Azim Premji University

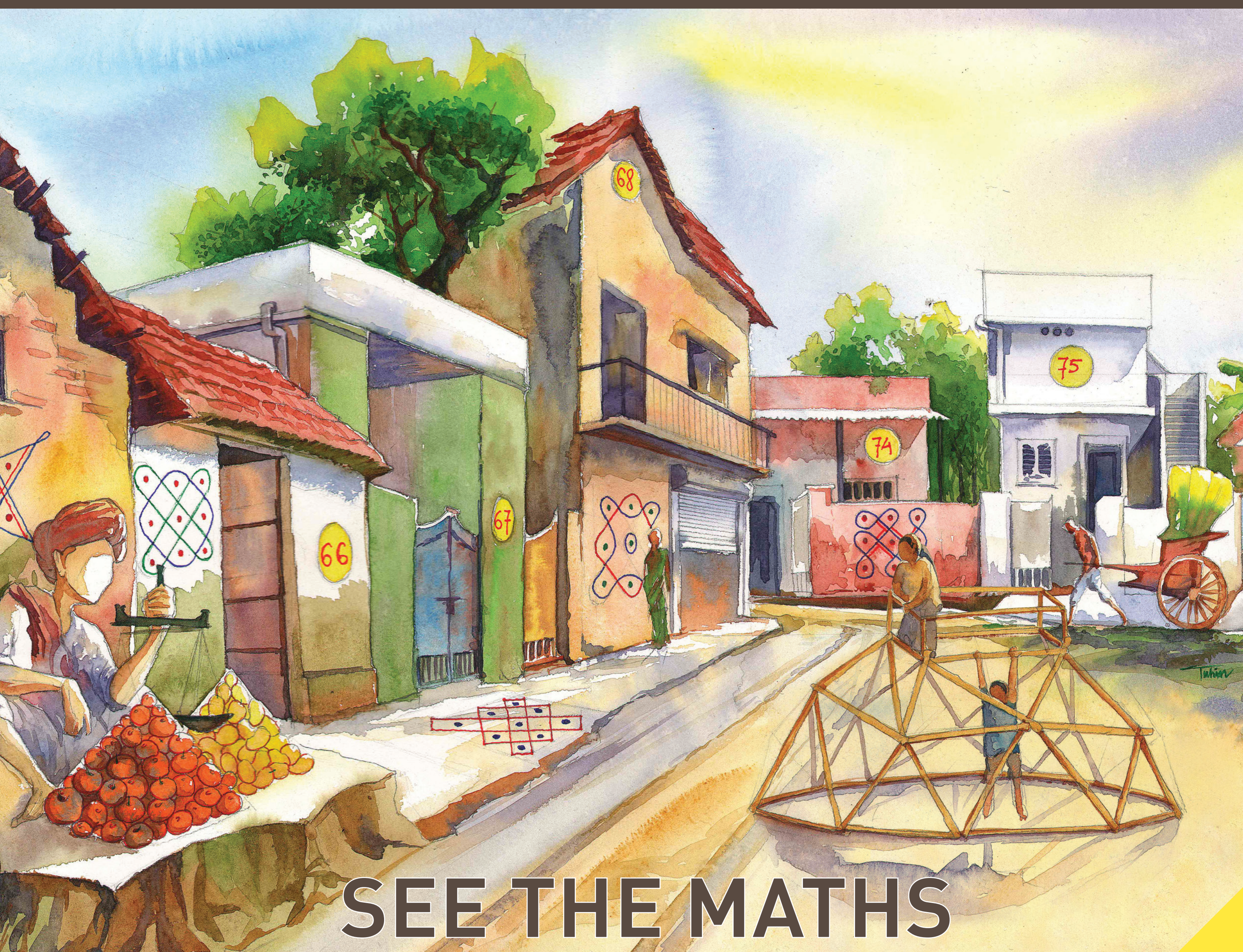
A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

COMMUNITY MATH CENTRE
RISHI VALLEY SCHOOL (KFI)

Volume 7 | Issue 1 | March 2018
Triannual | Bengaluru

Azim Premji University
At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS



SEE THE MATHS

PULLOUT
INTRODUCTION TO ALGEBRA

Rustling leaves, swirling dust motes, long lazy afternoons........our cover picture brings to mind rural India at its rustic best. What could be more enchanting and relaxing? Take a closer look and you will be instantly captivated - there are mathematical patterns in the house numbers, in the rangoli designs, in the wooden contraption, even in the pile of fruits! You will find more about these patterns in the articles in this issue, we hope they encourage you to go out and find beautiful math in the world around us!

From the Editor's Desk . . .

About this issue…

Celebrating the artist and his craft is not often associated with the science of mathematics. This issue of *At Right Angles* opens with just that – a series of problems based on Triangular Numbers presented by three mathematicians at the Institute of Mathematical Sciences, Chennai. This is followed by an article on S Ramanujan by Utpal Mukhopadhyay in which several well-known (thanks in part to the movie *The Man Who Knew Infinity*) incidents in this mathematical genius' life are peppered with his innovative solutions to these problems among others. Look out for a quick refresher on continued fractions. Later in the magazine, our student contributor, Bodhideep Joardar, also presents an innovative solution to the door number problem. Features continues with three articles by V. G. Tikekar, Shailesh Shirali and K.D. Joshi, these mathematicians introduce us to different aspects of the Power Triangle and its applications. More power to our readers as they go beyond the still fascinating but too well-known Pascal Triangle.

The Conjecturing Classroom is a hands-on recount of Anushka Rao Fitzherbert's experience with this pedagogical strategy – the article is packed with useful leads for teachers. Next, K. Subramaniam talks about an all-too familiar error made by students struggling with Algebra. This is followed by *Misconceptions in Fractions*. All very useful material for teachers and teacher educators.

Now for some nice math activities – we have a lovely construction for the Harmonic Mean as well as a *Proof Without Words* for the Arithmetic Mean- Geometric Mean inequality. Later in the magazine, Shailesh Shirali addresses the same inequality in the fourth part of his *Inequalities* series. And this article concludes with an interesting strategy to devise your own inequalities! Tejash Patel shares an innovative strategy on finding the GCD and LCM of two numbers and Seetha Rama Raju explains Euclid's method for calculating the same. There's more – the ClassRoom section will definitely add to your repertoire of mathematics and mathematics pedagogy.

Elementary Cellular Automata by Jonaki Ghosh and Rohit Adsule brings TechSpace to the cutting edge with investigations in mathematics using technology. And Michael de Villiers continues his discoveries in the land of quadrilaterals, it's the cyclic Kepler quadrilateral this time. The usual features in Problem Corner with an interesting add-on: AtRiA now has a GeoGebra Tube account called **AtRiATechSpace**! We have launched it with two dynamic worksheets based on the Middle School extension problems. The link is given in the article; do give us your feedback! Going forward, we plan to provide explorations to illustrate and supplement more of our articles.

Everyone is talking about the book that features in Review – *Beautiful, Simple, Exact, Crazy,* by Apoorva Khare and Anna Lachowska. This is a book that Rema Krishnaswamy (one of the reviewers, along with Jishnu Biswas) actually used as a resource while teaching a pre-calculus course at Azim Premji University. Their review is

From the Editor's Desk . . . (Contd.,)

sure to make you want to read this book. In *Mini Reviews*, Swati Sircar shares her opinion about *Weird Numbers*, a short film available on YouTube. It's a toss-up about which is better – the book or the movie!

Padmapriya Shirali takes on a tough one this time, introducing Algebra at the Upper Primary level. This PullOut is sure to be a keeper.

Feedback as usual to AtRiA.editor@apu.edu.in or on our FaceBook page AtRiUM, looking forward to hearing from you.

Sneha Titus
Associate Editor

Chief Editor

Shailesh Shirali
Sahyadri School KFI and
Community Mathematics Centre,
Rishi Valley School KFI

Associate Editor

Sneha Titus
Azim Premji University

Editorial Committee

Athmaraman R
Association of Mathematics Teachers of India, Chennai

Prithwijit De
Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research, Mumbai

Hriday Kant Dewan
Azim Premji University

Jonaki B Ghosh
Lady Shri Ram College for Women, University of Delhi

Sangeeta Gulati
Sanskriti School, Delhi

Giridhar S
Azim Premji University

Shashidhar Jagadeeshan
Centre for Learning, Bangalore

DD Karopady
Azim Premji University

A. Ramachandran
Formerly of Rishi Valley School KFI

Tanuj Shah
Rishi Valley School

Padmapriya Shirali
Sahyadri School KFI

Swati Sircar
Azim Premji University

K. Subramaniam
Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research, Mumbai

Sneha Kumari
Azim Premji University

Design

Zinc & Broccoli
enquiry@zandb.in

Print

SCPL
Bangalore - 560 062
www.scpl.net

Please Note:
All views and opinions expressed in this issue are those of the authors and Azim Premji Foundation bears no responsibility for the same.

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Azim Premji University

A Publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley and Sahyadri School, Pune

Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Azim Premji University

A Publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley and Sahyadri School, Pune

Contents contd.

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

TRIANGULAR NUMBERS

AMRITANSHU PRASAD & VIJAY RAVIKUMAR

Illustrated by MADHUSHREE BASU

Here is a popular story about the famous German mathematician Carl Friedrich Gauss (1777–1855). Hoping to get some rest while keeping the students busy, Gauss's mathematics teacher asked them to add up the numbers $1 + 2 + \cdots + 100$. The seven-year-old Gauss instantly found out the answer to be 5050. (*Comment.* The historical accuracy of this legendary story is questionable. In [1], Brian Hayes tries to find the origins of this story.)

Figure 1

Keywords: Gauss, triangular numbers

How could Gauss have added a hundred numbers so quickly? One possibility is, he wrote the sum *twice*, once in the forward direction and then in the backward direction:

$$\begin{array}{ccccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 2 & + & 1 \end{array}$$

Observe that $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, and so on. So adding up the numbers in each column gives:

$$101 \quad + \quad 101 \quad + \quad 101 \quad + \quad \cdots \quad + \quad 101 \quad + \quad 101 \quad (100 \text{ times})$$

which shows that the sum of the first hundred numbers is half of 100×101, which is 5050.

This trick can be used to add up as many numbers as we like. For example, $1 + 2 + \ldots + 1000$ would be $(1000 \times 1001)/2$, which is 500500.

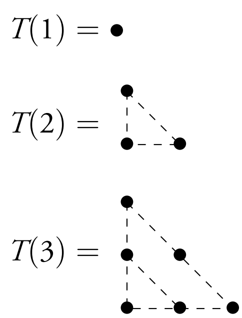
The sum of the first n natural numbers is called the n-th *triangular number*:

$$T(n) = 1 + 2 + \ldots + n.$$

The first few triangular numbers are:

$$\begin{aligned} T(1) &= 1 \\ T(2) &= 1 + 2 = 3 \\ T(3) &= 1 + 2 + 3 = 6 \\ T(4) &= 1 + 2 + 3 + 4 = 10 \end{aligned}$$

They are called *triangular numbers* because they count the number of objects that can be stacked up in triangles:



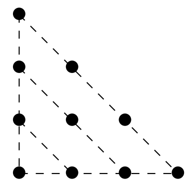
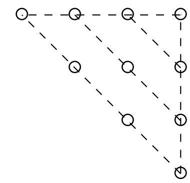
A formula for the n-th triangular number. Using Gauss's trick:

$$\begin{array}{rcccccccccc} T(n) & = & 1 & + & 2 & + & \cdots & + & n-1 & + & n \\ T(n) & = & n & + & n-1 & + & \cdots & + & 2 & + & 1 \\ \hline 2T(n) & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) \end{array}$$

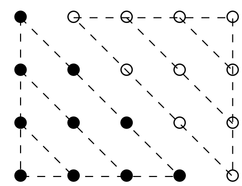
So

$$T(n) = \frac{n(n+1)}{2}. \tag{1}$$

Explaining the formula using pictures. This formula for $T(n)$ can be explained with pictures as well: For example, the dots in $T(4)$ can be visualized as

$T(4) =$  , or $T(4) =$  ,

so that

$T(4) + T(4) =$  ,

The rectangle on the right has 4 rows and 5 columns. So

$$T(4) + T(4) = 4 \times 5,$$

or $T(4) = (4 \times 5)/2$.

Similarly, for any n, $2T(n)$ points can be arranged to form a rectangular array with n rows and $n + 1$ columns, giving us $2T(n) = n \times (n + 1)$, which we had earlier obtained using Gauss's trick.

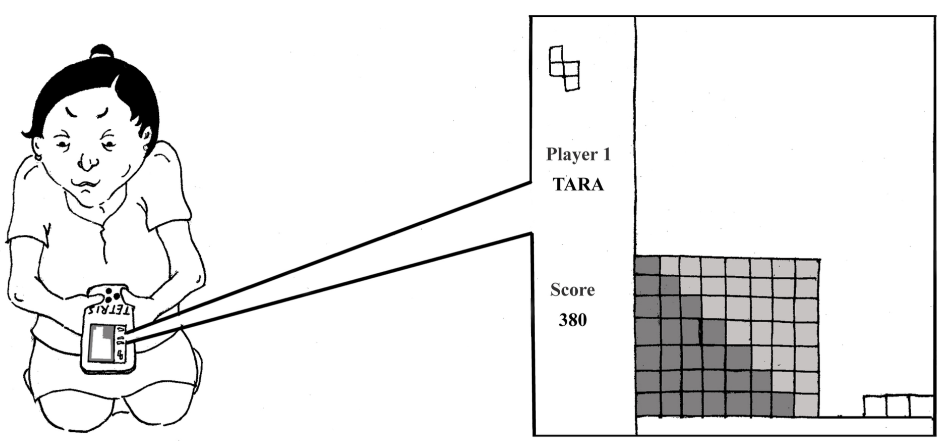


Figure 2

Why would we be interested in triangular numbers? Besides outwitting sadistic teachers, are there other reasons why we should be interested in triangular numbers? To see a couple of examples, try these exercises:

Exercise. In a *round robin tournament* each team plays every other team exactly once. For example, the 32 teams in the FIFA world cup are divided into eight groups of 4 teams each. In the qualifying round, each group plays a round-robin tournament. For example, Group A in the 2014 FIFA world cup had Brazil, Croatia, Mexico and Cameroon. How many matches were played in Group A during the qualifying round? What if the world cup format is changed so that each group has six teams. How many matches will have to be played within each group in the qualifying round?

Solution. Brazil must play each of Croatia, Mexico and Cameroon. This requires three matches. Now let's come to Croatia—we have already taken care of Brazil versus Croatia. It remains for Croatia to play Mexico and Cameroon—that's two more matches. Finally we need a match between Mexico and Cameroon to complete the round robin. Each ✓ in the following table represents a game to be played in this group:

	Brazil	Croatia	Mexico	Cameroon
Brazil		✓	✓	✓
Croatia			✓	✓
Mexico				✓
Cameroon				

Looks familiar, doesn't it? This is the triangular number $T(3)$. Similarly, if there were six teams in a group, the number of games would be $T(5)$.

Exercise. Tara's job is to set up a high-speed network of 24 computers at the university. Each of these computers is required to be connected directly to all the other computers in the network. How many cables will Tara need?

Figure 3

Solution. The first computer will need to be connected to 23 other computers. Once this is done, the second computer will need to be connected to 22 other computers (it is already connected to the first). After this, the third computer will need to be connected to 21 other computers (it is already connected to the first two), and so on. The total number of cables needed will be

$$23 + 22 + \cdots + 2 + 1 = T(23) = \frac{23 \times 24}{2} = 276.$$

How do we know if a number is triangular? Here is a recipé: if $8N + 1$ is a perfect square, then N is triangular. Otherwise N is not triangular. For, example, for the first few triangular numbers, we have:

N	1	3	6	10	15	21
$8N + 1$	9	25	49	81	121	169

Indeed, if $8N + 1$ is the square of a positive integer M, then

$$8N + 1 = M^2,$$

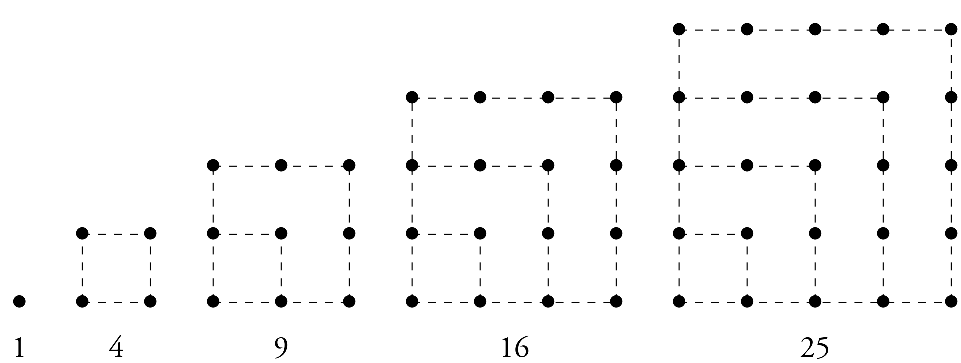
and M has to be odd, because its square is odd. It follows that $M - 1$ is even, so $n = (M - 1)/2$ is a positive integer. We have

$$\begin{aligned} T_n &= \frac{n(n+1)}{2} \\ &= \frac{M-1}{2} \times \frac{M+1}{2} \times \frac{1}{2} \\ &= \frac{M^2 - 1}{8} \\ &= N. \end{aligned}$$

This shows that each odd perfect square corresponds to a triangular number, and each triangular number corresponds to an odd perfect square.

Numbers based on other shapes

There are other sequences of integers based on geometrical shapes. The square numbers:



are given by the simple formula

$$S(n) = n^2,$$

while the pentagonal numbers:

$$1,\ 5,\ 12,\ 22,\ 35,\ \ldots$$

come from drawing pentagons of increasing size.

Figure 4

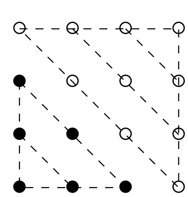
Pentagonal numbers also come with a nice formula:

$$P(n) = \frac{n(3n-1)}{2}.$$

These number sequences sometimes have relationships between them. The sum of two consecutive triangular numbers is always a square number:

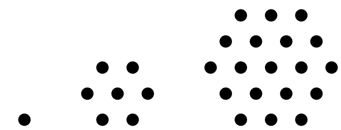
$$T(n) + T(n+1) = (n+1)^2.$$

This can be illustrated geometrically by merging triangles. For example, the figure below illustrates that $T(3) + T(4) = 4^2$.


,

Exercise. When Gauss dashed his mathematics teacher's hopes for a long break with his quick thinking, the teacher came up with another problem for Gauss to solve. "OK Smart Alec!" he said. "Let's see how you get around this one. Add up the first hundred *odd numbers*." So now they had to add $1 + 3 + 5 + \cdots + 199$. Barely a moment had passed when Tara's eyes lit up and a smile crossed her face. She whispered something to Gauss, who laughed out loud when he realized they'd outwitted the teacher again! Can you figure out what Tara told Gauss?

Exercise. Let's explore two more number sequences that arise from counting the dots in geometric figures. The first is the sequence of *centered hexagonal* numbers:



1 7 19

Can you find a formula for the n-th hexagonal number in terms of the triangular numbers?

Exercise. Have you ever seen a fruit stand selling guavas, *mosambis*, or pomegranates? Can you remember how the vendor stacked the fruits? It's very likely they were stacked in square-based pyramids. How many fruits are needed to make a square pyramid with n levels of fruits? Do you know of a closed formula for this number?

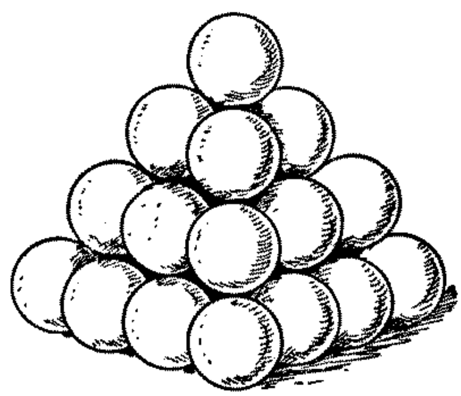


Figure 5

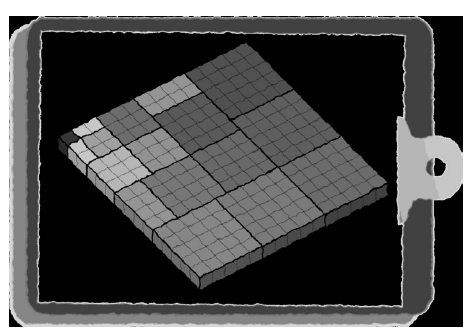
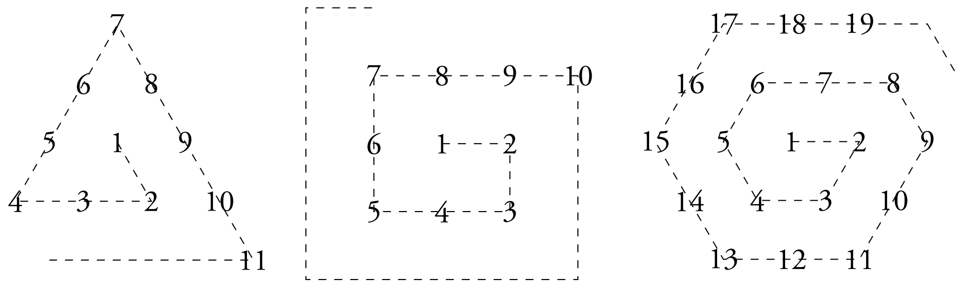


Figure 6

Exercise. Gauss and Tara spend the entire night discussing integer sequences, and by morning they have come up with a startling formula for the sum of the first n cubes: $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3$. The sum is just $\left(T(n)\right)^2$, the square of the n-th triangular number! They decide it's their turn to quiz the teacher, so the next day they present their findings and demand a proof from him. Can *you* help the hapless teacher, using the hint from Tara's notes (Figure 6)?

Exercise. After the school year ends, Tara gets a job as an urban planner for soon-to-be-built *Smart Spiral City*. Her boss wants the city plan to have equally spaced buildings, all lying on a single road that spirals outwards. He says that way they'll be identifiable via a single unique address number, with the numbers increasing outwards from the center. Tara considers three possible city plans:



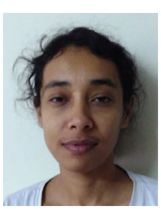
Plan A Plan B Plan C

Can you help Tara complete the three plans, giving addresses up to the number 50 in each?

Exercise. Tara's boss decides to go with plan A, and gives Tara another task. She must add avenues – roads that are perfectly straight – that will cut through the spiral road, so that the higher numbered houses are easier to reach. These avenues can be oriented in any direction. It turns out Tara's boss and his friends have already purchased houses whose addresses are triangular numbers. How many avenues need to be built in order that all the houses with 'triangular' addresses lie on avenues. How many avenues would have been required in plan B to touch all the square numbers, and in plan C to touch all the hexagonal numbers?

References

1. Brian Hayes, "Sides and Area of Pedal Triangle", *The American Scientist*, Vol. 94, No. 3, May-June 2006, page 200. http://dx.doi.org/10.1511/2006.3.200



MADHUSHREE BASU is a dancer and freelance illustrator based in Chennai. She completed a PhD in Mathematics at the Institute of Mathematical Sciences, Chennai in 2013. She may be contacted at pupai.madhusree@gmail.com.

AMRITANSHU PRASAD is a mathematician at the Institute of Mathematical Sciences, Chennai. His research in mathematics has recently revealed unexpected connections between Representation Theory and graph-theoretic properties of Pascal's Triangle and the Fibonacci numbers. He may be contacted at amri@imsc.res.in.

VIJAY RAVIKUMAR is a mathematician at the Chennai Mathematical Institute. He enjoys working with children, exploring science and art. He may be contacted at vijay.ravikumar@gmail.com.

RAMANUJAN AND SOME ELEMENTARY MATHEMATICAL PROBLEMS

UTPAL MUKHOPADHYAY

The year 1914 is a milestone in the history of Indian mathematics. On 17 March of that memorable year, the legendary Indian mathematician Srinivasa Ramanujan (1887–1920) set off on his historic voyage boarding the ship S S Nevasa for pursuing mathematical research under the supervision of the famous Cambridge mathematician, Professor G. H. Hardy (1877–1947). A detailed account of the historical background of the nourishment of the budding genius Ramanujan has been given by the present author elsewhere (*Resonance*, June 2014). The purpose of the present article is to narrate some interesting episodes of Ramanujan's life and to provide a few elementary problems solved by him to demonstrate how he attacked those problems and using his intuition generalized the results in some cases. Since the two words 'Ramanujan' and 'mathematics' cannot be separated, let us start with an episode involving an interesting mathematical problem.

Srinivasa Ramanujan

Prof G H Hardy

Keywords: Ramanujan, Hardy, Mahalanobis, door number problem, continued fraction, triangular number, square number, FRS, Cambridge, Trinity

In December 1914, when P. C. Mahalanobis (1893-1972) was in England for his studies, he showed Ramanujan, who was also engaged at that time in mathematical research under the supervision of Professor G. H. Hardy, a mathematical problem entitled 'Puzzles at a Village Inn' published in the popular English magazine *Strand*. The problem goes like this:

> *The houses on one side of a street are numbered from 1 to n. The sum of the numbers of the houses lying to the left of a particular house numbered m is equal to the sum of the numbers of the houses lying to the right of that particular house. If n lies between 50 and 500, then calculate the values of m and n.*

It may be mentioned here that being a member of a conservative Brahmin family, Ramanujan cooked his own meals. When Mahalanobis posed the problem to Ramanujan, the latter was engaged in preparing a meal. Without interrupting his work, Ramanujan instantly solved the problem. When Mahalanobis asked Ramanujan how he had arrived at the solution, he answered: "*Immediately I heard the problem it was clear that the solution should be a continued fraction; I then thought, which continued fraction? And the answer came to my mind*". Before presenting the solution of the problem, let us introduce the idea of a 'perfect median'.

The definition of perfect median as given by Brian Hayes in *American Scientist* (Vol. 96, page 36, 2008) is this: *A number m is a perfect median of n consecutive natural numbers* $1, 2, 3, \ldots, m, \ldots n$ *if* $1 + 2 + 3 + \ldots + (m-1) = (m+1) + (m+2) + (m+3) + \ldots + n$, *i.e., if*

$$m^2 = \frac{n(n+1)}{2}. \tag{1}$$

From (1), it is clear that for a perfect median, a square number and a triangular number (see Box 1) must be equal.

Multiplying both sides of (1) by 8 and adding 1 to both sides we get,

$$(2n+1)^2 - 8m^2 = 1 \tag{2}$$

Put:

$$2n + 1 = p, 2m = q. \tag{3}$$

From (2) we get $p^2 - 2q^2 = 1$, or by factorisation,

$$\left(p + \sqrt{2}q\right)\left(p - \sqrt{2}q\right) = 1. \tag{4}$$

Of the infinitely many solutions of (4), the one with the least values of p and q are $p_1 = 3$ and $q_1 = 2$. Using these values in (3), we get: $n_1 = 1$, $m_1 = 1$.

Putting these smallest values of p and q in (4) we obtain,

$$\left(3 + 2\sqrt{2}\right)\left(3 - 2\sqrt{2}\right) = 1. \tag{5}$$

Squaring each bracketed expression of (5) separately, we get,

$$\left(17 + 12\sqrt{2}\right)\left(17 - 12\sqrt{2}\right) = 1.$$

So, the next solution is: $p = 17$, $q = 12$, which yields $n_2 = 8$, $m_2 = 6$.

Again, by cubing (5) we get,

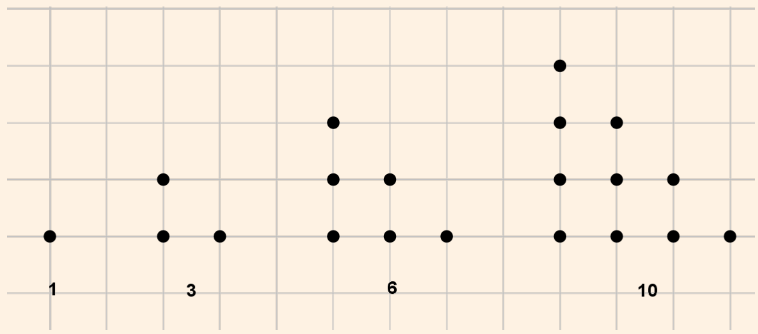
$$\left(99 + 70\sqrt{2}\right)\left(99 - 70\sqrt{2}\right) = 1.$$

So $p_3 = 99$, $q_3 = 70$, and $n_3 = 49$, $m_3 = 35$.

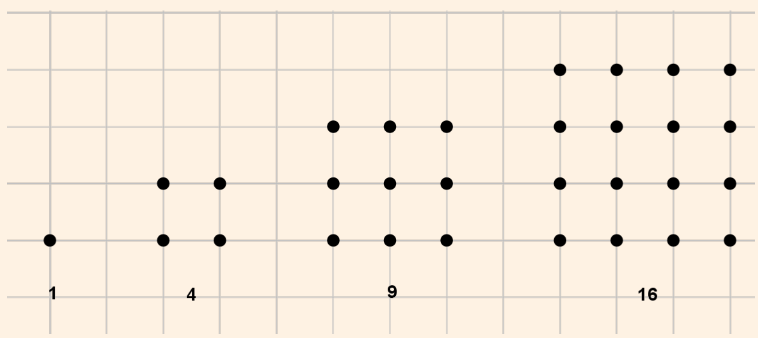
Proceeding in this way, by taking higher powers of (5), we get more solutions of (4).

Triangular and Square Numbers

Numbers which can be geometrically represented by triangular arrays of dots are called triangular numbers. For example, 1, 3, 6, 10, etc., are triangular numbers.



Similarly, numbers whose geometric representations are in the form of a square are square numbers, e.g. 1, 4, 9, 16, etc.



— If the n-th triangular number be T_n then $T_n = n(n+1)/2$.

— If the n-th square number be S_n then $S_n = n^2$.

Box 1

It may be mentioned here that in 1733, Leonhard Euler (1707-1783) in his paper *On the Solutions of Problems of Diophantus About Large Numbers* posed the problem of finding triangular numbers which also happen to be square numbers. His answer was the following: "Triangular numbers with $n = 1, 8, 49, 288, 1681, 9800$, etc., are square numbers corresponding to $m = 1, 6, 35, 204, 1189, 6930$, etc., respectively". For example, if we choose $n = 8$ and $m = 6$, we get the number 36 which is the eighth triangular number and also the sixth square number.

Now let us come back to Ramanujan's solution of the problem. He mentally calculated the first two values 1 and 6 of m and then constructed the following continued fraction (see Box 2 for what 'continued fraction' means):

$$\frac{1}{6-}\,\frac{1}{6-}\,\frac{1}{6-}\,\frac{1}{6-}\,\frac{1}{6-}\cdots$$

From each convergent (see Box 2) of the above continued fraction, we get two values of m. So the first convergent is 1/6 and the two values of m are 1 and 6. The second convergent is $6/35$ and the corresponding values of m are 6 and 35. The next convergent is 35/204 and the corresponding values of m are 35 and 204, and so on. If we assume

$$x = \frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\dots,$$

then $x = 1/(6-x)$ and therefore $6x - x^2 = 1$, i.e.,

$$x^2 - 6x + 1 = 0. \tag{6}$$

Solving (6), we get $x = 3 \pm 2\sqrt{2}$.

As $x < 1$, the plausible value of x is $3 - 2\sqrt{2}$.

Analyzing the above problem, we get a glimpse into how Ramanujan's mind worked!

Note from the editors: Another way of solving the door number problem is described in the article by Bodhideep Joardar, elsewhere in this issue.

Continued Fractions

An expression of the form shown below is called a **continued fraction**:

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \dots}}}.$$

The same continued fraction can also be written more conveniently as

$$1 + \frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\dots;$$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ etc. are called the elements of the continued fraction.

The value of the continued fraction obtained by truncating it at a certain stage is called a **convergent**. For the continued fraction shown above, the successive convergents are:

$$1, \frac{3}{2}, \frac{10}{7}, \dots.$$

Box 2

Ramanujan numbers

The Ramanujan number 1729 is well known to us. So, it is not necessary to repeat the story involving Ramanujan and his mentor Hardy and the taxicab. It suffices to say that this is the smallest positive integer which can be expressed as a sum of two cubes in two different ways. This property of the number 1729 was discovered by the French mathematician Bernard Frenicle de Bessy (1604-1674) in 1657. The number is found in one of Ramanujan's notebooks some years before the Hardy-Ramanujan incident. So most probably Ramanujan was already familiar with the special properties of that number. It should be added here that if we consider only cubes of positive integers, then 1729 is indeed the smallest number with that particular property. However, if cubes of negative integers are also taken into consideration, then 91 is the smallest such number, because

$$91 = 6^3 + (-5)^3 = 4^3 + 3^3.$$

Incidentally, 91 is a divisor of 1729. Now one may ask, which numbers larger than 1729 have the same property, i.e., they can be expressed in two different ways as a sum of two cubes? Such numbers are now known as *Ramanujan numbers*. Here are two other numbers with this property:

$$4104 = 2^3 + 16^3 = 9^3 + 15^3;$$
$$20683 = 10^3 + 27^3 = 19^3 + 24^3.$$

Note that if n is a Ramanujan number, then (trivially) so is nk^3 for any positive integer k; so $1729k^3$, $4104k^3$ and $20683k^3$ are Ramanujan numbers for any positive integer k.

A general rule for generating Ramanujan numbers is given below. Let m, n be integers. Define the integers a, b, c, d by using the following formulas:

$$a = 1 - (m - 3n)\left(m^2 + 3n^2\right),$$
$$b = -1 + (m + 3n)\left(m^2 + 3n^2\right),$$
$$c = (m + 3n) - \left(m^2 + 3n^2\right)^2,$$
$$d = -(m - 3n) + \left(m^2 + 3n^2\right)^2.$$

Then the following is identically true:

$$a^3 + b^3 = c^3 + d^3.$$

By appropriate rearrangement of terms (in case any of a, b, c, d turn out to be negative numbers), we get a Ramanujan-type relation involving only positive integers. Of course, we suitably cancel out on common factors. Some examples are shown below.

Example 1
Let $m = 2, n = 1$; then we get $a = 8, b = 34, c = -44, d = 50$. This yields the relation $8^3 + 34^3 = (-44)^3 + 50^3$. This in turn yields the interesting relation

$$4^3 + 17^3 + 22^3 = 25^3.$$

Example 2
Let $m = 3, n = 1$; then we get $a = 1, b = 71, c = -138, d = 144$. This yields the relation

$$1^3 + 71^3 + 138^3 = 144^3.$$

Example 3
Let $m = 4, n = 1$; then we get $a = -18, b = 132, c = -354, d = 360$. On division by 6, we get the following relation: $(-3)^3 + 22^3 = (-59)^3 + 60^3$, which may be rewritten as

$$59^3 + 22^3 = 60^3 + 3^3.$$

This yields the Ramanujan number 216027.

Example 4
Let $m = 1, n = 1$; then $a = 3, b = 5, c = -4, d = 6$ and hence the following relation:

$$3^3 + 4^3 + 5^3 = 6^3.$$

So if three solid metallic spheres of radii 3 cm, 4 cm and 5 cm are melted to form a larger sphere, then the radius of the new sphere will be 6 cm.

A Geometric Corollary of Ramanujan

Now, let us cite a particular work of Ramanujan which may be useful for school students. Ramanujan deduced the following result as a corollary of Pythagoras theorem.

Two segments BQ and CP are cut off from the hypotenuse BC of the right angled triangle ABC such that $BQ = BA$ and $CP = CA$. Then $PQ^2 = 2BP \cdot QC$.

Proof: We have (see Figure 1):

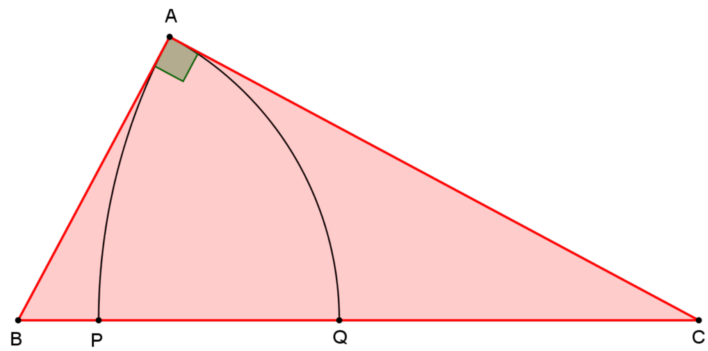


Figure 1

$BA = BQ = BP + PQ$,

so

$$AB^2 = BP^2 + PQ^2 + 2BP \cdot PQ. \tag{7}$$

Similarly,

$$AC^2 = CQ^2 + PQ^2 + 2CQ \cdot PQ. \tag{8}$$

Since BC is the hypotenuse of the right-angled triangle ABC, $BC^2 = AB^2 + AC^2$, so:

$$\begin{aligned} BC^2 &= BP^2 + PQ^2 + 2BP \cdot PQ + CQ^2 + PQ^2 + 2CQ \cdot PQ \\ &= BP^2 + 2PQ^2 + CQ^2 + 2BP \cdot PQ + 2CQ \cdot PQ. \end{aligned} \tag{9}$$

Again, $BC^2 = (BP + PQ + QC)^2$, so:

$$BC^2 = BP^2 + PQ^2 + CQ^2 + 2BP \cdot PQ + 2CQ \cdot PQ + 2BP \cdot CQ. \tag{10}$$

From (9) and (10) it follows that $PQ^2 = 2BP \cdot QC$, as required.

An algebraic identity

Using the above corollary, Ramanujan arrived at an algebraic identity. Let us put $AB = a$ and $AC = b$. Then $BC = (a^2 + b^2)^{1/2}$. Next, $BQ = AB = a$ and $CP = AC = b$, so

$$\begin{aligned} PQ &= BQ + CP - BC = a + b - \left(a^2 + b^2\right)^{1/2}, \\ QC &= BC - BQ = \left(a^2 + b^2\right)^{1/2} - a. \end{aligned}$$

A similar expression can be given for BP. Since $PQ^2 = 2BP \cdot QC$, one obtains,

$$\left(a + b - \left(a^2 + b^2\right)^{1/2}\right)^2 = 2\left(\left(a^2 + b^2\right)^{1/2} - a\right) \cdot \left(\left(a^2 + b^2\right)^{1/2} - b\right). \tag{11}$$

Extending the result (11), Ramanujan arrived at the following identity of the third degree:

$$\left((a + b)^{2/3} - \left(a^2 - ab + b^2\right)^{1/3}\right)^3 = 3\left(\left(a^3 + b^3\right)^{1/3} - a\right) \cdot \left(\left(a^3 + b^3\right)^{1/3} - b\right). \tag{12}$$

It can be shown that both sides of (12) are equal to

$$3\left(a^3+b^3\right)^{2/3}-(a+b)\left(a^3+b^3\right)^{1/3}+ab.$$

The above results are nice examples of the interrelationship between two branches of mathematics (in this case, geometry and algebra), as demonstrated by the great genius Ramanujan. Such examples may be cited during classroom teaching to inspire students in their search for deeper mathematical results.

Two problems posed and solved by Ramanujan

Next we reproduce below two problems and their solutions published in *Indian Journal of Mathematics* and their solutions by Ramanujan to show his ingenuity in elementary mathematics as well.

Problem 1

Evaluate
$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\ldots}}}}}.$$

Ramanujan's Solution

We know that $n+2=\sqrt{1+(n+1)(n+3)}$; hence:

$$n(n+2)=n\sqrt{1+(n+1)(n+3)}.$$

Let $f(n)=n(n+2)$. The above relation may then be written as:

$$f(n)=n\sqrt{1+f(n+1)}.$$

This substitution may be repeated iteratively:

$$\begin{aligned} f(n) &= n\sqrt{1+f(n+1)} \\ &= n\sqrt{1+(n+1)\sqrt{1+f(n+2)}} = \ldots \\ &= n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+\ldots}}}}. \end{aligned}$$

Putting $n=1$ in the above relation, we get:

$$3=\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\ldots}}}}}.$$

Problem 2

Evaluate:

$$\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\ldots}}}}.$$

Ramanujan's Solution

We know that $n(n+3)=n\sqrt{(n+5)+(n+1)(n+4)}$.

Let $f(n)=n(n+3)$; then

$$f(n)=n\sqrt{(n+5)+f(n+1)}.$$

We now use this identity iteratively:

$$\begin{aligned} f(n) &= n\sqrt{(n+5)+f(n+1)} \\ &= n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+f(n+2)}} = \dots \\ &= n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+(n+2)\sqrt{(n+7)+(n+3)\sqrt{(n+8)+\dots}}}}. \end{aligned}$$

Putting $n = 1$ in the above relation, we get:

$$4 = \sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\dots}}}}.$$

These two solutions demonstrate the deep insight and high level of intuition possessed by Ramanujan. We should mention here that in the book '*Wonders of Numbers*' by Clifford A. Pickover, the following formula is described as the most beautiful formula discovered by Ramanujan:

$$1+\frac{1}{1.3}+\frac{1}{1.3.5}+\frac{1}{1.3.5.7}+\dots+1+\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\dots = \sqrt{\frac{\pi e}{2}}.$$

Ramanujan becoming an FRS

Let us end our discussion by mentioning an episode related to Ramanujan and some information about his research work.

