Volume 7 | Issue 2 | July 2018
Triannual | Bengaluru

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Making Sense of Mathematics

PULLOUT
INTRODUCTION TO ALGEBRA-2

Sense-making in mathematics is all too necessary if we need to keep our wits about us in today's world, particularly with the onslaught of information that we face every day. NCF 2005 speaks about the importance of developing reasoning skills in children and this is the foundation of mathematical competency. A school course that does not develop this skill in children is surely shortchanging them.

What are mathematical reasoning and sense making? Reasoning is the process of manipulating and analyzing objects, representations, diagrams, symbols, or statements to draw conclusions based on evidence or assumptions. Sense making is the process of understanding ideas and concepts in order to correctly identify, describe, explain, and apply them.'[i] If a student cannot make sense of the mathematics that he or she learns, then disinterest and disengagement surely follow. A teacher needs careful planning and awareness of the different ways that students think in order to develop reasoning skills and beginning with this issue, we will be carrying articles that can help a teacher do the same.

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[i] Mathematical Reasoning and Sense - Making: Michael T Battista

From the Editor's Desk . . .

The issue you hold in your hand may look the same as the previous eighteen issues but there are significant changes which I take pleasure in bringing to your notice. For one, ***At Right Angles*** is now a *whole* school math resource, instead of just a *High* School math resource, with only the PullOut catering to the primary section. To mark this change, we have included a discreet colour band at the top of each article; the code is given in the Contents Page. It indicates whether the article is best suited for Primary (1-5), Middle School (6-8), High School (9-10) or Pre-University (11-12). Please don't restrict your reading to just the section you teach; there is plenty there for all those interested in mathematics and as usual, we do our best to make the content accessible and engaging. You may notice that some of the ClassRoom articles also have a boxed item on the first page that indicates the scope of the article and the different ways that the teacher can make use of it. Again, don't let us contain your ideas and imagination- and do write to us if you think of innovative take-offs on our articles!

Within the ClassRoom section itself, you will find some inspirational pedagogical tools. If you are a teacher, you may have longed to outsource the task of creating an innovative and engaging worksheet for your students. This is our offering to the math teachers who don't have the time to do this- a TearOut section: the first two pages are a ready-made worksheet for students and the next two, a helpful note on the activities for the teacher/facilitator. And for the teacher educator and parent, we have started a new section called Sense-Making in Mathematics. The name says it all, how can we use visuals, manipulatives and activities to make sense of mathematical concepts, rules and procedures. We have three different articles by Swati Sircar, Rupesh Gesota and Shailaja Sharma in this series. More in the ClassRoom section- Jagathapu Kasi Rao teaches us how to bisect an angle using a ruler and A S Rajagopalan takes us through a lovely lesson on counting the number of integer sided triangles. A Ramachandran describes an innovative way of mapping triangle shapes. There's much, much more- a small sample -CoMaC teaches us about 'characterisation' in mathematical parlance, Shailesh Shirali talks about the 'magic' of magic triangles and in Low Floor, High Ceiling, we discuss isosceles trapeziums this time.

Our Features section is particularly exciting this time too, for we have not one, but two articles by students here; Aradhana Anand talks about Creating Trigonometric Tables based on her study of Sanskrit texts, and Shuborno Das introduces us to Functional Equations in the first of a two-part series on the same. We've reviewed a book and a set of board games this time and the PullOut on Design Language is simply stunning. Your teaching of algebra is never going to be the same.

Enjoy! And send us feedback on AtRiA.editor@apu.edu.in

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SCPL
Bangalore - 560 062
www.scpl.net

Please Note:
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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Azim Premji University

A Publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley and Sahyadri School, Pune

Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

LEGENDS - Primary School | Middle School | High School | Pre-University

At Right Angles
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Contents contd.

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Student Corner – Featuring articles written by students.

Creating Trigonometric Tables

ARADHANA ANAND

Have you tried to create your own trigonometric tables and thus join the likes of several mathematician astronomers of the past who created tables of trigonometric functions for their astronomical calculations?

