At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

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Azim Premji University

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PULLOUT
THE WORLD OF INTEGERS



Ask a child (or even the average adult) what his/her image of a mathematician is and you will more often than not be subjected to words such as nerdy, boring, not-connected-to-reality, difficult-to-understand, abstract…….in fact the words very often tend to describe their impression of mathematics, almost as if the subject has literally absorbed the student. Visual images are even more predictable- glasses, a general air of dishevelment, total unconcern about clothes…….very often, movies about mathematicians are to blame for painting such unattractive images. Which is why, when one comes upon a mathematician such as Marcus du Sautoy, one would prefer him to be the brand ambassador of such an intriguing, absorbing and beautiful subject. Marcus du Sautoy is the Charles Simonyi Professor for the Public Understanding of Science at the Oxford University, a chair he holds jointly at the Department of Continuing Education and the Mathematical Institute. But he is also an accomplished author, speaker, television game show presenter, Super Geek consultant for a series of children's books, newspaper column writer, mathematical advisor for several theatre productions, presenter of concerts which explore the connection between mathematics and music……..…….and he has received awards for promotion of mathematics to the public. Small wonder that in in 2004 Esquire Magazine chose him as one of the 100 most influential people under 40 in Britain and in 2008 he was included in the prestigious directory Who's Who. Recognition of his multifaceted talents has been public and plentiful – in his personal life, he plays the trumpet and football. Like Beckham he also plays in a prime number shirt, no 17, for Recreativo FC based in the Hackney Marshes. Born in 1965, he lives in London with his wife, three children and cat Freddie Ljungberg.

I am sure that most mathematicians live rich and varied lives. Would that the public become as aware of this!

Biographical details retrieved from
https://www.simonyi.ox.ac.uk/about-marcus/

From the Editor's Desk . . .

About this issue…

The word *'heuristics'* has always left me fumbling for the perfect definition. *Rule of thumb* seems adequate but too informal, as a teacher-trainer, I've always preferred to illustrate rather than define this word. And the first article in the November issue does just that, Gaurav Bhatnagar leads you gently into the **Exponential Series- a heuristic definition!** The relaxed tone continues with V. G. Tikekar setting the stage to arrive at formulae which are usually given and proved using mathematical induction. He uses the narrative mode to set the context which explains and then proves the sum of squares and sum of cubes formulae. From story we move to song, V. Ramprashanth's article **Tonic Ambiguity** is a delightful depart from our usual articles and it explains very simply, using just the concepts of ratios, how two hearers can place the same tones at different positions on the musical scale. CoMaC and then Shailesh Shirali, wrap up the Features section with a bouquet of proofs.

ClassRoom opens with Prithwijit De's trademark investigative style – the title simply says **A Triangle Problem** – read on to find out more. Student Corner is our happy space and this time, we have Vinay Nair's students talking about **Happy Numbers** and Bodhideep Joardar **Investigating Pi**. Next, we have Naveen Yadav combining Set Theory, Geometry and Combinatorics in **Scalene Triples** and Ujjwal Rane following up on his previous article on **Fagnano's Theorem** with a very visual, very practical alternative proof. A video of this result has also been uploaded by him on YouTube, it is available at https://youtu.be/5MrNM-VxXd8. **Low Floor High Ceiling** features a modification of an investigation into triangles having the same area and perimeter and the implied conditions for congruency. Learn about **G Numbers** in Haragopal's article. We're particularly delighted to bring in yet another new sub-section to ClassRoom – Course Correction features vignettes from pedagogical experience, ways in which pitfalls leading to misconceptions can be addressed and used to advantage to strengthen student understanding. This time, we feature the solution of a problem from a traditionally ambivalent topic, **Combinatorics**.

On to TechSpace, we have not one but two articles this time. Thomas Lingefjard describes the use of technology to introduce concepts in **Calculus** and Sangeeta Gulati describes changes in the **Desmos** software and how it can now be used to build activity kits. Problem Corner is a treasure trove of interesting problems in addition to the usual Middle School, Senior School and Adventures in Problem Solving articles.

Review features an old classic – R. Ramanujam takes us through a lovely guided tour of Hardy's *A Mathematician's Apology*. And there is an unexpected bonanza, starting with this issue, we will feature a mini-review, it may be of a video clip, a game, a story……..all things mathematical in essence. This time, J. Shashidhar talks about Marcus du Sautoy's *Story of Mathematics*. Read the facing page to know more about this fascinating mathematician.

From the Editor's Desk . . . (Contd.,)

PullOut has graduated to Upper Primary – with this issue, we bring you a compendium of information about teaching **Integers** – surely the point when students begin to have negative thoughts about mathematics. I'm sure you'll enjoy reading how to combat those tendrils of fear.

Enjoy your read and do send feedback to AtRiA.editor@apu.edu.in

Sneha Titus
Associate Editor

Chief Editor

Shailesh Shirali
Sahyadri School KFI and
Community Mathematics Centre,
Rishi Valley School KFI

Associate Editor

Sneha Titus
Azim Premji University

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Fundamental Research, Mumbai

Sneha Kumari
Azim Premji University

Design

Zinc & Broccoli
enquiry@zandb.in

Print

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Rishi Valley and Sahyadri School, Pune

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Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

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This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

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We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

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PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Padmapriya Shirali
Integers

On the sums of powers of NATURAL NUMBERS Part 1

V.G. TIKEKAR

Introduction

The formulae to obtain the sum of the first n natural numbers, the sum of the squares of the first n natural numbers and the sum of the cubes of the first n natural numbers are generally introduced to students during their final school year. Generally, these formulae are derived using the celebrated technique of mathematical induction. But the defect of this approach is that we must know in advance the formula to be proved. This is clearly a great handicap.

This article consists of two parts. In Part 1, we construct situations or 'stories'; in solving the problems posed in these narratives, we derive the different formulae. By relating the mathematics to real-life situations, teaching and learning become lively and enjoyable. It is hoped that this method will encourage mathematics teachers to create relevant stories to introduce some topics of mathematics, right from the early years of mathematics education.

In Part 2, we present a remarkable triangular arrangement of numbers — somewhat like the famous Pascal triangle — which helps in obtaining an expression for the sum of the k-th powers of the first n natural numbers, for any given positive integer k.

Keywords: Sums of powers, natural numbers, squares, cubes, choice, combinatorics, narrative, story

Sum of the squares of the first n natural numbers

Problem 1. Suppose a patient has to undergo, on the advice and prescription of an expert physician, three medical tests A, B and C (say a urine test, a blood test, and an X-ray, to be specific). The physician has imposed certain conditions on these tests:

(i) Tests B and C must be done only after test A is done.

(ii) Test A takes a full day; no other test can be carried out that day.

(iii) Each test can be completed on a single day; its implementation does not spill over to the next day.

(iv) Tests B and C can be carried out on the same day, or on separate days, in either order. When two tests are done together on a single day, we consider them to be a single test. The order in which the tests are performed is of no importance in this case.

(v) The tests must be completed within $n+1$ days numbered $1, 2, 3, \ldots, n+1$.

The question now is: In how many different ways can tests A, B, C be performed?

Solution. Test A cannot be done on day $n+1$, as then no day will be left for the other tests, in view of conditions (i) and (ii). Let test A be done on the k-th day ($k = 1, 2, \ldots, n$). Having made this choice, the days available for tests B and C are $k+1$, $k+2$, …, $n+1$, i.e., $n+1-k$ in number. So having fixed day k for test A ($k = 1, 2, \ldots, n$), tests B and C can each be performed in $n+1-k$ different ways. As there is no restriction on tests B and C (in terms of which is done first), they can be done in $(n+1-k) \times (n+1-k) = (n+1-k)^2$ different ways for each value of k (from $k = 1, 2, \ldots, n$). So the total number of different ways in which all the tests A, B, C can be carried out is equal to

$$\sum_{k=1}^{n} (n+1-k)^2 = n^2 + (n-1)^2 + \cdots + 2^2 + 1^2, \tag{1}$$

i.e., the sum of the squares of the first n natural numbers.

Now let us look at the problem in a different way. We can choose either 2 days for the tests or 3 days (from the available $n+1$ days). Since 2 days can be chosen in $\binom{n+1}{2}$ different ways, this gives the number of ways of carrying out the tests in the event that we do the tests on 2 days. Similarly, 3 days can be chosen in $\binom{n+1}{3}$ different ways. Let the three days be u, v, w where $u < v < w$. Tests A, B, C can be conducted on days u, v, w **or** on days u, w, v respectively. Thus each such selection of three days gives rise to **two** different ways in which the tests can be carried out. This implies that the number of different ways in which tests A, B, C can be conducted is

$$\binom{n+1}{2} + 2 \cdot \binom{n+1}{3}. \tag{2}$$

From (1) and (2), we see that we have obtained two different expressions for the number of ways in which we can conduct the tests. The two expressions must yield the same number, which means that

$$n^2 + (n-1)^2 + \cdots + 2^2 + 1^2 = \binom{n+1}{2} + 2 \cdot \binom{n+1}{3}. \tag{3}$$

By simplifying the expression on the right side, we get an expression for the sum of the squares of the first n natural numbers:

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{4}$$

Thus by casting the problem in a relevant setting, we have derived the result.

The readers will, we hope, appreciate this derivation as very different from the 'standard derivation' based on mathematical induction, and will also see the merit of this new way of deriving the result.

Sum of the cubes of the first n natural numbers

Problem 2. A devotee is going to stay for $n+1$ days in a dormitory attached to a temple. During these $n+1$ days, the devotee proposes to offer four different rituals in the shrine: A, B, C, and D. Taking into consideration the traditions associated with these offerings, some restrictions have been placed on the rituals:

(i) Ritual A must precede rituals B, C, and D.

(ii) No other ritual must be performed on the day that ritual A is done.

(iii) Any ritual can be completed on a single day; no ritual will spill over to the next day.

(iv) Rituals B, C, D can be performed together on a single day or on separate days; or two of them can be performed on one day and the remaining on some other day. When two or more rituals are done together on a single day, we consider them to be a single ritual. The order in which the rituals are performed is of no importance in this case.

(v) The rituals must be completed within $n+1$ days numbered $1, 2, 3, \ldots, n+1$.

The question now is: In how many different ways can rituals A, B, C, D be done?

Solution. The solution to Problem 2 is obtained by following arguments similar to those used in the solution to Problem 1.

Suppose that ritual A is done on the k-th day ($k = 1, 2, \ldots, n$). This fixed, there are now $n+1-k$ choices for the days on which *each* of the remaining rituals (B, C, D) can be done. Hence, the total number of different ways in which all the four rituals A, B C, D can be performed is equal to

$$\sum_{k=1}^{n} (n+1-k)^3 = 1^3 + 2^3 + \cdots + (n-1)^3 + n^3. \tag{5}$$

Now let us look at the problem in the following way.

(i) Out of the available $(n+1)$ days, two different days can be chosen in $\binom{n+1}{2}$ ways. Let the chosen days be u, v, with $u < v$. We now do ritual A on day u and the remaining three rituals B, C, D on day v. (The order in which the rituals are done on that day is not of any consequence.)

(ii) Out of the available $(n+1)$ days, three different days (say u, v, w, in that order) can be chosen in $\binom{n+1}{3}$ ways. For this choice, ritual A is done on day u, and rituals B, C, D can be performed in any of the six ways depicted below.

Day v	B	C, D	C	B, D	D	B, C
Day w	C, D	B	B, D	C	B, C	D

(iii) Out of the available $(n+1)$ days, four different days (say u, v, w, x, in that order) can be chosen in $\binom{n+1}{4}$ ways. For each such choice, we do ritual A on day u and B, C, D on days v, w, x. Assigning rituals B, C, D to days v, w, x can be done in $3! = 6$ different ways.

So, collecting the arguments just made in (i), (ii) and (iii), the total number of different ways in which the rituals A, B, C, D can be done is equal to

$$\binom{n+1}{2} + 6 \cdot \binom{n+1}{3} + 6 \cdot \binom{n+1}{4}. \tag{6}$$

On simplification, we get

$$1^3 + 2^3 + \cdots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4}. \tag{7}$$

As earlier, we have derived the known formula by narrating a situation.

Remark. The reader should be able to work out for himself or herself that by composing similar 'stories' with larger numbers of 'tests' or 'rituals,' expressions can be found which give the sum of the 4-th powers or 5-th powers or 6-th powers (or still higher powers) of the first n positive integers. (The stories may however get progressively more complicated!) We leave the challenge of composing these stories to the reader.



PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1995. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.

How to discover the EXPONENTIAL FUNCTION e^x

GAURAV BHATNAGAR

If a function is such that its derivative is the function itself, then what would it be? Some interesting mathematical objects appear while trying to answer this question, including a power series, the irrational number e and the exponential function e^x. The article ends with a beautiful formula that connects e, π, the complex number $i = \sqrt{-1}$, 1 and 0.

A function whose derivative is the function itself

If a function is such that its derivative is the function itself, then what would it be? If it is anything like a polynomial, it will be of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots.$$

Assume further that $f(0) = a_0 = 1$, and try to find a_1, a_2, a_3, ... Now

$$f(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots.$$

Its derivative is of the form

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots.$$

Since the derivative is the same as the function itself, compare the two to obtain:

$$\begin{aligned} & a_1 = 1, \\ 2a_2 = a_1 \implies & a_2 = \frac{1}{2}, \\ 3a_3 = a_2 \implies & a_3 = \frac{1}{3 \times 2}, \\ 4a_4 = a_3 \implies & a_4 = \frac{1}{4 \times 3 \times 2}, \\ 5a_5 = a_4 \implies & a_5 = \frac{1}{5 \times 4 \times 3 \times 2}. \end{aligned}$$

Keywords: Power series, derivative, polynomial, irrational number

Evidently,

$$a_n = \frac{1}{n \times (n-1) \times (n-2) \times \cdots \times 2} = \frac{1}{n!},$$

where we use the factorial notation

$$n! = n \times (n-1) \times \cdots \times 2 \times 1.$$

Note the extra factor of 1 to write the denominator as $n!$. Further, make a_1 fit the pattern by writing $a_1 = 1 = 1/1!$.

Thus

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$

The function $f(x)$ appears to be a polynomial of 'infinite' degree. This is because the derivative of a polynomial of degree n has degree $n-1$. If the function and its derivative have to be equal, we have to make sure there is no 'largest' degree term. In other words, the polynomial never ends.

This kind of 'infinite polynomial' is called a **power series**.

The number e

For a given value of x, this power series contains an infinite number of additions. As you know, we can only perform a finite number of additions in this lifetime. So the idea is to sum only the first few terms, and use that as a guide to the real thing.

Try $f(1)$. Use a calculator (or a spreadsheet like MS-Excel) to find the following approximations to $f(1)$.

$$\begin{aligned}
1 &= 1 \\
1 + \frac{1}{1} &= 2 \\
1 + \frac{1}{1} + \frac{1}{2} &= 2.5 \\
1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} &= 2.67 \\
1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} &= 2.71 \\
&\vdots \\
1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{10!} &= 2.71828\ldots
\end{aligned}$$

The approximations seem to settle down to a number whose first few digits are $2.71828\ldots$. This number is called e.

Henceforth, we take $f(1) = e$.

The connection with exponents

Given that $f(x)$ is like a polynomial, you can expect many rules of the algebra of polynomials to apply. What happens, for instance, when you multiply $f(x)$ with $f(y)$? Here goes:

$$\begin{aligned} f(x)f(y) &= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right) \\ &= 1 + \left(\frac{x}{1!} + \frac{y}{1!}\right) + \left(\frac{x^2}{2!} + \frac{xy}{1!\,1!} + \frac{y^2}{2!}\right) + \left(\frac{x^3}{3!} + \frac{x^2y}{2!\,1!} + \frac{xy^2}{1!\,2!} + \frac{y^3}{3!}\right) + \cdots \\ &= 1 + \frac{1}{1!}(x+y) + \frac{1}{2!}\left(x^2 + 2xy + y^2\right) + \frac{1}{3!}\left(x^3 + 3x^2y + 3xy^2 + y^3\right) + \cdots \\ &= 1 + \frac{1}{1!}(x+y) + \frac{1}{2!}(x+y)^2 + \frac{1}{3!}(x+y)^3 + \cdots. \end{aligned}$$

From these first few terms, it looks like

$$f(x)f(y) = f(x+y),$$

which should remind you of the rule for exponents:

$$a^x a^y = a^{x+y}.$$

Indeed, the following calculations suggest that f is an exponential too.

From the relation $f(x+y) = f(x)f(y)$, it follows that

$$f(2) = f(1+1) = f(1)f(1) = (f(1))^2.$$

Thus $f(2) = e^2$, since $f(1) = e$. Similarly, you can show that $f(3) = e^3$. More generally, by induction, it follows that

$$f(n) = e^n \text{ for every natural number } n.$$

It seems reasonable to guess that $f(x) = e^x$, for any real number x.

This is the famous exponential function, and we have the power series representation:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$

We derived this from the two conditions $f(x)$ satisfies: Its derivative is the function itself; and its value at 0 is 1.

Notes

- The number e is named after the famed mathematician Euler (pronounced *Oiler*). It is similar to π in many ways. Both are irrational, i.e., cannot be written in the form p/q, where p and q are integers, $q \neq 0$. Moreover, both are transcendental numbers, which means that they are not solutions of any polynomial equations with rational coefficients. (**Note.** Like π and e, the number $\sqrt{2}$ is irrational, but it **is** a solution of a polynomial equation, namely $x^2 - 2 = 0$.)
- Just like π, the number e appears in a wide variety of contexts. Here are a few examples:
  - If you put 1 rupee in a bank account which pays 100% interest, *compounded continuously*, then at the end of one year, you will have Rs. e in the account!

– Shorn of all the financial-ese, the above statement simply says that:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

– Suppose n letters are written to n people, and are inserted randomly into n (previously addressed) envelopes. The probability that **no** letter is placed in the correct envelope approaches $1/e$, as $n \to \infty$.

- There are many mathematical rules I have broken in this article. I argued on the basis of a few examples, and jumped from the finite to the infinite case with abandon. All our computations can be justified, but they require the development of more theory, and clarifications of ideas of convergence, power series, etc.

  A lot of math is discovered this way. Mathematicians regularly guess, or conjecture, properties of objects they study on the basis of calculations (done by hand or computer). The computations serve to build their intuition which is useful when it comes to knowing which rules are possible to break or extend. Then they prove their conjectures, or perhaps, develop a new theory to allow for the extended rules.

  Progress in mathematics (and indeed in almost any field) often happens when old rules are broken and new ones are made.

- We didn't even bother explaining what it means to raise e to the power of a real number. The expression e^n is easy to understand—here e is multiplied with itself n times. So is $e^{p/q}$, where p/q is a rational number, where we take the qth root, and then the pth power. But e^x for x an irrational number? Think of taking a rational number r very 'close' to x, and taking e^r as an approximation to e^x. Indeed, one can take a sequence of rational numbers $x_1, x_2, x_3, \ldots$ that come closer and closer to x, in other words, converge to x, and define e^x as the limit of e^{x_n} as n goes to infinity.

- You can use the approach of this article to find the power series expansion of $\sin x$. Recall that the derivative of $\sin x$ is $\cos x$, and its derivative is $-\sin x$. Further, $\sin 0 = 0$ and $\cos 0 = 1$. Now assume that

  $$\sin x = a_0 + a_1 x + a_2 x^2 + \cdots,$$

  differentiate it twice and figure out a_0, a_1, a_2, and so on. In this manner show that:

  $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

  Differentiate both sides (again, by reinterpreting the 'derivative of sum is sum of derivatives' rule to allow for the sum of an infinite number of functions!) to find the power series corresponding to $\cos x$:

  $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

  Using these power series, prove that

  $$e^{i\theta} = \cos\theta + i\sin\theta,$$

  where $i = \sqrt{-1}$, and we assume the power series expansion holds even when x is a complex number. Finally, set $\theta = \pi$ to obtain the sublime formula

  $$e^{\pi i} + 1 = 0.$$

  This formula connects e with π, i, 1 and 0, the other fundamental numbers of math!

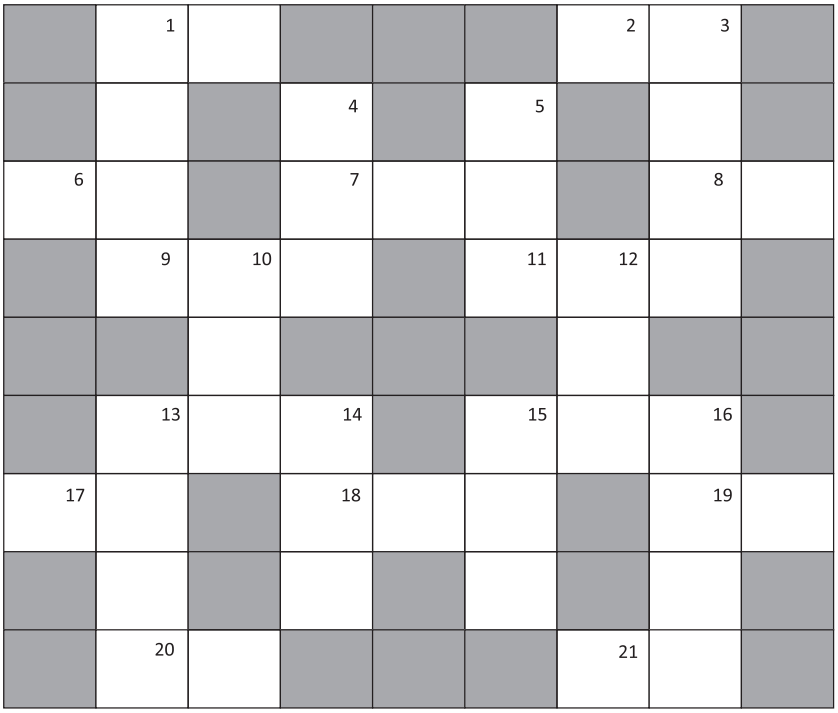


GAURAV BHATNAGAR is a mathematician who is interested in q-series and special functions, and especially interested in formulas of Ramanujan. At present, he is a post-doctoral researcher in the Department of Mathematics, University of Vienna, Austria. He is the author of *Get Smart! Maths Concepts*, Penguin (2008), a book on middle school mathematics. Links to articles by him can be found on his web-site www.gbhatnagar.com. He may be contacted at bhatnagarg@gmail.com.

NUMBER CROSSWORD

Solution on Page 36

D.D. Karopady & Sneha Titus



CLUES ACROSS	CLUES DOWN
1 Four score and ten	1 The highest two digit number times the first three digit prime
2 Quarter of 7A minus 2	3 Cube of 6 times 17
6 Three dozen and three	4 Two raised to (1A divided by 10)
7 Number of zeroes in a googol	5 17A times 6 plus 2A
8 Second in a twin prime pair with its reverse also a prime	10 James Bond's number
9 Multiple of 11	12 Medical emergency call number in the US
11 9A plus 1A	13 Kaprekar Constant
13 Digits in arithmetic progression	14 13A in reverse
15 Product of 1 more than 10 and 1 less than 30	15 Third term of a GP in which the first number and common ratio are both the highest single digit prime
17 Fourth power of a prime number	16 Anagram of year of Indian independence
18 Hundreds digit is the sum of the units and tens digit	
19 Six times the unlucky number	
20 Double of 2A	
21 A number which is a square, a cube and a sixth power	

The EXPONENTIAL SERIES an Addendum

SHAILESH SHIRALI

In the accompanying article, the author (Gaurav Bhatnagar) presents a heuristic derivation of the well-known series for the exponential function,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots . \qquad (1)$$

Here, we depict the same steps graphically and present the material in a different way. This article may thus be regarded as an addendum to the article mentioned above.

The intention is to find a smooth function f, mapping the set of real numbers $\mathbb{R}$ into itself, and having the following properties: (i) the function equals its own derivative, i.e., $f(x) = f'(x)$ for all x; (ii) $f(0) = 1$. It is remarkable that these two simple requirements completely fix the function.

Note the following consequence of these properties: since $f(x) = f'(x)$, we obtain, by repeated differentiation:

$$f(x) = f'(x) = f''(x) = f'''(x) = \cdots . \qquad (2)$$

Keywords: Exponential series, derivative, polynomial, partial sum, convergence, analytic function

Thus, all its derivatives are identical to itself. Since $f(0) = 1$, a particular consequence of this is:

$$f(0) = f'(0) = f''(0) = f'''(0) = \cdots = 1. \tag{3}$$

That is, at $x = 0$, every derivative equals 1 (including the zeroth derivative which is the function itself).

Remarkably, condition (3) by itself (which contains within itself infinitely many requirements) allows us to make substantial inroads into the problem: it allows us to find a power series for the function $f(x)$. What we do is to consider, at each stage, only a finite number of these requirements, reading from the left. Moreover, we confine our search to the family of polynomials, these being the simplest functions of all. Here is how we argue the case:

- Let condition (3) be replaced by: $f(0) = 1$. What is the simplest polynomial that satisfies this condition? Clearly, it is the constant polynomial

$$f(x) = 1 \quad \text{for all } x \in \mathbb{R}. \tag{4}$$

  (We must explain our use of the word 'simplest'. The meaning here is: *having the smallest possible degree*. In our particular case, this requirement is equivalent to: *with as few terms as possible*.)

- Now let condition (3) be replaced by: $f(0) = 1$ and $f'(0) = 1$. What is the simplest polynomial that satisfies these conditions? A moment's thought will reveal that it is the polynomial of degree 1 given by

$$f(x) = 1 + x \quad \text{for all } x \in \mathbb{R}. \tag{5}$$

  (To see why, note that since there are two conditions to be satisfied, we need two unknown coefficients to play around with. Now consider the general linear polynomial, $f(x) = a + bx$. By substituting the given conditions we obtain $a = 1 = b$; this yields $f(x) = 1 + x$, as stated.)

- Next, let condition (3) be replaced by these: $f(0) = 1$ and $f'(0) = 1$ and $f''(0) = 1$. (Note how we are gradually adding more and more conditions.) What is the simplest polynomial that satisfies these conditions? Calculation reveals (arguing as suggested above) that it is the polynomial of degree 2 given by

$$f(x) = 1 + x + \frac{x^2}{2} \quad \text{for all } x \in \mathbb{R}. \tag{6}$$

  (Exercise: Please fill in the algebraic details in the above derivation.)

- Next, let condition (3) be replaced by these: $f(0) = 1$ and $f'(0) = 1$ and $f''(0) = 1$ and $f'''(0) = 1$. What is the simplest polynomial that satisfies these conditions? Calculation reveals that it is the polynomial of degree 3 given by

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \quad \text{for all } x \in \mathbb{R}. \tag{7}$$

  (Exercise: Please fill in the algebraic details in the above derivation.)

- From the above, it is an easy step to guess — and then to prove using the principle of mathematical induction — that the simplest polynomial $f(x)$ satisfying the $n + 1$ conditions

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad f'''(0) = 1, \quad \ldots, \quad f^{(n)}(0) = 1, \tag{8}$$

  is the following polynomial of degree n:

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R}. \tag{9}$$

  Filling in the details makes for a good exercise.

Having reached this far, it is easy to see that the power series given by

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{for all } x \in \mathbb{R} \tag{10}$$

satisfies the requirements stated at the start, (2) and (3). It may be shown that the power series in (10) converges for all real values of x. (We shall not attempt to prove this statement. It belongs to a branch of mathematics called *real analysis*, in which criteria are worked out under which a power series converges. To learn more about analysis, which typically is studied in college rather than in school, you could refer to a standard text such as [1], chapters 5 and 7.) Hence, (10) defines a proper, well-behaved function ("analytic function" is the technical term; again, we shall not try to give a technical definition here; please consult [1]). Indeed, it may be shown that this is the *only* function which satisfies requirements (2) and (3).

The beautiful discovery that we now make about this function, as described in the accompanying paper, is that the function defined by (10) is the exponential function; namely, if we define the constant e by $e = f(1)$, so that

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots \approx 2.71828\ldots, \tag{11}$$

then the function is given by

$$f(x) = e^x. \tag{12}$$

It is fascinating at this stage to compare the graphs of the various functions that we have listed till now. In each of the plots below, we show two graphs: the graph of e^x, and the graph of a partial sum of the power series given in (10).

Note that owing to the extreme rapidity with which the exponential curve rises, we have had to use different scales for the x-axis and the y-axis.

Figure 1 displays the graphs of $y = e^x$ and $y = 1$, and Figure 2 displays the graphs of $y = e^x$ (red) and $y = 1 + x$ (blue). And so on for the other graphs.

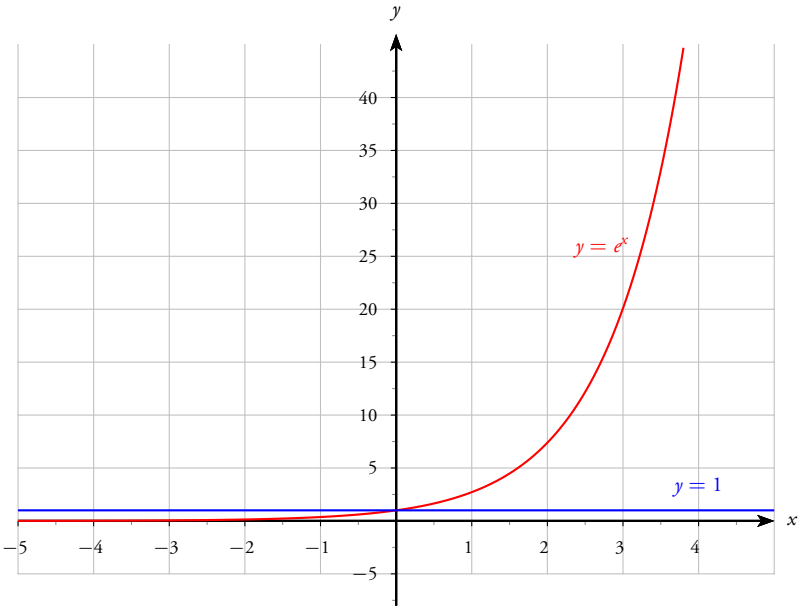
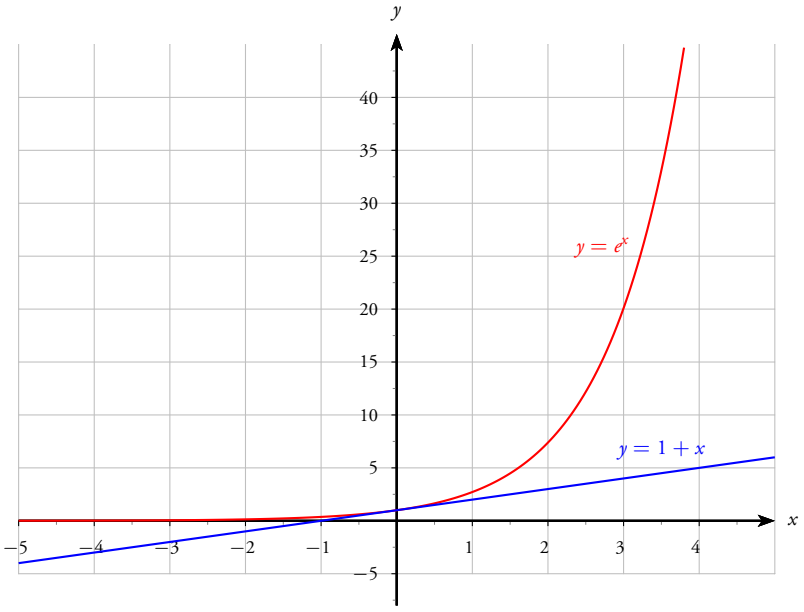


Figure 1. Graphs of $y = e^x$ (red) and $y = 1$ (blue)



Note: The slope of the line $y = 1 + x$ may not "look right" (it should be 1); but this is only because of the different scales used on the two axes.

Figure 2. Graphs of $y = e^x$ (red) and $y = 1 + x$ (blue)

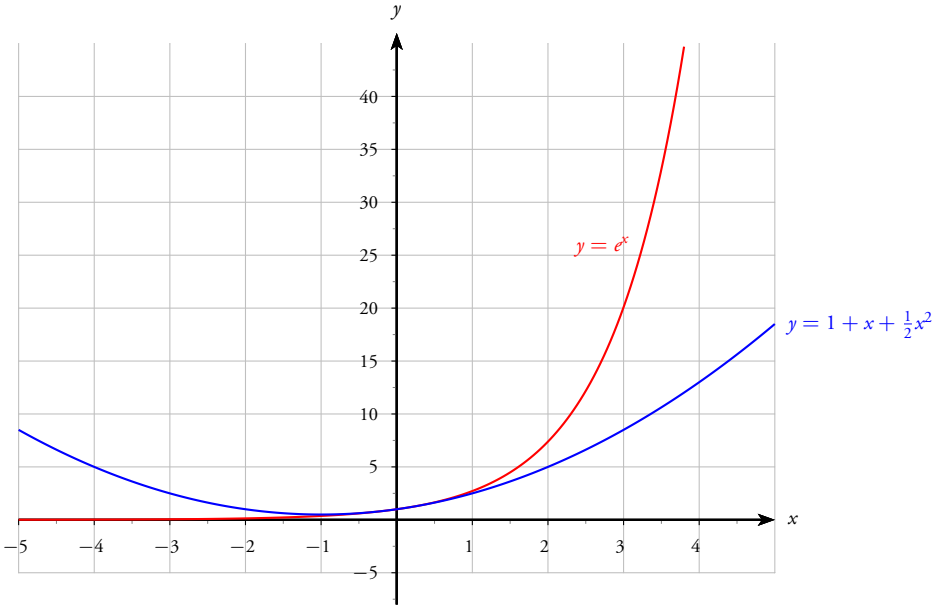


Figure 3. Graphs of $y = e^x$ (red) and $y = 1 + x + \frac{x^2}{2!}$ (blue)

As the degree of the partial sum gets larger, the closeness of the two graphs gets more marked; the range of values of x for which the two graphs seem to coincide and cannot be distinguished visually steadily gets larger.

To reinforce this point, we display in Figure 7 the graphs (with expanded scales) of $y = e^x$ and

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!}.$$

The two graphs are practically indistinguishable from each other over the range of x-values shown (except for a small bit at the left-hand side)!

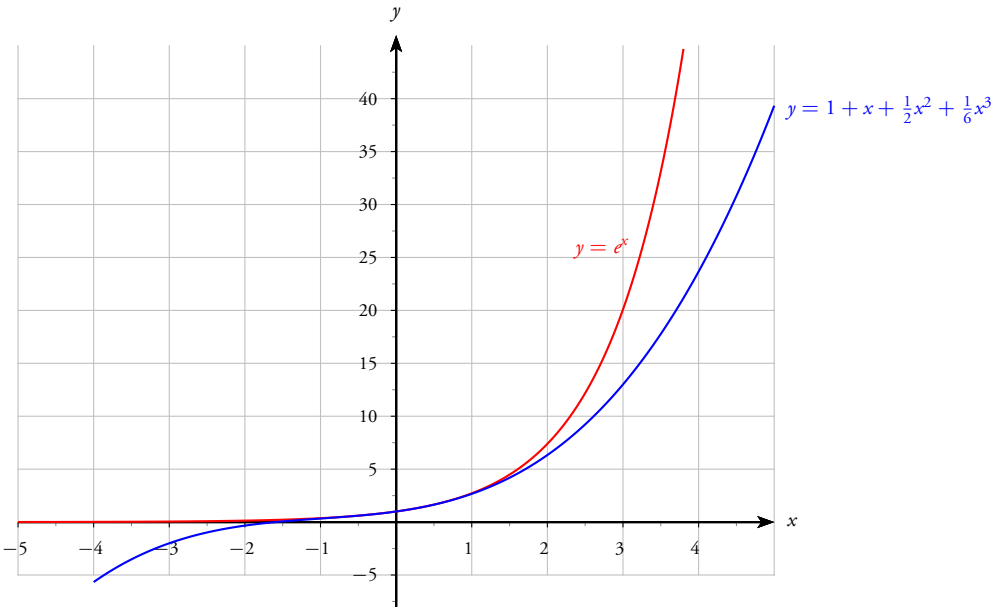


Figure 4 .Graphs of $y = e^x$ (red) and $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ (blue)

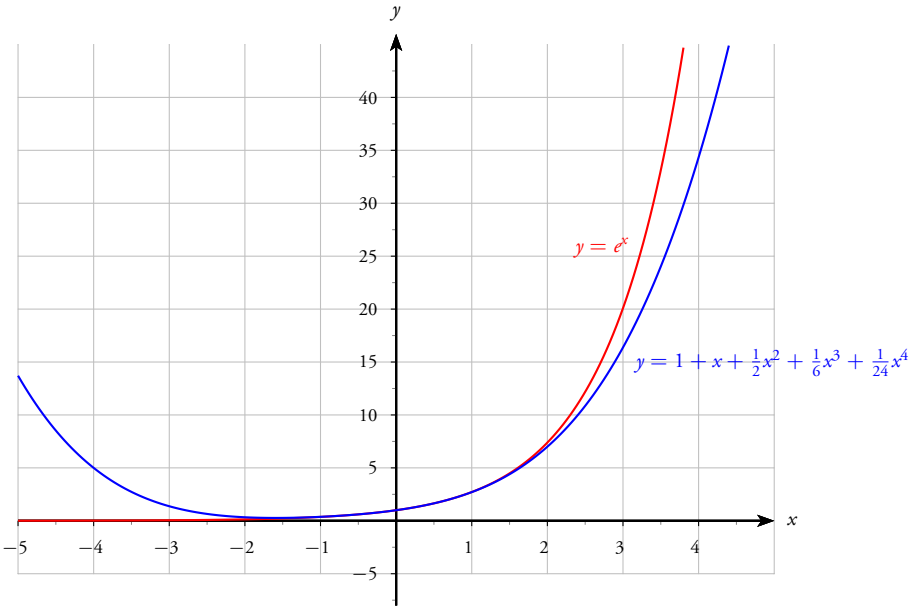


Figure 5. Graphs of $y = e^x$ (red) and $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ (blue)

Another approach is to display the graph of the following function:

$$\frac{1 + x + x^2/2! + x^3/3! + \cdots + x^n/n!}{e^x} \tag{13}$$

for different values of n (say $n = 3, 6, 10, 13$). Figures 8, 9, 10 and 11 do just this.

As n increases, we see that the graph of the function stays closer and closer to the line $y = 1$, for an ever-increasing range of values of x. Note, however, for any fixed value of n, there is only a finite interval of values of x where we see such a marked closeness. Outside this interval, the divergence between the two graphs becomes more visible, particularly at the negative end of the interval (where e^{-x} assumes extremely large values). You can see this happening in Figure 9 as well as Figure 10.

