h}$ by $f'(x)$.

The scaling factor idea then helps us understand both the chain rule and the derivative of $f^{-1}(x)$ in terms of $f(x)$. (See Figure 5.)

The book does cover integration although the bulk of it is used to introduce basic precalculus notions and differentiation.

End Note

Obviously The Cartoon Guides cannot replace textbooks in the subjects. I think they serve as excellent supplementary material. They fulfill different purposes for the teacher and the student. Teachers will find, as I mentioned earlier, ideas to introduce concepts, illustrations that can be used in the classroom, ideas to lighten one's teaching and, perhaps, even examples and problems that can be used in class.

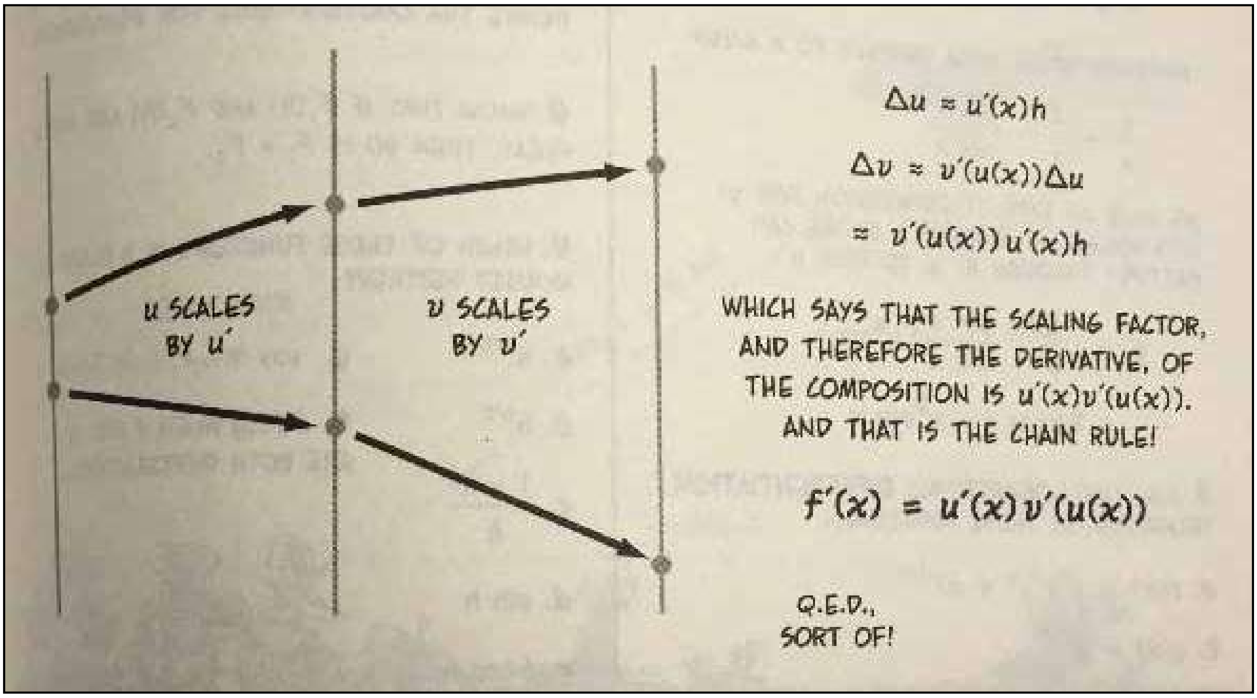


Figure 5

For students of either an algebra or a calculus course, I would use the material selectively. Perhaps ask them to read particular chapters after the material has been covered. The books will not only bring a smile onto their faces, its graphical approach will strengthen both intuitive and formal understanding. Once the material has been learned, the books will definitely serve as excellent resources for the reinforcement of concepts. They will enjoy the last chapter of both books which give previews of the more advanced topics that they will encounter, if they decide to learn higher mathematics.

I strongly recommend that school libraries stock these books. Before signing off, I would warn readers that, although the book is written with humour and wit and has some really amusing graphics, it is not a casual read. The mathematics has been treated very rigorously, so be prepared to learn and smile, and not merely be entertained!

Reference:

1. Gonick, Larry. *The Cartoon Guide to Algebra*. William Murrow, 2015.
2. Gonick, Larry. *The Cartoon Guide to Calculus*. William Murrow, 2012.