Sine tables in Bhāskarācārya's Siddhānta-Śiromaṇi

The great 12th century Indian mathematician and astronomer, Bhāskarācārya II, describes the creation of sine tables in his magnum opus, the Siddhānta-Śiromaṇi, in a section named Jyotpatti [1]. The very name Jyotpatti means 'jyānām utpattiḥ', i.e., creation or generation of sine tables. Jyotpatti is a part of Golādhyāya (dealing with Trigonometry), which is one of the four major parts of the Siddhānta-Śiromaṇi, the other three being Līlāvatī (dealing with Arithmetic), Bījagaṇita (dealing with Algebra) and Grahagaṇita (dealing with Planetary Motion).

Bhāskarācārya describes methods for creating several sine tables of varying granularities (i.e., different angle intervals: 3°, 3.75° and so on) [2], [3]. Among these is a table of approximate sine values for each integral angle in the quadrant and a table of exact sine values for all angles that are multiples of 3° in the quadrant. Bhāskara states that sine values can be determined in several ways and lists various identities to illustrate the point. Among these, he specifically highlights the usefulness of the following identities in the creation of these tables

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$ (1) (verse 21)
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$ (2) (verse 22)

Keywords: trigonometry, table, sine table, astronomy, Bhāskarācārya II, Siddhānta-Śiromaṇi

Creating Sine tables for angles at 3° intervals

Let us see how we can create a sine table for angles that are multiples of 3° using just identity (2) and the values of sin 90°, sin 45° and sin 30°.

Now, suppose the exact value of sin 3° were known. Substituting A=90 and B=3 in identity (2), the value of sin 87° can be determined (since we already know that sin 90° = 1). Now, substituting A=87 and B=3 will yield the value of sin 84°. Proceeding in this manner, the sine values for the other angles can be successively found.

The next obvious question is: how do we find the value of sin 3°?

Using A=45 and B=30 in identity (2), we can find the value of sin 15°.

If we could find the value of sin 18°, then substituting A=18 and B=15 in identity (2) would give us the value of sin 3°.

The rest of this article deals with a very elegant geometrical method to find the exact value of sin 18° [4].

The Sine of 18°

The value of sin18° is given by Bhāskarācārya in verse 9 of the Jyotpatti. The verse is:

त्रिज्याकृतीषुघातान्मूलं त्रिज्योनितं चतुर्भक्तम् ।
अष्टादशभागानां जीवा स्पष्टा भवत्येवम् ॥ ९ ॥

(Please see the appendix for the translation of this verse and another verse which appears later in the article.)

The verse tells us that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$. Let us now see how to arrive at this result.

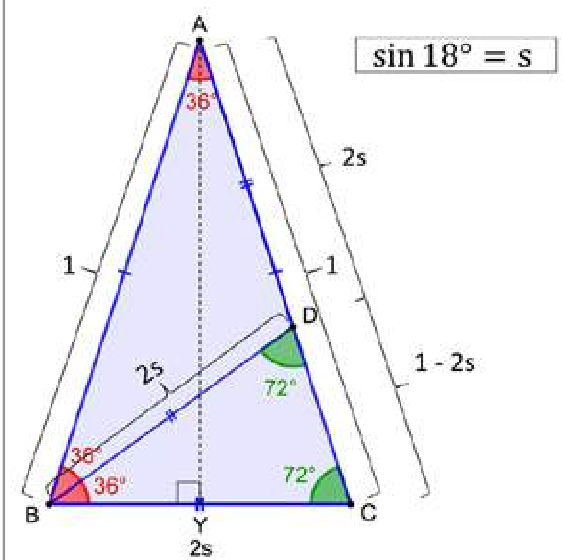

Figure 1.

Consider an isosceles triangle ΔABC (Figure 1) in which ∠BAC = 36° and AB = AC = 1 unit. Using the Angle Sum Property, we get ∠ABC = ∠ACB = 72°.

Construct altitude AY from vertex A to side BC. It is easy to see that BY = YC = s, where s = sin 18°, and thus BC = 2s.

Construct BD, the angular bisector of ∠ABC. This gives us two triangles, ΔABD and ΔBDC, both of which are isosceles. BD and AD are the equal sides of ΔABD and BD and BC are the equal sides of ΔBDC. Therefore, AD = BD = BC = 2s and DC = AC - AD = 1 - 2s.