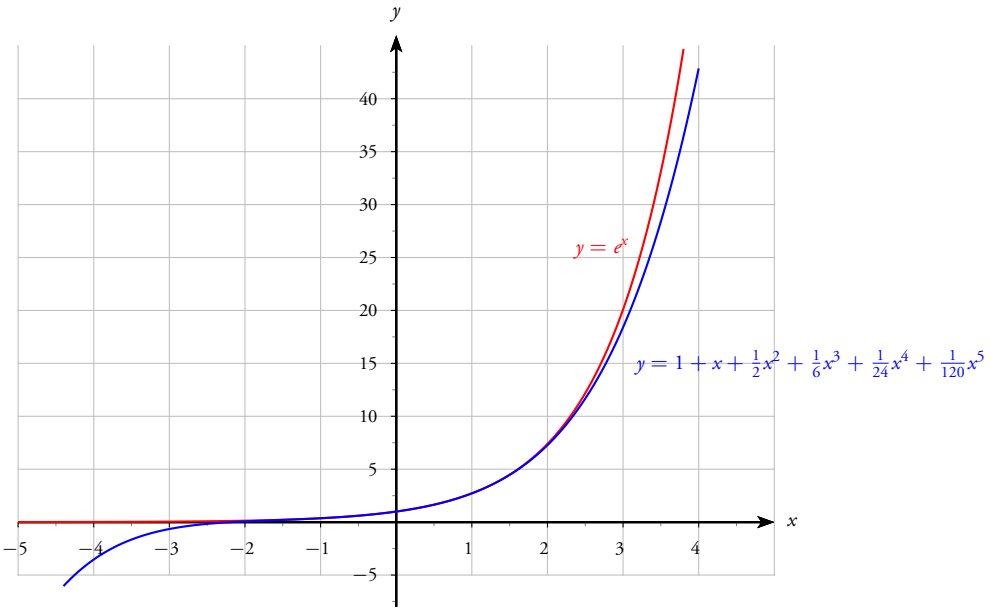


Figure 6. Graphs of $y = e^x$ (red) and $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$ (blue)

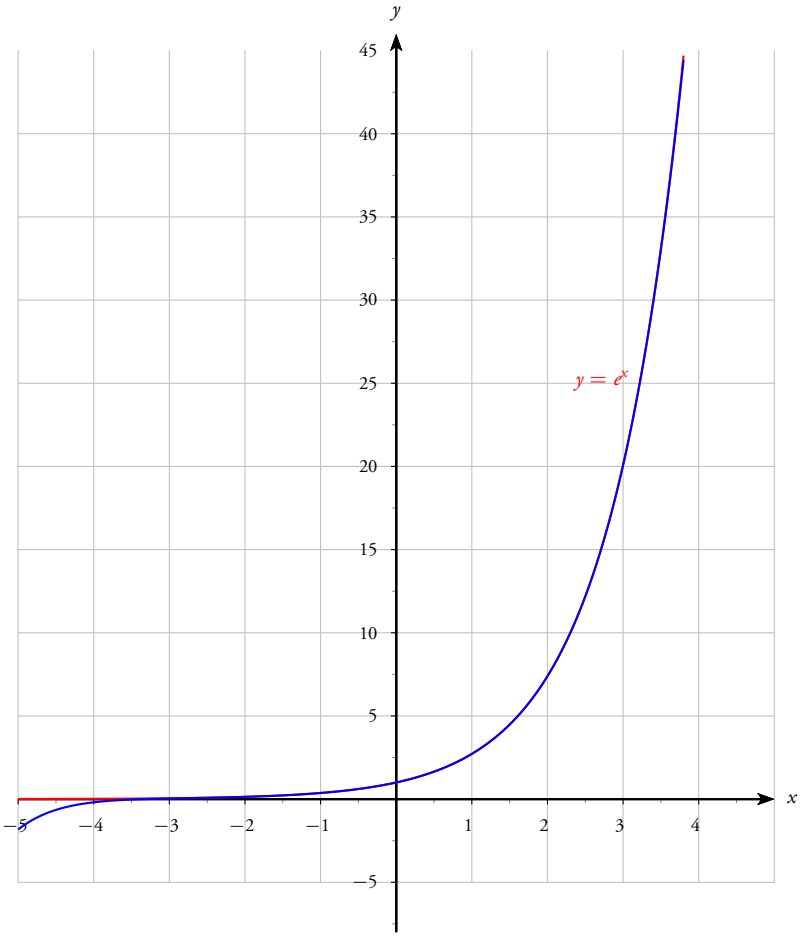


Figure 7. Graphs of $y = e^x$ (red) and $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!}$ (blue)

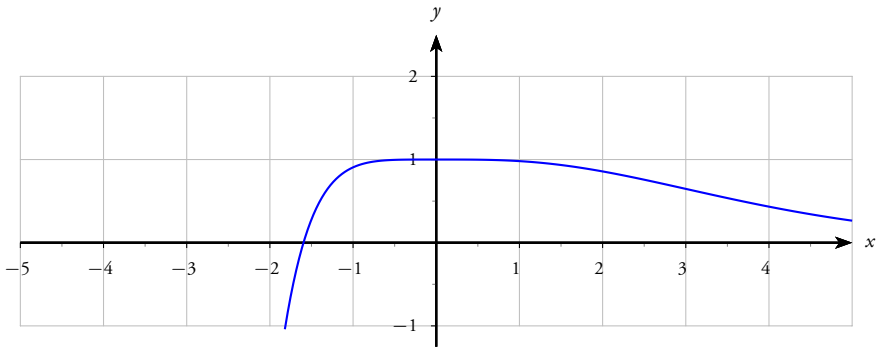


Figure 8. Graph of $y = e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$ for $n = 3$

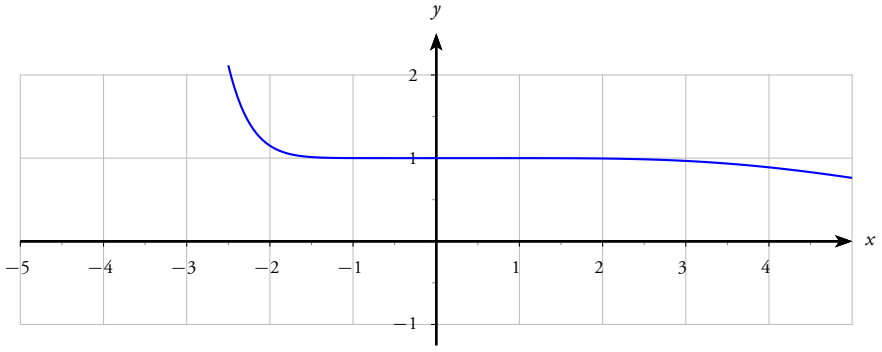


Figure 9. Graph of $y = e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$ for $n = 6$

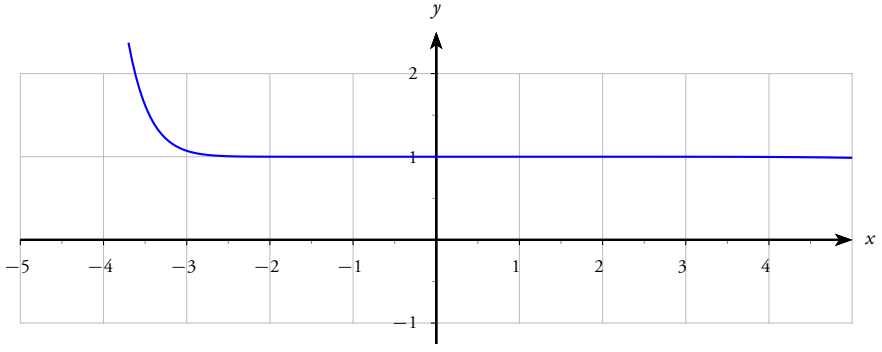


Figure 10. Graph of $y = e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$ for $n = 10$

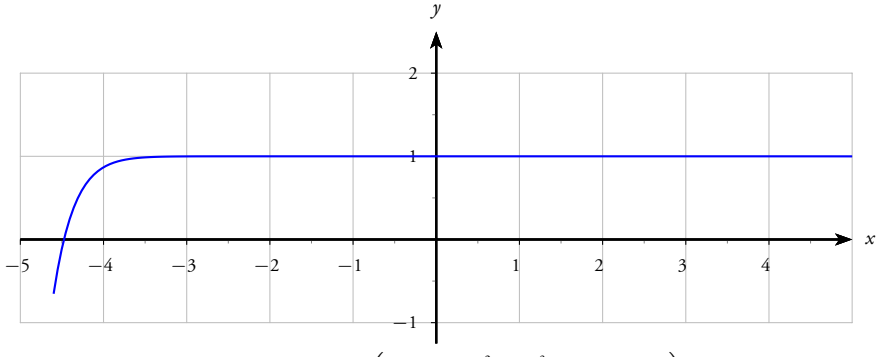


Figure 11. Graph of $y = e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$ for $n = 13$

Closing remark

The power series for the exponential function:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{14}$$

has become a familiar sight even at the high-school level, and perhaps students do not give it a second look when they encounter it; the price of familiarity! But we see that even in this familiar setting, looking at the matter through graphs rather than through equations throws fresh light on the topic and it does so in an extremely pleasing manner. Is it not truly remarkable that by adding more and more monomials, we begin to get closer and closer to an exponential function?

Exercises

(1) Show that the polynomial $f(x)$ with the smallest possible degree that satisfies the conditions $f(0) = 1, f'(0) = 1$ is

$$f(x) = 1 + x.$$

(2) Show that the polynomial $f(x)$ with the smallest possible degree that satisfies the conditions $f(0) = 1, f'(0) = 1, f''(0) = 1$ is

$$f(x) = 1 + x + \frac{x^2}{2}.$$

(3) Show that the polynomial $f(x)$ with the smallest possible degree that satisfies the conditions $f(0) = 1, f'(0) = 1, f''(0) = 1, f'''(0) = 1$ is

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

References

1. Richard Courant and Fritz John, *Introduction to Calculus and Analysis*, Volume 1. Interscience Publishers, John Wiley & Sons, 1965.



SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Tonic AMBIGUITY

V. RAMPRASHANTH

Musical tones

A musical tone is the musical perception of a physical sound frequency. It has been empirically found out that there are seven whole tones and five semi-tones. Most of the musical cultures of the world work with these or a sub-set of these. They arrange them into scales, which are hierarchically placed musical tones. Solfege is the notational system for the musical tones. The Indian solfege system denotes the whole tones as Sa, Ri, Ga, Ma, Pa, Dha, Ni and the western counter-part is Do, Re, Mi, Fa, So, La, Ti. A musical note however, is a symbol in the sign system that refers to musical tones.

Tonic pitch

Pitch is the quality of sound that is directly related to the number of vibrations per second of a sound wave. The unit of measurement of pitch is Hertz (Hz). The Tonic Pitch is the pitch against which all other pitches of musical tones are hierarchically referenced. Any pitch can be the Tonic Pitch. Hence it is variable. The musical note referring to this variable tonic-pitch is called 'Do' in the Western solfege system and 'Sa' (short for *Shadjam*) in the Indian solfege system.

Frequency ratios of musical tones

It has been found (right from Pythagorean times) that the frequency of the tonic and the frequencies of the rest of the tones and semi-tones form a simple ratio. A particular musical tone always has the same frequency ratio relationship with the tonic. The western solfege syllables corresponding to Sa, Ri, Ga, Ma, Pa, Dha, Ni are Do, Re, Mi, Fa, So, La, Ti respectively.

Keywords: music, tone, solfege, pitch, ambiguity

These frequency ratios are given in Table 1.

Sa	Ri	Ga	Ma	Pa	Dha	Ni
1	9/8	5/4	4/3	3/2	5/3	15/8

Table 1

For example if the variable Sa has a frequency value of 440 Hertz, then Ga will have a frequency value of 440 * (5/4) = 550 Hz.

If the variable Sa has a frequency value of 330 Hz, then Ma will have a frequency value of 330 * (4/3) = 440 Hz.

Why these particular ratios?

This has been a speculation right from Pythagorean times. Pythagoras considered that musical tones are good to hear because of their simple ratios. Later Euler formulated a whole theory of consonance and dissonance based on 'degree of agreeableness'. He classified all possible ratios into classes/degrees. But his theory does not seem to predict certain agreeable harmonies.

Ambiguity in perception of tones

There is a possibility that a particular pitch be perceived as different musical tones by different hearers even though the frequency is the same. This arises when the hearers assume different tonic pitches, i.e., in each of their minds the variable Sa has different frequency values.

Let P_1 and P_2 be two frequencies. Let x and y be frequency ratios of two different tones to the tonic pitch.

There can be cases of ambiguity when the same frequency is interpreted as different musical tones. For example,

The pitch 550 Hertz can be with different frequencies of tonic pitches (say 440 Hz and 330 Hz) different musical tones (say Ga and Dha).

550 Hz = 440 Hz * 5/4 = 330 Hz * 5/3

Hence in a case of ambiguity, the same frequency is interpreted as different musical tones (frequency ratios x and y) with frequencies of tonic pitches as P_1 and P_2 respectively.

$$P_1 * x = P_2 * y$$

Let there be another set of frequencies which are also interpreted as different musical tones (with ratios a and b) by two different hearers assuming frequencies of tonic pitches as P_1 and P_2 respectively.

$$P_1 * a = P_2 * b$$

From the above two equations,

$$\frac{x}{y} = \frac{a}{b} \quad \text{or} \quad x * \frac{1}{a} = y * \frac{1}{b}$$

The product grid

To compute the product of x and $1/a$ or y and $1/b$, we make a table with row headings as ratios of frequencies of a tone to its tonic pitch and the reciprocals of the ratios as column headings.

We now make the 'product grid' where value of each cell is the product of the ratio of a particular tone to the tonic and its reciprocal.

For example, the product of Ga and Ri' is same as Dha and Pa'. This means that the tones Ga and Ri of Hearer 1 correspond to Dha and Pa of Hearer 2.

	Ri' (8/9)	Ga' (4/5)	Ma' (3/4)	Pa' (2/3)	Dha' (3/5)	Ni' (8/15)
Ri(9/8)	1	9/10	27/32	3/4	27/40	3/5
Ga(5/4)	10/9	1	15/16	5/6	3/4	2/3
Ma(4/3)	32/27	16/15	1	8/9	4/5	32/45
Pa(3/2)	4/3	2/15	9/8	1	9/10	4/5
Dha(5/3)	40/27	4/3	5/4	10/9	1	8/9
Ni(15/8)	5/3	3/2	45/32	5/4	9/8	1

Table 2: The cells having the same value are colored the same. The column headings are the reciprocals (denoted by an apostrophe)

Let P_1 = 440 Hz, x = Ri (9/8), y = Pa (3/2)

$9/8 * 440 = 3/2 * P_2$

$9*55*2/3 = 330$ Hz $= P_2$

So there exists a frequency of tonic pitch 330 Hz, which along with 440Hz produces ambiguity when 495 Hz (Ri or Pa) and 550 Hz (Ga or Dha) are played.

The cells having the same value are colored the same. These pairs of cells correspond to the tone pairs (x, a) and (y, b) which are interpretations of Hearers 1 and 2 respectively with frequencies of tonic pitches P_1 and P_2.

Permutations

The equation

$$x * \frac{1}{a} = y * \frac{1}{b}$$

Can also be written as

$$x * \frac{1}{y} = a * \frac{1}{b}$$

This means that (x, a) of Hearer1 can be interpreted as (y, b) by Hearer2. So the frequency ratio pair (x, y) can be interpreted as (a, b).

For example, if Ri and Ga of Hearer1 are Pa and Dha of Hearer2 then Ri and Pa of Hearer1 are Ga and Dha of Hearer2.

Semitones

The frequency ratios of the semitones are:

ri	ga	ma	dha	ni
16/15	6/5	7/5	8/5	7/4

Table 3

The product grid including the semitones with the whole tones is:

	ri' 15/16	Ri' 8/9	ga' 5/6	Ga' 4/5	Ma' 3/4	ma' 5/7	Pa' 2/3	dha' 5/8	Dha' 3/5	ni' 4/7	Ni' 8/15
ri 16/15	1	128/135	8/9	64/75	4/5	16/21	32/45	2/3	16/25	64/105	128/225
Ri 9/8	135/128	1	15/16	9/10	27/32	45/56	3/4	45/64	27/40	9/14	3/5
ga 6/5	9/8	16/15	1	24/25	9/10	6/7	4/5	3/4	18/25	24/35	48/75
Ga 5/4	75/64	10/9	25/24	1	15/16	25/28	5/6	25/32	3/4	5/7	2/3
Ma 4/3	5/4	32/27	10/9	16/15	1	20/21	8/9	5/6	4/5	16/21	32/45
ma 7/5	21/16	56/45	7/6	28/25	21/20	1	14/15	7/8	21/25	4/5	56/75
Pa 3/2	45/32	4/3	5/4	6/5	9/8	15/14	1	15/16	9/10	6/7	4/5
dha 8/5	3/2	64/45	4/3	32/25	6/5	8/7	16/15	1	24/25	32/35	64/75
Dha 5/3	75/48	40/27	25/18	4/3	5/4	25/21	10/9	25/24	1	20/21	8/9
ni 7/4	105/64	14/9	35/24	7/5	21/16	5/4	7/6	35/32	21/20	1	14/15
Ni 15/8	225/128	5/3	25/16	3/2	45/32	75/56	5/4	75/64	9/8	15/14	1

Table 4

In this product grid, it is surprising that we find more than 2 cells with the same value. This correspondingly means that there are more interpretations for the two pitch frequencies.

- There are 2, 3 and 5 cells with the same value.
- There are no 4 cells with the same value.

Glossary

- *Pitch*: Distinctive quality of a sound, dependent primarily on the frequency of the sound waves produced by its source.
- *Note*: A symbol for a tone, indicating pitch.
- *Swara*: It is a Sanskrit word that denotes a note in the successive steps of the octave. Swaras are the selected pitches from which the musician constructs the scales, melodies. The seven notes of the musical scale in Indian classical music are shadjam (Sa), rishabham (Ri), gandharam (Ga), madhyamam (Ma), panchamam (Pa), dhaivatham (Dha) and nishadham (Ni).



RAMPRASHANTH V is a student of mathematics and linguistics. He is also a student of Carnatic music and has been learning the violin. He is interested in the mathematical aspects of music. He has penned a few other articles such as 'swara-latin squares' and 'control structures in music.' He may be contacted at ramprashanth.venkatakrishnan@gmail.com.

There are INFINITELY MANY PRIME NUMBERS

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

Some theorems, it seems, are evergreen. New proofs keep turning up for them. One such is the theorem of Pythagoras (the current number of proofs stands at over 300). Another is the claim that the square root of 2 is irrational. A third example is the statement that *there exist infinitely many prime numbers*. This is the one on which we will dwell in this short article. The proof that we feature appeared in *The American Mathematical Monthly*, in its December 2006 issue [1].

Before presenting the new proof by Filip Saidak, let us take a look at the original proof given by Euclid. We present it below. Note that it is presented in a modern form and not the form originally given by Euclid in his book *The Elements*.

Remark. In pre-algebra times, proofs were presented in a purely verbal form, using highly stylised language. Most modern readers find such proofs difficult to follow. Perhaps this is because algebra and symbolic reasoning in general have made such deep inroads into our thinking. The following example of the use of stylised language illustrates the point perfectly. It is Proposition 6 from Euclid's Elements, Book II:

> *If a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line.*

Try to decipher this statement!

Keywords: Prime number, Euclid, proof by contradiction, infinite

Theorem (Euclid). ***There exist infinitely many prime numbers.***

Euclid's proof (in modern form)

To set up a contradiction, we assume that there are only finitely many prime numbers; say there are just k prime numbers where k is some finite positive integer. Let the complete collection of prime numbers be $p_1, p_2, p_3, \ldots, p_k$. Define Q to be the number obtained by adding 1 to the product of all these prime numbers. That is,

$$Q = p_1 p_2 p_3 \cdots p_k + 1. \tag{1}$$

It follows from the definition that Q leaves remainder 1 under division by each of the primes $p_1, p_2, p_3, \ldots, p_k$. That is,

$$\gcd(Q, p_1) = 1, \gcd(Q, p_2) = 1, \\ \gcd(Q, p_3) = 1, \ldots, \gcd(Q, p_k) = 1. \tag{2}$$

This implies that Q has a prime factor which is different from all the existing primes. (This prime number could be Q itself.) This is, of course, a contradiction. Hence the assumption that there are finitely many prime numbers is self-contradictory and therefore must be false. Hence there exist infinitely many prime numbers.

Proof by Filip Saidak

The new proof which we now feature shows in effect that *there can be no upper bound to the total number of prime numbers*. In short, there must exist infinitely many prime numbers. The argument is an absurdly simple one and it seems amazing that no one has found it earlier. (Perhaps this is true of any beautiful discovery.)

Here is how the argument runs. Let $N_1 > 1$ be any positive integer. Since N_1 and $N_1 + 1$ are consecutive integers, they are co-prime. This means that the set of primes that divide N_1 is disjoint from the set of primes that divide $N_1 + 1$. Hence the number N_2 defined by

$$N_2 = N_1 (N_1 + 1) \tag{3}$$

must have at least two different prime factors.

We now continue the argument with N_2 in place of N_1. Since N_2 and $N_2 + 1$ are consecutive integers, they are coprime. So the set of primes that divide N_2 is disjoint from the set of primes that divide $N_2 + 1$. Hence the number N_3 defined by

$$N_3 = N_2 (N_2 + 1) \tag{4}$$

must have at least three different prime factors.

We now continue the argument with N_3 in place of N_2 and deduce that the number $N_4 = N_3 (N_3 + 1)$ must have at least four different prime factors; and so on. It should be obvious that these steps can be continued indefinitely. It follows that the number of primes is infinite.

Closing remark.

In [1], the author notes that this proof is conceptually simpler than Euclid's original proof, as it is not based on 'proof by contradiction.' Moreover, the proof is constructive, in that it provides an explicit way of exhibiting an integer having more than any given number of different prime factors.

References

1. Filip Saidak, "A New Proof of Euclid's Theorem", *The American Mathematical Monthly* (Mathematical Association of America), Vol. 113, No. 10 (Dec., 2006), pp. 937-938. http://www.jstor.org/stable/27642094.

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Complex ..... NAPOLEON'S THEOREM ..... Made Simple

Part 2

SHAILESH SHIRALI

In an earlier issue of At Right Angles, we had studied a gem of Euclidean geometry called Napoleon's Theorem, a result discovered in post-revolution France. We had offered proofs of the theorem that were computational in nature, based on trigonometry and complex numbers. We continue our study of the theorem in this article, and offer proofs that are more geometric in nature; they make extremely effective use of the geometry of rotations.

Napoleon's Theorem states the following. Let ABC be an arbitrary triangle. With the three sides of the triangle as bases, construct three equilateral triangles, each one outside $\triangle ABC$. Next, mark the centres P, Q, R of these three equilateral triangles. Napoleon's theorem asserts that $\triangle PQR$ is equilateral, irrespective of the shape of $\triangle ABC$. (See Figure 1.)

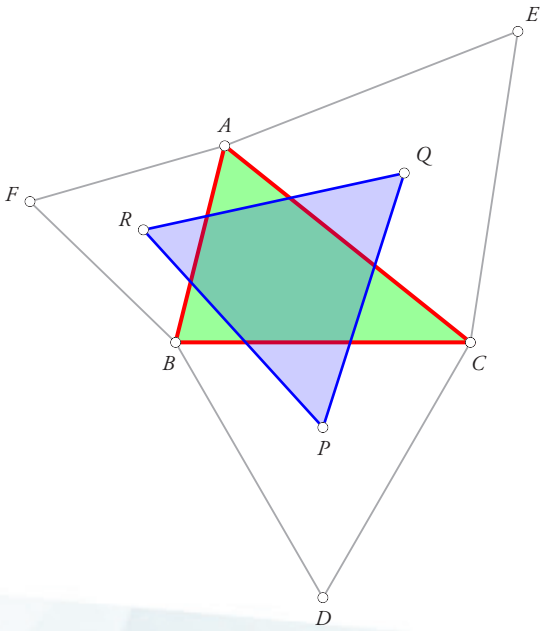


Figure 1

Keywords: *Napoleon, equilateral triangle, rotation, parallelogram, basic proportionality theorem*

In Part 1 of this article, we had considered computational proofs of Napoleon's theorem. In the trigonometric proof, we derived an expression for the length of one side of $\triangle PQR$ in terms of the sides and the angles of the $\triangle ABC$ (i.e., in terms of a, b, c, A, B, C). After going through the computations, we discovered that the resulting expression is symmetric in the parameters of the parent triangle. This fact suffices to prove that triangle PQR is equilateral.

Now we study an extremely elegant pure geometry proof of Napoleon's theorem; it makes very effective use of rotational geometry. In the literature, it is ascribed to an Irish mathematician, MacCool [1].

Before proceeding, we make a comment about rotations. Figure 2 shows a segment AB being subjected to two different rotations, both centred at a point O. The first one is through an angle of $+30^\circ$ (the positive sign tells us that the rotation is in a *counterclockwise direction*); it takes segment AB to segment A_1B_1. The second one is through an angle of -30° (the negative sign tells us that the rotation is in a *clockwise direction*); it takes segment AB to segment A_2B_2. Note that segments AB, A_1B_1 and A_2B_2 have equal length.

Now we get back to the proof of Napoleon's theorem. Consider a rotation through an angle of -30°, centred at B (see Figure 3; the rotation is in a clockwise direction). Our interest is in what this rotation 'does' to points R and P, i.e., where it takes these two points. Since $\angle ABR = 30^\circ$ and $\angle DBP = 30^\circ$, it follows that the image R_1 of R lies on side AB, and the image P_1 of P lies on side BD.

We argue as follows. The steps of the reasoning are laid out in itemised form at the right side of the diagram.

To see why $BR_1/BA = 1/\sqrt{3} = BP_1/BD$, you will first need to understand why $BR/BA = 1/\sqrt{3} = BP/BD$. But this follows from the basic geometry of an equilateral triangle. We leave the details for you to fill in.

From the fact that $BR_1/BA = BP_1/BD$, we deduce (using the basic proportionality theorem) that

$$R_1P_1 \parallel AD, \qquad \frac{R_1P_1}{AD} = \frac{1}{\sqrt{3}}. \tag{1}$$

Since $RP = R_1P_1$, it follows that:

$$\frac{RP}{AD} = \frac{1}{\sqrt{3}}. \tag{2}$$

In just the same way, we consider a rotation through an angle of $+30^\circ$, centred at C. Then, if the rotation takes Q and P to Q_2 and P_2, respectively, it follows that Q_2 lies on side AC, and P_2 lies on side CD; and arguing as earlier, we conclude that

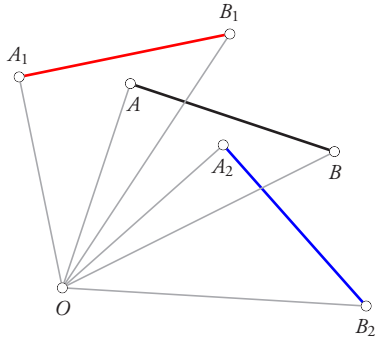
$$Q_2P_2 \parallel AD, \qquad \frac{Q_2P_2}{AD} = \frac{1}{\sqrt{3}}, \tag{3}$$

and

$$\frac{QP}{AD} = \frac{1}{\sqrt{3}}. \tag{4}$$

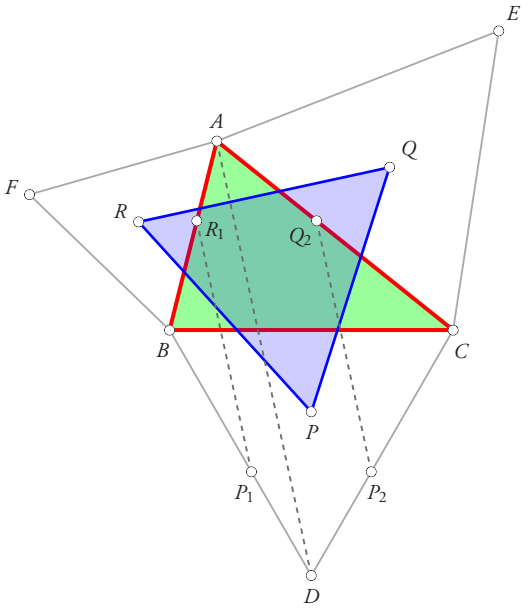
From (2) and (4), we conclude that $RP = QP$.

At this stage, we can proceed in two different ways. One way is to say that the same argument can be repeated for another pair of sides of $\triangle PQR$ and to conclude that equality therefore holds for the lengths of that pair of sides of $\triangle PQR$, and to



- $\angle AOA_1 = +30^\circ = \angle BOB_1$
- $\angle AOA_2 = -30^\circ = \angle BOB_2$
- Segments AB, A_1B_1 and A_2B_2 have equal length

Figure 2



(a) $R_1P_1 = RP$

(b) $BR_1 = BR$

(c) $BR_1/BA = 1/\sqrt{3}$

(d) $BP_1 = BP$

(e) $BP_1/BD = 1/\sqrt{3}$

(f) $BR_1/BA = BP_1/BD$

(g) $R_1P_1 \parallel AD$

(h) $R_1P_1/AD = 1/\sqrt{3}$

Figure 3

conclude from this that all three sides of the triangle have the same length. From this it follows that $\triangle PQR$ is equilateral. (We do not actually have to repeat all the steps of the argument. All we need to say is that since the argument worked for this particular pair of sides, it will also work for another pair of sides. Note that this is an appeal to symmetry.)

Another way is to say that $RP = QP$ and $\measuredangle RPQ = 60°$; this is so because R_1P_1 is parallel to Q_2P_2, and we had obtained these two segments by rotations of segments RP and QP through 30° and 30° respectively, the first one through a rotation of $-30°$ (i.e., 30° in a clockwise direction), and the second one through a rotation of $+30°$ (i.e., 30° in an anticlockwise direction). So the two rotations are in *opposite directions*. After the two rotations, the resulting segments are parallel to each other, which means that *prior* to the rotations they must have been inclined at an angle of $(+30)° - (-30)° = 60°$ to each other. This suffices to prove that $\triangle PQR$ is equilateral. □

This proof is to be admired for its elegance and its compactness! It shows just how much can be accomplished using arguments belonging only to elementary geometry.

In Part 3 of this series, we will consider generalisations and further aspects of Napoleon's theorem.

References

1. M. R. F. Smyth, "MacCool's Proof of Napoleon's Theorem", Irish Math. Soc. Bulletin 59 (2007), 71–77, http://www.maths.tcd.ie/pub/ims/bull59/M5903.pdf



SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

A TRIANGLE PROBLEM and THREE VARIANTS

PRITHWIJIT DE

In this short note we start with a simple problem concerning a triangle and then analyze new problems derived from it by changing the hypothesis. Our starting point is the following:

Problem. *In triangle ABC, the largest angle is twice the smallest angle and the lengths of the sides are consecutive positive integers. Determine the lengths of the sides.*

The simplest way to tackle this problem is to use the sine rule. If θ is the smallest angle and if the side-lengths are $x-1$, x and $x+1$, then by the sine rule,

$$\frac{x-1}{\sin\theta} = \frac{x}{\sin(180^\circ - 3\theta)} = \frac{x+1}{\sin 2\theta}. \tag{1}$$

From this it follows, by using standard identities for $\sin 2\theta$ and $\sin 3\theta$, that

$$\begin{aligned} \frac{x+1}{x-1} &= 2\cos\theta; \\ \frac{x}{x-1} &= 3 - 4\sin^2\theta = (2\cos\theta)^2 - 1. \end{aligned} \tag{2}$$

Therefore:

$$\frac{x}{x-1} = \left(\frac{x+1}{x-1}\right)^2 - 1, \tag{3}$$

which yields the equation $x(x-5) = 0$, from which follows $x = 5$, since x is a positive integer. Thus the side-lengths are 4, 5 and 6. We note in passing that $\theta \approx 41.41^\circ$ (computed using the fact that $\cos\theta = 3/4$).

Keywords: Triangle, angle, AP, GP, HP, similarity, golden ratio, intermediate value theorem

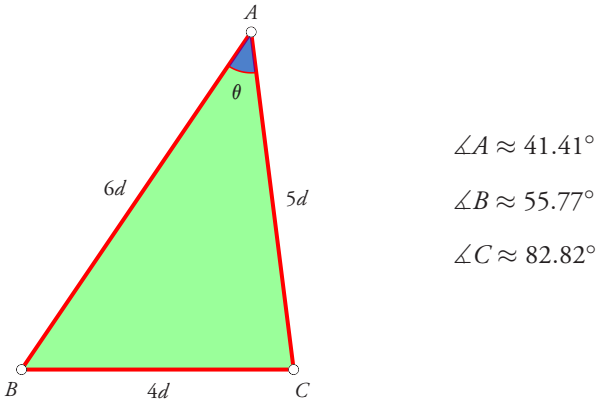


Figure 1. Sides in AP

Variation 1. We now modify the hypothesis of this problem. Instead of requiring that the side-lengths be consecutive positive integers, suppose we demand that they be in arithmetic progression (AP). We keep the angle condition unchanged. What are the side-lengths? Let them be $x - d$, x and $x + d$ where d is the common difference and $0 < d < x$. A triangle exists with these side-lengths if and only if the sum of the two smaller lengths exceeds the largest length, i.e.,

$$x - d + x > x + d. \tag{4}$$

The inequality is equivalent to $x > 2d$. By an argument similar to the one used to analyze the given problem, we have

$$\frac{x}{x-d} = \left(\frac{x+d}{x-d}\right)^2 - 1 \tag{5}$$

which gives $x = 5d$. Thus the side-lengths are $4d$, $5d$ and $6d$. Therefore the effect of enlarging the difference between the side-lengths is that of scaling the side-lengths in the original problem by the common difference. The resulting triangle is similar to the one obtained earlier, with side-lengths 4, 5 and 6. The smallest angle naturally remains unchanged ($\theta \approx 41.41°$). The relevant triangle is depicted in Figure 1.

Variation 2. What if the side-lengths are in geometric progression (GP) and the largest angle is (as earlier) twice the smallest angle? Let the side-lengths be x, xr and xr^2 where r is the common ratio (we may assume that $r > 1$ by agreeing to denote the smallest side-length by x). A triangle exists with these side-lengths if and only if the sum of the two smaller lengths exceeds the largest length, i.e.,

$$x(1 + r) > xr^2, \quad \text{i.e.,} \quad r^2 - r - 1 < 0. \tag{6}$$

This inequality yields $1 < r < \phi$ where $\phi = \frac{1}{2}(1 + \sqrt{5})$, the Golden Ratio. (Recall that the Golden Ratio ϕ is the positive solution of the quadratic equation $x^2 = x + 1$. Its value is approximately 1.618034.) Invoking the sine rule, we get:

$$\frac{x}{\sin\theta} = \frac{xr}{\sin(180° - 3\theta)} = \frac{xr^2}{\sin 2\theta}, \tag{7}$$

whence

$$r^4 - r - 1 = 0. \tag{8}$$

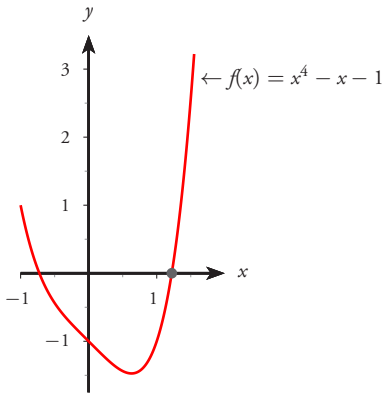


Figure 2. Graph of the function $f(r) = r^4 - r - 1$

Let $f(r) = r^4 - r - 1$. We need to estimate the positive roots of this polynomial. The relevant portion of the graph of this function is shown in Figure 2.

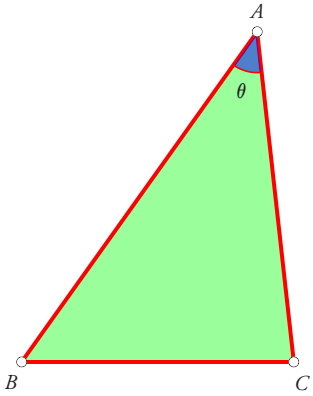
We observe that $f(r)$ has a positive root r_0 between 1 and 2. We can see why without appealing to the graph. First, note that $f(r)$ is strictly increasing for $r > 1$. For, the slope function is $f'(r) = 4r^3 - 1$, which is positive for $r > 1$. Next, note that $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Indeed, we also have $f(1.5) = 2.5625 > 0$. Hence, by the intermediate value theorem, there exists a root between 1 and 1.5, and it is the sole such real root. (Editor's note: The *intermediate value theorem* states that if f is a continuous function defined on an interval I, and its values at the endpoints of I have opposite sign, then it must take a value of 0 somewhere in I.) The use of a computer algebra system such as *Mathematica* or *Derive* reveals the root r_0 to be approximately equal to 1.2207. The corresponding value of θ is given by:

$$2\cos\theta = r_0^2, \quad \therefore \ \theta \approx \cos^{-1}\frac{1.2207^2}{2} \approx 41.83^\circ. \tag{9}$$

The relevant triangle is depicted in Figure 3.

Variation 3. The third case is a triangle with side-lengths in harmonic progression (HP) and the largest angle is (as earlier) twice the smallest angle. (Recall that a harmonic progression is obtained by taking the reciprocals of the terms of an arithmetic progression whose terms are all of one sign.) Let the side-lengths be

$$\frac{1}{x-d}, \qquad \frac{1}{x}, \qquad \frac{1}{x+d}, \tag{10}$$



$\measuredangle A \approx 41.83^\circ$

$\measuredangle B \approx 54.51^\circ$

$\measuredangle C \approx 83.66^\circ$

Figure 3. Sides in GP

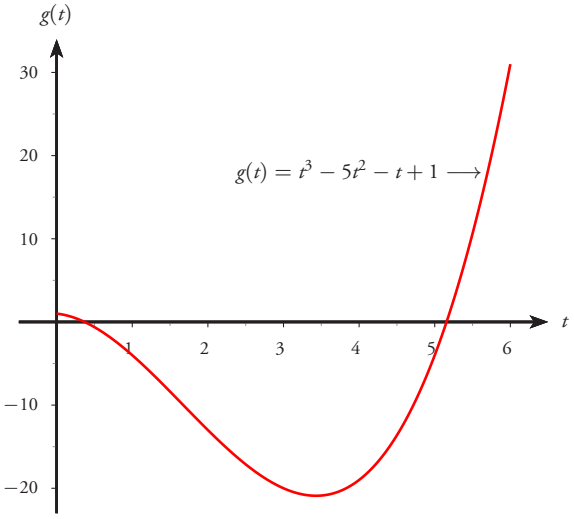


Figure 4. Graph of the function $g(t) = t^3 - 5t^2 - t + 1$

where $0 < d < x$. A triangle exists with these side-lengths if and only if the sum of the two smaller lengths exceeds the largest length, i.e.,

$$\frac{1}{x} + \frac{1}{x+d} > \frac{1}{x-d}. \tag{11}$$

This yields:

$$\frac{1}{x} > \frac{2d}{x^2 - d^2}, \quad \text{i.e.,} \quad x^2 - d^2 > 2dx, \tag{12}$$

which may be expressed as:

$$t^2 - 2t - 1 > 0, \tag{13}$$

where $t = x/d > 1$. This inequality may be expressed as $(t-1)^2 > 2$; hence it is valid if and only if $t > 1 + \sqrt{2}$. After going through the usual route of sine rule followed by algebraic manipulations, we are left with:

$$\frac{x+d}{x-d} = 2\cos\theta, \quad \frac{x+d}{x} = 4\cos^2\theta - 1. \tag{14}$$

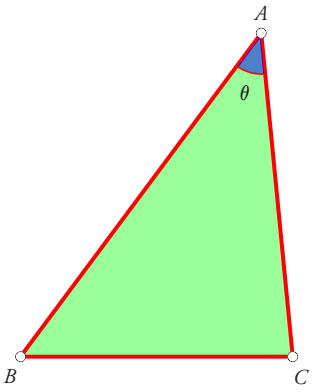
Eliminating θ and setting $t = x/d$, we obtain:

$$t^3 - 5t^2 - t + 1 = 0. \tag{15}$$

Let $g(t) = t^3 - 5t^2 - t + 1$. Figure 4 shows a graph of $g(t)$ for $0 < t < 6$. We see that g has a root between 0 and 1, and another root between 5 and 6. (Alternative proof: $g(0) = 1 > 0$, $g(1) = -4 < 0$, $g(5) = -4 < 0$, $g(6) = 31 > 0$. Now invoke the intermediate value theorem, as earlier.) The former root is not of interest to us, as we know that t exceeds 1; but the latter is, and use of a computer algebra software reveals the root to be approximately equal to 5.1563.

The corresponding approximate magnitude of θ is

$$\theta = \cos^{-1}\frac{t+1}{2(t-1)} \approx 42.22^\circ. \tag{16}$$

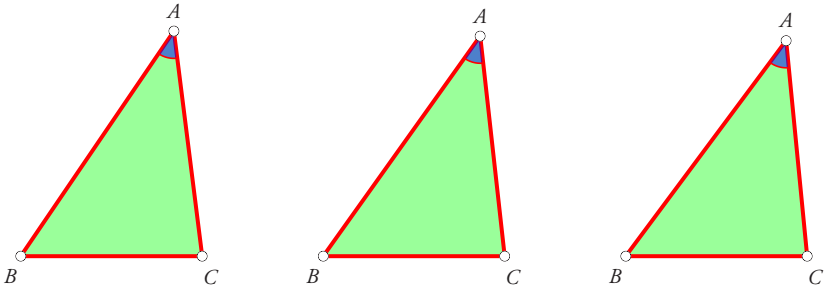


$\angle A \approx 42.22°$

$\angle B \approx 53.35°$

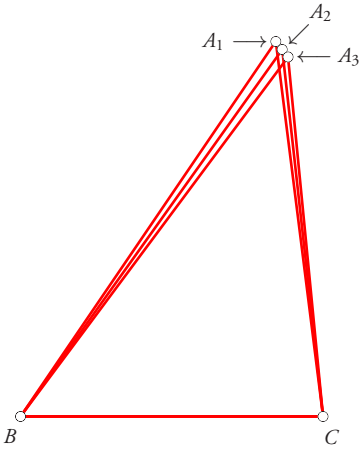
$\angle C \approx 84.44°$

Figure 5. Sides in HP



Sides in AP Sides in GP Sides in HP

Figure 6



$\triangle A_1BC$: sides in AP

$\triangle A_2BC$: sides in GP

$\triangle A_3BC$: sides in HP

Figure 7

Since the side-lengths are in the ratio $1/(t+1) : 1/t : 1/(t-1)$, by substituting the value of t, we see that the side-lengths of the required triangle are approximately in the ratio $0.1624 : 0.1939 : 0.2405$. The relevant triangle is depicted in Figure 5.