SHASHIDHAR JAGADEESHAN received his PhD from Syracuse in 1994. He is a teacher of mathematics with a belief that mathematics is a human endeavour; his interest lies in conveying the beauty of mathematics to students and looking for ways of creating environments where children enjoy learning. He may be contacted at jshashidhar@gmail.com

The Closing Bracket . . .

It is 'examination marks' season as I write this piece. Over the past few weeks, the results of various board examinations have been released – CBSE, ISC, ICSE; some other boards as well. We have seen the country go through its annual absurdity of worshipping the 'toppers'; their photographs have appeared in the newspapers, and their 'secrets of success' revealed in interviews and sound-bites. At the other end of the spectrum, there are those who either fared very poorly or did not pass. Perhaps their households are enveloped in an atmosphere of shame. Who knows what tragedies are to be enacted in the coming days and weeks in these households, how many tears will be shed.

It is sad that we worship the successful in so undignified a manner. It is a different matter that our board examinations are of no value in any deep sense, based as they are primarily on 'exam preparation'. (The same is true of our numerous entrance examinations. The fact that we have so many coaching institutes is a testimony to how meaningless they are.) To worship those who do well in such examinations seems so mindless, so silly. But what if the examinations are designed more intelligently, with more interesting questions, with questions that are more open-ended and more imaginative, questions that really test our understanding? Would we not then be 'justified' in making a fuss about those who do well in such examinations? Of course not; we are still caught in worship – the worship of success in the intellectual field. It is more sophisticated than earlier, but at bottom it is still very crude – for it is essentially the worship of power. I wonder when we are going to confront this basic fact, this feudal attitude towards power. In this country, we have it to an extraordinary degree.

Probably it is only we who are school teachers who have any hope of making a dent in this problem, because we engage with children on a daily basis and have an opportunity every day of confronting this mindset. If this be so, then school teaching becomes an incredibly precious activity, for we have the opportunity to engage with the future of our species. (This may sound like hyperbole, but it is so.) But if we do not grasp the chance and we continue to perpetuate the values of society at large, we are squandering one of our most precious resources.

Alongside the teaching of mathematics, must not we as teachers also communicate to the student and help him or her understand the shallowness of success? Must we also not help him or her understand how dangerous it is to worship power and fame? Of course we must. If we do not – and if all we do is to teach mathematics – then we are not really doing what a teacher must be doing.

Teachers may wonder whether this is going to be an additional burden; and how many such burdens can we carry, when there is already a burdensome syllabus to cover? The answer is that we really do not have a choice; or rather, the choice is corruption and ultimate destruction – of humanity and of the Earth itself. When we grasp the magnitude of the problem and the danger of the situation, it will not seem a burden; it is something that we have to do, simply because it is necessary, and we do so out of our love of life.

Let us therefore work together, all of us, to bring to life before the student, both the beauty of living and the vital importance of ending our present way of thinking.

Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

Printed and Published by Manoj P on behalf of Azim Premji Foundation for Development;
Printed at Suprabha Colorgrafix (P) Ltd., No. 10, 11, 11-A, J.C. Industrial Area, Yelachenahalli, Kanakapura Road, Bangalore 560 062.
Published at Azim Premji Foundation for Development, Azim Premji University Pixel B Block, PES College of Engineering Campus, Electronic City, Bangalore 560 100; **Editor:** Shailesh Shirali

Call for Articles

At Right Angles welcomes articles from math teachers, educators, practitioners, parents and students. If you have always been on the lookout for a platform to express your mathematical thoughts, then don't hesitate to get in touch with us.

Suggested Topics and Themes

Articles involving all aspects of mathematics are welcome. An article could feature: a new look at some topic; an interesting problem; an interesting piece of mathematics; a connection between topics or across subjects; a historical perspective, giving the background of a topic or some individuals; problem solving in general; teaching strategies; an interesting classroom experience; a project done by a student; an aspect of classroom pedagogy; a discussion on why students find certain topics difficult; a discussion on misconceptions in mathematics; a discussion on why mathematics among all subjects provokes so much fear; an applet written to illustrate a theme in mathematics; an application of mathematics in science, medicine or engineering; an algorithm based on a mathematical idea; etc.

Also welcome are short pieces featuring: reviews of books or math software or a YouTube clip about some theme in mathematics; proofs without words; mathematical paradoxes; 'false proofs'; poetry, cartoons or photographs with a mathematical theme; anecdotes about a mathematician; 'math from the movies'.

Articles may be sent to :
AtRiA.editor@apu.edu.in

Please refer to specific editorial policies and guidelines below.

Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to 'At Right Angles'.

While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

An in-depth, serious magazine on mathematics and mathematics education.