Further, we have the triangle similarity ΔBDC ~ ΔABC. Using the similarity ratio, we get,

$$\frac{DC}{BC} = \frac{BC}{AC}$$
$$\Rightarrow \frac{1-2s}{2s} = \frac{2s}{1},$$
$$\Rightarrow 4s^2 = 1-2s,$$
$$\Rightarrow 4s^2 + 2s - 1 = 0.$$

Thus, we have a quadratic equation whose root is s.

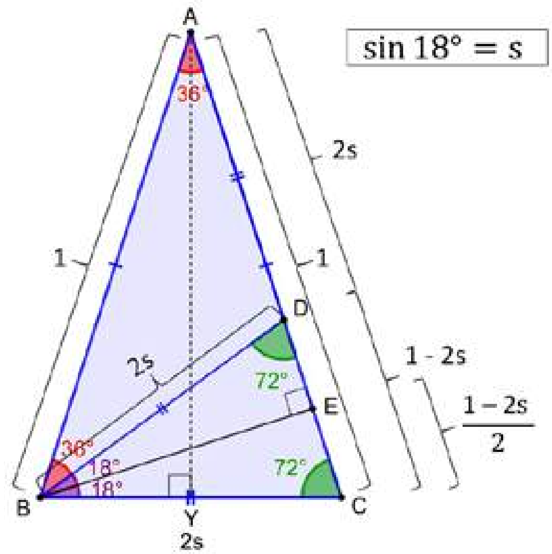

Figure 2.

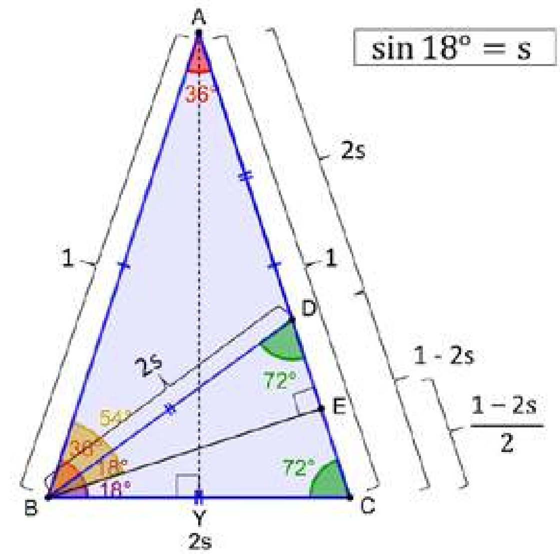

Figure 3.

Applying the quadratic formula, we get s, which is the value of sin 18°, to be $\frac{\sqrt{5}-1}{4}$.

Using a slight modification to the previous method, we can arrive at the quadratic equation in a different way.

Construct altitude BE of ΔBDC (Figure 2). Consider the right triangle ΔBEC. In this triangle,

$$\sin\ 18^{\circ} = s = \frac{EC}{BC} = \frac{1-2s}{2} \times \frac{1}{2s}.$$

Rearranging the terms, we once again arrive at the quadratic equation $4s^2 + 2s - 1 = 0$.

This second method yields us something more; we can find the exact value of sin 54° too! Further, we get a nice relation between sin 18° and sin 54°.

In the right triangle ΔAEB, (Figure 3)

$$\sin 54^{\circ} = \frac{AE}{AB} = 2s + \frac{1-2s}{2} = \frac{1}{2} + s = \frac{1}{2} + \sin 18^{\circ} = \frac{\sqrt{5}+1}{4}.$$

Conclusion

It is quite exciting to note that we can create our own trigonometric tables using simple high school mathematics!

It is also fascinating to see how these results were captured so compactly and beautifully in verse form by the great Indian mathematicians of the past.

Drawing inspiration from the verses in the Jyotpatti section, we conclude with a humble attempt at a verse* capturing the value of sin 54°as well as its relation with sin18°.