Figure 6 displays the three triangles next to one another.

Figure 7 shows the three triangles superimposed upon each other, with a common segment as base. Observe how close are the three vertices to one another: A_1, A_2, A_3.

We conclude by saying that in the three different cases, we obtain three triangles whose angles do not differ much from each other.



PRITHWIJIT DE is a member of the Mathematical Olympiad Cell at Homi Bhabha Centre for Science Education (HBCSE), TIFR. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. His other interests include puzzles, cricket, reading and music. He may be contacted at de.prithwijit@gmail.com.

SOLUTIONS
NUMBER CROSSWORD

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Are you interested in creating the number crossword for the next issue? Send a mail to AtRiA.editor@apu.edu.in and we can help you puzzle your fellow readers.

Student Corner – Featuring articles written by students.

HAPPY NUMBERS

DEV CHANDAN
ADHIRAJ PATIL

In this short note, we talk about ***Happy Numbers***. As this has to do with the digital representation of a number, we must specify right at the start which number base we are working in. So please note: *We will be working throughout in base ten*.

The iteration. Start with any positive integer and compute the sum of the squares of its digits. Do the same for the number that results from this operation; then repeat for the new number, and continue this iteration. (Note from the editor: An 'iteration' is simply some operation or some step that you do over and over again. Mathematicians have studied many different kinds of iterations. A particularly well-known iteration is the one used in Euclid's GCD algorithm.) Let us call this the ***SSQ iteration*** (SSQ for 'sum of squares').

For example, suppose we start with the number 2375. The sum of the squares of its digits is $2^2 + 3^2 + 7^2 + 5^2 = 87$. The sum of the squares of the new number is $8^2 + 7^2 = 113$. Continuing, here is what we get:

$$2375 \to 87 \to 113 \to 11 \to 2 \to 4 \to 16 \to 37 \to 58$$
$$\to 89 \to 145 \to 42 \to 20 \to 4 \to 16 \to 37 \to 58$$
$$\to 89 \to 145 \to 42 \to 20 \to 4 \to 16 \to 37 \to 58$$
$$\to \cdots \to \cdots$$

Keywords: Happy number, iteration, cycle

Observe that we have got caught in a cycle: the sequence of numbers 89, 145, 42, 20, 4, 16, 37, 58 recurs indefinitely. (You will see why we call it a **cycle**; it is a closed loop.)

Let us now start with some other number, say 55. Here is the sequence of numbers which it yields:

55, 50, 25, 29, 85, 89, 145, 42, 20, 4, 16, 37, 58, ...

We are caught in the very same cycle again (89, 145, 42, 20, 4, 16, 37, 58, . . .)!

Experimentation with other starting numbers shows that we end up in this cycle very often. However, it is not the only possible outcome. For example, let us start with the number 49. Here is what it yields:

49, 97, 130, 10, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...

Note what has happened: at some point, the SSQ value turns out to be 1, and once this happens, we will never budge from 1; for: $1 \to 1 \to 1 \to \cdots$.

Here is another example which yields the same outcome. Let us start with the number 193; here is what it yields:

193, 91, 82, 68, 100, 1, 1, 1, 1, 1, 1, 1, 1, ...

At this point, we ask: *For which starting numbers does it happen that we ultimately get stuck at* 1?

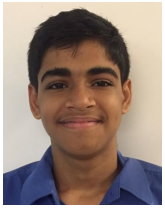
Interval	Happy numbers in this interval
1–100	1, 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 94, 97, 100
101–200	103, 109, 129, 130, 133, 139, 167, 176, 188, 190, 192, 193
201–300	203, 208, 219, 226, 230, 236, 239, 262, 263, 280, 291, 293
301–400	301, 302, 310, 313, 319, 320, 326, 329, 331, 338, 356, 362, 365, 367, 368, 376, 379, 383, 386, 391, 392, 397
401–500	404, 409, 440, 446, 464, 469, 478, 487, 490, 496
501–600	536, 556, 563, 565, 566
601–700	608, 617, 622, 623, 632, 635, 637, 638, 644, 649, 653, 655, 656, 665, 671, 673, 680, 683, 694, 700
701–800	709, 716, 736, 739, 748, 761, 763, 784, 790, 793
801–900	802, 806, 818, 820, 833, 836, 847, 860, 863, 874, 881, 888, 899
901–1000	901, 904, 907, 910, 912, 913, 921, 923, 931, 932, 937, 940, 946, 964, 970, 973, 989, 998, 1000

Table 1. List of happy numbers from 1 to 1000

Numbers with this property — that the SSQ iteration starting with that number ultimately reaches 1 and gets 'stuck' there — have been called *Happy Numbers*.

Is there a formula to generate happy numbers? There does not seem to be any formula. Table 1 lists all the happy numbers that do not exceed 1000. We generated the list manually!

Note the large number of happy numbers between 301 and 400, between 601 and 700, and between 901 and 1000. Note also the small number of happy numbers between 501 and 600. The wide variation is very puzzling!



DEV VINOD CHANDAN is currently studying in Grade 10 at SAS Billabong High School, Mahim, Mumbai. He worked on Happy Numbers as part of Vinay Nair's "Vedic Maths Camp" at Chinmaya Ashram, Pune in May 2016. He lives with his parents and younger brother in Mumbai. He is an avid maths and science enthusiast. He enjoys playing cricket, table tennis and football. He is an eager yoga student and also learns the keyboard. Dev may be contacted at devvchandan@gmail.com.

Note from the editor:

Adhiraj Patil is from SAS Billabong High School, Mahim, Mumbai. We have been unable to get his photograph and key biographical information; hence this brief write-up.

The above article was written as a result of a Vedic Maths workshop done by Vinay Nair at SAS Billabong High School. Dev Chandan and Adhiraj Patil were assigned the task of exploring the notion of Happy Numbers. The project was to be completed within 2-3 days. Further, the investigators were to use manual calculations only; no use was allowed of either calculators or cell phones or any kind of computer devices. So they got down to their task by dividing the numbers between them and then physically checking each and every number and cross-checking the same. This article has arisen as a result.

HOW to PROVE it

SHAILESH SHIRALI

Elsewhere in this issue, two students report on their search for Happy Numbers. This is an interesting topic in recreational mathematics which throws up questions that are not easy to answer. We describe the essential proof techniques used in the study of such topics.

In our column for this issue, we deal with the notion of Happy Numbers. This has to do with the digital representation of a number, so we specify right at the start that we will be working throughout in base ten.

The SSQ iteration. Start with any positive integer and compute the sum of the squares of its digits. Do the same for the resulting number, then repeat for the new number, and continue the iteration. We call this the SSQ iteration. For example, the number 2375 yields:

$$2375 \to 87 \to 113 \to 11 \to 2 \to 4 \to 16 \to 37 \to 58$$
$$\to 89 \to 145 \to 42 \to 20 \to 4 \to 16 \to 37 \to 58$$
$$\to \cdots \to \cdots,$$

and we see that we have got caught in a cycle, (4, 16, 37, 58, 89, 145, 42, 20). As the cycle has 8 numbers, we refer to it as an 8-cycle.

On the other hand, observe what happens if we start with the number 91:

$$91 \to 82 \to 68 \to 100 \to 1 \to 1 \to 1 \to 1 \to \cdots.$$

Having reached the number 1, we never leave it. Once again, we have reached a cycle, but this time the cycle has just one number; it is called a 1-cycle. The number in a **1-cycle** is called a **fixed point** of the iteration under study.

Keywords: Happy number, iteration, cycle, programming, Mathematica, Python

The two cycles we have found may be depicted in a more striking way as follows:

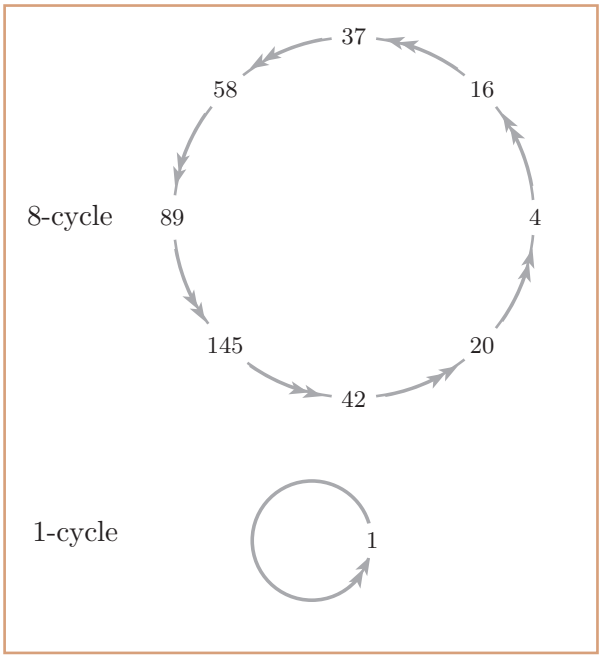


Figure 1

As noted in the companion article, numbers which ultimately map to the fixed point 1 (under the SSQ iteration) are referred to as *Happy Numbers.*

In this note, we shall establish that the SSQ iteration yields just the two cycles listed above. This conclusion is valid regardless of the choice of the initial number.

Proof of the claim that the SSQ iteration yields precisely 2 cycles. Suppose that n is a six-digit number. What is the largest that its SSQ value could be? Clearly, the largest value is attained when $n = 999999$, i.e., it is composed wholly of 9's. So the largest possible SSQ value for a six-digit number is $6 \times 81 = 486$. Observe that this is very much smaller than the original number.

If n were a seven-digit number, this discrepancy would be even larger. For, the largest possible SSQ value for a seven-digit number is $7 \times 81 = 567$.

In general, if n is a k-digit number, the largest its SSQ value can be is $k \times 9^2 = 81k$.

The smallest possible k-digit number is 10^{k-1}, and it is easy to check that $81k < 10^{k-1}$ for all $k \geq 4$.

We display here the values of $81k$ and 10^{k-1} for a few values of k:

k	1	2	3	4	5	6	...
$81k$	81	162	243	324	405	486	...
10^{k-1}	1	10	100	1000	10000	100000	...

The claim that $81k < 10^{k-1}$ for $k \geq 4$ is easy to prove using induction; we leave the details to the reader.

What the inequality proves is that if n has four or more digits, then its SSQ value is strictly less than n. Therefore, *no matter how large this initial number n is, by applying the* SSQ *operation repeatedly, we eventually obtain numbers with no more than three digits.*

It is possible to make a stronger statement. The largest possible SSQ value for a three-digit number is 243; the largest possible SSQ value for numbers not exceeding 243 is $1 + 9^2 + 9^2 = 163$ (achieved by 199); and the largest possible SSQ value for numbers not exceeding 163 is $9^2 + 9^2 = 162$ (achieved by 99). This chain of statements cannot be further continued, because 99 lies below 162. So we conclude by making the following statement: *No matter how large the initial number is, by applying the SSQ operation repeatedly, we eventually obtain numbers no larger than* 162.

Therefore, to list the cycles corresponding to this iteration, it suffices to study what the SSQ operation does to numbers between 1 and 162. This being a finite set, we are faced with a relatively simple task; all we need to do is to list the SSQ values of all the numbers between 1 and 162 and trace out the various trajectories (also called 'orbits'). For example, 3 leads to the trajectory 3, 9, 81, 65, 61, 37, ... We do not need to continue beyond 37, because we know that 37 is part of the 8-cycle listed earlier; following 37, we simply cycle around that 8-cycle indefinitely. Similarly, 5 leads to the trajectory 5, 25, 29, 85, 89, ... As in the earlier case, we do not need to continue beyond 89, because 89 is part of that same 8-cycle.

Proceeding in this way, we obtain all the cycles corresponding to this iteration. In the end, we

find that there are just two cycles — the ones that we had listed earlier. We do not give the details of this listing here; it can be done in a straightforward manner. But it would be a good idea if *you* go ahead and trace out all the possible trajectories! If you do so, you will find that what we have claimed is true.

You may be slightly unhappy at the fact that we have left the last part of the proof incomplete; or rather, we have left the details of the verification to the reader. But you will find that this is true, in general, of all computer-assisted proofs. Most such proofs end with some such statement: "the details of the computation are left to the reader"; or: "the algebraic verification is left to the reader"; and so on. In some cases, the missing details can be filled in by pencil-and-paper computation. But if the setting is more complex, we may need to make use of sophisticated computer algebra software (for example, *Mathematica*) or a general purpose computer programming language such as Python.

In the case of computer-assisted proofs, questions may arise such as: "How do we know that the program we have written does not contain any logical errors?" Or: "How do we know that the computer algebra software is error-free?" This is not the place to discuss such questions. But they do arise, inevitably, whenever we have a computer-assisted proof. (Such questions have indeed arisen in some cases, for example, the proof of the four-colour theorem, which depends in a critical way on a gigantic amount of machine computation. See [5] and [6] for details.)

A remark on the distribution of happy numbers. Extended computations using high-speed computers reveal some curious facts about the distribution of happy numbers; here are two of these facts (see [2] for details):

- It appears that roughly 1/7 of all the positive integers are happy! What is striking is that this fraction seems to persist as numbers get larger.
- It appears that the longest run of consecutive numbers which are all happy has length 5. Once again, this statistic does not seem to change as the numbers get larger.

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SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of several mathematics books for high school students, and serves as an editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

An Approximation of π by using THE LAW OF COSINES

BODHIDEEP JOARDAR

The approximation of π is a popular pastime of students of mathematics and I have started with the familiar age-old way, of fitting a regular polygon tightly in a circle of radius r. The vertices of the n-sided polygon are joined to the centre of the circle so that the angle subtended at the centre by each segment is $360^\circ/n$. If n is sufficiently large, the perimeter of the polygon approaches the circumference $2\pi r$ of the circle and this approximation improves as the number of segments increases. My object is to find a formula for π that is independent of the radius r.

The change that I have made is that I have used the cosine rule and a few trigonometric identities to calculate the perimeter of the polygon. Once I got a formula for π, my approximation has been improved with the use of a scientific calculator.

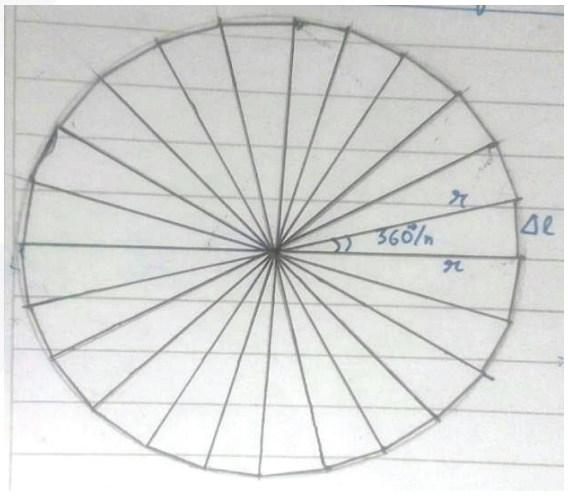


Figure 1. Up to this it is similar to Archimedes' construction, but instead of using Pythagoras' Theorem as he did, I want to use the law of cosines.

Keywords: π, *approximation, circle, polygon, circumference, law of cosines, difference of cosines, limits*

In Figure 1, the polygon is divided into n isosceles triangles, having equal arms of length r (the radius of the circle) and congruent to one another. Using the *law of cosines* the base $\triangle l$ of each triangle satisfies the relation,

$$(\triangle l)^2 = r^2 + r^2 - 2.r.r.\cos\frac{360^\circ}{n} = 2r^2\left(1 - \cos\frac{360^\circ}{n}\right)$$

Therefore, $\triangle l = r\sqrt{2\left(1 - \cos\frac{360^\circ}{n}\right)}$

For sufficiently large n, the perimeter of the polygon, i.e., $n.\triangle l$ will approximate the circumference of the circle.

So, the circumference $2\boldsymbol{\pi} r \approx n.\triangle l = nr\sqrt{2\left(1 - \cos\frac{360^\circ}{n}\right)}$

Therefore, $\boldsymbol{\pi} \approx \frac{n}{2}\sqrt{2\left(1 - \cos\frac{360^\circ}{n}\right)}$, which is independent of the radius r as it should be. *This is the desired formula for* $\boldsymbol{\pi}$, which can be simplified further by using the identity

$$\cos A - \cos B = 2\sin\frac{B+A}{2}.\sin\frac{B-A}{2}.$$

So,

$$\boldsymbol{\pi} \approx \frac{n}{2}\sqrt{2\left(1 - \cos\frac{360^\circ}{n}\right)} \approx \frac{n}{2}\sqrt{2\left(\cos 0^\circ - \cos\frac{360^\circ}{n}\right)}$$
$$\approx \frac{n}{2}\sqrt{2\left(2\sin\frac{180^\circ}{n}.\sin\frac{180^\circ}{n}\right)}$$

That is, $\boldsymbol{\pi} \approx n\sin\frac{180^\circ}{n}$, *a simple formula*.

Taking $n = 24$, the value of $\boldsymbol{\pi}$ is 3.132628613281238…..

Taking higher and higher values of n and applying this formula, we can get better and better approximations of $\boldsymbol{\pi}$. Let us look at Table 1.

n	$\boldsymbol{\pi}$
24	3.132628613281238…
96	3.14103195089051…
576	3.141577077703974…
20736	3.141592641571345…
124416	3.141592653255947…
746496	3.141592653580528…

Table 1. Improved approximations for $\boldsymbol{\pi}$



BODHIDEEP JOARDAR (born 2005) is a student of South Point High School, Calcutta, who is a voracious reader of all kinds of mathematical literature. He is interested in number theory, Euclidean geometry, higher algebra, foundations of calculus and infinite series. He feels inspired by the history of mathematics and by the lives of mathematicians. His other interests are in physics, astronomy, painting and the German language. He may be contacted at ch_kakoli@yahoo.com.



Maryam Mirzakhani: A Tribute

Maryam Mirzakhani has gone the way of all too many great mathematicians who died young: Evariste Galois (1811–1832), Niels Abel (1802–1829), Srinivasa Ramanujan (1887–1920), Bernard Riemann (1826–1866), Gotthold Eisenstein (1823–1852), Frank Ramsey (1903–1930),... The list is already much too long, and now Maryam Mirzakhani (1977–2017) – to date, the only woman and the only mathematician from Iran to win a Fields medal (the top medal in mathematics and often referred to as the "Nobel prize of mathematics") – has joined it. She died in July 2017 of breast cancer, which she had had since 2013. She was 40 years old at the time of her death.

She won the Fields medal in 2014. The official citation read in part: [for] "... her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces."

She was a professor in Stanford University from 2009 till the time of her death. We give below a few extracts from the obituary which appeared in the Stanford newsletter [1]:

"Maryam is gone far too soon, but her impact will live on for the thousands of women she inspired to pursue math and science."

"Maryam was a brilliant mathematical theorist, and also a humble person who accepted honours only with the hope that it might encourage others to follow her path. Her contributions as both a scholar and a role model are significant and enduring, and she will be dearly missed."

She was a self-professed 'slow' mathematician. Her colleagues describe her as "resolute and fearless in the face of problems others would not, or could not, tackle. She denied herself the easy path, choosing instead to tackle thornier issues. Her preferred method of working on a problem was to doodle on large sheets of white paper, scribbling formulas on the periphery of her drawings. Her young daughter described her mother at work as 'painting.'"

"You have to spend some energy and effort to see the beauty of math," she told one reporter. In another interview, she said: "I don't have any particular recipe [for developing new proofs]... It is like being lost in a jungle and trying to use all the knowledge that you can gather to come up with some new tricks, and with some luck you might find a way out."

Reference

[1] http://news.stanford.edu/2017/07/15/maryam-mirzakhani-stanford-mathematician-and-fields-medal-winner-dies/

On Scalene Triples

NAVEENKUMAR YADAV

Given three positive integers a, b, c such that $a \geq b \geq c$, we say that the triple (a, b, c) has the **triangular property** if the sum of any two of the numbers exceeds the third one (i.e., $a + b > c$ and $b + c > a$ and $c + a > b$). The terminology draws from the well-known property of any triangle: the sum of any two of its sides exceeds the third side. We say further that the triangular triple (a, b, c) is **scalene** if a, b, c are distinct positive integers satisfying the triangular property. For example, the triple $(4, 3, 2)$ has the triangular property and it is scalene. However, the triples $(3, 2, 1)$ and $(5, 3, 1)$ do not have the triangular property.

Fix an upper limit n, and let a, b, c take all possible positive integer values from 1 to n (i.e., $1 \leq a, b, c \leq n$). Clearly, n^3 triples are possible. In this article, we study the following question:

> *Given a fixed positive integer n, how many scalene triples exist?*

Notation.

- $S(n)$ denotes the set of all triples (a, b, c) with $1 \leq a, b, c \leq n$. This may be regarded as the "universal set" within which the set we are studying resides (the triples satisfying the triangular property and ordered so that $a \geq b \geq c$). The number of triples in $S(n)$ is clearly n^3, i.e., $|S(n)| = n^3$.
- $T(n)$ denotes the set of triples in $S(n)$ which possess the triangular property.
- $\triangle(n)$ denotes the number of triples in $T(n)$ which are scalene, i.e., $\triangle(n)$ counts the scalene triples in $T(n)$.
- $s(n)$ denotes the number of scalene triples added when a positive integer n is introduced to the range. That is, $s(n) = \triangle(n) - \triangle(n-1)$ for $n > 1$. Stated otherwise, $\triangle(n) = s(1) + s(2) + s(3) + \cdots + s(n)$.

Keywords: Triangular property, scalene triples

Counting the number of scalene triples

To calculate the number of scalene triples formed by integers $1, 2, 3, \ldots, n$, we first enumerate the values of $\triangle(n)$ for a few small values of n.

Since we need three distinct positive integers to form a scalene triangle, it follows that $\triangle(1) = 0$ and $\triangle(2) = 0$. Further, for $n = 3$, the triple $(3, 2, 1)$ does not satisfy the triangular property; hence $\triangle(3) = 0$ too. The values of $\triangle(n)$ for $n = 4, 5, 6, 7, 8$ are computed as shown below, by actually listing all the triples which satisfy the stated property.

n	Scalene triples added	Number of triples added, i.e., $s(n)$	$\triangle(n)$
4	$(4, 3, 2)$	1	1
5	$(5, 4, 3)$, $(5, 4, 2)$	2	3
6	$(6, 5, 4)$, $(6, 5, 3)$, $(6, 5, 2)$ $(6, 4, 3)$	4	7
7	$(7, 6, 5)$, $(7, 6, 4)$, $(7, 6, 3)$, $(7, 6, 2)$ $(7, 5, 4)$, $(7, 5, 3)$	6	13
8	$(8, 7, 6)$, $(8, 7, 5)$, $(8, 7, 4)$, $(8, 7, 3)$, $(8, 7, 2)$ $(8, 6, 5)$, $(8, 6, 4)$, $(8, 6, 3)$ $(8, 5, 4)$	9	22

Counting the scalene triples for larger n. For larger n, let us list the possible scalene triangular triples (a, b, c) in some systematic manner. Remember that we need $c < b < a \leq n$ and $b + c > a$.

Suppose that $a = 2k$, where k is a positive integer. Since the entries must be unequal, we must have $b < 2k$; so the largest possible value that b can take is $2k - 1$. What about the least possible value? Since $b + c > a$ and $c < b$, we must have $b \geq k + 1$. (If $b = k$, then $b + c \leq 2k - 1 < 2k$, which violates the triangle inequality.) So if $a = 2k$, then b can take the values $2k - 1$, $2k - 2$, …, $k + 1$. For each of these b-values, we list the possible c-values which fit the requirement. For example, if $b = 2k - 1$, then c can take values from 2 till $2k - 2$, or $2k - 3$ values in all. Similarly, if $b = 2k - 2$, then c can take values from 3 till $2k - 3$, or $2k - 5$ values in all. In this manner, we construct the following table of possibilities corresponding to $a = 2k$:

$$(2k, 2k-1, 2k-2),\ (2k, 2k-1, 2k-3),\ (2k, 2k-1, 2k-4),\ \ldots,\ (2k, 2k-1, 2),$$

$$(2k, 2k-2, 2k-3),\ (2k, 2k-2, 2k-4),\ (2k, 2k-2, 2k-5),\ \ldots,\ (2k, 2k-2, 3),$$

$$\vdots$$

$$(2k, k+2, k+1),\ (2k, k+2, k),\ (2k, k+2, k-1),$$

$$(2k, k+1, k).$$

It follows that the total number of triples in this case is

$$s(2k) = (2k-3) + (2k-5) + \cdots + 5 + 3 + 1 = (k-1)^2. \tag{1}$$

(To see why, recall the formula for the sum of an arithmetic progression; or recall that the sum of the first n odd positive integers is equal to n^2.) By reasoning in exactly the same manner, we construct the following table of possibilities corresponding to $a = 2k+1$:

$$(2k+1, 2k, 2k-1),\ (2k+1, 2k, 2k-2),\ (2k+1, 2k, 2k-3),\ \ldots,\ (2k+1, 2k, 2),$$

$$(2k+1, 2k-1, 2k-2),\ (2k+1, 2k-1, 2k-3),\ \ldots,\ (2k+1, 2k-1, 3),$$

$$\vdots$$

$$(2k+1, k+3, k+2),\ (2k+1, k+3, k+1),\ (2k+1, k+3, k),\ (2k+1, k+3, k-1),$$

$$(2k+1, k+2, k+1),\ (2k+1, k+2, k).$$

It follows that the total number of triples added is

$$s(2k+1) = (2k-2) + (2k-4) + \cdots + 6 + 4 + 2 = k^2 - k. \tag{2}$$

(As earlier, we may use the formula for the sum of an arithmetic progression; or we may note that the required sum is twice the sum of the first $k-1$ positive integers.)

Theorem. *For fixed n, the number of scalene triples formed by positive integers $1, 2, \ldots, n$ is*

$$\triangle(n) = \begin{cases} \dfrac{k(k-1)(4k-5)}{6}, & \text{if } n = 2k, \\ \dfrac{k(k-1)(4k+1)}{6}, & \text{if } n = 2k+1. \end{cases} \tag{3}$$

Proof. For an even positive integer $n = 2k$,

$$\begin{aligned} \triangle(n) &= s(1) + s(2) + \cdots + s(2k-1) + s(2k) \\ &= \sum_{i=1}^{k-1} (i^2 - i) + \sum_{i=1}^{k} (i^2 - 2i + 1) \\ &= \sum_{i=1}^{k-1} (2i^2 - 3i + 1) + k^2 - 2k + 1 \\ &= 2 \cdot \frac{(k-1)k(2k-1)}{6} - 3 \cdot \frac{(k-1)k}{2} + k - 1 + k^2 - 2k + 1 \\ &= \frac{k(k-1)(4k-5)}{6}, \text{ on simplification.} \end{aligned}$$

Similarly, for an odd positive integer $n = 2k + 1$,

$$\begin{aligned}\triangle(n) &= s(1) + s(2) + \cdots + s(2k-1) + s(2k) + s(2k+1)\\ &= \sum_{i=1}^{k}(i^2 - i) + \sum_{i=1}^{k}(i^2 - 2i + 1)\\ &= \sum_{i=1}^{k}(2i^2 - 3i + 1)\\ &= 2 \cdot \frac{k(k+1)(2k+1)}{6} - 3 \cdot \frac{k(k+1)}{2} + k\\ &= \frac{k(k-1)(4k+1)}{6}, \text{ on simplification.}\end{aligned}$$

Hence the stated result.



NAVEENKUMAR YADAV has a Master's degree in Mathematics from Maharaja Sayajirao University of Baroda, India. At present, he is working as an Assistant Professor at BKM Science College, Valsad. He may be contacted at ynav411@gmail.com.

Fagnano's Theorem

ALTERNATIVE PROOF

UJJWAL RANE

Fagnano's Problem: In 1775, Giovanni Fagnano posed and solved the problem - "For a given acute angled triangle, determine the inscribed triangle of minimum perimeter." Using calculus, Fagnano showed the solution to be the Orthic Triangle – a triangle formed by the feet of the three altitudes. A different proof was given in At Right Angles, Vol. 6, No. 1, March 2017, available at http://azimpremjiuniversity.edu.in/SitePages/resources-ara-march-2017-fagnanos-problem.aspx

Let's see if it can be solved using geometry and a result from Physics called Fermat's principle.

Fermat's principle states that between two points, light always follows the path that takes the shortest time. In case of pure reflection, it is the path of the shortest length too.

Hence, if the given acute triangle is formed using three mirrors PQ, QR, RP and a ray of light is '*perpetually trapped*' between them by getting repeatedly reflected at points A, B and C, the loop CBAC will be the shortest path that reflects a ray from point C back to itself. See Figure 1.

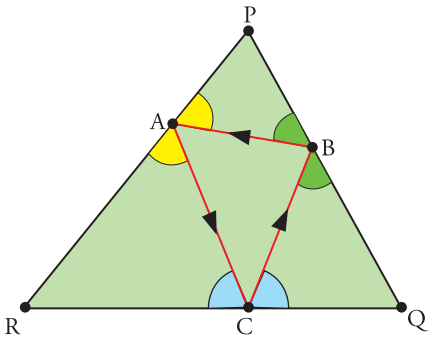


Figure 1. By **Fermat's Principle,** a ray of light perpetually trapped between mirrors PQ, QR and RP gives the shortest path from point C to itself.

Keywords: Geometry, Fagnano's Theorem, Optimization, Minimization, Perimeter

The path of such a repeated reflection can be plotted on the original triangle, its image P′Q′R′ (in mirror PQ) and an image of that image – or its double image - P″Q″R″ (in mirror P'R'). See Figure 2.

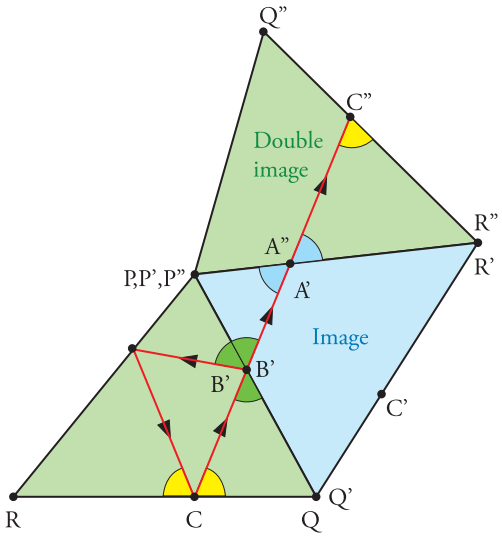


Figure 2. The loop CBAC of repeated reflection of light can be unfolded on the given triangle and its two successive images. It connects point C to its double image C" with a straight line.

Note that the double reflection P″Q″R″ is merely a rotation of the original triangle PQR!

The closed path CBAC then unfolds into a straight line from point C to its double image C″.

The rectilinear nature of the unfolded loop shows that such a loop must be unique and shortest in length as well.

Of course, there would be infinitely many such loops or straight lines – one for every starting point C.

So, which of these loops would be the triangle Fagnano was looking for?

To find out, let's consider an arbitrary point D, its double image D″ and the triangle PDD″ (see Figure 3).

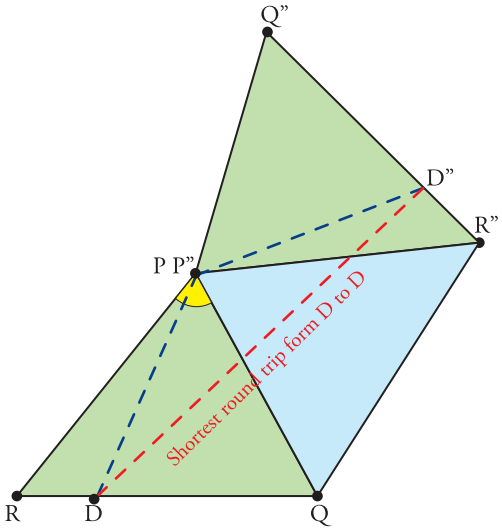


Figure 3. All triangles PDD″ are isosceles with identical vertex angle *2 γ a*nd hence similar. For base DD″ to be the shortest, side PD should also be minimized.

Sides PD and PD″ must be identical, since PD″ is simply the double image of side PD.

Thus, triangle PDD″ must be isosceles.

Moreover, since a double reflection really amounts to a simple rotation through twice the angle RPQ = γ, the vertex angle DPD″ for all such isosceles triangles must be the same, viz., 2γ, making them all similar.

The triangle we seek would have the shortest possible base DD″.

Since similar triangles are scaled replicas of each other, the shortest possible base belongs to the smallest of the triangles, which in turn will have the shortest possible sides PD or PD″ too. And this condition is met when PD is the perpendicular dropped from P on to side RQ (shortest distance of a point from a line).

Hence the shortest path is obtained when point D is the foot of the perpendicular. By symmetry, this applies to all the three points A, B and C, making the shortest path the so called Orthic triangle! See Figure 4.

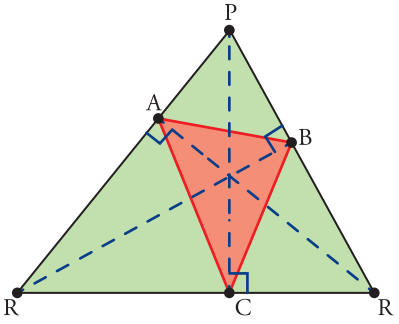


Figure 4. The minimum perimeter inscribed triangle connects the feet of the three altitudes forming an **Orthic Triangle**

Please see the YouTube video at https://youtu.be/5MrNM-VxXd8 for an animation of the above.



UJJWAL RANE is a Mechanical Engineer by training and has master's degrees from IIT Madras (Machine Dynamics) and Arizona State University, USA (Computer Aided Geometric Design). He has a particular interest in finding visual representations and solutions of mathematical and physical phenomena and in using this approach in classroom teaching of Engineering, Physics and Math, and also online via his channel on YouTube (https://www.youtube.com/user/UjjwalRane). He can be reached on ujjukaka@gmail.com.

Low Floor High Ceiling Tasks

If you reason, you can begin with…

APC - AREA, PERIMETER AND CONGRUENCY

PRITHWIJIT DE,
SWATI SIRCAR,
SNEHA TITUS

We continue our Low Floor High Ceiling series in which an activity is chosen – it starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that students are pushed to their limits as they attempt their work. There is enough work for all, but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

The topic of Perimeter and Area provides rich ground for teachers to examine the truth values of statements and then introduce the crucial 'What-If' which can change a situation around completely. While the topics of area, perimeter and congruency are addressed in class 9 and 10, students will also need to know some trigonometry including the relationship between the sines of supplementary angles, area in terms of the sine of the enclosed angle and the cosine rule to do these tasks. In case these are not familiar to the students, we give them below.

1. $\sin\theta = \sin(180 - \theta)$
2. Area of a triangle $= \frac{1}{2}\, a\, b \sin\theta$, where a, b are the lengths of the sides enclosing the angle θ
3. $\cos\theta = \frac{a^2 + b^2 - c^2}{2\, ab}$ in a triangle with sides of length a, b, c where c is the side opposite the angle θ

Keywords: Polygon, area, perimeter, angles, congruency, mensuration, trigonometry, conditional statement, exploration, skill development.

The tasks are in the nature of explorations and give students plenty of opportunity to do ruler and compass constructions or create GeoGebra sketches. In each of the tasks, the underlying question is: Given some equal parameters, are two polygons congruent? In exploring, conjecturing, justifying and proving, students are able to create conditional statements that illustrate the delicate art and science of mathematical reasoning.

Task 1
Consider two identical (congruent) rectangles.

- Do they have the same perimeter?
- Do they have the same area?
- Explain why this is so.

Now consider two rectangles with the same perimeter.

- Do they have the same area?
- Prove your answer.

Consider two rectangles with the same area.

- Do they have the same perimeter?
- Justify your stance.

Finally, consider two rectangles with the same perimeter and the same area.

- Are they identical?
- Prove your answer.

Teacher's Notes: A nice easy start to proof in the mathematics class! Rectangles provide a gentle introduction to mensuration and students are also able to justify their stance in words and in some cases, simply with a numerical example that is an exception to the statement. An 'if and only if' statement can be framed and the teacher can use it as an example for the same.

Task 2

- The congruence of two triangles implies equality of their areas and perimeters. True or False? Justify your stance.
- Find the area and perimeter of the two triangles with sides 20,21,29 and 17,25,28. Are they congruent? Justify.
- Use the above questions and their answers to frame a statement that connects the congruency of two triangles with their area and perimeter.
- If two similar triangles have the same perimeter, will they be congruent?
- If two similar triangles have the same area, will they be congruent?

Teacher's Note: A nice refresher course for mensuration formulae. Students should be facilitated to frame the statement that 'if two triangles are congruent, then their area and perimeter are the same, but if they have the same area and perimeter, they need not be congruent.'

In this task, students get practice in framing a conditional statement. In addition, they are able to deepen their understanding of the concept of 'if and only if' with both examples as well as counter-examples.

Students' understanding of similarity can lead them to conclude that when either the area or the perimeter is the same for similar triangles, the common ratio is 1, implying congruency.

Task 3
Now consider two triangles with equal area and perimeter and having one pair of equal sides. Are they congruent? Prove your statement.

Teacher's Note: This task may need some support from the teacher, but with a little support in writing up the given data in symbolic form and using the appropriate mensuration formulae, students should be able to arrive at a satisfactory conclusion with a rigorous proof. One proof has been given below.

Area, perimeter and one side equal

Let the sides of the triangle be a, b, c and u, v, c. Since the perimeters are equal, let $p = a + b + c = u + v + c$. Since the areas are equal we have (by Heron's formula),

$$p(p-2a)(p-2b)(p-2c) = p(p-2u)(p-2v)(p-2c). \quad (1)$$

Upon simplification and observing that $a + b = u + v$ we obtain

$$ab = uv. \quad (2)$$

Thus we now need to ascertain whether the two equations: $a + b = u + v$ and $ab = uv$ imply $\{a, b\} = \{u, v\}$. Observe that

$$(x-a)(x-b) - (x-u)(x-v) = 0 \quad (3)$$

for all real values of x. By substituting $x = a$ and $x = b$ in turn shows that $\{a, b\} = \{u, v\}$. Hence the triangles are congruent by SSS.

Task 4

Is there any other condition under which triangles with the same area and the same perimeter become congruent?

Teacher's Note: Now that the class has got one condition for two triangles with the same area and the same perimeter to be congruent, experiment with sides and angles to find other such conditions. Here is one of our discoveries.

Area, perimeter and one angle equal

Let the sides of the triangle be a, b, c and u, v, w. Let the common angle be θ and in the first triangle let the sides a and b include θ, and in the second triangle let the sides that include θ be u and v. As the areas are equal we have

$$\frac{1}{2}ab\sin\theta = \frac{1}{2}uv\sin\theta \quad (4)$$

whence $ab = uv$. The cosine rule gives us

$$\cos\theta = \frac{a^2+b^2-c^2}{2ab} = \frac{u^2+v^2-w^2}{2uv} \quad (5)$$

Therefore

$$a^2 + b^2 - c^2 = u^2 + v^2 - w^2, \quad (6)$$

and upon adding the equal quantities $2ab$ and $2uv$ to the left hand side and right hand side respectively of the previous equation we obtain

$$(a+b)^2 - c^2 = (u+v)^2 - w^2. \quad (7)$$

Factoring both sides and using the fact that $a + b + c = u + v + w$ leads to

$$a + b - c = u + v - w, \quad (8)$$

whence $c = w$ and we end up as in the previous case (area, perimeter and one side equal). Thus the two triangles are congruent.

Task 5

- If two triangles have the same perimeter and two pairs of equal sides, are they congruent?
- Now, construct two triangles having the same area and with two pairs of sides the same.
- Are they congruent?
- Prove your statement.