For teachers, teacher educators and students connected with the subject.

In this magazine, teachers and students can:

- Access resources for use in the classroom or elsewhere
- Read about mathematical matters, possibly not in the regular school curriculum
- Contribute their own writing
- Interact with one another, and solve non-routine problems
- Share their original observations and discoveries
- Write about and discuss results in school level mathematics.

Publisher

Azim Premji University together with Community Mathematics Centre, Rishi Valley.

Editors

Currently drawn from Rishi Valley School, Azim Premji Foundation, Homi Bhabha Centre for Science Education, Lady Shri Ram College, Association of Math Teachers of India, Vidya Bhavan Society, Centre for Learning.

You can find At Right Angles here:

Free download from

http://azimpremjiuniversity.edu.in/SitePages/resources-at-right-angles.aspx

AtRiA is also available on http://teachersofindia.org/en/periodicals/at-right-angles

At Right Angles is available as a free download in both hi-res as well as lowres versions at these links. Individual articles may be downloaded too.

Hard Copy

At Right Angles magazine is published in March, July and November each year. If you wish to receive a printed copy, please send an e-mail with your complete postal address to at AtRightAngles@apu.edu.in

The magazine will be delivered free of cost to your address.

On FaceBook

https://www.facebook.com/groups/829467740417717/

AtRiUM (At Right Angles, You and Math) is the Face - Book page of the magazine which serves as a platform to connect our readers in e-space. With teachers, students, teacher educators, linguists and specialists in pedagogy being part of this community, posts are varied and discussions are in-depth.

On e-mail: AtRiA.editor@apu.edu.in

We welcome submissions and opinions at AtRiA.editor@apu.edu.in The policy for articles is published on the inside back cover of the magazine.

Your feedback is important to us. Do write in.

Azim Premji University

PES Institute of Technology Campus, Pixel Park, B Block, Electronics City, Hosur Road (Beside NICE Road), Bangalore – 560100 India
Email: reachus@apu.edu.in

At Right Angles
A Resource for School Mathematics

Vol. 6, No.2, August 2017

THE *SECRET* WORLD OF LARGE NUMBERS

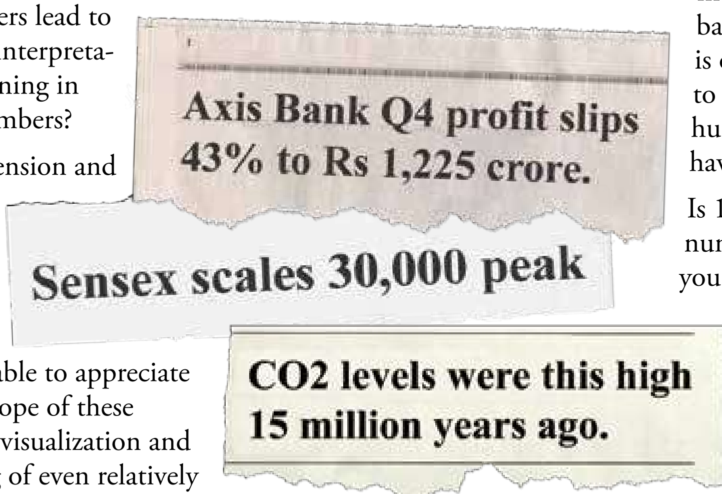
PADMAPRIYA SHIRALI

Azim Premji University

A publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley

The secret world of large numbers.

Increasingly, we live in a world where large numbers seem to bombard us from all directions. Thanks to globalization and knowledge explosion, encounters with very large numbers happen on a regular basis. Yet, one pertinent question is: 'Does increased exposure necessarily lead to an understanding of the size of these numbers?' Or is it possible that we underestimate the size of these due to their constant use! Does excessive familiarity prevent right perspective? Also, will not a lack of understanding of large numbers lead to an ineffective interpretation and reasoning in using such numbers?


Axis Bank Q4 profit slips 43% to Rs 1,225 crore.

Sensex scales 30,000 peak

CO2 levels were this high 15 million years ago.

Our comprehension and the students' understanding of the world remain incomplete unless we are able to appreciate the size and scope of these numbers. Yet, visualization and understanding of even relatively small numbers like ten thousand and a lakh (hundred thousand) may be difficult for many people. When does one see ten thousand units of some object? Do one's every day experiences include such numbers?