त्रिज्याकृतीषुघातात्
मूलं त्रिज्याधिकं चतुर्भक्तम् ।
त्रिज्यार्धं वसुविधुलव-
गुणसहितं युगशरांशज्या॥

(*) Thanks to Prof. K. Ramasubramanian and Dr. K. Mahesh (both of IIT Bombay) for reviewing and correcting this verse

The first two lines of this verse capture the fact that sin 54°= $\frac{\sqrt{5}+1}{4}$ and the last two lines capture the relation that sin 54° = $\frac{1}{2} + \sin 18^{\circ}$.

Appendix: Translations of the Sanskrit Verses

त्रिज्याकृतीषुघातान्मूलं त्रिज्योनितं चतुर्भक्तम् ।
अष्टादशभागानां जीवा स्पष्टा भवत्येवम् ॥ ९ ॥

त्रिज्या - Radius of the circle (R)

कृति – square (here, square of the radius)

इषु – 5 (in the Bhūtasaṃkhyā system [5])

घात – multiplied by

त्रिज्याकृतीषुघात – five times the square of the radius

मूलम् – square root (here, square root of five times the square of the radius)

त्रिज्या-ऊनितम् – reduced by R

चतुर्भक्तम् – divided by 4

अष्टादश - 18

जीवा – Indian Sine = Rsine

अष्टादशभागानां जीवा – Rsine 18°

Translation: The square root of five times the square of the radius, reduced by the radius, divided by four, is the exact value of the Rsine of 18°.

$$\text{i.e., Rsine } 18^\circ = \frac{\sqrt{5R^2} - R}{4}$$

$$\Rightarrow \sin 18^\circ = \frac{\sqrt{5} - 1}{4}$$

त्रिज्याकृतीषुघातान्मूलं त्रिज्याधिकं चतुर्भक्तम् ।
त्रिज्यार्धं वसुविधुलवगुणसहितं युगशरांशज्या॥

त्रिज्या – Radius of the circle (R)

कृति – square (here, square of the radius)

इषु – 5 (in the Bhūtasaṃkhyā system [5])

घात – multiplied by

त्रिज्याकृतीषुघात – five times the square of the radius

मूलम् – square root (here, square root of five times the square of the radius)

त्रिज्या-अधिकम् – increased by R

चतुर्भक्तम् – divided by 4

त्रिज्या-अर्धम् – half of R

वसुविधु – 18 (in the Bhūtasaṃkhyā system)

गुण – Indian Sine (Rsine)

वसुविधुलवगुण – Rsine 18°

सहितम् – along with (added to)

त्रिज्यार्धं वसुविधुलवगुणसहितं – half of R added to Rsine 18°

युगशर – 54 (in the Bhūtasaṃkhyā system)

ज्या - Indian Sine (Rsine)

युगशरांशज्या – Rsine 54°

Translation: The square root of five times the square of the radius, increased by the radius, divided by four, is the value of the Rsine of 54°; and this is equal to the Rsine of 18° added to half the radius.

$$\text{i.e., Rsine } 54^\circ = \frac{\sqrt{5R^2} + R}{4} = \frac{R}{2} + \textbf{Rsine } \mathbf{18^\circ}$$

$$\Rightarrow \sin 54^\circ = \frac{\sqrt{5} + 1}{4} = \frac{1}{2} + \mathbf{\sin 18^\circ}$$

References

1. Siddhānta-Śiromaṇi of Bhāskarācārya with his autocommentary Vāsanābhāṣya and Vārttika of Nṛsiṃha Daivajña, edited by Dr. Murali Dhara Chaturvedi, Varanasi 1981, published by Lakshmi Narayan Tiwari, Librarian, Sarasvati Bhavana Library, Sampurnanand Sanskrit Vishvavidyalaya, Varanasi
2. Lecture 33 (YouTube video)– Trigonometry and Spherical Trigonometry 1, Prof. M. S. Sriram, NPTEL Course on "Mathematics in India - From Vedic Period to Modern Times"
3. Lecture 34 (YouTube video)– Trigonometry and Spherical Trigonometry 2, Prof. M. S. Sriram, NPTEL Course on "Mathematics in India - From Vedic Period to Modern Times"
4. Translation of the Sūrya Siddhānta by Pundit Bāpū Deva Sāstri, and of the Siddhānta Śiromaṇi by the late Lancelot Wilkinson, ESQ., C. S., revised by Pundit Bāpū Deva Sāstri, From the Sanskrit, Printed by C. B. Lewis, at the Baptist Mission Press, Calcutta, 1861
5. Bhūtasaṃkhyā system https://en.wikipedia.org/wiki/Bhutasamkhya_system