Teacher's Note: Do discuss with your students how each statement differs from the previous statements – it's a good exercise in subtle alteration of constraints!

A little bit of thinking is required before students do the construction. We used the theorem that triangles with the same base and between the same parallel lines have the same area. Constructions could be done with ruler and compass or on GeoGebra.

Through D draw a line parallel to AB (see Figure 1). With A as centre and AD as radius draw an arc of a circle. Let this arc intersect the line parallel to AB through D again at E. Join BD. The triangles under consideration are $\triangle ABD$ and $\triangle ABE$.

The construction (by use of a counter-example) will give students an indication that the triangles need not be congruent; we give the justification below but strongly advise you to allow the students to explore and conjecture before going into a rigorous proof.

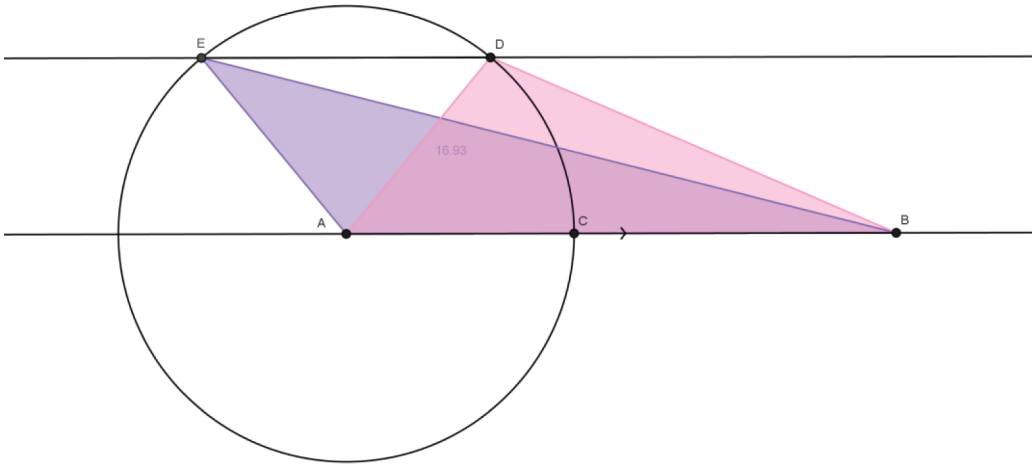


Figure 1. Triangles of the same area with two sides in common

Let the two common sides of the two triangles be $AB = a$ and $AD = AE = b$. If $\triangle$ denotes the common area then the angle included by the common sides, θ_1 in the first triangle and θ_2 in the second triangle, satisfy

$$\sin\theta_1 = \sin\theta_2 = \frac{2\triangle}{ab}. \tag{9}$$

Therefore, either $\theta_1 = \theta_2$ or $\theta_1 = 180^\circ - \theta_2$, and hence in this case we cannot conclude, barring when $\theta = 90^\circ$, that under the given conditions the triangles are congruent. Note that the diagram makes this obvious.

Task 6
Sketch two triangles with the same area, having one side and one angle in common. Are they congruent?

Teacher's Note: Here the challenge for the students is to identify three cases, one in which the common angle has the common side opposite it for both triangles, the second in which the common side is an arm of the common angle for both triangles and a third in which the common angle is opposite the equal side in one triangle and adjacent to the equal side in another.

A proof for the first two cases is given below.

If the common side is one of the sides that include the common angle then the triangles are congruent. To see this, let the common side be a and the common angle θ. In the first triangle let the sides that include θ be a and x, and the same in the second triangle be a and y.

Then equality of area forces

$$\frac{1}{2}ax\sin\theta = \frac{1}{2}ay\sin\theta \tag{10}$$

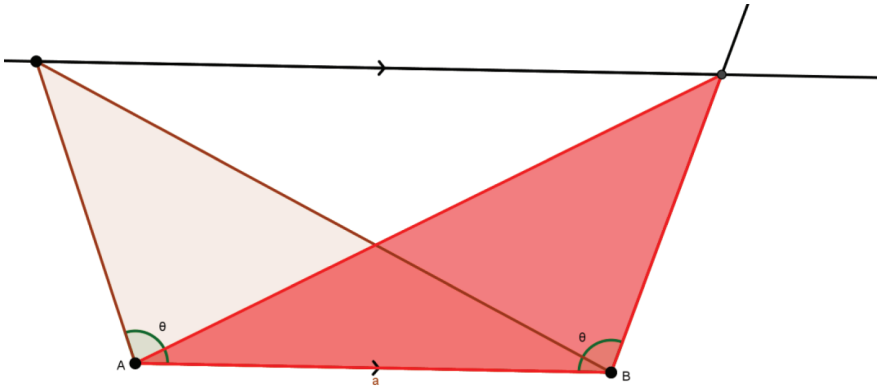


Figure 2. Two triangles of the same area with one common side and one equal angle adjacent to the side

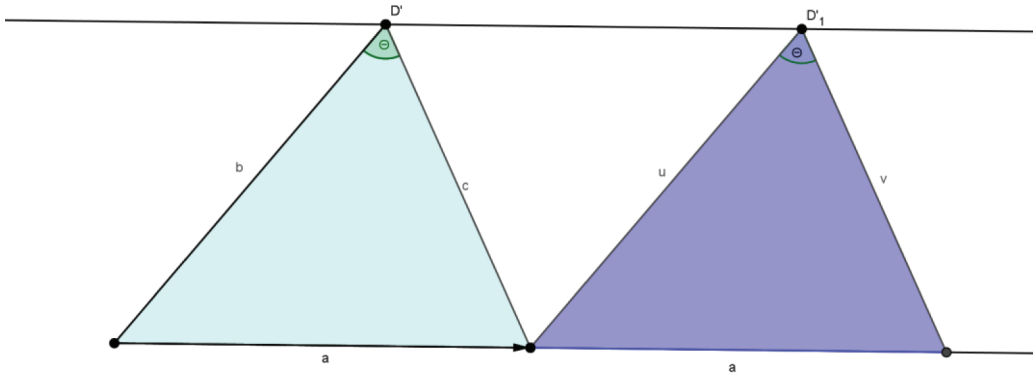


Figure 3. Triangles with the same area having one pair of sides equal and one equal angle opposite the equal sides

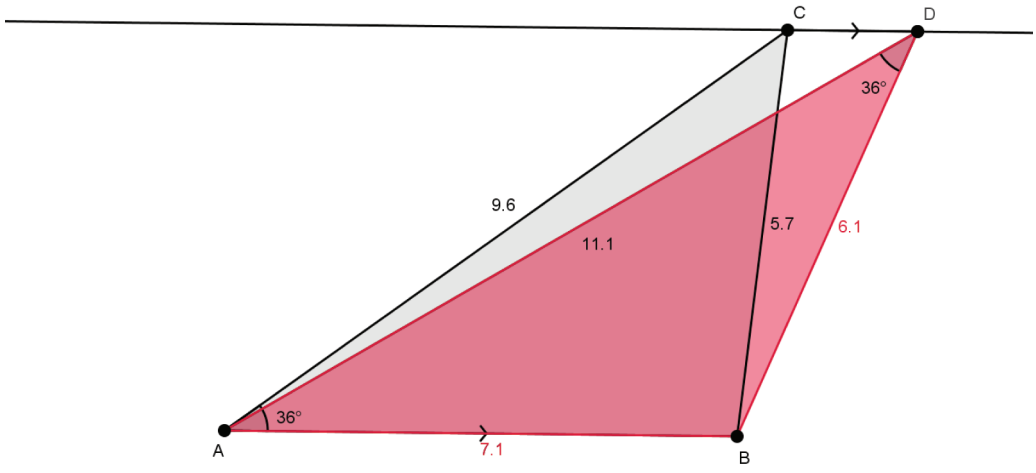


Figure 4. Triangles with the same area, one common side.
One common angle is situated adjacent to the common side in one triangle and opposite the common side in the next

whence $x = y$ and the triangles are congruent by SAS.

What if the common side is situated opposite the common angle? Let b and c be the sides that include θ in the first triangle and let u and v be the corresponding sides in the second triangle.

Then using the equality of areas,

$$bc = uv \tag{11}$$

and

$$\begin{aligned} \frac{b^2 + c^2 - a^2}{2bc} &= \frac{u^2 + v^2 - a^2}{2uv} \\ \implies b^2 + c^2 &= u^2 + v^2 \\ \implies (b + c)^2 &= (u + v)^2 \\ \implies (b + c) &= (u + v), \end{aligned} \tag{12}$$

and we see that the two triangles have equal perimeter. Since they have equal area, equal perimeter and a side in common, they are congruent (refer to task 2).

Construction of the third situation gives us two triangles which are not congruent. In Figure 4, $\triangle ABC$ and $\triangle ABD$ have the same area, common base and $\angle A = \angle D$. The corresponding heights (from C and D respectively) must be equal. The figure shows that the triangles are not congruent; the exception proves that these conditions do not imply congruency. An interesting tangential investigation is to show that this construction is not possible for all values of $\angle A$; the reader may want to find which values, at leisure.

Task 7
Write up your conclusions from all the tasks above.

Teacher's Note: With so many options considered, the high ceiling in this case, is for students to collate all their learning from the tasks and present it in a compiled form.

Conclusion: This exploration has scope for practising the skills of visualization, logical reasoning, proof and communication, among others. Students may enjoy devising other cases to study, such as replacing equal area with equal perimeter in Task 6. We look forward to hearing from you as you break the class ceiling!



PRITHWIJIT DE is a member of the Mathematical Olympiad Cell at Homi Bhabha Centre for Science Education (HBCSE), TIFR. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. His other interests include puzzles, cricket, reading and music. He may be contacted at de.prithwijit@gmail.com.



SWATI SIRCAR is Senior Lecturer and Resource Person at the School of Continuing Education and University Resource Centre, Azim Premji University. Math is the second love of her life (first being drawing). She has a B.Stat-M.Stat from Indian Statistical Institute and a MS in math from University of Washington, Seattle. She has been doing mathematics with children and teachers for more than 5 years and is deeply interested in anything hands on, origami in particular. She may be contacted at swati.sircar@apu.edu.in.



SNEHA TITUS works as Asst. Professor in the School of Continuing Education and University Resource Centre, Azim Premji University. Sharing the beauty, logic and relevance of mathematics is her passion. Sneha mentors mathematics teachers from rural and city schools and conducts workshops in which she focusses on skill development through problem solving as well as pedagogical strategies used in teaching mathematics. She may be contacted on sneha.titus@azimpremjifoundation.org.

G-Numbers

HARAGOPAL R

Notation. Given two digits a and b, by $\overline{ab}$ we mean the number $10a + b$.

Definition. Consider an n-digit number N. Suppose it happens that N is equal to *the sum of all permutations of* $(n-1)$*-digit numbers whose digits are taken from the original number*; then we call N a **G-number**. (Here we take $n > 1$, as the definition becomes meaningless for $n = 1$.)

For example, consider a three-digit number $\overline{abc}$. Suppose it is equal to the sum $\overline{ab} + \overline{ba} + \overline{ac} + \overline{ca} + \overline{bc} + \overline{cb}$; then it is a G-number.

Two-digit G-numbers

It is easy to check that there can be no 2-digit G-number. For, if $\overline{ab} = 10a + b$ were such a number, where a, b are digits and $a > 0$, then the definition leads to the following relation:

$$10a + b = a + b, \quad \therefore\ 10a = a. \tag{1}$$

But this clearly has no solution with $a > 0$. Hence there are no 2-digit G-numbers.

Three-digit G-numbers

We next check whether there are any 3-digit G-numbers. Let $\overline{abc} = 100a + 10b + c$ be such a number; here a, b, c are digits and $a > 0$. Applying the definition, we see that

$$\overline{abc} = \overline{ab} + \overline{ba} + \overline{ac} + \overline{ca} + \overline{bc} + \overline{cb},$$

$$\therefore\ 100a + 10b + c = 11(a + b) + 11(a + c) + 11(b + c),$$

$$\therefore\ 100a + 10b + c = 22(a + b + c),$$

$$\therefore\ 78a = 12b + 21c,$$

$$\therefore\ 26a = 4b + 7c. \tag{2}$$

Keywords: *Place value, digits, selections, permutations, powers of ten*

The solutions of this equation can be worked out by hand, as follows.

(i) Suppose that $a = 1$. Then we have $4b + 7c = 26$. Since $4b$ and 26 are even numbers, it follows that c is even. Also, we must have $c < 4$ (since $7 \times 4 > 26$); so the only feasible value for c is 2. And indeed this yields a solution: $b = 3$, $c = 2$. So we have our first example of such a number: 132. To verify that this is correct, note that $132 = 12 + 21 + 13 + 31 + 23 + 32$.

Another way of writing this is

$$132 = (12 + 32) + (13 + 31) + (21 + 23) = 3 \times 44.$$

(ii) Suppose that $a = 2$. Then we get $4b + 7c = 52$. Since $4b$ and 52 are multiples of 4, it follows that c is a multiple of 4 as well. Since we must have $c < 8$, the only feasible value for c is 4. And this yields a solution: $b = 6$, $c = 4$. This yields the number 264; note that $264 = 2 \times 132$.

Another way of writing this is

$$264 = (24 + 64) + (26 + 62) + (42 + 46) = 3 \times 88.$$

(iii) Suppose that $a = 3$; then we end up getting the number 396 which happens to be equal to 3×132. Note that

$$396 = 3 \times 132 = (36 + 96) + (39 + 93) + (63 + 69).$$

(iv) If $a \geq 4$, then $26a \geq 104$. However, $4b + 7c$ cannot exceed $(4 \times 9) + (7 \times 9) = 99$. Hence there can be no solutions with $a \geq 4$.

It follows that there are precisely three such 3-digit numbers: 132, 264 and 396.

Four-digit G-numbers

We next check whether there are any 4-digit G-numbers. Let $\overline{abcd}$ be such a number; here a, b, c, d are digits, $a > 0$ and $\overline{abcd} = 1000a + 100b + 10c + d$. Applying the definition, we see that

$$\overline{abcd} = \overline{abc} + \overline{acb} + \cdots + \overline{bcd} \qquad (24 \text{ such terms}).$$

(The number 24 comes from $4 \times 3! = 4!$. To understand why, note that there are 3! terms corresponding to the selection $\{a, b, c\}$; another 3! terms corresponding to the selection $\{b, c, d\}$, etc.; and there are 4 such selections.) Now we have:

$$\overline{abc} + \overline{acb} + \overline{bac} + \overline{bca} + \overline{cab} + \overline{cba} = 222(a + b + c).$$

The sum of 4 such quantities is:

$$222(a + b + c) + 222(a + b + d) + 222(a + c + d) + 222(b + c + d),$$

which equals $666(a + b + c + d)$. Hence we get:

$$1000a + 100b + 10c + d = 666(a + b + c + d),$$

$$\therefore \quad 334a = 566b + 656c + 665d. \tag{3}$$

We must solve this equation in digits a, b, c, d; i.e., $a, b, c, d \in \{0, 1, 2, 3, \ldots, 8, 9\}$. It turns out that there are no solutions at all. This means that there are no 4-digit G-numbers. (Editor's note: You could try finding a proof of this claim on your own. In case you give up, a proof has been given in Box 1, by CoMaC.)

Five-digit G-numbers

Now we check whether there are any such numbers having 5 digits. Let

$$\overline{abcde} = 10^4 a + 10^3 b + 10^2 c + 10d + e$$

be such a number; here a, b, c, d, e are digits and $a > 0$. Applying the definition, we see that

$$\overline{abcde} = \overline{abcd} + \overline{abdc} + \cdots + \overline{bcde} + \cdots \qquad (120 \text{ such terms}).$$

(The number 120 comes from $5 \times 4! = 5!$, the explanation being as earlier.) Now we have:

$$\overline{abcd} + \overline{abdc} + \overline{acbd} + \cdots + \overline{dcba} = 6 \times 1111(a + b + c + d).$$

The sum of 5 such quantities is:

$$6666(a + b + c + d) + 6666(a + b + c + e) + 6666(a + c + d + e) + \cdots$$
$$= 4 \times 6666(a + b + c + d + e) = 24 \times 1111(a + b + c + d + e).$$

Hence we get:

$$10000a + 1000b + 100c + 10d + e = 24 \times 1111(a + b + c + d + e) \tag{4}$$

We must solve this equation in digits a, b, c, d, e. But it will be immediately obvious that there can be no solution in positive integers, as the coefficients on the right side uniformly exceed the corresponding coefficients on the left side ($24 \times 1111 > 10000$ and so on). Hence there are no 5-digit G-numbers.

General treatment of numbers with 4 or more digits

The situation for numbers with still more digits remains exactly the same as for 5-digit numbers. Namely, we get an equation which has no solutions in positive integers. It is possible to express this argument symbolically as follows. For $n > 2$ let

$$\overline{a_n a_{n-1} a_{n-2} \ldots a_2 a_1} = 10^{n-1} a_n + 10^{n-2} a_{n-1} + \cdots + 10a_2 + a_1 \tag{5}$$

be an n-digit number with the stated property; here $a_n, a_{n-1}, \ldots, a_2, a_1$ are digits and $a_n > 0$. This n-digit number should be equal to the sum of $n!$ numbers each having $(n-1)$ digits which are taken from the digits of the original n-digit number:

$$10^{n-1} a_n + 10^{n-2} a_{n-1} + \cdots + 10a_2 + a_1$$
$$= \text{sum of } n! \text{ numbers each having } (n-1) \text{ digits, as noted above.}$$

In these $n!$ numbers, each digit will appear exactly $(n-1)!$ times in each place (i.e., units, tens, hundreds, …). Hence the sum of all the $n!$ numbers is

$$(n-1)! \underbrace{(111\ldots11)}_{(n-1)} (a_n + a_{n-1} + \cdots + a_2 + a_1).$$

Hence the equation which we have to solve is:

$$10^{n-1} a_n + 10^{n-2} a_{n-1} + \cdots + 10a_2 + a_1$$
$$= (n-1)! \underbrace{(111\ldots11)}_{(n-1)} (a_n + a_{n-1} + \cdots + a_2 + a_1).$$

This may be rewritten as follows:

$$\sum_{k=1}^{k=n} \left((n-1)! \underbrace{(111\ldots11)}_{(n-1)} - 10^{n-k} \right) a_{n-k+1} = 0. \tag{6}$$

If a solution can be found for this equation, in digits, then $\overline{a_n a_{n-1} a_{n-2} \ldots a_2 a_1}$ is a G-Number.

But as we argued earlier, there are no solutions for $n \geq 4$. This is so simply because for $n \geq 4$, each bracketed coefficient is a positive number.

It follows that there are precisely three G-numbers, namely: 132, 264 and 396. It is curious and interesting that each one has just three digits.

Comment from the editors. Teachers may wonder about the pedagogical value of such an exploration (which some may find rather whimsical!). Note that it demands reasoning, rigour and an understanding of topics such as permutations, factorial notation and solution of equations with more than three variables. Note also the fact that an important aim in mathematics education is *nurturing the ability to think mathematically*. In that sense, embarking on such explorations seems a very worthy exercise.

Finding all 4-digit G-numbers

We are required to solve the equation

$$334a = 566b + 656c + 665d \tag{7}$$

in digits a, b, c, d; i.e., with $a, b, c, d \in \{0, 1, 2, 3, \ldots, 8, 9\}$. The exercise may look tedious at first sight, in view of the large numbers involved, but we can reduce the labour greatly by means of a few observations.

(i) It cannot be that both c and d are 0; for this leads to the equation $334a = 566b$, or $157a = 283b$. This cannot have any solution in digits since 157 and 283 are coprime. Hence $c + d > 0$.

(ii) From the relation $334a = 566b + 656c + 665d$, we get $334a > 566b + 566c + 566d$, which yields $a > b + c + d$.

(iii) Since $a \leq 9$, the above inequality implies that $b + c + d \leq 8$, and hence that $a + b + c + d \leq 17$.

(iv) Reading equation (7) modulo 2, we get: $0 \equiv d \pmod 2$. Hence d is even.

(v) Reading equation (7) modulo 5, we get: $-a \equiv b + c \pmod 5$, which implies that $a + b + c$ is a multiple of 5.

(vi) Reading equation (7) modulo 9, we get: $a \equiv -b - c - d \pmod 9$, which implies that $a + b + c + d$ is a multiple of 9. Combining this with observation (iii), we deduce that $a + b + c + d = 9$, and hence that $a + b + c = 5$ and $d = 4$.

(vii) Since $a + b + c + d = 9$ and $a > b + c + d$, we deduce that $2a > 9$, i.e., $a \geq 5$ and $b + c + d \leq 4$. Since $a + b + c = 5$, this leads to $b = 0 = c$ and $a = 5$.

(viii) Therefore the conditions lead to just one set of values for the variables, namely: $(a, b, c, d) = (5, 0, 0, 4)$. These values satisfy all the necessary conditions obtained above, but unfortunately do not satisfy the defining equation itself, because $5 \times 334 \neq 4 \times 665$. So they do not lead to a solution.

Therefore, equation (7) has no solution in digits. It follows that there is no 4-digit G-number.

— $\mathcal{C} \otimes \mathcal{M}\alpha\mathcal{C}$



HARAGOPAL R. in an enthusiastic Mathematics teacher in a Municipal School in Kurnool, Andhra Pradesh. He loves to explore properties in numbers and strongly believes that if we are able to develop the ability of creating the ability to explore, then students can do wonders. He is also a faculty member of 'Raising a Mathematician Foundation' (a non-profit organisation for the promotion of Mathematics) and does teacher training programs. Haragopal may be contacted at rharagopal@gmail.com.

https://www.youtube.com/watch?v=Gh8h8MJFFdI&feature=share

Follow this link to find the following problem: 36 is a triangle-square number, i.e.,

$$36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8,$$

and it is a square number as well, i.e.,

$$36=6^2.$$

Are there other triangle-square numbers and is there a formula to generate such numbers? Send in your answer to AtRiA.editor@apu.edu.in

Course Correction
WHAT TO DO WHEN ONE GETS TWO DIFFERENT ANSWERS TO A COUNTING PROBLEM

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

This short article narrates a real-life classroom episode: a situation where two different answers were obtained to a counting problem and the class was nonplussed for a while. Ultimately, good sense prevailed and we were able to discover the error.

The problem studied was this:

> *Eight tennis players wish to split up into four pairs to play four singles games. In how many ways can they do this?*

Note that the pairs do not have any identifying names; it does not matter which pair plays on which court. What is of interest is only who gets paired with whom, and in how many different ways this pairing can be done.

We present three different approaches, just as it happened in the classroom.

First approach. Recall, firstly, that the number of ways that a group of $2n$ individuals can be partitioned into two subgroups of n individuals each is equal to

$$\frac{1}{2} \times \binom{2n}{n}.$$

Keywords: Counting, selection, combination, binomial coefficient

Division by 2 is needed to avoid over-counting. (The division is needed precisely because the subgroups are not identified by a name. What is of interest is only the manner in which the full group is split up. If we assign names to the subgroups to distinguish them from each other, then this step is not needed.)

Partition the group of 8 players into two subgroups of 4 players each. The number of ways in which this can be done is

$$\frac{1}{2} \times \binom{8}{4} = 35.$$

Now partition each group of 4 players into two subgroups of 2 players each. Each such partition can be achieved in $\frac{1}{2} \times \binom{4}{2} = 3$ ways. That is, each group of 4 players can be subdivided into two pairs in 3 different ways. As the two partitions are done independently, the number of possibilities is 3 × 3. Hence the desired number of ways (i.e., of subdividing a set of eight objects into four subsets with two objects each) is

$$35 \times 3 \times 3 = 315.$$

Second approach. Choose any 2 players from the set of 8 players; this can be done in $\binom{8}{2}$ ways. Remove this pair from consideration. Choose any 2 players from the remaining set of 6 players; this can be done in $\binom{6}{2}$ ways. Remove this pair too from consideration. Choose any 2 players from the remaining set of 4 players; this can be done in $\binom{4}{2}$ ways. As earlier, remove this pair from consideration. The remaining two players now automatically partner each other. These selections can be done in $\binom{8}{2} \times \binom{6}{2} \times \binom{4}{2}$ ways. However, there is a great deal of over-counting which has happened, so the number obtained needs to be divided by a suitable divisor. There are 4 pairs which have emerged at the end, but the same pairs could have been selected in a different order. As there are 4 pairs, this could have happened in 4! different ways. So the required divisor is 4!. Hence the number of ways in which the pairings can be made is equal to

$$\frac{1}{4!} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2}$$

$$= \frac{1}{24} \times \frac{8 \times 7}{1 \times 2} \times \frac{6 \times 5}{1 \times 2} \times \frac{4 \times 3}{1 \times 2} = 105.$$

Third approach. Arrange the eight players in a line. Start with the leftmost player. We must assign a partner to him; this can be done in 7 ways, as there are 7 players to choose from. Make a selection and then let the two players who have been 'processed' leave the line. Once again start with the leftmost player. We must assign a partner to him; this can be done in 5 ways. Make a selection and then (as earlier) let the two players who have been processed leave the line. There are now 4 players left.

Once again start with the leftmost player and assign him a partner; this can be done in 3 ways. Make a selection. The remaining two players automatically partner each other. It follows from this description that the number of ways the selections can be done is 7 × 5 × 3 = 105.

Remarks. As can be seen, the second and third approaches yield the same answer (105), whereas the first approach yields a different answer (315, which interestingly is 3 × 105). Which answer is correct?

If you think about it carefully, you will realise that the correct answer is 105 and not 315; we have inadvertently over-counted in the first approach. Here is why this happens.

Call the eight players *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*. (Not very imaginative names, I admit.) Suppose that the first partitioning into two groups of four each results in {*A*, *B*, *C*, *D*} in one subgroup and {*E*, *F*, *G*, *H*} in the other subgroup. Suppose next that when these two subgroups are subdivided into pairs, we obtain the pairs {*A*, *B*}, {*C*, *D*}, {*E*, *F*} and {*G*, *H*}.

Now note that the same final pairing could have been obtained if the initial sub- division into two subgroups of four each had been {*A*, *B*, *E*, *F*}, {*C*, *D*, *G*, *H*}. This observation immediately shows

that over-counting has happened, and that the actual answer must therefore be less than 315. A closer look reveals that the actual answer is 1/3 of 315, because the pair {*A*,*B*} can be combined with three different pairs, namely: {*C*, *D*}, {*E*, *F*} and {*G*, *H*}. That is to say, the following three different ways of doing the initial subdivision could all result in the same final pairings (i.e., the pairs {*A*, *B*}, {*C*, *D*}, {*E*, *F*} and {*G*, *H*}) as listed above:

{*A*, *B*, *C*, *D*} and {*E*, *F*, *G*, *H*},

{*A*, *B*, *E*, *F*} and {*C*, *D*, *G*, *H*},

{*A*, *B*, *G*, *H*} and {*C*, *D*, *E*, *F*}.

Hence to obtain the correct answer, we must divide 315 by 3, which yields 105, as it should.

The contradiction has thus been resolved. We have uncovered where the extra factor of 3 entered into the reckoning in the first approach.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

INEQUALITIES in Algebra and Geometry Part 3

SHAILESH SHIRALI

This article is the third in the 'Inequalities' series. In this part, we explore the validity and application of the AM-GM inequality for three numbers and four numbers respectively.

AM-GM inequality for three numbers and four numbers. In the previous two parts of this series, we studied the AM-GM inequality for two numbers in some detail. This is the statement that

$$\frac{a+b}{2} \geq \sqrt{ab} \tag{1}$$

for any two nonnegative numbers a and b. Moreover, the equality sign holds if and only if $a = b$. Using this inequality, we proved various other inequalities in algebra as well as geometry.

Now we consider the extension of the AM-GM inequality to three nonnegative numbers and four nonnegative numbers, respectively. Rather oddly, the extension to four numbers is simpler than the extension to three numbers!

Theorem 1 (AM-GM inequality for four nonnegative numbers)**.** *Let a, b, c, d be any four nonnegative numbers. Then we have the following inequality:*

$$\frac{a+b+c+d}{4} \geq (abcd)^{1/4}. \tag{2}$$

Moreover, the equality sign holds if and only if $a = b = c = d$.

Proof. We accomplish the desired result through three applications of the inequality (1) for two numbers. This takes place as follows. We have, firstly:

$$\left.\begin{aligned} \frac{a+b}{2} &\geq \sqrt{ab}, \\ \frac{c+d}{2} &\geq \sqrt{cd}, \end{aligned}\right\} \tag{3}$$

and since

$$\frac{a+b+c+d}{4} = \frac{\frac{a+b}{2} + \frac{c+d}{2}}{2},$$

Keywords: AM-GM inequality for three numbers, box problem, maxima, minima

it follows that

$$\frac{a+b+c+d}{4} \geq \frac{\sqrt{ab}+\sqrt{cd}}{2}. \tag{4}$$

We now apply the AM-GM inequality to the expression on the right side of (4):

$$\frac{\sqrt{ab}+\sqrt{cd}}{2} \geq \left(\sqrt{ab}\times\sqrt{cd}\right)^{1/2} = (abcd)^{1/4}. \tag{5}$$

The desired inequality now follows, i.e.:

$$\frac{a+b+c+d}{4} \geq (abcd)^{1/4}. \tag{6}$$

Moreover, for equality to hold in (6), the inequality signs in the preceding would *all* have to be replaced by equality signs. Starting at the top, we see that for equality to hold in (3), we would need to have $a = b$ and $c = d$. And for equality to hold in (5), we would need to have $ab = cd$. Combining these conditions, we see that equality will hold in (6) if and only if $a = b = c = d$. This proves Theorem 1. □

Remark. When we are studying the topic of inequalities, it is good practice to identify the *exact conditions* under which an equality sign can replace the inequality sign. We have consistently tried to do this in our study of inequalities, and we shall continue this practice.

We now take up the inequality for three nonnegative numbers. The curious thing is, in proving this result, we make use of the inequality for four nonnegative numbers!

Theorem 2 (AM-GM inequality for three nonnegative numbers)**.** *Let a, b, c be any three nonnegative numbers. Then we have the following inequality:*

$$\frac{a+b+c}{3} \geq (abc)^{1/3}. \tag{7}$$

Moreover, the equality sign holds in (7) if and only if $a = b = c$.

Proof. As already noted, we are going to make use of Theorem 1. As we are given only three nonnegative numbers a, b, c, we need to produce a suitable fourth number d before we can apply the theorem. Our choice for this number will be

$$d = \frac{a+b+c}{3}, \tag{8}$$

that is, d is the arithmetic mean of a, b, c. With this choice we obtain, on the left side of (6):

$$\frac{a+b+c+d}{4} = \frac{a+b+c+\frac{1}{3}(a+b+c)}{4} = \frac{a+b+c}{3} = d.$$

(This makes sense. Given a list of numbers, if you append the arithmetic mean of the numbers to the same list, the arithmetic mean will naturally remain unaltered.) On the right side of (6), we obtain:

$$(abcd)^{1/4} = (abc)^{1/4}\times d^{1/4}.$$

Therefore, the substitution $d = \frac{1}{3}(a+b+c)$ in Theorem 1 results in the following:

$$d \geq (abc)^{1/4}\times d^{1/4}.$$

This may be simplified to yield $d^{3/4} \geq (abc)^{1/4}$. Now raising both sides to power $4/3$, we get:

$$d \geq (abc)^{1/3},$$

i.e.,

$$\frac{a+b+c}{3} \geq (abc)^{1/3}. \tag{9}$$

We have proved the desired inequality. The conditions under which an equality sign can replace the inequality sign can be deduced exactly as earlier; the conditions are $a = b = c$. □

Remark. The path we have taken to prove Theorem 2 is indeed curious; we proved Theorem 1 first (the four-numbers case), and then came down to the three-numbers case. While this is not the only way to prove Theorem 2, it is one of the simplest.

Some applications of the AM-GM inequality for three numbers

The inequality in Theorem 2 can also be stated in the following form: if a, b, c are any three nonnegative numbers, then

$$a + b + c \geq 3 \cdot (abc)^{1/3}. \tag{10}$$

This form of stating the inequality turns out to be more useful in problem-solving and in establishing various results.

We now consider a few problems and their solutions.

Problem 1: Find the positive number x which minimises the value of $x^2 + \dfrac{16}{x}$.

Solution. We use (10) with $a = x^2$, $b = \dfrac{8}{x}$, $c = \dfrac{8}{x}$. This yields:

$$x^2 + \frac{16}{x} = x^2 + \frac{8}{x} + \frac{8}{x} \geq 3 \times \left(x^2 \cdot \frac{8}{x} \cdot \frac{8}{x}\right)^{1/3} = 3 \times 64^{1/3} = 3 \times 4 = 12.$$

This proves that

$$x^2 + \frac{16}{x} \geq 12,$$

with equality if and only if

$$x^2 = \frac{8}{x}, \quad \text{i.e.,} \quad x = 2.$$

So the minimising number is $x = 2$, and the minimum value of the expression is 12. More generally, if k is a positive number, then the positive number x which minimises the value of

$$x^2 + \frac{k}{x}$$

is $x = \sqrt[3]{k/2}$. This result would usually be proved using derivatives.

Problem 2: Show that among all triangles that share the same perimeter, the one with the largest area is equilateral. (This is called the **isoperimetric problem** for triangles. Earlier, we had solved the isoperimetric problem for rectangles.)

Solution. Let a, b, c be the sides of the triangle, and let $s = \frac{1}{2}(a + b + c)$ be the semi-perimeter; then the area K is given by

$$K = \sqrt{s(s-a)(s-b)(s-c)}. \tag{11}$$

We are told that the perimeter $2s$ is a constant; this implies that $a + b + c$ is a constant. And since

$$(s-a) + (s-b) + (s-c) = 3s - (a+b+c) = s = \text{constant},$$

the sum of the numbers $s-a$, $s-b$ and $s-c$ is a constant as well. Note also that each of these three numbers is strictly positive; for, by the triangle inequality, we have:

$$b+c>a, \quad \therefore \ a+b+c>2a, \quad \therefore \ 2s>2a, \quad \therefore \ s-a>0,$$

and similarly for $s-b$ and $s-c$. So we are in a position to apply the AM-GM inequality to these three numbers. We obtain:

$$(s-a)+(s-b)+(s-c) \geq 3 \times ((s-a)(s-b)(s-c))^{1/3}.$$

This yields:

$$(s-a)(s-b)(s-c) \leq \frac{s^3}{27}.$$

From this we deduce that

$$s(s-a)(s-b)(s-c) \leq \frac{s^4}{27},$$

and hence, by taking square roots, that

$$K \leq \frac{s^2}{3\sqrt{3}}. \tag{12}$$

For the upper bound to be attained, we must have $s-a=s-b=s-c$, i.e., $a=b=c$. Hence the largest area is attained by the equilateral triangle. This justifies the statement that was to be proved.

Problem 3: Given: a square sheet of paper, measuring 30 cm by 30 cm. From each corner we snip off a square shape (all squares of the same size), and then fold the remaining piece of paper into an open box with a square base (open at the top; we tape the sides at the corners). What should be the dimensions of the parts snipped off for the volume of the resulting box to be largest? (See Figure 1. In the literature, this is referred to as the **Box Problem**.)

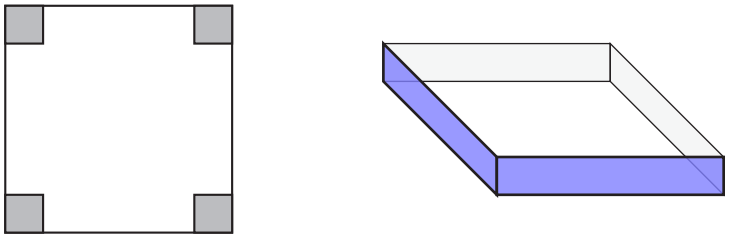


Figure 1.

Solution. The dimensions of the square are 30×30. Let the squares snipped away from the corners have dimensions $a \times a$. Then the base of the box has dimensions $(30-2a) \times (30-2a)$, and its height is a. Therefore, its volume V is given by

$$V = a(30-2a)^2. \tag{13}$$

To maximise V we typically use derivatives; but we shall use Theorem 2 in the form given in (10). First, however, we need to make a few adjustments in the expression for V. We have:

$$\begin{aligned} a(30-2a)^2 &= 4a \times (15-a) \times (15-a) \\ &= 2 \times (2a \times (15-a) \times (15-a)). \end{aligned}$$

In order to maximise the expression $a(30-2a)^2$, we might as well maximise the expression $2a \times (15-a) \times (15-a)$, as the second expression is equal to the first expression divided by a constant (namely, 2). The second expression is in a form that allows us to apply Theorem 2. For, the quantities $2a$, $15-a$, $15-a$ are nonnegative, and their sum is 30. Therefore the following is true:

$$(2a \times (15-a) \times (15-a))^{1/3} \leq \frac{2a+(15-a)+(15-a)}{3} = \frac{30}{3} = 10,$$

and so:

$$2a \times (15-a) \times (15-a) \leq 10^3.$$

Hence the volume of the box cannot exceed 2000 cm^3. For this upper bound to be attained, we must have $2a = 15-a$, i.e., $a = 5$. So the squares snipped off must have dimensions 5×5.

Solutions to problems from July 2017 issue

(1) Let a, b, c be positive real numbers. Show that:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.$$

Solution. We use (10). We have:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 \times \left(\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}\right)^{1/3} = 3 \times 1 = 3.$$

Equality holds if and only if $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$, i.e., if and only if $a = b = c$.

(2) Let a, b, c be positive real numbers. Show that:

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 3,$$

with equality if and only if $a = b = c$.

Solution. We again use (10). We have:

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 3 \times \left(\frac{a^2}{bc} \cdot \frac{b^2}{ca} \cdot \frac{c^2}{ab}\right)^{1/3} = 3 \times 1 = 3.$$

Equality holds if and only if $\frac{a^2}{bc} = \frac{b^2}{ca} = \frac{c^2}{ab}$, i.e., if and only if $a = b = c$.

(3) Let a, b, c be positive real numbers. Show that:

$$\left(a^2b + b^2c + c^2a\right) \cdot \left(ab^2 + bc^2 + ca^2\right) \geq 9a^2b^2c^2,$$

with equality if and only if $a = b = c$.

Solution. Yet again we use (10). We have:

$$a^2b + b^2c + c^2a \geq 3 \times \left(a^2b \cdot b^2c \cdot c^2a\right)^{1/3} = 3\left(a^3b^3c^3\right)^{1/3} = 3abc.$$

Similarly,

$$ab^2 + bc^2 + ca^2 \geq 3 \times \left(ab^2 \cdot bc^2 \cdot ca^2\right)^{1/3} = 3\left(a^3b^3c^3\right)^{1/3} = 3abc.$$

The desired result now follows by multiplication. Equality holds if and only if $a^2b = b^2c = c^2a$ and $ab^2 = bc^2 = ca^2$, i.e., if and only if $a = b = c$.

(4) Let a, b, c be positive real numbers. Show that:

$$a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a,$$

with equality if and only if $a = b = c$.

Solution. This is a very challenging problem! We are going to use a clever trick to solve it. We apply the AM-GM inequality to the following three lists of numbers: $\{a^3, a^3, b^3\}$; $\{b^3, b^3, c^3\}$; $\{c^3, c^3, a^3\}$. Note the symmetry in these lists. Invoking (10) for each list, we get:

$$a^3 + a^3 + b^3 \geq 3 \times \left(a^3 \cdot a^3 \cdot b^3\right)^{1/3} = 3a^2 b,$$

$$b^3 + b^3 + c^3 \geq 3 \times \left(b^3 \cdot b^3 \cdot c^3\right)^{1/3} = 3b^2 c,$$

$$c^3 + c^3 + a^3 \geq 3 \times \left(c^3 \cdot c^3 \cdot a^3\right)^{1/3} = 3c^2 a.$$

By adding the three inequalities, we immediately get the desired result. The condition for equality is easily deduced.

A small modification of the lists yields the following inequality: If a, b, c are three positive real numbers, then

$$a^3 + b^3 + c^3 \geq ab^2 + bc^2 + ca^2,$$

with equality if and only if $a = b = c$.

The problem and solution have been taken from [1].

Using exactly the same idea, we can prove results such as the following:

Theorem. *If a, b, c are three positive real numbers, then*

$$a^4 + b^4 + c^4 \geq a^3 b + b^3 c + c^3 a,$$

$$a^5 + b^5 + c^5 \geq a^3 b^2 + b^3 c^2 + c^3 a^2,$$

and so on, with equality in each case if and only if $a = b = c$.