Prof. Hardy not only shaped Ramanujan's research career, but also exercised his powers to bring recognition for Ramanujan. He believed that his selection as a Fellow of the Royal Society (FRS) was a necessity for Ramanujan to boost his spirit. In Ramanujan's FRS application form, Hardy proposed his name, and the proposal was seconded by Major P. A. McMahon (1854–1929). There were eleven other signatories.[1] Among them, all except McMahon were Wranglers of the Cambridge Mathematical Tripos (with six Senior Wranglers). It is interesting to note that before writing his historic letter (dated 16 January, 1913) to Prof. Hardy, Ramanujan had sent letters to Baker and Hobson mentioned above without receiving any response from them. Anyway, Ramanujan's application form was submitted to the Royal Society on 18 December, 1917. His deteriorating health condition was a source of anxiety for Hardy. So he wasted no time in trying to convince high ranking persons of the Royal Society regarding Ramanujan's mathematical talent. He communicated detailed information to the then President of the Royal Society and Nobel Prize winning physicist Sir J. J. Thompson (1856–1940). Hardy also mentioned in his letter the physical condition of Ramanujan and warned that if the Royal Society delayed Ramanujan's selection, then "*The Society would have to live forever with its failure to honour him.*" All these efforts of Hardy culminated with the selection of Ramanujan as FRS in the meeting of Royal Society on 28 February, 1918. After 2 May, 1918 Ramanujan was entitled to write FRS after his name, becoming the second Indian to achieve this honour. It may be mentioned here that prior to his selection as FRS, Ramanujan was selected Member of London Mathematical Society on 6 December, 1917 and Member of Cambridge Philosophical Society on 18 February, 1918. On 10 October, 1918, Ramanujan was selected as a Fellow of Trinity College, being the first Indian to achieve this honour. In Ramanujan's selection as a Fellow of Trinity College, Littlewood played a leading role for nullifying the racial issues raised against Ramanujan.

[1] J. H. Grace (1873-1958), Joseph Larmour (1857-1942), T. J. Bromwich (1875-1929), E. W. Hobson (1856-1933), H. F. Baker (1866-1956), J. E. Littlewood (1885-1977), J. W. Nicholson (1881-1955), W. H. Young (1863-1942), E. T. Whittaker (1873-1956), A. R. Forsyth (1858-1942) and A. N. Whitehead (1861-947)

When questions arose whether Ramanujan was mentally fit, Littlewood produced two medical certificates to prove Ramanujan's mental fitness. The main argument placed in favour of Ramanujan was "*For a Fellow of Royal Society to be denied a Trinity Fellowship would be a scandal*." All these honours encouraged Ramanujan in his mathematical research. According to Srinivas Rao, "*These awards acted as great incentives to Ramanujan who discovered some of the most beautiful results in mathematics subsequently*." Justification of this statement can be judged from the fact that shortly before his selection to the Royal Society, Ramanujan jumped in front of a running train to kill himself but was saved somehow by the alertness of the train driver. So, no doubt Ramanujan was suffering from some kind of depression before achieving the honours mentioned above. He was arrested by Scotland Yard Police but was released through Hardy's intervention.

Some Statistical Information

We may get an idea of Ramanujan's mathematical work during his student life and immediately after his failure in F. A. Examination. In 1902, Ramanujan mastered the method of solving cubic equations and used it to develop his own method of solving quadratic equations. In 1903, he failed to solve equations of fifth degree due to his ignorance regarding the impossibility of its solution in simple form. During 1904, Ramanujan paid attention to the summation of series of the form $\sum \frac{1}{n}$ and calculated the value of the Euler constant to 15 decimal places. In the year 1908, he engaged himself in mathematical works on continued fractions and convergent series. Before going abroad, five research papers of Ramanujan were published in the *Journal of The Indian Mathematical Society*. The title of Ramanujan's first published paper was *Some properties of Bernoulli Numbers* (vol. 3, pp 219 – 234, 1911). The other four papers were *On question 330 of Prof. Sanjana* (vol. 4, pp 59 – 61, 1912), *Notes on a Set of Simultaneous Equations* (vol. 4, pp 94 – 96, 1913), *Irregular Numbers* (vol. 5, pp 105 – 106, 1913) and *Squaring the Circle* (vol. 5, page 132, 1913). Afterwards, the number of papers of Ramanujan published in the years 1914, 1915, 1916, 1917, 1918, 1919, 1920 and 1921 were 1, 9, 3, 7, 4, 4, 3 and 1 respectively. Ramanujan's last paper *Congruence Properties of Partitions* was published posthumously in *Mathematische Zeitschrift* (vol.93, pp 147 – 150, 1921). In all, 37 papers of Ramanujan were published in his research career, seven of them being joint papers with Prof. Hardy. Among those 37 papers, 11 were published in the *Journal of The Indian Mathematical Society*, 7 in *Proceedings of the London Mathematical Society*, 6 in *Messenger of Mathematics*, 5 in *Proceedings of the Cambridge Philosophical Society*, 3 in *Quarterly Journal of Mathematics*, 2 in *Transactions of the Cambridge Philosophical Society*, and 1 each in *Proceedings of the Royal Society*, *Mathematische Zeitschrift* and *Comptes Rendus*.

References

1. Robert Kanigel, *The Man Who Knew Infinity*; Abacus (2010)
2. Satyabachi Sar, Notebooks of Srinivasa Ramanujan; News. Bull. Cal. Math. Soc., Vol. 35, Nos. 7 – 9 (July, August, September), 2012
3. Anko niye khela, Bishseswar Panigrahi; Ananda Publishers, Kolkata, 2010 (in Bengali)
4. Satyabachi Sar, *Ganit Jagater Bismoy Ramanujan*; Jnan Bichitra, Agartala (2006) (in Bengali)

UTPAL MUKHOPADHYAY is a teacher of Barasat Satyabharati Vidyapith. His areas of research are general relativistic cosmology and history of science. He is actively engaged in science popularisation and has published several popular and semi-popular articles in various English and Bengali Journals of India and abroad including Resonance. He may be contacted at utpalsbv@gmail.com.

Features

On the sums of powers of NATURAL NUMBERS Part 2

V.G. TIKEKAR

Introduction

In Part I of this article, we had considered how to find the formulas for the sums of the squares and the cubes of the first n natural numbers. We had noted that these formulas are routinely encountered in the high school syllabus; they are typically proved using the principle of mathematical induction. In contrast, we had presented a context where the number of ways of carrying out a certain procedure needs to be computed. Two different ways of finding this number were presented. On juxtaposing the results, the desired formulas were obtained as mere corollaries.

Now in Part II of the article, we present a unified approach by which the formula for the sum of the k-th powers of the first n natural numbers can be obtained, for positive integral values of k. The method makes use of a triangular arrangement of numbers which bears a close similarity to the well-known Pascal Triangle.

Power Triangle – The Pascal Triangle

We all know the triangular arrangement of numbers displayed in Figure 1. It is known as the ***Pascal Triangle*** or the *Arithmetic Triangle*, and it yields a simple way of obtaining binomial expansions.

Keywords: Pascal triangle, power triangle, sums of powers

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	2	1			
$n=3$	1	3	3	1		
$n=4$	1	4	6	4	1	
$n=5$	1	5	10	10	5	1

Figure 1. The first few rows of the Pascal Triangle or Arithmetic Triangle

Its rows are numbered $0, 1, 2, 3, 4, \ldots$, and the numbers in the n-th row give us the coefficients of the successive terms in the expansion of $(a+b)^n$, for $n = 0, 1, 2, 3, \ldots$; i.e., the numbers of the Pascal triangle are the binomial coefficients $\binom{n}{r}$. The rule of formation of the numbers is this:

(i) the 0-th row has just one number, namely, 1;

(ii) the n-th row has $n+1$ numbers;

(iii) the first and last numbers in each row are 1, i.e.,

$$\binom{n}{0} = 1 = \binom{n}{n}, \quad \text{for } n = 0, 1, 2, 3, \ldots;$$

(iv) every other number of the Pascal triangle is given by the sum of the number immediately above it, and the number immediately to the left of that number (this is the well-known property of the Pascal Triangle), i.e.,

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad \text{for } n = 1, 2, 3, \ldots \text{ and } r = 1, 2, 3, \ldots, n-1.$$

(v) if the element at any position is absent, it is taken to be 0.

The Power Triangle. Now we display another such triangular array. As noted, it bears a certain resemblance with the Pascal Triangle. It is of comparatively recent origin and it helps in finding a formula for the sum of the k-th powers of the first n natural numbers, for any given positive integer k. It is called the *Power Triangle*. Its first few rows are given in Figure 2, and we can extend it indefinitely.

Rules governing the formation of the Power Triangle. Denote the number in row n and column r by $T(n,r)$; here $n = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots, n+1$. Then:

Rule 1: Row n has $n+1$ numbers, $T(n,1)$, $T(n,2)$, $T(n,3)$, ..., $T(n,n+1)$. We adopt the convention that $T(n,r) = 0$ if $r < 1$ or if $r > n+1$. (In words: if the element at any position is absent, it is taken to be 0.)

Rule 2: The first number of every row is 1; so $T(n,1) = 1$ for $n = 0, 1, 2, \ldots$.

Rule 3: The numbers in the successive rows of the power triangle are determined recursively as follows: for $n = 1, 2, 3 \ldots$ and $r = 1, 2, 3, \ldots, n+1$,

$$T(n,r) = (r-1) \cdot T(n-1, r-1) + r \cdot T(n-1, r). \qquad (1)$$

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	3	2			
$n=3$	1	7	12	6		
$n=4$	1	15	50	60	24	
$n=5$	1	31	180	390	360	120

Figure 2. The first few rows of the Power Triangle

Many nice properties of the T-numbers can be derived using the recursive property repeatedly. For example,

$$T(n,2) = 2^n - 1,$$
$$T(n, n+1) = n!\,,$$

for all positive integers n.

Using the Power Triangle

Now we show how this number array can be used to find a formula for the sum of the k-th powers of the first n natural numbers. The formula used is this:

$$1^k + 2^k + \cdots + n^k = \binom{n}{1} \cdot T(k,1) + \binom{n}{2} \cdot T(k,2) + \cdots + \binom{n}{k+1} \cdot T(k,k+1),$$

i.e.,

$$1^k + 2^k + \cdots + n^k = \sum_{r=1}^{k+1} \binom{n}{r} \cdot T(k,r). \tag{2}$$

As simple as that! Now let us see this remarkable formula in action.

Sum to power 0: Here $k = 0$; so:

$$1^0 + 2^0 + \cdots + n^0 = \binom{n}{1} \cdot T(0,1) = n \cdot 1 = n. \tag{3}$$

Clearly true!

Sum to power 1: Here $k = 1$; so:

$$1^1 + 2^1 + \cdots + n^1 = \binom{n}{1} \cdot T(1,1) + \binom{n}{2} \cdot T(1,2)$$
$$= n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}. \tag{4}$$

We have recovered the familiar formula.

Sum to power 2: Here $k = 2$; so:

$$1^2 + 2^2 + \cdots + n^2 = \binom{n}{1} \cdot T(2,1) + \binom{n}{2} \cdot T(2,2) + \binom{n}{3} \cdot T(2,3)$$

$$= n + 3 \cdot \frac{n(n-1)}{2} + 2 \cdot \frac{n(n-1)(n-2)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}, \quad \text{on simplification.} \qquad (5)$$

Once again, we have recovered the familiar formula.

Sum to power 3: Here $k = 3$; so

$$1^3 + 2^3 + \cdots + n^3$$

$$= \binom{n}{1} \cdot T(3,1) + \binom{n}{2} \cdot T(3,2) + \binom{n}{3} \cdot T(3,3) + \binom{n}{4} \cdot T(3,4)$$

$$= n + 7 \cdot \frac{n(n-1)}{2} + 12 \cdot \frac{n(n-1)(n-2)}{6} + 6 \cdot \frac{n(n-1)(n-2)(n-3)}{24}$$

$$= \frac{n^2(n+1)^2}{4}, \quad \text{on simplification.} \qquad (6)$$

Yet again, we have recovered the known formula. We can continue in this vein indefinitely and produce more and more such formulas.

Closing remark. The power triangle provides a simple and extremely convenient way of recovering the formulas for the sums of the k-th powers of the first n natural numbers, for any given positive integer k.

But note, however, that we have not provided any *proof* of the method. Interested readers may refer to the companion article by Prof K D Joshi, in which a proof has been sketched.

PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1995. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.

Features

Stirling SET NUMBERS & Powers of INTEGERS

SHAILESH SHIRALI

Introduction

This article introduces a remarkable class of combinatorial numbers, the ***Stirling set numbers***. They are also known as *Stirling numbers of the second kind*. (In the literature, you will find many references to this term. Note that there are also *Stirling numbers of the first kind*.) However, we shall not use this name; the name 'Stirling set number' seems more natural. These numbers are named after the 18th century Scottish mathematician James Stirling, and they are the natural counterparts of the binomial coefficients. Some readers may recall the name 'Stirling'. Indeed, it is the same Stirling whose name features in "Stirling's approximation for the factorial function."

Stirling set numbers feature prominently in the problem of finding a formula (in terms of n) for the sum of the k-th powers of the first n natural numbers, for given positive integer values of k, which is why we have included this article in this issue. (See the articles by Prof Tikekar and by Prof Joshi.)

Defining the Stirling set numbers

We start by defining the numbers. For positive integers k and r, the ***Stirling set number*** $S(k,r)$, is the *number of ways of partitioning the set* $\{1,2,3,\ldots,k\}$ *into* r *non-empty subsets* (order does not matter). Equivalently, $S(k,r)$ is the number of ways to put k distinct objects into r non-distinct boxes in such a way that no box is empty. Thus, for example:

Keywords: Stirling set number, Stirling number of the second kind, sums of powers

- $S(3,2)$ is the number of ways to partition a 3-element set into two nonempty subsets. It is easy to see that $S(3,2)=3$, by simply listing all the possibilities. For, if the set is $\{a,b,c\}$, then it can be partitioned into two nonempty subsets in the following ways: $\{a,b\}\cup\{c\}$; $\{a,c\}\cup\{b\}$; and $\{b,c\}\cup\{a\}$.
- $S(4,2)$ is the number of ways to partition a 4-element set into two nonempty subsets. Let us compute by hand the value of $S(4,2)$. Let the set be $\{a,b,c,d\}$. Since 4 can be written as a sum of two positive integers as $2+2$ and $3+1$, there are two kinds of partitions: (i) the two subsets have 2 elements each; (ii) one subset has 3 elements and the other subset has 1 element. It is not difficult to see that the first option contains three possibilities (namely: $\{a,b\}\cup\{c,d\}$; $\{a,c\}\cup\{b,d\}$; $\{a,d\}\cup\{b,c\}$), while the second option contains four possibilities (namely: $\{a,b,c\}\cup\{d\}$; $\{a,b,d\}\cup\{c\}$; $\{a,c,d\}\cup\{b\}$; $\{b,c,d\}\cup\{a\}$). Hence $S(4,2)=7$.

More such values can be found. More generally, we have, for any positive integer k,

$$\left.\begin{aligned} S(k,1) &= 1,\\ S(k,k) &= 1,\\ S(k,k-1) &= \binom{k}{2}. \end{aligned}\right\} \tag{1}$$

The first two equalities are obvious.

To see why the third is true, observe that if $r=k-1$, then one of the $k-1$ subsets has two elements, while the other subsets have one element each. Since the subsets are indistinguishable, this can be done in as many ways as the number of ways of selecting two elements from the given set. Hence the stated equality.

Next:

$$S(k,2)=2^{k-1}-1. \tag{2}$$

For: $r=2$ means that we partition the given set of k objects into two non-empty subsets; the order does not matter. Arbitrarily put aside any one object (it does not matter which one); call it X. Each of the remaining $k-1$ objects can choose to partner with X or not; this yields 2^{k-1} choices. However, we cannot have all of the objects partnering with X, as that would result in only one subset and not two, as required. So we need to subtract 1 from the number obtained. Hence the number of ways is $2^{k-1}-1$. That is, $S(k,2)=2^{k-1}-1$. (Observe that this is consistent with our earlier finding that $S(4,2)=7$.)

Values of $S(k,r)$ for $2<r<k-1$ may be found using the following very convenient recurrence relation:

$$S(k,r)=r\,S(k-1,r)+S(k-1,r-1). \tag{3}$$

To see why (3) is true, consider where object k belongs. There are two possibilities:

- If k is in a subset all by itself, with no partners, then the remaining subsets correspond to a way of partitioning the set $\{1,2,\ldots,k-1\}$ into $r-1$ nonempty subsets. This can be done in $S(k-1,r-1)$ ways.
- If on the other hand, k lies in a subset with more than 1 element, then by hiding just that element, we get a partition of the set $\{1,2,\ldots,k-1\}$ into r nonempty subsets. This can be done in $S(k-1,r)$ ways. Now bring back k. It can be put into any of the r subsets; so we get $rS(k-1,r)$ ways.

Hence $S(k,r)=r\,S(k-1,r)+S(k-1,r-1)$, as claimed.

Values of the Stirling set numbers have been displayed in Figure 1. They have been computed using the recursive formula (3).

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
$k=1$	1							
$k=2$	1	1						
$k=3$	1	3	1					
$k=4$	1	7	6	1				
$k=5$	1	15	25	10	1			
$k=6$	1	31	90	65	15	1		
$k=7$	1	63	301	350	140	21	1	
$k=8$	1	127	966	1701	1050	266	28	1

Figure 1. The Stirling set numbers $S(k, r)$

Observe that if we put $r = 2$ in this recurrence relation, then we get

$$S(k, 2) = 2\,S(k-1, 2) + S(k-1, 1) = 2\,S(k-1, 2) + 1,$$

and this allows us to prove inductively that $S(k, 2) = 2^{k-1} - 1$, thereby yielding another proof of that relation.

Connection between Stirling set numbers and powers of integers

There is an important relation that allows us to express the quantity n^k in terms of the Stirling set numbers:

Theorem 1. *For positive integers n and k, the following identity is true:*

$$n^k = \sum_{r=1}^{k} r! \cdot S(k, r) \cdot \binom{n}{r}. \tag{4}$$

This relation needs to be understood. It helps if we consider small values of k.

k = 1: The sole Stirling set number for $k = 1$ is $S(1, 1) = 1$, so the claim reduces to the trivial statement $n = 1 \cdot n$.

k = 2: The Stirling set numbers for $k = 2$ are $S(2, 1) = 1$, $S(2, 2) = 1$. Hence the claim reduces to the following statement:

$$n^2 = 1 \cdot 1 \cdot \binom{n}{1} + 2 \cdot 1 \cdot \binom{n}{2},$$

i.e.,

$$n^2 = \binom{n}{1} + 2 \cdot \binom{n}{2}, \tag{5}$$

which may be verified algebraically or using combinatorial reasoning. (Here is a combinatorial proof. Consider all ordered pairs (x, y), where both x and y are elements of $\{1, 2, 3, \ldots, n\}$. The total number of such pairs is n^2. Now subdivide the pairs (x, y) into two categories: those in which $x = y$, and those in which $x \neq y$. The number of pairs of the first kind is $\binom{n}{1} = n$, and the number of pairs of the second kind is $2 \cdot \binom{n}{2}$. Hence the stated result.)

k = 3: The Stirling set numbers for $k = 3$ are $S(3, 1) = 1$, $S(3, 2) = 3$, $S(3, 3) = 1$. Hence the claim reduces to the following statement:

$$n^3 = 1 \cdot 1 \cdot \binom{n}{1} + 2 \cdot 3 \cdot \binom{n}{2} + 6 \cdot 1 \cdot \binom{n}{3},$$

i.e.,

$$n^3 = \binom{n}{1} + 6 \cdot \binom{n}{2} + 6 \cdot \binom{n}{3}, \qquad (6)$$

which too may be verified algebraically or using combinatorial reasoning. (Here is a combinatorial proof, just like the one presented above. Consider all triples (x, y, z), where $x, y, z \in \{1, 2, 3, \ldots, n\}$. The total number of such triples is n^3. Now subdivide the triples (x, y, z) into three categories according to the number of different numbers used in the triple; this could be 1, 2, or 3. The number of triples of the first kind is $\binom{n}{1}$, the number of triples of the second kind is $\binom{n}{2} \times \binom{2}{1} \times \frac{3!}{2!} = 6 \cdot \binom{n}{2}$, and the number of triples of the third kind is $6 \cdot \binom{n}{3}$. Hence the stated result.) (*Remark*. The second of these three claims may need some justification but we leave the details to the reader.)

k = 4: The Stirling set numbers for $k = 4$ are $S(4, 1) = 1$, $S(4, 2) = 7$, $S(4, 3) = 6$, $S(4, 4) = 1$. Hence the claim reduces to the following statement:

$$n^4 = 1 \cdot 1 \cdot \binom{n}{1} + 2 \cdot 7 \cdot \binom{n}{2} + 6 \cdot 6 \cdot \binom{n}{3} + 24 \cdot 1 \cdot \binom{n}{4},$$

i.e.,

$$n^4 = \binom{n}{1} + 14 \cdot \binom{n}{2} + 36 \cdot \binom{n}{3} + 24 \cdot \binom{n}{4}. \qquad (7)$$

Again, this may be verified algebraically or using combinatorial reasoning. (Here is a combinatorial proof. Consider all four-tuples (x, y, z, w), where $x, y, z, w \in \{1, 2, 3, \ldots, n\}$. The total number of such four-tuples is n^4. Now subdivide these four-tuples into four categories according to the number of different numbers used in the four-tuple; this could be 1, 2, 3, or 4. The number of four-tuples of the first kind is $\binom{n}{1}$, the number of four-tuples of the second kind is $\binom{n}{2} \times \left(\frac{4!}{2!2!} + 2 \times \frac{4!}{3!}\right) = 14 \cdot \binom{n}{2}$, the number of four-tuples of the third kind is $\binom{n}{3} \times \binom{3}{1} \times \frac{4!}{2!} = 36 \cdot \binom{n}{3}$, and the number of four-tuples of the fourth kind is $24 \cdot \binom{n}{4}$. Hence the stated result.)

We prove the general result (for the higher powers) later in the article.

There is another set of important relations that we make use of:

Theorem 2. *The following relations are true for any positive integer n:*

$$\binom{n+1}{2} = \binom{n}{1} + \binom{n-1}{1} + \binom{n-2}{1} + \cdots + \binom{1}{1},$$

$$\binom{n+1}{3} = \binom{n}{2} + \binom{n-1}{2} + \binom{n-2}{2} + \cdots + \binom{1}{2},$$

$$\binom{n+1}{4} = \binom{n}{3} + \binom{n-1}{3} + \binom{n-2}{3} + \cdots + \binom{1}{3}, \quad \ldots.$$

The number of terms on the right is n in each case. However, recall that $\binom{r}{s} = 0$ when $s > r$. (So, in effect, the summations stop earlier.)

The relations in Theorem 2 are easily proved using combinatorial reasoning. We do not give the proofs here. Please try to find them for yourself.

Formulas for the sums of powers

Bringing together the results of Theorem 1 and Theorem 2, we obtain formulas for the sums of the squares, the cubes, and the fourth and higher powers of the first n natural numbers. First, the squares. We know that

$$n^2 = \binom{n}{1} + 2 \cdot \binom{n}{2}.$$

Hence, using the identities from Theorem 2:

$$\begin{aligned}\sum_{i=1}^{n} i^2 &= \sum_{i=1}^{n} \binom{i}{1} + 2 \cdot \sum_{i=1}^{n} \binom{i}{2} \\ &= \binom{n+1}{2} + 2 \cdot \binom{n+1}{3}. \qquad (8)\end{aligned}$$

On simplifying the expressions on the right side, we get the familiar formula for the sum of the squares of the first n natural numbers.

Similarly, for the sum of the cubes: we know that

$$n^3 = \binom{n}{1} + 6 \cdot \binom{n}{2} + 6 \cdot \binom{n}{3}.$$

Hence, using Theorem 2 as earlier:

$$\begin{aligned}\sum_{i=1}^{n} i^3 &= \sum_{i=1}^{n} \binom{i}{1} + 6 \cdot \sum_{i=1}^{n} \binom{i}{2} + 6 \cdot \sum_{i=1}^{n} \binom{i}{3} \\ &= \binom{n+1}{2} + 6 \cdot \binom{n+1}{3} + 6 \cdot \binom{n+1}{4}. \qquad (9)\end{aligned}$$

As earlier, on simplifying the expressions on the right side, we get the familiar formula for the sum of the cubes of the first n natural numbers.

Similarly we get, for the sum of the fourth powers:

$$\sum_{i=1}^{n} i^4 = \binom{n+1}{2} + 14 \cdot \binom{n+1}{3} + 36 \cdot \binom{n+1}{4} + 24 \cdot \binom{n+1}{5}. \qquad (10)$$

We can continue in this vein indefinitely.

Proof of Theorem 1

The derivations presented above rest critically on Theorem 1, which we proved only for the cases $k = 2, 3$, and 4. So it is incumbent upon us, now, to provide the full proof of the theorem. Let us recall the statement here.

Claim (Theorem 1). *For positive integers n and k, the following identity is true:*

$$n^k = \sum_{r=1}^{k} r! \cdot S(k, r) \cdot \binom{n}{r}.$$

Proof. Consider all k-tuples whose elements are drawn from the set $\{1, 2, 3, \ldots, n\}$. The number of such k-tuples is n^k. We subdivide these k-tuples according to the number r of different numbers used in the k-tuple, $r \in \{1, 2, \ldots, k\}$.

To motivate the proof, we take a close look at the cases $r = 1, 2, 3$.

r = 1: This means that a single number has been used in the k-tuple, repeated k times. The number of such k-tuples is therefore the same as the number of ways of choosing that number and is thus equal to $\binom{n}{1}$. Note that we may write this in the form $1! \times S(k, 1) \times \binom{n}{1}$

r = 2: This means that just two numbers (call them a and b) have been used repeatedly in the k-tuple. These two numbers can be chosen in $\binom{n}{2}$ ways. In the k-tuple thus obtained, every number is either a or b. The placement of these a's and b's creates a natural partition of the set $\{1, 2, \ldots, k\}$ into two nonempty subsets. The number of such partitions is $S(k, 2)$. For each such partition, we have a choice of deciding which part will be occupied by the a's and which part by the b's; there are 2! ways of making this choice.

Hence the total number of ways corresponding to $r = 2$ is

$$2! \times S(k, 2) \times \binom{n}{2}.$$

r = 3: This means that just three numbers (call them a, b, c) have been used repeatedly in the k-tuple. These numbers can be chosen in $\binom{n}{3}$ ways. In the k-tuple thus obtained, every number is either a or b or c. Their placement creates a natural partition of the set $\{1, 2, \ldots, k\}$ into three nonempty subsets. The number of such partitions is $S(k, 3)$. For each such partition, we have a choice of deciding which part will be occupied by the a's, which part by the b's, and which part by the c's; there are 3! ways of making this choice.

Hence the total number of ways corresponding to $r = 3$ is

$$3! \times S(k, 3) \times \binom{n}{3}.$$

The reasoning described for the cases $r = 2$ and $r = 3$ is perfectly general; clearly, we do not need to repeat it for other values of r. Theorem 1 thus follows. $\square$

As a consequence of the above and the reasoning used earlier, we see that the truth of the following theorem has been established:

Theorem 3. *For positive integers n and k, the following identity is true:*

$$1^k + 2^k + \cdots + n^k = \sum_{r=1}^{k} r! \cdot S(k, r) \cdot \binom{n+1}{r+1}. \qquad (11)$$

Concluding remarks

The Stirling set numbers provide a well-knit and intellectually very satisfying way of arriving at formulas for the sums of the k-th powers of the first n natural numbers, for any positive integer k. What is pleasing about this approach is its combinatorial nature and the way it 'hangs together.'

For further study, the reader could refer to the readings suggested. Reference [2] is a particularly good read.

References

1. Wikipedia, *Stirling numbers of the second kind*, https://en.wikipedia.org/wiki/Stirling_numbers_of_the_second_kind
2. Khristo N. Boyadzhiev, "Close Encounters with the Stirling Numbers of the Second Kind", *Mathematics Magazine*, Math. Assoc. America, Vol. 85, No. 4 (Oct 2012), pp. 252-266; https://www.maa.org/sites/default/files/pdf/upload_library/2/Boyadzhiev-2013.pdf
3. School of Mathematics and Statistics, University of St Andrews, Scotland, "James Stirling", http://www-history.mcs.st-andrews.ac.uk/history/Biographies/Stirling.html

SHAILESH SHIRALI is the Director of Sahyadri School (KFI), Pune, and heads the Community Mathematics Centre based in Rishi Valley School (AP) and Sahyadri School KFI. He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

THE POWER TRIANGLE

PROF. K. D. JOSHI

In the companion article by Prof V G Tikekar, a method is described for finding a formula for the sum of the k-th powers of the first n positive integers, for any positive integer k, using a triangular array of numbers—the ***Power Triangle***. This article shows why the method works. The proof makes use of a family of combinatorial numbers called the ***Stirling set numbers*** (also called *Stirling numbers of the second kind*; they are dwelt upon in the other companion article by Dr Shailesh Shirali). By exploiting a relation between these numbers and the power functions, we are led to the formulas we seek.

Recalling the definition. For positive integers k and r, the ***Stirling set number*** $S(k, r)$, is the *number of ways of partitioning the set* $\{1, 2, 3, \ldots, k\}$ *into* r *non-empty subsets*; order does not matter. Note that the definition implies that $S(k, r) = 0$ for $k < r$. For example, $S(3, 2) = 3$ and $S(4, 2) = 7$, as the reader can verify. The following is true about the Stirling set numbers:

- For all positive integers k, we have
$$S(k, 1) = 1, \quad S(k, k) = 1, \quad S(k, 2) = 2^{k-1} - 1.$$
- For all positive integers $k > 1$, we have
$$S(k, k-1) = \binom{k}{2}.$$
- For all positive integers k and r, $1 < r < k - 1$, we have:
$$S(k, r) = r\,S(k-1, r) + S(k-1, r-1). \qquad (1)$$

Keywords: Power triangle, Stirling set number, sums of powers

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
$k=1$	1							
$k=2$	1	1						
$k=3$	1	3	1					
$k=4$	1	7	6	1				
$k=5$	1	15	25	10	1			
$k=6$	1	31	90	65	15	1		
$k=7$	1	63	301	350	140	21	1	
$k=8$	1	127	966	1701	1050	266	28	1

Figure 1. The Stirling set numbers $S(k,r)$

A few values of the Stirling set numbers are displayed in Figure 1. They have been computed using the above recursive relation.

Connection between Stirling set numbers and powers of integers. There is an important relation connecting the quantity n^k with the Stirling set numbers:

Theorem 1. *For positive integers n and k,*

$$n^k = \sum_{r=1}^{k} r! \cdot S(k,r) \cdot \binom{n}{r}. \tag{2}$$

This leads naturally to Theorem 2 which gives a formula for the sum of the k-th powers of the first n natural numbers, for any positive integer k. Both Theorem 1 and Theorem 2 are proved in Shirali's article.

Theorem 2. *For positive integers n and k,*

$$1^k + 2^k + \cdots + (n-1)^k + n^k = \sum_{r=1}^{k} r! \cdot S(k,r) \cdot \binom{n+1}{r+1}. \tag{3}$$

Connection with the Power Triangle. We now recall the method presented in Prof Tikekar's article, involving the power triangle. The rules governing its formation are:

(i) Row n has $n+1$ numbers, $T(n,1)$, $T(n,2)$, $T(n,3)$, ..., $T(n,n+1)$; we adopt the convention that $T(n,r) = 0$ if $r < 1$ or if $r > n+1$.

(ii) $T(n,1) = 1$ for $n = 0, 1, 2, \ldots$.

(iii) For $n = 1, 2, 3 \ldots$ and $r = 1, 2, 3, \ldots, n+1$,

$$T(n,r) = (r-1) \cdot T(n-1, r-1) + r \cdot T(n-1, r). \tag{4}$$

Figure 2 shows the first few rows of the Power Triangle, computed using relation (4).

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	3	2			
$n=3$	1	7	12	6		
$n=4$	1	15	50	60	24	
$n=5$	1	31	180	390	360	120

Figure 2. The first few rows of the Power Triangle

Formula for the sum of the powers. For each positive integer k, define the following function f_k on the set of positive integers $\mathbb{N}$:

$$f_k(n) = 1^k + 2^k + \cdots + (n-1)^k + n^k.$$

A formula for $f_k(n)$ is claimed to be:

$$f_k(n) = \sum_{r=1}^{k+1} \binom{n}{r} \cdot T(k, r). \tag{5}$$

This is what we must now set out to prove.

A formula connecting the Stirling set numbers and the Power Triangle numbers. To make progress, we must find a connection between the Stirling set numbers and the Power triangle numbers. The connection is easy to spot when one places the two sets of numbers next to each other, as in Figure 3:

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$k=1$	1					
$k=2$	1	1				
$k=3$	1	3	1			
$k=4$	1	7	6	1		
$k=5$	1	15	25	10	1	
$k=6$	1	31	90	65	15	1

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	3	2			
$n=3$	1	7	12	6		
$n=4$	1	15	50	60	24	
$n=5$	1	31	180	390	360	120

Figure 3. Stirling set numbers (left) and Power triangle numbers (right)

The formula is easily seen to be this: for all integers $k \geq 1$, $0 \leq r \leq k$,

$$T(k, r) = (r-1)! \cdot S(k+1, r). \tag{6}$$

Once seen, it is easy to prove the relationship using the principle of induction. Thus, assuming that the relation $T(i, r) = (r-1)!\ S(i+1, r)$ is true for $i = 0, 1, \ldots, k-1$ and $r = 1, 2, \ldots, i+1$, we have, by repeatedly using the recursive properties of the Stirling set numbers and the T-numbers,

$$\begin{aligned} T(k, r) &= (r-1) \cdot T(k-1, r-1) + r \cdot T(k-1, r) \\ &= (r-1) \cdot (r-2)! \cdot S(k, r-1) + r \cdot (r-1)! \cdot S(k, r) \\ &= (r-1)! \cdot (S(k, r-1) + r \cdot S(k, r)) \\ &= (r-1)! \cdot S(k+1, r). \end{aligned}$$

This proves the property that had been claimed.

Proof of the Power Triangle formula. With all the pieces now in place, the proof of the claimed relation (5) is now not too difficult to work out. It has been displayed as a separate boxed item (Box 1).

Remarks from the editor.

- It is quite often the case, in combinatorial settings, that once a property has been seen, it is relatively easy to prove it using the principle of induction. Finding the property poses much more of a challenge than proving it! The case of (6) illustrates this remark perfectly.
- Proving (5) using (6) and the recursive properties of the Stirling set numbers and the T-numbers involves a considerable amount of algebraic manipulation; in particular, manipulation of the summation symbol and indices. We have opted to display the proof as a separate boxed item (Box 1).

  It is important to keep in mind that "doing mathematics" involves a great many things: experimentation and generating data; studying the data for patterns, using all the techniques one possesses; conjecturing and hypothesizing; systematically checking out our conjectures and hypotheses; and, finally, proving our conjectures and hypotheses. Sometimes, one or more of these steps requires a substantial amount of algebra and symbol manipulation. *All these are part and parcel of the subject.*

PROF. KAPIL D JOSHI received his PhD from Indiana University in 1972 He served in the Department of Mathematics at IIT Powai, 1980–2013. He has written numerous mathematics textbooks, among them *Introduction to General Topology*, *Calculus for Scientists and Engineers – An Analytical Approach* and *Educative JEE (Mathematics)*. He was keenly involved for many years in the GATE and JEE. He may be contacted at kdjoshi314@gmail.com.

Proof of relation (5)

We are required to prove the following relation:

$$f_k(n) = \sum_{r=1}^{k+1} \binom{n}{r} \cdot T(k,r),$$

where $f_k(n) = 1^k + 2^k + \cdots + n^k$. We have, from (3):

$$\begin{aligned}
f_k(n) &= \sum_{r=1}^{k} r! \cdot \binom{n+1}{r+1} \cdot S(k,r) = \sum_{r=1}^{k} r! \cdot \left(\binom{n}{r} + \binom{n}{r+1}\right) \cdot S(k,r) \\
&= \sum_{r=1}^{k} r! \cdot \frac{T(k-1,r)}{(r-1)!} \cdot \left(\binom{n}{r} + \binom{n}{r+1}\right) \\
&= \sum_{r=1}^{k} r \cdot T(k-1,r) \cdot \binom{n}{r} + \sum_{r=1}^{k} r \cdot T(k-1,r) \cdot \binom{n}{r+1} \\
&= \sum_{r=1}^{k} \Big(T(k,r) - (r-1) \cdot T(k-1,r-1)\Big) \cdot \binom{n}{r} + \sum_{r=1}^{k} r \cdot T(k-1,r) \cdot \binom{n}{r+1} \\
&= \sum_{r=1}^{k} T(k,r) \cdot \binom{n}{r} - \sum_{r=1}^{k} (r-1) \cdot T(k-1,r-1) \cdot \binom{n}{r} + \sum_{r=1}^{k} r \cdot T(k-1,r) \cdot \binom{n}{r+1} \\
&= \sum_{r=1}^{k} T(k,r) \cdot \binom{n}{r} - \sum_{r=0}^{k-1} r \cdot T(k-1,r) \cdot \binom{n}{r+1} + \sum_{r=1}^{k} r \cdot T(k-1,r) \cdot \binom{n}{r+1} \\
&= \sum_{r=1}^{k} T(k,r) \cdot \binom{n}{r} + k \cdot T(k-1,k) \cdot \binom{n}{k+1} \\
&= \sum_{r=1}^{k} T(k,r) \cdot \binom{n}{r} + k \cdot (k-1)! \cdot \binom{n}{k+1} = \sum_{r=1}^{k} T(k,r) \cdot \binom{n}{r} + k! \cdot \binom{n}{k+1} \\
&= \sum_{r=1}^{k+1} T(k,r) \cdot \binom{n}{r}.
\end{aligned}$$

Box 1. Details of the proof of relation (5)

The Conjecturing CLASSROOM

ANUSHKA RAO FITZHERBERT

I first came across the idea of a conjecturing classroom during my teacher training (PGCE) in the UK. We were watching a video clip of 6 and 7 year old students in a mathematics class. The classroom looked ordinary, children sitting in rows, a bit of chatting and minor disruptions, books and pencils and erasers strewn across desks, the teacher at the front without any spectacular resources, holding an unfussy whiteboard pen. Nowhere were the exciting new ideas I had heard during my course thus far, such as creative ways of organizing children and resources, the magic of group work, the use of manipulatives, etc. It looked like a mundane, regular class. Then the teacher began with this: "Who can remember Lucy's conjecture from yesterday?" Several hands sprang up and the teacher wrote a version of what they were saying, on the board, in quotations: "Every number can be written as the sum of consecutive numbers" and above that she wrote "Lucy's conjecture."

I was blown away. The language used by the teacher and these 6 and 7 year olds was nothing like I imagined in a mathematics classroom. They talked about how to 'test the conjecture' and 'strategies' for how to 'prove or disprove' it. As the class progressed, the flow was often interrupted by the teacher adding other students' related conjectures: "It only works for positive numbers" or "Every number can be written as a multiplication of other numbers." Every time she would write above the statement "Eli's conjecture" or "Kian's conjecture." There were frequent shouts such as, "I've disproved it – doesn't work for 6" and then another child might say after a few minutes "It does!" and they would rush to compare their findings. As the lesson came to a close, a few children had noticed that the conjecture seemed to be

Keywords: Pedagogy, class protocol, student engagement, conjecture, verification, contradiction, proof

true for all odd numbers, and were even close to a proof of sorts, which one of them tried to explain on the board. And that was it.

While reflecting on the lesson, I realised what had appealed to me was the agency of the children. It was by any standard quite chaotic and even a bit noisy, but it did seem, at least to a bright-eyed new teacher watching from the safe distance of a screen, that the children were all involved, and they seemed to be involved in creating mathematics. They also seemed to be comfortable with trying new things, without being sure of where they were heading exactly. And, I thought, they felt important, their conjectures were being named after them, others in the class were giving those conjectures due consideration. They really looked like a bunch of mini mathematicians, another word that the teacher used often to describe her students or to encourage them – "How would a good mathematician record his/her work?"

I knew then this was the kind of classroom I wanted, and I began in earnest when I had my own class, armed with carefully thought out lesson plans full of investigations and open-ended tasks. Of course, it was a disaster. It would take me a long time to realize how much thought and work that teacher must have had to put in to get the conjecturing classroom of my dreams. She seemed all laidback and easy-going, but in fact she was doing a huge amount; it was just all so subtle.

I believe now that in order to create a conjecturing classroom, I, the teacher must be fully invested in this. If I am, then I will value a thoughtful conjecture as much as the one I know to be true. I must be equally joyous when a conjecture is proven or disproven. I must be willing to let students go off in the direction they have become interested in, even if it is different from what the rest of the class are doing. To have a conjecturing classroom is to value curiosity and allow children to pursue their curious questions to the full. Of course, not every lesson can or even should be like this; there are times when I need to guide children more pointedly, for the sake of the curriculum or to prepare them for a whole new concept. However I want my students to feel completely comfortable and at ease and full of questions when exploring mathematics, and for that there needs to be a lot of opportunity for conjecturing. Also it is necessary to have a set of guidelines in the class. The tasks that promote conjecturing require a fair amount of independent work by students, and having clear guidelines helps them stay on task. The guidelines should also address respecting fellow classmates. Children are all on board and agree with guidelines like "always listen when someone is talking/explaining" and "allow someone to finish an explanation;" if you think someone is wrong, use respectful language like "I think there might be an error in your working" or "I have noticed that what you say doesn't work for such and such, I think it is because such and such." In the heat of the moment, when someone shouts out "you're totally wrong!" with much glee, a quick reminder of the guidelines and children are more than willing to comply. Particularly if they were all part of and agreed with the class guidelines at the start of the school year.

I began teaching the 7th standard in Rishi Valley School last year. Here is a brief account of how the year progressed, with snippets from three lessons, in pursuit of a conjecturing classroom. I was teaching both sections of class 7 - class 7A and class 7B – in parallel, often teaching the same lesson back-to-back. The accounts are an amalgamation of things that happened in both classes, although in fact very often the lessons unfolded in similar ways. Hence I will talk about both 7A and 7B as one class.