Do such numbers arise in the child's experience of life? If not, how does the teacher help the child to mentally visualize such numbers? Large numbers are incomprehensible to students unless they are able to relate these figures to their own experiences, say of crowds, etc. We often hear statements like 'there were thousands of people at the rally' when the numbers may well have been a few hundreds. Is that a lack of skill of estimation? Is it a lack of sense of a thousand?

Now we will get to the question: When does a number get viewed as a large number? That is relative and it is context dependent. Is 10 a large number? It doesn't seem so. However we are aware that the brain can recognize up to 4 or 5 objects without counting. It would not be able to recognize 10 objects at a glance. 10 is too big for the brain to do that!

However, all of us recognize that 10 is a very important number for us as our decimal system is based on 10. This is obviously due to the fact that human beings have 10 fingers.

Is 100 a large number? For many young students 100 seems quite large. And they are proud of being able to rattle off the numbers up to 100. A hide and seek game is often started off by a count of 100. 100 runs is an achievement if one is a batsman. 100 certainly is impressive! 100 does give us a sense of largeness and yet it is humanly attainable. One can skip 100 times, run a 100 metre stretch easily. We routinely talk about distances in hundreds of kilometers. We purchase clothes for hundreds of rupees.

How about a 1000? Nobody would embark on counting up to a 1000. It is too large to do that. 1000 is also a huge deal in our world and is called by different names: a kilo, and k or G. A 1000 rupee note (in the days before demonetization) was considered too big to be used while paying for a bus

Keywords: numbers, magnitude, large, comprehension, lakh, crore, million, billion, trillion, light year, place value

ticket or an auto ride. However many things are priced in thousands. It is large, yes, but it is also fairly common-place.

Interestingly the Roman numerals go up to a thousand (I V X L C D M).

Is a lakh a large number? Or is a million a large number? Yes, these are very big numbers but they are not beyond human comprehension. We talk about populations in millions, distances in the solar system in millions of kilometers, prices of houses in lakhs and crores.

Yet at some point as students begin to explore the secret world of numbers, they will meet billions, trillions and even bigger, fearsome, awe inspiring numbers which have majestic names and are too difficult to comprehend. They will meet such numbers in physical sciences (astronomical distances, atoms), in life sciences (number of cells, number of blood cells, RBC or WBC in a drop of blood). There is a need for benchmarks to comprehend the magnitudes of such large numbers.

So, how do we help students in developing a sense of large numbers?

While introducing large numbers to students what are our aims?

- Skill of reading and writing large numbers, of course. However, merely helping students with procedural skills of reading and writing large numbers does not aid in building a number sense. They need to appreciate the size of a number and an understanding of the relative sizes of numbers.
- Understanding the relationships between different places. Our number system is a language describing quantities and relationships between quantities. Students need to be able to understand and use these relationships flexibly and with ease. For example: They need to understand that 10 thousands = 100 hundreds = 1000 tens = 10,000 ones. A million is 1000 thousands. It is a thousand times a thousand.
- To make sense of large numbers in context, whether it is in science, social science or business.
- To meet students' natural fascination for large numbers.
- More importantly to develop their mathematical and logical power in building an appreciation of such numbers.

Here is a set of questions we can work with to enhance procedural skills, relational skills and a sense of the size.

How does our knowledge of place value relate to the way we read large numbers? How is the value of a digit determined in a place value system? How are the different places in a place value system related to one another? How much bigger is a crore than a lakh? How much smaller is a thousand than a million? In what ways does estimation assist in appreciation of the size of a number? In what way does the usage of a bench-mark, a reference point, useful in understanding the magnitude of a number?

Introduction of Numbers at the Upper Primary Level.

Revisit quickly the usage of standard materials (cubes, flats, longs, units) to represent numbers up to a thousand (by drawing them on the board). It is important not to assume that students will remember the conceptual basis of the place value system. Remind them that the value attributed to a digit depends on its place with reference to the rightmost place (units). Review and establish the principle of relationship, that each place is ten times the place preceding it.

Once students have understood the grouping pattern till ten thousand, they will not have difficulty in extending the place value system. In the Indian system, the naming is in blocks of two (thousand, ten thousand and lakh, ten lakh, etc.) Are the students learning to apply the rules of the place value system (each place is ten times the place to its right) to understand the value of the new places?

In the international system, the naming of these places follows a simple pattern in blocks of three (thousand, ten thousand, hundred thousand and million, ten million, hundred million, etc.) However, note that the term billion has two different meanings. A billion is 1000 million in the American system. The European billion is a million million.

Students will naturally become curious and ask 'What is the next place called? How far can we go?'

At this point they can use the internet to find the names of some interesting numbers. What number is one followed by a hundred zeroes?