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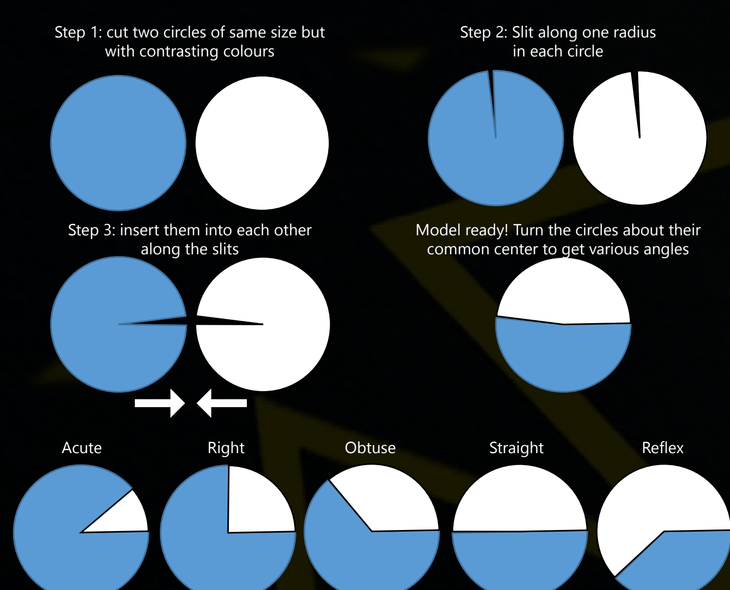
Merry Go Round Angles
Contributed by Kayan Gurung studying at Gangtok DIET

This teaching aid is good to introduce angles and requires only some paper, compass or a bowl to draw the circles, a pair of scissors and a pen to shade one circle.



An added bonus: It also gives a sense of the remaining i.e. that reflex angles get generated for acute, right and obtuse angles.

Student Corner – Featuring articles written by students.

Functional Equations

Part 1

A *functional equation* is an equation where the unknown is a function rather than a variable. It may happen that one knows only a certain property of a function; e.g. that it is even, i.e., $f(x) = f(-x)$ for all x; or that it is additive, i.e., $f(x+y) = f(x) + f(y)$ for all x, y; and so on. The question that then arises is, what functions exist with the stated property? Is there just one such function? In this two-part article, we answer such questions for certain types of functional equations. Functional equations are used to model behaviour in engineering fields (e.g., Shannon's entropy in Information Theory) and the social sciences. They are also of use in the study of difference equations.

SHUBORNO DAS

In Part I of this two-part article, student Shuborno Das goes into the topic of Functional Equations in which the unknown quantities are functions rather than numbers. He draws deeply on concepts related to the topic of Functions which is covered in Standards 11-12. Each new concept that is introduced is clearly defined, with plenty of examples and explanations. A delightful way to learn more about functions and operate flexibly with them.

What are functional equations?

Functions. A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output [1]. In other words, a function is a mapping between two sets A and B where each element in A maps to a unique element in B. This tells us that $f(a)$ cannot be b and c at the same time, where $b \neq c$. A function can be considered as an input-output machine, which takes an input and gives an output (Figure 1). The set of values which the machine can take as input is the *domain*, and the set of values which the machine gives as output is the *range*. Mathematically, a function f which maps elements of set A into set B is written as $f : A \to B$.

Example.

- $f(x) = x^2$ for all real x. Here, the domain is the set of real numbers and the range is the set of all nonnegative real numbers.
- $f(x) = \sin(x)$ for all real x. Here, the domain is the set of real numbers and the range is $[-1, 1]$.

Co-domain of a function is the possible values which a function can give as output. Range is a subset of co-domain.