Lesson 1: Hiccup Numbers

The very first lesson I had with them was an investigation on what I called 'hiccup numbers.' It turned out that some of them had worked on this investigation, although in a different way, the previous year, but it was still a good introduction to conjecturing. I told the class to choose a three-digit number. I then introduced a new (and made up) verb 'to hiccup'. I illustrated this with an example: 142 'hiccupped' is 142142. I asked them to hiccup their chosen three-digit number and then divide it by 13, then divide the answer by 11, and finally divide that answer by 7. Their little heads were all bent over in serious concentration, furiously applying the long division algorithm that most seemed to be fairly comfortable with (I had a chance to go around and notice and help those who were struggling). Soon enough, one little head looked up with an expression of surprised elation. And a few others bobbed up too. They had remembered my instructions while setting our class guidelines, to never shout out the answer or "I'm done" or "I've got it" but to catch my eye and show me a thumbs-up sign to let me know they were done, without putting off the rest of the class. These little rituals are important in a conjecturing classroom, because the aim is for the entire class to be conjecturing and to do that, everyone in the class must keep to a minimum the usual habits in a classroom that might be demoralizing for others. I went over and they exclaimed in excited whispers that they had got back their original number. "Hmm interesting...try another?" I offered. This task led very quickly to one of the students conjecturing "it always works." I insisted on a well-worded conjecture that had clarity and was not superfluous and after several attempts and with the help of others in the class we had a conjecture: "A three digit number that is hiccupped will give back the original number when divided by 13, 11 and 7 successively." I wrote this on the board and above it I wrote "Ajay's conjecture." I gave a little speech about conjectures and proofs and how the aim of a mathematician, after noticing patterns and formulating a conjecture is to prove or disprove it. These 11-year olds scoffed and laughed a bit nervously at this prosaic and grand language and my referring to them as mathematicians, but I didn't laugh. I was deadly serious. The children took up this language themselves remarkably quickly!

[A little aside, the act of naming a conjecture after the student might seem gimmicky, but I really think this is a wonderful thing to do, to hand ownership of the lesson and the maths to students. After initial excitement over 'their' conjecture at the beginning of the year, students got used to it, and there were delightful instances of students referring to their classmates' conjecture from a previous lesson, overheard talk of 'so and so's conjecture' on the way to lunch, a child excitedly telling me they had disproved their own conjecture. It did seem to me that students felt less proud of having their conjectures named after them, as the novelty wore off, but what remained was a sense of courage for students to create their own ideas.]

Ajay's conjecture was quickly verified by others, one or two children said "it doesn't work for such and such number" then others would dive-in to check that it did indeed work for that number. A few trivial variations of the conjecture were introduced, for example "it doesn't matter in which order you do the divisions." Many students were able to explain why the order didn't matter. Many then also noticed that the divisor was in fact the number 1001 and a few went on to appreciate how this magic trick worked and so proved the conjecture. A new conjecture, "the hiccup number for a 4-digit number is 10001" was proposed.

The mathematical content in this lesson was well within the grasp of students in this class, but my aim for the lesson was to introduce a behaviour – a conjecturing behaviour. The lesson wasn't about adding to their mathematical knowledge. It did, however, prove to be a good exercise to practice long division, which some students were struggling with, and this self-checking task gave them a lot of 'practice with purpose.' I liked this task because although Ajay was the first to articulate his conjecture, nearly everyone in the class had come to the same conclusion or rather suspicion that this would always happen. It was good for their self-esteem and confidence – they could make conjectures.

Throughout the year, I used the magnificent site nrich (www.nrich.maths.org) and the curriculum mapping on it to delve into tasks where children could explore mathematics related to the topic we were doing. Here are a few examples.

1. A few investigations that led to rich discussions, conjecturing and playing around with expressions:
   a. Perimeter Expressions (https://nrich.maths.org/7283);
   b. Number Pyramids (https://nrich.maths.org/2281);
   c. Always a Multiple (https://nrich.maths.org/7208).
2. Some investigations which were great at the end of the year after introducing quadratics:
   a. Multiplication Square (https://nrich.maths.org/2821);
   b. Always Perfect (https://nrich.maths.org/2034).
3. Investigations that proved very useful in the teaching of geometry:
   a. Changing Areas, Changing Volumes (https://nrich.maths.org/7535);
   b. Warmsnug Double Glazing (https://nrich.maths.org/4889);
   c. Painted Cube (https://nrich.maths.org/2322); this is a classic and remains one of my favorites; it encouraged much discussion and conjecturing on area and volume;
   d. Cyclic Quadrilaterals (https://nrich.maths.org/6624);
   e. Tilted Squares (https://nrich.maths.org/2293) (more on this one later).

The last two investigations (3d and 3e) were particularly important in the experience of turning conjectures into proofs.

Lesson 2: A Silent Lesson

In the middle of the school year, after a fair number of these tasks and many conjectures, I did a 'silent lesson.' A silent lesson is another way of getting students to predict what is coming and then articulate it with words and algebra (no speaking allowed) and although the conjectures from this lesson aren't wordy, students are encouraged to test others' theories or formulae as well as make their own. I will describe what happened in the lesson: I entered the class and instead of doing my usual countdown to get the class quiet, I stood silently, with a calm and composed face, not revealing any emotion, in front of the class, looking down at the children with a beatific smile. "Shall we get our books, *Akka*?" and all other questions were met with a smile and a finger to my lips. The class fell silent as well, any chatty stragglers were reprimanded by other students and all I had to do was to wait patiently. When I had every single eye focused on me, wondering what on earth I was up to, I put my finger to my lips again, held up my piece of chalk, and made a slow and dramatic movement to the board. On it I wrote, slowly and deliberately, pausing to think after each arrow:

I looked around happily at the class, then wrote:

$1 \rightarrow 7$

$2 \rightarrow 12$

$3 \rightarrow ?$

I looked puzzled then, and offered up my chalk to the class. Someone asked a question and I dramatically put my finger to my lips. A hand went up and I gave the child my chalk. He wrote 17 where the question mark was. I walked slowly to the board, and drew a smiley face next to the 17. The class laughed and I immediately put my finger to my lips. I then wrote:

4 → ?

A different child wrote 22. I drew a smiley face and after going through 5, 6 and 7, with nearly all hands up, I wrote:

37 → ?

Until now, children were noticing how the pattern was changing horizontally (add 5 each time), I wanted them to think now about the formula or general term applied vertically. After a few incorrect answers and sad faces next to these, a few children had worked out the formula and were beginning to get the answers when I put down different numbers. I then drew a box on the right hand corner of the board and wrote: "Give hints to your fellow classmates." With some of these hints more children felt confident about the formula and then I wrote:

n → ?

We carried on in this vein, with the formulae getting more and more complex. Towards the end of the lesson, after finding a particularly tricky formula involving squares, I held out the chalk and when a student came to tentatively take it, I went and sat in her place, thus indicating she could make up a formula!

The reason for the silence is (apart from the novelty of it) to focus children's attention on the pattern. Although the entire class was conducted in absolute silence, the children were exhausted at the end, because they were constantly testing their and other children's formulae, constantly adapting their idea with every sad face. The aim of the lesson was, apart from conjecturing, to gain fluency in writing algebraic expressions.

Lesson 3: Tilted Squares

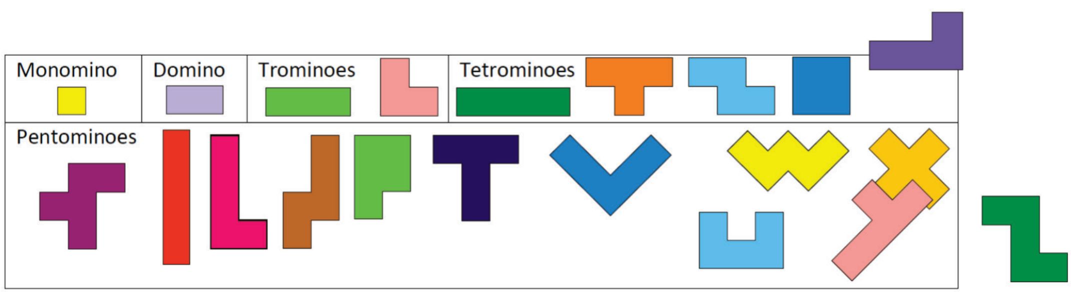
A third lesson I will briefly describe is one we did towards the end of the year, an investigation that leads to the discovery of the Pythagoras theorem. On nrich it is called Tilted Squares (https://nrich.maths.org/2293). It is a very rich investigation, but too much to describe in detail here. The nrich site has a video of it being introduced that is enlightening to watch.

In the course of working on the task, students made up their own notation to describe different types of tilted squares. For example, '3 ↑ 1 →' meant "3 up and 1 across" or a tilted square that can be drawn on dotty paper by going 3 dots up and 1 across from each corner; another students' notation to describe the same square was + + + –. Students made conjectures about ways to find the areas of different sizes or 'tilts' of squares (some looking at the 'straight up' square around the tilted one and subtracting triangles and others looking at the square inside the tilted square and adding triangles). In the course of the lesson they agreed, after some whole-class sharing of strategies, that a table would be an efficient way to keep track of any emerging patterns, and soon the board was full of conjectures! When students finally used their recently honed skills of expanding brackets to come up with the Pythagoras' theorem, there was a real sense of achievement. Many students came to the theorem independently, at different times, and using varied methods, some with the help of fellow classmates or with some guidance from me. When I revealed to them that this conjecture/theorem already had a name and was one of the most famous ones in mathematics, they were having none of it. That one belonged to them, if anything it was, and always would be to them, 7A's Theorem.

ANUSHKA RAO FITZHERBERT is a maths teacher at Rishi Valley School, AP. She did her teacher training, PGCE, in Oxford University and taught in comprehensive schools in the UK before moving back to India. She is keenly interested in all things math education related. She lives at the school with her husband and three children. Here is a blog which she occasionally maintains on teaching 11th and 12th maths: https://taminindiablog.wordpress.com/. She can be contacted at anushkarao@gmail.com.

'POLYOMINOES' – REVISITED

In our first article, *Investigations with Pentominoes,* in the **Low Floor High Ceiling** series in **At Right Angles**, Vol 4, Issue 1, we proved by induction that every polyomino has an even number of sides. It was a long visual proof considering various cases and subcases. Here is a shorter proof of the same result, sparked by discussions with the B. Math students at Indian Statistical Institute.



Any polyomino is made of squares of the same size. So each internal angle of a polyomino must be an odd multiple of a right angle, i.e., either 90° or 270°. This means that we can always arrange any polyomino on the Cartesian plane so that its sides are parallel to the axes. We will refer to the sides parallel to the x-axis as 'horizontal' and those parallel to the y-axis as 'vertical.' Now horizontal and vertical sides must alternate, i.e., between any two consecutive horizontal sides there must be a vertical side, and vice versa. Since polyominoes are polygons with closed curves, the number of horizontal sides must equal the number of the vertical sides. Therefore, if any polyomino has k horizontal sides, then it must have k vertical sides as well, making it a $2k$-gon, i.e., a polygon with even number of sides.

The same argument extends to polyominoes with holes. A n-omino with a hole can be thought as a $(n + m)$-omino from which a m-omino has been taken out, for some $m < n$, where the $(n + m)$-omino and the m-omino are without any holes and have $2k$ and $2l$ number of sides respectively. So the number of sides for this n-omino is the sum of the number of sides of the $(n + m)$-omino and that of the m-omino, i.e., the number of sides for the n-omino with a hole = $2k + 2l = 2(k + l)$, which is an even number. A similar argument can be extended for a polyomino with multiple holes.

– Swati Sircar

ClassRoom

The 'CONJOINING' error in SCHOOL ALGEBRA

K. SUBRAMANIAM

Many students who become reasonably proficient in arithmetic face great difficulties with school algebra, which may lead to a cascading spiral of low performance and eventually to their giving up mathematics. Understanding why these difficulties occur is the first step in changing one's teaching to deal effectively with them. If students can be helped through these difficulties, it can lead to fewer students dropping out of mathematics or out of learning tracks like science that need mathematics. This article briefly discusses a particular error in algebra that is quite common and the possible reasons for this error. Even though our focus is just on one particular error, analysing the error leads to insights about deeper issues that students have with algebra. Our discussion draws on research in mathematics education on the learning of algebra. Such research, done in many places across the world including in India, not only analyses various kinds of errors that students make but also develops better approaches to the teaching of algebra. A discussion on this research would be too long to include here. However, we provide references to articles that describe one such approach to the teaching of algebra that was developed at the Homi Bhabha Centre.

The Conjoining Error

The conjoining error, which is quite common, is seen in responses to the task of simplifying algebraic expressions.

$$1)\ 5 + 2a = 7a^{*} \qquad 2)\ 5a + 2b = 7ab^{*}$$

(The asterisk to the right indicates that the response is incorrect.)

Many teachers would recognize this as one of the most frequent errors that they see in students' work. The name "conjoining

Keywords: Algebra, Error Analysis, Cognition, Visuospatial, Expression, Equality, Simplification, BODMAS

error" refers to the incorrect joining of the two terms. Before we proceed, we need to clarify that both sentences are actually *not* incorrect if they are interpreted as equations rather than identities. The sentence (1), interpreted as an equation, would be true when $a = 1$. Similarly sentence (2) would be true for all pairs of *a* and *b* given by the function $b = 5a/(7a - 2)$ *with* $a \neq 2/7$. The sentences are false only when they are interpreted as identities, i.e., a sentence that is true for all values of the variables. A *simplification* of the expression on the left to the one on the right is possible only if the LHS is identically equal to the RHS.

How would you deal with this error if you were a teacher? A frequent suggestion is that we must stress the concept of 'like' terms. You can add like terms just as you can add apples to apples but not apples to bananas. But it is very hard to remember this, especially when faced with (2) above. A student might think that we can always put 5 apples and 2 bananas together in a basket to get 7 fruits, which are apples and bananas. Thus the student responds with $5a + 2b = 7ab$ *, and the 'fruit salad' algebra breaks down. Moreover, the student is led to think that '*a*' in the expression stands for things like apples, rather than standing for a number, which is an even more serious misunderstanding.

Another way to think about the error is to see if there is a counterpart in arithmetic. We can indeed find a counterpart to (1), which is presented below in (1a), but it is difficult to think of a counterpart for (2).

1) $5 + 2a = 7a$ *

1*a*) $5 + 2 \times 3 = 7 \times 3 = 21$ *

The error in (1a) is that the convention for the order of operations or the BODMAS rule has been broken. Since the error in the algebraic sentence looks very similar to the arithmetic error, one may think that the right way of dealing with this is to remind students of the BODMAS rule and give them practice in applying it. However, the underlying reasons for the errors in (1) and (1a) may be very different, as we will see. In other words, although the error in (1) and (1a) are *mathematically* similar, the *cognitive* aspects that lead to the errors may be very different.

First, let us think about why the rule of order of operations is taught. Many mathematicians might say that such a rule is unnecessary. In fact, the LHS expression in (1a) is ambiguous because we have not put brackets. We can put brackets in two ways to get two different values:

$$(5 + 2) \times 3 = 7 \times 3 = 21 \quad \text{or} \quad 5 + (2 \times 3) = 5 + 6 = 11$$

Once we put brackets, both of these are correct. So a mathematically correct view would be that we must put brackets whenever we have two or more binary operations in a single expression. Otherwise, the expression would be ambiguous, except when the binary operations are all addition or all multiplication, in which case the different ways of putting brackets lead to the same result.

$$(5 + 2) + 3 = 5 + (2 + 3) = 10 \quad \text{and}$$
$$(5 \times 2) \times 3 = 5 \times (2 \times 3) = 30$$

This, of course, is because of the associative property of addition and multiplication.

In the light of the above, it is clear that the BODMAS or any other rule for the order of operations is a convention that allows us to interpret an expression with multiple binary operations *when brackets are not written*. (Incidentally, the BODMAS rule when applied precisely to an expression like 30 – 10 + 10, leads to an error since it suggests that addition is to be done before subtraction. 'BODMSA' is more faithful to the convention than 'BODMAS'.) Why do we need such a convention? Why not simply put brackets for all numerical expressions and do away with the need for such rules, which anyway are difficult for students to remember and apply correctly? It is worth thinking about this proposal.

It appears that the actual reason for teaching a rule like BODMAS is that it prepares students to interpret and work with algebraic expressions. The rules for simplifying and manipulating algebraic expressions will break down if the expressions do not have a definite and unambiguous value when the variables are substituted with numbers.

What about the suggestion to put lots of brackets to make the expression unambiguous? Putting brackets in algebraic expressions makes them hard to read and interpret and so we must minimize the use of brackets. Of course, the BODMAS rule also applies to algebraic expressions and multiplication gets priority over addition and subtraction, just as in numerical expressions. However, the BODMAS rule is rarely invoked while parsing algebraic expressions. This is because algebraic expressions use visuospatial and reading conventions to encode which operations get priority. For example, the sign for the multiplication operation is omitted both in writing (and reading) algebraic expressions to signal the priority of multiplication over addition or subtraction. The convention for writing (and reading) exponents signals the priority of this operation over others. Note that these conventions are very different from the BODMAS convention for numerical expressions. Hence it is unlikely that violating the BODMAS rule, which underlies the error in (1a) above, also causes the error in (1). Further, even if a student knows the BODMAS rule perfectly well, she or he cannot use the rule to simplify the expression in (1) because of the presence of the letter variable instead of a number. Thus, the presence of letter variables constrains the use of rules of order of operations and limits their usefulness.

There is also a noticeable difference between the BODMAS convention and the conventions for algebraic expressions. The former is a verbal rule that states which operations precede which and is encoded through an abbreviation or a mnemonic. In contrast, the conventions for algebraic expressions are visuospatial and based on ways of writing. It suggests that working with the BODMAS rule may not be of great help in learning to parse algebraic expressions correctly. Is it possible to work with numerical and algebraic expressions using similar conventions for parsing both kinds of expressions? Indeed, an approach to working with numerical expressions that uses visual parsing of "terms" supported by appropriate naming has been found to be helpful in bridging the gap between arithmetic and algebra (Banerjee & Subramaniam, 2012).

Let us now return to the conjoining error. One of the explanations proposed for why students make this error is that they find "unclosed expressions" unacceptable as answers. In other words, their experiences in arithmetic leads them to think that a simpler looking expression needs to be written to the right of the "=" sign. A response such as the following

$$5 + 2a = 5 + 2a \quad \text{or} \quad 5a + 2b = 5a + 2b$$

may appear as a kind of cheating – writing the question again as the answer because the unclosed expression on the right "$5 + 2a$" looks like a question rather than an answer. In contrast, the "closed" expression "$7a$" looks compact and like an answer. The reluctance to accept unclosed expressions as answers brings to the fore a move that is at the heart of algebraic thinking, which is to actually accept a question as an answer! That is, we allow the expression that shows the operations to be carried out also to stand for the result of the operation. Thus $5 + 2$ or $2 \times 3 + 1$ are not only expressions that tell us to carry out certain operations, but may also stand for the answer that is obtained as the result, namely, the number 7. Thus $5 + 2a$ and $7a$ stand for numbers, and these numbers are in general not equal when the same value of a is used in both expressions. Seeing this depends on seeing that the sentence in (1) or (2) is about the equality of numbers on the left and right sides of the "=" sign, numbers which are "variable" and become "fixed" once the variables a and b are substituted. Indeed, many students interpret the "=" sign as asking one to "do something and write the answer" rather than as stating the equality of expressions on either side of the sign. This interpretation leads to errors when faced with a question like "11 + 7 = __ + 9" to which students may respond with "11 + 7 = **18** + 9 = 27*". Such students also think that there is something very wrong with a sentence like "3 = 3".

The reification principle

The algebraic principle of allowing an expression to stand for the result of the expression, sometimes called the "reification principle," is already used in arithmetic. For example, the result of dividing 5 by 7 is expressed as 5/7. In general, $a \div b = a/b$, which is very much like "writing the question as the answer". Similarly the square root of 2 (or any number a) is simply written as $\sqrt{2}$ (or $\sqrt{a}$ in general). This is a very important aspect of algebra that students generally miss out on and it helps enormously if teachers point it out to them. Understanding the reification principle allows them, for example, to recognize that $(a+b)^2$ and $a^2 + 2ab + b^2$ are different expressions for the same number (when a and b are substituted with numbers). It allows them to read and grasp expressions, because they contain information about the number represented. Thus, 48 + 47, 45 + 45 + 3 + 2, and 50 + 50 – 2 – 3 are different expressions for the same number and hence equivalent, but encode different ways in which the number 95 is 'composed' from other numbers. (We have described the information encoded by an expression as the *operational composition* of a number in Subramaniam and Banerjee, 2011.) An expression such as 199 + 70 × 0.5, written to show how a cellphone tariff is calculated, suggests that there is a fixed cost of Rs 199 and a rate of Rs 0.5 per minute. Even the numeral "536", is a short form of the expression 5 × 100 + 3 × 10 + 6, which shows the operational composition of the number 536 in terms of multi-units which are different powers of ten. The capability to "read" expressions in this manner is important in algebra and requires the understanding of the reification principle as its foundation. Again we refer to Banerjee and Subramaniam (2012) for a way in which such ability can be developed while working with numerical expressions as a preparation for algebra.

References

1. Banerjee, R. & Subramaniam, K. (2012). Evolution of a teaching approach for beginning algebra. *Educational Studies in Mathematics*. 80:351–367.
2. Subramaniam, K. & Banerjee, R. (2011). The arithmetic-algebra connection: A historical-pedagogical perspective. In Cai, J. & Knuth, E. (Eds). *Early Algebraization: A Global Dialogue from Multiple Perspectives*. Springer, 87-107.
3. Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 1–36.

Misconceptions in FRACTIONS

RICHA GOSWAMI

Introduction

Mathematics is notorious for being a difficult subject. Algebra, as a whole, is feared and despised; so also is a topic like Fractions. There are many reasons that make mathematics a difficult subject. One of them is its already earned reputation. Teachers, parents and children (as a result of the other two) start the process of teaching and learning mathematics with a pre-conceived notion of the subject being difficult.

There are other reasons too, rooted in the nature of the subject, that make it difficult to grasp and grapple with, unlike other school subjects. It is a highly abstract area of study, based on assumptions and logical derivations. Because of its logically derived subject content, it is hierarchical in nature. So the knowledge of previous concepts is essential for further study in the subject. For example, to understand the concept of multiplication, a learner needs to understand and be comfortable with the concept of addition.

One of the most abstract concepts introduced in primary classes is Fractions. Unlike other number sets introduced until now (natural numbers and whole numbers), fractional numbers are not used for counting. They basically denote a proportion. There is much research and writing around difficulties in learning of fractions and also about its pedagogy. In this article, we shall focus only on some of the misconceptions related to fractions that children develop.

Misconceptions or errors

The words misconceptions, errors, mistakes, alternative frameworks, etc., are often used interchangeably. Here, I make

Keywords: Misconceptions, Errors, Fractions, Part-Whole, LCM, Representation, Pedagogy

two categories to understand them better. The words 'misconceptions' and 'alternative frameworks' are close to each other, as are the words 'errors' and 'mistakes.' Though there may be small differences in words which have been placed together, for the purpose of this paper, I will not engage with that.

An error is a result of carelessness, misrepresentation of symbols, a lack of knowledge of that particular area, or a task that is far too demanding of the child's current level. A misconception, on the other hand, implies that the learner's conception of a particular idea or topic, of a rule or algorithm, is in conflict with its accepted meaning and understanding in mathematics (Barmby et al., 2009). It could be a wrong application of a rule, an over- or under-generalization, or an alternative conception of the situation. For example, the rule that "a number with three digits is bigger than a number with two digits" works only in some cases. When you compare 35 and 358, the rule gives the correct answer, but not when you compare 35 and 3.58.

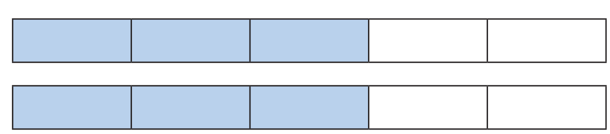
Misconceptions, unlike errors, are a sign of what the child knows, or a sign of the child's present level of understanding. Thus, from a teacher's perspective, an attempt to uncover the misconceptions of students is a very productive activity, as it is a guide to the future teaching-learning process.

Some misconceptions in Fractions

Fractions have often been considered one of the significant culprits in scaring people away from mathematics. As discussed above, the abstractness of fractions as numbers is difficult to grasp and, to top this, the introduction of the algorithms of operations widens the understanding gap. At a time when children require more experience in visualizing fractions as numbers, complicated procedures are introduced to carry out operations. As a result, children often make efforts to remember the procedure without understanding why it is being done, and experience of teacher training indicates that often the teachers do the same. Since children often remember the procedures and not the reasons, they end up applying them in incorrect situations. Example: Both students and teachers know that to add fractions, you need to take the LCM of the denominators. Is it correct? Yes. But is it enough? No. A child who knows this as a rule does not understand why the LCM is needed to add and subtract fractions. As a result, it is common to see children extrapolate this 'rule' to multiplication as well.

- **Missing the importance of equal parts:** The denominator in a fraction not just represents the parts into which the whole has been divided, it also implies that the whole has been divided into as many equal parts. A common misconception is to focus only on the number of parts, and not at their being equal. This misconception is often propagated by the colloquial usage of phrases like a 'half-glass of milk' or a 'half-roti.' In such usages, it is not essential for the glass to be exactly half-full or empty, but just less than full. This misconception shows itself when students are asked for the pictorial representation of a fraction.

- **Representation of a situation in fractions and presenting a fraction:** A common misconception about fractions is that a fraction represents part of a whole. This comes out when asked to pictorially represent an improper fraction. For example, represent 3/5 and 7/4:



A person who is able to represent 3/5 knows that the denominator indicates the number of parts into which the whole has been divided, but when it comes to 7/4, the person reverses the understanding. Why? Perhaps because it is impossible to show 7 out of 4. So it must be 4 out of 7.

In a test given to 9th class students, the following question was asked, "A watermelon is cut into 16 parts and of those, I ate 7 parts and my friend ate 4 parts. How much watermelon was eaten by both of us? Represent as a fraction." In a class of more than 20 students, only 2 got the answer and the rest made a variety of errors, some of which indicate misconceptions about fractions.

A child while attempting to answer this question wrote as follows:

- Watermelon: 16
- Pieces: 7
- Friend: 4
- Total: 52

Another child wrote: 16 + 7 + 4 = 27

These responses indicate problems at two levels: the first is about how children read and comprehend/unpack a word problem and the second is about the conception of number sets, which, in these cases seem to be limited to integers or perhaps only to whole numbers.

The responses indicate that some children while reading a question are only trying to ascertain given numeric data and then applying some operation to this. The randomness of the operation to be applied is evident in this case. It seems that these children have an understanding that everything else written in the question is either distraction or meaningless frills. They do not seem to infer from the question which operation needs to be applied.

The second misconception that seems evident from the above responses is about the conception of number sets. The number system for many children is restricted to positive integers or whole numbers. The double-decker numbers (fractions) are not seen as a part of the number system. Thus children with such misconceptions would attempt to work with fractional numbers, when presented as fractional numbers (i.e., double-decker form), but would not be able to represent situations or pictures in p/q form. Another example of this misconception was found in a response where the child wrote 11 watermelons (which is the sum of 7 and 4) instead of 11/16 which represents the portion of watermelon consumed.

Some responses reveal that the understanding of fractions emerges in stages. So the first stage as discussed above is where there is no familiarity with the p/q form. And the second stage is when there is familiarity but the child is still unable to represent the situation correctly in the fractional form. Two responses to the above question, such as 16/5 and 7/4, indicate this particular misconception. In the first response the child has written 16 (the total number of pieces) as the numerator (instead of denominator) and in the second case, the two numbers, which indicate parts of watermelon have been written in the p/q form.

- Believing that in fractions, numerators and denominators can be treated as separate whole numbers.

  It is common to see children add or subtract fractions by treating numerator and denominator as separate whole numbers. In the above mentioned question itself, a child who could correctly represent the two fractions involved as 7/16 + 4/16, wrote the final answer as 11/32.

- **Misconceptions related to simplifying:** Dividing top and bottom by common factors is often loosely referred to as canceling common factors or numbers which then leads to responses as follows:

$$\frac{7}{16}+\frac{4}{16}=\frac{7+4}{16}=\frac{7+1}{4}=\frac{8}{4}=2.$$

  Another issue highlighted by the response is that following algorithms and getting an answer has no connection with the question at hand. As a result, the absurdity of the response does not bother the child.

- **Failing to find a common denominator when adding or subtracting fractions with unlike denominators:** Students often do not understand why it is important to make the fractions like before adding or subtracting them. For example,

$$\frac{2}{3}+\frac{4}{7}=\frac{6}{10} \quad \text{or} \quad \frac{1}{6}+\frac{2}{3}=\frac{3}{9}.$$

Without understanding the reason or the need, procedural knowledge of taking the LCM and proceeding poses its own misconceptions as seen in the following example:

$$\frac{7}{16}+\frac{4}{16}=\frac{1+7+1+4}{16}=\frac{8+5}{16}=\frac{13}{16}.$$

The learner here took the LCM, which is 16 and then instead of multiplying with 1 (obtained by dividing LCM with the denominator) has added it to both the numerators, thus obtaining an incorrect answer.

- **Leaving the denominator unchanged in multiplication of like fractions:** an overgeneralization of the addition rule to multiplication leads to responses as follows:

$$\frac{2}{5}\times\frac{1}{5}=\frac{2}{5}.$$

This misconception is basically an overgeneralization of the algorithm for adding like fractions.

- **Failing to understand the invert-and-multiply procedure for solving fraction division problems:** The procedure for division of fraction numbers seems to cause many problems to children. The following response indicates one such:

$$\frac{7}{4}\div\frac{3}{2}=\frac{7}{4}\times\frac{2}{3}=\frac{21\times 8}{12}=\frac{168}{12}.$$

The child here did invert the divisor, but after that, instead of simply multiplying them, did a cross multiplication, followed by another multiplication between the two numbers in the numerator.

Another division related misconception, which is perhaps an overgeneralization of the multiplication procedure, is to cancel before inverting. For example,

$$\frac{2}{3}\div\frac{6}{7}=\frac{2}{1}\div\frac{2}{7}=\frac{2}{1}\times\frac{7}{2}=\frac{1}{1}\times\frac{7}{1}=\frac{7}{1}.$$

Implication for teachers

A study of misconceptions is both productive and interesting. It reveals patterns of learning of a particular child but, in most likelihood, is not unique to him/her, and thus it becomes a window to understand how children learn.

A discussion on identified misconceptions will not only help the particular child but all the children. An acceptance of misconceptions as a process of evolving understanding would help us in developing learners who are confident about their learning and process of learning. This would obviously require teachers to develop a culture of discussion and analysis of errors and not just a discussion of the correct process of solving a problem.

RICHA GOSWAMI worked with Vidya Bhawan Education Resoure/Centre from 2008 to 2015. She has done her Masters in Elementary Education from Tata Institute of Social Sciences, Mumbai and is currently pursuing PhD on parental perception of quality education. She may be reached at goswami84@gmail.com.

ClassRoom

An easy construction for the HARMONIC MEAN

MOSHE STUPEL &
VICTOR OXMAN

We are given two positive numbers a and b. We wish to show how to construct a length corresponding to the harmonic mean of a and b, namely, the quantity

$$\frac{2ab}{a+b}$$

Construct $\triangle ABC$ in which $BC = a$ and $AC = b$. The third side c (alternatively, the included $\measuredangle ACB$) can be chosen arbitrarily; see Figure 1. Next, draw the internal bisector of $\measuredangle ACB$. Let it intersect side AB at D. Draw DE parallel to side BC, with E on side AC. Let x be the length of DE. Then:

$$x = \frac{ab}{a+b} = \text{half the harmonic mean of } a \text{ and } b$$

This can be checked using the applet https://www.geogebra.org/m/etMhyzaE. For the proof, please examine Figure 1.

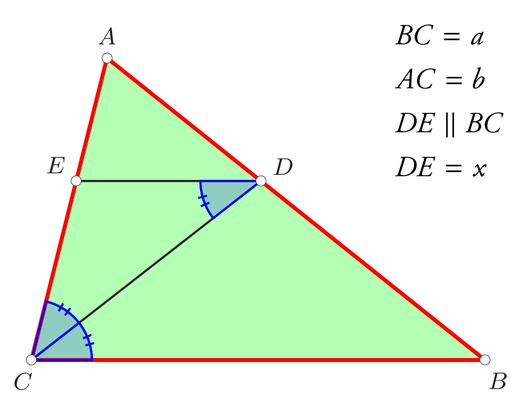


Figure 1

Keywords: Proof without words (PWW), harmonic mean

DR. MOSHE STUPEL is Professor, Head of Mathematics Department, a researcher, and a pre-service teacher educator in the Shaanan Academic College and in the Gordon Academic College. Earlier he was a high school principal. He has published 40 papers on mathematics and mathematics education. Recently, his research is focused on methods of problem solving and on variance and invariance property in dynamic geometrical environments (DGEs) in mathematics. He may be contacted at stupel@bezeqint.net.

DR. VICTOR OXMAN is a Professor at Western Galilee College, Acre, Israel. He teaches courses on mathematics and statistics. His research interests are in geometry and mathematics education. In these fields, he has published many books and papers for researchers, teachers, and students. At present, he focuses on the study of methods of teaching mathematics using dynamic geometry software. He may be contacted at victor.oxman@gmail.com.

PRIME MAGIC SQUARES

— C⊗MαC

Do you know any magic squares composed only of prime numbers?
Here is one that we found, of order 3:

47	29	101
113	59	5
17	89	71

Its magic sum is 177.

And here is a magic square of order 4, composed only of prime numbers:

79	89	53	37
47	67	61	83
31	59	71	97
101	43	73	41

Its magic sum is 258.

Needless to say, it is quite a challenge to find such magic squares. We invite you to find another magic square of order 3, composed only of prime numbers.

ClassRoom

Visual Proof of THE TWO-VARIABLE AM-GM INEQUALITY

PROF. K. D. JOSHI

The AM-GM inequality which is normally stated in the following form: *"If a and b are any two non-negative real numbers, then*

$$\frac{a+b}{2} \geq \sqrt{ab},$$

with equality holding if and only if $a = b$*,"* may be stated in the following equivalent form, where we have used the numbers a^2 and b^2 rather than a and b: For any two positive real numbers a and b, we have

$$\frac{a^2+b^2}{2} \geq ab, \qquad (1)$$

with equality holding if and only if $a = b$.

The algebraic proof of (1) consists of recognising that $a^2 + b^2 - 2ab$ is a perfect square, namely, $(a - b)^2$ which is always non-negative as a and b are real numbers. This is probably the simplest and the most direct proof. But to justify the word 'geometric' in the definition of the 'geometric mean' and hence in the name of the inequality, it is desirable to have a geometric proof. One such proof has appeared on pp.42–43 of *At Right Angles*, August 2017, in an article by Shailesh Shirali. In addition to proving the inequality, the proof also gives a geometric construction for the geometric mean.

But if merely proving (1) geometrically is the goal, there is a much more direct proof which we now give.

Without loss of generality, assume $a \geq b$. Construct right-angled isosceles triangles OAA' and OBB' with legs a and b respectively, with B' lying on OA' as shown in Figure 1. Extend BB' to meet AA' at C.

Keywords: AM-GM inequality, visual proof

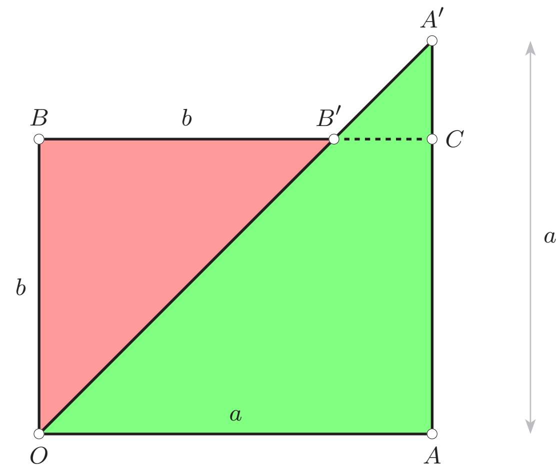


Figure 1

It is clear that the union of the two triangles OAA' and OBB' covers the rectangle $OACB$ and hence has a higher area except when $B' \equiv A'$. The inequality (1) follows by taking the areas of these two triangles and of the rectangle $OACB$.

PROF. KAPIL D JOSHI received his PhD from Indiana University in 1972. He served in the Department of Mathematics at IIT Powai, 1980–2013. He has written numerous mathematics textbooks, among them *Introduction to General Topology, Calculus for Scientists and Engineers – An Analytical Approach and Educative JEE (Mathematics)*. He was keenly involved for many years in the GATE and JEE. He may be contacted at kdjoshi314@gmail.com.

Student Corner – Featuring articles written by students.

Solution of Ramanujan's DOOR NUMBER PROBLEM by using VARGAPRAKRITI

BODHIDEEP JOARDAR

The Strand Magazine Door Number Problem, now eternally associated with Ramanujan, may be stated more generally as follows: *A street has n houses numbered consecutively, the numbers starting from 1, and there is a house numbered x such that the sums of the house numbers on each side of x are the same. Find n and x, given that n lies within some specified range.*

As per the stated information, $1 + 2 + \cdots + (x-1) = (x+1) + (x+2) + \cdots + n$. Add $1 + 2 + \cdots + (x-1) + x$ to both sides:

$$2(1 + 2 + \cdots + (x-1)) + x = 1 + 2 + \cdots + n.$$

The expression on the left side simplifies to $x(x-1) + x = x^2$. Hence:

$$x^2 = \frac{n(n+1)}{2}$$

To solve this equation, Ramanujan used continued fractions. His approach has become part of history ever since!

Our question is: *Is there an approach apart from continued fractions?* The answer is: **Yes**. I demonstrate such an approach. First I rearrange the above equation into a more familiar shape. Multiply both sides by 8 and then add 1 to both sides; I get:

$$8x^2 + 1 = 4n(n+1) + 1,$$

i.e.,

$$(2n+1)^2 - 8x^2 = 1.$$

This is nothing but an instance of the Indian mathematician Brahmagupta's *Vargaprakriti*, the indeterminate quadratic equation $y^2 - kx^2 = 1$ on which he worked in the 6th century CE.

Keywords: Door number problem, Vargaprakriti, Ramanujan, Brahmagupta, bhāvanā.

Here it is of the form $y^2 - 8x^2 = 1$, where $y = 2n + 1$. (See Box 1 for the history of this equation.)

Therefore, by finding the solutions (which are infinite in number) of $y^2 - 8x^2 = 1$, I should get solutions to the generalised Strand Magazine Door Number Problem.

According to the composition law (***bhāvanā***) found by Brahmagupta, if (y_1, x_1) and (y_2, x_2) are solutions to $y^2 - kx^2 = 1$, then so is $(y_1y_2 + kx_1x_2, y_1x_2 + x_1y_2)$. By applying *bhāvanā* again and again, infinitely many solutions can be generated.

The most obvious solution of $y^2 - 8x^2 = 1$ is $y = 3$, $x = 1$, i.e., $n = 1$, $x = 1$. This solution corresponds to there being just one house in the street.

Starting with the pair (3, 1) and applying *bhāvanā* on itself, I get the solution

$$(3^2 + 8 \cdot 1^2, 2 \cdot 3 \cdot 1) = (17,6),$$

i.e., $n = 8$, $x = 6$, or 8 houses; the desired one is the 6th one. Next, applying *bhāvanā* on the pairs (17,6) and (3,1), I get the solution

$$(17 \cdot 3 + 8 \cdot 6 \cdot 1, 17 \cdot 1 + 6 \cdot 3) = (99, 35),$$

i.e., $n = 49$, $x = 35$, or 49 houses; the desired one is the 35th one. Thus I generate infinitely many solutions of $y^2 - 8x^2 = 1$ and find the answer according to the specified range of n. Some pairs of solutions obtained in this manner and the corresponding values of n are listed in the table below.

$y = 2n + 1$	x (desired Door Number)	n (no. of houses)
3	1	1
17	6	8
99	35	49
577	204	288

The Strand Magazine problem states that $50 \le n \le 500$, hence $n = 288$, $x = 204$. *So there are 288 houses, and the desired house number is 204.*

Note from the editors: Another method for solving this problem has been described in the article on Ramanujan by Utpal Mukhopadhyay, elsewhere in this issue.

BODHIDEEP JOARDAR (born 2005) is a student of South Point High School, Calcutta who is a voracious reader of all kinds of mathematical literature. He is interested in number theory, Euclidean geometry, higher algebra, foundations of calculus and infinite series. He feels inspired by the history of mathematics and by the lives of mathematicians. His other interests are in physics, astronomy, painting and the German language. He may be contacted at ch_kakoli@yahoo.com.

Vargaprakriti

Vargaprakriti refers to the second-degree indeterminate equation in two variables, $y^2 - Nx^2 = 1$, which must be solved over the positive integers. Here N is an arbitrary positive integer. For example, the equation that arises in the door number problem is $y^2 - 8x^2 = 1$.

Literally, Vargaprakriti means 'equation of the multiplied square'; Varga means 'coefficient' and refers to the number N in the equation $y^2 - Nx^2 = 1$.

The equation is better known as the ***'Pell equation'*** (after John Pell, a 17th century English scholar), but the name is now known to be a historical inaccuracy. These equations were first studied in detail by Brahmagupta in the 6th century CE, and later by Bhaskaracharya II (who developed a so-called *'Chakravala'* or cyclic method of solution) and others. A more appropriate name for the equation would therefore be the Brahmagupta-Bhaskara equation.