Discussion can also lead to 'What is the largest number?' The discovery that numbers will never end can come as a great surprise! Let students come upon it on their own.

Sometimes students hold a mistaken notion that infinity is the largest number. Help them to understand and distinguish between the concept of infinity and very large numbers! Infinity is not a number. It means that numbers continue endlessly. Link it to other contexts of infinity. Example: How many points can be found between any two given points?

As students begin to explore astronomical distances, they will encounter a new measure: the light year. Discuss the problem of expressing the huge distances between stars and galaxies and the historical need to invent light years. A light year is the distance light travels in a year; it is slightly less than 10 trillion kilometers.

Exposure to exponential numbers, number representation in calculators and scientific notation will be the next step on the ladder.

TEACHING AIDS

Aid A: Reading in stages!

Objective: To build the skill of reading number names.

Materials: Large number cards as shown in the picture (The example here is in Indian system.)

Note: Thousands and ten thousands together are considered as a family of thousands, similarly lakhs and ten lakhs together belong to the family of lakhs, etc. They are read together.

Students are already familiar with reading numbers up to ten thousand or a lakh by this stage. While introducing lakh and crore, one can start by helping students to read numbers with which they are already familiar, and then name the new places (lakh and crore).

Open the card in stages as shown, revealing hundreds, tens and units first. (017) Read seventeen.

Follow it up by opening thousands block (18,017) (note: both thousands and ten thousands are read together). Read it as eighteen thousand and seventeen.

Follow it up further by opening up the lakhs block (both lakhs and ten lakhs are read together).

(02,18, 017) Read it as two lakh, eighteen thousand and seventeen.

Finally open the full number (3,02,18,017). Read it as three crore, two lakh, eighteen thousand and seventeen.

Aid B: Reading in blocks (Example in international system.)

Objective: To facilitate reading in blocks.

This is according to the American system where one billion equals 1000 million.

Materials: Place value charts as shown in the picture.

Use structured coloured reading charts to help in learning, to read in blocks (in stages).

Note: In the international system thousands, ten thousands and hundred thousands belong to the family of thousands. Similarly millions, ten millions, hundred millions belong to the family of millions, etc.

6,247,148.

Help the child to read the digit in the millions place first (from the millions family). In order to identify the place value of 6, students will need to name the places in order from the right side. Read it as six millions.

Thousands family members are read together. Two hundred and forty seven thousand. Read it as six millions, two hundred and forty seven thousand.

Finally the full number is read. Read it as six million, two hundred and forty seven thousand, one hundred and forty eight.

Aid C: Writing in blocks!

Objective: To facilitate writing in blocks.

Materials: Place value charts as shown in the picture.

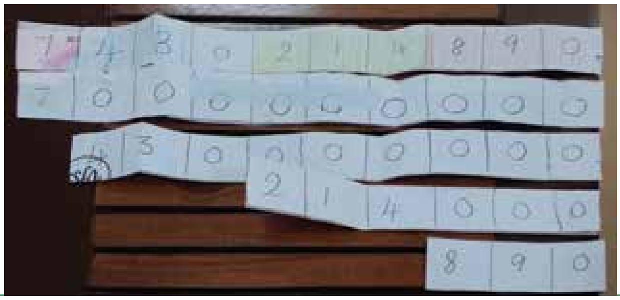
They will place zeroes in the missing spaces.

Students should initially be encouraged to use place value charts to record the numbers read out.

Aid D: Decompose!

Materials: Expanded notation cards as shown in the picture.

Objective: To break up the number into its components.



Numbers can be initially decomposed using family groups together.

Later they can be decomposed separately, i.e., hundred thousand separately, ten thousand separately, thousand separately, etc.

Aid E: Build and read!

Objective: Scaffolding for learning to read large numbers

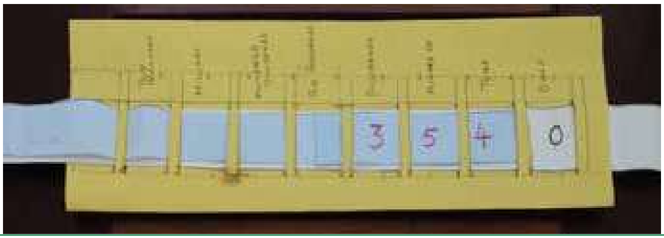
Materials: 3 digit number groups

Students can build numbers and practise reading them. Each set of three digits will be treated as a family while reading.

Aid F: Slide fun!