Keywords: Functional equation, function, domain, range, injective function, surjective function

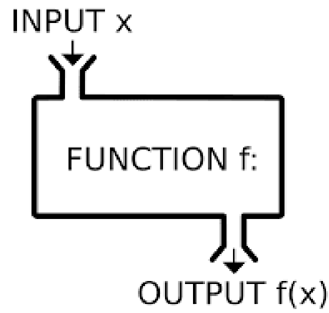


Figure 1. Function as input-output machine [1]

Functional Equations. A functional equation is an equation in which the unknown is a function of one or more variables. Often, the functional equation relates the value of a function (or functions) at some point with its values at other points [1]. Also, since it involves functions, the domain and range must also be specified unlike the case of more common equations.

Example.

- Suppose $f: A \to B$ such that $f(x+1) = f(x) + 1$ for all x in A.
- Suppose $f: \mathbb{R} \to \mathbb{R}$ such that $f(2f(x) + f(y)) = 2x + f(y)$ for all $x, y \in \mathbb{R}$.

Some properties of functions
We will discuss some properties of functions which are useful in solving functional equations.

Injective. An *injective function* or *injection* or *one-to-one function* is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its co-domain. In other words, every element of the function's co-domain is the image of one element of its domain [1].

Definition. If a function $f: A \to B$ has the property that the equality $f(a) = f(b)$ implies $a = b$, then f is said to be injective.

Let's get to some basic examples of recognising the injective property of a function.

- Problem 1: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = \sin x$. Is f injective?

Let $f(a) = f(b)$.

$$\sin a = f(a) = f(b) = \sin b.$$

Consider $a = 30°$ and $b = 150°$, $\sin a = \sin b$ but $a \neq b$. Hence f is not injective.

- Problem 2: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = x^3$. Is f injective?

Let $f(a) = f(b)$.

$$a^3 = f(a) = f(b) = b^3.$$

Since a, b are real, it follows that $a = b$. Hence f is injective.

Some facts regarding injectivity:

- Noticing a term of x not inside f(some variable) is pretty helpful. For example, in Problem 2, RHS had a term of x not involving any f, which helped us show the injectivity of f.

- It is important to check the range and domain while proving whether a function is injective or not. If the domain and range of f was complex in the previous example, then f wouldn't be injective.
- Using injectivity, we can reduce composite functions. For example, if we have $f(f(x^2)) = f(f(x)^2)$ and if f is injective then we can get $f(x^2) = f(x)^2$. That is, function value of x^2 is equal to the square of the function value of x.

Surjective. A *surjective function* or *onto function* is a function in which the elements of the domain map to the entire set of elements in the range [1].

Definition. If a function $f: A \to B$ has the property that for every b in B, there exists a value of a in A such that $f(a) = b$, then f is said to be surjective.

This tells us that for any element in the co-domain, we will have at least one element in the domain mapping to it. Let's get to some basic examples of recognising the surjective property of a function.

- Problem 3: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = \sin x$. Is f surjective?

We have

$$-1 \leq \sin x \leq 1,$$

therefore f is not surjective.

- Problem 4: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = x^3$. Is f surjective?

  Suppose $a^3 = f(a) = b$, then $a = \sqrt[3]{b}$ (we are dealing with reals). Therefore there exists a real number a such that $f(a) = b$ for every real b.

Some facts regarding surjectivity:

- Noticing a term of x not inside f(some variable) is pretty helpful. In Problem 3, RHS had a term of x not involving any f which helped us show the surjectivity of f.
- It is important to check the range and domain while proving whether a function is surjective or not.
- Surjectivity tells us that for all elements in the range, there is an element in the domain mapping to it. This is helpful in assigning arbitrary values to the function (e.g., assume $f(x) = 0$ if $f(x)$ is surjective and co-domain is the set of reals), if that helps to simplify the functional equation.

Definition. A function which is both injective and surjective is known as *bijective* function.