For more on this topic, the reader is directed to the following excellent reference material which we have used freely: http://www-groups.dcs.st-and.ac.uk/history/Miscellaneous/Pearce/Lectures/Ch8_6.html

ClassRoom

Using Co-Ordinate Geometry to find the GCD and LCM of TWO NUMBERS *a* & *b*

TEJASH PATEL

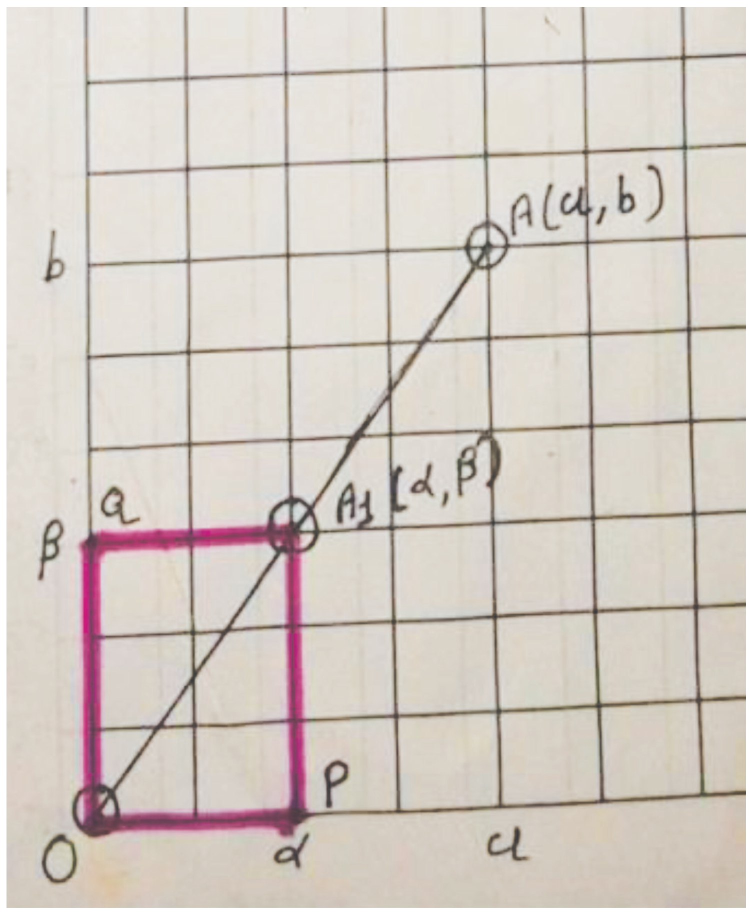


Figure 1.

Claim: Let ***a, b*** be natural numbers. If we plot the point (a, b) on a square grid and draw the line joining (0, 0) to (a, b), then the GCD of a and b is given by the number of grid points on this line decreased by 1.

Keywords: Natural Numbers, LCM, HCF, Coordinate Geometry, Lattice Points, Visualization, Representation

Illustration: Find the GCD of 4 and 6.

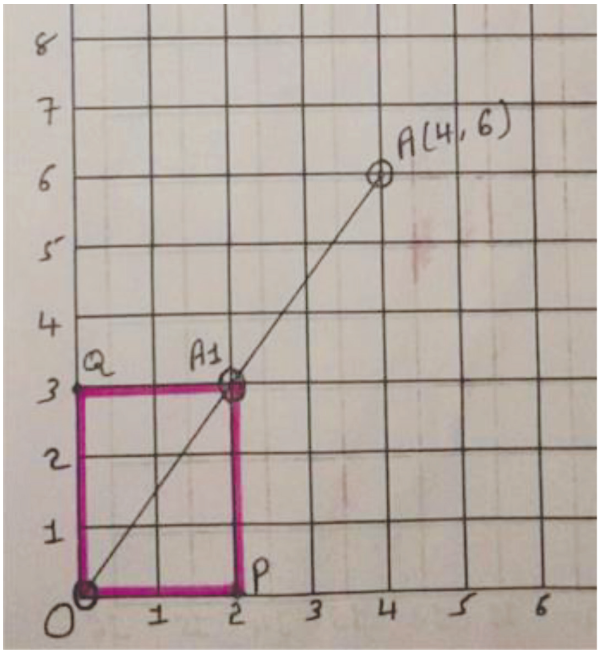


Figure 2

- Represent point A (4, 6) in the sheet.
- Construct line segment $\overline{OA}$ whose end points are O (0, 0) and A (4, 6).
- Find the number of lattice points on $\overline{OA}$. Here, the number is 3.
- **GCD = number of lattice points – 1 = 3 – 1 = 2.**

  Thus, GCD of 4 and 6 is 2.

Claim: In Figure 1, choose the lattice point O (0, 0) and the lattice points next to it on the same line, say, A_1, construct the rectangle OPA_1Q, and then find the number of squares contained in the rectangle OPA_1Q. Then:

LCM (*a, b*) = GCD (*a, b*) × number of squares contained in Rectangle OPA_1Q.

Illustration: Find the LCM of 4 and 6

- In Figure 2, choose lattice point O (0, 0) and the 'next' lattice point, A_1.
- Construct rectangle OPA_1Q. Find the number of squares in OPA_1Q.
- The LCM can be found with the help of the following formula

LCM = GCD × number of squares contained in rectangle OPA_1Q = 2 × 6 =12.

Explanation

1. Consider point A (*a, b*) in the sheet (Figure 1). Write the equation of $\overleftrightarrow{OA}$ as follows.

   $$\therefore \overleftrightarrow{OA} : y = \frac{b}{a}x$$

If k is a common divisor of a and b, then $a = k\alpha$ and $b = k\beta$ for some positive integers α and β.

If we substitute $x = \alpha$ in the equation of $\overleftrightarrow{OA}$, we get $y = \beta$ and hence point $A_1(\alpha, \beta)$ is on line segment $\overline{OA}$.

The equation of $\overleftrightarrow{OA}$ can therefore be simplified to $y = \frac{\beta}{\alpha}x$.

We see that y has an integer solution only if x is a multiple of α.

Now $0 \le x \le a$, therefore $0 \le x \le k\alpha$. There are $k + 1$ multiples of α starting with 0 and ending with $k\alpha$.

   ∴ Number of lattice points = k + 1

   ∴ k = number of lattice points – 1

   ∴ GCD (*a,b*) = number of lattice points – 1

2. Now LCM (a,b) = LCM$(k \cdot \alpha,\ k \cdot \beta)$

   ∴ LCM (a,b) = k · LCM(α, β) where α *and* β are relatively prime.

   ∴ LCM$(a,b) = k \cdot \alpha\beta$

   ∴ LCM (a,b) = GCD (a,b) × Area of rectangle OPA_1Q.

   ∴ LCM (a,b) = GCD (a,b) × number of squares in rectangle OPA_1Q.

TEJASH PATEL is from Patan in Gujarat. He teaches in Chanasma Primary School No. 2. He has an M.Sc in Mathematics and a B.Ed. His interests are in Coordinate Geometry and Statistics. At present, he is working towards a post graduate diploma in Applied Statistics from IGNOU. He may be contacted at tejash_82@yahoo.com.

Why does Euclid's GCD ALGORITHM work?

SEETHA RAMA RAJU SANAPALA

We start with the definition of the *GCD* of two numbers. (Throughout this article, 'number' means 'integer.')

Definition: The *GCD* or 'Greatest Common Divisor' of two numbers, also called the Highest Common Factor *(HCF)*, is:

- A divisor of both the numbers, i.e., it is a common divisor.
- Of all the common divisors, it is the greatest.

Note that the *GCD* is a multiple of every other common factor of the two numbers.

Example 1: Consider the numbers 8 and 20. Notice that some factors are common to both 8 and 20, namely: 1, 2 and 4. Of these common factors, 4 is the greatest and it is called the *GCD* of 8 and 20. We write: *GCD* (8, 20) = 4.

Example 2: Consider the numbers 24 and 36.

- The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24
- The factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18 and 36.

The common factors of 24 and 36 are 1, 2, 3, 4, 6 and 12. The largest of these, 12, is the *GCD*. We write: *GCD* (24, 36) = 12. Notice that 12 is a multiple of the other common factors.

The above method of enumeration for finding the *GCD* is cumbersome and error-prone. The genius Euclid came up with an efficient and less vulnerable algorithm for this problem. Before we discuss this, we look at some facts about the *GCD* (we encourage you to justify these claims).

Keywords: Positive Integer, GCD, Algorithm, Reasoning, Proof

1. GCD (0, 0) is not defined.

2. When x, y are both non-zero, $GCD(x, y) = GCD(y, x)$.

3. When $x \neq 0$, $GCD(x, 0) = GCD(0, x) = |x|$.

4. $GCD(x, y) = GCD(x, -y) = GCD(-x, y) = GCD(-x, -y) = GCD(|x|, |y|)$.

In view of these facts, we may consider both x and y to be non-negative and $x \leq y$, with no loss of generality.

Euclid's algorithm is based on the following two properties:

(i). $GCD(x, y) = GCD(y - xk, x)$;

(ii). $GCD(x, xk) = |x|$,

where k is any integer and where neither x nor y is zero.

Using these properties, the algorithm converts the problem of finding the GCD of a given pair of numbers, x and y, to finding the GCD of a smaller pair of numbers, $y - xk$ and x. This is repeated till we reach situation (ii) above.

Consider these examples:

- $GCD(12, 32) = GCD(8, 12) = GCD(4, 8) = 4$;
- $GCD(25, 60) = GCD(10, 25) = GCD(5, 10) = 5$.

Proof of property (i): $GCD(x, y) = GCD(y - xk, x)$, for any integer k.

It is clearly enough if we prove the following:

(a) Every common divisor of x and y is a common divisor $y - xk$ and x;

(b) Every common divisor of $y - xk$ and x is a divisor of x and y.

Proof of (a): Suppose that d is a common divisor of x and y, i.e., $d|x$ and $d|y$. Let k be any integer; then $d|xk$. Hence $y - xk$ is a difference of two multiples of d and is therefore a multiple of d. Hence d is a common divisor of $y - xk$ and x.

Proof of (b): Let $d|y - xk$ and $d|x$; then $d|xk$. So $(y - xk) + xk = y$ is a sum of two multiples of d and is therefore a multiple of d. Hence d is a common divisor of y and x.

And that's why Euclid's algorithm works! QED

References

1. https://answers.yahoo.com/question/index?qid=20070625061113AAPWfC1

SEETHA RAMA RAJU SANAPALA completed his BE from Andhra University (1984) and his ME from Osmania University (1986). He worked in DRDO as a scientist for 16 years and at AT&T Bell Labs Innovations for 11 years. Currently, he teaches engineering students. He has publications in popular science magazines and technical journals. He may be contacted at manjuvenamma@yahoo.co.in.

On DIVISIBILITY BY 27

SHAILESH SHIRALI

Notation. We denote the sum of the digits of a positive integer *n* by SD(*n*). The notation $a \mid b$ means: '*a* is a divisor of *b*', i.e., '*b* is a multiple of *a*.' Throughout, we work in base 10.

Two well-known statements. The following two statements are very well-known:

(1) A positive integer *n* is divisible by 3 if and only if SD(*n*) is divisible by 3.

(2) A positive integer *n* is divisible by 9 if and only if SD(*n*) is divisible by 9.

Or, more compactly: $3|n \Leftrightarrow 3|SD(n)$, $9|n \Leftrightarrow 9|SD(n)$. Both the statements are true and easy to prove. Examining them, one may be tempted to generalise:

$$\boxed{27|n \Leftrightarrow 27|\text{SD}(n).}$$??? Status unclear ???

But is this claim true? Note that it consists of two sub-claims: *For any positive integer n,*

(1) *If n is divisible by* 27*, then* SD*(n) is divisible by* 27*;*

(2) *If* SD*(n) is divisible by* 27*, then n is divisible by* 27*.*

How do we check whether either of these claims is true?

Notion of a counterexample. A standard way of proceeding when confronted with a statement about which we are uncertain (regarding whether it is true or false) is to actively look for a

Keywords: Divisibility by 3, divisibility by 9, divisibility by 27

counterexample; i.e., a situation where the given statement is falsified. If we find such a counterexample, then the given statement must be false.

Illustrations. Here are some illustrations of this notion.

(1) Claim: I*f n is a positive integer, then* $4n^2+1$ *is a prime number*. This is true for $n = 1,2,3$ but false for $n = 4$. So $n = 4$ provides a counterexample to the stated claim.

(2) Claim: *If p is a prime number, then* 2^p-1 *is a prime number*. This is true for $p = 2,3,5,7$ but false for $p = 11$. So $p = 11$ provides a counterexample to the stated claim.

Finding a counterexample is clearly a very effective way of disposing off false claims. It is an extremely important notion in the study and exploration of mathematics.

Looking for a counterexample. Armed with this notion, let us look for a counterexample to the statement *"If a positive integer n is divisible by* 27, *then* SD(n) *is divisible by* 27."

Once we set ourselves this task, the question becomes absurdly simple to answer. Indeed, the very first (positive) multiple of 27 (namely, 27 itself) is a counterexample! For, 27 is a multiple of 27. On the other hand, SD(27) = 9, which is not a multiple of 27. *So the statement under study is not true in general.* (That got resolved rather quickly, didn't it? Too quickly, perhaps?)

What about the converse: "*If* SD(n) *is divisible by* 27, *then n is divisible by* 27"? Is this true?

What combinations of digits yield a sum of 27? The simplest such combination is 9 + 9 + 9= 27; and we find that 999 is a multiple of 27; indeed, 999 = 27 × 37. Next in simplicity we have the combination 1 + 8 + 9 + 9 = 27. And 1899 turns out to be not divisible by 27! Indeed, 1899 = (70 × 27) + 9, so 1899 ≡ 9 (mod 27). *So the condition that* SD(n) *is divisible by* 27 *is not enough to force n to be divisible by* 27.

The counterexample just found may have been found a little too easily (it is rather disappointing when things happen too easily, isn't it?); we may wonder whether we could have proceeded in a more systematic way. Indeed we can. Having seen the number 999 above, our mind may naturally turn to the number 888. This number is divisible by 3 but not by 9 and therefore not by 27 either. We find by actual division that 888 ≡ 24 (mod 27) ≡ −3 (mod 27). On multiplication by 10, we obtain:

$$8880 \equiv -30 \pmod{27} \equiv -3 \pmod{27}.$$

We infer that the numbers 888, 8880, 88800, . . . all leave the same remainder (namely, 24) under division by 27. Continuing, we infer that the numbers 8883, 88803, 88830, 888030 . . . are all divisible by 27. Note that each of these numbers has sum-of-digits equal to 27. And since $111 \equiv 3 \pmod{27}$, exactly the same statement can be made for the numbers 888111, 8880111, 8881110, 88811100 . . . : each of them is divisible by 27, and each of them has sum-of-digits equal to 27.

On the other hand, note that 1011 ≡ 12 (mod 27) and 1101 ≡ 21 (mod 27). (Please check these computations.) So 1011 and 1101 do not leave remainder 3 under division by 27. It follows that the numbers 8881011 and 8881101 are *not* divisible by 27 (on division by 27, they leave remainders of 9 and 18, respectively). However, each of them has sum-of-digits equal to 27. Therefore, each of these numbers contradicts the claim that if SD(n) is divisible by 27, then n is divisible by 27.

More counterexamples can be generated by arguing in this manner. It may be a good exercise for you to do so.

What we have found is that both the statements under study are false, and in both cases, we have determined that this is so by finding counterexamples. Our conjecture has thus met with a sorry end!

In much the same way, we can put to rest the following claim,

$$\text{If } 81 \mid \text{SD}(n), \text{ then } 81 \mid n,$$

just in case anyone ever made such a claim. But we shall leave the task of finding the counterexample to you. (See [1] for more on this.)

Is there any test for divisibility by 27?

After the disappointing experience above, we may wonder whether there is any worthwhile test for divisibility by 27. There is, and it is provided by the observation that 999 is divisible by 27 (check: $999 = 27 \times 37$, remainder 0). From this, it follows that $1000 \equiv 1 \pmod{27}$.

This observation gives rise to the following test of divisibility. Let n be the given positive integer. We assume to start with that $n \geq 1000$. Let b denote the number formed by the last three digits of n, and let a denote the 'rest' of the number (with those three digits deleted); so $n = 1000a + b$. For example, if $n = 123456$, then $b = 456$ and $a = 123$. We now replace n by $n_1 = a + b$ and continue the computations with n_1 in place of n. It is not difficult to see that if n has four or more digits, then n_1 is substantially smaller than n. The crucial fact now is: $n \equiv n_1 \pmod{27}$. On this observation rests the algorithm.

Continuing in this manner, we ultimately obtain a number with three or fewer digits. There are two ways to proceed at this stage. The first is based on recognition: we assume that we are able to *recognise* all three-digit multiples of 27. (This is not as difficult as it sounds. I'm sure that we can all rise to the challenge!)

The other approach, which we use if n has three or fewer digits, uses the digits of n. It is based on the observation that $27 \times 4 = 108 = 100 + 8$, and therefore that $100 \equiv -8 \pmod{27}$.

Let a be the hundreds digit of n and let b be the two-digit number formed by the remaining two digits (i.e., the tens digit and the units digit) of n. Note that this means that $n = 100a + b$. For example, if $n = 453$, then $a = 4$ and $b = 53$. Now replace n by the number $n_1 = b - 8a$. *Then n is divisible by 27 if and only if n_1 is divisible by 27.* Observe that at the end, we again fall back on recognition: we assume that we are able to recognise all the two-digit multiples of 27. But that, surely, is not asking for too much!

Examples. In the examples shown below, we use the symbol $\rightsquigarrow$ to denote the following number replacement operation:

- If n has four or more digits and $n = 1000a + b$ where b has three or fewer digits $(\text{i.e., } b < 100)$, then $n \rightsquigarrow a + b$.
- If n has three or fewer digits and $n = 1000a + b$ where b has two or fewer digits $(\text{i.e., } b < 100)$, then $n \rightsquigarrow b - 8a$.

(1) $n = 123456$. We now obtain:

$$123456 \rightsquigarrow 123 + 456 = 579.$$

Next:

$$579 \rightsquigarrow 79 - (8 \times 5) = 79 - 40 = 39.$$

Since 39 is not a multiple of 27, we conclude that 123456 is not a multiple of 27.

(2) $n = 11134233$. Then we have:

$$\begin{aligned} 11134233 &\rightsquigarrow 11134 + 233 = 11367 \\ &\rightsquigarrow 11 + 367 = 378 \\ &\rightsquigarrow 78 - (8 \times 3) = 78 - 24 = 54. \end{aligned}$$

Since 54 is a multiple of 27, we conclude that 11134233 is a multiple of 27.

We now offer a formal proof that this algorithm works correctly.

Proof of correctness of algorithm. Let a and b be any two integers, and let n and n_1 be defined as follows:

$$n = 1000a + b, \; n_1 = a + b.$$

Then we observe that:

$$n - n_1 = 999a = 27 \times 37a,$$

i.e., $n - n_1 \equiv 0 \pmod{27}$ Since the difference between n and n_1 is a multiple of 27, if either of them is a multiple of 27, so must be the other one.

Similarly, if n and n_1 are defined as follows:

$$n = 100a + b, \; n_1 = b - 8a,$$

then we observe that:

$$n - n_1 = 108a = 27 \times 4a,$$

i.e., $n - n_1 \equiv 0 \pmod{27}$. Since the difference between n and n_1 is a multiple of 27, if either of them is a multiple of 27, so must be the other one. The conclusion here is identical to the earlier one.

So the replacement of n by n_1 does not alter the status of divisibility by 27. This proves the correctness of the algorithm. But more can be said. Since in each case we get $n - n_1 \equiv 0 \pmod{27}$, it follows that n and n_1 *leave the same remainder under division by* 27. So this algorithm also provides a way of computing the remainder when we divide a large integer by 27.

References

1. Alexander Bogomolny, "Division by 81 from Interactive Mathematics Miscellany and Puzzles", https://www.cut-the-knot.org/Generalization/81.shtml, Accessed 17 November 2017

SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Sahyadri School/Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for A*t Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

ClassRoom

Low Floor High Ceiling Tasks

Here we go round…

NEWER AND NEWER SPIRALS - OPEN AND SHUT CASES

SWATI SIRCAR,
SNEHA TITUS

We continue our Low Floor High Ceiling series in which an activity is chosen – it starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that students are pushed to their limits as they attempt their work. There is enough work for all, but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

In the March 2017 issue, Khushboo Awasthi had described an investigation of the familiar Square Root Spiral, which had taken her along unexpected paths filled with mathematical discoveries. At the end of the article, she posed some questions for the reader to investigate and we did just that! We share our bonanza of findings with you, and as usual, the tasks are arranged from Low Floor to High Ceiling. This time, we include some investigations with the free dynamic geometry software GeoGebra; regular constructions with compass and ruler will do the job just as well!

This is what the Square Root Spiral looks like (see Figure 1).

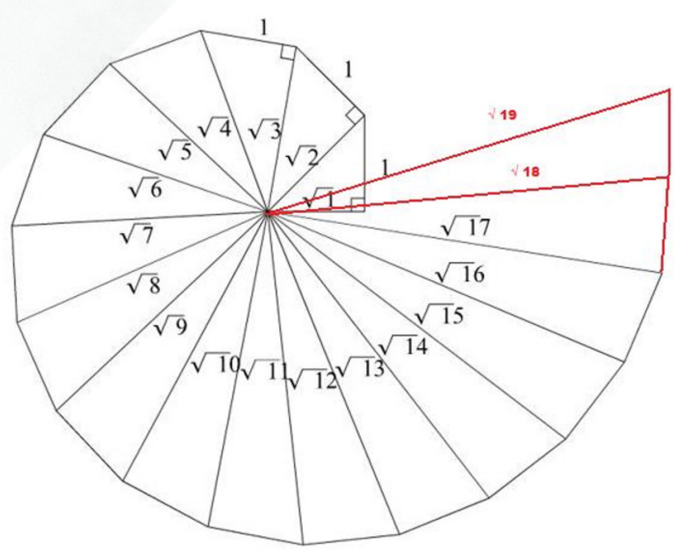


Figure 1: The Square Root Spiral

Keywords: Right-angled triangles, Pythagoras, similarity, reflections, construction

The question posed at the end of the article was this:

What will happen to the spiral if, at every iteration, we vary the length of the opposite side and make it equal to the base?

Let's explore!

Steps to create Square Root Spiral Version 2.0

1. On a blank sheet of paper, draw a line segment AB, of unit length, in the middle of the paper. At point B, construct a perpendicular line segment of unit length (same as AB), named BC. The hypotenuse AC will hence be of length$\sqrt{2}$. (Refer Figure 2).

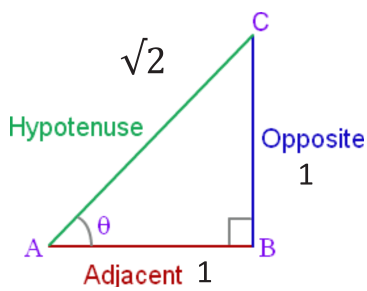


Figure 2

2. Construct a line at point C perpendicular to segment AC. Construct line segment CD of length $\sqrt{2}$ (same as AC) on this line. Then, draw AD to form the hypotenuse with AC as base and CD as the opposite side of the new right angled triangle. So, **AD** = $\sqrt{4}$ = 2.
3. Similarly, construct a line segment DE of length AD = 2, from point D perpendicular to AD. Join AE to form the hypotenuse of the right angled triangle with AD as base.

Repeat this process to get more right angled triangles. The only point to remember is that the perpendicular side of the new triangle has to be the same length as the hypotenuse of the previous triangle. (Ref. Figure 3).

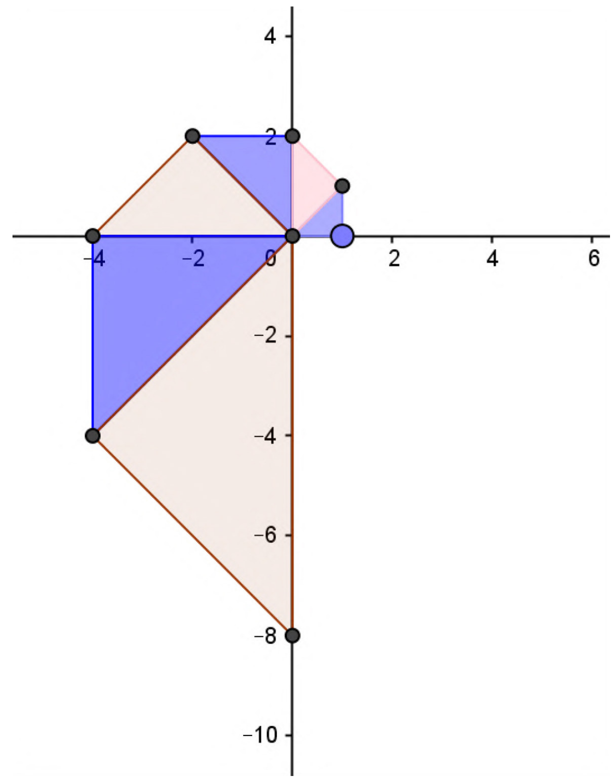


Figure 3

Task 1

Briefly describe the characteristics that are common to each of the triangles.

Teacher's Notes: The low floor start simply requires students to recognize that the triangles are all similar, being right angled isosceles triangles. Some of the students may recognize that if the base and height are both a, then the hypotenuse of the triangle is $\sqrt{2}$a. This is a chance for the teacher to point out triangles that are similar but not congruent.

Task 2

How are the triangles related to each other? Compare the lengths of the sides and the areas.

Teacher's Notes: Building on the previous question, the teacher can facilitate the students to record their findings and recognize patterns in a table such as the one given below:

No. of Triangle (n)	Base (b)	Opposite side (a)	Hypotenuse (c)	Area
1	1	1	√2	½
2	√2	√2	2	1
3	2	2	√8	2
4	√8	√8	4	4
5	4	…..	…..	
n	…...	……	…..	

Table 1: Length of base, opposite side and hypotenuse

Clearly, the sides of each new triangle are √2 times the sides of the old triangle and the area is twice the area of the previous triangle.

Task 3

Does the spiral close in this case? Justify your stance. Describe what happens in the second loop of the spiral.

How do the sides and area of the first triangle in the second loop compare with those of the first triangle in the first loop? What about succeeding loops? Does any pattern emerge?

Teacher's Notes: As usual, we ramp up the difficulty level with the need to *justify*. As students construct the triangles, they will realise that the spiral closes simply because the angle at the centre of the spiral is 45° in each case and so after 8 triangles the angle adds up to 360° and the spiral closes. Communicating this in words or writing is, however, a skill that students may need to practise.

Students may complete the table to obtain the base and height of the 9^{th} triangle (the first in the second loop of the spiral) to be 16, the hypotenuse to be 16√2 and the area to be 128. So the sides are 16 (= 2^4) times the sides of the first triangle in the first loop and the area is 256 (= 2^8) times the area of this triangle. A great chance to see that the ratio of the areas is the square of the ratio of the sides. And a chance to reiterate and see that similarity of triangles means that the corresponding angles are the same but the sides are in the same ratio. If a table is made of the ratio "the side of the first triangle in n^{th} loop : the side of the first triangle", the following pattern emerges: 1, 2^4, $(2^4)^2$, $(2^4)^3$ … $(2^4)^{(n-1)}$. A similar pattern arises for the areas.

Task 4

These triangles proceed in the anti-clockwise direction. Construct the triangles which proceed in the clockwise direction, with each new height being equal to the previous hypotenuse.

Conjecture about what would happen to the ratio of the areas if the triangles proceeded clockwise.

Teacher's Notes: Construction of these triangles is a more difficult task requiring the understanding that the right angle is now opposite the base of the previous triangle. There are several ways of doing this construction and so this question is open ended and an opportunity to encourage experimentation. What we have done is to use the fact that the height is now 1/√2 the height of the previous triangle. We started with a triangle of area 1 and reflected the mid-point of the hypotenuse of the first triangle in its base, thus arriving at the vertex of the second triangle. Do encourage your students to observe several congruent triangles, apart from the similar triangles. See Figure 4.

Conjecturing is an important mathematical skill and this is easily done in this task with students simply realizing from the figure that the table now reads to be read backwards, the areas are halved instead of being doubled. Students should be encouraged to record their conjectures, the objective being to realise that where the areas were increasing in the ratio 2^1, they are now decreasing in the ratio 2^{-1}.

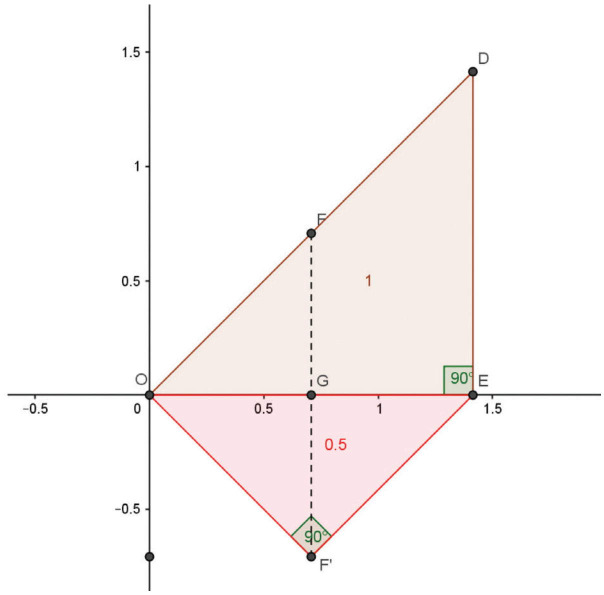


Figure 4

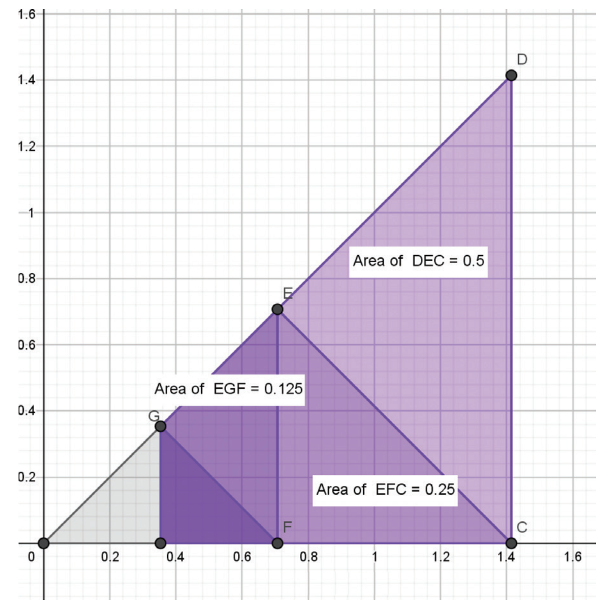


Figure 5

Task 5

Is it possible to tile the original triangle of area 1 with triangles whose areas are being halved? Will the original triangle be filled with these triangles or will the triangles 'spill out' as more and more are generated?

Teacher's Notes: A very preliminary, very visual approach to limits using a geometrical progression without mentioning either of these. This is intended for the more visual student who can use the ideas from the construction above to fill up the original triangle of area 1.

From the way the triangle fills up, it becomes clear that the construction of each new triangle is by finding the mid-point of the hypotenuse of the previous triangle and then making this half segment the base of the next triangle. Theoretically, it is possible to continue this process infinitely, without exiting from the first triangle. We would encourage teachers to write down this finding in the form of a mathematical statement; an example would be:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+ \ldots\ldots\ldots\ldots\ldots\ldots = 1$$

Verification is possible with the use of a calculator and this should be an exciting observation for students.

Task 6

Starting with a triangle of base 1 and height 2, generate a spiral of right angled similar triangles, building the base of each new triangle on the hypotenuse of the previous triangle.

What is the ratio of the sides and of the areas?

How can you generate a spiral of right angled similar triangles, starting with a triangle of base 1 such that the areas are in the ratio n (where n is a rational number)?

For what values of n does the spiral close?

Teacher's Note: The hypotenuse of the first triangle is $\sqrt{5}$, its area is 1 unit. For the next triangle to be similar, the sides have to increase in the ratio $\sqrt{5} : 1$ (as the hypotenuse is the new base). This makes the height $2\sqrt{5}$ units and the area will then be 5 square units.

Successive triangles will also have corresponding sides in the ratio $\sqrt{5} : 1$ with areas increasing in the ratio 5 : 1.

If the ratio of the areas of the similar triangles is to be $n : 1$, then $n = \dfrac{A_2}{A_1} = \dfrac{\frac{1}{2}b_2h_2}{\frac{1}{2}b_1h_1} = \sqrt{n} \times \sqrt{n}$

The first triangle has base 1, hypotenuse = $\sqrt{n}$, then the height has to be $\sqrt{(n-1)}$.

Area will be $\dfrac{\sqrt{(n-1)}}{2}$

The next triangle has base $\sqrt{n}$, height $\sqrt{[n(n-1)]}$ and hypotenuse n, area will be $n\dfrac{\sqrt{(n-1)}}{2}$.

Clearly, we have to start by constructing the hypotenuse of the triangle with base 1, to be $\sqrt{n}$, and then generate succeeding triangles. Since $\sqrt{n}$ is constructible, we use the standard construction to generate the first triangle. See Figure 6, which is from the Std. 9 NCERT Text Book.

Task 7

Is it possible to construct new triangles in the clockwise direction for this new spiral?

What is the ratio of the sides in this case? What about the ratio of the areas?

Teacher's Note: Entirely possible, using similar triangles – the angle at the origin is reflected in the base of the first triangle and then the triangle is completed using the original base as the new hypotenuse. See Figure 7.

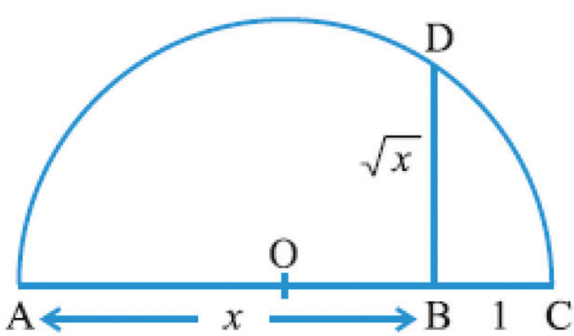


Figure 6

The sides will be in the ratio 1 : $1/\sqrt{n}$ and the areas in the ratio 1 : $1/n$. So while anti-clockwise rotations produce ratios of areas which are positive powers of n, the clockwise rotations produce ratios of areas which are negative powers of n.

Task 8

For what values of n does the spiral close?

Teacher's Note: Since the triangles are similar, the angles at the vertex are all equal for a given value of the ratio of areas (n : 1). The following table helps students to inspect the angle for values of n from 1 to 10. It becomes clear that the spiral closes for n = 2 and 4 for which the angles at the origin are 45° and 60° respectively. Both these are factors of 360°, the other factor in this range (remember that the angle is increasing (why?) but has to be acute) being 72°, but its cos value is not of the form $1/\sqrt{n}$.

Do encourage students to investigate beyond the first complete rotation for the spirals that close (n = 2 and n = 4). It is particularly interesting for them to note down the ratio of the sides and of the areas for the first triangle in each new loop!

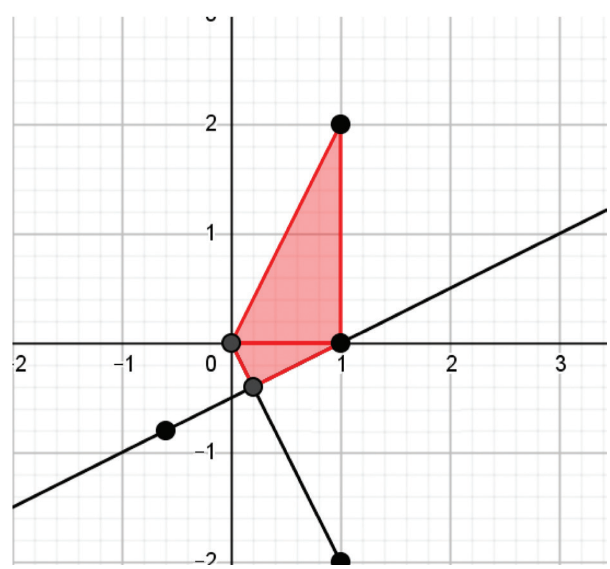


Figure 7

n [Ratio of Areas $n:1$]	Base (b)	Opposite side (a)	Hypotenuse (c)	Cos θ = $\frac{b}{c}$	θ
2	1	1	$\sqrt{2}$	$\frac{1}{\sqrt{2}}$	45°
3	1	$\sqrt{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	54.7°
4	1	$\sqrt{3}$	2	$\frac{1}{2}$	60°
5	1	2	$\sqrt{5}$	$\frac{1}{\sqrt{5}}$	63.4°
6	1	…..	…..		
n	…...	……	…..		

Table 2: Angles at the origin

Conclusion

Isn't it amazing how going round and round the same question can make our investigation spiral? We hope that you will enjoy this investigation as much as we did creating it!

SWATI SIRCAR is is Senior Lecturer and Resource Person at the School of Continuing Education and University Resource Centre, Azim Premji University. Mathematics is the second love of her life (first being drawing). She has a B.Stat-M.Stat from Indian Statistical Institute and an MS in mathematics from University of Washington, Seattle. She has been doing mathematics with children and teachers for more than 5 years and is deeply interested in anything hands on, origami in particular. She may be contacted at swati.sircar@apu.edu.in.

SNEHA TITUS is Asst. Professor at the School of Continuing Education and University Resource Centre, Azim Premji University. Sharing the beauty, logic and relevance of mathematics is her passion. Sneha mentors mathematics teachers from rural and city schools and conducts workshops in which she focusses on skill development through problem solving as well as pedagogical strategies used in teaching mathematics. She may be contacted at sneha.titus@azimpremjifoundation.org.

Is a Parallelogram ever NOT A PARALLELOGRAM?

$\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$

A parallelogram may appear to be a very simple and basic shape of plane geometry, but its simplicity is deceptive; indeed, it possesses a lot of richness of structure. Much of this richness is revealed when we ask the following question: What characterises a parallelogram? In other words:

> What minimal properties must a quadrilateral have for us to know that it is actually a parallelogram?

The fact that a parallelogram can be defined in several different ways that turn out to be equivalent to each other is indicative of this richness. There is no other class of geometric objects which can be defined in so many different but equivalent ways.

The basic definition of a parallelogram is: A plane four-sided figure whose opposite pairs of sides are parallel to each other. That is, a plane four-sided figure $ABCD$ is a parallelogram if and only if $AB \parallel CD$ and $AD \parallel BC$ (see Figure 1).

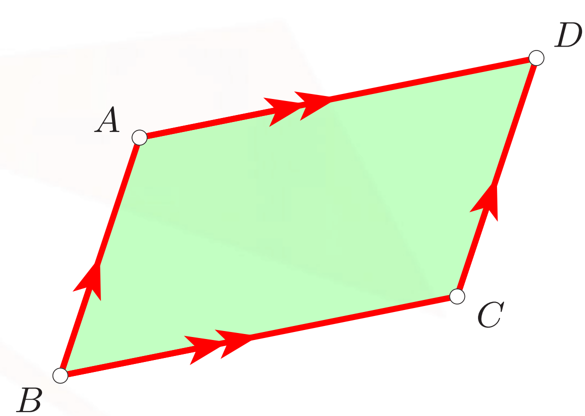


Figure 1

Keywords: Quadrilateral, parallelogram, SAS congruence, ASA congruence, SSS congruence

Here is a definition of a parallelogram which the reader may find unfamiliar, as it has been framed in the language of transformation geometry: A parallelogram is a quadrilateral with rotational symmetry of order 2. That is, if there exists a point O in the plane of a quadrilateral $ABCD$ such that a half-turn centred at O maps $ABCD$ back to itself, then $ABCD$ is a parallelogram.

Other alternative definitions

Here are some other ways in which a parallelogram can be defined. Each of these is equivalent to the basic definition given above. In each case, we have given a one-line indication of the proof. Throughout, we use '**iff**' as a short form for 'if and only if.' Also, throughout, 'quadrilateral' means 'planar quadrilateral.'

1. A four-sided figure $ABCD$ is a parallelogram iff $AB = CD$ and $AD = BC$. In words: A quadrilateral is a parallelogram iff both pairs of opposite sides have equal length. Proof: Use SSS congruence on an appropriate pair of triangles.

2. A four-sided figure $ABCD$ is a parallelogram iff $\measuredangle A = \measuredangle C$ and $\measuredangle B = \measuredangle D$. In words: A quadrilateral is a parallelogram iff both pairs of opposite angles have equal measure. Proof: Use ASA congruence on an appropriate pair of triangles.

3. A four-sided figure $ABCD$ is a parallelogram iff $AB \parallel CD$ and $AB = CD$. In words: A quadrilateral is a parallelogram iff for one pair of opposite sides, the sides are both parallel to each other and of equal length. Proof: Use SAS congruence on an appropriate pair of triangles.

4. A four-sided figure $ABCD$ is a parallelogram iff diagonals AC and BD bisect each other. In words: A quadrilateral is a parallelogram iff the diagonals bisect each other. Proof: Use SAS congruence on an appropriate pair of triangles.

These alternative definitions are well-known, so we shall not dwell on them further. Instead, we consider some fresh possibilities.

Do the following conditions characterise a parallelogram?

We offer below five different properties possessed by a parallelogram and ask in each case whether the property in question characterises a parallelogram; i.e., *if a planar quadrilateral possesses that property, is it necessarily a parallelogram?*

5. If $ABCD$ is a parallelogram, then each of its diagonals divides it into a pair of triangles with equal area. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that each of its diagonals divides it into two triangles that have equal area, is $ABCD$ necessarily a parallelogram?

6. If $ABCD$ is a parallelogram, then $AB = CD$ and $AD \parallel BC$. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that $AB = CD$ and $AD \parallel BC$, is $ABCD$ necessarily a parallelogram?

7. If $ABCD$ is a parallelogram, then $AB = CD$ and $\measuredangle A = \measuredangle C$. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that $AB = CD$ and $\measuredangle A = \measuredangle C$, is $ABCD$ necessarily a parallelogram?

8. If $ABCD$ is a parallelogram, then the sum of the squares of the sides equals the sum of the squares of the diagonals. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that

   $$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2,$$

   is $ABCD$ necessarily a parallelogram?

9. If $ABCD$ is a parallelogram, then the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point, is $ABCD$ necessarily a parallelogram?