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1. Brilliant, "Applying the Arithmetic Mean Geometric Mean Inequality", https://brilliant.org/wiki/applying-the-arithmetic-mean-geometric-mean/. (Home page: https://brilliant.org/.)



SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

DIVISIBILITY BY 7

ANANT W VYAWAHARE

This article deals with a simple test for divisibility by 7 for natural numbers having a minimum of four digits. Here, a case of a six-digit number is proved initially and similar proofs follow for other higher-digit numbers.

Six-digit numbers

Let n be a six-digit natural number, $n = \overline{abcdef}$. That is,

$$n = 100000a + 10000b + 1000c + 100d + 10e + f. \quad (1)$$

Theorem. 7 divides n if and only if 7 divides $|p - q|$, where p is the number formed by the first three digits of n and q is the number formed by the last three digits of n, i.e.,

$$\left.\begin{aligned} p &= \overline{abc} = 100\text{a} + 10\text{b} + \text{c}, \\ q &= \overline{def} = 100d + 10\text{e} + \text{f}. \end{aligned}\right\} \quad (2)$$

Proof. Let $n = \overline{abcdef}$. Assume that 7 divides $|p - q|$; we shall prove that 7 divides n.

We are told that 7 divides $|p - q|$; hence $p - q = 7k$ where k is an integer. This yields:

$$p - q = (100a + 10b + c) - (100d + 10e + f) = 7k. \quad (3)$$

Keywords: Divisibility by 7, place value, digits, factors, multiples

Next,

$$\begin{aligned} n &= 100000a + 10000b + 1000c + (100d + 10e + f) \\ &= 100000a + 10000b + 1000c + (100a + 10\text{b} + c) - 7k, \text{ (from equation (3))} \\ &= 100100a + 10010b + 1001c - 7k = 1001(100a + 10b + c) - 7k \\ &= 143 \times 7(100a + 10b + c) - 7k. \end{aligned}$$

From this, we see that n is a multiple of 7. That is, 7 divides n. Next, assume that 7 divides n; to prove that 7 divides $|p - q|$, we follow much the same steps as used above (in a slightly different order; please fill in the steps).

Corollary. If 7 divides the number $\overline{abcdef}$, then 7 divides the number $\overline{defabc}$.

Examples

(1) Let $n = 976213$; then $p = 976$ and $q = 213$. Hence $p - q = 976 - 213 = 763$.

Since 7 divides 763, it follows that 7 divides 976213. (The quotient in the division is 139459.)

(2) Let $n = 123782$. Here $p = 123$ and $q = 782$. Hence $|p - q| = |782 - 123| = 659$.

Now, 7 does not divide 659. Hence, 7 does not divide 123782.

Remarks

- Similar statements are true for four-digit numbers, five-digit numbers and higher-digit numbers. The only condition to be kept in mind, in all cases, is that the q-block should be made up of the last three digits from the right side. Then p will be formed by all the digits from the left except the last three, as mentioned. That is, q is the number made up of the last three digits of the number, and p is the number made up of the remaining digits. The statement made in the theorem now holds, regardless of how many digits p has: *The original number is divisible by 7 if and only if $p - q$ is divisible by 7. For a short proof of this claim, please see Box 1.*
- Interestingly, all the cyclic numbers formed from the number 976213 obey the statement given above; hence, all these numbers are divisible by 7. For example, 762139 is divisible by 7.

Some more examples

Shown below are some more examples to illustrate this method:

Four-digit numbers:

- Let $n = 1239$; here, $p = 1$, $q = 239$, $|p - q| = 239 - 1 = 238$. Now, 7 divides 238. Hence 7 divides 1239.
- Let $n = 9321$; here, $p = 9$, $q = 321$, $|p - q| = 321 - 9 = 312$. Now, 7 does not divide 312. Hence 7 does not divide 9321.

Five-digit numbers:

- Let $n = 34426$; here, $p = 34$, $q = 426$, $|p - q| = 426 - 34 = 392$. Now, 7 divides 392. Hence 7 divides 34426.
- Let $n = 12345$; here, $p = 12$, $q = 345$, $|p - q| = 345 - 12 = 333$. Now, 7 does not divide 333. Hence 7 does not divide 12345.

Nine-digit numbers:

- Let $n = 258469232$; here, $p = 258469$, $q = 232$, $|p - q| = 258237$. If we repeat the same step for the new number, we get 258–237 = 21. Now, 7 divides 21. Hence 7 divides 258237. Hence 7 divides 258469232.

Proof of the General Claim - Box 1

Given a positive integer n, let q be the number made up of the last three digits of n, and let p be the number made up of the remaining digits. Then the following is true: n is divisible by 7 if and only if $p - q$ is divisible by 7. With p and q as defined above, we have the relation $n = 1000p + q$. For example,

$$2345 = (2 \times 1000) + 345,$$

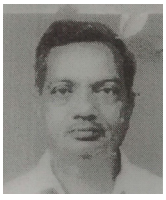
$$12345 = (12 \times 1000) + 345.$$

Hence the following is equivalent to the claim made above:

$$7 \text{ divides } 1000p + q \iff 7 \text{ divides } p - q. \quad (5)$$

Statement (5) may be proved as follows.

Proof. Let $a = 1000p + q$ and $b = p - q$. Then $a + b = 1001p$, which is a multiple of 7 (since $1001 = 7 \times 11 \times 13$). Hence if a is a multiple of 7, so must be b; and if b is a multiple of 7, so must be a. This is exactly the claim made above.



ANANT W VYAWAHARE has a M Sc in Mathematics and a Ph D in Number Theory. He is a retired Professor of Mathematics from M. Mohota Science College, Nagpur University. He has taught degree classes and postgraduate classes for over 40 years. His areas of specialisation are History of Mathematics, Vedic Mathematics, Fuzzy Mathematics and Operations Research. He may be contacted at anantngp@gmail.com.

Teaching Concepts of CALCULUS

THOMAS LINGEFJÄRD & DJAMSHID FARAHANI

The topic of calculus is an integral part of the senior secondary mathematics curriculum. The concepts of limits and derivatives, which form the foundation of Calculus, are often hard to teach. In this article, we suggest a dynamic way of teaching the concept of the derivative through practical examples which may be easily explored through GeoGebra applets available through the Internet.

Introduction

Calculus may be described as the mathematical study of change. It provides us with the tools to study and analyse change of specific phenomena in the real world. The change may be with respect to time or with respect to another quantity. Let us consider the example of a liquid being poured into a cylindrical can. The volume of the liquid in the can varies with respect to time. The height (level) of the liquid also varies with respect to time. Further, the volume and the height of the liquid also vary with respect to each other. Thus, calculus enables us to describe change represented by variables which vary with respect to each other. Calculus may be described as the branch of mathematics used to study any phenomena involving change, which may involve any pair of dimensions such as time, force, mass, length, displacement, temperature, etc.

Much of the calculus taught in school is procedural in nature, as a great deal of emphasis is laid on learning the manipulative skills for computing limits and derivatives of different functions. Often, students believe that calculus is mainly about applying formulae and rules, and computing the correct symbolic expressions for derivatives and integrals. There is a strong need to help students see the relevance of calculus to daily life by highlighting appropriate examples.

One way to start a lesson in calculus is to connect it to something the students are familiar with. Riding a bicycle, travelling in a car or skiing down a mountain slope may be situations which students can easily relate to and may be used to build on their intuitive idea of change in velocity. These experiences probably

Keywords: Variables, change, graphs, GeoGebra, slope, tangent, derivative

occur long before they encounter the concepts of calculus in upper secondary school. Thompson (1994) argued that

> *The development of images of rate starts with children's image of change in some quantity (e.g., displacement of position, increase in volume), progresses to a loosely coordinated image of two quantities (e.g., displacement of position and duration of displacement), which progresses to an image of the covariation of two quantities so that their measures remain in constant ratio. (p. 128)*

Thus teaching of calculus could begin with a discussion about change of velocity. Changes may be represented graphically as a function. Nevertheless, to understand the relation between a function and its derivative through graphs is by no means trivial. In *The development of students' graphical understanding of the derivative* (Asiala, Cottrill, Dubinsky & Schwingendorf, 1997, p. 402), the authors write: "many authors discuss specific problems students have with the graphical interpretation of the derivative."

Classroom activity: A daily life situation

Let us consider a classroom activity where students are required to investigate a motorcycle in motion. This may be introduced graphically as shown in Figure 1. The motorcycle's motion starts at time $t = 0$ and spans a period of 4 hours.

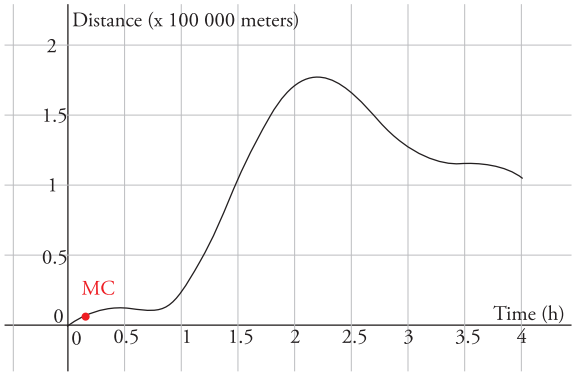


Figure 1. A motorcycle and its motion illustrated by GeoGebra.

Figure 1 shows the graphical representation of the motorcycle's displacement from its position at time t. Students may explore the dynamic version of the activity through the GeoGebra file https://www.geogebra.org/m/SVDrYnvQ

Using this applet the teacher may facilitate a discussion by moving the red point labeled MC (which represents the position of the motorcycle) and asking students to explain what change in displacement means (they should also discuss the displacement at any instant of time and understand the difference between the displacement at an instant and the change in displacement over an interval).

> *Students should be encouraged to express their ideas in their own words before a formal explanation is provided.*

Usually, it is difficult for students to translate the mental image of a motorcycle into a point on a graph. In addition, they should be able to visualize that movement away from the starting point (the *origin*) is related to a positive velocity while movement back to the starting point is related to a negative velocity. We measure the displacement of the motorcycle by a vertical line for a time t_1 and then we identify where that line meets the graph.

A good deal of research has been conducted to investigate students' alternative conceptions about graphical concepts (Elby, 2000; Hammer, 2000; McDermott & Schaffer, 2005; Lingefjärd & Farahani, 2017). Research indicates that it is often quite difficult for students to make interpretations of a graph in a coordinate system, especially if the graph describes a real situation in some way. Many students have alternative conceptions that compete with and interact with scientific interpretations. (We prefer to use the term 'alternative conception' instead of 'misconception' which is associated with inaccuracy and mistakes.)

Learning from graphical representations requires students to understand the syntax of each representation. The format of a graph includes

attributes such as labels, number of axes, and line shapes. Interpretations require finding the gradients of lines, minima and maxima, as well as intercepts (Ainsworth, Bibby & Wood, 1997). Students must also understand which parts of the domain of the function are represented in the graph.

If we look carefully at the graphical representation in Figure 1, we may observe that the motorcycle is running rather slowly at first since it only covers a distance of 25 kilometres in the first hour. It also appears as if the motorcycle partly turns around and moves back between 25 minutes and 45 minutes after the start of the journey. Thereafter it changes its course once again.

The teacher may ask students to reason about why the motorcycle rider is steering the motorcycle in that way. Perhaps the rider was searching for the correct road. A motorcycle that is in motion for 4 hours will also have a velocity. How can we view velocity in a displacement time graph? What is velocity? We believe that students have a general understanding of velocity, which they interpret as a measurement of how fast they are able to move from position A to position B. "The faster I am moving from position A to position B, the higher is my velocity." At this point of the lesson it might be suitable to introduce the concept of the tangent line based on the slope of the graph.

Tangent Line

For visualizing the derivative it is important to introduce the concept of slope of a curve at a specific point. Is the motorcycle going forward or backward or changing its direction at a specific point on the graph? How do we translate that into common behavior when steering a motorcycle? As it passes through the point where the tangent line and the curve coincide, called the point of tangency, the tangent line is going in the same direction as the curve, and is thus the best straight-line approximation to the curve at that point. Such an example enables us to develop the concept image (Tall & Vinner, 1981) of derivative in the context of graphical representation; see Figure 2.

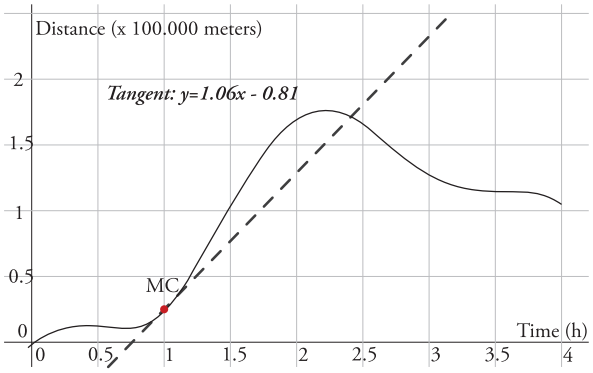


Figure 2. Introducing the concept of tangent line illustrated by GeoGebra.

While teaching the concept of a tangent line in relation to the motion, and velocity, of a motorcycle, it could be beneficial for students to explore the dynamic version of Figure 2. The GeoGebra applet available at https://www.geogebra.org/m/gnzPK9QH allows the user to move the point MC, and observe the tangent line as it moves along the curve. The symbolic representation (equation) of the tangent gets updated as the tangent moves. We argue that this dynamic approach is a more efficient way to explain and discuss the meaning of slope of the tangent than a pen and paper approach.

The teacher may move the point MC and ask students to explain the information given by the tangent line. They may be asked to interpret the positive and negative values of the slope of the tangent as it moves along the curve. They should be encouraged to express their ideas in their own words.

Velocity

The next step is to allow GeoGebra to illustrate how the velocity varies when the motorcycle is in motion. The velocity may be viewed as the trace of a point (h,k) where we define h as the x-value of the point MC and k is the slope of the tangent line at $x = h$ (in GeoGebra we enter $h = x$(MC) and k = Slope[Tangent] in the Input bar). If we furthermore allow the point velocity to be traced, we obtain the green curve as shown in Figure 3.

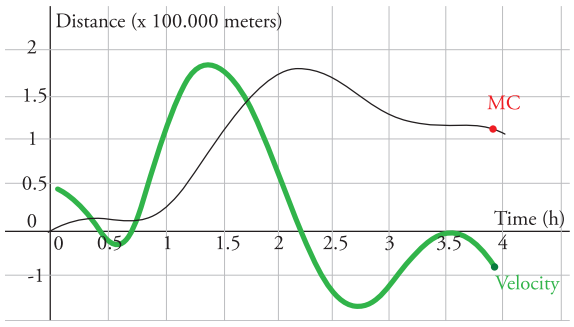


Figure 3. GeoGebra illustrates how the motorcycle's velocity is related to its position.

The GeoGebra file can be found at https://www.geogebra.org/m/pJqtPZyM. While using this applet, we can move the point MC to show how velocity is changing with respect to the motion of the motorcycle.

Students should now be given the challenge to interpret how the graphical representation of the velocity is related to the motorcycle's position. They may attempt this task alone at first, then discuss in pairs and finally participate in a classroom discussion. Maintaining students' written records would give the teacher an insight into their thinking. Students' interpretations of graphs and other representations can shed empirical light on a longstanding theoretical debate about learning of scientific concepts. Students' intuitive knowledge about science concepts may consist of unarticulated, loosely-connected knowledge elements, the activation of which depends sensitively on context.

Learning is not a matter of replacing bad mini-generalizations with good ones. Instead, it's partly a matter of tweaking those mini-generalizations into a more articulate, unified, coherent structure.

The green velocity graph (as shown in Figure 3) is the derivative of the blue graph of the motorcycle's position over time. So velocity can be interpreted as derivative of displacement with respect to time. The significant details of one representation can be used to interpret a different representation. This may result in interpretations such as viewing a graph as a picture (see Lingefjärd & Farahani, 2017). Students should be given ample time to understand the relationship between the displacement graph and the velocity graph.

Acceleration

Finally we can let GeoGebra illustrate a graphical representation of how the acceleration is changing over time when the motorcycle is in motion.

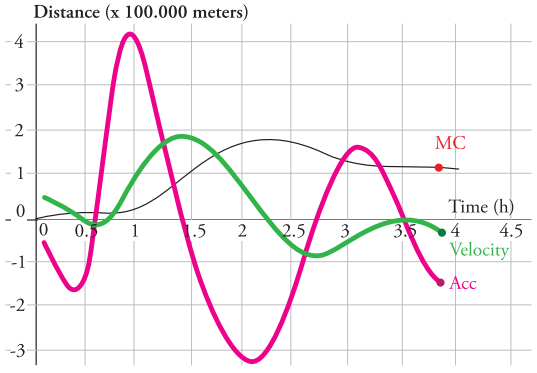


Figure 4. GeoGebra shows how the motorcycle's displacement, velocity and acceleration change over time.

The GeoGebra file can be found at https://www.geogebra.org/m/u56mCPFy.

While using this applet, we can once again move the point MC and show how velocity and acceleration change with respect to the motorcycle's motion. It is important to help students to understand the relationship between the original function (the displacement graph shown in blue), its derivative (the velocity graph shown in green) and the second derivative (the acceleration graph shown in red).

There must be scaffolding to translate between the graphical and symbolic representations, of velocity, acceleration, and displacement. Students need sufficient time to understand that velocity is the rate of change of displacement (symbolically written as $v(t) = d'(t)$) , and that acceleration is the rate of change of velocity ($a(t) = v'(t)$). Acceleration is therefore defined as the second derivative of displacement which is expressed as $a(t) = d''(t)$.

Conclusion

In this article we have highlighted the fact that a dynamic geometry software like GeoGebra can enable students to visualize and explore the concept of the derivative dynamically through graphical and symbolic representations. Research has shown that students face major difficulties in understanding graphical representations even in a dynamic environment. However we believe that with proper facilitation, the GeoGebra applets of the motorcycle in motion can be used to help the students develop a more coherent concept image of a function, slope at a point and derivative.

Students need an understanding of time as a variable in a changing process. They also need to develop an understanding of the slope of a curve in a graphical representation, something that is not always explicitly clear. Dynamic representations of realistic examples, such as that of the motion of a motorcycle or other similar phenomena can help students visualise concepts better than static iconic interpretations.

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THOMAS LINGEFJÄRD is Professor of Mathematics Education at the University of Gothenburg, Sweden, and is deeply interested in both mathematical and educational issues. The impact of ICT on learning is something he has been interested in for 25 years. He visited India in 2011 for the India-Swedish Math Education initiative, and has since then been collaborating and working with Jonaki Ghosh. He may be contacted at thomas.lingefjard@ gu.se.



DJAMSHID FARAHANI is a lecturer in Mathematics Education at the University of Gothenburg, Sweden. He is a former upper secondary school teacher in physics and mathematics. His area of interest concerns cognitive aspects of the learning of mathematical representations. He may be contacted at djamshid.farahani@ gu.se.

Desmos Activity Builder

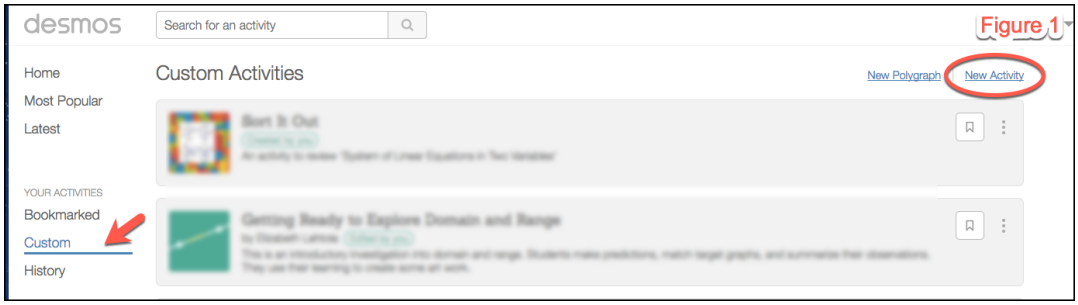
CREATE YOUR OWN INTERACTIVE ACTIVITY

SANGEETA GULATI

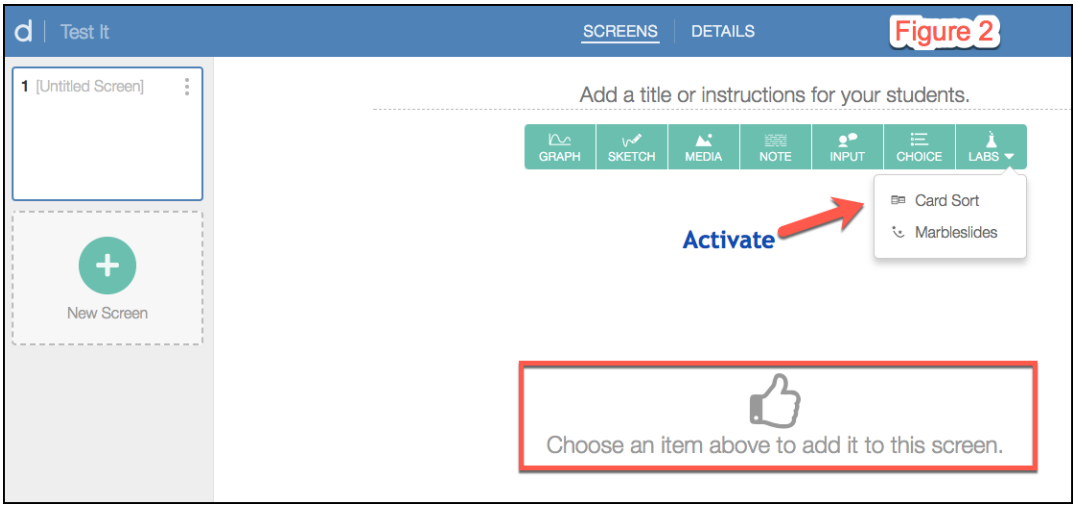
In the ever growing, ever changing field of educational technology, the effectiveness of a tool is often measured in terms of the new features being added over a period of time and their effectiveness in improving teaching in classrooms.

Desmos (www.desmos.com), an online graphing calculator, is one such tool; it has made great progress over the past few years and is no longer just a graphing calculator. In the July 2014 issue I had discussed the features of Desmos as an online graphing calculator and had also shared my experience of using it in the classroom as a teaching tool, in the article titled *Graphing with Desmos*. Later in the March 2016 issue I had discussed the features of Desmos Classroom Activities in the article *Play and learn with Desmos*. Desmos has evolved a great deal since then. In this article I will highlight the features of the Desmos Activity Builder - which allows the user to create a customized Classroom Activity - and then take you through the process of creating one such activity. To get started (Figure 1), login to www.teacher.desmos.com (prior registration is required; registration is free and no software needs to be downloaded to your own computer); select Custom and then New Activity.

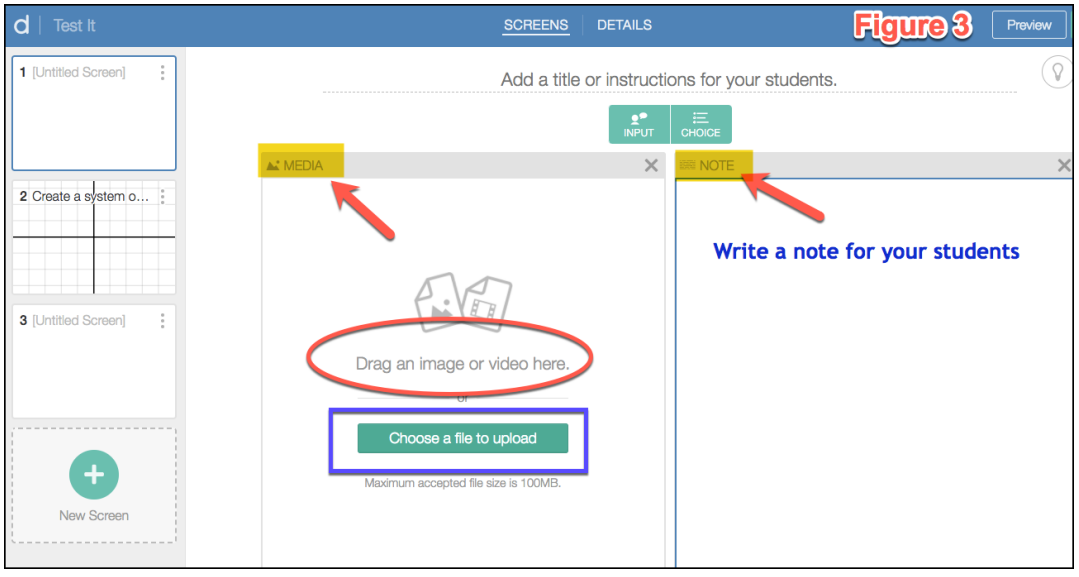
Keywords: Technology, pedagogy, graphing calculator



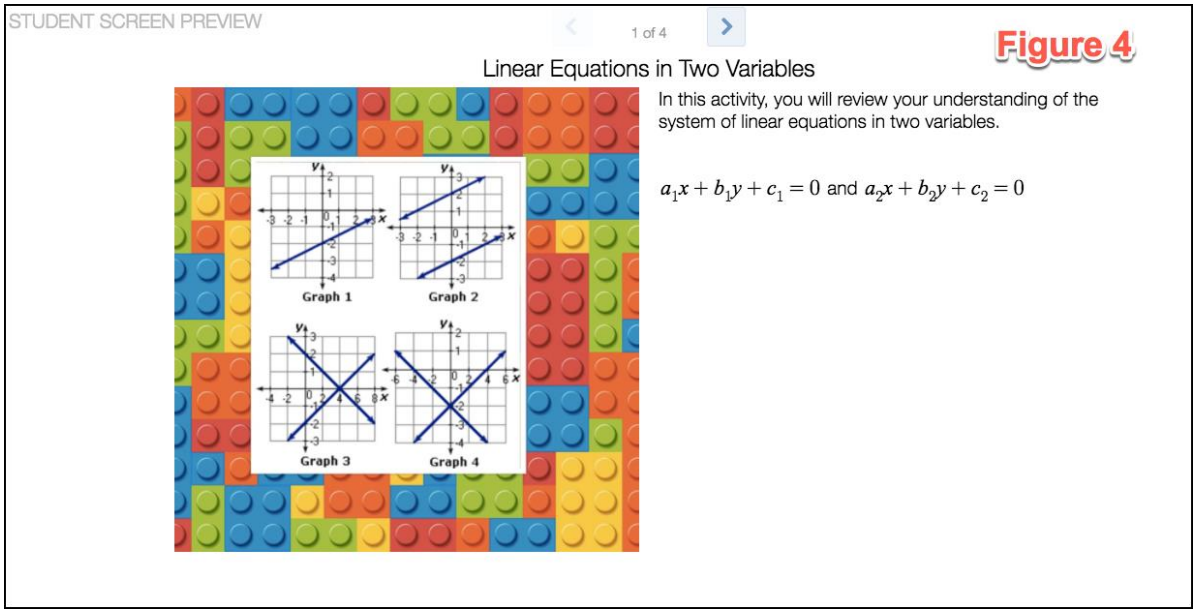
Each activity consists of multiple screens, which the students will progress through as directed. These screens can be made using 'Graph', 'Sketch', 'Media', 'Note', 'Input', 'Choice' or 'Labs'. One needs to activate 'Card Sort' and 'Marbleslides' under the option 'Labs' in order to use these special features (Figure 2).



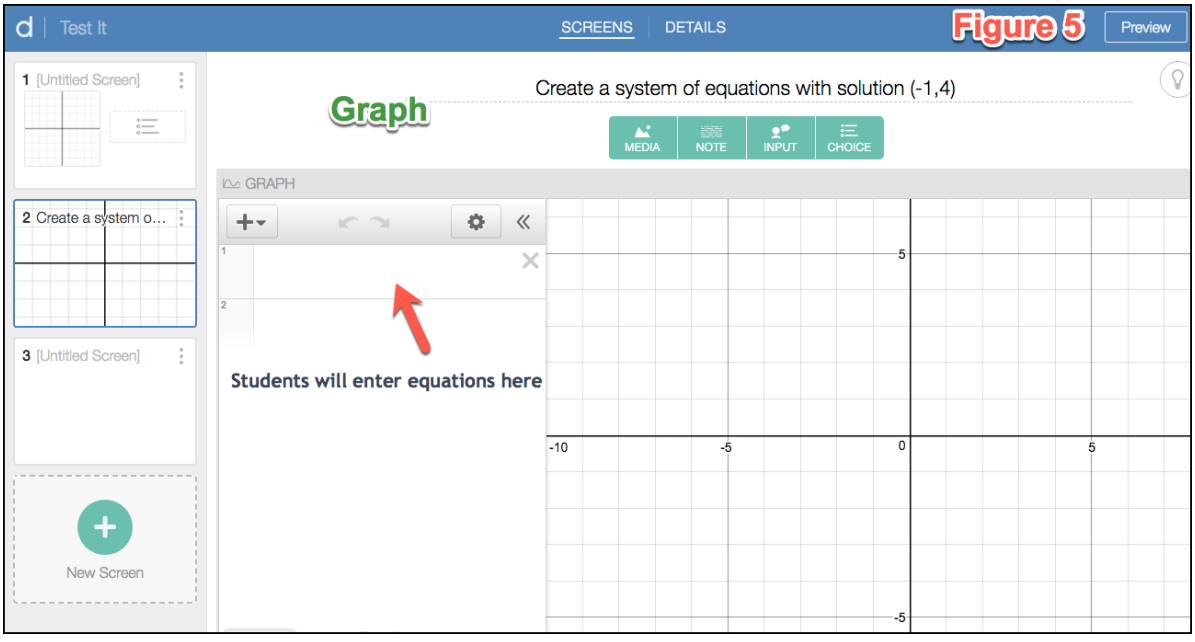
The first screen may be used in multiple ways such as giving instructions to the students, as a warm up exercise using Graph or inserting a video or an image under Media. One can use the option of 'Note' to write the instructions for students and explain what they will be learning through this activity (Figure 3).



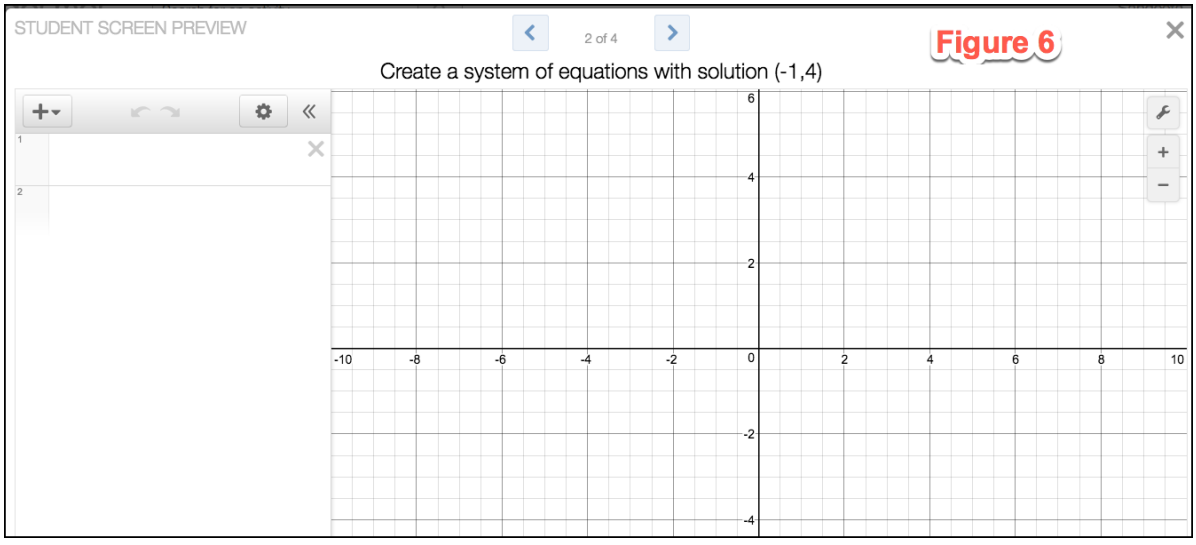
This (Figure 4) is what the screen will look like to a student, when 'Media' and 'Note' option is used to create the first screen.



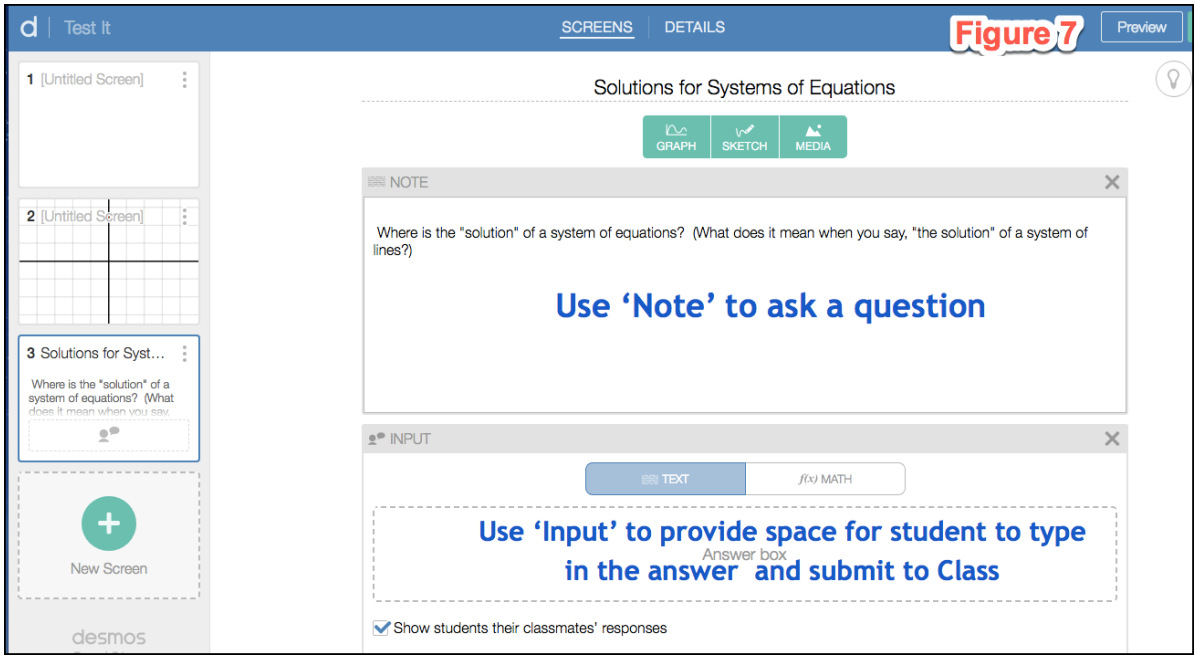
Click on the 'Add' sign to get a new screen. The next screen will require students to type in multiple equations with the requirement that (-1,4) is the solution of the system so generated. Selecting the 'Graph' option to get the 'Desmos' graph inserted in the screen allows students to type in the equations and see the graph plotted alongside (Figure 5).



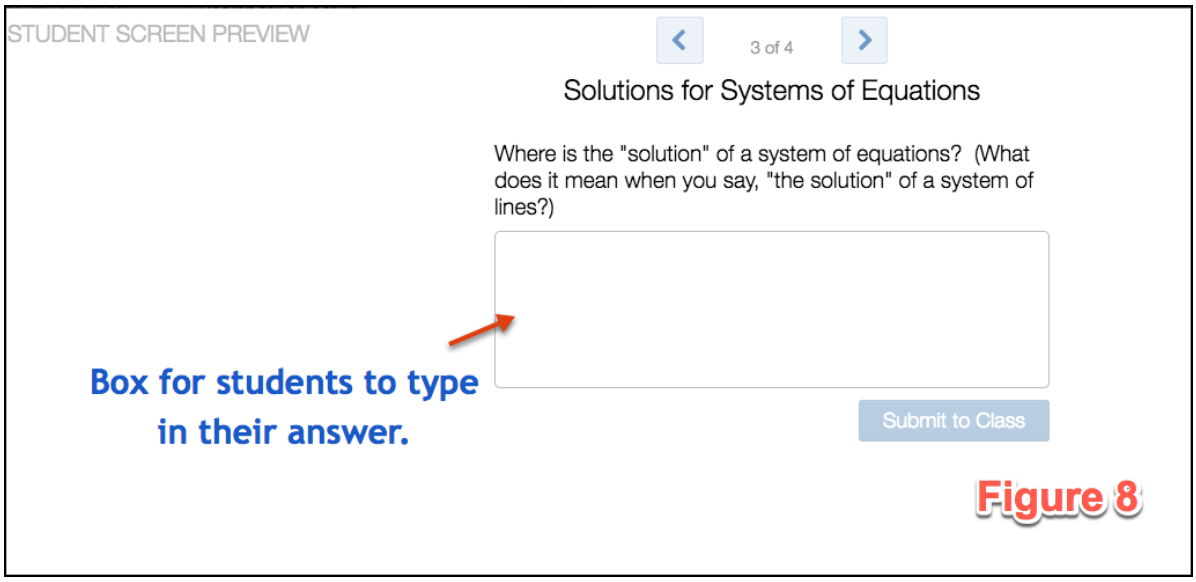
The second screen will look like this (Figure 6) to students.



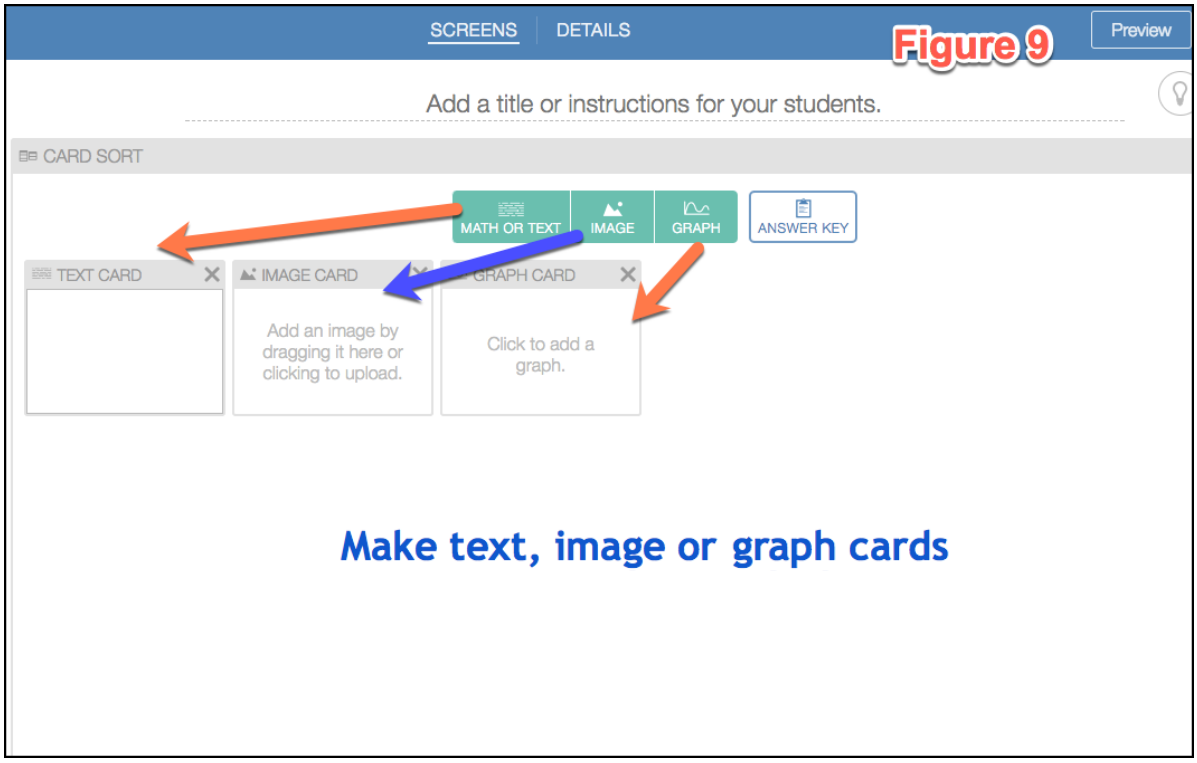
The Desmos activities can be made thought provoking by posing questions to the students and letting them submit their responses to the class. Students can see other responses when at least three responses have been submitted. As the activity progresses, the teacher has the option to pause the activity, review the collective responses and start a discussion. Students can be given an opportunity to clarify and explain their answers or discuss the validity of the response of a peer. In our third screen (Figure 7), we will add a 'Note' to pose questions and use 'Input' to provide a box for students to write in their answers and submit to Class.