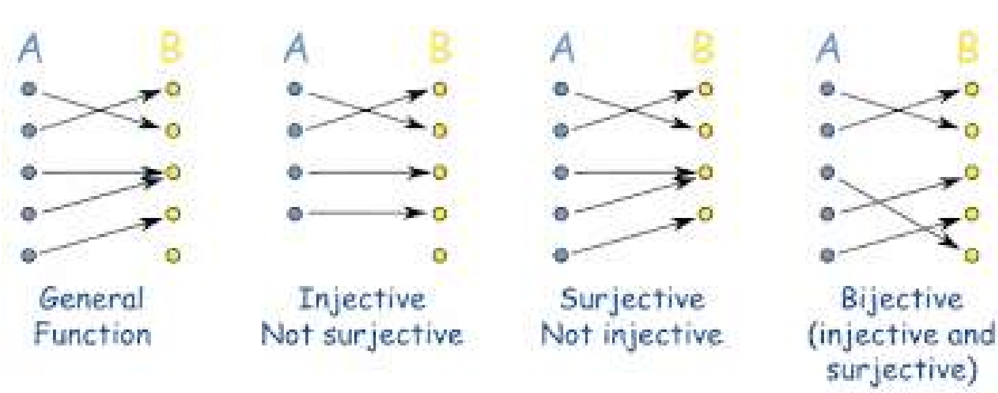


Figure 2. Injective, surjective and bijective function [2]

Monotonic. A monotonic function (or monotone function) is a function between ordered sets that preserves or reverses the given order. In order words, a monotonic function is an increasing or a decreasing function [1].

Definition. If $f: A \to B$ be a function and $f(x) \geq f(y)$ for all $x > y$ where x, y are in the set A, then the function is said to be increasing. If $f(x) > f(y)$ for all $x > y$ where x, y are in the set A, then the function is said to be strictly increasing.

Definition. If $f: A \to B$ be a function and $f(x) \leq f(y)$ for all $x > y$ where x, y are in the set A, then the function is said to be decreasing. If $f(x) < f(y)$ for all $x > y$ where x, y are in the set A, then the function is said to be strictly decreasing.

Let's try to recognize the monotonic property of functions.

- Problem 5: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = \sin x$. Is f monotonic?

  Observe that $\sin 90^\circ > \sin 30^\circ$ and $\sin 150^\circ < \sin 90^\circ$. From these relations, we realise that the sin function is not monotonic. The graph of the $\sin x$ function (Figure 3) illustrates this feature: the graph is rising in some regions and falling in other regions.

- Problem 6: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = x^3$. Is f monotonic?

  We shall show that f is an increasing function. Suppose that $x > y$, then $f(x) - f(y) = x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. We have assumed that $x - y > 0$ and

$$x^2 + xy + y^2 = \frac{2x^2 + 2xy + 2y^2}{2} = \frac{(x^2 + y^2) + (x + y)^2}{2} \geq 0.$$

  Combining the results, we have $f(x) > f(y)$. The graph of the function (Figure 3) illustrates this property.

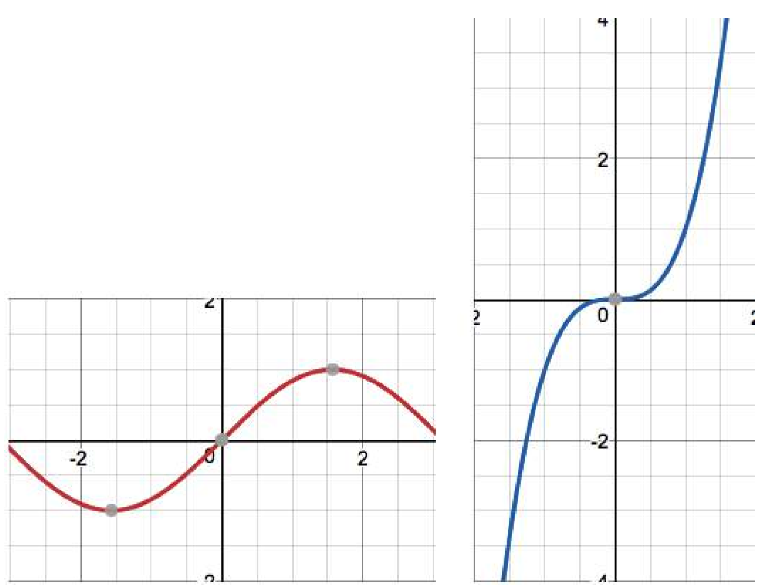


Figure 3. Graphs of $f(x) = \sin x$ and $f(x) = x^3$

Continuous. A function for which sufficiently small changes in the input result in arbitrarily small changes in the output is said to be *continuous*. Otherwise, it is said to be a *discontinuous* function [1].