Objective: To understand the relationships between different places.

Materials: Slide, one background card with zeroes written on it, changeable large number cards in the front as shown in the picture.



Use a slide as shown in the picture. Slide the front card to demonstrate multiplication by 10 or 100 or 1000.

Note: Are all numbers read with place values? Think of phone numbers, registration numbers and identification numbers. They look like very big numbers, but they are actually code words rather than numbers, and place value has no meaning for such objects.

Problem Solving

Here is a set of problems which can help build quantitative literacy and increased number sense. The problems are posed. They require students to do mathematics! (This means: make some assumptions, discuss and use resources, if necessary, and then compute). The students see that the results apply to real world situations and are obtained by estimation and approximations. They can also use the estimations to check the reasonableness of a given answer. While solving these problems, students should develop their own steps and methods. The process of organized thinking in developing strategies to reach an estimate becomes more important than the actual answer itself.

'Learning to think algorithmically builds mathematical literacy.' (NCTM)

I have suggested a possible approach which students could follow only for activity 1 and 3. It is of the utmost importance to let the students work out the strategy. It is also equally important to record the assumptions made, the strategy adopted, the approach (the in-between steps) and findings.

ACTIVITY 1: **HOW MANY GRAINS OF RICE DO I EAT EVERY DAY?**

Objective: To build an understanding of the relative magnitude of numbers to a thousand.

Materials: A cupful of rice grains, paper plates

Place 1 grain on the first plate.

Count out 10 grains and place on the second plate.

Pick up approximately 10 grains at a time. (Not counting at this point). Do this ten times to place approximately 100 grains on the third plate.

Now pick up approximately 100 grains at a time. (Not counting). Do this ten times to place approximately 1000 grains on the fourth plate.

At the end the teacher could share the normal measure of rice a person eats per day. Students can pour their 1000 grains into a measuring cup to judge how much they eat in a day.

ACTIVITY 2: **CAN WE GO ROUND THE WORLD?**

Fact: The diameter of the earth is 12,756 kilometers.

Objective: To build an understanding of ten thousand.

Problem: Consider ten thousand students holding hands and standing in a line.

How far would they reach?

Would they go across a football field?

Across your state?

Across India?

Around the world at the equator?

ACTIVITY 3: **HOW MANY ARE WATCHING CRICKET?**

Fact: Before its renovation in 2011, India's Salt Lake Stadium seated 1,20,000.

Objective: To build an understanding of a lakh.

Approach: Students can count the number of people their classroom can seat or estimate the number of people the school assembly hall can seat. They could find out the capacity of a theatre or auditorium they are familiar with and proceed from there. They can use rounding where necessary.

Discuss students' suggestions for finding out the answers.

Encourage them to record their findings in writing.

1,20,000 is roughly the same as 2,000 of our classrooms.

1,20,000 is roughly equal to the number of seats in 600 theatres.

ACTIVITY 4: **MESSY HAIR!**

Fact: People have about 100,000 hairs growing on their heads!

Objective: To build an understanding of a hundred thousand.

Materials: A few graph papers from a millimeter graph book

Give students a millimeter graph paper.

Problem: Find out how many sheets would be needed to have 100,000 little (millimeter) squares.

ACTIVITY 5: **REACHING THE MOON!**

Fact: Our moon is at a distance of 3,84,400 km from the earth.

Objective: To build an understanding of a lakh in the context of height.

Problem: Consider 1 lakh students, each standing on the shoulders of another.

How high would they reach?

Would they be as high as the Qutub Minar?

As high as the tallest building in the world?

As high as a satellite?

As high as the moon?

ACTIVITY 6: **HOW MUCH SPACE?**

Objective: To build an understanding of a million in the context of capacity.

Materials needed: A set of cubes

Problem: Estimate how much space a million cubes would occupy.

Would they fit in the cupboard?

Would they fit in the class room?

Would they fit in the assembly hall?

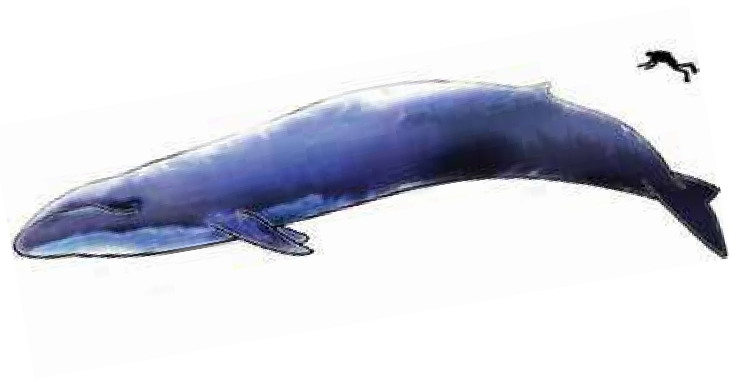
ACTIVITY 7: **HOW HEAVY?**

Fact: The blue whale weighs approximately 140,000 kilograms.