You may be intrigued to learn that three of these five conditions are genuine characterisations of a parallelogram; but two of them are not! We will leave it to you to find the two errant conditions, the ones that 'do not fit.' (Of course, in each of the cases, the assertion made in the first sentence is true. You may not be familiar with the last two assertions, items 8 and 9.)

For those who are impatient to know the answers, we study these five conditions in greater detail in the "How To Prove It" column, elsewhere in this issue.

References

[1] Jonathan Halabi, "Puzzle: proving a quadrilateral is a parallelogram" from *JD2718*, https://jd2718.org/2007/01/10/puzzle-proving-a-quadrilateral-is-a-parallelogram/

[2] Wikipedia, "Parallelogram" from https://en.wikipedia.org/wiki/Parallelogram

---

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

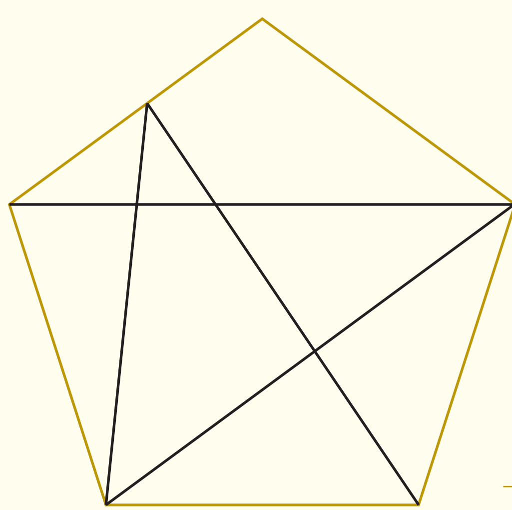
HOW MANY TRIANGLES CAN YOU SPOT?

How many triangles can you spot in the figure shown below?

Let's see if you can make an accurate count!

Much more challenging:

See if you can prove that your count is correct!



— CoMaC

INEQUALITIES in Algebra and Geometry Part 4

ClassRoom

SHAILESH SHIRALI

This article is the fourth in the 'Inequalities' series. This time, we present a novel proof of the general AM-GM inequality, based on iteration. Following this, we present some applications of the inequality.

The arithmetic mean-geometric mean inequality (generally referred to as the AM-GM inequality; said to be part of the daily diet for aspiring mathletes, and routinely used in many branches of mathematics) is well-known. New proofs come up once in a while. The following iterative proof is highly unusual and will be of interest to some readers.

Statement of the theorem

The arithmetic mean A and the geometric mean G of n given positive numbers $a_1, a_2, \ldots, a_n$ are defined as follows:

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{\sum_{i=1}^{n} a_i}{n}, \tag{1}$$

$$G = (a_1 a_2 \cdots a_n)^{1/n} = \left(\prod_{i=1}^{n} a_i\right)^{1/n}. \tag{2}$$

Theorem (AM-GM inequality)**.** *Let $a_1, a_2, \ldots, a_n$ be positive numbers with arithmetic mean A and geometric mean G. Then $A \geq G$. Moreover, the equality $A = G$ holds if and only if $a_1 = a_2 = \cdots = a_n$.*

Proof of the theorem

For convenience, we use the short forms AM for arithmetic mean and GM for geometric mean.

We assume for convenience that the a_i's are indexed in increasing order, so that $a_1 \leq a_2 \leq \cdots \leq a_n$; this implies that $a_1 \leq A \leq a_n$. If $a_1 = A$ or $a_n = A$, then it means that all the a_i's are equal. In this case we have $A = G$, so there is nothing to prove; so we may assume that $a_1 < A < a_n$. Note that in this case, the quantity $X = a_1 + a_n - A$ is positive.

Keywords: AM-GM inequality, iteration, exponents, limits, graphs, sandwich principle, pinch principle

We now replace the numbers a_1 and a_n by A and X, respectively. The n numbers we now have are: $A, a_2, a_3, \ldots, a_{n-1}, X$. The AM of these numbers is exactly the same as that of the numbers in the earlier list, because $A + X = a_1 + a_n$. However, their GM is strictly greater than the earlier value, because A and X lie strictly between a_1 and a_n (i.e., $a_1 < A, X < a_n$), implying that $AX > a_1 a_n$. To see why, observe that the inequality $AX > a_1 a_n$ is equivalent to

$$A\left(a_1 + a_n - A\right) - a_1 a_n > 0,$$

i.e., to

$$(A - a_1)(a_n - A) > 0,$$

and this is clearly true since $a_1 < A < a_n$.

So the replacement preserves the AM but results in an increased value for the GM.

The procedure is now iterated: at each stage, we replace the least and greatest numbers in the latest list by the AM of the collection and the value of

$$\text{least number} + \text{greatest number} - \text{AM},$$

respectively, and we continue this as long as the numbers in the collection are not all equal.

After each iteration we obtain a list of numbers in which the number of entries equal to A has increased. Therefore, after no more than $n - 1$ iterations we reach a stage when all entries are equal to A. So the iteration definitely comes to an end after a finite number of steps.

Let G_i represent the geometric mean at the i-th stage; then $G_0 = G$ and

$$A = G_{n-1} \geq G_{n-2} \geq \cdots \geq G_1 \geq G_0 = G,$$

so $A \geq G$, as required. □

An example using numbers. It helps if we show the working of the algorithm using actual numbers. Let us start with the list $1, 2, 3, 4, 10$ and see what the algorithm accomplishes.

We display the working in the form of a table as shown below. In the second column, we always display the list in sorted form, i.e., in non-decreasing order.

Step	Latest list	Min element	Max element	AM	X	GM
1	$1, 2, 3, 4, 10$	1	10	4	7	$240^{1/5}$
2	$2, 3, 4, 4, 7$	2	7	4	5	$672^{1/5}$
3	$3, 4, 4, 4, 5$	3	5	4	4	$960^{1/5}$
4	$4, 4, 4, 4, 4$	4	4	4	4	$1024^{1/5}$

Observe the steady increase in the values of the GM, while the AM stays fixed. At the end, when all the numbers are equal, we have AM = GM.

Some applications of the AM-GM inequality

In this section, we invert the usual procedure. Rather than start with a problem from some Olympiad collection or the other, we try to *create* interesting inequalities by applying the AM-GM inequality to various lists of numbers. It can become quite a nice game to play! Here are some inequalities that we obtain as a result.

Result 1: Start with the list of numbers $1, 2, \ldots, n$, where n is any positive integer ($n > 1$). Their arithmetic mean is

$$\frac{1+2+\cdots+n}{n} = \frac{n(n+1)}{2} \times \frac{1}{n} = \frac{n+1}{2},$$

and their geometric mean is

$$(1 \times 2 \times \cdots \times n)^{1/n} = (n!)^{1/n}.$$

It follows that

$$(n!)^{1/n} < \frac{n+1}{2}. \tag{3}$$

The inequality is strict, since the numbers in the list $1, 2, \ldots, n$ are not all equal (indeed, they are all unequal).

Result 2: Start with the list of n numbers $1, 1, \ldots, 1, 2$, with $(n-1)$ repetitions of 1 and a solitary 2; here n is any positive integer ($n > 1$). Their arithmetic mean is

$$\frac{1}{n}\left(\underbrace{1+1+\cdots+1}_{n-1 \text{ repetitions}}+2\right) = \frac{n+1}{n} = 1 + \frac{1}{n},$$

and their geometric mean is

$$(1 \times 1 \times \cdots \times 1 \times 2)^{1/n} = 2^{1/n}.$$

It follows that

$$2^{1/n} < 1 + \frac{1}{n}. \tag{4}$$

The inequality is strict, since the numbers in the list $1, 1, \ldots, 1, 2$ are not all equal.

Corollary. Here is an interesting result which follows from the above inequality. It is obvious that $2^{1/n} > 1$. Hence the following is true for all positive integers $n > 1$:

$$1 < 2^{1/n} < 1 + \frac{1}{n}. \tag{5}$$

In this double inequality, we let n increase without bound (i.e., $n \to \infty$). The number at the extreme left is a constant (equal to 1), while the numbers at the extreme right tend to 1 (in the limit). Therefore, by the so-called ***sandwich principle*** or ***pinch principle***, the following statement is true:

$$\lim_{n\to\infty} 2^{1/n} = 1. \tag{6}$$

Result 3: Similarly we may prove: for all positive integers $n > 1$,

$$1 < 3^{1/n} < 1 + \frac{2}{n}, \tag{7}$$

and hence:

$$\lim_{n\to\infty} 3^{1/n} = 1. \tag{8}$$

Result 4: Now consider of the list $1, 1, \ldots, 1, 1+x$, with $n-1$ repetitions ($n > 1$) of 1 and a solitary $1+x$, where $x > -1$. (This restriction is needed to avoid having a negative value for $1+x$.) The arithmetic mean of the numbers is

$$\frac{1}{n}\left(\underbrace{1+1+\cdots+1}_{n-1 \text{ repetitions}}+(1+x)\right) = \frac{n+x}{n} = 1 + \frac{x}{n},$$

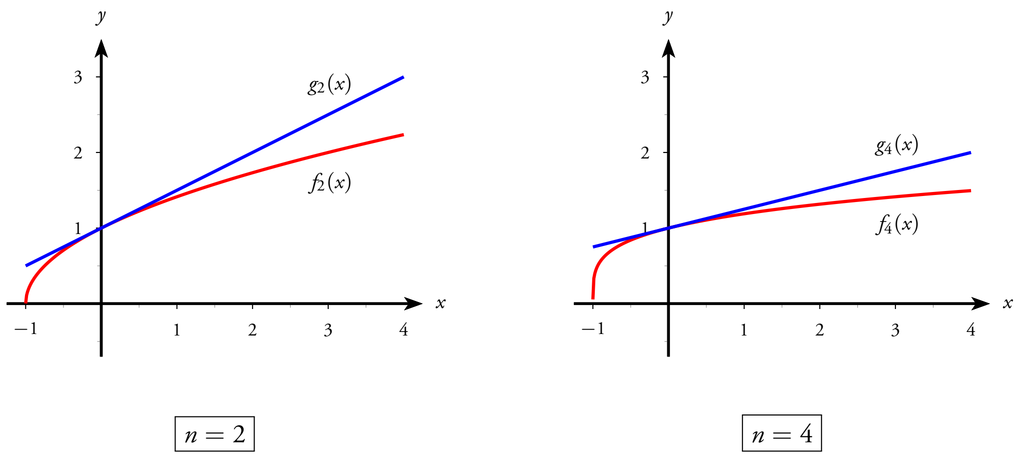


Figure 1

and their geometric mean is

$$(1 \times 1 \times \cdots \times 1 \times (1+x))^{1/n} = (1+x)^{1/n}.$$

It follows that

$$(1+x)^{1/n} \leq 1 + \frac{x}{n}. \qquad (9)$$

The inequality is true for all integers $n > 1$ and for all real numbers $x > -1$. Equality holds precisely when $x = 0$. The inequality is strict provided that x is non-zero.

It is interesting to look at this relation graphically. Figure 1 displays the graphs of the following two functions for $x \geq -1$:

$$f_n(x) = (1+x)^{1/n} \qquad \text{(red)},$$

$$g_n(x) = 1 + \frac{x}{n} \qquad \text{(blue)},$$

for $n = 2$ and $n = 4$. Observe that in both cases, the graph of g (shown in blue colour) is a straight line tangent to the graph of f (shown in red colour) at the point $(0, 1)$. The tangent line lies entirely above the curve, touching it only at the indicated point.

Result 5: A particularly interesting inequality is obtained by considering the list of $n+1$ numbers:

$$1,\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ \ldots,\ 1+\frac{1}{n}, \qquad (10)$$

where n is any positive integer. (So there are n repetitions of the number $1 + \frac{1}{n}$.) Their arithmetic mean is

$$\frac{1}{n+1}\left(1 + n \cdot \frac{n+1}{n}\right) = \frac{n+2}{n+1} = 1 + \frac{1}{n+1},$$

and their geometric mean is

$$\left(1 + \frac{1}{n}\right)^{n/(n+1)}.$$

We therefore get the following inequality which is true for all positive integers n:

$$1 + \frac{1}{n+1} > \left(1 + \frac{1}{n}\right)^{n/(n+1)},$$

that is,

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n}. \tag{11}$$

This establishes, as a mere corollary, the result that the following sequence

$$1, \quad \left(1+\frac{1}{2}\right)^{2}, \quad \left(1+\frac{1}{3}\right)^{3}, \quad \left(1+\frac{1}{4}\right)^{4}, \quad \ldots, \tag{12}$$

is *strictly increasing*. This result is needed in the proof of the claim that the sequence of numbers (12) has a limit. Some of you may know that the limit is the very well known number e whose approximate value is 2.71828.

Closing remarks. In the latter part of this article, we have tried to show how one can find mathematical results on one's own. This is in fact how mathematics is created! We invite you to find some interesting inequalities of your own by applying the AM-GM inequality to various lists of numbers.

SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

ClassRoom

On an OPTIMIZATION PROBLEM

PRITHWIJIT DE

Consider a solid sphere of radius r. Suppose it is melted and the molten mass is recast into two solid spheres of radii r_1 and r_2. Does the total surface area increase, decrease or remain the same, if it is assumed that there is no loss of matter in the whole process? Let us answer this question. Since the total volume remains the same, it must be that

$$\frac{4\pi}{3}r^3 = \frac{4\pi}{3}(r_1^3 + r_2^3), \tag{1}$$

and we are interested in knowing whether the ratio

$$\alpha = \frac{r_1^2 + r_2^2}{r^2} \tag{2}$$

is greater, less than or equal to one. Writing $x = r_1/r$ and $y = r_2/r$ we readily see that

$$x^3 + y^3 = 1. \tag{3}$$

Since $0 < x, y < 1$ it follows that $x^2 > x^3$ and $y^2 > y^3$, and these two inequalities combine to yield

$$\alpha = x^2 + y^2 > x^3 + y^3 = 1. \tag{4}$$

Thus the surface area increases. How large can α be? Can it be arbitrarily large, or is there a number it doesn't grow beyond? Since x and y are proper fractions, it is clear that α cannot be arbitrarily large. So, what is the upper bound? The algebraic identity

$$(x+y)\left(x^3+y^3\right) - \left(x^2+y^2\right)^2 = xy(x-y)^2 \tag{5}$$

Keywords: Area, volume, Cauchy-Schwarz inequality

shows that $\alpha^2 \leq x + y < 2$. That is, $\alpha < \sqrt{2}$, so $\sqrt{2}$ is an upper bound for α. But note that this upper bound is not attained as x and y are strictly less than 1. Can we improve this upper bound? Observe that

$$(x+y)^2 \leq 2\left(x^2+y^2\right) = 2\alpha, \tag{6}$$

so

$$\alpha^2 \leq x + y \leq \sqrt{2\alpha} \tag{7}$$

and this leads to

$$\alpha \leq \sqrt[3]{2}. \tag{8}$$

Also, if $x = y = 1/\sqrt[3]{2}$, then $\alpha = \sqrt[3]{2}$. This shows that in order to maximize the total surface area of the two spheres derived from the original sphere, one needs to make them of equal size.

At this juncture, we naturally ask the following question:

> *If one wishes to maximise the total surface area of n spheres derived from a given sphere, then should they all be of the same radius?*

If $r_1, r_2, \ldots, r_n$ are the radii of the n spheres and we set $x_i = r_i/r$ for $i = 1, 2, \ldots, n$ and $s_n =$ the ratio of the sum of their surface areas to the surface area of the given sphere, then

$$s_n = x_1^2 + x_2^2 + \ldots + x_n^2 \tag{9}$$

and the problem reduces to maximizing s_n, subject to

$$x_1^3 + x_2^3 + \ldots + x_n^3 = 1. \tag{10}$$

Note that $\alpha = s_2$. This problem can be solved using multi-variable calculus, but there is a neat algebraic approach which uses the Cauchy-Schwarz inequality. For the benefit of the reader, we will state it and provide a sketch of proof.

Theorem (Cauchy-Schwarz Inequality)**.** *Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be any two sets of real numbers. Then*

$$\left|\sum_{j=1}^{n} a_j b_j\right| \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}. \tag{11}$$

Equality holds if and only if there exists a constant λ such that $a_j = \lambda b_j$, for $1 \leq j \leq n$.

Proof. Write $A = \sum_{j=1}^{n} a_j^2$, $B = \sum_{j=1}^{n} b_j^2$, $C = \sum_{j=1}^{n} a_j b_j$. We need to show $C^2 \leq AB$.

If $B = 0$, then $b_j = 0$ for all j, so $C = 0$. Hence the inequality holds, trivially. We may therefore assume that $B \neq 0$. Since B is the sum of squares of real numbers, B must be positive.

Using the fact that the square of a real number is non-negative, we have

$$\begin{aligned} 0 &\leq \sum_{j=1}^{n} (Ba_j - Cb_j)^2 \\ &= B^2 \sum_{j=1}^{n} a_j^2 + C^2 \sum_{j=1}^{n} b_j^2 - 2BC \sum_{j=1}^{n} a_j b_j \\ &= B(BA - C^2). \end{aligned} \tag{12}$$

Since $B > 0$, it follows that $C^2 \leq AB$. We also infer that equality holds if and only if $Ba_j - Cb_j = 0$ for all j, $1 \leq j \leq n$. Taking $\lambda = C/B$, the condition for equality that $a_j = \lambda b_j$ for $1 \leq j \leq n$ is obtained. □

Application to the original problem. Returning to the original problem, we see that if we put $a_i = x_i^{1/2}$ and $b_i = x_i^{3/2}$ for $1 \le i \le n$, then by Cauchy-Schwarz we get

$$\begin{aligned} s_n^2 &= \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^2 \\ &\le \left(x_1 + x_2 + \cdots + x_n\right)\left(x_1^3 + x_2^3 + \cdots + x_n^3\right) \\ &= x_1 + x_2 + \cdots + x_n. \end{aligned}$$

Another application of Cauchy-Schwarz with $a_i = 1$ and $b_i = x_i$ for $1 \le i \le n$ yields

$$\left(x_1 + x_2 + \cdots + x_n\right)^2 \le \left(1 + 1 + \cdots + 1\right)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) = ns_n. \tag{13}$$

Combining the last two inequalities, we get

$$s_n^2 \le \sqrt{ns_n} \Rightarrow s_n \le \sqrt[3]{n}, \tag{14}$$

with equality if and only if $x_i = 1/\sqrt[3]{n}$ for $1 \le i \le n$.

Therefore, to maximize the total surface area, the spheres must be of radius $r/\sqrt[3]{n}$.

To get a feel of how fast the maximum value of the total surface area increases as n grows, consider the values of n from the set $\{10^{3k} : k = 1, 2, 3, \ldots\}$ and compute the maximum value of $s_{10^{3k}}$. It is 10^k. If $k = 1$, a ten-fold increase in the total surface area can be achieved by making 1000 spheres, each of radius $r/10$. The striking observation is that as n grows large, the volume of each sphere goes to zero, but the total surface area gets arbitrarily large. Perhaps in the limit the spheres are reduced to specks of dust and the entire mass is distributed as a two-dimensional surface.

References

1. Venkatachala, B J.. *Inequalities: An Approach Through Problems*, Hindustan Book Agency.

PRITHWIJIT DE is a member of the Mathematical Olympiad Cell at Homi Bhabha Centre for Science Education (HBCSE), TIFR. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. His other interests include puzzles, cricket, reading and music. He may be contacted at de.prithwijit@gmail.com.

FRACTIONAL TRIPLETS

VINAY NAIR

Consider this puzzle. *Three brothers – Youngest, Middle and Oldest – each receive some money in the form of some inheritance. Youngest keeps half the amount he receives and divides the balance equally among Middle and Oldest. After Middle receives his share, he too keeps half the amount and distributes the balance equally among Youngest and Oldest. Oldest in turn keeps half of the amount he now has (after receiving the shares from his brothers) and divides the remaining equally among Youngest and Middle. At the end, if all three of them have equal amounts, can we determine how much each of them had at the start? If so, how do we do so? If not, why not?*

This is an old puzzle with a slight modification. Let's explore the puzzle using numbers. We assume at the start that there is a bank from where we can draw as much money as we wish (natural numbers only; no fractions). We consider three players A, B and C. The order in which they play the game is A, then B, then C. Each player in turn takes a certain amount from the bank. For the players to act as required, A must take a multiple of 4. Let's say that A takes 4 units. How much should B take? Remember, B gets 1 from A, as A divides half of his amount equally among B and C. Obviously, B cannot take an even number, because an even number and 1 make an odd number. B also cannot take odd numbers like 1, 5, 9, 13, etc., because if we add 1 to these numbers, the resulting number will be an even number which is not divisible by 4. So B must take odd numbers like 3, 7, 11,… Suppose that B takes 7 units. B gets 1 from A, resulting in a total of 8. He keeps 4 for himself and divides the remaining amount equally between A and C. A now has 4 units, B has 4 units, and C has his own money plus 1 unit that he receives from A and 2 units that he receives from B. Now what quantity should C take so that he can complete the iterative steps? Like B, C too must have a quantity that is a multiple of 4 so that he can keep half for himself

Keywords: Sharing, factors, multiples, estimation

and divide the other half equally among A and B. So C's options are numbers like 1, 5, 9, 13,… Suppose that C takes 5 units; the result will be as shown below.

	A	B	C
	4 − 2	7 + 1	5 + 1
Balance	2 + 2	8 − 4	6 + 2
Balance	4 + 2	4 + 2	8 − 4
Balance	6	6	4

So if A, B and C start with 4, 7 and 5, they end up with 6, 6 and 4 respectively.

To understand any game better, it is a good idea to play it yourself. So take a pause and try playing the game with three players. Each player in turn chooses a certain number. At the end of every game, write down the quantities chosen at the start by the three players and the quantities that each has at the end of three rounds. For example, if A, B and C start with 4, 11 and 8, then this is what would happen.

	A	B	C
	4 − 2	11 + 1	8 + 1
Balance	2 + 3	12 − 6	9 + 3
Balance	5 + 3	6 + 3	12 − 6
Balance	8	9	6

At the end of the game, A is left with 8 units, B with 9, and C with 6.

Suppose the puzzle was, if the numbers left with A, B and C after the exchanges are 8, 9 and 6 respectively, how much did they have initially? How is this to be solved? Let's call this **Puzzle #2** and the puzzle at the beginning of this article **Puzzle #1**. We will return to this after some time. In the meantime, you can give some thought to how this can be solved.

After we play some games and observe the results, we find some relations and patterns. Table 1 shows the different results if A chooses 4 and B chooses 11.

Initial Amounts			**Final Amounts**		
A	B	C	A	B	C
4	11	4	7	8	4
4	11	8	8	9	6
4	11	12	9	10	8
4	11	16	10	11	10
4	11	20	11	12	12
4	11	24	12	13	14
4	11	28	13	14	16

Table 1

B could have taken 3, 7, 11, 15 or any number in this arithmetic progression. Table 1 shows the choices that C has, once A and B make a choice. We can see some patterns in the 'final amounts' table: *the initial amounts for A and B being the same, an increase of 4 units for C leads to increases in the final amounts of A, B and C by 1, 1 and 2, respectively.*

Let's fix rules for 'winning' the game. Say that the person who scores *least* in the end wins the game. Once A and B have chosen 4 and 11, C can choose 4, 8 or 12 to score the least. If he chooses 16, he ties with A (10 – 10). If he chooses 20, he ties with B. If he chooses any number beyond 24, he ends up with the maximum amount at the end of the game.

Let's try changing the rules for winning the game. The person who makes the maximum profit (final amount - initial amount) wins the game. Or the person who makes the maximum loss wins the game. Who gets an advantage in the game? Or is it a fair game?

A closer look at a few more game scores may help us see some more patterns and make more rules.

Initial Amounts			**Final Amounts**		
A	B	C	A	B	C
4	3	2	4	3	2
4	3	6	5	4	4
4	3	10	6	5	6
4	3	14	7	6	8

Table 2

Initial Amounts			Final Amounts		
A	B	C	A	B	C
8	6	0	7	5	2
8	6	4	8	6	4
8	6	8	9	7	6
8	6	12	10	8	8

Table 3

Initial Amounts			Final Amounts		
A	B	C	A	B	C
4	3	2	4	3	2
8	6	4	8	6	4
12	9	6	12	9	6
$4x$	$3x$	$2x$	$4x$	$3x$	$2x$
8	10	11	11	10	8
16	20	22	22	20	16
$8x$	$10x$	$11x$	$11x$	$10x$	$8x$
4	7	13	8	8	8
8	14	26	16	16	16
$4x$	$7x$	$13x$	$8x$	$8x$	$8x$

Table 4

The above tables show the patterns when different numbers are chosen as initial amounts. Let's generalise the game. When A chooses a number of the form $8a - 4$, B must choose a number of one of these forms: $16b + 3$, $16b + 7$, $16b + 11$, $16b + 15$. Accordingly, C must choose a number of one of these forms: $4c + 2$, $4c + 1$, $4c$, $4c + 3$. Similarly, when A chooses a number of the form $8a$, B must choose a number of one of these forms: $16b + 2$, $16b + 6$, $16b + 10$, $16b + 14$. Accordingly, C must choose a number of one of these forms: $4c + 1$, $4c$, $4c + 3$, $4c + 2$, respectively. Table #5 summarises this observation:

A	B	C
$8a - 4$	$16b + 3$	$4c + 2$
$8a - 4$	$16b + 7$	$4c + 1$
$8a - 4$	$16b + 11$	$4c$
$8a - 4$	$16b + 15$	$4c + 3$
$8a$	$16b + 2$	$4c + 1$
$8a$	$16b + 6$	$4c$
$8a$	$16b + 10$	$4c + 3$
$8a$	$16b + 14$	$4c + 2$

Table 5

Solving puzzles related to triplets

Let's go back to Puzzle #2, where it is given that after three exchanges, A, B and C are left with 8, 9 and 6, and we need to find the initial amounts. One can do it algebraically by assuming the initial amounts to be a, b and c and forming three equations. Solving the equations, we deduce the initial amounts. However, the answer can also be arrived at by studying patterns. Here's what one can do.

A must have started with a multiple of 4. In this example, A must have started with one of the following numbers: 4, 8, 12, 16, 20.

Case #1: Can A have started with 16 or 20? No, because after the first exchange itself, he would be left with 8 or more, and after the second and third exchanges, he would have still more at the end of the game. However, it is given that A has 8 left with him at the end of the game. So we rule out this possibility.

Case #2: Can A have started with 12? If so, then he will be left with 6 after the first exchange and would need only 2 more to reach 8 at the end of the game. For this to happen, B and C should give A 1 each, or one of them should give 0 and the other one should give 2. If B gives 1, then B should have 4. Then C will have 7 (23 – 12 – 4 = 7). With A's and B's share of 1 each, C will now have 9 which cannot be divided. Hence B cannot give 1 to A. Let's now consider B giving 2 to A. In that case, B should have 8. If A has 12 and B has 8, then C will have 3. Since C has 3, anything added to C will result in a minimum share of 1 going to A, which is not what we want. So we rule out this possibility as well.

Case #3: Can A have started with 8? If so, then B will have a number of one of these forms: $16x + 2$, $16x + 6$, $16x + 10$, $16x + 14$, and C will have a number of one of these forms: $4c + 1$, $4c$, $4c + 3$, $4c + 2$ (refer Table #5). Considering these points, B can have 2, 6, 10 or 14, which leaves C with the options (from 23 – A's amount – B's amount) 13, 9, 5 or 1, respectively.

- If B has 2, then after getting the share of 2 from A, B will have to give 1/4 of the share to A and 1/4 to C. Since C should have 13 (23 – A's amount – B's amount), after the second exchange, C will have 16 but he should have only 12 because he is left with 6 at the end. So this does not work out.

- If B has 6, then C will have 9 and after the second exchange, C will be left with 13 instead of 12.
- If B has 10, then C will have 5 and after the second exchange, C will be left with 10 instead of 12.
- We don't even have to check if B has 14 because after observing the earlier case, we are sure that C will be left with less than 10.

We conclude that A cannot have 8. Hence A has 4.

- **Case #4:** A has 4. According to Table #5, B will have 3, 7, 11 or 15 and accordingly C will have four different values. Again by taking four cases, we will be able to eliminate three of them and will be left with only one case which is the initial amount of 4, 11 and 8 respectively.

Using the general form, we can create puzzles where the final result after three exchanges is given and one has to work out the initial amounts that the three people had. There can be other strategies to tackle the puzzle which the reader will discover on his or her own.

Some points to ponder

Before we conclude, here are a few points to ponder.

1. The first few rows in Table #4 show that when we consider three numbers of the form $4x$, $3x$ and $2x$, the game will go into a loop. Let us call (4, 3, 2) a ***Repeating Fractional Triplet***.
2. A triplet of the form ($4x$, $7x$, $13x$) results in ($8x$, $8x$, $8x$) after three exchanges. Let's call (4, 7, 13) a ***Uniform Fractional Triplet***. Is there a ***Uniform Fractional Triplet*** not of the form ($4x$, $7x$, $13x$)? If yes, how many? If not, why not?
3. A triplet of the form ($8x$, $10x$, $11x$) results in ($11x$, $10x$, $8x$) after three exchanges. Let's call (8, 10, 11) a ***Reverse Fractional Triplet***. Is there a ***Reverse Fractional Triplet*** not of the form ($8x$, $10x$, $11x$)? If yes, how many? If not, why not?
4. The triplet (4, 15, 3) results in (8, 10, 4) after three exchanges. But it doesn't stop there. We can continue further and reach (7, 6, 9) after five exchanges. Is there a triplet for which more than five exchanges can be done? If so, what is the maximum number of exchanges that can be done with a given triplet? How many such triplets are there? Can we support it with a proof?
5. Instead of a triplet, let us consider a quadruplet (6, 5, 4, 3) where four persons play the game and four exchanges need to be done instead of three. After four iterations, (6, 5, 4, 3) results in (6, 5, 4, 3), i.e., the same quadruplet; so it is a ***Repeating Fractional Quadruplet***. Similarly, (8, 7, 6, 5, 4) is a ***Repeating Fractional Quintuplet***. Can we find more like these?
6. Can we find the general form for a ***Fractional Quadruplet*** and ***Fractional Quintuplet?***
7. Can we find new types of triplets other than ***Uniform, Repeating, Reverse?***

Can we solve Puzzle #1 when the final numbers (or the total of what remains with the three people) are not given? What do you think?

VINAY NAIR is the co-founder of *Raising a Mathematician Foundation*. He conducts various online and offline programs in different parts of India on exploratory learning in Mathematics and ancient Indian Mathematics. He aspires to create a research mentality in the minds of school children. He can be reached at vinay@sovm.org.

HOW to PROVE it

ClassRoom

SHAILESH SHIRALI

In this episode of "How To Prove It," we study a number of possible characterisations of a parallelogram, as listed in the article 'Parallelogram' elsewhere in this issue.

The following question was posed in the article 'Parallelogram': What characterises a parallelogram? In other words:

What minimal properties must a quadrilateral have for us to know that it is actually a parallelogram?

The basic definition of a parallelogram is: A plane four-sided figure whose opposite pairs of sides are parallel to each other. That is, a plane four-sided figure $ABCD$ is a parallelogram if and only if $AB \parallel CD$ and $AD \parallel BC$ (see Figure 1).

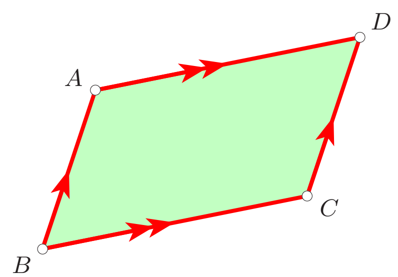


Figure 1

Here is an alternative definition, but framed in the language of transformations: A parallelogram is a quadrilateral with rotational symmetry of order 2.

Which of these properties characterises a parallelogram?

We went on to list five different properties possessed by a parallelogram and asked in each case whether the property in question characterises a parallelogram; i.e., if a planar quadrilateral possesses that property, is it necessarily a parallelogram?

Keywords: Parallelogram, characterisation, congruence, one-way implication

Note that we have retained the numbering of the items from the original article.

5. If $ABCD$ is a parallelogram, then each of its diagonals divides it into a pair of triangles with equal area. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that each of its diagonals divides it into two triangles that have equal area, is $ABCD$ necessarily a parallelogram?

6. If $ABCD$ is a parallelogram, then $AB = CD$ and $AD \parallel BC$. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that $AB = CD$ and $AD \parallel BC$, is $ABCD$ necessarily a parallelogram?

7. If ABCD is a parallelogram, then AB = CD and $\measuredangle A = \measuredangle C$. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that $AB = CD$ and $\measuredangle A = \measuredangle C$, is $ABCD$ necessarily a parallelogram?

8. If ABCD is a parallelogram, then the sum of the squares of the sides equals the sum of the squares of the diagonals. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2,$$

is $ABCD$ necessarily a parallelogram?

(9) If ABCD is a parallelogram, then the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point, is $ABCD$ necessarily a parallelogram?

The characterisations which do not work

It turns out that the statements numbered 6 and 7 are not characterisations of a parallelogram. How do we show this? In general, how do we show that any statement is not true?

Notion of a counter example. One way to disprove a statement is to exhibit a counterexample. This notion is discussed in detail in the article "Divisibility by 27," elsewhere in this issue. Nevertheless, we give a few illustrations of the notion here. Consider the following statements:

Statement 1: On observing that the odd numbers 3, 5 and 7 are prime, we may be tempted into making the following (very rash) conjecture: "All odd numbers exceeding 1 are prime." But we quickly discover a counterexample: the number 9. So the conjecture is false.

Statement 2: It is quite easy to see that if n is composite, then $2^n - 1$ is composite as well. For example, $2^{10}-1$ is a composite number (it is divisible by 3). The reader should be easily able to prove the following statement: if $n = rs$, where r and s are positive integers greater than 1 (i.e., r and s are proper divisors of n), then both $2^r - 1$ and $2^s - 1$ are proper divisors of 2^n-1. With this established, we may be tempted to make the following conjecture: If p is a prime number, then 2^p-1 is prime as well. The evidence is encouraging to start with, for the numbers

$$2^2 - 1 = 3,\ 2^3 - 1 = 7,\ 2^5 - 1 = 31,\ 2^7 - 1 = 127$$

are all prime. However, the very next number in the sequence, $2^{11} - 1$, turns out to be composite:

$$2^{11} - 1 = 2047 = 23 \times 89.$$

This means that we have found a counterexample to the stated claim, and therefore the claim is false.

Counterexample to statement 6

The question under examination is this: If $ABCD$ is a quadrilateral such that $AB = CD$ and $AD \parallel BC$, is $ABCD$ necessarily a parallelogram? The reader will readily see that the answer must be No, and that a counterexample is readily at hand; namely, an isosceles trapezium ($ABCD$ in Figure 2). Here $AD \parallel BC$, $AB = DC$ and $\measuredangle ABC = \measuredangle DCB$. (The figure may be constructed as follows. Start with a nonrectangular parallelogram $ABED$; by assumption, $\measuredangle ABE \neq 90^\circ$. There is no harm in assuming that $\measuredangle ABE < 90^\circ$. Extend BE and drop perpendicular DF to line BE. Extend BF further to C so that $FC = EF$. Join DC.)

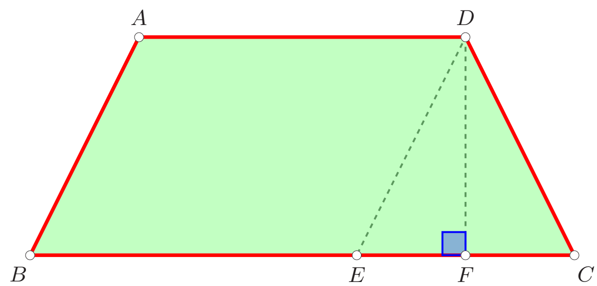


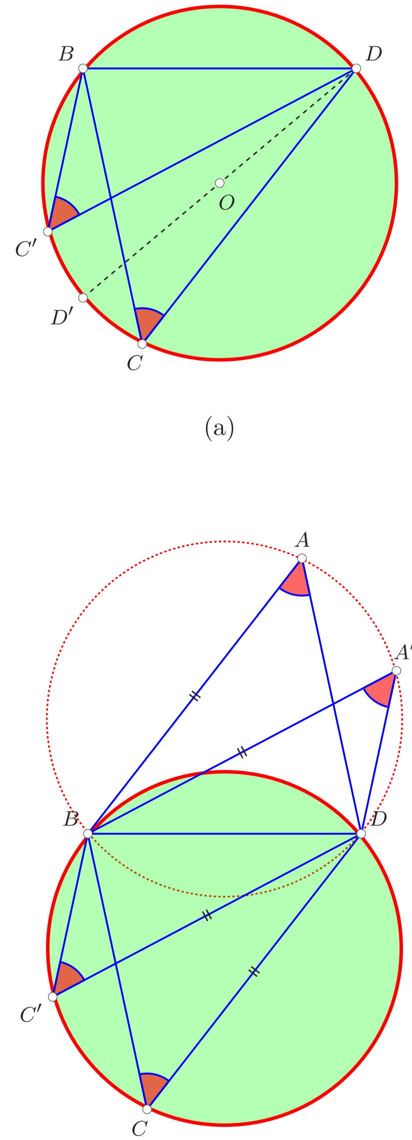
Figure 2

Counterexample to statement 7

The question under examination is this: If $ABCD$ is a quadrilateral such that $AB = CD$ and $\measuredangle A = \measuredangle C$, is $ABCD$ necessarily a parallelogram? We shall exhibit a figure which shows that the answer is again 'No.' But finding a counterexample is more challenging now than earlier! (Please try to find one on your own before reading on.)

We make use of the symmetries of the circle. Consider the configuration shown in Figure 3 (a). It shows a chord BD of a circle with centre O; here it is important that BD is *not* a diameter of the circle. Infinitely many pairs of points C, C' can now be located on the circle, on the same side of BD as O, with the property that $CD = C'D$. One way to do this is to draw the diameter DD' through D and choose a suitable point C on the circle, on the same side of BD as O (a few restrictions need to be placed on the position of C, but we will leave it to you to work out these restrictions); then reflect CD in diameter DD'. Its image is $C'D$, with C' also on the circle. This does the needful. Note that in this configuration we have $CD = C'D$ and $\measuredangle BCD = \measuredangle BC'D$.

Now we locate point A in such a way that $ABCD$ is a parallelogram as shown in Figure 3 (b). Observe now that in $ABC'D$, we have $AB = C'D$ and $\measuredangle BAD = \measuredangle BC'D$. But $ABC'D$ is clearly not a parallelogram.



(a)

(b)

Figure 3

Equally, we could locate point A' in such a way that $A'BC'D$ is a parallelogram; then $A'BCD$ has $A'B = CD$ and $\measuredangle BA'D = \measuredangle BCD$, yet it is not a parallelogram.

Remark. Points A, B, D, A' lie on a circle which is the reflection in BD of the circle through points B, C', C, D. This brings out an unexpected and elegant symmetry of the figure: if you rotate the entire configuration through 180° about the midpoint of BD, it gets mapped to itself (points B, D exchange places, as do points A, C and points A', C').

Statements 5, 8 and 9

We see that statements 6 and 7 do not provide characterisations of a parallelogram. What about statements 5, 8 and 9? They do provide the asked-for characterisations! We shall give proofs for these claims in a follow-up article.

References

1. Jonathan Halabi, "Puzzle: proving a quadrilateral is a parallelogram" from JD2718, https://jd2718.org/2007/01/10/puzzle-proving-a-quadrilateral-is-a-parallelogram/
2. Wikipedia, "Parallelogram" from https://en.wikipedia.org/wiki/Parallelogram

SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

A Collection of Prime Facts...

1 The only prime number which is 1 less than a perfect 2^{nd} power is 3.

2 The only prime number which is 1 less than a perfect 3^{rd} power is 7.

3 The only prime number which is 1 less than a perfect 5^{th} power is 31.

4 The only prime number which is 1 less than a perfect 7^{th} power is 127.

5 The only prime number which is 1 less than a perfect 13^{th} power is 8191.

6 The only prime number which is 1 less than a perfect 17^{th} power is 131071.

Two Questions

How would you check that these claims are true (not that the stated numbers are prime, but that they are the **only** such prime numbers)?

Note the exponents in this sequence of statements: 2, 3, 5, 7, 13 and 17. They are themselves all prime numbers. Note also that there is one prime number absent from the list – the prime 11. Why should this be so?

Note also that the numbers 3, 7, 31, 127, 8191,… are all 1 less than powers of 2. In no case is a power of 3 involved, nor a power of 4, nor a power of 5, nor a power of 6,… Do you see why this must be the case?

Hint: Think of the factorisations of such numbers.

— $\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

A CYCLIC KEPLER QUADRILATERAL & THE GOLDEN RATIO

MICHAEL DE VILLIERS

Introduction

A recent paper by Bizony (2017) discussed the interesting golden ratio properties of a Kepler triangle, defined as a right-angled triangle with its sides in geometric progression in the ratio $1 : \sqrt{\phi} : \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$.

This inspired me to wonder what would happen if one similarly defined a 'Kepler quadrilateral' with sides in geometric progression with common ratio $\sqrt{\phi}$. (This is an example of the mathematical process of 'constructive defining,' whereby a new concept or object is defined by extension or modification of the definition of an already existing concept or object; compare De Villiers, 2017a & 2017b.) Would such a quadrilateral perhaps also exhibit some golden ratio, or other interesting properties?