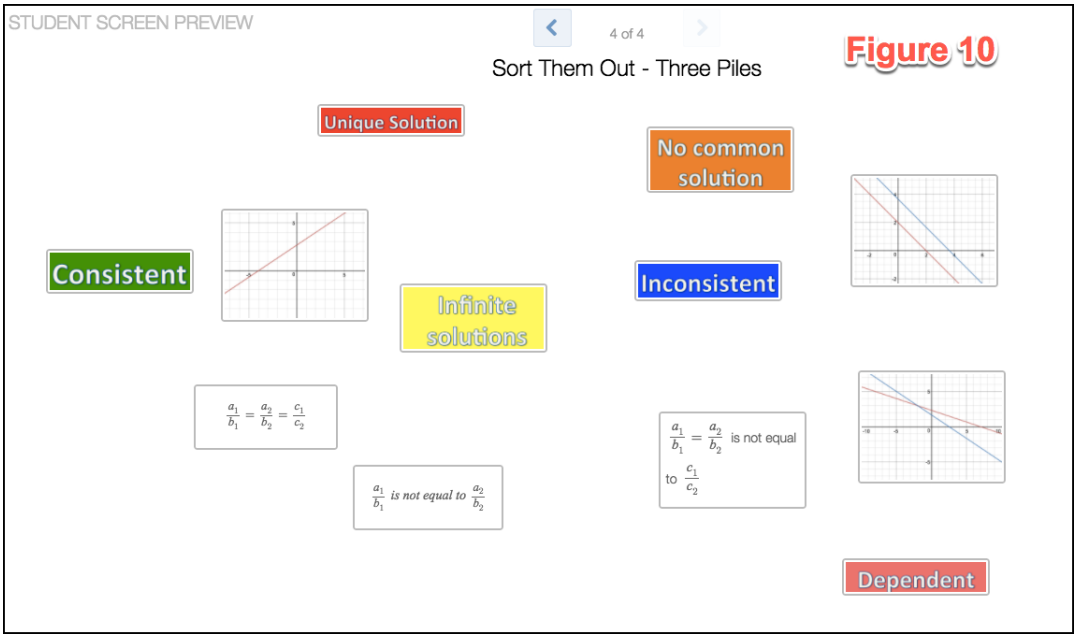
The screen will look like this (Figure 8) to students:



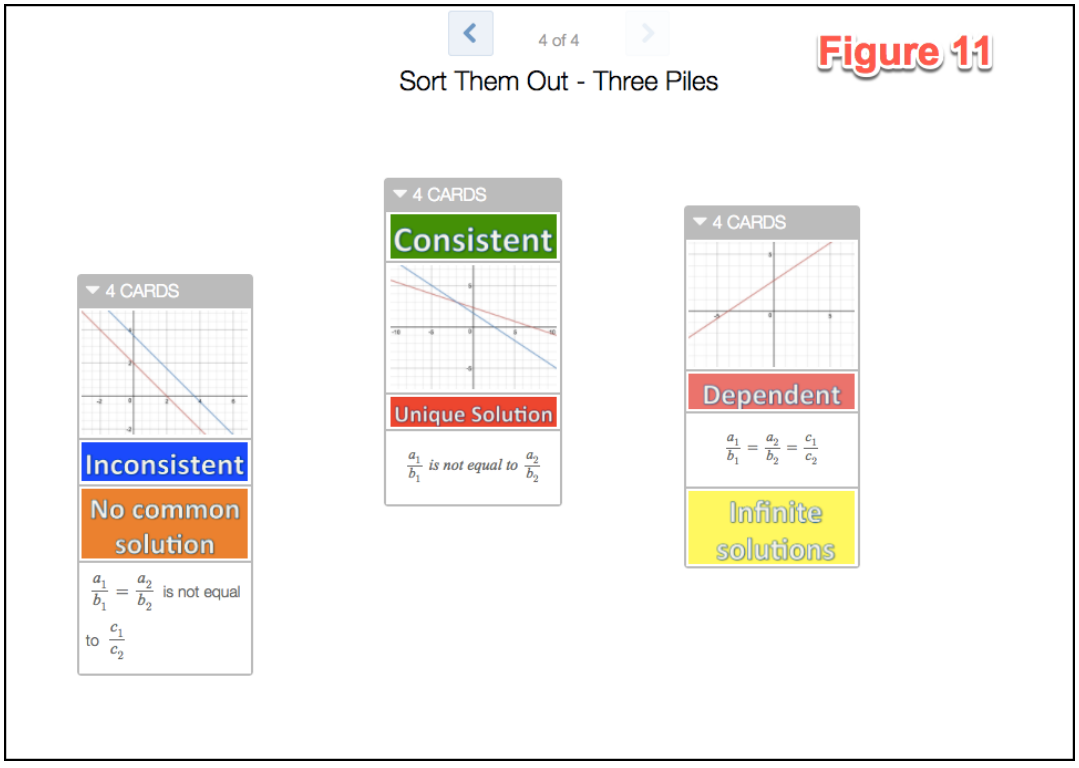
Adding in 'Card Sort' can further enhance the activities. To use 'Card Sort' one must enable it under the 'Labs' option in the screen. Cards using Text (includes Math text), Image and Graph (Figure 9) can be used to design activities which allow students to review their math vocabulary, definitions, concepts and work with multiple representations.



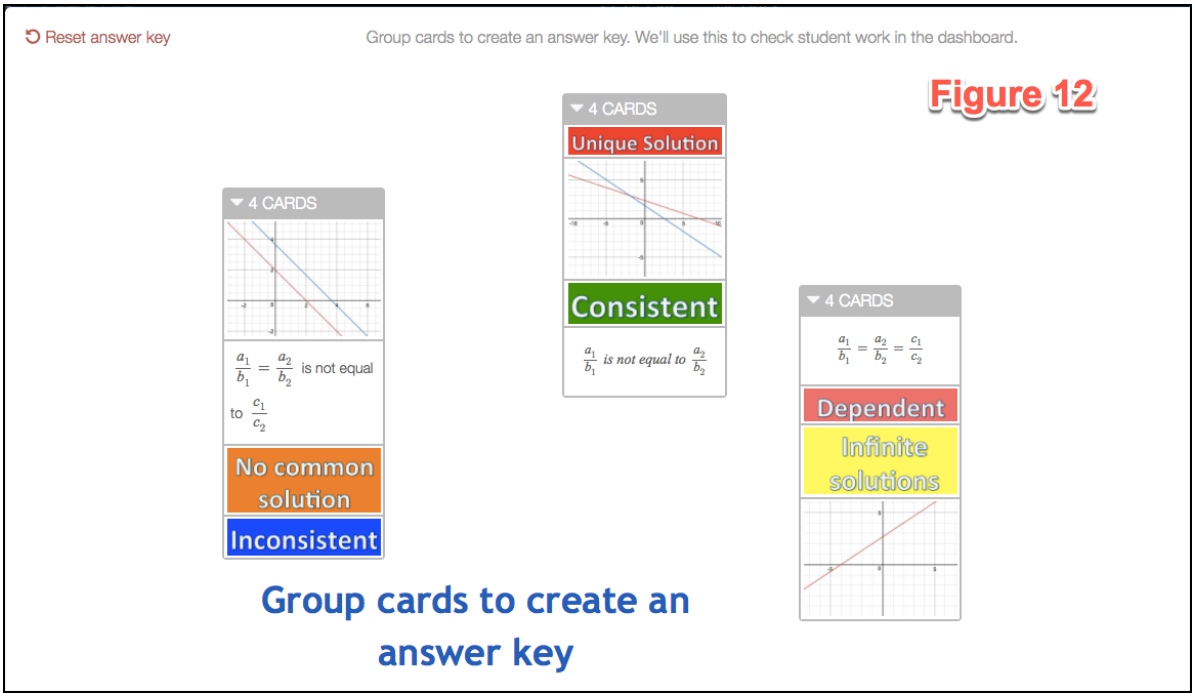
In this activity (Figure 10), students are required to sort the twelve cards into three piles according to the nature of the solutions of the pairs of linear equations. For each pile, text, image and graph cards have been made. You can make as many cards as you want but too many cards to sort and pile may confuse the students.



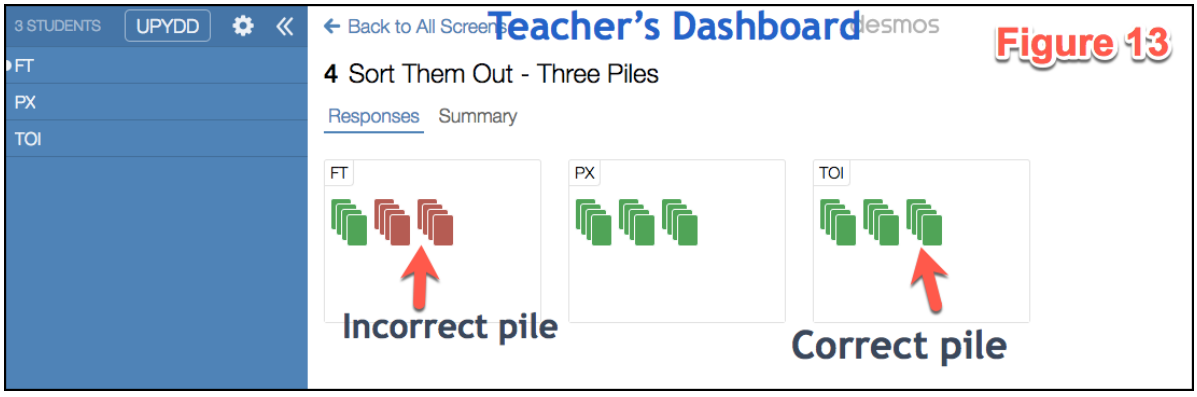
On clicking, the card magnifies so that a small text can also be easily read. Students can drag and drop cards to get them linked and create piles (Figure 11).



Before concluding the activity, click on Answer Key (Figure 12) and create the key which will be used to check students' work in the dashboard.



As the activity progresses the teacher is able to see individual screens and identify students who need help. Through the dashboard (Figure 13) an overall view of each screen helps to identify common mistakes and incorrect responses.



The greatest merit of the activities created via the Desmos Activity Builder is that students can be given multiple learning opportunities. It could be using the 'Graph' feature to plot, reflect and express their understanding in the form of responses to the questions asked, share their work with peers, and review their learning. In addition, such activities are often looked at as a game by many students, thus making it fun to learn maths.

The teacher has the advantage of seeing the work of students 'live' during the class. She also has the option to pause the class and to discuss students' responses, identify those who need individual attention,

and give each student a chance to learn at her own pace. The teacher is empowered with this tool as it allows her to customize activities exclusively for her students and also cater to their individual needs.

Go ahead, create one for yourself and try it with your students.

Resources for Activity Builder:

https://teacher.desmos.com/

http://learn.desmos.com/create

https://tackk.com/desmosab



MS SANGEETA GULATI is Head of Mathematics Department, Sanskriti School, New Delhi. A teacher of mathematics for the past twenty-six years, she has been teaching students in grade 11 and 12. Awardee of 'National Award for using ICT for Innovation in Teaching 2016' and a Google Education Certified Innovator, Sangeeta has been actively involved in exploring the use of technology in teaching and learning of mathematics and has contributed to several Professional Development and Technology Orientation Program all over India. She conducts workshop on GeoGebra, Desmos, Google Apps and Online Resources. Sangeeta is a resource person with NCERT in developing ICT material and has developed video lessons for class XI & XII with Central Institute of Educational Technology, NCERT. An Edmodo Certified Trainer and Global Teacher Accreditation (British Council) award winner, Sangeeta was awarded the Fulbright Distinguished Award in Teaching in 2011. The wikispace: dynamath.wikispaces.com was started as part of her fellowship and is a great resource for math teachers. She may be contacted at sangeetagulati92@gmail.com.

HARMONIC TRIANGULAR TRIPLES

SHAILESH SHIRALI

The equation $1/u + 1/v = 1/w$, where u, v, w are permitted to take positive integral values only, has been studied earlier in this magazine, on more than one occasion. Triples (u, v, w) which serve as solutions of the equation are known as *harmonic triples*. We have seen that there is a simple algorithmic way of listing all such triples. (See the July 2013 issue of *At Right Angles*.)

It is of interest to consider a variation of this equation in which u, v, w are permitted to take values only in some specified set of numbers. Depending on what that set is, interesting mathematics can be uncovered. For example: Can there be any solution in which u, v, w are all odd integers? A simple parity argument shows that such solutions cannot exist. (Please fill in the details for yourself.) More interesting: Can there be any solution in which u, v, w are all prime numbers? It is easy to show that there cannot be any solutions under this condition either. For, the equation leads to the equality $w(u + v) = uv$, showing that w is a divisor of uv. As u, v, w are all prime, this implies that w is a divisor of at least one of the primes u, v and hence is equal to one of them. In turn, this leads to one of u, v being zero. But this is absurd. Hence there can be no solution under the imposed condition.

In this article, we consider the case where u, v, w must all be *triangular numbers*. Recall that the sequence of triangular numbers is $1, 3, 6, 10, 15, \ldots$. The n-th triangular number T_n for any positive integer n is the sum

$$1 + 2 + 3 + \cdots + n.$$

Keywords: Harmonic triple, harmonic triangular triple (HTT), parity, prime, triangular number, square, mathematical induction, quadratic, completing the square, Pell, Brahmagupta

It is well-known that $T_n = \frac{1}{2}n(n+1)$. So we seek positive integral values for a, b, c such that $1/T_a + 1/T_b = 1/T_c$, i.e.,

$$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} = \frac{1}{c(c+1)}. \tag{1}$$

This problem was first studied and solved by R. Venkatachalam Iyer in *The American Mathematical Monthly*, Vol. 67, No. 10 (Dec., 1960), pp. 1034-1035. As we shall see below, it gives rise to rich mathematics.

The equation is well-suited for computer-assisted investigation, using a general-purpose language like *Python* (which is gaining in power and popularity these days), or a computer algebra software like *Mathematica*. As the equation is symmetric in a and b, and we do not wish to count two solutions as different if a and b have merely swapped places, let us impose the condition that $a \leq b$. Writing the appropriate code and running the program, we find that the following are the only solutions in which a, b, c do not exceed 100:

$$(a, b, c) = (3, 3, 2),\ \ (4, 5, 3),\ \ (20, 20, 14),\ \ (28, 29, 20).$$

(Thus we have:

$$\frac{1}{6} + \frac{1}{6} = \frac{1}{3}, \quad \frac{1}{10} + \frac{1}{15} = \frac{1}{6}, \quad \frac{1}{210} + \frac{1}{210} = \frac{1}{105}, \quad \frac{1}{406} + \frac{1}{435} = \frac{1}{210}.$$

The last relation may come as a surprise.)

We note right away that in each case, *a and b are either equal to each other or differ by* 1. This is a striking observation, and it turns out to be true in general. That is, we have the following result:

Theorem (Iyer)**.** *If a, b, c are positive integers such that*

$$\frac{1}{T_a} + \frac{1}{T_b} = \frac{1}{T_c},$$

then either $a = b$ or $a = b \pm 1$.

For the proof, we shall exploit the well-known connection between the triangular numbers and the square numbers: $8T_n + 1 = (2n+1)^2$ for any positive integer n. (We had given a visual proof of this identity on page 57 of the March 2014 issue of *At Right Angles*. For the reader's convenience, we display the figure in Box 1, at the end of the article.) The given equation implies that

$$\frac{1}{(2a+1)^2 - 1} + \frac{1}{(2b+1)^2 - 1} = \frac{1}{(2c+1)^2 - 1}. \tag{2}$$

Let $x = 2a + 1$, $y = 2b + 1$, $z = 2c + 1$. Then we have:

$$\frac{1}{x^2 - 1} + \frac{1}{y^2 - 1} = \frac{1}{z^2 - 1}. \tag{3}$$

This yields on cross multiplication and simplification:

$$\begin{aligned} \left(z^2 - 1\right)\left(x^2 + y^2 - 2\right) &= \left(x^2 - 1\right)\left(y^2 - 1\right), \\ &= x^2y^2 - 1 - \left(x^2 + y^2 - 2\right), \\ \therefore\ z^2\left(x^2 + y^2 - 2\right) &= x^2y^2 - 1. \end{aligned} \tag{4}$$

What follows now is pure algebraic jugglery! We add/subtract the same quantity to both sides, twice. From the last line we get:

$$z^2\left(x^2+y^2-2xy-2\right)+1=x^2y^2-2xyz^2,$$
$$\therefore\ z^4+z^2\left(x^2+y^2-2xy-2\right)+1=x^2y^2-2xyz^2+z^4,$$
$$\therefore\ z^4+z^2\left((x-y)^2-2\right)+1=\left(xy-z^2\right)^2. \tag{5}$$

Now we invoke familiar logic concerning quadratic expressions. The expression on the right-hand side is identically a perfect square. Treating the expression on the left-hand side as a quadratic expression in z^2, we deduce that its discriminant must be 0. That is:

$$\left((x-y)^2-2\right)^2-4=0. \tag{6}$$

Hence we have: $(x-y)^2-2=\pm 2$, giving $x-y=0$ or ± 2. This yields $a-b=0$ or $a-b=\pm 1$. Thus a and b are either equal to each other or differ by 1, as had been claimed. The numerically observed pattern has been explained. □

Solving the equation

Now we shall show how to solve the given equation, $1/T_a+1/T_b=1/T_c$, given the result we have just found. We consider the two cases ($a=b$; $a-b=\pm 1$) separately. As we shall see, both cases lead to instances of a well-known equation which we have studied earlier (November 2014 issue): the Brahmagupta-Fermat equation. (It is referred to in most textbooks as the 'Pell equation' but as it is now known that this name represents a historical inaccuracy, we shall not use it.)

Case I, a = b: The condition yields $T_b=2T_c$. This amounts to finding pairs of triangular numbers in the ratio 2 : 1. From the relation $b(b+1)=2c(c+1)$, we get:

$$b^2+b=2\left(c^2+c\right),$$
$$\therefore\ \left(b+\frac{1}{2}\right)^2-\frac{1}{4}=2\left(c+\frac{1}{2}\right)^2-\frac{1}{2},$$
$$\therefore\ (2b+1)^2-1=2(2c+1)^2-2,$$
$$\therefore\ X^2-2Y^2=-1,\quad \text{where } X=2b+1,\ Y=2c+1. \tag{7}$$

Case II, a = b − 1: The condition yields

$$\frac{1}{T_{b-1}}+\frac{1}{T_b}=\frac{1}{T_c},$$
$$\therefore\ \frac{2}{(b-1)b}+\frac{2}{b(b+1)}=\frac{2}{c(c+1)},$$
$$\therefore\ \frac{2}{(b-1)(b+1)}=\frac{1}{c(c+1)},$$
$$\therefore\ 2b^2-2=4c^2+4c=(2c+1)^2-1,$$
$$\therefore\ (2c+1)^2-2b^2=-1, \tag{8}$$
$$\therefore\ X^2-2Y^2=-1,\quad \text{where } X=2c+1,\ Y=b. \tag{9}$$

We see that both cases lead to the same underlying equation, $X^2 - 2Y^2 = -1$. Hence each solution (X, Y) to the equation $X^2 - 2Y^2 = -1$ yields two solutions to the original equation, one of each type.

Now it is not hard to see that in any integral solution to this equation, both X and Y must be odd. For: $X^2 = 2Y^2 - 1$, from which we see that X^2 is odd, and hence X too. Now if Y were even, say $Y = 2m$ where m is an integer, then we would get $X^2 = 8m^2 - 1$. However, no square is of the form $8k - 1$. Hence Y must be odd.

A numerical illustration. We use the solution $(X, Y) = (7, 5)$ of the equation $X^2 - 2Y^2 = -1$ to demonstrate how each solution to this equation yields two solutions to the original problem, one of each type. Thus we have:

- A Type I solution with $(2b + 1, 2c + 1) = (7, 5)$, giving $b = 3$ and $c = 2$. From this we get the solution $(a, b, c) = (3, 3, 2)$, i.e., $1/T_3 + 1/T_3 = 1/T_2$.
- A Type II solution with $(2c + 1, b) = (7, 5)$, giving $c = 3$ and $b = 5$. From this we get the solution $(a, b, c) = (4, 5, 3)$, i.e., $1/T_4 + 1/T_5 = 1/T_3$.

Patterns in the solutions of the Brahmagupta-Fermat equation

Listing the solutions of the equation $X^2 - 2Y^2 = -1$ is an interesting as well as rewarding exercise because of the rich patterns to be found. A computer assisted search yields the following solution pairs:

X	1	7	41	239	1393	$\cdots$
Y	1	5	29	169	985	$\cdots$

Now consider the sequence of the X values alone:

$$1,\ 7,\ 41,\ 239,\ 1393,\ \ldots$$

This sequence obeys a Fibonacci-type recurrence relation! Namely: each term after the second one equals 6 times the one before it minus the term before that. That is, if the sequence is $X_1, X_2, X_3, \ldots$, then

$$X_n = 6X_{n-1} - X_{n-2}, \quad \text{for all } n > 2. \tag{10}$$

Remarkably, the sequence of Y values obeys the same recurrence; please check.

It is a nice exercise in mathematical induction to show that if the first two (X, Y) pairs satisfy the equation $X^2 - 2Y^2 = -1$, and we generate the X- and Y-sequences as per the recurrence noted above, then *all* the (X, Y) pairs satisfy the equation. That is, if the following are true:

$$\left.\begin{aligned} X_1^2 - 2Y_1^2 = -1, &\quad X_2^2 - 2Y_2^2 = -1, \\ X_n = 6X_{n-1} - X_{n-2}, &\quad \text{for all } n > 2, \\ Y_n = 6Y_{n-1} - Y_{n-2}, &\quad \text{for all } n > 2, \end{aligned}\right\} \tag{11}$$

then $X_n^2 - 2Y_n^2 = -1$ for all $n > 2$. We leave the algebraic verification to you.

An irrational way of generating the solutions of the Brahmagupta-Fermat equation

Here is another curious way, baffling at first encounter, of generating the positive integer solutions of the equation $X^2 - 2Y^2 = -1$.

Take the irrational quantity $\alpha = 1 + \sqrt{2}$. If we raise α to any integral power n, we necessarily get a quantity of the type $a + b\sqrt{2}$ where a and b are integers whose values depend on n. If we raise α to an *odd* integral power greater than 1 and extract the coefficients a and b from the resulting expression, we find exactly what we want! Here are the data for the first five such exponents n:

n	$(1+\sqrt{2})^n$	(a, b)
3	$7 + 5\sqrt{2}$	$(7, 5)$
5	$41 + 29\sqrt{2}$	$(41, 29)$
7	$239 + 169\sqrt{2}$	$(239, 169)$
9	$1393 + 985\sqrt{2}$	$(1393, 985)$
11	$8119 + 5741\sqrt{2}$	$(8119, 5741)$

Have a look at the third column!

This discovery enables us to find a fresh recurrence relation for the solutions, one which turns out to be more convenient than the one shown above. Suppose that

$$\left(1 + \sqrt{2}\right)^n = a + b\sqrt{2} \tag{12}$$

for some pair (a, b) of integers. Then we have:

$$\begin{aligned}\left(1 + \sqrt{2}\right)^{n+2} &= \left(a + b\sqrt{2}\right) \cdot \left(1 + \sqrt{2}\right)^2 \\ &= \left(a + b\sqrt{2}\right) \cdot \left(3 + 2\sqrt{2}\right) \\ &= (3a + 4b) + (2a + 3b)\sqrt{2}. \end{aligned} \tag{13}$$

This yields the following recurrence relation connecting (X_{n+1}, Y_{n+1}) with (X_n, Y_n):

$$\left.\begin{aligned} X_{n+1} &= 3X_n + 4Y_n, \\ Y_{n+1} &= 2X_n + 3Y_n. \end{aligned}\right\} \tag{14}$$

Note that this is a single-step recurrence relation, as distinct from the two-step recurrence relation we had found earlier. It is more convenient to use this relation for computational purposes than the earlier one.

The following important question may have occurred to you after reading the above paragraphs. It is easy to justify that the procedures described do indeed generate infinitely many solutions to the equation $X^2 - 2Y^2 = -1$. (The simplest proof, as suggested above, is via the use of mathematical induction.) But it is far from clear whether they generate *every possible solution*. The question left unresolved by this approach is: Do some solutions get left out? The answer is: the two procedures do indeed generate every possible solution; nothing gets left out! But the full proof of this is rather intricate, and we shall not go into the details at this point.

Back to harmonic triangular triples

Let us now list the triples which are solutions to the original problem proposed: the triples (a, b, c) with $a \leq b$, for which

$$\frac{1}{T_a} + \frac{1}{T_b} = \frac{1}{T_c}.$$

We have shown above that:

- Each such triple derives from a solution (X, Y) to the Brahmagupta-Fermat equation $X^2 - 2Y^2 = -1$;
- From a solution $(X, Y) = (2b + 1, 2c + 1)$ to the equation $X^2 - 2Y^2 = -1$, we get a Type I solution (b, b, c) as well as a Type II solution $(2c, 2c + 1, b)$.

Here are the triples generated by the solution pairs (X, Y) listed above. We omit the solution $(X, Y) = (1, 1)$ as it leads to a trivial solution for (a, b, c).

	(X, Y)	$(7, 5)$	$(41, 29)$	$(239, 169)$	$(1393, 985)$
Type I	(a, b, c)	$(3, 3, 2)$	$(20, 20, 14)$	$(119, 119, 84)$	$(696, 696, 492)$
Type II	(a, b, c)	$(4, 5, 3)$	$(28, 29, 20)$	$(168, 169, 119)$	$(984, 985, 696)$

The relationships between corresponding solutions of Types I and II are striking. Thus for example, we have:

$$\frac{1}{T_{20}} + \frac{1}{T_{20}} = \frac{1}{T_{14}},$$
$$\frac{1}{T_{28}} + \frac{1}{T_{29}} = \frac{1}{T_{20}},$$

which together imply that:

$$2\left(\frac{1}{T_{28}} + \frac{1}{T_{29}}\right) = \frac{1}{T_{14}}.$$

Similarly for the other cases.

Closing remark

To conclude this article, we mention a connection between harmonic triangular triples and the famous house number puzzle which too has been studied in an earlier issue of this magazine (March 2014, page 36). That puzzle essentially requires the determination of all pairs (m, n) of positive integers such that $m < n$ and

$$1 + 2 + 3 + \cdots + (m - 1) = (m + 1) + (m + 2) + \cdots + n.$$

An example of such a pair is $(6, 8)$; for $1 + 2 + 3 + 4 + 5 = 7 + 8$, both sides being equal to 15. More generally, observe that the above equation gives rise to the following:

$$\frac{1}{2}(m - 1)m = \frac{1}{2}n(n + 1) - \frac{1}{2}m(m + 1),$$
$$\therefore \quad m^2 = \frac{1}{2}n(n + 1),$$
$$\therefore \quad 8m^2 = 4n^2 + 4n = (2n + 1)^2 - 1,$$
$$\therefore \quad (2n + 1)^2 - 2(2m)^2 = 1.$$

Let $X = 2n + 1$ and $Y = 2m$. Then we have $X^2 - 2Y^2 = 1$, and we see yet again an instance of the Brahmagupta-Fermat equation (but note that the sign of the 1 on the right-hand side is positive rather than negative). Its solutions can be generated in a similar way: take the irrational quantity $\alpha = 1 + \sqrt{2}$ and raise it to an *even* integral power larger than 2. (Raising it to power 2 yield a trivial solution.) The coefficients in the resulting expression then yield a solution to our problem. Thus:

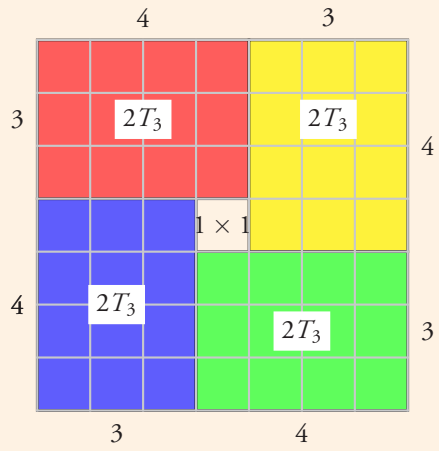
n	$(1+\sqrt{2})^n$	(a, b)	(m, n)	Corresponding equality
4	$17 + 12\sqrt{2}$	$(17, 12)$	$(8, 6)$	$1 + 2 + \cdots + 5 = 7 + 8$
6	$99 + 70\sqrt{2}$	$(99, 70)$	$(49, 35)$	$1 + 2 + \cdots + 34 = 36 + 37 + \cdots + 49$
8	$577 + 408\sqrt{2}$	$(577, 408)$	$(288, 204)$	$1 + 2 + \cdots + 203 = 205 + 206 + \cdots + 288$
10	$3363 + 2378\sqrt{2}$	$(3363, 2378)$	$(1681, 1189)$	$1 + 2 + \cdots + 1188 = 1190 + 1191 + \cdots + 1681$

The column of (m, n) values is obtained from the (a, b) column by using the relations $m = (a - 1)/2$, $n = b/2$.

You may wonder whether it will always be the case that a is odd and b is even. The answer is Yes, but we leave the proof for you to complete. (It can be done using the principle of induction.)

Visual proof of an identity concerning the triangular numbers

The identity being referred to is the following: $8T_n + 1 = (2n + 1)^2$ for any positive integer n. The following figure illustrates the identity for the case $n = 3$.



Box 1. Illustrating `why' $8T_3 + 1$ is a perfect square



SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

A TRIANGLE PROBLEM

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

A reader once sent me the following nice problem:

> *Two sides of a triangle have lengths* 6 *and* 10*, and the radius of the circumcircle of the triangle is* 12*. Find the length of the third side.*

Let the triangle be labelled ABC, with sides $a = BC = 6$ and $b = CA = 10$. Let the radius of the circumcircle be $R = 12$. We must find c, the length of side AB.

Attempts to solve the problem using trigonometry will probably prove challenging for most high school students. To get a handle on the problem, it is useful to study the situation using GeoGebra. Figure 1 depicts the situation. We draw a circle Γ with radius $R = 12$, centred at an arbitrary point O. Next, we select any point on Γ and label it B. Then we draw a circle ω_B centred at B, with radius $a = 6$. This circle intersects Γ at two points; we arbitrarily select one of them and label it C. (No essential difference will result if we were to select the other point of intersection. Please verify this for yourself.) Next, we draw a circle ω_C centred at C, with radius $b = 10$. Circle ω_C too intersects Γ at two points; call them A_1 and A_2. *Both these points serve as solutions to our problem. That is, we have* ***two*** *triangles A_1BC and A_2BC, both of which satisfy the given conditions.* The 'algebra' window of GeoGebra reveals the two possible values of c; namely, 4.23 and 15.14, respectively.

Let us now see if we can derive these figures using trigonometry.

The trigonometric approach for solving the problem is the following. We use the cosine rule twice, to find the angles subtended by the two given sides (a and b) at the centre O of the circumscribing circle Γ. Then we use the addition and subtraction formulas to find the cosines of the sum and difference of these two angles. Finally, we use the cosine rule again to find the third side.

Keywords: Triangle, circumradius, sine rule, cosine rule

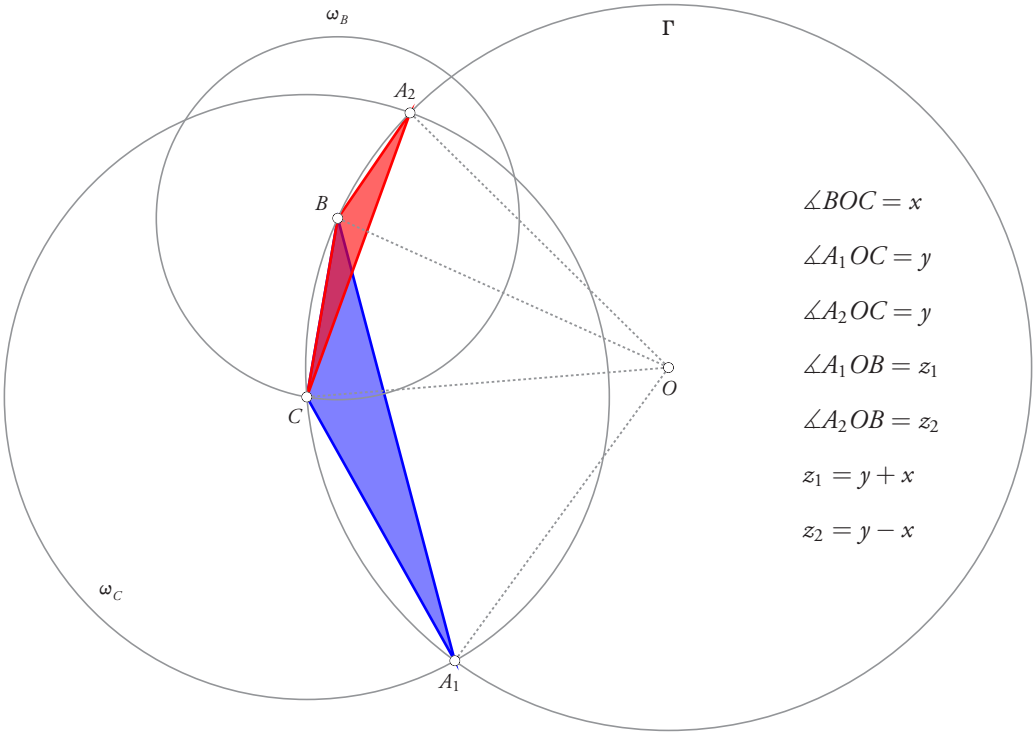


Figure 1.

Let $\angle BOC = x$ and $\angle A_1OC = y = \angle A_2OC$. Using the cosine rule, we get:

$$\cos x = \frac{12^2 + 12^2 - 6^2}{2 \times 12 \times 12} = \frac{7}{8},$$
$$\cos y = \frac{12^2 + 12^2 - 10^2}{2 \times 12 \times 12} = \frac{47}{72}. \qquad (1)$$

From this we get:

$$\sin x = \frac{\sqrt{8^2 - 7^2}}{8} = \frac{\sqrt{15}}{8},$$
$$\sin y = \frac{\sqrt{72^2 - 47^2}}{72} = \frac{5\sqrt{119}}{72}. \qquad (2)$$

These in turn yield:

$$\cos(y + x) = \cos y \cos x - \sin y \sin x$$
$$= \frac{329 - 5 \cdot \sqrt{1785}}{576}, \qquad (3)$$

$$\cos(y - x) = \cos y \cos x + \sin y \sin x$$
$$= \frac{329 + 5 \cdot \sqrt{1785}}{576}. \qquad (4)$$

We need both these values, i.e., $\cos(y + x)$ and $\cos(y - x)$, precisely because of the two possibilities indicated in Figure 1. The first possibility arises when angles x and y do not overlap with each other (point A_1); the second possibility arises when they do overlap with each other (point A_2).

The first possibility yields, by using the cosine rule in $\triangle OA_1B$:

$$c^2 = 2R^2(1 - \cos(y + x))$$
$$\therefore \quad c = \frac{5\sqrt{15} + \sqrt{119}}{2}, \quad \text{on simplification,}$$
$$\therefore \quad c \approx 15.137.$$

The second possibility yields, by using the cosine rule in $\triangle OA_2B$:

$$c^2 = 2R^2(1 - \cos(y - x))$$
$$\therefore \quad c = \frac{5\sqrt{15} - \sqrt{119}}{2}, \quad \text{on simplification,}$$
$$\therefore \quad c \approx 4.228.$$

These values agree with what GeoGebra had told us earlier. Just as well!

Remark. It is an interesting commentary on how our minds work that when I initially solved the problem using trigonometry, I considered only the possibility where angles x and y do not overlap; so I only obtained the first possible value of c (the larger value). It is only when I drew the diagram using GeoGebra that I noticed the other possibility! Of course, I then back-checked the calculation and spotted the point at which I had missed a possibility. The results then tallied.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Problems for the MIDDLE SCHOOL

Problem Editors: Sneha Titus and R. Athmaraman

A popular pastime for children – and adults - is doing puzzles. Here is a puzzle with a seemingly difficult solution. In how many ways can you trace this figure without lifting your hand from the paper or retracing any path?

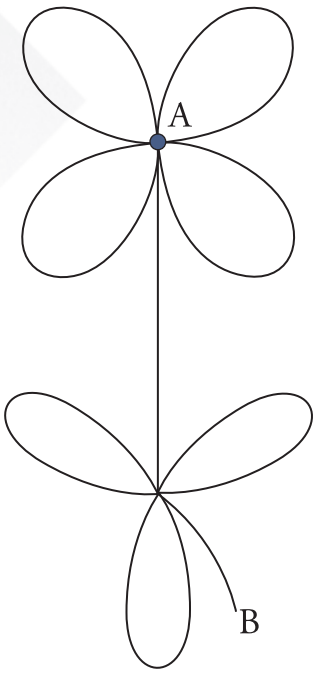


Figure 1: A counting problem
Problem Source: Dr. S. Muralidharan, TCS

Keywords: Counting, permutations, combinations.

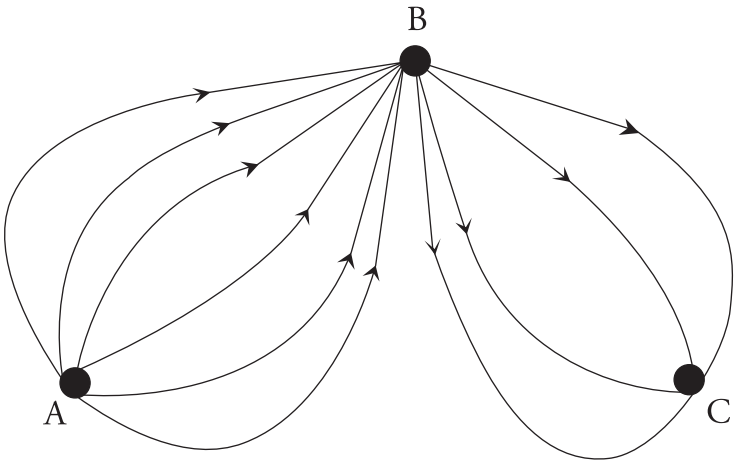


Figure 2: A simpler problem
Source: Mathematical Circles

First of all, is it possible to trace this figure without lifting your hand or retracing any path? It is easily done and can be done starting either from point A or point B. So clearly, there is more than one way of tracing the figure. And depending on the order in which the petals are traced and whether they are traced clockwise or anticlockwise, we can see the number of options increasing. How can we systematically count the number of options? We start with some simpler problems.

How many ways are there to get from A to C? (See Figure 2)

Solution: We can see that to get from A to C, one has to go via B. There are 6 ways to go from A to B and for each of these 6 ways, there are 4 ways to go from B to C. Hence, the total number of ways is $6 \times 4 = 24$. This is called the multiplication principle in counting.

Problem VI-3-M.1

How many two-digit numbers are there whose digits are both odd? How many such numbers have different digits (both odd)?

Problem VI-3-M.2
How many factors does 24 have?

Problem VI-3-M.3

How many 5 digit multiples of 5 can be formed from the digits 1, 2, 3, 4, 5?

We use exactly the same technique as in the example for all the problems. Taking up the first problem, there are 5 odd digits and so there are 5 options for the units place. Having placed one of the odd digits in the units place, there are again 5 options for the tens place. So there are totally 5×5 two-digit numbers whose digits are both odd. On the other hand, if the digits are different, there are only 4 options for the tens place and so there are totally $5 \times 4 = 20$ two-digit numbers whose digits are different from each other and are both odd. Do encourage students to create variations of this problem and strengthen their understanding of the multiplication rule.

Though the second problem was addressed in the previous issue, this is again a great illustration of the multiplication rule. $24 = 2^3 \times 3$, so the factors can be formed with 0 to 3, i.e., zero, one, two or three 2's, and 0 to 1, i.e., zero or one 3's. The total number of factors is therefore $4 \times 2 = 8$.