An Asian elephant weighs 5,500 kilograms.

Objective: To build an understanding of a hundred thousand in the context of weight.

Can students use knowledge of their own weight to understand these stupendous weights?



ACTIVITY 8: **AM I WELL-READ?**

Objective: To build an understanding of a hundred million.

Materials: Printed paper from any book.

Give students a paper printed fully with text.

Problem: Find out how many books would be needed to have read a hundred million letters (characters).

Once the activity is over, they can assess how well-read they are!

ACTIVITY 9: **HOW OLD AM I?**

Objective: To build an understanding of a million and billion in the context of time.

Problem: How old are you if you have lived for a million seconds?

How many seconds old will you be on your 10th birthday?

Does anyone live for one billion seconds?

ACTIVITY 10: **IF I WERE RICH!**

Objective: To practise writing big numbers while writing cheques.

Problem: How will your class spend a million rupees?

A problem like this can be placed in a context. Your class is going on an expedition to Antarctica for one week. What do you need for the expedition? Food? Clothing? Gadgets?

Students can get data about prices of different things from a wish list.

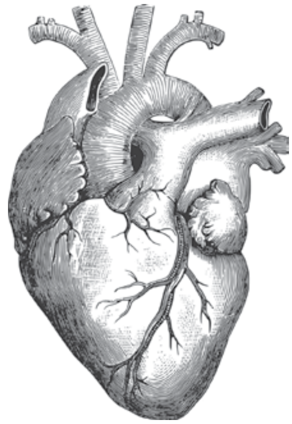
ACTIVITY 11: **WALKING IS THE BEST!**

Objective: To develop a sense of million in the context of distance.

Problem: Will I ever walk a total of a million km?

How will students find out their walking rate? How many hours can a person walk per day? Are the students realistic in their problem solving approach?

ACTIVITY 12: **MY HEART IS BEATING!**

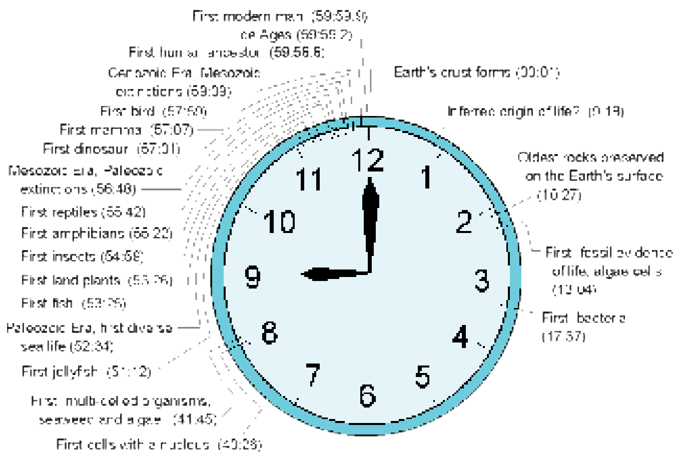


Objective: To develop a sense of billion in the context of time.

Problem: How many times would my heart beat if I live up to 80 years?

What information is needed? How will students use scaling as an approach?

Can the information be obtained by counting beats per minute? Is there an exact answer? Does it vary?



4.6 billion years in one hour

ACTIVITY 13: **HUMAN HISTORY TICKING AWAY!**

Objective: To develop the skill of using a 24 hour clock to represent the arrival of human beings on earth. (There is an implicit usage of the notions of scaling and proportion here).

Problem: Students can construct a 24 hour clock to depict history of earth and the recent arrival of man on earth.

What information is needed? How many years are represented by each hour?

When did the earth have dinosaurs? In terms of hours, how long ago is that?

What is the relationship between a place value system and the number of digits it uses?

The Hindu- Arabic numeration system uses base 10, notion of place value and zero in a dual role, as a digit as well as a place holder. The Hindu-Arabic numeration system allows very large numbers to be written.

Point out the way the Roman number system works. In the Roman system a few letters represent certain quantities. Each letter stands for the same value no matter where it is placed. Zero is not represented in any form.