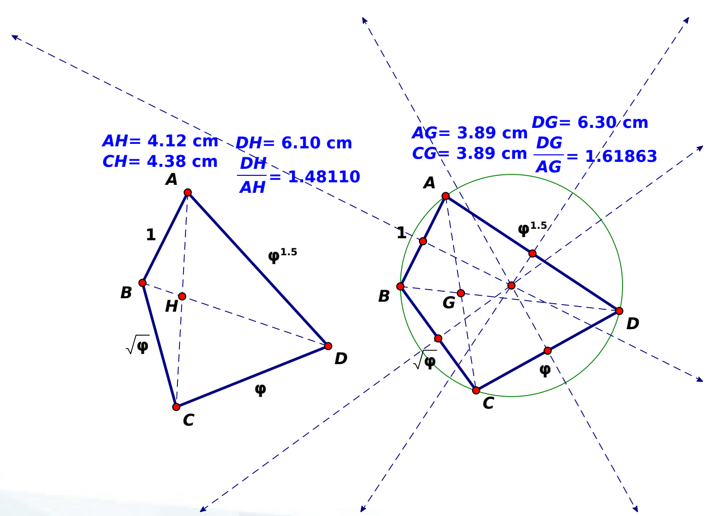


Figure 1

Keywords: Dynamic geometry, Kepler triangle, Kepler quadrilateral

A Conjecture about a Cyclic Kepler Quadrilateral

Proceeding to construct such a 'Kepler quadrilateral' $ABCD$ with sides in geometric progression as indicated by the first figure in Figure 1 produces a flexible quadrilateral with a changing shape. No interesting, invariant properties seemed immediately apparent. However, if $ABCD$ is dragged so that the perpendicular bisectors of the sides become concurrent (i.e., so that it becomes cyclic), as indicated by the second figure in Figure 1, it was observed as shown by measurements that not only did it seem that diagonal AC appeared to be bisected by diagonal BD, but also that $DG : AG = \phi$.

The reader is invited to explore these conjectures dynamically by visiting an interactive sketch at: http://dynamicmathematicslearning.com/cyclic-kepler-quadrilateral.html.

Though the dynamic geometry software is quite convincing about the truth of the conjectures, it doesn't *explain why* the result is true. For a satisfactory explanation, a deductive proof is required. Readers are now challenged to first try and prove the result themselves before reading further.

Proof

With reference to the second figure in Figure 1, we have from the sine area formula that

$$\text{Area } \Delta ABD = \frac{1}{2}\phi^{3/2} \sin A,$$
$$\text{Area } \Delta BCD = \frac{1}{2}\phi^{3/2} \sin C.$$

But since it is given that $ABCD$ is cyclic, angles A and C are supplementary, hence the sines of the two angles are equal. It follows that $area\Delta ABD = area\Delta BCD$. Therefore, diagonal BD bisects the area of the cyclic Kepler quadrilateral. It turns out that this is a necessary and sufficient condition for BD to bisect diagonal AC (compare Pillay & Pillay, 2010, pp. 16-17; Josefsson, 2017, p. 215), and, in addition, a proof is given as an Addendum at the end of this paper. Also note from the proof above that this bisecting diagonal property obviously generalizes to any cyclic quadrilateral with sides $AB : BC : CD : DA$ in geometric progression with common ratio r.

For the second property, again apply the sine area formula to obtain

$$\text{Area } \Delta ACD = \frac{1}{2} \cdot \phi \cdot \phi^{3/2} \sin D,$$
$$\text{Area } \Delta ABC = \frac{1}{2} \cdot 1 \cdot \phi^{1/2} \sin D.$$

Hence,

$$\frac{\text{Area } \Delta ACD}{\text{Area } \Delta ABC} = \phi^2 = \frac{\text{perpendicular height from } D \text{ to } AC}{\text{perpendicular height from } B \text{ to } AC} = \frac{DG}{BG},$$

since the two right triangles with respective hypotenuses DG and BG respectively formed by the perpendiculars from D and B to AC are similar (two pairs of corresponding angles are equal). Therefore,

$$BG = \frac{DG}{\phi^2}. \qquad (1)$$

From the intersecting chords theorem, $BG \times DG = AG^2$ and by substitution of (1) above into this equation, the desired result $\frac{DG}{AG} = \phi$ is obtained. The proof is now complete.

It is left to the reader to explore additional properties of a cyclic Kepler quadrilateral, which is called a 'bisect-diagonal quadrilateral' (because one diagonal is bisected by the other) by Josefsson (2017), who proves a long list of interesting results related to such quadrilaterals. For example, for the cyclic Kepler quadrilateral in Figure 1, diagonal BD can be determined from either one of the following equivalent formulae:

$$\sqrt{\frac{1+\phi+\phi^2+\phi^3}{2}} = \sqrt{\frac{\phi^4-1}{2(\phi-1)}} = \sqrt{\frac{(\phi^2+1)(\phi+1)}{2}}.$$

Concluding Remark

This little exploration would be suitable for high school level students knowing some trigonometry and circle geometry. It therefore provides an accessible, non-routine challenge as well as illustrating how new results can sometimes be discovered experimentally by the 'what-if' extension of older known results, and using dynamic geometry as an effective investigative tool.

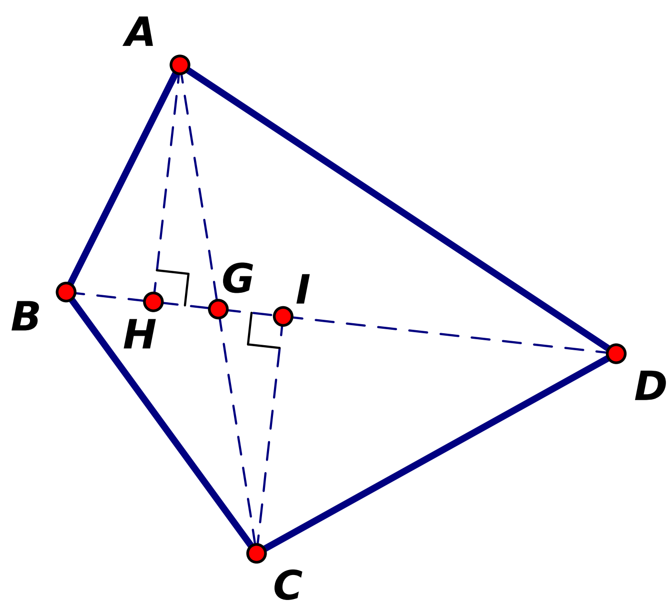


Figure 2

Addendum: Lemma

The area of a quadrilateral $ABCD$ is bisected by diagonal BD if and only if BD bisects AC.

Proof. Consider Figure 2. If area ABD = area BCD, it follows that the perpendicular heights AH and CI to the common base BD have to be equal. Hence, triangles AHG and CIG are congruent and therefore, $AG = CG$. The converse follows similarly and is left to the reader. (Note: the result is also true for a concave quadrilateral $ABCD$).

References

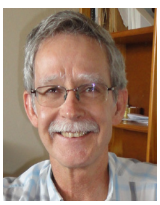
1. Bizoni, M. (2017). The Golden Ratio Unexpectedly. *At Right Angles*, Vol. 6, No. 1, March, pp. 29-31.

2. De Villiers, M. (2017a). An example of constructive defining: From a golden rectangle to golden quadrilaterals & beyond: Part 1. *At Right Angles*, Vol. 6, No. 1, March, pp. 64-69.

3. De Villiers, M. (2017b). An example of constructive defining: From a golden rectangle to golden quadrilaterals & beyond: Part 2. *At Right Angles*, Vol. 6, No. 2, August, pp. 74-81.

4. Josefsson, M. (2017). Properties of bisect-diagonal quadrilaterals. *The Mathematical Gazette*, Vol. 101, No. 551, pp. 214-226.

5. Pillay, S. & Pillay, P. (2010). Equipartitioning and Balancing Points of Polygons. *Pythagoras*, 71, July, pp. 13-21.



MICHAEL DE VILLIERS has worked as researcher, mathematics and science teacher at institutions across the world. Since 1991 he has been part of the University of Durban-Westville (now University of KwaZulu-Natal). He was editor of Pythagoras, the research journal of the Association of Mathematics Education of South Africa, and has been vice-chair of the SA Mathematics Olympiad since 1997. His main research interests are Geometry, Proof, Applications and Modeling, Problem Solving, and the History of Mathematics. His home page is http://dynamicmathematics-learning.com/homepage4.html. He maintains a web page for dynamic geometry sketches at http://dynamic-mathematicslearning.com/JavaGSPLinks.htm. He may be contacted on profmd@mweb.co.za.

On Quadratic Equations:

An Observation

We discuss the quadratic equation

$$ax^2 + bx + c = 0 \ (a \neq 0).$$

Though it is a well-known fact at the Class 11 level, it is nevertheless remarkable that we can state the sum of the roots and the product of the roots of this quadratic equation without knowing the roots! Indeed, the sum of the roots is $-b/a$ and the product of the roots is c/a. This leads to the following two simple observations. Let k be any non-zero real number. Then:

- The sum of the roots of the equation $x^2 - kx + a = 0$ is k for any real number a.
- The product of the roots of the equation $x^2 + ax + k = 0$ is k for any real number a.

– Contributed by **Ranjitbhai Desai**

TechSpace

The Elementary Cellular Automata

A Journey into the COMPUTATIONAL WORLD

JONAKI B GHOSH & ROHIT ADSULE

"It's always seemed like a big mystery how nature, seemingly so effortlessly, manages to produce so much that seems to us so complex." – Stephan Wolfram

The topic of cellular automata has many interesting and wide ranging applications to real life problems emerging from areas such as image processing, cryptography, neural networks, developing electronic devices to modelling biological systems. In fact cellular automata can be a powerful tool for modelling many kinds of systems. They may be described as mathematical models for systems in which simple components act together to produce complicated patterns of behaviour. A large number of cellular automata (CA) models are used to study physical, biological, chemical or social phenomena involving interacting entities. Examples of such CA applications may be as diverse as modelling traffic flow, crystal growth, ant colony activity or forest fires. A popular and well-known example of CA is The Game of Life which was originally developed by John Conway, a British Mathematician in 1970. Though the game appears to be a simple 'toy' played by applying a few simple rules in a two-dimensional grid of square cells, it has been the springboard for the study of 'artificial life' systems because of the amazingly complex behaviour displayed by some of the patterns which emerge when this CA evolves. More details about the Game of Life may be found in [2].

Keywords: *Cellular automata, neural network, ECA, Mathematica, computer algebra system, Boolean expression, Karnaugh map*

The topic of Cellular Automata lends itself to interesting investigations which are well within the reach of high school students. As we hope to illustrate in this article, the ideas are simple and yet powerful. We shall describe briefly our attempts to investigate this unique and interesting topic.

Briefly defined, a cellular automaton is a collection of cells on a grid of a specified shape that evolves through discrete time steps according to a set of rules based on the state (or color) of the neighbouring cells. Cellular Automata may be one, two or three-dimensional. Here we will describe an exploratory project where we attempted to explore the one-dimensional Elementary Cellular Automata (ECA) as defined by Stephan Wolfram using Mathematica, a Computer Algebra System. A part of the explorations were also done using NICO [5] an open source software. Mathematica's extensive numeric as well as easy-to-use graphic capabilities along with inbuilt commands for cellular automata make it very conducive for exploring this topic. Readers with access to Mathematica may explore the topic using the commands described in a later section. Others may use the NICO software.

The aim of the project was to

a) Explore the evolution of the ECAs;

b) Represent the ECA rules in decimal, binary, and Boolean forms;

c) Classify all the ECAs into specific categories depending on the patterns emerging from their evolution. Mathematica programming and the NICO software were used to generate pictures (graphics) of the 256 ECAs and observe their patterns;

d) Explore the sensitivity of the ECAs to initial conditions.

Some Mathematical Preliminaries

As mentioned earlier, a cellular automaton is a collection of coloured cells on a grid of a specified shape that evolves through discrete time steps according to a set of rules based on the state (or color) of the neighbouring cells. One-dimensional cellular automata are usually found on triangular, square or hexagonal grids. We shall limit our discussion to cellular automata on a square grid, that is, on a grid of square cells.

The defining characteristics of cellular automata are as follows.

i) A cellular automaton develops on a grid of cells.

ii) Each cell has a state – dead or alive. Live cells are coloured black or assigned a value 1 whereas dead cells are white and assigned a value of 0. Colours other than black or white may also be used.

iii) Each cell in the grid has a neighbourhood. A neighbourhood of a given cell is a set of cells which are adjacent to it. This may be chosen in various ways. For example, if we consider a linear grid of square cells, then the neighbourhood of each cell may be the two adjacent cells – one to its left and the other to its right.

iv) Finally every cellular automaton must have a defining rule based on which it grows and evolves in discrete time steps. For example, in a square grid, each row of cells may be considered as a different generation of cells. Thus the first row is the initial generation (or generation 0) where each cell has a state (0 or 1). The state of each cell in the second row must be a function of its neighboring cells in the row above it (that is the initial row). This may be written as

$$(\text{Cell state}_t) = f(\text{Neighboring Cell state}_{t-1})$$

To begin with let us consider a linear grid of 8 cells where every cell has state 0 except the 5th cell which has state 1.

Figure 1: A linear grid of 8 cells where the 5^{th} cell is a live cell.

This linear grid of square cells will be referred to as **generation 0** (or row 0). The states of cells in

generation 1(that is row 1) will be determined by the neighborhood of each cell in row 0 which comprises of the three cells just above it. Clearly the states of these three cells may be any one of the following

000 001 010 100 011 101 110 111

The fact that there are three cells and the state of each cell is either 0 or 1 implies that there are $2^3 = 8$ ways of colouring these cells. Thus there are 8 neighbourhood configurations described by the triples of 0's and 1's as shown above. Conventionally, while defining an ECA, these neighbourhoods are taken in the following specific order.

111 110 101 100 011 010 001 000

Each of these configurations will determine the state of the middle cell of the three cells just below it in the next row, which may again be either 0 or 1. However the state of the leftmost corner cell in row 1 will be determined by the state of the cell just above it in row 0, its right neighbour, and the last cell in row 0. Similarly, the state of the rightmost corner cell in row 1 will be determined by the state of the cell just above it in row 0, its left neighbour and the first cell in row 0.

Let us now arbitrarily assign 0's and 1's to all 8 neighbourhood configurations as follows

111	110	101	100	011	010	001	000
0	0	0	1	1	1	1	0

Pictorially this may be represented as

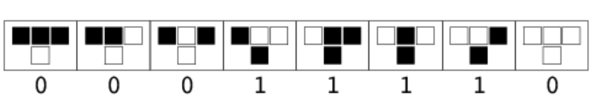


Figure 2: A rule set for a one-dimensional cellular automaton

This arbitrary assignment (also known as the rule set) will be the defining rule which will determine how this particular automaton will evolve. Note that this defining rule 00011110 may be treated as a binary number whose decimal representation may be obtained as follows

$$0 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 30$$

This kind of a rule set generates an elementary cellular automaton. The ECA which evolves from this rule set is referred to as **Rule 30**. However a different assignment of 0's and 1's would lead to a different rule set and a different ECA. Since each of the 8 groups of three cells may be assigned 0 or 1, this leads to $2^8 = 256$ possible assignments. Thus, in all, there are 256 ECA rules.

Equivalent expressions of ECA rules

We have seen that each ECA rule has a decimal representation as well as a binary representation. For example, let us consider rule 110. Its binary representation may be obtained by expanding 110 in powers of 2 as follows

$$0 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 110$$

Here 110 has been written in base 2. Reading off the coefficients of the powers of 2, we get 01101110, which is the binary representation of 110.

Pictorially this translates to the following

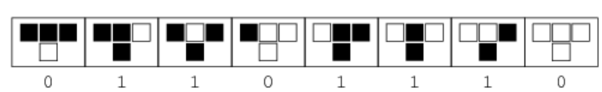


Figure 3: The elementary Cellular Automaton 110

Boolean expressions for ECA rules

Sometimes it is useful to represent the Cellular Automata rules using logical expressions. In such a case we need to find the Boolean expressions equivalent to the rule. This section is for the reader who is interested to understand the details of converting the ECA rules to Boolean expressions. However, the reader may skip this section and move to the next section on Mathematica explorations in case he or she wishes to omit the details. We will now highlight the method of finding Boolean expression for a given rule, say rule 110, using the concept of Karnaugh maps [4]. A detailed treatment of the method is described in [5].

In this example three input variables can be combined in 8 different ways. Thus a truth table with three arguments p, q and r as input variables will have 8 rows. The 8 possible configurations of the truth values of p, q and r (where 1 corresponds to true and 0 corresponds to false) correspond to the 8 neighbourhood configurations of rule 110 as discussed above. Further the truth value of each row corresponds to the binary digits of 110 as follows:

Row number	p	q	r	f (p, q, r)
1	1	1	1	0
2	1	1	0	1
3	1	0	1	1
4	1	0	0	0
5	0	1	1	1
6	0	1	0	1
7	0	0	1	1
8	0	0	0	0

Table 1: Truth table for three arguments p, q and r

There are two different unsimplified boolean expressions representing the function f(p, q, r), using the Boolean variables p, q, r and their inverses. These are

$f(p, q, r) = \sum m_i,\ i \in \{2,3,5,6,7\}$ where m_i are the miniterms to map (that is, rows that have output 1 in the truth table).

$f(p, q, r) = \sum M_i,\ i \in \{1,4,8\}$ where M_i are the maxterms to map (that is, rows that have output 0 in the truth table).

Using the concept of Karnaugh maps (which are used to simplify Boolean expressions), we obtain Table 2 (placing 1's in the cells which correspond to 1 in the f(p, q, r) column). For example, row 7 corresponds to 001 (values of p, q and r respectively) and has truth value 1. Thus, in the following grid, we place 1 in the cell which corresponds to 001 (that is, in cell (2, 3)). Similarly we fill in the cells which correspond to rows with truth value 1 (note that there are five such rows and hence the following table has five 1's).

p/qr	00	01	11	10
0		1	1	1
1		1		1

Table 2: The Karnaugh map table

Now that the Karnaugh table has been constructed, we shall try to find the simplest Boolean expression to define the rule 110. To achieve this, we will group the 1's inside the table in sets of 2, 4, 8 respectively (that is in powers of 2). These groupings, referred to as miniterm groupings, must be done in such a way that adjacent 1's in the Karnaugh table are encircled (or grouped in rectangles). Thus the set of 2 groupings (since 4, 8, etc., are not possible) must include all the five 1's in the table and may be taken as follows:

Group 1: These are the two 1's in column 3

Group 2: These are the two 1's in column 5

Group 3: These are the two 1's appearing in the 3rd and 4th cells of row 2. (Instead we could have also taken the 1's appearing in the 2nd and 3rd cells of row 2. The groups may intersect as groups 2 and 3 here).

These translate to q'r (group 1- since the value of q is 0 and the value of r is 1), qr' (group 2 - since the value of q is 1 and the value of r is 0) and p'r (group 3 – value of p is 0 and that of r changes from 1 to 0).

The simplified Boolean expression is = p'r + q'r + qr'

Using the 'and', 'or' and 'not' notations of logic denoted by the symbols $\wedge$, $\vee$, and $\sim$ respectively the above expression may also be expressed as

$$(\sim p \wedge r) \vee (\sim q \wedge r) \vee (q \wedge \sim r)$$

(note that '+' is replaced by $\vee$, ' . ' is replaced by $\wedge$ and ' is replaced by ~)

This may be further simplified using the XOR notation as

$$(\sim p \wedge r) \vee (q \oplus r)$$

(Note that XOR is referred to as the exclusive or argument in logic and is represented by the symbol ⊕. A ⊕ B is true when either A or B is true (or has value 1)).

In a similar manner using the concept of Karnaugh maps we were able to find Boolean expressions corresponding to several other ECA rules.

A *Mathematica* based exploration

Mathematica is a powerful Computer Algebra System which can be used to explore the Elementary Cellular Automata. In this section we shall use Mathematica to obtain the graphic (pictorial) representations of all the 256 ECAs. The aim is to observe the evolutionary pattern of each ECA through the first 100 iterations and to categorise them into specific classes based on the patterns manifested by them.

The inbuilt *Mathematica* command for exploring cellular automata is

```
CellularAutomaton[rule, init, t]
```

Here:

`rule` stands for the decimal representation of the rule set in binary form. For example, 26 is the decimal representation of the rule 0 0 0 1 1 0 1 0.

`init` represents the initial condition or generation 0.

`t` denotes the number of steps

For example, the **rule 30** elementary cellular automata after 10 iterations, with an initial condition of 9 cells, where only the 5th cell is alive may be computed as follows

```
CellularAutomaton[30, {0, 0, 0, 0, 1, 0, 0, 0, 0}, 10]
```

Note that the output is displayed as a list of lists where each list is the state of the automaton at time t.

```
{{0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 1, 1, 1, 0, 0, 0}, {0, 0, 1, 1, 0, 0, 1, 0, 0}, {0, 1, 1, 0, 1, 1, 1, 1, 0}, {1, 1, 0, 0, 1, 0, 0, 0, 1}, {0, 0, 1, 1, 1, 1, 0, 1, 1}, {1, 1, 1, 0, 0, 0, 0, 1, 0}, {1, 0, 0, 1, 0, 0, 1, 1, 0}, {1, 1, 1, 1, 1, 1, 1, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 1, 1}, {0, 1, 0, 0, 0, 0, 1, 1, 0}}
```

By adding the //`TableForm` command at the end of the code, the output is obtained in the form of a table or matrix.

```
CellularAutomaton[30, {0, 0, 0, 0, 1, 0, 0, 0, 0}, 10]//TableForm
```

```
000010000
000111000
001100100
011011110
110010001
001111011
111000010
100100110
111111100
100000011
010000110
```

However, if we wish to create the automaton with a larger number of cells in row 0 having a single live cell in the middle we may replace `init` by `{{1}, 0}`. By doing this Mathematica adjusts the number of cells in row 0 according to the number of iterations required. Thus

```
CellularAutomaton[30, {{1}, 0}, 10]//TableForm
```

outputs 10 rows (iterations) with an initial row consisting of a single live cell in the centre (with value 1) and 10 dead cells (value 0) on either side.

```
000000000010000000000
000000000111000000000
000000001100100000000
000000011011110000000
000000110010001000000
000001101111011100000
000011001000010010000
000110111100111111000
```

```
00110010001110000010 0
01101111011001000111 0
11001000010111101100 1
```

Whereas

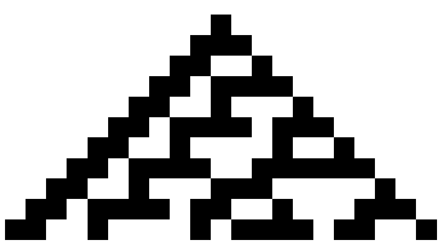
```
CellularAutomaton[30, {{1}, 0},
20]//TableForm
```

will display the output with the initial row having 1 live middle cell and 20 cells of value 0 on either side.

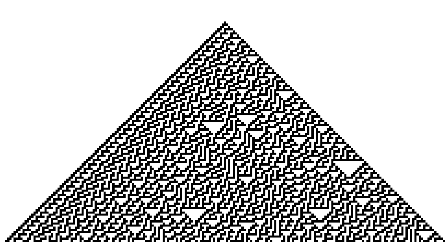
The above codes display the output only in binary form and do not help us to visualise them. For obtaining the visual image of the automata we use the **`ArrayPlot`** command.

Thus the first 10 iterations of Rule 30 with a single live cell in the initial condition is as follows. Note that all cells with state 1 are black whereas the ones with state 0 remain white.

```
ArrayPlot[CellularAutomaton[30,
{{1}, 0}, 10]]
```

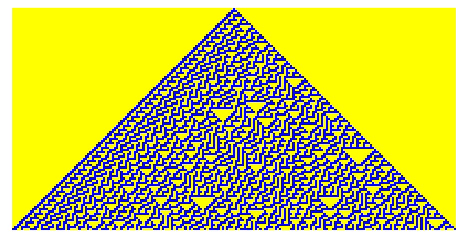


Increasing the iterations to 50 and 100 (by changing 10 to 50 and 100 respectively in the above code) yields the following outputs:



We can also add colour to the output by using the **`ColorRules`** option within the **`ArrayPlot`** command as follows

```
ArrayPlot[CellularAutomaton[30,
{{1}, 0}, 100], ColorRules->
{1→Blue, 0->Yellow}]
```



We observe that the evolution of rule 30 is random. There appears to be a continuous border on the left-hand side of the 'triangle'. However there is no fixed pattern in the yellow triangles which are randomly spread across the pattern.

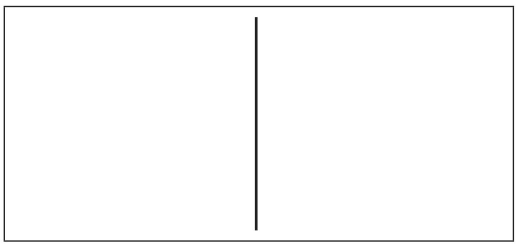
Classification and sensitivity analysis of the ECAs

Using *Mathematica* and NICO software (details are mentioned later in the article) we were able to explore the evolution of all the 256 ECA rules and classify them into four major categories which are also mentioned in the research literature associated with cellular automata.

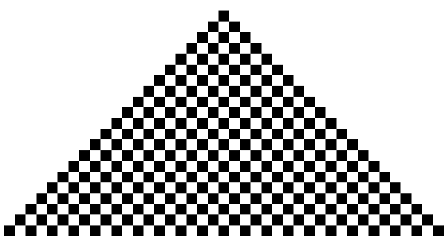
1. **Uniform:** where all cells are either black or white.
2. **Repetitive:** Some patterns are repetitive having a regular alternating pattern or a block of cells which repeat themselves throughout.
3. **Nested or Fractal–like:** These automata lead to Sierspinski triangle like fractal patterns exhibiting clear self- similarity or other nested patterns.
4. **Random or chaotic:** These are patterns which cannot be placed in any of the above three categories. There is no fixed pattern in these automata and their evolution is highly unpredictable.

Here are some examples of ECAs which evolve from one live cell in the centre of the top row of the grid

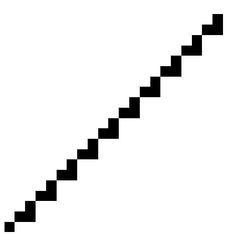
Rule 151: Uniform – all cells are black



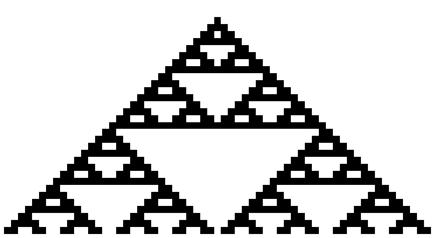
Rule 4: Repetitive: stationary – all cells are black in the same location



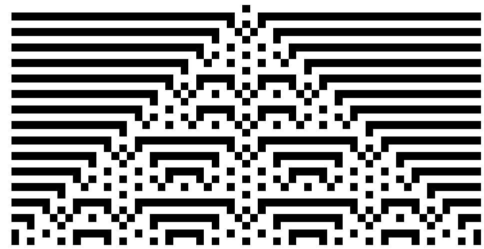
Rule 50: Repetitive: alternating black and white cells remain the same throughout



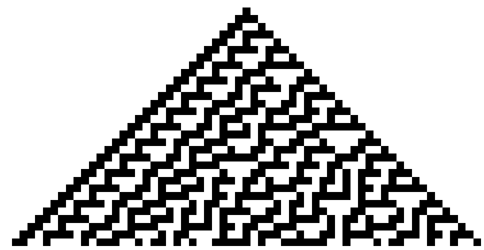
Rule 6: Repetitive: non-stationary: the same pattern is repeated in a different location.



Rule 126: Nested: looks like the Sierpinski triangle pattern



Rule 105: Nested: nested behaviour appearing in symmetrical patterns



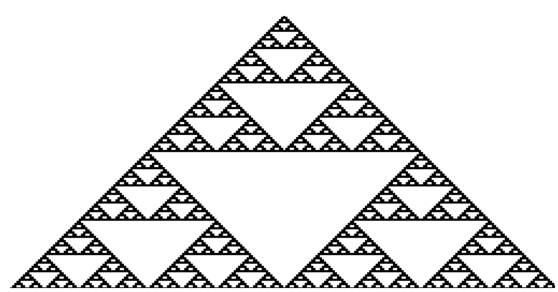
Rule 30: Random: completely random behaviour

Exploring the ECAs online

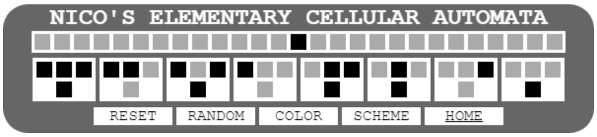
The NICO'S ELEMENTARY CELLULAR AUTOMATA software may be accessed through link https://sciencevsmagic. net/eca/#. It provides a wonderful opportunity to explore the ECAs even if one doesn't have access to a sophisticated computer algebra system such as Mathematica.

In order to explore a particular ECA rule one may enter the rule number after the # symbol.

Thus https://sciencevsmagic. net/eca/#126 will lead to the Sierpinski triangle like pattern shown below.



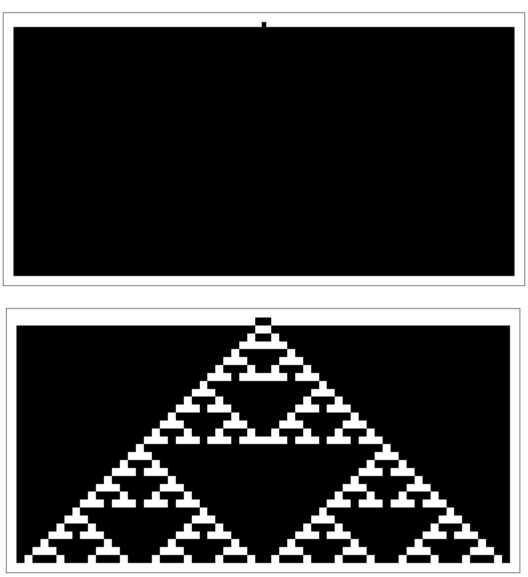
The user may also obtain an ECA by specifying the initial condition and the defining rule of the automaton by clicking on the relevant cells in the options provided at the bottom of the screen as shown.



Sensitivity analysis of the ECAs

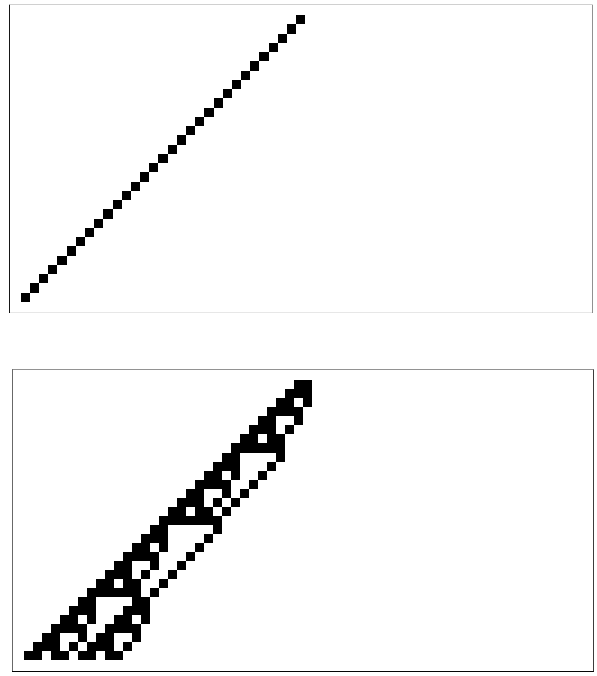
We tried to explore the sensitivity of each ECA rule to the specified initial conditions using both *Mathematica* as well as NICO. In this section we have included the Mathematica images. In the previous section the automata have been generated by considering one live cell positioned in the centre (value 1) of a linear grid of cells. We were interested to see if making a slight variation in the initial condition significantly impacted the evolution of the cellular automata.

For example, Rule 151 with one black cell in its initial row leads to all black cells (as we have seen above). However if we add another black cell to the initial condition we get a Sierpinski-like nested pattern.



Rule 151 for different initial conditions which differ by only one black cell.

Similarly Rule 106 displays a repetitive non-stationary evolution where the same pattern is repeated in a different location. However by making a slight change in its initial condition we get a random behaviour.



Rule 106 for different initial conditions which differ by only one black cell.

We tested all 256 ECAs for sensitivity using Mathematica and NICO. The findings have been summarised in Table 1.

Category	Rule Number(s)	Characteristics	Sensitivity to Initial Conditions
Uniform	0, 8, 32, 40, 64, 72, 96, 104, 128, 136, 160, 168, 192, 200, 224, 232	All cells are white.	Not sensitive to initial conditions.
Uniform	151, 159, 183, 191, 215, 223, 233, 235, 237, 239, 247, 249, 251, 253, 255	All cells are black.	Rule 151 is very sensitive and leads to randomness and Sierpinski-like structures. Rule 183 leads to nestedness.
Repetitive (Stationary)	1, 4, 5, 7, 12, 19, 21, 23, 29, 31, 33, 36, 37, 44, 51, 55, 63, 68, 71, 76, 87, 91, 95, 100, 108, 119, 123, 127, 132, 140, 164, 172, 196, 201, 203, 204, 205, 207, 217, 219, 221, 228, 236	The same pattern is repeated in the same location.	Same structure retained with minor variations.
Repetitive (Single Triangular pattern)	50, 54, 58, 77, 94, 109, 114, 122, 133, 147, 158, 163, 177, 178, 179, 186, 190, 214, 222, 242, 246, 250, 254	A single triangle is formed in which the same pattern continues throughout although inner variations may occur.	109 and 133 are very sensitive to initial conditions, and lead to randomness. Rule 122 leads to Sierpinski-like structure.
Repetitive (Non-Stationary)	2, 3, 6, 9, 10, 11, 14, 15, 16, 17, 20, 24, 25, 27, 34, 35, 38, 39, 41, 42, 43, 46, 47, 48, 49, 52, 53, 56, 59, 61, 65, 66, 67, 74, 80, 81, 83, 84, 85, 88, 97, 98, 103, 106, 107, 111, 112, 113, 115, 116, 117, 120, 121, 125, 130, 134, 138, 139, 142, 143, 144, 148, 152, 155, 162, 166, 170, 171, 173, 174, 175, 176, 180, 184, 185, 187, 189, 194, 202, 208, 209, 211, 212, 213, 216, 226, 227, 229, 231, 234, 240, 241, 243, 244, 245, 248	The same pattern is repeated in a different location.	Rules 106, 120 are very sensitive and lead to randomness.
One-Sided Pattern	13, 28, 60, 69, 70, 78, 79, 92, 93, 102, 110, 124, 137, 141, 153, 156, 157, 188, 193, 195, 197, 198, 199, 206, 220, 230, 238, 252	Forms a definite shape on only one side.	Some rules, such as 124 and 137 displayed chaotic behaviour.
Nested	89, 105, 150	Shows nested patterns	
Sierpinski Triangle	18, 22, 26, 82, 90, 126, 129, 146, 154, 161, 165, 167, 181, 182, 210, 218	Lead to a Sierpinski-Triangle structure	Rules 22 and 182 are very sensitive to initial conditions and lead to randomness.
Multiple Patterns	57, 62, 73, 99, 118, 131, 145	Contains two patterns in the same rule. Many of these are symmetric.	In Rule 73, the symmetry disappears.
Random	30, 45, 75, 86, 89, 101, 135, 149, 169, 225	The evolution seems to be random.	Randomness is retained

Table 3: Classification and sensitivity analysis of the 256 ECAs

Closing Remarks

In this article we have discussed the basics of the one-dimensional Elementary Cellular Automata as described by Stephan Wolfram. Our explorations have convinced us that simple rules can lead to interesting and complicated evolution patterns. These can be classified into the four categories: Uniform, Repetitive, Nested and Random, as described in the earlier section. However some of these ECAs are quite sensitive to initial conditions. Changing the state of one single cell in the initial row of the automata may lead to a completely different pattern. Thus the ECA may show transition from one category to another.

The topic of cellular automata offers tremendous scope for investigations. In the subsequent article, we shall detail elementary cellular automata which we have created by defining our own CA rules and we hope to highlight some of their interesting properties.

References

1. Gage, D., Laub, E., McGarry, B. Cellular Automata: Is rule 30 random? extracted from https://www. cs. indiana. edu/~dgerman/2005midwestNKSconference/dgelbm. pdf
2. Conway's Game of Life extracted from https://en. wikipedia. org/wiki/Conway%27s_Game_of_Life
3. Georgiadis, E., (2007). A Note on Minimal Boolean Formula Size of One-Dimensional Cellular Automata, Massachusetts Institute of Technology, Cambridge, MA 02139, U. S. A
4. Karnaugh maps retrieved from https://en. wikipedia. org/wiki/Karnaugh_map
5. House, A. (2003). Introduction to Karnaugh Maps extracted from http://www. eecg. toronto. edu/~ahouse/mirror/engi3861/kmaps. pdf
6. Nico's Elementary Cellular Automata retrieved from https://sciencevsmagic. net/eca/#176
7. Wolfram, S., (1985). Two-Dimensional Cellular Automata In *Journal of Statistical Physics*, 38, 901- 946.
8. Wolfram, S., (1986). Cellular Automata and Cryptography In Crypto '85 Proceedings, Lecture Notes in ComputerScience, 218, 429-432 (Springer-Verlag, 1986).
9. Yang, Y., Billings, S. A. (2000). Extracting Boolean Rules from CA Patterns In *IEEE Transactions on Systems, Man and Cybernetics - Part B: Cybernetics,* 30 (4), 575 - 581.

JONAKI GHOSH is an Assistant Professor in the Dept. of Elementary Education, Lady Sri Ram College, University of Delhi where she teaches courses related to math education. She obtained her Ph.D. in Applied Mathematics from Jamia Milia Islamia University, New Delhi, and her M.Sc. from IIT Kanpur. She has taught mathematics at the Delhi Public School, R K Puram, where she set up the Math Laboratory & Technology Centre. She has started a Foundation through which she conducts professional development programmes for math teachers. Her primary area of research interest is in the use of technology in mathematics instruction. She is a member of the Indo Swedish Working Group on Mathematics Education. She regularly participates in national and international conferences. She has published articles in journals and authored books for school students. She may be contacted at jonakibghosh@gmail.com.

ROHIT ADSULE is a 12th grade student from The Shri Ram School Aravali, Gurugram, Haryana. His hobbies include chess, piano and football. He loves integrating his skills in mathematics with concepts related to computer science, and it was thus he came across the topic of cellular automata. He wishes to explore and understand the connection of mathematics to seemingly unrelated fields such as music theory and behavioural mechanics.

Problems for the MIDDLE SCHOOL

R. ATHMARAMAN & SNEHA TITUS

One of the scarier words in a math student's lexicon is the word *locus*! The definition (*A path traced by a point when it moves under certain condition*) seems amorphous, difficult to pin down and much too open-ended! This topic is usually introduced in high school; we are deliberately presenting problems on locus which will give students a gentler introduction to the same. By incorporating constructions, we hope to give students practice in a topic which is often taught in recipe mode; following a series of construction steps usually cooks up a figure designed to satisfy your teacher and get you the marks you need! Why do constructions work? Can one design constructions? These are the questions which lead to skill-based learning and problem solving in the mathematics class.

Problem VII-1-M.1

Construct the locus of a point which moves so that it is always at a given distance d from a given point O. Describe the locus in words.

Problem VII-1-M.2

Construct the locus of a point which moves so that it is at a fixed distance from a given straight line l. Describe the locus in words.

Problem VII-1-M.3

A and B are two fixed points. Construct the locus of a point which moves so that at every instant it is equidistant from both A and B. Describe the locus in words. Justify your construction.

Keywords: Geometry, path, locus, constraint, constructions, area

Problem VII-1-M.4

AB is a line segment of length 5 cm. Find the locus of a point C which moves so that it is the third vertex of triangle ABC whose area is 10 cm^2.

Problem VII-1-M.5

AB is a line segment of length 8 cm. Find the locus of a point C which moves so that it is the third vertex of parallelogram ABCD whose area is 40 cm^2.

Problem VII-1-M.6

Construct the locus of a point which moves so that it remains at equal distance from two given parallel straight lines l and m. Describe the locus in words.

Construct the locus of a point which moves so that it is a vertex of a trapezium whose base AB is on one of two parallel lines which are at a distance of 5 cm. from each other and whose area is 36 cm^2.

Problem VII-1-M.7

Given two intersecting straight lines, find the locus of the centre of the circle that touches (but does not intersect) both lines.

Note: When a straight line touches a circle, it is perpendicular to the radius of the circle at the point of contact.

Solutions

Problem VII-1-M.1

Construct the locus of a point which moves so that it is always at a given distance d from a given point O. Describe the locus in words.

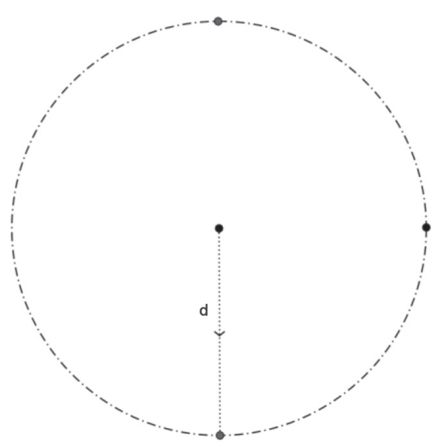


Figure 1

The locus is a circle of radius *d* centred at O. See Figure 1

Teacher's Note: This problem is deliberately set to reassure the student that the locus is nothing but an application of definitions already learnt. The very action of constructing the circle with the compass reinforces the constraint set by the locus.

Problem VII-1-M.2

Construct the locus of a point which moves so that it is at a fixed distance s from a given straight line l. Describe the locus in words.

The locus is either one of 2 straight lines parallel to *l* and at a distance of *s* units from it. See Figure 2.