In problems such as the third one above, we assume that the given digits can only be used once. The first part simply has the constraint that the units digit has to be 5. Then there is 1 option

for the units digit, 4 for the tens digit, 3 for the hundreds digit, 2 for the thousands digit and 1 for the ten thousands digit. This makes the total number of 5 digit multiples of 5 made from 1, 2, 3, 4, 5 equal

$$1 \times 2 \times 3 \times 4 \times 1 = 24.$$

We are now ready to tackle the flower tracing problem. Suppose we start from A. Then the 4 petals can be traced in $4 \times 3 \times 2 \times 1 = 24$ different orders. At each petal, we have the choice of tracing it clockwise or anti-clockwise. So we have 2 choices at each petal and so 24×2^4 ways to finish the first flower. Coming to the second flower, we have $3 \times 2 \times 1 = 6$ different orders to draw the petals and 2^3 choices for the direction, clockwise or anti-clockwise, for each petal. So there are 6×2^3 ways to draw the second flower and that means $24 \times 16 \times 6 \times 8 = 18432$ ways to draw the two flowers starting from A and ending at B! And twice that if you start from B and end at A!!

Problem VI-3-M.4

i. How many multiples of 5 can be formed from the same digits, i.e., 1, 2, 3, 4, 5?

ii. How many multiples of 5 can be formed from the digits 0, 2, 7, 5?

We assume that these digits cannot repeat in any of these numbers.

Now a layer of complexity is added with each question because the number of digits in the multiples of 5 is not specified.

i. With the given digits,

- There is 1 single digit multiple of 5
- There are $4 \times 1 = 4$ two digit multiples of 5
- There are $3 \times 4 \times 1 = 12$ three digit multiples of 5
- There are $2 \times 3 \times 4 \times 1 = 24$ four digit multiples of 5
- And 24 five digit multiples of 5

Now, we **add** all these options and get the total number of multiples of 5 from the digits to be 65. Notice that the operation of addition is used when we have several alternative ways in which we can comply with the conditions.

ii. In this case, the numbers can end in 0 or 5. So, we again have to add the possible options:

- Numbers ending in 0: 1 single digit + 3 two digit + 6 three digit + 6 four digit = 16
- Numbers ending in 5: 1 single digit + 2 two digit + 2 three digit (with 0 in tens place) + 2 three digit (no 0) + 2 four digit (with 0 in tens place) + 2 four digit (0 in hundreds place) = 11
- The total number of multiples of 5 made with the digits 0, 2, 7, 5 is 27.

This problem becomes simple once the options are enumerated and students should be able to see that there is no scope for double counting.

Problem VI-3-M.5
Source: Mathematical Circles

The map of a town is depicted in Figure 3. All its streets are one-way, so that you can drive only "east" or "north". How many different ways are there to reach point B starting from point A?

Systematic counting is again called for and again we start with some simpler problems.

Problem VI-3-M.6

n people meet in a room and shake hands all around. How many handshakes are there in total?

Problem VI-3-M.7

How many diagonals does a *n*- sided polygon have?

How do these problems differ from the previous problems? Look at the handshake problem. If there were 2 people X and Y in the room, common sense tells us that there would be only one handshake. If we go by the multiplication principle, there are 2 options for the first handshake (either X or Y to initiate the

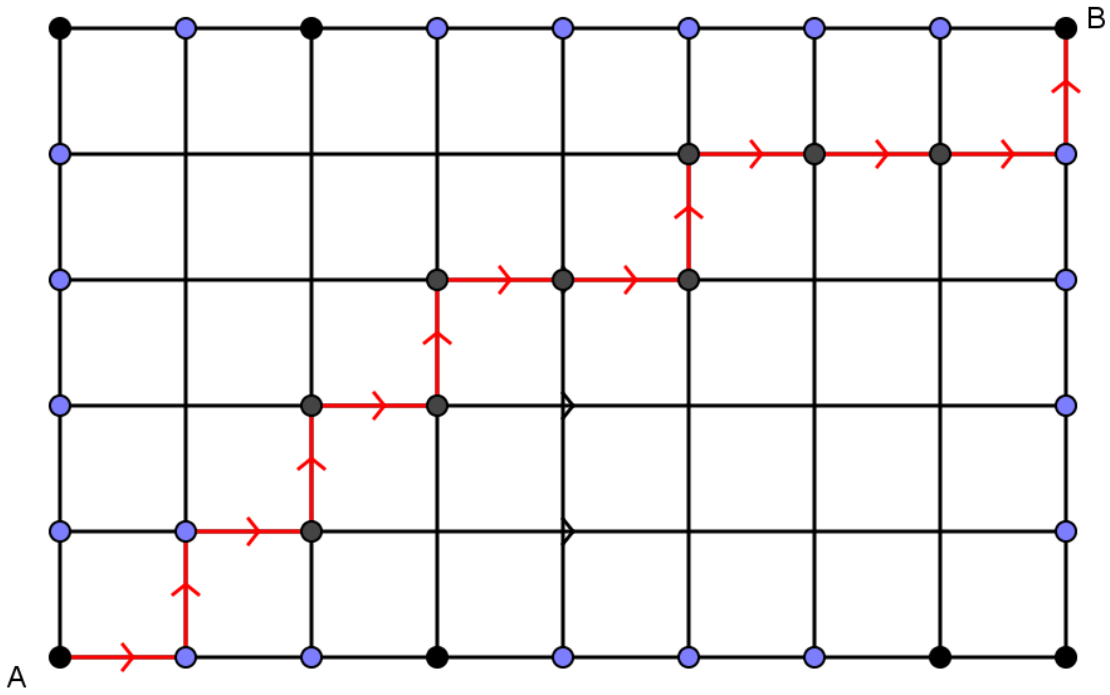


Figure 3 One possible route from A to B

handshake) and then 1 option for the second handshake. But here, a handshake between X and Y is the same as a handshake between Y and X. So we halve the number of handshakes.

Similarly, if there are 3 people X, Y and Z in the room, then the possible handshakes are: X and Y, X and Z, Y and Z. Again, 3 options for the initiator, 2 options for the receiver and 3x2 is divided by 2 to avoid replication, giving 3 handshakes in all.

What if there are four people in the room? There are 4 options for the initiator, 3 options for the receiver and we divide 4 × 3 by 2 to get the total. We can extend this pattern to get 5 × 4 /2 = 10 handshakes if there are 5 people in the room. If there are n people in the room, the number of handshakes is $\frac{n(n-1)}{2}$.

The diagonal problem would be almost the same as the handshake problem with one difference. Now, we have to subtract the number of lines which are not diagonals but sides of the polygons. You should be easily able to see that the number of diagonals would be $\frac{n(n-1)}{2} - n$.

Problem VI-3-M.8

In how many ways can a team of 3 students be selected from a group of 5?

This problem differs slightly from the previous one because there are more duplications. The team ABC is identical to the team ACB, BAC, BCA, CAB and CBA. We easily see that the number of repetitions has 3 options for the first place, 2 for the second and 1 for the third, i.e., number of repetitions when a team of 3 is picked is 3 × 2 × 1 = 6.

So the number of 3 member teams that can be constituted is 5 × 4 × 3 divided by 6, i.e., 10 possible teams of 3 students each.

Problem VI-3-M.9

When 5 coins are tossed simultaneously, how many outcomes can there be?

The options are

i. 5 HEADS: 1 way

ii. 4 HEADS, 1 TAIL: Out of the 5 spots, we select 1 spot for the TAIL - can be done in 5 ways.

iii. 3 HEADS, 2 TAILS: Out of the 5 spots, we select 2 spots for the TAILs -, can be done in

$\frac{5 \times 4}{2} = 10$ ways.

iv. 2 HEADS, 3 TAILS: Out of the 5 spots, we select 3 spots for the TAILs - can be done in

$\frac{5 \times 4 \times 3}{3 \times 2} = 10$ ways.

v. 1 HEAD, 4 TAILS: Out of the 5 spots, we select 4 spots for the TAILs - can be done in

$\frac{5 \times 4 \times 3 \times 2}{4 \times 3 \times 2} = 5$ ways

vi. 5 TAILS: 1 way.

Totally, 32 ways!

Now, we are ready to address Problem VI-3-M.5.

Let each point on the grid be connected to the point on the north or the east by a street of length 1. So to traverse from A to B, we need to cover 13 streets, which run either East (E) or North (N). Also, out of these 13 streets, exactly 5 have to be N and 8 have to be E. The route shown is ENENENEENEEEN. How many such routes are there? The answer is simple- we just have to choose 5 spots out of the 13 for the Ns.

This can be done in

$$n\frac{13 \times 12 \times 11 \times 10 \times 9}{5 \times 4 \times 3 \times 2 \times 1} = 1287 \text{ ways.}$$

Students may ask if we can choose 8 spots out of 13 for the E instead, they will be delighted to find that this is equivalent to the above.

Combinatorics provides delightful problems which develop a student's skills of visualization, pattern recognition and logical thinking. Do encourage students to show their reasoning with sketches or in words, instead of just aiming for the correct answer.

Happy problem solving! Do send in interesting discoveries to AtRiA.editor@apu.edu.in

Sources

We provide a list of sources for interesting problems on combinatorics. Most of them can be attempted by students at the middle school level.

https://nrich.maths.org/public/search.php?search=permutations

https://www.khanacademy.org/math/precalculus/prob-comb/combinatorics-precalc/e/permutations_1

https://www.youtube.com/watch?v=siFBqH-LaQQ An unforgettable talk by Manjul Bhargava.

References

Shirali, S. (2002). *Adventures in Problem Solving*. India: Universities Press. Chapter 9, The Art of Counting

Fomin, D. Genkin, S. Itenberg, I. (1998). *Mathematical Circles*. India: Universities Press. Chapter 2, Combinatorics -1.

Problems for the SENIOR SCHOOL

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem VI-3-S.1
The numbers

$$\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2017}$$

are written on a blackboard. A student chooses any two numbers from the blackboard, say x and y, erases them and instead writes the number $x + y + xy$. She continues to do so until there is just one number left on the board. What are the possible values of the final number?

Problem VI-3-S.2
The numbers $1, 2, 3, \ldots, n$ are arranged in a certain order. One can swap any two adjacent numbers. Prove that after performing an *odd* number of such operations, the arrangement of the numbers thus obtained will differ from the original one.

Problem VI-3-S.3
At each of the eight corners of a cube, write $+1$ or -1. Then, on each of the six faces of the cube, write the product of the numbers written at the four corners of that face. Add all the fourteen numbers so written down. Is it possible to arrange the numbers $+1$ and -1 at the corners initially in such a way that this final sum is zero?

Keywords: parity, integers, combinatorics, polynomials, identities

Problem VI-3-S.4
At a party, it was observed that each person knew exactly twenty others. Also, for each pair of persons who knew one another, there was exactly one other person whom they both knew. Further, for each pair of persons who did not know one another, there were exactly six other persons whom they both knew. Also, if A and B were present in the party and A knew B, then B knew A. Determine, with proof, the number of people at the party.

Problem VI-3-S.5
Suppose there are k teams playing a round-robin tournament; that is, each team plays against every other team. Assume that no game ends in a draw. Suppose that the i-th team loses l_i games and wins w_i games. Show that

$$\sum_{i=1}^{k} l_i^2 = \sum_{i=1}^{k} w_i^2.$$

Solutions to Problems in Issue-VI-2 (July 2017)

Solution to problem VI-2-S.1
A teacher wrote the quadratic $x^2 + 10x + 20$ *on the board. Then each student either increased by* 1 *or decreased by* 1 *either the constant or the linear coefficient. Finally* $x^2 + 20x + 10$ *appeared. Did a quadratic expression with integer zeros necessarily appear on the board in the process?* [From *Polynomials* by Ed Barbeau]

Solution. The key observation to make, which solves the problem, is that the quadratic expression $x^2 + (k+1)x + k$ has integer zeros for any integer value of k. (Namely: its zeros are $-k$ and -1.)

Suppose that at some stage of the process, the quadratic expression on the board is $x^2 + ax + b$. What happens at the next stage? The ordered pair (a, b) is replaced by the ordered pair (a', b') which is one of $(a+1, b)$, $(a, b+1)$, $(a-1, b)$, $(a, b-1)$. Now observe that $a' - b' = a - b \pm 1$. That is, the value of $a - b$ changes by 1 at each stage (it either decreases or increases) . Initially $a - b = -10$; at the end $a - b = 10$. So the difference between a and b climbs from -10 to 10, changing by ± 1 at each step. If we can guarantee that $a - b = 1$ at some stage, then we will be done. But that will certainly happen at some stage, as each change is ± 1. So the answer is: a quadratic expression with integer zeros does necessarily appear on the board, at some stage.

Solution to problem VI-2-S.2
Let $p(t)$ be a monic quadratic polynomial. Show that, for any integer n, there exists an integer k such that $p(n)p(n+1) = p(k)$. [From *Polynomials* by Ed Barbeau]

Solution. Let $p(t) = t^2 + bt + c$. Then $p(n) = n^2 + bn + c$ and $p(n+1) = (n+1)^2 + b(n+1) + c$. Then

$$\begin{aligned}(p(n) - c)(p(n+1) - c) &= n(n+b)(n+1)(n+1+b)\\ &= (n(n+1) + bn)(n(n+1) + bn + b)\\ &= m(m+b),\end{aligned}$$

where $m = n(n+1) + bn$. But

$$p(n) + p(n+1) = 2(n^2 + n + bn) + 1 + b + 2c = 2m + 1 + b + 2c.$$

Therefore

$$\begin{aligned} p(n)p(n+1) &= m^2 + (b+2c)m + c(1+b+c) \\ &= (m+c)^2 + b(m+c) + c \\ &= p(m+c) = p(n(n+1) + bn + c). \end{aligned}$$

Thus $k = n(n+1) + bn + c$. This value of k "works."

Solution to problem VI-2-S.3
Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers. [From Round 1, British Mathematical Olympiad, 2011]

Solution. Suppose there exist positive integers m and n such that

$$m(m+1) = n(n+1)(n+2)(n+3).$$

Then

$$\begin{aligned} m(m+1) + 1 &= (n^2 + 3n)(n^2 + 3n + 2) + 1 \\ &= u(u+2) + 1 = (u+1)^2, \end{aligned}$$

where $u = n^2 + 3n$. However, $m^2 < m^2 + m + 1 < (m+1)^2$. That is, the number $m^2 + m + 1$ lies between squares of two consecutive positive integers and hence cannot be a square. This means that the stated equality, that $m(m+1) = n(n+1)(n+2)(n+3)$ for some two positive integers m, n, cannot take place.

Solution to problem VI-2-S.4
Find all integers n for which $n^2 + 20n + 11$ is a perfect square. [From Round 1, British Mathematical Olympiad, 2011]

Solution. Let $n^2 + 20n + 11 = y^2$. Then $(n+10)^2 - y^2 = 89$, hence

$$(n + 10 + y)(n + 10 - y) = 89.$$

Since 89 is a prime number, the only ways that it can be written as a product of two integers are $\pm 89 \times \pm 1$ and $\pm 1 \times \pm 89$. So the following cases arise.

(i) Case 1: $n + 10 + y = 89$, $n + 10 - y = 1$, giving $n = 35$.

(ii) Case 2: $n + 10 + y = 1$, $n + 10 - y = 89$, giving $n = 35$.

(iii) Case 3: $n + 10 + y = -1$, $n + 10 - y = -89$, giving $n = -55$.

(iv) Case 4: $n + 10 + y = -89$, $n + 10 - y = -1$, giving $n = -55$.

So there are just two possible values of n, namely -55 and 35.

Solution to problem VI-2-S.5

Find all integers x, y, z such that $x^2 + y^2 + z^2 = 2(yz + 1)$ and $x + y + z = 4018$. [From Round 1, British Mathematical Olympiad, 2009]

Solution. From the first equation we have $x^2 + (y - z)^2 = 2$. Therefore $x = \pm 1$ and $y - z = \pm 1$.

(i) Case 1: $x = 1$. This leads to $y + z = 4017$ and $y - z = \pm 1$. Thus $(x, y, z) = (1, 2009, 2008), (1, 2008, 2009)$.

(ii) Case 2: $x = -1$. This leads to $y + z = 4019$ and $y - z = \pm 1$. Thus $(x, y, z) = (-1, 2010, 2009), (-1, 2009, 2010)$.

These are the only solutions in integers of the given pair of equations.

ADVENTURES IN PROBLEM SOLVING

Some Problems from the pre-RMO 2017

SHAILESH SHIRALI

In this edition of 'Adventures' we study a few problems from the pre-RMO, which was held in various centres across India, on 20 August 2017.

As usual, we pose the problems first and give the solutions later in the article, thereby giving you an opportunity to work on the problems.

(The problems are numbered as in the actual paper.)

Problems

Problem 8: A pen costs ₹ 11 and a notebook costs ₹ 13. Find the number of ways in which a person can spend exactly ₹ 1000 to buy pens and notebooks.

Problem 14: Suppose x is a positive real number such that $\{x\}$, $[x]$ and x are in geometric progression. Find the least positive integer n such that $x^n > 100$. (Here $[x]$ denotes the integer part of x and $\{x\} = x - [x]$.)

Problem 16: Five distinct 2-digit numbers are in geometric progression. Find the middle term.

Problem 20: What is the number of triples (a, b, c) of positive integers such that (i) $a < b < c < 10$ and (ii) $a, b, c, 10$ form the sides of a quadrilateral?

Problem 21: Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$.

Keywords: integers, exponents, real numbers, quadrilaterals, factors, primes, combinatorics

Solutions

Problem 8. *A pen costs* ₹11 *and a notebook costs* ₹13. *Find the number of ways in which a person can spend exactly* ₹1000 *to buy pens and notebooks.*

Solution. Let x be the number of pens and y the number of notebooks. Then we have $11x + 13y = 1000$. Hence we must find the number of nonnegative integer solutions to the equation $11x + 13y = 1000$. Note that this is a single equation with two unknowns. So its general solution must involve one free parameter.

To find the parametric equation, we need any one integer solution to the equation; we can find this by inspection or trial. Here is one such: $x = 20$, $y = 60$ (corresponding to $220 + 780 = 1000$). Next, we need a parametric integer solution to the homogeneous equation $11x + 13y = 0$ (note that the number on the right side is 0). An obvious solution is $x = 13m$, $y = -11m$, where m is any integer. This captures all integer solutions to the homogeneous equation. Combining these two solutions, we see that a general integer solution to the equation $11x + 13y = 1000$ is the following:

$$x = 20 + 13m, \quad y = 60 - 11m.$$

where m is an arbitrary integer. *Importantly, this parametric solution captures all integer solutions to the given equation.*

As our interest is only in nonnegative integer solutions, we must have

$$20 + 13m \geq 0, \quad 60 - 11m \geq 0.$$

The two inequalities yield:

$$m \geq -\frac{20}{13}, \quad m \leq \frac{60}{11},$$

i.e., $m \geq -1$ and $m \leq 5$, since m is an integer. The integer values allowed by these constraints are $-1, 0, 1, 2, 3, 4, 5$, or 7 values in all. Hence the required number of ways is 7.

Problem 14. *Suppose x is a positive real number such that $\{x\}$, $[x]$ and x are in geometric progression. Find the least positive integer n such that $x^n > 100$.*

Solution. Let $[x] = m$ and $x - [x] = t$; then $m \geq 0$, $0 \leq t < 1$, and $t, m, m + t$ are in GP. Note that this means that $m \neq 0$; a GP cannot have a 0 in it. Hence $m \geq 1$ and $x \geq 1$. Let r be the common ratio of the GP; then

$$r = \frac{m}{t}, \quad r = \frac{m+t}{m}.$$

Since $0 \leq t < 1$, we have

$$r > m, \quad r = 1 + \frac{t}{m} < 1 + \frac{1}{m}.$$

These relations imply that

$$m < 1 + \frac{1}{m}.$$

This inequality is true for $m = 1$ and is false for all $m \geq 2$. Hence $m = 1$. So the GP is $t, 1, 1 + t$. Hence we have $t(1 + t) = 1$, i.e., $t^2 + t - 1 = 0$, and so

$$t = \frac{-1 \pm \sqrt{5}}{2}.$$

The negative sign cannot apply since $t \geq 0$; hence $t = \frac{1}{2}(\sqrt{5} - 1)$ and

$$x = \frac{\sqrt{5}+1}{2} \approx 1.618.$$

So x is the famous Golden Ratio. Its defining property is the relation $x^2 = x + 1$. We now observe the following about the powers of x:

$$x^2 = 1x + 1,$$
$$x^3 = x \cdot x^2 = x(x+1) = x^2 + x = 2x + 1,$$
$$x^4 = x \cdot x^3 = x(2x+1) = 2x^2 + x = 3x + 2,$$
$$x^5 = x \cdot x^4 = x(3x+2) = 3x^2 + 2x = 5x + 3,$$

and so on. From the last two relations we get:

$$\begin{aligned} x^9 &= (3x+2)(5x+3) = 15x^2 + 19x + 6 \\ &= 15(x+1) + 19x + 6 = 34x + 21, \\ x^{10} &= (5x+3)^2 = 25x^2 + 30x + 9 \\ &= 25(x+1) + 30x + 9 = 55x + 34. \end{aligned}$$

Note the striking pattern: *each power of x may be expressed in the form $ax + b$ where a and b are two consecutive Fibonacci numbers*. From the last two relations, we observe that:

$$x^9 \approx 34(1.62) + 21 < 100,$$

$$x^{10} \approx 55(1.62) + 34 > 100.$$

Hence the desired value of n is 10.

Problem 16. *Five distinct 2-digit numbers are in geometric progression. Find the middle term.*

Solution. There is no loss of generality in assuming that the numbers of the GP are in increasing order; if not, we simply reverse the order of the numbers (the GP remains a GP!). We may therefore assume that the common ratio of the GP exceeds 1. Let a be the first term of the GP, $a \geq 10$, and let $r = m/n > 1$ be the common ratio, where m, n are positive integers with no factors in common ($m > n$). The fifth term of the GP is $ar^4 \geq 10r^4$. Since the fifth term is a two-digit number, we have $10r^4 < 100$, hence $r < 2$. It follows that r is a rational number lying between 1 and 2, and therefore that $n \geq 2$ and $m \geq 3$. The terms of the GP are:

$$a, \quad \frac{am}{n}, \quad \frac{am^2}{n^2}, \quad \frac{am^3}{n^3}, \quad \frac{am^4}{n^4}.$$

As each of these quantities is an integer, and m^4 and n^4 have no factors in common, it follows that n^4 divides a. Hence $a/n^4 \geq 1$. Since $am^4/n^4 < 100$, it follows that $m^4 < 100$ and therefore that $m \leq 3$. Combining this with the fact that $m \geq 3$, we deduce that $m = 3$ and hence that $n = 2$; so $r = 3/2$. Therefore a is a multiple of 16. Focusing attention on the fifth term, we see that

$$\frac{a \times 3^4}{2^4} < 100, \qquad \therefore \quad a < 100 \times \frac{16}{81} < 20.$$

Hence $a = 16$ and the GP is $16, 24, 36, 54, 81$. The middle term is 36.

Problem 20. *What is the number of triples (a, b, c) of positive integers such that (i) $a < b < c < 10$ and (ii) $a, b, c, 10$ form the sides of a quadrilateral?*

Solution. The triangle inequality shows that a quadrilateral with prescribed sides exists if and only if the sum of the lengths of the three shortest sides exceeds the longest side. Therefore we must find the number of triples (a, b, c) of positive integers such that $a < b < c < 10$ and $a + b + c > 10$. Since $a + b + c$ is at most equal to $c - 2 + c - 1 + c = 3c - 3$, we must have $3c - 3 > 10$, i.e., $c > 13/3$. Since c is an integer, this tells us that $c \geq 5$. Therefore the possible values that c can take are $5, 6, 7, 8, 9$. We consider each value in turn, starting from the top.

c = 9: We must have $a < b \leq 8$ and $a + b > 1$. The inequality will be satisfied for all possible choices of a, b, hence the number of choices is $\binom{8}{2} = 28$.

c = 8: We must have $a < b \leq 7$ and $a + b > 2$. The inequality will be satisfied for all possible choices of a, b, hence the number of choices is $\binom{7}{2} = 21$.

c = 7: We must have $a < b \leq 6$ and $a + b > 3$. The inequality will be satisfied for all choices of a, b except for $(a, b) = (1, 2)$, hence the number of choices is $\binom{6}{2} - 1 = 15 - 1 = 14$.

c = 6: We must have $a < b \leq 5$ and $a + b > 4$. The inequality will be satisfied for all choices of a, b except for $(a, b) = (1, 2)$ and $(1, 3)$, hence the number of choices is $\binom{5}{2} - 2 = 10 - 2 = 8$.

c = 5: We must have $a < b \leq 4$ and $a + b > 5$. The inequality will be satisfied only by $(a, b) = (3, 4)$ and $(2, 4)$, hence the number of choices is 2.

So the total number of choices available is $28 + 21 + 14 + 8 + 2 = 73$.

Problem 21. *Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$.*

Solution. Since the prime factorisation of 108 is $108 = 2^2 \times 3^3$, the only primes that divide a, b, c must be 2 and 3. We must 'distribute' two 2's and three 3's across a, b, c. The ways to distribute two 2's across a, b, c are the following:

- $2 + 0 + 0$, which has 3 permutations;
- $1 + 1 + 0$, which too has 3 permutations.

Hence there are $3 + 3 = 6$ ways of distributing two 2's across a, b, c.

The ways to distribute three 3's across a, b, c are the following:

- $3 + 0 + 0$, which has 3 permutations;
- $2 + 1 + 0$, which has 6 permutations;
- $1 + 1 + 1$, which has just 1 permutation.

Hence there are $3 + 6 + 1 = 10$ ways of distributing three 3's across a, b, c.

Any of the ways of distributing the twos can be combined with any of the ways of distributing the threes, hence the total number of ways is $6 \times 10 = 60$. This is the required answer.

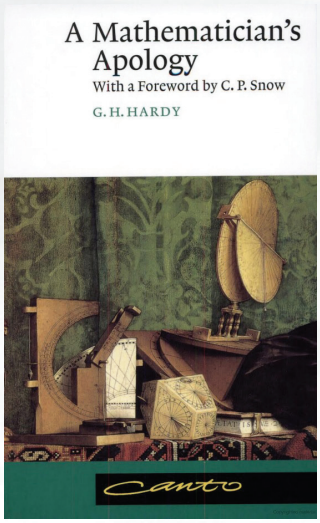


SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Review of G H Hardy's A MATHEMATICIAN'S APOLOGY

Reviewed by: R Ramanujam

R RAMANUJAM



Why an apology?

G. H. Hardy (1877–1947), a mathematician known for his deep contributions to Analysis and Number Theory, wrote this book in 1940, when he was 62 years old. It is more a long essay than a book, and remains interesting to this day, more than 75 years after it was published.[1] In what way is it relevant to us, especially mathematics teachers and students of today? To Indians, Hardy is best known for his discovery of Srinivasa Ramanujan and subsequent collaboration with him. For many all over the world, Kanigel's book *The Man Who Knew Infinity*, and a popular Hollywood film released last year based on the book, have made Hardy famous among the general public. The gaunt solitary Cambridge don that Hardy was, entirely intellectual, cricket loving, and awkward in social interactions, fits the popular image of a professor and mathematician. The style of the book reinforces the image in many ways.

The essay raises the question:

I shall ask, then, why is it really worth while to make a serious study of mathematics? What is the proper justification of a mathematician's life?

[1] In 1967, Hardy's book was re-published with a long foreword by C. P. Snow; it's a delightful biography of Hardy, and discusses in detail Hardy's collaboration with Littlewood and Ramanujan.

Keywords: Pure mathematics, usefulness, employability, aesthetics, truth, technique

In trying to answer these questions, he raises more: what criteria go into deciding what constitutes *good* mathematics? Should we distinguish pure and applied mathematics? Is it important that mathematics be *useful*? The essay is a long argument that attempts to answer these questions honestly and sincerely, in an autobiographical manner. It is an apology, not in the sense of seeking society's forgiveness, but in providing a detailed justification to society, for not only seeking to spend one's productive life on what seems a useless self-indulgent preoccupation, but also asking that society provide support for it. In the 21st century, when it is widely acknowledged that mathematics has tremendous life changing applications, one may be tempted to think that mathematics needs no justification. But one has to only look at modern science policy and its obsession with 'thrust areas' and 'applicability' to realise that "pure" mathematics cannot take public financing for granted. The recent stress on 'STEM education'[2] is indeed positive on mathematics education but it envisions mathematics as enhancing 'employability' and contributing to industrial development. It is hardly clear whether pure mathematics with no apparent applications would be considered justifiable in such a vision. Therefore, some justification is worthwhile.

Content

Quickly, Hardy disposes of two reasons why a person might do something all his life. The first is, "I do X because X is the only thing I do well." The second is, "This just came my way, so I do it." These might suffice for individuals, but perhaps not for the social worth of an activity. In essence, Hardy wishes to consider only those who do something well, and then ask, is the activity worth doing well? In that sense, why is mathematics worthwhile? I do not wish to go through all the hypotheses proposed by Hardy, with arguments and counter-arguments. In brief, and not necessarily in the order in which Hardy discusses them, he asserts the following.

1. Mathematical achievement is *permanent*. Unlike all other forms of knowledge, there is a certainty and permanence to mathematical truth, which makes pursuit of mathematical knowledge worthwhile.

2. Leaving behind something of permanent value is a noble ambition for anyone to undertake.

3. Mathematics is worth doing because its patterns are *beautiful*.

4. The best mathematics is not only beautiful, it is also *important*, as opposed to 'trivial' mathematics. In that sense, there is a seriousness to it, which lies in the *significance* of the ideas which it connects.

5. Seriousness of mathematical truth is characterized by its *generality* and its *depth*.

6. Good proofs are characterized by a high degree of surprise,[3] combined with *inevitability* and *economy*.

7. Mathematics is often considered to be important because it is *useful*. This (for Hardy) is a misconception. Pure mathematics is all about aesthetics, but mathematical *technique* is taught mainly through pure mathematics, and it is technique that is most useful.

8. Applied mathematics deals with reality, and hence is more useful than pure mathematics, but in some sense, applied mathematics is not 'real' mathematics (which deals only with abstract patterns).

There are many interesting notions here to reflect on: the character of mathematics, what constitutes good mathematics, and its usefulness.

[2] STEM stands for Science, Technology, Engineering and Medicine.
[3] Hardy uses the term 'unexpectedness'.

(i) Usefulness. This is what bothers Hardy most, and he builds elaborate arguments and draws surprising conclusions from them. For Hardy, pure mathematics is an *unprofitable, perfectly harmless and innocent occupation*. To him, it is 'unprofitable' in the sense of being unlike medicine where knowledge of the subject leads directly to making money. (In our society, with its craze for engineering and medicine, this is easy to understand.) It is 'harmless' in being not directly useful in warfare.

Poor Hardy was proved hopelessly wrong in just a few years after publication of the book. His beloved number theory was important in decoding the German Enigma machine during the Second World War. Worse, quantum mechanics (listed among 'useless' mathematics by Hardy in Section 26) was critically used in the making of the atomic bomb that caused unprecedented destruction in Hiroshima and Nagasaki. For a professed anti-war peace-activist like Hardy, this was heart-breaking.

In the century since, we have seen dramatic applications and use of what Hardy called 'pure' mathematics, in unanticipated areas. The theorems of Hardy and Ramanujan find newer applications to this day. So when Hardy says, *I have never done anything 'useful'*, he could not have been more wrong.

Therein lies a lesson for policy makers and governments who demand applications in 'thrust areas.' Mankind has been notoriously bad at predicting what knowledge will be useful in future.

There is another side to Hardy's distinctions between 'pure' and 'applied' mathematics. For Hardy, those aspects of mathematics that serve as tools for engineers and applied scientists are precisely the "boring" ones.

(ii) Character. Hardy is at his best when he explains his view of mathematics as a creative art. Perhaps the most famous quote from the essay is also one that beautifully expounds this view (section 10):

> *A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with **ideas**. . . . The mathematician's patterns, like the painter's or the poet's, must be **beautiful** ; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test; there is no permanent place in the world for ugly mathematics.*

Hardy admits that it may be difficult to define mathematical beauty but that does not prevent us from recognizing such beauty when we see it. Where does beauty reside in mathematics? In the intricacy of connections, in the surprise that such connections reveal, and in the significance and depth of such revelations. The significance is not in terms of the consequences, but in terms of its impact on mathematics itself. In all this, there is an emphasis on technique, which is critical. Here is Hardy again (section 8):

> *(The mathematician's) subject is the most curious of all – there is none in which truth plays such odd pranks. It has the most elaborate and the most fascinating technique, and gives unrivaled openings for the display of sheer professional skill.*

The latter statement might well apply to music, but the former, in relating to truth- seeking, distinguishes mathematics. This principally aesthetic character of mathematics is the central thesis of Hardy's essay.

(iii) Seriousness. Where does beauty reside in mathematics? Is it only in intellectual pursuit of abstraction? That is not so, and Hardy cites the example of Chess. The tremendous difficulty in playing Chess also consists of intricate work on abstractions. But this is 'trivial' mathematics for Hardy. This is often misunderstood and dismissed as arrogance.[4] For Hardy, what distinguishes chess problems and mathematics is that the

[4] There are many discussions on Hardy's book on the Internet, and many take this attitude.

former is particular and the solution (winning) is an end in itself, while the latter is universal and every solution leads to new questions, extensions and generalisations. It is this inherently unending quest that helps the mathematician build layer upon layer of abstraction, which in turn characterises *depth* of mathematical thought.

Hardy presents two theorems and their proofs to illustrate such aesthetics. Both are familiar to children in high school: one is Euclid's demonstration that there are infinitely many primes, and the second is Pythagoras's proof that $\sqrt{2}$ is irrational. Both proofs demonstrate the element of surprise and the virtues of inevitability and economy of notions that Hardy considers essential for what constitutes beauty and elegance in mathematics.

Relevance

All this sounds very philosophical, so one might well ask: in what way is Hardy's apology relevant to the mathematics teacher of today? In my opinion, the relevance is direct and often unacknowledged.

Schooling is compulsory and mathematics is a compulsory subject. Yet, barring a few, every child asks: why should I learn mathematics? It is important to grant the child the right to ask this question, and acknowledge that the question is asked not in a bright and inquisitive mood but in one of immense frustration. In this context, there are some questions that we need answer honestly and sincerely.

- Why should one study **mathematics?**
- Why should **I** study mathematics?
- What can a (reasonably sincere) student expect to learn from ten years of school mathematics?

If the first is answered by pointing to mathematics being useful in everyday life, the truth is that, except in a few professions, everyday use of mathematics rarely goes beyond what is taught in elementary school.[5] If the answer is 'use in science' (and this is indeed critical), calculators are able to take on that role nowadays.

We usually go farther, point to the use of mathematics in all disciplines, its immense applicability in life, and so on. But that is where the second question above is relevant. Even if I were to grant that mathematics is important, I could wonder why I should learn it, since others who like it could very well take on that important work, leaving me to do unimportant things that I like. Importance and usefulness to society are no source of comfort.

For those who get past the first two, who consent to learn mathematics, the third is still relevant, since what one learns through ten years of schooling may yet not be the important mathematics that was used to justify learning it in the first place! In what way is factoring monomials and manipulating trigonometric identities to be seen as useful either for my everyday life or for advancing applications in the world?

What we need is *A Mathematics Educator's Apology*, and Hardy provides us with a part of it.

The National Curriculum Framework (NCF) [NCERT, 2005] speaks of the mathematization of the child's thought processes as the main goal of mathematics education. The position paper of NCF [PP 2005] goes on to cite George Polya and distinguish the 'narrow aims' and the 'higher aims' of mathematics education. The former relate to knowledge and skills that contribute to economic development, what Hardy would call 'useful' mathematics. The latter relate to the aesthetic dimension Hardy is at pains to elucidate.

In its vision, NCF wants children to learn *important mathematics*, and asserts: "Understanding when and how a mathematical technique is to be used is always more important

[5] Sadly, the kind of mathematics that could be of immense use in most professions on an everyday basis like optimization and expectation are not taught in school at all.

than recalling the technique from memory." Once again, Hardy's distinction between 'real' mathematics and 'trivial' mathematics comes to mind.

The best answers we can give to children's questions above lie in viewing mathematics as a way of thinking, in the tremendous enrichment of "inner resources" it offers. This is not a luxury, but a process of realising one's potential. The aesthetic dimension of mathematics is often missed in school, and this impoverishes all of us. This dimension is reflected not in the content areas of mathematics, but in the wide range of processes at work in doing mathematics: formal problem solving, use of heuristics, estimation and approximation, optimization, use of patterns, visualization, representation, reasoning and proof, making connections, mathematical communication, etc. Indeed, according to the NCF:

> *Giving importance to these processes constitutes the difference between doing mathematics and swallowing mathematics, between mathematization of thinking and memorizing formulas, between trivial mathematics and important mathematics, between working towards the narrow aims and addressing the higher aims.*

Critical remarks

We cannot end this review without remarking on certain aspects of Hardy's style. There is much to offend in it too. It is a product of its time, and we cannot judge style across 70 years, but it is good to warn potential readers that it is implicitly sexist and highly Oxbridge-centric. There is a certain air of misanthropy ("Most people can do nothing at all well"), arrogance ("Is not the position of the ordinary applied mathematician a little pathetic?"), and contempt ("the intolerably ugly and incredibly dull school mathematics").

But Hardy does give us a real glimpse of what pure mathematics is all about, and ultimately, accomplishes what he sets out to do: a justification for doing mathematics.

Reference:

1. (NCF, PP): Position paper on teaching of mathematics by the National Focus Group, National Curriculum Framework – 2005, National Council for Educational Research and Training (NCERT), New Delhi 2006. http://www.ncert.nic.in/html/focus_group.htm.



R RAMANUJAM is a researcher in mathematical logic and theoretical computer science at the Institute of Mathematical Sciences, Chennai. He has an active interest in science and mathematics popularization and education, through his association with the Tamil Nadu Science Forum. He may be contacted at jam@imsc.res.in.

The Story of Maths
A BRIEF REVIEW

SHASHIDHAR JAGADEESHAN

If you want a whirlwind tour of the history of mathematics in four hours, where you will be taken to marvellous sites all around the globe, meet historians of mathematics, mathematicians, curators of museums, descendants of famous mathematicians and also learn some mathematics, then you must see the four-part series called The Story of Maths. This series, produced by BBC Four (http://www.bbc.co.uk/bbcfour) and presented by Professor Marcus du Sautoy, aired in October 2008. Prof Sautoy is the Charles Simonyi Professor for the Public Understanding of Science, and Professor at the University of Oxford, and is familiar to regular readers of *AtRiA*: his book *Symmetry* was reviewed in the March 2014 issue.

Part I, *The Language of the Universe*, covers the ancient mathematics of Egypt, Babylonia and Greece. Part II covers mathematics from China, India and the Middle East, and is called, predictably, *The Genius of the East*! We return to Europe in Part III (*The Frontiers of Space*) and Part IV (*To Infinity and Beyond*), with brief sojourns to both Russia and America.

The series attempts a general sweep of the history of mathematics from its very beginning to the work of the great French mathematician Alexander Grothendieck. This is an onerous task, because the body of knowledge that needs to be covered is vast, and one must avoid gender and ethnic bias. What impressed me the most is that the series avoids Euro-centrism by giving a decent account of contributions from various parts of the globe. The history of math

Keywords: Math resources, history, museums, mathematicians, Marcus du Sautoy

beyond antiquity ends up being the history of mathematicians. While we certainly meet many male mathematicians, the series does take pains to include women mathematicians like Hypatia, Emma Noether and Julia Robinson. I felt disappointed that we did not get to meet Andrew Wiles who proved Fermat's last theorem (which gets a very cursory mention) or Srinivasa Ramanujan. I liked the fact that even while eulogising some of the greats like Descartes, Newton and Gauss, we are not left with any illusions about what they were like as human beings!

A very strong feature of the series is the encounter with historians of mathematics, who share how they use resources to make interpretations. It is fascinating to learn, for example, from Dr Eleanor Robson of Cambridge University, about the intricacies of the Mesopotamian tablets that were actually exercises meant for students of mathematics, and a reinterpretation of what the famous tablet Plimpton 322 reveals about Pythagorean triples! As a student of history it is also thrilling to be handling classical resources (albeit vicariously), like the Rhind papyrus and original manuscripts of great mathematicians.

In terms of mathematics itself, the series gives us a bird's eye view of the most important mathematical ideas from the number system to the Riemann hypothesis. This is done mostly by introducing us to mathematicians and the great problems they were trying to solve. Interspersed are Marcus's notions of what mathematics is and why he finds it beautiful. I don't know if it was conscious, but the series takes great pains to portray Marcus as a dashing man of the world, who can ride horses, drive fast cars and hold his alcohol. A far cry from the nerdish stereotypical images of mathematicians!