I II III IV
V VI VI VIII
IX X XI XII
L C D M

ACTIVITY 14: **CONTRAST ROMAN NUMBERS AND HINDU ARABIC NUMBERS!**

Objective: To appreciate the efficiency of the Hindu-Arabic system.

Pre-requisite: Familiarity with Roman numbers

I = 1, V = 5, L = 50, C = 100 and M = 1000.

If a heavy bar is placed over the numeral that means it is multiplied by 1000.

For example a V with a bar over it would stand for 5000. An M with a bar would be 1,000,000.

Problem: Students can be given a few numbers (up to a lakh) in words and asked to represent them in both the systems. In the process of writing and reading the numbers they will understand the efficiency as well as simplicity of the Hindu-Arabic system.

How would 50,000 be written in the Roman system?

They can also be given some addition problems to solve in the Roman system first and verify using the decimal system.

Example: 23 + 58

How would we write these as Roman numerals?

23 is XXIII and 58 is LVIII.

If we bring them together they become XXIII LVIII.

Now, we have to arrange these in descending order. It becomes LXXVIIIIII.

We notice that there are six I's. That is the same as VI.

Now the number is rewritten as LXXVVI.

Again, there are two V's. That is an X.

Now the number is rewritten as LXXXI.

We can now use the Hindu-Arabic system to check if our answer is right.

Note: At this point, talk about Aryabhata and his greatest contribution, the concept of zero. While he did not use the symbol which we use now for zero, he did develop the place value system which is based on zero.

PROJECTS

Let students research on the following topics:

Dictionary: How many words in the dictionary?

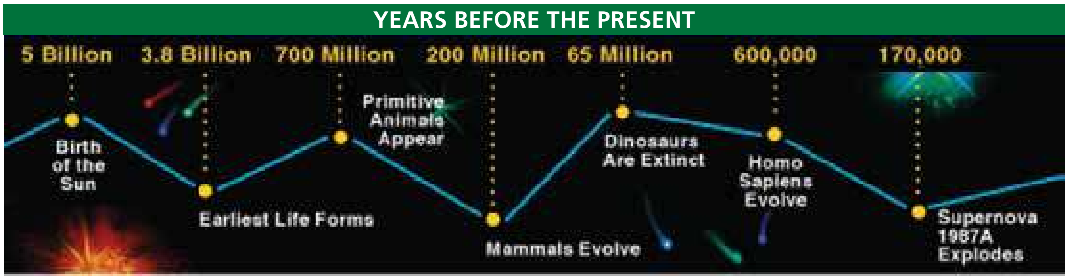
There are about 8,000,000 words in the English language!

Universe: How many stars in a galaxy?

On a perfectly clear night you can see about 2500 stars in the night sky!

Solar system: Building a solar system model (scaled version with all distances and diameters proportional to the actual).

Earth: Age of the earth



Population: Population figures of different states in India.

Human body: Number of cells

One drop of blood can contain about five million red blood cells!

Budget: Allocation of funds to different projects.

Games: Use Monopoly or Business game to give practice in using large numbers in the context of money.

More problems to explore!

How many times is the Qutub Minar taller than you?

How many twelve year olds are there in India?

How many kilometers of railroad tracks are there in India?

How many ants are there in an ant colony?

GAME 1: **Make the number**

Objective: Practice in reading 8 digit numbers

Materials: 3 sets of Number cards with digits 0 to 9, Task cards (about 10)

Tasks can be ‘Make the number closest to 5,00,00,000’, ‘Make the largest 8 digit number’, ‘Make a 8 digit number which has the largest digit in ten thousands place’.

Number of players: 4

Both number cards and task card are placed face down. The first player selects 8 number cards and one task card from the pile. The player arranges the cards to fit the task best. If another player can suggest a better way of performing the task the first player does not get a point. Else the first player gets a point. Each player takes turns to play. After a few rounds the player with maximum points wins.

Game 2: **Hear! Build!**

Objective: Practice in reading and writing 8 digit numbers in Indian system

Materials: Challenging eight-digit number cards

Number of players: 2

The number cards are placed face down. The first player selects a number card from the pile and reads it out without letting the other player see the number. The second player writes the number down. (Restrictions can be placed: No corrections are allowed after the number is written down). This is followed by the second player selecting a number and the first player writing it down. For every round, a correct entry gets the player one point.

There were 100 zeroes in this pullout; now it's 102; now 103; now 104;...

Reference: Please also refer to: https://sites.google.com/site/largenumbers/home/1-1/new_intro

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at padmapriya.shirali@gmail.com