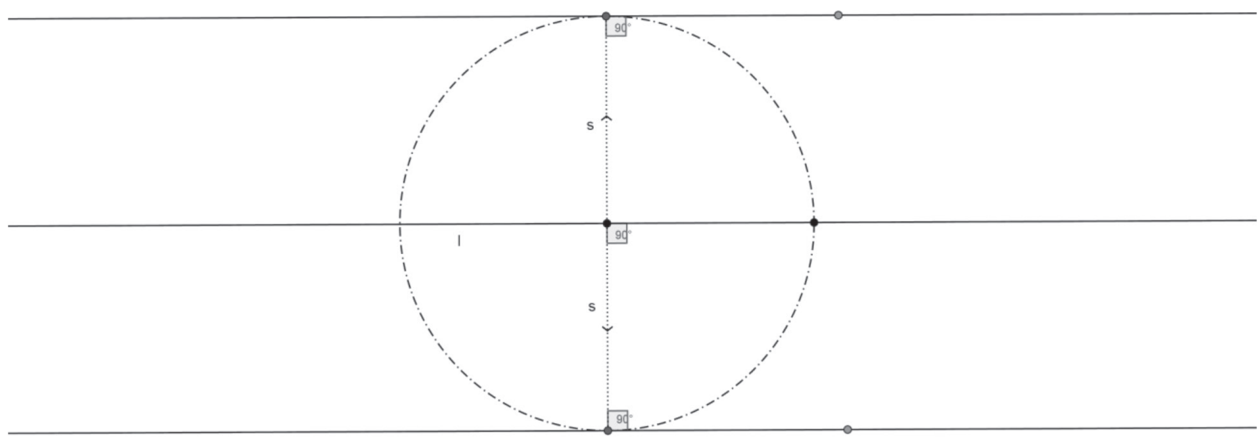


Figure 2

Teacher's Note: Again, easy enough but students get to practise the construction of two perpendiculars and also to figure out how to make the distance between the parallel lines fixed at *s*.

Problem VII-1-M.3

A and B are two fixed points. Construct the locus of a point which moves so that at every instant it is equidistant from both A and B. Describe the locus in words. Justify your construction.

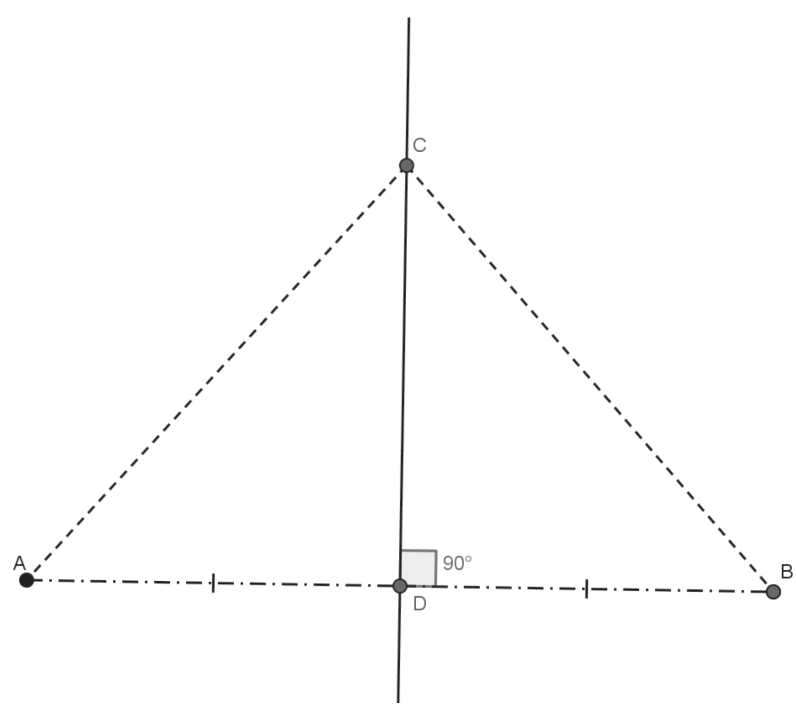


Figure 3

The locus is the perpendicular bisector of the segment joining A and B.

Teacher's Note: The construction of a perpendicular bisector is actually based on the fact that the diagonals of a rhombus bisect each other at right angles. So we start with equal sides and arrive at the perpendicular bisector. Students who have done congruency, can justify that triangles CAD and CBD are congruent by SAS and that this ensures that AC = BC. This justification starts with the perpendicular bisector and arrives at the equal sides. Talk about appreciating a view point from two angles!! Note the jump in the 'ask' of the problem, the layer of *justification* requires students to think about why their construction will work.

Problem VII-1-M.4

AB is a line segment of length 5 cm. Find the locus of a point C which moves so that it is the third vertex of triangle ABC whose area is 10 cm^2.

Since the area and the base are given, the height of the triangle is fixed and will be equal to 4 cm. Since the height has to remain constant, vertex C will have to be at a constant perpendicular distance of 4 cm. from the base. Referring to problem 2 above, we see that the vertex C will have to move on a line parallel to AB and at a distance of 4 cm. from it. There are two such lines on opposite sides of AB and these represent the required locus as shown in Figure 4.

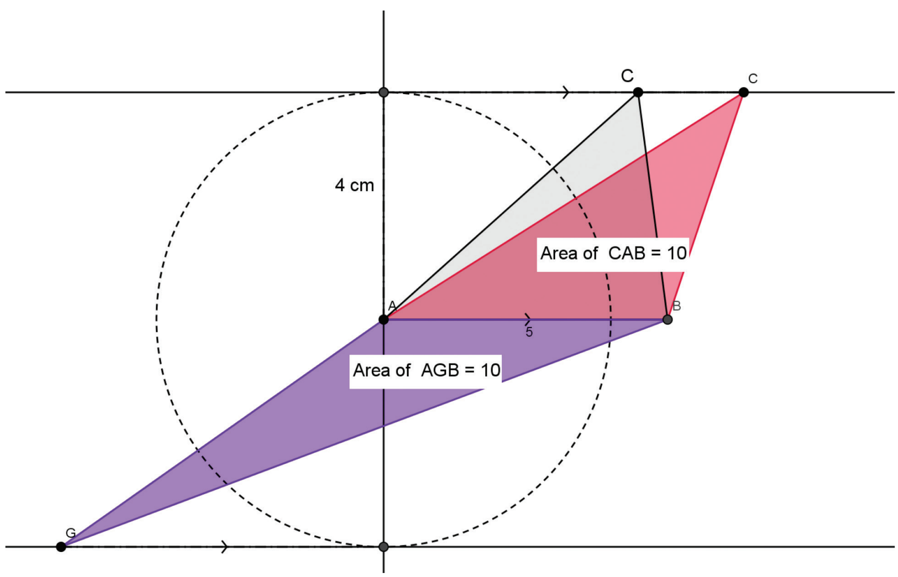


Figure 4

Teacher's Note: In this problem, there is an interesting juxtaposition of the formulae learnt in mensuration with constructions in geometry. Do give students the time to mull over the problem. An additional point - The idea of two possibilities for the locus is something that may not occur to students.

Problem VII-1-M.5

AB is a line segment of length 8 cm. Find the locus of a point C which moves so that it is the third vertex of parallelogram ABCD whose area is 40 cm^2.

Teacher's Note: The difference in this problem is that only 2 points of the parallelogram are given. Two more points are required to be found: this, however, is not a difficult task as, finding just one of the points will immediately fix the other also. Since the area of the parallelogram is fixed and so is the base, the locus of C is one of the lines parallel to the base and at a perpendicular distance of 5 cm. from AB. We show only one of the possible loci, to avoid visual clutter.

An interesting twist on this problem would be: find the locus of vertex D. (The answer is that D travels on exactly the same line as vertex C; so its locus is identical to the locus of C.)

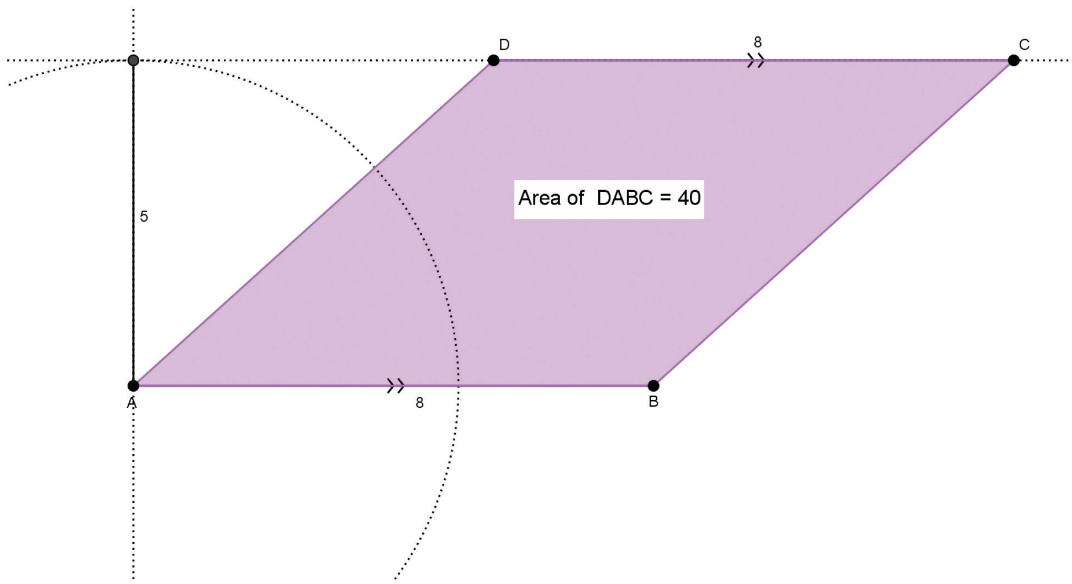


Figure 5

We leave it as an interesting variation for the reader to get the locus of the vertex of the trapezium.

Problem VII-1-M.6

Given two intersecting straight lines, find the locus of the centre of the circle that touches (but does not intersect) both lines.

The required locus is either one of the angular bisectors of the two angles between the lines.

Teacher's Note: The note is given for the benefit of middle school students, who may not know this fact. This question is slightly more difficult and will require careful facilitation by the teacher, who may start by asking students to sketch a line on which the centre will lie. Next, students may be asked to fill in all the constraints imposed by the problem. A sample figure which may emerge is given below.

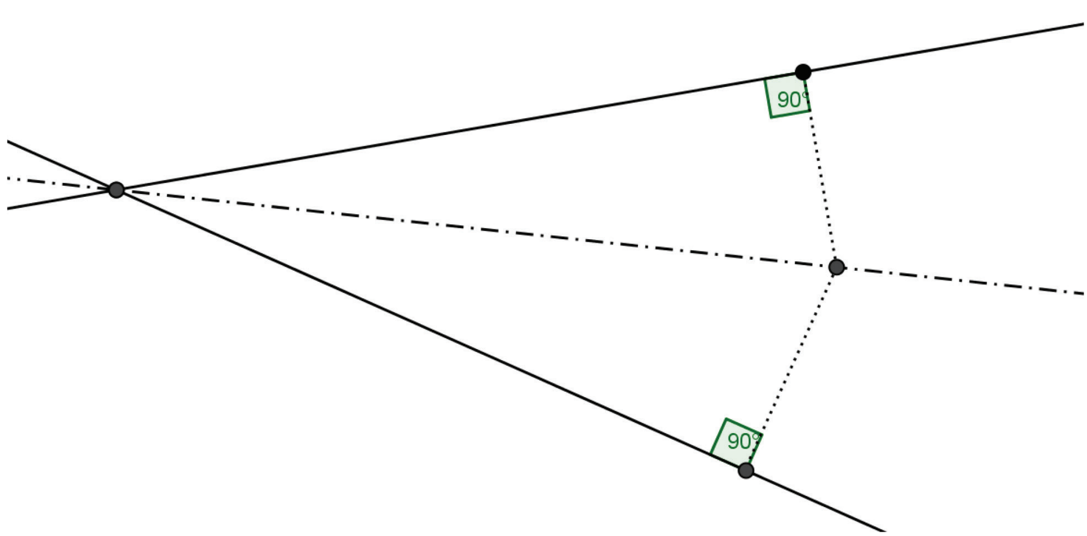


Figure 6a

Next, the teacher may ask students to shade in the two triangles that have emerged.

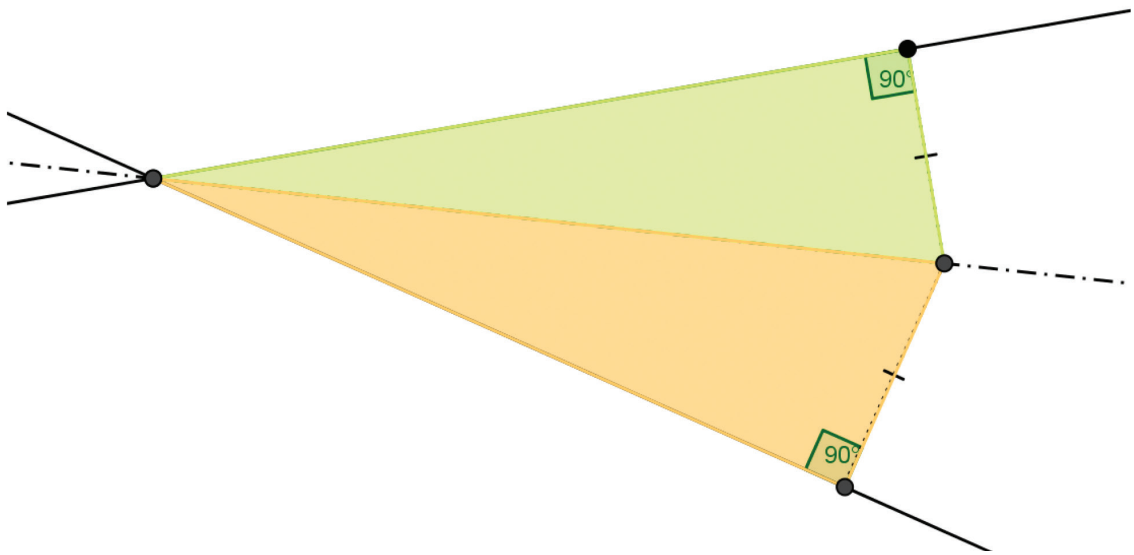


Figure 6b

Careful questioning will allow students to realise that the two triangles should, in fact, be identical- the suggestion of folding the paper along the line on which the centre lies may help this realization. From here, the idea that the locus is the angular bisector of one of the angles formed by the two lines should be a logical step. Some students may point out that there is a second locus, that is, the bisector of the other two angles at the vertex.

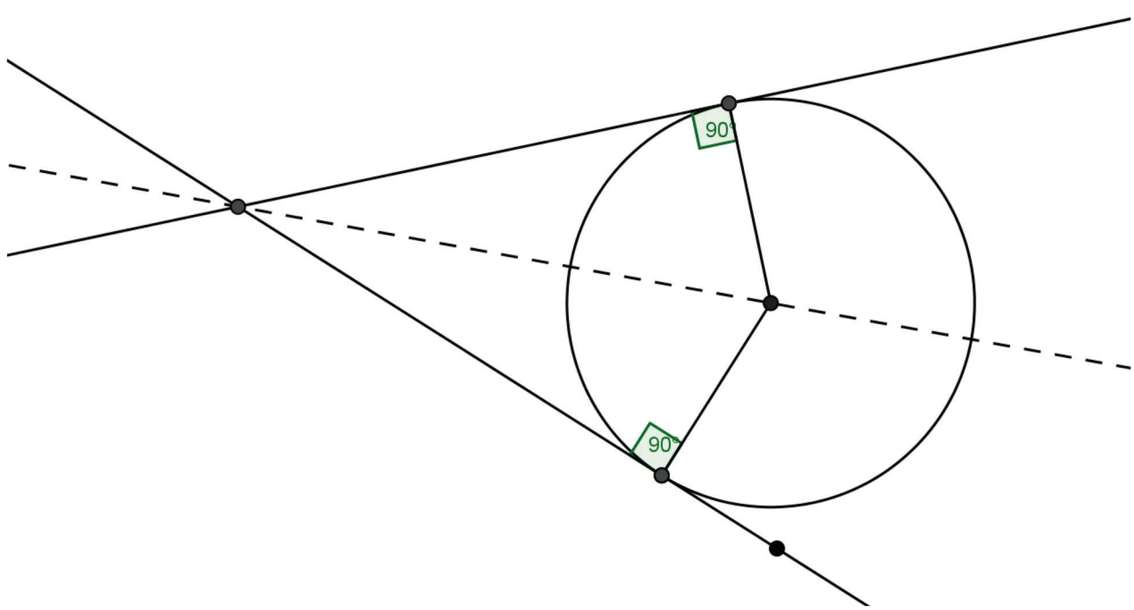


Figure 6c

Pedagogy: Problem solving strategies that have been imparted with this problem set are visualization, representation, trial and error, and logical reasoning. These exercises will develop the skills of mathematical communication, justification and proof, and give students plenty of scope for drill and practice in simple constructions in an interesting manner.

Some locus problems for students who have done circle properties are given below, their solutions have been uploaded https://www.geogebra.org/m/hzMBVtnv, https://www.geogebra.org/m/FwmeaSKV.

Exercises:

1. A is a fixed point on a given circle, and B is a variable point on the circle; it travels around the circle. M is the midpoint of AB. What is the locus of M?
2. A ladder leans against a vertical wall. A cat is seated on its central rung (i.e., at the midpoint of the ladder). The ladder, for some reason, starts to slip away from the wall. The cat stays on the central rung. What is the locus described by the cat?
3. The same question as Q2, but this time the cat is not seated at the midpoint of the ladder. What is the locus described by the cat in this situation?

GIMPS Finds a New Mersenne Prime!

The Great Internet Mersenne Prime Search (GIMPS for short) has just discovered a new Mersenne prime. It is the following number:

$$\mathbf{2^{77,232,917} - 1}$$

It is currently the largest known prime number and has more than 23 million digits. The discovery was made in late December 2017, using the free GIMPS software available at the following website: www.mersenne.org/download/.

It is the 50th such prime number known. Mersenne primes have the form $2^n - 1$ where n itself is a prime number. The first few such primes are the following:

$$2^2 - 1 = 3, \quad 2^3 - 1 = 7, \quad 2^5 - 1 = 31, \quad 2^7 - 1 = 127$$

These prime numbers are named after the French monk Marin Mersenne who lived in the 17th century.

For more details, please refer to the following press release: https://www.mersenne.org/primes/press/M77232917.html.

For more details on this fascinating class of prime numbers, please refer to: https://en.wikipedia.org/wiki/Mersenne_prime.

Some of you may consider becoming part of this collaborative search for Mersenne primes, GIMPS. You would need to download the free software mentioned at the top.

– $\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$

Problem Corner

ADVENTURES IN PROBLEM SOLVING
Miscellaneous Problems from the RMOs

SHAILESH SHIRALI

In this edition of 'Adventures' we study a few miscellaneous problems, some from past RMOs.

As usual, we pose the problems first and give the solutions later in the article, thereby giving you an opportunity to work on the problems.

Problems

(1) Show that the equation

$$a^2 + (a+1)^2 + (a+2)^2 + (a+3)^2 + (a+4)^2 + (a+5)^2 + (a+6)^2 = b^4 + (b+1)^4$$

has no solutions in integers a and b. (RMO 2017, #2)

(2) Given that x is a non-zero real number such that $x^4 + 1/x^4$ and $x^5 + 1/x^5$ are rational numbers, prove that $x + 1/x$ is a rational number. (RMO 2013, Paper 4, #4)

(3) Let ABC be an isosceles triangle in which $\measuredangle A = 100^\circ$ and $\measuredangle B = 40^\circ = \measuredangle C$. Let side AB be extended to a point D such that $AD = BC$. Find $\measuredangle BCD$.

Solutions

(1) *Show that the equation*

$$a^2 + (a+1)^2 + (a+2)^2 + (a+3)^2 + (a+4)^2 + (a+5)^2 + (a+6)^2 = b^4 + (b+1)^4$$

has no solutions in integers a and b.

Keywords: Rational number, isosceles, triangle, exterior angle, trigonometric identity, modulus

Solution. The fact that on the left side of this equation we have the sum of the squares of **seven** consecutive integers is a strong clue to how we must proceed. Namely, we can examine both sides of the equation modulo 7. (This strategy offers us a good chance of succeeding. Of course, nothing can be guaranteed….) On the left side, the coefficient of a^2 is 7, which is $0 \pmod 7$. The coefficient of a is $2(1+2+3+4+5+6)$ which also is $0 \pmod 7$. Finally, the constant term is

$$1^2+2^2+3^2+4^2+5^2+6^2=\frac{6\times 7\times 13}{6},$$

and this yet again is $0 \pmod 7$. So the quantity on the left is identically $0 \pmod 7$.

Now we must check what happens on the right side of the equation, where we see the sum of the fourth powers of two consecutive integers. Let us first list the residues left when we divide the sequence of fourth powers (0, 1, 16, 81, 256, …) by 7; we get:
$0,\ 1,\ 2,\ 4,\ 4,\ 2,\ 1,\ 0,\ 1,\ 2,\ 4,\ 4,\ 2,\ 1,\ \ldots.$ We see a repeating sequence of residues:

$$\underbrace{0,\ 1,\ 2,\ 4,\ 4,\ 2,\ 1},\ \underbrace{0,\ 1,\ 2,\ 4,\ 4,\ 2,\ 1},\ \ldots.$$

The string $0,1,2,4,4,2,1$ of length 7 repeats indefinitely. To verify that it does repeat, we only need to verify that $(n+7)^4-n^4$ is a multiple of 7 for all integers n. But this verification is routine.

Finally, it is easy to verify that no two consecutive members of the string 0, 1, 2, 4, 4, 2, 1 yield a sum which is $0 \pmod 7$. It follows that the expression $b^4+(b+1)^4$ is not a multiple of 7 for any integer b. Hence the given equation has no integer solutions.

Remark. In the same way, we can show that the following equations have no solutions in integers a and b:

$$a^2+(a+1)^2+\cdots+(a+5)^2+(a+6)^2=b^2+(b+1)^2,$$

$$a^2+(a+1)^2+\cdots+(a+9)^2+(a+10)^2=b^2+(b+1)^2,$$

$$a^2+(a+1)^2+\cdots+(a+9)^2+(a+10)^2=b^2+(b+1)^2+(b+2)^2+(b+3)^2.$$

(2) *Given that x is a non-zero real number such that x^4+1/x^4 and x^5+1/x^5 are rational numbers, prove that $x+1/x$ is a rational number.*

Solution. Write S_n to denote the quantity x^n+1/x^n; then $S_0=2$ and $S_{-n}=S_n$. Note the following identity:

$$\left(x^m+\frac{1}{x^m}\right)\cdot\left(x^n+\frac{1}{x^n}\right)=x^{m+n}+\frac{1}{x^{m+n}}+x^{m-n}+\frac{1}{x^{m-n}}.$$

From this we see that

$$S_m\cdot S_n=S_{m+n}+S_{m-n},$$

$$S_{2m}=S_m^2-2.$$

From these relationships, we see that if any three of the four quantities S_m, S_n, S_{m+n}, S_{m-n} are rational numbers, then so is the fourth one. Also, if S_m is rational, then so is S_{2m}. Now observe the following:

- Since S_4 and S_5 are rational (given), so are S_8 and S_{10}.

- Since $S_4 S_2 = S_6 + S_2$ and $S_8 S_2 = S_{10} + S_6$, it follows that

$$\begin{aligned} S_{10} &= S_8 S_2 - S_6 \\ &= S_8 S_2 - S_4 S_2 + S_2 \\ &= S_2 \left(S_8 - S_4 + 1 \right). \end{aligned}$$

  Since S_4, S_8 and S_{10} are rational, so is S_2.

- Since $S_4 S_2 = S_6 + S_2$ and S_2, S_4 are rational, so is S_6.
- Finally, since $S_5 S_1 = S_6 + S_4$ and S_4, S_5, S_6 are rational, so is S_1.

It would be of interest to explore the following question:

Let x be a nonzero real number, and let $S_n = x^n + 1/x^n$. Suppose that S_m and S_{m+1} are rational numbers, for some positive integer $m > 1$. Does it necessarily follow that S_1 is rational?

(3) *Let ABC be an isosceles triangle in which $\measuredangle A = 100^\circ$ and $\measuredangle B = 40^\circ = \measuredangle C$. Let side AB be extended to a point D such that $AD = BC$. Find $\measuredangle BCD$.*

Solution. We offer three solutions; the first one uses trigonometry, while the other two use only ideas from pure geometry.

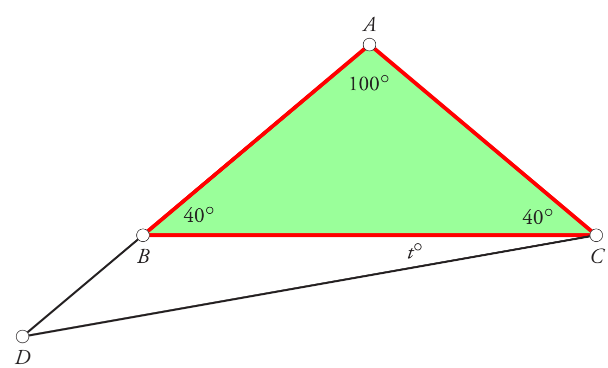


Figure 1

Trigonometric solution. Let t° denote the measure of $\measuredangle BCD$. Then we have, via the sine rule:

$$\text{From } \triangle ADC: \quad \frac{AD}{AC} = \frac{\sin(40+t)^\circ}{\sin(40-t)^\circ},$$

$$\text{From } \triangle ABC: \quad \frac{BC}{AC} = \frac{\sin 100^\circ}{\sin 40^\circ}.$$

Since $AD = BC$ (given), we obtain:

$$\frac{\sin(40+t)^\circ}{\sin(40-t)^\circ} = \frac{\sin 100^\circ}{\sin 40^\circ}.$$

Since $\sin 100^\circ = \sin 80^\circ = 2 \cdot \sin 40^\circ \cdot \cos 40^\circ$, this yields:

$$\frac{\sin(40+t)^\circ}{\sin(40-t)^\circ} = 2\ \cos 40^\circ.$$

This equation needs to be solved for t. The solution can be effected in many ways; I'm sure that readers will find some ways of their own. Here is one that occurred to me. Using the fact that $\sin 30^\circ = 1/2$ and $\cos 40^\circ = \sin 50^\circ$, we rewrite the above relation as:

$$\sin(40+t)^\circ \cdot \sin 30^\circ = \sin(40-t)^\circ \cdot \sin 50^\circ.$$

Next, using the identity $2 \cdot \sin x \cdot \sin y = \cos(x-y) - \cos(x+y)$, we get:

$$\cos(10+t)^\circ - \cos(70+t)^\circ = \cos(10+t)^\circ - \cos(90-t)^\circ.$$

Hence:

$$\cos(70+t)^\circ = \cos(90-t)^\circ.$$

Since $t^\circ < 40^\circ$ ("exterior angle of a triangle strictly exceeds each of the interior opposite angles"), both $70+t$ and $90-t$ lie in the interval $[0, 180]$; and since the cos function does not repeat any values in the interval $[0^\circ, 180^\circ]$, the above relation yields $70+t = 90-t$. Hence $t = 10$, i.e., $\measuredangle BCD = 10^\circ$.

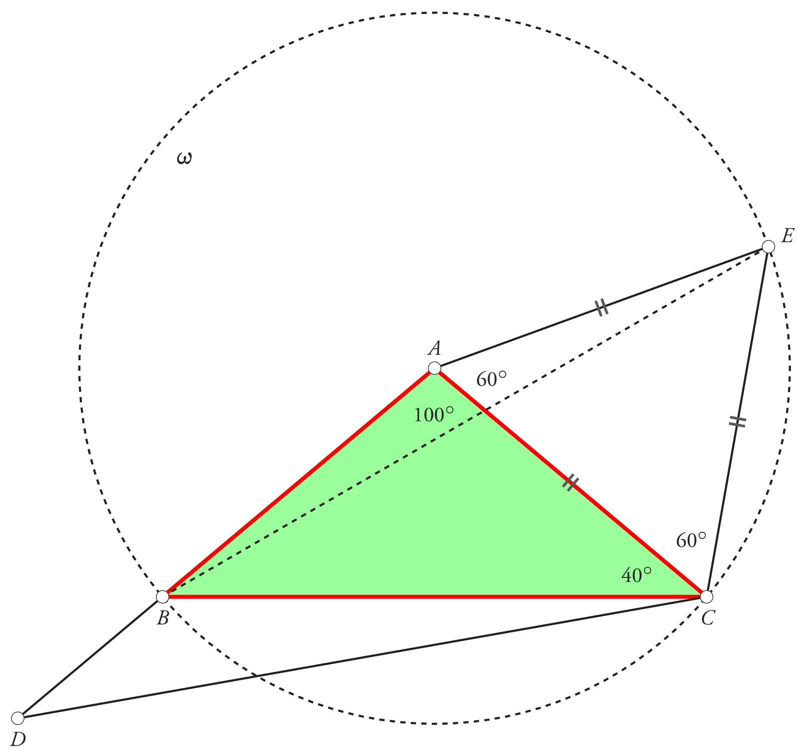


Figure 2

Geometric solution–I. Draw a circle ω centred at A and passing through C. Draw equilateral triangle ACE with AC as a side, lying outside $\triangle ABC$. (See Figure 2.) Naturally, E lies on circle ω. Join BE.

Then $\triangle BCE \cong \triangle DAC$ (side-angle-side or 'SAS' congruence: $BC = DA$, $CE = AC$ and $\measuredangle BCE = \measuredangle DAC$), hence $\measuredangle ACD = \measuredangle CEB = 100^\circ \div 2 = 50^\circ$, hence $\measuredangle BCD = 10^\circ$.

(Solution sent in by Dipak Bidap of Bangalore.)

Geometric solution–II. Draw the angle bisector CP of $\measuredangle ACB$, with P on side AB. Locate Q on side BC such that $CQ = CP$. Also locate R on side BC such that $\measuredangle PRC = 100^\circ$. (See Figure 3.)

We have now: $\triangle APC \cong \triangle RPC$, hence $AP = RP$.

Next, $\measuredangle PQB = 100^\circ$, so $\measuredangle PQR = 80^\circ = \measuredangle PRQ$, hence $PQ = PR$, hence also $AP = PQ$.

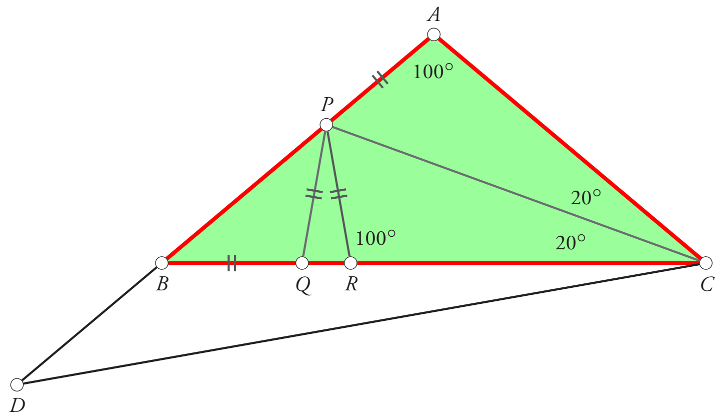


Figure 3

Again, $\triangle QPB$ has angles 100°, 40°, 40°, so $QP = QB$, i.e., $BQ = AP$. Since $BC = AD$, we get by subtraction $CQ = PD$. Hence $PD = PC$.

Since $\measuredangle DPC = 120^\circ$, it follows that $\measuredangle PCD = (180^\circ - 120^\circ)/2 = 30^\circ$. Hence $\measuredangle BCD = 10^\circ$.

(Solution sent in by K V Bhagwat of Mumbai.)

Remark. Note the streamlined elegance of the pure geometry solutions, in comparison with the trigonometric solution. This is a typical scenario: solutions based on pure geometry always look extremely elegant when placed alongside solutions based on trigonometry or coordinate geometry. But this elegance is highly deceptive; finding such solutions is no easy matter.

SHAILESH SHIRALI is the Director of Sahyadri School (KFI), Pune, and heads the Community Mathematics Centre based in Rishi Valley School (AP) and Sahyadri School. He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Problems for the SENIOR SCHOOL

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem VII-1-S.1
Two hundred students are positioned in 10 rows, each containing 20 students. From each of the 20 columns thus formed, the shortest student is selected, and the tallest of these 20 (short) students is labelled A. These students return to their initial places. Next, the tallest student in each row is selected, and from these 10 (tall) students, the shortest is labelled B. Who is taller, A or B?

Problem VII-1-S.2
Given 13 coins, each weighing an integral number of grams. It is known that if any coin is removed, then the remaining 12 coins can be divided into two groups of 6 with equal total weight. Prove that all the coins are of the same weight.

Problem VII-1-S.3
Show that there are infinitely many positive integers A such that $2A$ is a square, $3A$ is a cube and $5A$ is a fifth power.

Problem VII-1-S.4
An infinite sequence of positive integers $a_1, a_2, \ldots, a_n, \ldots$ satisfies the condition $\sum_{k=1}^{m} a_k^3 = \left(\sum_{k=1}^{m} a_k\right)^2$, i.e.,

$$a_1^3 + a_2^3 + a_3^3 + \cdots + a_m^3 = (a_1 + a_2 + a_3 + \cdots + a_m)^2$$

for each positive integer m. Determine the sequence.

Problem VII-1-S.5
The function $f(n) = an + b$, where a and b are integers, is such that for every integer n, the numbers $f(3n+1)$, $f(3n)+1$ and $3f(n)+1$ are three consecutive integers in some order. Determine all such functions $f(n)$.

Keywords: Integer, coin, square, cube, infinite, sequence, function, inversion, round-robin tournament

Solutions of Problems in Issue-VI-3 (November 2017)

Solution to problem VI-3-S.1

The numbers 1, 1/2, 1/3, …, 1/2017 are written on a blackboard. A student chooses any two numbers from the blackboard, say x and y, erases them and instead writes the number $x + y + xy$. She continues to do so until there is just one number left on the board. What are the possible values of the final number?

Observe that

$$xy + x + y = (x + 1)(y + 1) - 1,$$

and if

$$x = (u + 1)(v + 1) - 1, \quad y = (p + 1)(q + 1) - 1,$$

then

$$xy + x + y = (u + 1)(v + 1)(p + 1)(q + 1) - 1.$$

Thus at the end of the process the number on the board is

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2018}{2017} - 1 = 2018 - 1 = 2017.$$

Solution to problem VI-3-S.2

The numbers $1, 2, 3, \ldots, n$ are arranged in a certain order. One can swap any two adjacent numbers. Prove that after performing an ***odd*** *number of such operations, the arrangement of the numbers thus obtained will differ from the original one.*

Let $a_1, a_2, a_3, \ldots, a_n$ be a random permutation of the numbers $1, 2, 3, \ldots, n$. We say that the numbers a_i and a_j give rise to an *inversion* if $i < j$ but $a_i > a_j$. After every swap, the number of inversions either increases or decreases by 1. Thus the parity of the number of inversions in the arrangement is changed. Therefore, after an odd number of operations, the parity of the number of inversions in the final arrangement will be different from the parity of the number of inversions in the initial arrangement, hence the arrangement must differ from the original one.

Solution to problem VI-3-S.3

At each of the eight corners of a cube, write $+1$ or -1. Then, on each of the six faces of the cube, write the product of the numbers at the four corners of that face. Add all the fourteen numbers so written down. Is it possible to arrange the numbers $+1$ and -1 at the corners initially in such a way that this final sum is zero?

Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 and x_8 be the numbers written at the corners. Then, the final sum is given by

$$\sum_{i=1}^{8} x_i + x_1x_2x_3x_4 + x_5x_6x_7x_8 + x_1x_4x_5x_8 + x_2x_3x_6x_7 + x_1x_2x_5x_6 + x_3x_4x_7x_8.$$

As there are fourteen terms in the above sum and each term is $+1$ or -1, the sum will be zero only if some seven terms are $+1$ each and the remaining seven terms are -1 each. But, the product of the fourteen terms is

$$(x_1x_2x_3x_4x_5x_6x_7x_8)^4 = (\pm 1)^4 = +1.$$

Therefore, it is not possible to have an odd number of -1's in the above sum. We conclude that the desired arrangement is not possible.

Solution to problem VI-3-S.4

At a party, it is observed that each person knows 20 *others. Also, for each pair of persons who know one another, there is exactly one other person whom they both know. Further, for each pair of persons who do not know one another, there are exactly* 6 *other persons whom they both know. Also, if A and B are present in the party and A knows B, then B knows A. Determine, with proof, the number of people at the party.*

Pick a person u. We count in two ways all pairs of persons (v, w) such that u and v are distinct, v knows w, v knows u, and u does not know w. First, there are 20 persons v that know u, and, for each such v, there are 19 persons (excluding u) who know v and of these, exactly one knows u, so there are 18 persons w that know v but do not know u. So the number of pairs (v, w) as above is $20 \times 18 = 360$.

On the other hand, if n is the total number of people at the party, there are $n - 21$ people $w(\neq u)$ that do not know u, and for each such w, there are 6 persons v that know both u and w. So the total number of pairs (v, w) as above is $6(n - 21)$. Hence $6(n - 21) = 360$, and so $n = 81$.

Solution to problem VI-3-S.5

Suppose there are k teams playing a round-robin tournament; that is, each team plays against every other team. Assume that no game ends in a draw. Suppose that the i-th team loses l_i *games and wins* w_i *games. Show that* $\sum_{i=1}^{k} l_i^2 = \sum_{i=1}^{k} w_i^2$.

Since there are no draws, we must have

$$\sum_{i=1}^{k} l_i = \sum_{i=1}^{k} w_i.$$

Also, $\sum_{i=1}^{k}(l_i + w_i) = k - 1$. Therefore

$$\begin{aligned}\sum_{i=1}^{k} l_i^2 - \sum_{i=1}^{k} w_i^2 &= \sum_{i=1}^{k} \left(l_i^2 - w_i^2\right)\\ &= \sum_{i=1}^{k} \left(l_i - w_i\right)\left(l_i + w_i\right)\\ &= (k-1)\sum_{i=1}^{k} \left(l_i - w_i\right) = 0.\end{aligned}$$

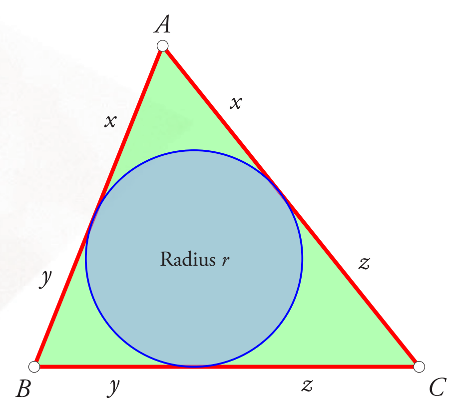
The desired conclusion follows.

Problem Corner

A Minimum PERIMETER PROBLEM

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

LinkedIn reader Peter Lovasz asks ([1]): *Among all triangles that share a given circle as incircle, which one has the smallest perimeter?* We give two approaches to the solution of this problem.



Solution I. Let the given circle have radius r. Let x, y, z denote the lengths of segments as marked in the figure. (Note that the figure makes implicit use of the theorem that the two tangents drawn from an external point to a circle have equal length.) Then the sides of the triangle are $y+z, z+x, x+y$; so the semi-perimeter is $s = x+y+z$. Therefore by Hero's formula, the area k of the triangle is given by the following relation:

$$k^2 = (x+y+z)xyz. \tag{1}$$

We also have $k = rs$ (this is another well-known formula), i.e.,

$$r(x+y+z) = \sqrt{(x+y+z)xyz};$$

hence:

$$r^2(x+y+z) = xyz. \tag{2}$$

Keywords: Triangle, incircle, perimeter

We now invoke the AM-GM inequality on the non-negative numbers x, y, z (proved and discussed in the article "Inequalities" elsewhere in this issue):

$$(xyz)^{1/3} \leq \frac{x+y+z}{3},$$
$$\therefore \; xyz \leq \frac{(x+y+z)^3}{27}, \tag{3}$$

with equality precisely when $x = y = z$ (i.e., precisely when the triangle is equilateral). Hence, from (2),

$$r^2(x+y+z) \leq \frac{(x+y+z)^3}{27},$$

which gives:

$$(x+y+z)^2 \geq 27r^2,$$

i.e.,

$$x+y+z \geq 3\sqrt{3}\,r. \tag{4}$$

Hence the perimeter of the triangle is not less than $6\sqrt{3}\,r$. Equality holds precisely when the triangle is equilateral. Therefore the equilateral triangle is the minimizing one. □

Solution II. This solution requires a good working knowledge of trigonometry and calculus. For convenience, we list a few relevant formulas at the end of the article (Box 1). We hope you will work your way through the solution!

Let the angles of the triangle be $2\alpha, 2\beta, 2\gamma$; then

$$0 < \alpha, \beta, \gamma < \frac{\pi}{2}, \qquad \alpha + \beta + \gamma = \frac{\pi}{2}, \tag{5}$$

and the perimeter p of the triangle is given by

$$p = 2r(\cot\alpha + \cot\beta + \cot\gamma). \tag{6}$$

Hence we must minimise $\cot\alpha + \cot\beta + \cot\gamma$ subject to (5). We have:

$$\cot\alpha + \cot\beta = \frac{\cos\alpha}{\sin\alpha} + \frac{\cos\beta}{\sin\beta} = \frac{\cos\alpha\sin\beta + \cos\beta\sin\alpha}{\sin\alpha\sin\beta}$$
$$= \frac{2\sin(\alpha+\beta)}{\cos(\alpha-\beta) - \cos(\alpha+\beta)}$$
$$= \frac{2\cos\gamma}{\cos(\alpha-\beta) - \sin\gamma} \geq \frac{2\cos\gamma}{1-\sin\gamma}. \tag{7}$$

with equality precisely when $\alpha = \beta = \frac{1}{2}(\pi/2 - \gamma)$; the inequality in the last line follows from the fact that the cosine of any angle cannot exceed 1. Hence

$$\cot\alpha + \cot\beta + \cot\gamma \geq \frac{2\cos\gamma}{1-\sin\gamma} + \cot\gamma, \tag{8}$$

with equality precisely when $\alpha = \beta = \frac{1}{2}(\pi/2 - \gamma)$.