The series really exploits the power of film as a medium. It is exciting to visit contemporary sites and learn about the ancient mathematics that was produced there, to see the Fort of Gwalior where zero is inscribed, to enter Newton's and Leibnitz's homes, and almost meet Grigori Perelman, the eccentric genius who proved the Poincaré conjecture. Graphics have been used in clever and creative ways to explain rather advanced mathematical concepts.

I would strongly recommend the series to all teachers of mathematics. With students of class VII and up, the series can be used in a variety of ways: as a very good introduction to the history of mathematics (with paper and pencil, and frequent breaks for explanations), to motivate students in learning new concepts and to give a flavour of how mathematics is done. Mini projects can be undertaken in various topics using the series as a starting point.

I would also recommend it to anyone intellectually curious, because it shows that mathematics does indeed belong to all human beings, and is a truly universal language.



SHASHIDHAR JAGADEESHAN received his PhD from Syracuse in 1994. He is a teacher of mathematics with a belief that mathematics is a human endeavour; his interest lies in conveying the beauty of mathematics to students and looking for ways of creating environments where children enjoy learning. He may be contacted at jshashidhar@gmail.com

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The Closing Bracket . . .

To love is one of the greatest gifts one can possibly have. As mathematics teachers, it would be wonderful if we can nurture in our students a love for the subject, a love that can last through life, a love that can protect us from the vicissitudes of life.

But what can we do in this regard? Do we have the means at our disposal that can nurture such a relationship with the subject?

There are at least two things that we *can* do. They are within the reach of every mathematics teacher, and the potential they hold for learning life lessons is very great.

The first is *to ensure that the subject makes sense to the student*. This may seem an utterly trivial suggestion, but there is more to it than meets the eye. A syndrome that plagues school mathematics is the lack of sense that it makes for students. Many students report that they feel quite capable of handling the intellectual challenge involved, but they simply do not see the point of it. They simply do not see the sense in it. This is not a recent development. The great mathematics educator WW Sawyer once wrote, many decades back, "In a class I was taking there was one boy who was much older than the rest. He clearly had no motive to work. I told him that, if he could produce for me, accurately to scale, drawings of the pieces of wood required to make a desk like the one he was sitting at, I would try to persuade the Headmaster to let him do woodwork during the mathematics hours – in the course of which, no doubt, he would learn something about measurement and numbers. Next day, he turned up with this task completed to perfection. This I have often found with pupils; it is not so much that they cannot do the work, as that they see no purpose in it."

Why is sense-making such an issue? Is it that in our hurry to get forward, our sense of urgency to transact the syllabus in the limited time available, we neglect to see how what we say connects with what the student already knows, how what we say connects with the world at large? Perhaps. If so, the price that we pay for this neglect is steep; and it is not the teacher who pays for it but the student.

The second is *to nurture a love for problem-solving*. The responsibility of the teacher is very great in this matter. To challenge students with just the right level of difficulty is itself a great challenge – for the teacher. It is utterly vital that the teacher starts from 'ground level' and inches his or her way forward, making sure at every point that the student stays connected with the problem. The importance of the experience of being able to successfully handle a challenging problem cannot be overstated. It can potentially make or break one's future relationship with the subject. This fact underscores the importance of choosing problems wisely, tailoring and customising them as needed. Clearly, the problem one poses must not be so easy that it presents no challenge; the exercise then becomes a bore, and we end up communicating quite the wrong message. Nor, to put it the way David Hilbert did, should the problem be "… so difficult that it mocks our efforts…"

Of equal importance is the role that the teacher plays: the conversational element, urging the student to frame the question in the most appropriate way, coaxing the student to organise his or her findings in the most useful way, challenging the student through the endeavour, staying in touch with the student when he or she gets stuck; all the while maintaining an affectionate, caring contact with the student.

It is worth pointing out here that we are not advocating problem-solving as an approach to teaching mathematics merely to become better at the subject; not at all. An intelligent use of problem-solving in the classroom can lead to an appreciation and love of the subject that perhaps cannot be nurtured using any other approach. And it can also connect with the first requirement listed above, that of sense-making. It can lead to a quality of engagement and a sense of happy aliveness that no other approach can bring. This aspect is of particular relevance at times like the present, when education has so largely lost its meaning, particularly at the higher level.

For example, consider this simple sounding problem which we can pose as a challenge to children in the middle school: given 10 dots spaced equally around a circle, find the number of triangles one can form with these dots as vertices. (See the webpage [1] for a similar problem.) The use of the number 10 here is not crucial, any reasonably large number will do. The number should be large enough that one cannot get the answer simply by listing all the possibilities; that would be too tedious. And it should be just large enough that different strategies are possible by which to get the answer. Consider the rich possibilities is that this problem opens up, in terms of group work, in terms of having to justify to a fellow student why the answer that we have obtained is reasonable, in terms of ensuring for oneself that we have not missed out any possibility and have not counted the same possibility more than once, in terms of working out the most systematic way of recording our results.

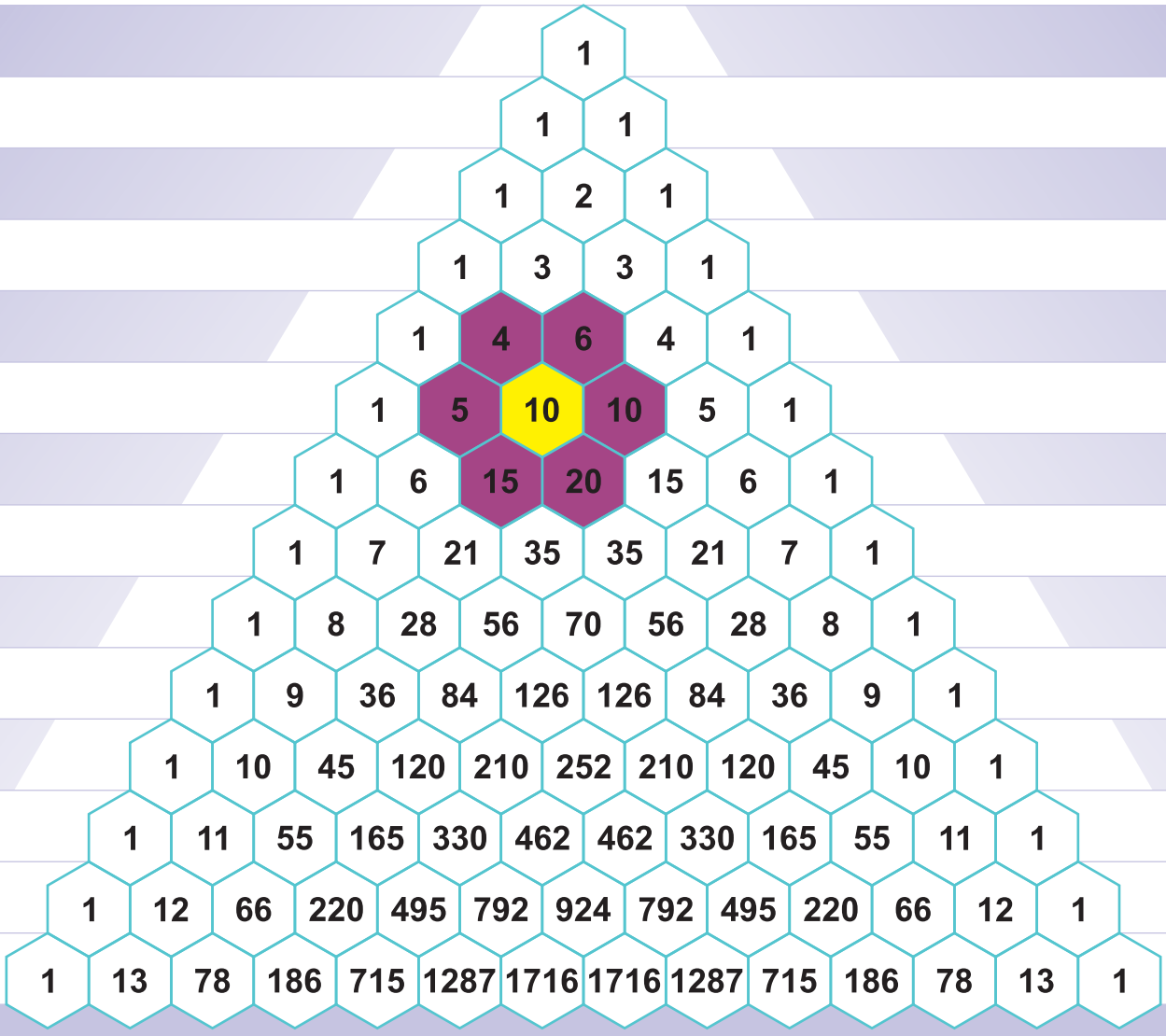
Ultimately, it is not the solution of the problem that matters, no matter how ingenious it may be; it is what the act of grappling with the problem has done to develop our inner resources and our stamina for sustained enquiry. And that matters, more than the mathematics…

Why don't we see more of these in our classrooms? I leave this question for us to ponder over.

Shailesh Shirali

[1] https://nrich.maths.org/5727

PASCAL'S PETAL



Here is an amazing bit of Number Theory based on Pascal's Triangle.

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Let us examine Pascal's Petal (or flower) where a petal is really 6 hexagons or numbers around a central 7th.

Observe any number that has 6 hexagons around it, like the number 10 shown in the diagram.

The highlighted number 10 is surrounded by 6 numbers or hexagons starting with 4.

Select every alternate number, i.e., (4, 10, 15).

Multiply these 3 numbers giving a product of 4 x 10 x 15 = 600.

What do you think happens when you select the other 3 remaining numbers in the ring of 6, i.e., (6, 20, 5) and you multiply these 3 numbers?

The answer 6 x 20 x 5 = 600 which is the same as before!

Does this neat property exist for all other Petals?
Can you prove why?

Send in your answer to AtRiA.editor@apu.edu.in

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5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.

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7. Provide a compact list of references, with short recommendations.

8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.

9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.

10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.

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14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.

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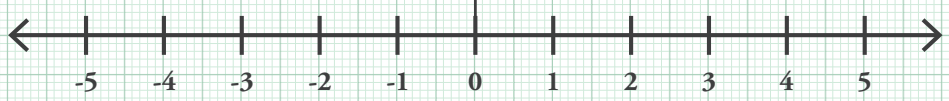
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At Right Angles

A Resource for School Mathematics

TEACHING INTEGERS

PADMAPRIYA SHIRALI



Azim Premji University

A publication of Azim Premji University
together with Community Mathematics Centre,
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INTEGERS

Preamble: It is important to state at the outset that manipulatives, which are designed to aid in the understanding of certain concepts, are not meant to be a replacement for the axiomatic development of the subject. Their purpose is essentially to enable students to construct their own mental models for mathematical concepts.

> Introduction of a new topic or a new concept is always a tricky issue. What is a good starting point? Should it begin with the known and then proceed to the new elements? That would make good pedagogic sense.
>
> Or should one probe students' exposure to the topic (applications) through their contact and familiarity with the world and begin there?
>
> Or should one pose questions for which answers do not exist in the past mathematical knowledge that students have acquired?
>
> Or can one create an artificial construct which models a concept?
>
> Perhaps there is no single best way. Every teacher bases such a decision on the knowledge level, age, exposure, readiness and inclination of the students.

The understanding of integers and negative numbers in particular poses certain challenges for many people. One difficulty lies with the fact that the negative sign appears along with a number to indicate an object, a number, but the same negative sign is also used for the operation of subtraction. The other difficulty lies in the way we read, for example: - (-2) as minus negative 2, which is not easily comprehensible. Some of these difficulties are addressed by making the transition from concrete to abstract slowly, as shown in this approach.

'Integers' as a topic is generally introduced in Class 6. Most text books talk about the contextual usage of integers (temperature, seabed, profit and loss, etc.) and use a number line to demonstrate addition and subtraction before progressing to multiplication and division. Some have begun to make use of counters of two colours as well.

I have shared here an approach which I have learnt from Shri P. K. Srinivasan. I have used it for many years successfully with many batches of students and shared it with many teachers in workshop settings. The approach has been received well and appreciated by several teachers who implemented it in their classrooms. It has been an easy task to share this approach in workshops where I could clarify various questions and doubts that were raised. I do hope that in this article, I am able to communicate clearly the various steps involved in this. I also hope that I manage to convey the charm and effectiveness of this approach.

Games have a great appeal for students in general and, in particular, for ten or eleven year olds. This approach begins with a game. It proceeds to teach addition and subtraction using the same materials and helps students to deduce the rules of addition and subtraction. Students find it easy to visualise the problem as it makes use of concrete materials in the initial stages. Also by persisting with the same material and approach for a few days, students are firmly rooted in a particular way of looking at it before encountering it in other forms of representation.

Introduction to Integers through a game

ACTIVITY 1: GAME OF OPPOSITES

Objective: To model the property of integers (opposites cancel each other).

Materials: Red and green buttons (20 of each), a small bag to hold them, a pair of dice.

Rule of the game: Two opposites cancel each other.

Aim: Who has the maximum number of pairs?

Students can be made into two teams.

The players in the first team throw the dice and collect that many buttons from the bag (without seeing the colour of the buttons). Then they lay out the buttons on the floor and remove one pair (a red and green pair) at a time. Then they count the pairs and record the number. Then they put back all the buttons into the bag.

In the same way, the players in the second team throw the dice, collect the required number of buttons, lay them out and count the pairs while removing them, then return the buttons to the bag.

The team with the maximum number of pairs gets a point. If they get the same number of pairs, neither team gets a point.



Figure 1



Figure 2: This shows 3 Pairs



Figure 3: This shows 5 Pairs

The game can be played for several rounds till some team gets all six pairs in a single round.

Students can be exposed to this game for a couple of days before moving on to the next step of teaching.

ACTIVITY 2

Objective: To model addition of integers

Materials: Red and green buttons (10 of each), two dice as shown in Figure 6 (one for colour and one for number), square ruled paper for each child to record results.

Aim: 12 green or red out!

This is a cooperative game. Students record the various steps in square ruled paper as shown in Figure 8.

At this point the teacher can introduce the words positive and negative, that the green button represents a positive and the red button represents a negative. "This is Positive 3." "This is Negative 4." And so on.



Figure 4



Figure 5

The red and green buttons are separated and placed separately on the table.



Figure 6



Figure 7

The first team throws the colour die followed by the number die to pick up the respective buttons and lay them out.

Example: If the colour die shows red and the number die shows 5, the team picks up 5 red buttons.

The second team throws the dice and lays out its buttons.

Example: If the colour die shows green and the number die shows 3, the team picks up 3 green buttons.

The buttons are brought together (combined - 3 greens and 5 reds) and read out by the teacher as: "Now we have positive 3 and negative 5".

Any pairs present are now removed. That leaves behind (2 reds) negative 2.The addition step is recorded as shown, both pictorially and in writing.

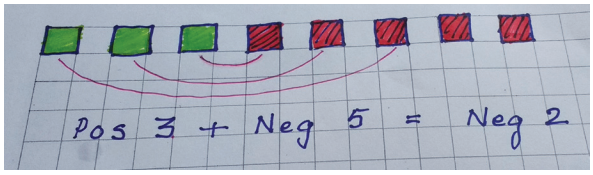


Figure 7a

If at any point the sum becomes (12 green or red buttons) positive 12 or negative 12, the game is over!

Note: This will give rise to all the three possible combinations – positive and positive, negative and negative, positive and negative (red and red, green and green, red and green). Since the numbers involved are small (all less than 7), students may be able to give the result without using the buttons. However, encourage them to record the results as it will help in building their ability to reconstruct it when necessary and aid as a visual memory.

As additional practice, the teacher can also write a few combinations on the board (using numbers less than 10) to get students to do individual work.

Positive 4, Negative 4

Negative 6, Positive 2

Positive 8, Negative 3

Some students who have grasped the concept may be able to give the answer without recourse to pictures. However, they should be encouraged to demonstrate their answers using pictorial models.

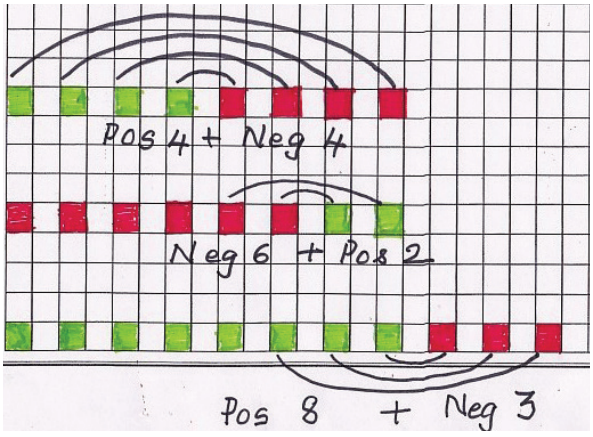


Figure 8

Note: In the initial stages, one uses the words positive and negative. At a later point, once the concept of addition and subtraction is understood thoroughly, the words can be replaced with the signs + and - . This is crucial, as students take time to feel comfortable with the new meaning assigned to + and - . If we can focus initially on the concept and introduce the sign gradually, that would make it easier for the students.

This effectively means that + (plus) and – (minus) will be used initially only to indicate the operations and not as a sign of the number. We use, instead, Pos and Neg to indicate the signs.

ACTIVITY 3

Objective: Oral work in addition of integers.

Aim: Practice in mental visualisation of addition of integers

Once the students have internalised the concept through visual means, they are ready for abstract thinking. The teachers can have oral sessions where larger rounded numbers can be given.

Example: Negative 75, Positive 40 or Positive 200, Negative 120, etc.

ACTIVITY 4

Objective: Rules of addition of integers.

Aim: Observing and deriving the rules of addition of integers

The teacher poses multiple questions for each type of addition situation in integers (Pos+Pos, Neg+Neg, Pos+Neg).

Example: Pos 5 + Pos 4, Pos 2 + Pos 3, Pos 7 + Pos 1.

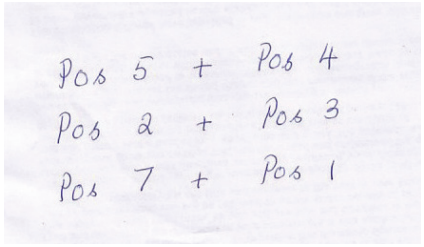


Figure 9

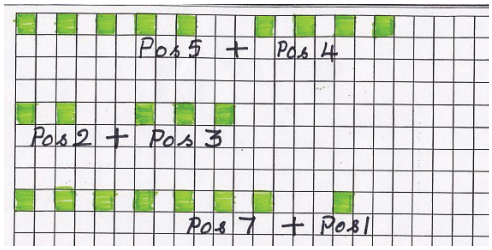


Figure 9a

Students use the materials and record the results on a square paper. They notice that a positive number added to a positive number results in an answer which is positive, and the number is the sum of the given two numbers.

Similarly questions are posed with two negative numbers.

Example: Neg 4 + Neg 6, Neg 2 + Neg 1, Neg 7 + Neg6

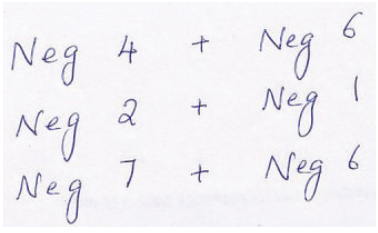


Figure 10

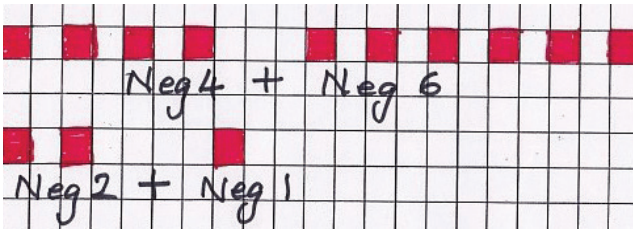


Figure 10a

Students use the materials and record the results on square paper. They notice that a negative number added to a negative number results in an answer which is negative, and the number is the sum of the given two numbers.

The next step is slightly challenging, that is, when we consider a positive and negative combination

Example: Pos 5 + Neg 4, Pos 1 + Neg 7, Neg 5 + Pos 2, Neg 4 + Pos 7

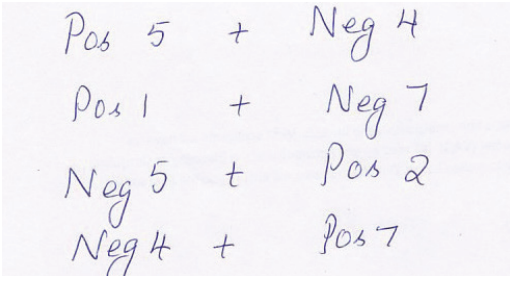


Figure 11

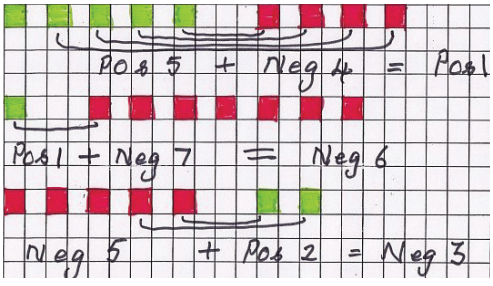


Figure 11a

Again, let them record all the problems pictorially.

Students will need to be guided with appropriate questions to discover the pattern.

The teacher can at first help the students to focus on the numerical part of the answer.

How does the answer connect with the two numbers?

They will be able to see that the number in the answer is the difference of the given two numbers.

How do we know if the result is positive or negative? The teacher points to a specific example and poses the questions, 'Are there more positives here or more negatives?' 'Does that affect the answer?'

Slowly, help the students to formulate the rule for addition of a positive and negative integer.

When a positive and a negative number are to be added, the number in the answer will be the difference of the two numbers and the sign will be that of the integer further from zero.

ACTIVITY 4A

Objective: To model subtraction of integers.

Materials: Red and green buttons (20 of each)

Aim: To be able to do subtraction using the materials.

Subtraction of integers can be classified into two types of situations. I have illustrated the two types using examples.

Situation 1: When there is enough to take away:

Pos 5 – Pos 2 or Neg 7 – Neg 1

Let students pick up buttons corresponding to Pos 5. Ask them to remove Pos 2 from them. There is no difficulty and the result is Pos 3.

Similarly, they pick up buttons corresponding to Neg 7 and remove Neg 1 resulting in Neg 6.

Again, students record these results on square paper.



Figure 12

Situation 2a: When there is not enough to take away and they are of opposite signs

Pos 3 – Neg 2

Ask students to pick up buttons corresponding to Pos 3. They will pick up 3 green buttons.

Now pose the question

'Is there any way of removing Neg 2 from these buttons?'

The initial answer will be a 'NO!'

However, persuade them to think further by giving some hints. 'I can remove Neg 2 from this only if I have Neg 2 here. Is there any way I can have Neg 2 here without changing the value of what I have here?'



Figure 13



Figure 14

'What would be the value of this (of Pos 3) if a pair of opposites is brought in here?'

Students see that it will not alter the value as pairs cancel out each other. Some student may come up with the suggestion that two pairs can be brought in and combined with these buttons.



Figure 14a

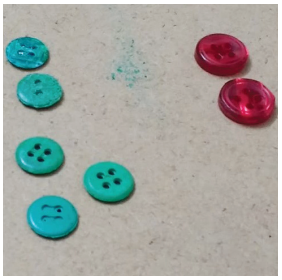


Figure 14b

Now there are Pos 5 and Neg 2 laid out. It is now possible to remove Neg 2.

What is left? Pos 5.

Hence the result for Pos 3 – Neg 2 turns out to be Pos 5.

It is of crucial importance to record this pictorially and write the corresponding steps numerically to understand the logic and the procedure that one follows later.

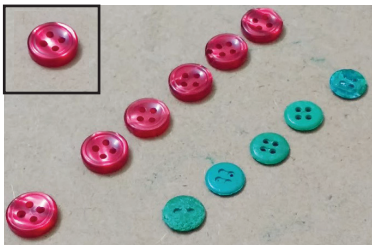


Figure 15a

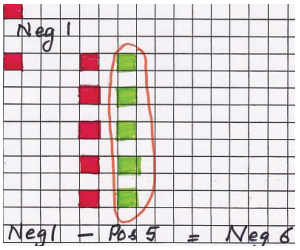


Figure 15b

The same would apply if the problem is Neg 1 – Pos 5.

Student picks up one red button to represent negative one. Now Pos 5 has to be removed.

How many pairs need to be brought in? 5 pairs.

Now there are Neg 6 and Pos 5 laid out. Once Pos 5 is removed there will be Neg 6 left.

Hence the result for Neg 1 – Pos 5 turns out to be Neg 6.

Situation 2b: When there is not enough to take away and they are of same sign

Pos 3 – Pos 5

What should be done here? Students will pick up 3 green buttons to represent Pos 3. How does one remove Pos 5? A few students will quickly see that they can apply the same approach as before, i.e., bring in 5 pairs. Now there are Pos 8 and Neg 5. Pos 5 is now removed.

Three pairs which are present are also removed. What is left?

Hence Pos 3 – Pos 5 turns out to be Neg 2.

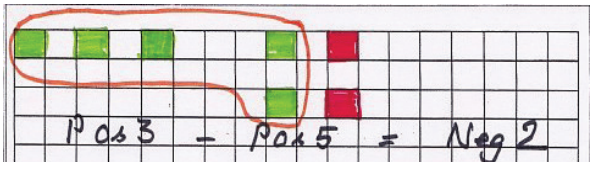


Figure 16

However, some others may suggest that it may be much simpler to bring in only 2 more pairs, as what needs to be removed is Pos 5. In that situation there are Pos 5 and Neg 2 buttons. Once Pos 5 is removed, Neg 2 is left.

So Pos 3 – Pos 5 is Neg 2.

Both approaches are valid; the second one is more efficient.

Students will need to practise doing several of these problems with materials, represent them pictorially and record the steps numerically.

While it is possible to derive rules specifically for different subtraction situations, it becomes cumbersome with too many rules and conditions to remember. Instead, once the students become comfortable with the subtraction concept as shown earlier, it is easier to show that the subtraction of an integer is the same as the addition of its additive inverse. This corresponds to the numerical procedure that we follow normally. This is established in the next activity.

ACTIVITY 5

Objective: To show the link between the concrete method and the numerical procedure

Materials: Square paper

Aim: To understand that subtraction is achieved by adding the additive inverse of the second number

Pose the question Pos 3 – Neg 2.

Let the students record Pos 3 at first in the drawing.

Now comes the crucial step of helping students notice that removing Neg 2 from Pos 3 is effectively leading to adding two pairs (Pos 2 and Neg 2).

Pos 3 is the same as Pos 3 + Pos 2 + Neg2.

Point out that Pos 2 is called the additive inverse of Neg 2, as Pos 2 and Neg 2 make zero.

Now show that Neg 2 can be removed to leave behind Pos 5.

Help them to articulate that subtraction can be achieved by adding the additive inverse of the integer that is to be subtracted.

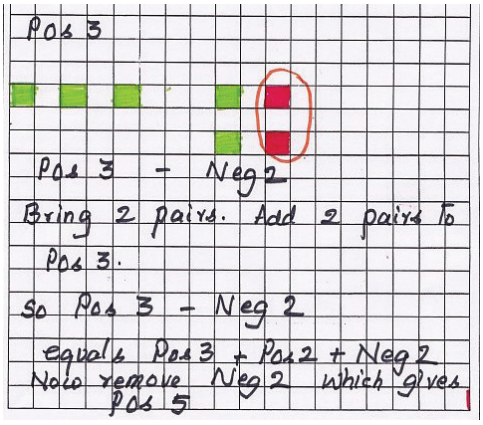


Figure 17

ACTIVITY 6

Objective: To show opposites on a number line and introduce the usage of + and - signs.

Materials: Chalk

Aim: To understand positive and negative numbers and that numbers with a negative sign are less than zero.

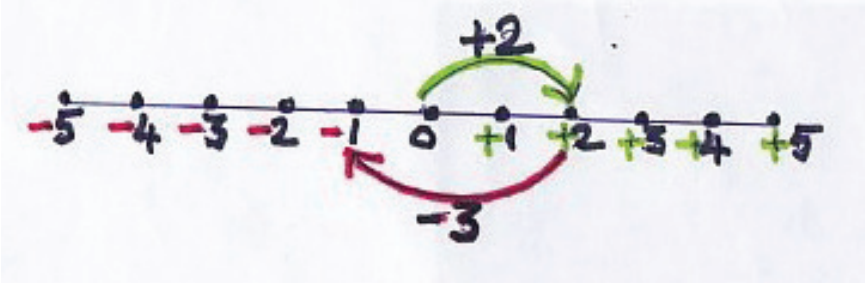


Figure 18

The teacher asks the student to take three steps. The student may take three steps forward without any further questions. The teacher can point out to the child that he or she has not yet specified a direction.

The teacher can now draw a line on the floor and mark 0, 1, 2, 3, 4 and 5. Let the child stand on zero and the teacher repeats the instruction 'take two steps forwards.' Student moves to Point 2. 'Take three steps backwards. ' Student finds that he has to go beyond zero and moves to a new position behind zero.

The teacher now explains that the new position is labelled as negative one. The number line is extended further now on both sides from Positive 10 to Negative 10.

At this point the teacher can introduce the signs for positive and negative, 'Positive' is indicated by a '+' sign and 'Negative' is indicated by a '-' sign'. Numbers are now marked on the number line using the signs.

Now the teacher can mention that '+' is often omitted, i.e., + 2 is written simply as 2.

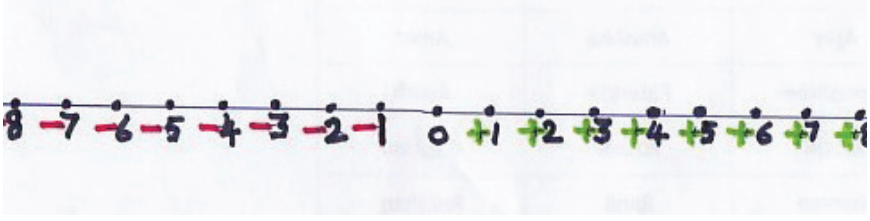


Figure 19

Point out that numbers get smaller as they move away from zero on the left and get bigger as they move away from zero on the right. Also for any two numbers on the number line, the one on the left is smaller than the one on the right. Therefore -3 < -1, although 1 < 3.

Now, using the number line students can compare numbers, determine the bigger and the smaller, successor and predecessor, and derive the rules for comparing numbers.

ACTIVITY 7

Objective: To show usage of integers in the context of temperature

Materials: Temperature chart of some cold places

Aim: To understand the positive and negative numbers in the context of temperature

The teacher discusses the concept of temperature and how it is measured with a thermometer in degrees Centigrade. Temperature chart of their area can be studied together at first. What is the normal temperature on a hot summer day? What would be the night temperature in winter? What would be the temperature in the refrigerator? The teacher can inform the students that water freezes at 0° Centigrade and boils at 100° Centigrade.

Now the teacher can mention that in very cold places the temperature often falls below 0° Centigrade. The students' attention should be drawn to the way the information is recorded, the usage of – sign and the letter C to denote Centigrade.

The teacher and students can discuss the temperature chart of some cold places.

Various comparative statements can now be made based on the drawing.

- Which is the coldest place?
- Which is the warmest place?
- Which places have temperatures below 0°C?
- Which place is 5° higher than -8°C?
- Which place is 2° lower than -10°C?
- How many degrees difference is there between the coldest and the warmest place?

ACTIVITY 8

Objective: To show addition of integers on a number line

Materials: Chalk

Aim: To associate forward movement with positive and backward movement with negative

Till now students have associated positive and negative with positions on the number line. They will also associate adding a positive with a forward movement and adding a negative with a backward movement.

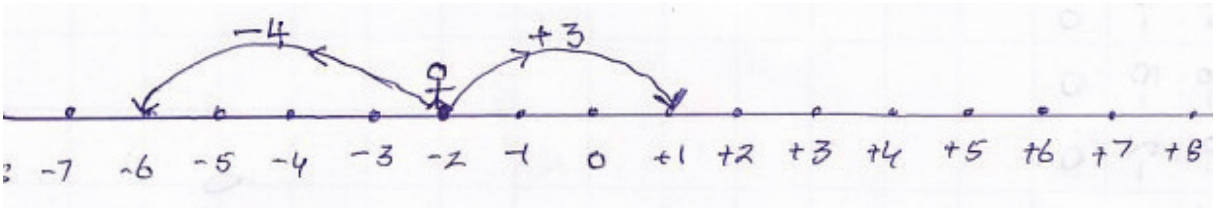


Figure 20

Let a number line be drawn on the ground depicting numbers from -10 to +10. Help the children to understand that adding +2 implies moving forward by two steps from the current position. Adding -3 implies moving backward by 3 steps from the current position.

Students can be asked to demonstrate a problem stated by the teacher.

Example: (-2) + (+3)

Student starts at -2 on the number line and takes 3 steps forward to show addition by +3 and reaches +1 position.

Example: (-2) + (-4)

Student starts at -2 on the number line and takes 4 steps backward to show addition by -4 and reaches -6 position.

Various such problems can be tried out and modelled till all students acquire understanding.

They could also do it the reverse way.

One student demonstrates a problem. Other students have to state the addition fact.

At this point the reader may expect to see subtractions on the number line as the next activity. While it is possible to demonstrate subtractions, procedurally it gets more complex. Subtraction would involve an inverse movement on the number line which becomes difficult for students to comprehend.

I have personally found the red and green button approach adequate. However, if the reader wants to pursue subtractions using a number line, they could consult the NCERT book for Class 6.

Note: It is necessary now to help the students to see the link between the earlier approach that has been followed and the standard methods employed while working with numbers. Teachers may choose to use brackets if it is helpful.

For example:

- Pos 5 + Neg 4 is now written as 5 + (-4) = 1
- Neg 4 + Pos 7 is now written as (-4) + 7 = 3
- Pos 3 – Neg 2 is written as 3 - (-2) = 3 + 2 = 5
- Pos 3 – Pos 5 is written as 3 – 5 = 3 + (-5) = -2

ACTIVITY 9

Objective: To show multiplication of integers using buttons

Materials: Red and green buttons (20 of each)

Aim: To demonstrate multiplication

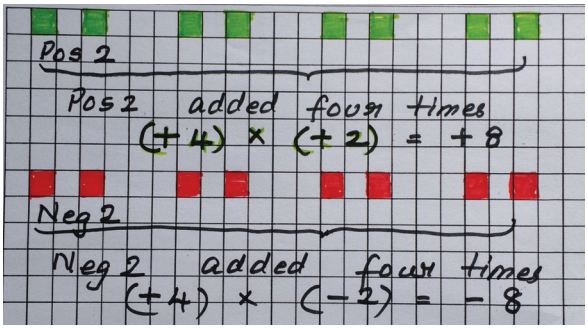


Figure 22

Example: (+4) x (+2)
This is to be read as Pos 2 added four times.
So (+4) x (+2) = +8

Example: (+4) x (-2)
This is to be read as Neg 2 added four times.
So (+4) x (-2) = -8

Example: (-4) x (-2)
This is to be read as Neg 2 subtracted four times.

There is nothing to start with to subtract from. So we bring in the required number of pairs.

Neg 2 is brought in four times along with its opposite Pos 2.

Now we remove (-2) four times. That leaves behind +8.
So (-4) x (-2) = +8

Example: (-4) x (+2)
This is read as Pos 2 subtracted four times.

Again we have nothing to start with. So we bring in two pairs four times.

Now we can remove Pos 2 four times leaving behind Neg 8.

So (-4) x (+2) = -8

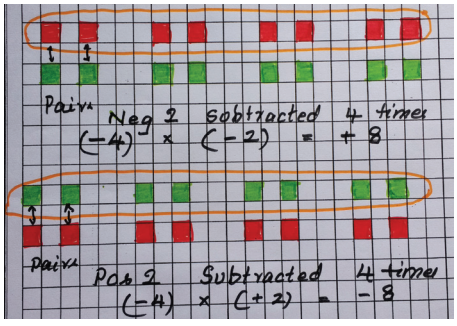


Figure 23

ACTIVITY 10

Objective: To show multiplication of integers

Materials: Square paper

Aim: To derive the rules of multiplication of integers through patterns

Let the students create a table as shown in Figure 21.

In the first step, students are expected to fill in the first quarter of the box.

In the second step, students observe each row (the way the number is decreasing) and continue the pattern beyond zero to complete each row.

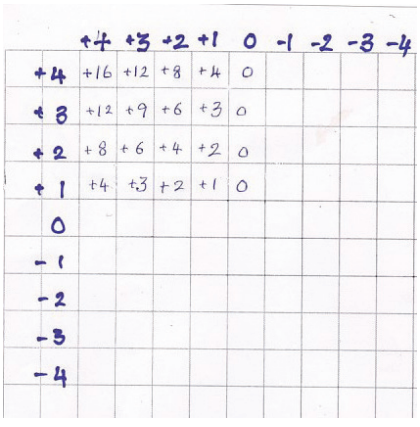


Figure 21



Figure 21a

In the third step, students observe the columns one by one from +4 to 0 and fill in the empty squares.

Now comes the crucial part. The pattern can be continued row wise or column wise. Whichever way it is done, students will see that numbers now start from negative side and increase in steps of 1 or 2 or 3, etc.

The teacher needs to take extra care to point out that now rows and columns start with negative numbers and increase gradually and that the numbers after zero turn positive.

Once the table is filled up and verified by the teacher students can begin to derive rules for multiplication of integers.



Figure 21b

- What happens when a positive number is multiplied by a positive number?
- What happens when a positive number is multiplied by a negative number?
- What happens when a negative number is multiplied by a positive number?
- What happens when a negative number is multiplied by a negative number?

In my experience, I have found the pattern approach simple and satisfactory; children appreciate it.

Activity 10A, below, describes yet another way of demonstrating multiplication (as suggested by PK Srinivasan).

ACTIVITY 10A

Objective: To show multiplication of integers using buttons

Materials: Red and green buttons (20 of each)

Rule: Positive stands for repeating in the same way (repeating the same colour)

Negative stands for repeating the opposite way (repeating the opposite colour)

Aim: To derive the rules of multiplication of integers

Example: Demonstration of (+4) x (+2)

How do we read this? → This can be read as positive 2 repeated positive 4 times.

How do we understand this? → Positive 2 is clear, it means 2 green buttons.

Repeating positive 4 times implies repeating four times in the same way.

So 2 green buttons are repeated four times which makes them equal 8 green buttons.



Figure 5

Example: Demonstration of (+2) x (-2)

This can be read as negative 2 repeated positive 2 times.

Repeating positive 2 times implies repeating two times **in the same way.**

Hence (+2) x (-2) equals -4.



Figure 22a

Example: Demonstration of (-2) x (+4)

Here 4 green buttons are repeated twice but in the opposite way (meaning that the colour is changed).

4 green buttons are replaced by 4 red buttons and repeated twice.

Hence (-2) x (+4) equals -8.



Figure 23a

Example: Demonstration of (-2) x (-2)

2 red buttons are repeated twice in the opposite way.

Hence we will get 4 green buttons.

(-2) x (-4) = +8.

ACTIVITY 11

Objective: To show division of integers

Aim: To apply the rules of multiplication of integers to solve division problems

Rules for division flow naturally from the multiplication rules.

Hence, division problems are approached basically by using related multiplication facts.

Example: -10 divided by +5 = -2 (using +5 x -2 = -10)

GAME1: RACE TO THE ENDS!

Objective: To practise addition of integers

Materials: Number line and counters, integer cards (-1 to -10, +1 to +10)

Aim: Who reaches +20 or -20 first?

Players use two differently coloured counters. Both of them start at zero. The integer cards are kept upside down. Each player picks up one integer card and moves that many steps either forward or backwards on the number line based on the card he gets.

Example:

If a player gets -2 first he moves his counter from zero to -2. In the next round if he gets -4 he moves his counter from -2 to -6. Whoever reaches + or – 20 first is the winner.

GAME 2: STRIKE OUT!

Objective: To practise addition and subtraction of integers

Materials: 2 number dice

Aim: To strike out all the numbers from -5 to +12

Players draw a number line marked from -5 to +12. They throw the dice and use the numbers on the dice along with addition or subtraction operation to make up as many numbers as possible between -5 and 12. They strike out the numbers they have been able to make. Each player takes turns to throw the dice and strikes out some numbers. Whoever strikes out all the numbers first is the winner.

Acknowledgement: Thanks to Ms Swati Sircar for her suggestions.



Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box' Padmapriya may be contacted at padmapriya.shirali@gmail.com