Write $g(\gamma)$ for the function $\dfrac{2\cos\gamma}{1-\sin\gamma} + \cot\gamma$; then we find that

$$g'(\gamma) = \frac{2}{1-\sin\gamma} - \csc^2\gamma = \frac{2}{1-\sin\gamma} - \frac{1}{\sin^2\gamma}. \tag{9}$$

From (9), we infer that $g'(\gamma)$ attains a minimum value in $(0, \frac{1}{2}\pi)$ when

$$2\sin^2\gamma + \sin\gamma - 1 = 0, \quad \text{i.e.,} \quad (\sin\gamma + 1)(2\sin\gamma - 1) = 0, \tag{10}$$

i.e., when $2\sin\gamma = 1$; this happens when $\gamma = \frac{1}{6}\pi$. To see why it is a minimum, we check the sign profile of $g'(\gamma)$ at $\gamma = \frac{1}{6}\pi$; it is $-$, 0, $+$, indicating that it is a minimum. (It is easier to perform this check in this situation than to compute the second derivative.) Since

$$g\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{1/2} + \sqrt{3} = 3\sqrt{3}, \tag{11}$$

it follows that for $0 < \gamma < \frac{1}{2}\pi$,

$$\frac{2\cos\gamma}{1 - \sin\gamma} + \cot\gamma \geq 3\sqrt{3} \tag{12}$$

with equality only at $\gamma = \frac{1}{6}\pi$; and therefore that

$$\cot\alpha + \cot\beta + \cot\gamma \geq 3\sqrt{3}, \tag{13}$$

with equality precisely when

$$\gamma = \frac{\pi}{6}, \quad \alpha = \beta = \frac{\pi/2 - \pi/6}{2} = \frac{\pi}{6}.$$

Hence the minimising figure is the equilateral triangle. □

References

1. https://www.linkedin.com/groups/1872005/1872005-6126047981431980033

List of relevant trigonometric formulas

For angles x, y, we have the following relationships:

- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $2\sin x \sin y = \cos(x - y) - \cos(x + y)$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$

Box 1

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Mini Review:
WEIRD NUMBER

Reviewed by Swati Sircar

This is a 13 minute story set in a village of natural numbers. A thief appears and is clearly not a natural number. It turns out that he is a (positive) rational number. Since all the characters (and hence the numbers) are positive, it is fair to say that this video is about natural numbers and fractions and their relation! The part-whole model is invoked and is used to define a (positive) rational number, explain equivalent fractions and that any natural number is also a rational number.

But more importantly, it indicates most of the difficulties children face when they encounter fractions. This includes going beyond counting into measuring; it mentions that fractions (and rational numbers) look like two-storeyed numbers and can't be simplified further. The reactions of some of the characters (read natural numbers) on seeing the first fraction are similar to how children feel when they first see fractions. The fact that any given fraction can have infinitely many equivalent forms has been brought out well along the story line.

It ends with an indication that there may be numbers beyond rational numbers, but the characters (i.e., positive rational numbers) find it unimaginable. This reminds the viewer of the Pythagorean School which refused to accept anything beyond the rational numbers and where the person who discovered the irrationality of $\sqrt{2}$ (i.e., the length of the diagonal of a unit square) faced dire consequences.

This is a great video to share with teachers. It can be shown to children as an introduction to fractions with some follow-up activities.

https://www.youtube.com/watch?v=SbjtIRp9C6A

Keywords: Natural Numbers, Rationals, Story, Video

BEAUTIFUL, SIMPLE, EXACT, CRAZY

By Apoorva Khare and Anna Lachowska

Reviewed By Jishnu Biswas and Rema Krishnaswamy

JISHNU BISWAS & REMA KRISHNASWAMY

This is a review of the book "Beautiful, Simple, Exact, Crazy" written by Apoorva Khare and Anna Lachowska. The authors write in the preface that this book arose out of an introductory course called *Mathematics in the Real World* which they co-designed (and taught at Stanford and Yale University, respectively). The target audience of that course consisted mainly of undergraduates of humanities and social sciences – students whose principal interests lay outside of mathematics and the sciences. The preface explains the choice of topics for the course and the book: the mathematics should be simple to explain, and it should "generate a wide range of practical, amusing or impressive real world applications."

Guided by these principles, the authors have put together a dazzling collection of mathematical ideas and their uses in the real (and abstract) world, and they have achieved this assuming very little (and in particular, without using any calculus). Not only is the mathematics interesting and inspiring but the style in which it is presented – with connections in almost every page to literature, history, art, music, physics, biology, medicine, geology, archaeology and many other subjects – is absolutely compelling. We feel that the material in the book will be of interest not only to the intended audience but to anybody who has even a passing interest in mathematics, from high school students to mathematicians and mathematics teachers.

Keywords: Fibonacci numbers, Golden ratio, Phyllotaxis, Mathematics in the Real World, Monty Hall problem, Seven Messengers.

After reading the book, we also did a search on the two writers. One of them, Apoorva Khare, is now a faculty member of the Indian Institute of Science, Bangalore. The other, Anna Lachowska, is now at the Ecole Polytechnique Federale de Lausanne in Switzerland (her homepage – the link [AL] given below – contains some more instances of mathematics applied to the real world). One of us (RK) also used this book as one of the sources to teach a pre-calculus course for first year undergraduate students at the Azim Premji University (during July-November, 2017).

Here is a brief discussion of the topics covered in the book.

The initial chapters deal with applications of the Newtonian laws of motion, rational and irrational numbers, the notions of simple and compound interests. Logarithms are introduced and real life situations where they arise are discussed: the brightness levels of stars, the Richter scale of earthquakes, the decibel level which measures the noise level of sound and the equal temperament scale in music. The exponential growth and decay models are defined, and the reader is introduced to several examples like population growth models, the Fibonacci sequence, the process of radioactive decay and radiocarbon dating (which leads to a very interesting account of the Voynich manuscript). Finite geometric series are used to calculate mortgage rates. Infinite geometric series lead to fractals and their dimensions and their uses (a very recent and interesting application of fractals to archaeology is discussed). Modular arithmetic is introduced with many examples (the final one is the Diffie-Hoffman public key exchange) and an interesting historical discussion on the modern (Gregorian) calendar, and the earlier (Julien) calendar. The last part of the book deals with counting and some probability (many applications of Bayes law) and statistics (how to analyse data and fit curves using the least squares method).

Here are a few of our favorite examples from the book in some detail.

The Seven Messengers, a novella written in the 1940's by Dino Buzzati, is the story of a prince who sets out with seven knights to explore his father's kingdom. The prince keeps moving in one fixed direction at a constant speed while the seven knights keep oscillating (at different points in space and at a constant speed, greater than that of the prince) between the prince and the capital, as the prince moves further and further away. The prince (who is also the narrator) makes several numerical statements in the story, which Khare and Lachowska explain and verify. But they go far beyond, and give a geographical explanation of how the prince could have travelled for so long in a fixed direction, given the earth is round. They introduce the International Date Line to explain the notion of losing or gaining a day of a traveler going round the globe. Finally, they consider the abstract concept underlying the story of an object moving away from an observer and messengers oscillating between them, and discuss the similarities with a space probe like the Voyager (the messengers are photon particles in this case). This wonderful example contains interesting mathematics, and alongside there is also literature, history, geography, physics and philosophy. This is mathematical story-telling at its best.

Another example deals with the golden ratio phi which is intimately connected with the arrangement of leaves, florets, and seeds in many plants. A hexagonal scale on the surface of a pineapple is part of three spirals (Figure 1A). The numbers of such spirals on most pineapples are five, eight, thirteen or twenty one, which are Fibonacci numbers. The florets in a sunflower (Figure 1B) form clockwise and anticlockwise spirals, the numbers of these two types of spirals appear in the ratios of consecutive Fibonacci numbers (34 to 55; 55 to 89; 89 to 144; depending on the size of the sunflower). The limit of the ratio of two consecutive Fibonacci numbers is the golden ratio phi. A mathematical reason for the golden ratio to appear in the formation of leaves and florets is that it is 'more irrational' (in a certain sense) than other irrational numbers. The leaves along a twig or the stems along a branch prefer to appear at irrational angles to each other in order to maximize exposure to sunlight, air and moisture. The angle between the two directions made by joining two successive leaves to the stem, known as the divergence angle, splits 360 degrees in the ratio 1 to phi. The book [ML] (referred to by Khare and Lachowska) also provides a physical explanation for the spiral patterns from an interesting experiment in physics where magnetic drops (that act as tiny magnetic dipoles or bar magnets) at the silicon oil-air interface settle into a spiral pattern to minimize their repulsive energy.

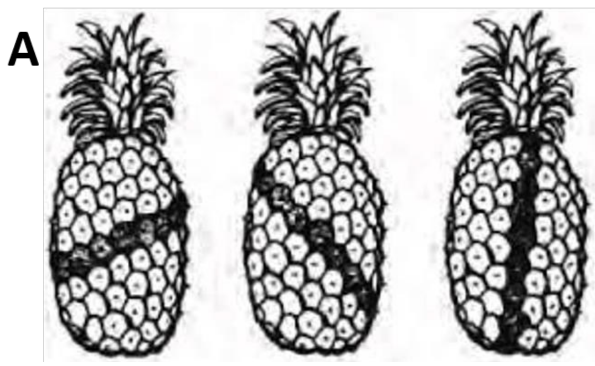


Figure 1A: Hexagonal scales on a pineapple form a part of three different spirals, *http://moziru.com/explore/Drawn%20pineapple%20golden%20ratio/#gal_post_8118_drawn-pineapple-golden-ratio-5.jpg.*

Figure 1B: Spiral patterns formed by the florets of a sunflower, *https://www.hxbenefit.com/are-sunflower-seeds-good-for-you.html.*

A third example is the famous Monty Hall problem. Here is a brief description, if you have not heard of it before. You appear on a television show and the host shows you three closed doors (named A, B, C), one of which conceals a car while the other two have goats behind them. You can win a car by choosing the right door, and you choose door A. The host then opens door B and shows you a goat behind it. He then gives you a chance to choose again. Should you choose C, or remain with your original choice of door A? The history of this very interesting problem is discussed, along with the (not at all obvious) answer. A minor modification of this problem, which completely changes the answer, is also discussed. In the problems at the end of this chapter, two generalizations of the Monty Hall problem are discussed.

All chapters, except the first, consist of two parts, a first part of a few pages containing the mathematics, and a much longer second part consisting of the applications of the math done initially. Each chapter also contains problems which are solved and practice problems with just the answers given. There is a list of problems at the end of each chapter and the solutions to odd numbered problems are provided. Six practice tests are given; the solutions of all the tests are given too. The two authors are not just wonderful expositors; they are sympathetic teachers as well! They have made sure that an interested person can read and learn this book completely on his or her own! At the end, the authors have written a small section ("The bigger story") in which connections between concepts appearing in different chapters are noted and a few books on different branches of interesting mathematics (Knot Theory, Graph Theory, Topology) are suggested for the reader who wants to do some further reading.

We have a few (very) minor quibbles about the book. It might have helped if the authors had included a short section on functions and their graphs, as these notions are completely basic and are used quite a few times in the book. The phyllotaxis part seemed a bit harder for us to read than the rest of the book, it might have been nicer to include a few more photos and pictures, and perhaps give a bit more explanation. We also found a total of three very minor typographical errors in the book.

In the preface, Khare and Lachowska write, "Our main goal is to convince our reader that mathematics can be easy, its applications are real and widespread, and it can be amusing and inspiring." There is no doubt in our minds that they have succeeded brilliantly. If you know a high school student who is even mildly interested in mathematics, please do give him or her a copy of this book as a gift!

References

1. [ML] Mario Livio, The Golden Ratio: The Story of Phi, the World's Most Astonishing Number, Broadway Books, 2002
2. [AL] http://dedis.cs.yale.edu/~anouchka/book/index.html

REMA KRISHNASWAMY teaches Physics at the School of Liberal Studies, Azim Premji University. Her area of expertise is in Experimental Soft Matter Physics. Besides teaching, she is also interested in developing an undergraduate science curriculum that is interdisciplinary and centred on experimentation. Rema may be contacted at rema.krishnaswamy@apu.edu.in.

JISHNU BISWAS works at the Indian Statistical Institute, Bangalore Centre. He may be contacted at jgbiswas@gmail.com.

The Closing Bracket . . .

A question largely absent from most teachers' minds is *why* we teach anything. If at all we ask the 'why' question, we answer it in terms of utility, by looking at the future, at college education and job prospects. This way of looking at things is very strong in our country, which is why, perhaps, teachers of history and literature come under pressure from students and parents; they are caught in trying to justify the study of these subjects from the point of view of utility.

I feel it is important that teachers ask the 'why' question and try to answer it, but not in terms of utility. In Why teach mathematics? (an article on the NCTM blog), the author Matthew Larson notes that how we answer this question "influences who we think should do mathematics and how we teach it." He lists three reasons (given by Paul Ernest) why we teach mathematics: *utility* (mathematics for employment; functional numeracy; practical and work-related knowledge; relevance for the economy); *personal development and social empowerment* (problem posing & solving; mathematical confidence & persistence); *appreciation of mathematics as an element of human culture* (importance of the subject in itself; its role in history, culture and society in general). Traditionally, the first of these has been cited as the main reason for teaching mathematics, and in recent years, the second one has grown in relation to the first one, as seen in the increased interest in the mathematical Olympiads (but note also that the Olympiads are now seen by many students and teachers as a passport for getting into good colleges! The utilitarian way of thinking is indeed very deeply rooted in us…). But the third reason continues to receive little or no consideration.

In this country, our thinking is much too strongly dominated by the utility factor, by the question of practicality. It was not always this way; but at the present moment, to do something out of a feeling of beauty and love has all but evaporated. If at all we do have that feeling, the forces of conditioning around us make sure that we give up all such 'impractical' ways of thinking. It is, surely, part of our work as teachers of mathematics to challenge this mindset, within the community of teachers, parents and students.

But this cannot be all. I think it is vital that teachers work towards bringing about a mindset that eagerly seeks out engagement with the world; that questions accepted wisdom and all the belief structures of society at large, including those of religion; that questions all accepted power structures; and, most of all, questions one's own personal assumptions and prejudices, one's experiences. All this is necessary not only for the healthy functioning of democracy, but equally, for us to discover inward richness in life. This too, surely, is part of a teacher's work. It has not been considered so, but we must challenge this mindset – not out of a feeling of cynicism, but of affection. So let us put our shoulder to the wheel and engage in this important task.

Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.

2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.

3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.

4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.

5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.

6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.

7. Provide a compact list of references, with short recommendations.

8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.

9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.

10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.

11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).

12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.

13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.

14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.

15. Submit articles in MS Word format or in LaTeX.

Printed and Published by Manoj P on behalf of Azim Premji Foundation for Development;
Printed at Suprabha ColorGrafix (P) Ltd., No. 10, 11, 11-A, J.C. Industrial Area, Yelachenahalli, Kanakapura Road, Bangalore 560062.
Published at Azim Premji University Pixel B Block, PES College of Engineering Campus, Electronics City, Bangalore 560100;
Editor: Shailesh Shirali

It all depends on the teacher

Azim Premji University

Join us in building the best teacher education programmes in the country.

Faculty for Teacher Education

B.Sc.B.Ed & M.Ed

Azim Premji University's B.Sc.B.Ed and M.Ed. programmes are being launched in July 2018 aiming to prepare outstanding teachers and teacher educators.

The University currently offers undergraduate programmes in the Sciences, Economics and Humanities and postgraduate programmes in Education, Development, Public Policy, and Law & Development.

We invite experienced Teacher Educators who resonate with the purpose of the University and have prior experience in Teacher Education institutions, to join us as faculty. Applicants without experience but meeting the qualifications are also encouraged to apply.

Qualifications

- M.A. or M.Sc. (Psychology, Sociology, Political Science, History, Geography, Mathematics, Physics, Chemistry, Biology, English) with minimum 55% marks
- M.Ed with minimum 55% marks
- M.Phil/PhD/NET in Education is desirable

Application Process

For details, please visit

azimpremjiuniversity.edu.in/workwithus-teacher-ed

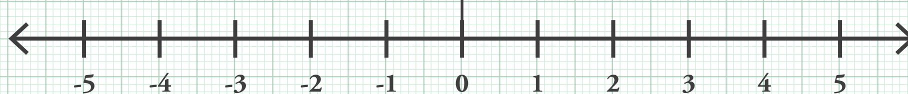
or write to us at

facultypositions@apu.edu.in

At Right Angles
A Resource for School Mathematics

INTRODUCTION TO ALGEBRA

PADMAPRIYA SHIRALI



Azim Premji University

A publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley

INTRODUCTION

Introduction to school algebra can happen through varied approaches. Some prefer to start with an unknown in an equation, while some prefer to start with a formula and some others may prefer to use a pattern based approach. Does it make a difference which approach one uses? Is one approach better than the others? These questions can be debated. However, each of these approaches relates to different conceptions of algebra.

The unknown in an equation conceives algebra as a study of procedures for solving certain kinds of problems requiring simplification; the formula approach conceives algebra as the study of relationship among quantities which vary. The pattern based approach conceives algebra as generalized arithmetic leading to generalization of known relationships among numbers.

Algebra thus is all of these: generalized arithmetic, a procedure for solving certain problems, and a means of understanding relationships and mathematical structures.

In school algebra, the term 'variable' typically appears first in the form of a letter that represents an unknown in an open sentence or an equation (e.g., $4 + x = 9$), followed by formulas (e.g., $A = L \times B$), as a generalized property (e.g., $a + b = b + a$), later as an identity (e.g., $(a + b)^2 = a^2 + 2ab + b^2$) and as a function (e.g., $y = 3x$). Students learn to use variables to solve various types of problems.

However, does algebraic thinking take place in a child's mind well before he/she encounters a variable? For instance, when a child says 'I have 6 toffees; if there were 4 more I would have 10" or when a child is able to abstract a pattern from numerical relationships, or when a child is able to guess the tenth figure in a pattern of figures, can one say that the child has begun to think algebraically?

The late Shri P. K. Srinivasan had developed an approach to the teaching of algebra titled *'Algebra – a language of patterns and designs'*. I have used it for several years at the Class 6 level and found it to be very useful in making a smooth introduction to algebra, to the idea and usage of concepts such as *variable* and *constant*, to performing operations involving terms and expressions. This approach steadily progresses from studying numerical patterns to line and 2-D designs, finally leading to indices and identities. Over the years, I have adapted this material to meet the needs and interests of the students. However, the basic structure has remained largely the same. I share here the adapted approach.

Patterns, numerical or visual, have an inherent appeal to children and adults alike. It may have to do with the aesthetic feeling present in the human psyche. We are able to recognise and sense patterns in nature, patterns in the movements of the heavenly bodies, patterns in time (seasons) – patterns on a macro-scale as well as on a micro-scale.

Patterns make a very good starting point for the introduction of algebra. They arise easily from the mathematical knowledge that students have already acquired by Class 6 (even and odd numbers, multiplication tables, behaviour of certain numbers, number relationships).

In this pull-out, I focus on the first step of working with patterns as an introductory step to the usage of concepts such as Variable, Constant, Term and Expression and also operations involving these concepts. In the second pull-out I will take up design language and depict the usage of the same concepts and operations. In the third pull-out I will take up indices and identities. Subsequently, approaches to equations will be taken up.

Keywords: Algebra, unknowns, equations, expressions, patterns, activities, manipulatives.

ACTIVITY 1

Objective: To expose students to different kinds of patterns

Pattern recognition is innate to the brain and happens quickly and naturally. However, if students have been taught earlier through a rote and mechanical approach, one may need to reawaken their observation and thinking powers.

Pattern problems in numbers and designs are available in plenty as resources. The teacher will need to make an appropriate graded selection suited to the needs of the upper primary kids.

I have given here a few model problems.

1. What is the pattern here? What goes into the blank space?
   a. 7, ____, 24, 34, 45, 57, 70
   b. 71, 70, 73, 72, 75, _____, _____, _____

2. Find the odd one out. Justify your answer.
   a. 252, 72, 1, 275, 24, 488

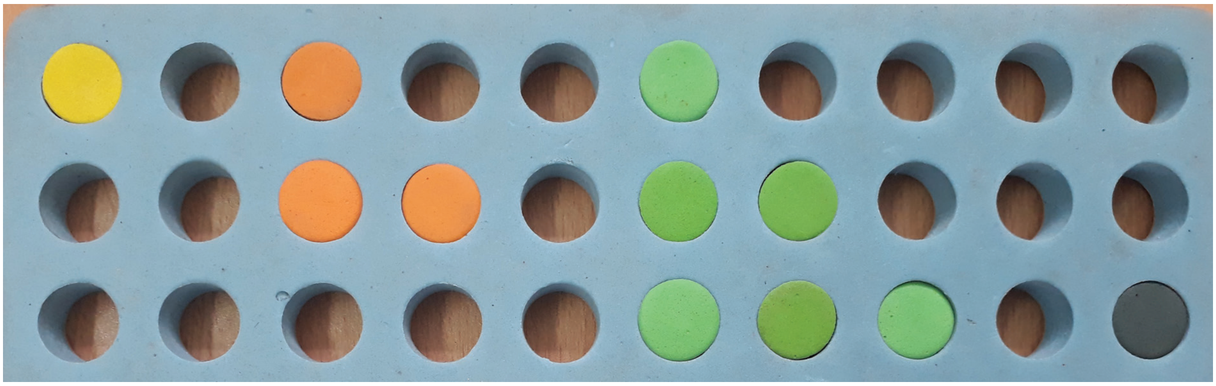


Figure 1

1. Here are the first five *triangular numbers*: 1, 3, 6, 10, 15.
2. Can you see a pattern?
3. Can you predict the next triangular number?
4. What would the tenth triangular number be?

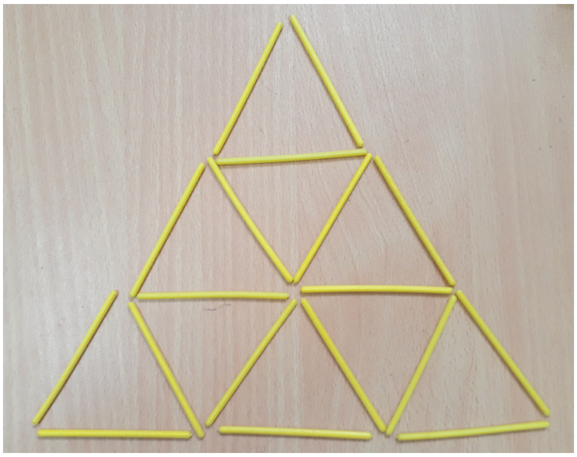
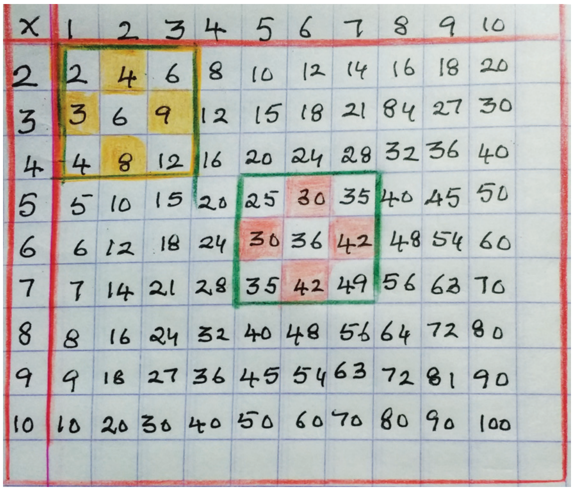


Figure 2

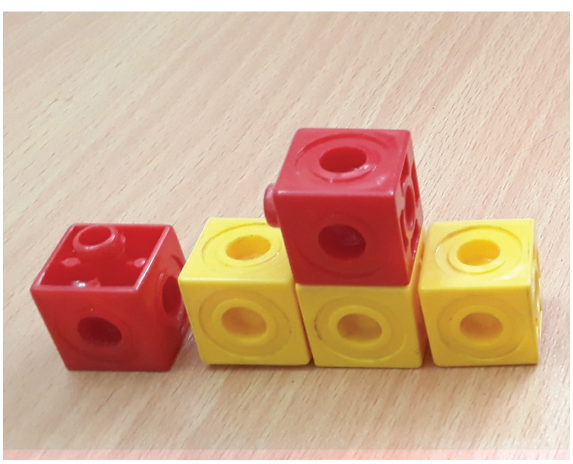
1. I used 3 matches to make 1 small triangle.
2. How many matches do I need to build a second row of triangles under that?
3. How many matches do I need to build a third row of triangles under that?
4. How many matches will I need to make the sixth row?
5. Can you make out how many matches I will need to make the twentieth row?



x	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	84	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Figure 3

1. Make a box around a set of nine numbers (a 3 × 3 square) in the tables square.
   a. Add the numbers in the shaded squares.
   b. Add the corner numbers.
   c. Multiply the centre number by 4. What happens?
2. Make a box around another set of nine numbers and try this again.



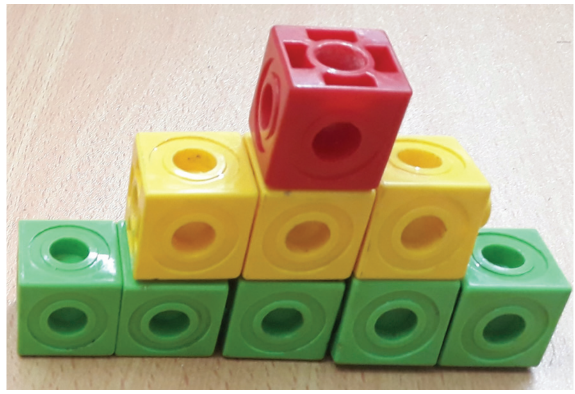


Figure 4

1 block is needed to make an up-and-down staircase, with 1 step up and 1 step down.

4 blocks are needed to make an up-and-down staircase with 2 steps up and 2 steps down.

How many blocks would be needed to build an up-and-down staircase with 5 steps up and 5 steps down?

Explain how you would work out the number of blocks needed to build a staircase with any number of steps.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 5

A hundred square has been printed on both sides of a piece of paper. One square is directly behind the other just like in the pages of a book.

What number lies on the other side of 100? The other side of 58? Of 23? Of 19?

Do you see a pattern?

ACTIVITY 2

Objective: Introduction to pattern language and the usage of a letter for changing numbers.

Introduction to the notion of a *changing number* (variable) and an *unchanging number* (constant) can begin in familiar settings. We introduce the words constant and variable, term and expression a little later.

Students already know about even numbers and about multiples and square numbers. This activity helps them to learn to write pattern language in the context of their prior knowledge of number relationships.

Give students a set of even numbers. For example, 12, 22, 8, 44.

Pose the question, "What are these?" They will notice that they are all even numbers.

What else can be said about them? They are all multiples of 2.

Now the teacher can rewrite all these numbers as multiples of 2.

$$22 = 2 \times 11$$

$$8 = 2 \times 4$$

$$44 = 2 \times 22$$

Now pose the question 'What do you notice about the right hand side?' What is the first number? It is always 2. What is the second number? It is changing each time.

So how can we describe an even number? It is 2 times some number.

Since the second number changes or varies, we represent it using a letter.

An even number can be now written as 2 times 'n' or $2 \times n$. (Mention to the students that we drop the multiplication sign as it looks like the letter 'x'. So $2n$ means '2 times n').

We can take up another example using multiples.

44,11, 220, 121.

What are these? They are all multiples of 11.

They can be written in this way.

$$44 = 11 \times 4$$

$$11 = 11 \times 1$$

$$220 = 11 \times 20$$

$$121 = 11 \times 11$$

What do we see on the right side? The first number is always 11. The second number is changing.

This pattern can be written as $11x$ or $11y$. (Tell the children that any letter can be used to stand for the changing number).

Let us take up a slightly different example where there is no constant factor.

16, 49, 4, 81.

What are these numbers? Square numbers.

They can be written in this way.

$$16 = 4 \times 4$$

$$49 = 7 \times 7$$

$$4 = 2 \times 2$$

$$81 = 9 \times 9$$

What can we say about the numbers on the right hand side? Help the students articulate this. *'The first number is changing. The second number too is changing. But the first number and the second numbers are always the same.'* So, how does one describe such a pattern?

It can be described as 'y' times 'y' or '$y \times y$'or 'yy'. (Note: At this point, we do not write yy as y^2 as we have not yet introduced indices to them.)

Let us take up another type of situation where both the factors are different variables.

Here are some numbers. Can you write them as product of two numbers without using 1 as a factor?

65, 14, 6, 77.

We write them as products:

$$65 = 5 \times 13$$

$$14 = 2 \times 7$$

$$6 = 2 \times 3$$

$$77 = 7 \times 11$$

On the right hand side, what can we say about the first number? The second number? *They are both changing*. The first changing number can be called 'x' and the second changing number can be called 'y'. The pattern here can be described as 'x' times 'y' or '$x \times y$' or 'xy'.

The teacher can ask the students to come up with more such examples of their own.

Students can also work in pairs. Each student can create a pattern using multiples of slightly larger numbers, say, between 10 and 20 and ask another to describe the pattern using pattern language. Or they can do the same with cube numbers.

ACTIVITY 3

Objective: Patterns with two terms and two operations.

Let us look at these numbers.

21, 43, 7, 101.

What are these numbers? They are odd. How do we describe them? Students may take time to respond to that question.

Another question which can help is: 'What is their relationship to even numbers?' They are either 1 more or 1 less than even numbers.

So we write them initially as follows:

$$21 = 20 + 1$$

$$43 = 42 + 1$$

$$7 = 6 + 1$$

$$101 = 100 + 1$$

At this point, we can describe them as $n + 1$. Is there something further we can do? How did we describe the even numbers earlier? So now we write these numbers as follows:

$$21 = 20 + 1 = 2 \times 10 + 1$$

$$43 = 42 + 1 = 2 \times 21 + 1$$

$$7 = 6 + 1 = 2 \times 3 + 1$$

$$101 = 100 + 1 = 2 \times 50 + 1$$

Now we describe the pattern as $2n+1$.

Students can be shown that the same numbers, expressed differently, can be described as $2n - 1$.

Note: At this point the teacher can introduce the words variable, constant, term and expression to the students.

Here is another pattern.

49, 69, 19, 89.

All the numbers end with a 9 in the units place. They can be written as follows.

$$49 = 10 \times 5 - 1$$

$$69 = 10 \times 7 - 1$$

$$19 = 10 \times 2 - 1$$

$$89 = 10 \times 9 - 1$$

The pattern here is $10n - 1$.

Let us look at another pattern which uses place value.

36, 75, 49, 81, 19.

What pattern can one see here? They are not all composite. They are not multiples of any single number. They are all double digit numbers. They can be written as follows:

$$36 = 10 \times 3 + 6$$

$$75 = 10 \times 7 + 5$$

$$49 = 10 \times 4 + 9$$

$$81 = 10 \times 8 + 1$$

$$19 = 10 \times 1 + 9$$

This pattern can be described as $10m+n$.

How about this set?

94, 99, 91, 95.

They could be expanded as follows:

$$94 = 100 - 6 = 10 \times 10 - 6$$

$$99 = 100 - 1 = 10 \times 10 - 1$$

$$91 = 100 - 9 = 10 \times 10 - 9$$

$$95 = 100 - 5 = 10 \times 10 - 5$$

Hence the pattern becomes $10 \times 10 - n$.

Students may also see it as $90 + n$.

Game: Pattern detective

Objective: To detect the pattern created by another.

Materials: Black board or blank paper

This game can be played by the whole class or by small groups of 5 students or even in pairs.

Student I calls out any number between 1 and 10, say 5. Student II performs any two operations on the given number to generate a new number, say 12. This exchange between student I and student II is repeated at least four times. Each time student II performs the same operations in the same order to generate corresponding numbers.

This is how it goes.

Student I	Student II
5	12
3	8
8	18
10	22

What is student II doing with the numbers given by student I?

The pattern needs to be detected by either student I or the group or the class that is watching.

Here student II is doubling the number and adding 2 to the product.

The pattern can be described as $2n+2$.

Note: Initially it is better for the students to use two specified operations, i.e., either '× and +' or '× and –'.

Here is another example of this game between the two students.

Student I	Student II
5	24
3	8
8	63
10	99

What is student II doing with the numbers given by student I?

Here student II is squaring the number and subtracting 1 from the product.

The pattern can be described as '$nn - 1$'.

Here is one more example of this game between two students.

Student I	Student II
1	3
2	7
3	11
4	15

What is student II doing with the numbers given by student I?

I will leave it to you to figure out!

ACTIVITY 4: PATTERNS IN EXPRESSIONS

Objective: To describe given patterns and create patterns for a given expression

To observe addition of like terms with a single variable

$2 \times 3 + 3 \times 3$

$2 \times 5 + 3 \times 5$

$2 \times 2 + 3 \times 2$

$2 \times 1 + 3 \times 1$

How do we describe the pattern here?

Let the students state that it is $2a + 3a$.

Now ask the students to work out the sum for each expression and write it as shown.

Ask them to find a pattern in the answers. They will see that they are multiples of 5.

Let them write the answer as multiples of 5.

2 × 3 + 3 × 3	15	5 × 3
2 × 5 + 3 × 5	25	5 × 5
2 × 2 + 3 × 2	10	5 × 2
2 × 1 + 3 × 1	5	5 × 1

How will this pattern be described? It will be 5*a*.

Teacher can point out to the students the fact that 2*a* and 3*a* have summed up to 5*a*.

Now pose the question: "What would 3*x* and 4*x* add up to?" Let the students guess and build patterns to verify their answer.

It is important at this point to show that when unlike terms are added, the answer cannot be 'simplified'.

Provide a number pattern like this:

3 × 3 + 2 × 4	17
3 × 5 + 2 × 2	19
3 × 2 + 2 × 7	20
3 × 1 + 2 × 3	9

How will the pattern on the left hand side be described? It is 3*a* + 2*b*.

Can the students find any pattern in the sums of these numbers?

They can now try to guess the answer for a subtraction situation, e.g., 5*x* – 2*x*, and build a pattern to verify the answer.

ACTIVITY 5: PATTERNS IN EXPRESSIONS

Objective: To describe given patterns and create patterns for a given expression

To observe addition of like terms with more than one variable

How will this pattern be described?

2 × 4 + 4 × 2

3 × 6 + 6 × 3

5 × 2 + 2 × 5

8 × 3 + 3 × 8

It is of the form *ab* + *ba*.

Here again the students can sum them and observe the results.

What is the pattern in the answers? They are all multiples of 2.

Let the students write them initially as multiples of 2 (16 = 2 × 8, etc).

As a second step, they can write the factors of the second number as well (16 = 2 × 2 × 4, etc).

2 × 4 + 4 × 2	2 × 8	16	2 × 2 × 4
3 × 5 + 5 × 3	2 × 15	30	2 × 3 × 5
6 × 7 + 7 × 6	2 × 42	84	2 × 6 × 7
3 × 3 + 3 × 3	2 × 9	18	2 × 3 × 3

What is the pattern of the answers in the final column?

It is 2*ab*.

Again draw the students' attention to the addition of *ab* + *ba* which equals 2*ab*.

Are *ab* and *ba* like terms? Why?

Discuss more examples of 'like' and 'unlike' terms in two variables.

As a practice exercise, students can be asked to set up a number pattern for a given pattern language, using only like terms initially.

Ex. Create number patterns for *xy* + *xy* + *xy*.

What does it become?

Would it be different for *xy* + *yx* + *xy*?

Create number patterns for 5*cd* – 2*cd*.

What does it become?

Let the students also create patterns for addition and subtraction of unlike terms.

Example: Create number patterns for each:

(i) *abc* – *cde* (ii) *ab* + *bc* + *ca*.

ACTIVITY 6: LAWS OF COMMUTATIVITY AND ASSOCIATIVITY

Objective: To establish commutativity and associativity

What do we notice here?

$$3 + 2 = 2 + 3$$

$$5 + 1 = 1 + 5$$

$$6 + 4 = 4 + 6$$

Property: $a + b = b + a$

Pose the question to the students: "Can I replace the + sign with – sign?" "Can I replace the + sign with ×?" "Can I replace the + sign with ÷?"

What do we notice here?

$$2 + (3 + 5) = (2 + 3) + 5$$

$$1 + (4 + 2) = (1 + 4) + 2$$

$$5 + (2 + 1) = (5 + 2) + 1$$

Property: $a + (b + c) = (a + b) + c$.

In a similar manner, the teacher can build patterns to demonstrate properties of multiplication and division, properties of 0 and 1 by studying the patterns.

Property: $a \times b = b \times a$, $a \times (b \times c) = (a \times b) \times c$, $a \times (b + c) = (a \times b) + (a \times c)$.

Properties of 1: $1 \times a = a$, $a \div a = 1$, $a \div 1 = a$.

Properties of 0: $a + 0 = a$, $a - 0 = a$, $a - a = 0$, $a \times 0 = 0$, $0 \div a = 0$.

ACTIVITY 7

Objective: To discover some number properties and describe them as expressions

Create a pattern with consecutive numbers.

Tell the students to sum the numbers in the pattern to discover and state the property using pattern language.

11 + 12

2 + 3

7 + 8

10 + 11

The sum of two consecutive numbers is always an odd number.

This pattern can be rewritten as follows:

11 + 12	11 + 11 + 1	2 × 11 + 1
2 + 3	2 + 2 + 1	2 × 2 + 1
7 + 8	7 + 7 + 1	2 × 7 + 1
10 + 11	10 + 10 + 1	2 × 10 + 1

It can be described as $n + n + 1$ which becomes $2n + 1$.

The students can set up patterns and discover the answers for the following questions. The answers can be stated as expressions.

What is the difference between any pair of consecutive numbers?

What is the sum of three consecutive numbers?

Can they state a property of the product of two consecutive numbers?

Can they state a property of the product of three consecutive numbers?

ACTIVITY 8

Objective: Exploring challenging problems through algebraic thinking

Let the students take any two digit number, say 53. Ask them to reverse the digits, i.e., 35. Let them find the difference between these numbers.

They can do this with some more numbers to spot a pattern.

$$53 - 35$$

$$74 - 47$$

$$21 - 12$$

$$63 - 36$$

Can they describe the pattern that emerges?

Here is one more question.

Ask the students to take a set of five numbers, say 5, 12, 4, 20, 6. Let them total it.

Now pose the following questions:

1. "If you take 2 away from each of those numbers what happens to the total? Why?"
2. "If you add 3 to each of those numbers what happens to the total? Why?"
3. "If you double each of those numbers what happens to the total? Why?"

Are they able to use expressions to answer these questions?

One final challenge!

Here is an interesting result.

$$55^2 - 45^2 = 1000$$

$$105^2 - 95^2 = 2000$$

$$85^2 - 65^2 = 3000$$

How do we describe this pattern?

Are there any other pairs which give multiples of 1000?

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box' Padmapriya may be contacted at padmapriya.shirali@gmail.com

Call for Articles

At Right Angles welcomes articles from math teachers, educators, practitioners, parents and students. If you have always been on the lookout for a platform to express your mathematical thoughts, then don't hesitate to get in touch with us.

Suggested Topics and Themes

Articles involving all aspects of mathematics are welcome. An article could feature: a new look at some topic; an interesting problem; an interesting piece of mathematics; a connection between topics or across subjects; a historical perspective, giving the background of a topic or some individuals; problem solving in general; teaching strategies; an interesting classroom experience; a project done by a student; an aspect of classroom pedagogy; a discussion on why students find certain topics difficult; a discussion on misconceptions in mathematics; a discussion on why mathematics among all subjects provokes so much fear; an applet written to illustrate a theme in mathematics; an application of mathematics in science, medicine or engineering; an algorithm based on a mathematical idea; etc.

Also welcome are short pieces featuring: reviews of books or math software or a YouTube clip about some theme in mathematics; proofs without words; mathematical paradoxes; 'false proofs'; poetry, cartoons or photographs with a mathematical theme; anecdotes about a mathematician; 'math from the movies'.

Articles may be sent to :
AtRiA.editor@apu.edu.in

Please refer to specific editorial policies and guidelines below.

Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to 'At Right Angles'.

While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

An in-depth, serious magazine on mathematics and mathematics education.

For teachers, teacher educators and students connected with the subject.

In this magazine, teachers and students can:

- Access resources for use in the classroom or elsewhere
- Read about mathematical matters, possibly not in the regular school curriculum
- Contribute their own writing
- Interact with one another, and solve non-routine problems
- Share their original observations and discoveries
- Write about and discuss results in school level mathematics.

Publisher

Azim Premji University together with Community Mathematics Centre, Rishi Valley.

Editors

Currently drawn from Rishi Valley School, Azim Premji Foundation, Homi Bhabha Centre for Science Education, Lady Shri Ram College, Association of Math Teachers of India, Vidya Bhavan Society, Centre for Learning.

You can find At Right Angles here:

Free download from

http://azimpremjiuniversity.edu.in/SitePages/resources-at-right-angles.aspx

AtRiA is also available on http://teachersofindia.org/en/periodicals/at-right-angles

At Right Angles is available as a free download in both hi-res as well as lowres versions at these links. Individual articles may be downloaded too.

Hard Copy

At Right Angles magazine is published in March, July and November each year. If you wish to receive a printed copy, please send an e-mail with your complete postal address to AtRightAngles@apu.edu.in

The magazine will be delivered free of cost to your address.

On FaceBook

https://www.facebook.com/groups/829467740417717/

AtRiUM (At Right Angles, You and Math) is the Face - Book page of the magazine which serves as a platform to connect our readers in e-space. With teachers, students, teacher educators, linguists and specialists in pedagogy being part of this community, posts are varied and discussions are in-depth.

On e-mail:
AtRiA.editor@apu.edu.in

We welcome submissions and opinions at AtRiA.editor@apu.edu.in The policy for articles is published on the inside back cover of the magazine.

Your feedback is important to us. Do write in.

Azim Premji University

PES Institute of Technology Campus, Pixel Park, B Block, Electronics City, Hosur Road (Beside NICE Road), Bangalore – 560100 India
Email: reachus@apu.edu.in