Definition. Suppose $f(x)$ is a function. It is continuous at a point c if and only if the following conditions are satisfied:

- $f(c)$ exists.
- $\lim_{x\to c} f(x)$ exists.
- $\lim_{x\to c} f(x) = f(c)$.

In simplistic terms, the graph of a continuous function will be an uninterrupted curve in the domain where the function is defined. We won't go into the depths of continuity, as the concept involves ideas outside the scope of this article.

Practice problems
Determine whether the following functions are injective, surjective and/or monotonic.

(1) $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = c$ for some real constant c for all x.

(2) $f\colon \mathbb{C} \to \mathbb{C}$ satisfying $f(x) = x^3$ for all x.

(3) $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = \dfrac{x^2 + x}{2}$ for all x.

(4) $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying $f(x) = |x|$ for all odd x and $f(x) = -|x|$ for all even x.

Solving Functional Equations

Now that we know the basic properties of functions, let's discuss the ways to approach a functional equation.

- Guessing the solution: It may seem strange but an important technique to solve a FE is to guess the solution if feasible. However mere guessing and plugging in the original equation to prove the guess works is not sufficient since there may be other solutions also. Still, guessing a solution may help to simplify the functional equation in a manner from where the final solution can easily be obtained.
- Finding values of elements in domain: Finding values of $f(0), f(1)$, etc. depending on the domain is important. These values help to derive other relations which may lead to the solution.
- Determining properties of the functions: If we can prove that the unknown function is injective (or surjective/bijective/monotonic), it allows additional manipulation to the original FE which may lead to the solution.
- Deriving equations and manipulating it to get the solution: Most often, FE are equations in more than one variable. If so, then constraining the FE for suitable values of those variables can help to derive more properties of the function and may even solve it completely.

Example 1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(3x+2) = 5x$$

for all $x \in \mathbb{R}$.

Solution 1. In this problem, the equation given is similar to $f(x) = ?$ Let's try to remove the $3x+2$ and replace it with a simpler term, say y. We want $3x+2 = y$ or $x = \frac{y-2}{3}$. Therefore plugging $x = \frac{y-2}{3}$, we get $f(y) = \frac{5(y-2)}{3}$ for all reals y ($3x+2$ covers all reals). The reasoning used shows that this function is the only one satisfying the given condition. Hence the solution is: $f(x) = \frac{5(x-2)}{3}$ for all $x \in \mathbb{R}$.

Example 2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x-y) = f(x) + f(y) - 2xy$$

for all $x, y \in \mathbb{R}$.

Solution 2. As discussed earlier, one way to start a FE is to find some values in the domain. In this problem the domain is $\mathbb{R}$. We can try to find $f(0)$.

Let $P(x,y)$ be the assertion of the problem statement. $P(0,0)$, i.e plugging $x = y = 0$ in the original equation gives $f(0) = f(0) + f(0) - 0 = 2f(0)$. This gives $f(0) = 0$.

Now that we have $f(0) = 0$, let's try to use this information. We may want to eliminate the LHS of the equation. $P(x,x)$ gives $f(0) = f(x) + f(x) - 2x^2 \implies f(x) = x^2$ for all x which is indeed a solution. The reasoning used shows that this function is the only one satisfying the given condition. Hence the solution is: $f(x) = x^2$ for all $x \in \mathbb{R}$.

Example 3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(2f(x) + f(y)) = 2x + f(y)$$

for all $x, y \in \mathbb{R}$.

Solution 3. One of the first steps in solving FE is to derive different properties of the unknown function like injectivity, surjectivity, etc. Once the property has been derived, this can then be applied to understand more about the function, sometimes even solving it completely.

In this problem, we notice an isolated x term in RHS. As indicated in section 2.1, this may help us to prove that the desired function is injective. Let's see how we can prove that f is injective.

Let $P(x,y)$ be the assertion of the problem statement, i.e $f(2f(x) + f(y)) = 2x + f(y)$.

Suppose $f(a) = f(b)$, $P(a,y)$ and $P(b,y)$ gives

$$2a + f(y) = f(2f(a) + f(y)) = f(2f(b) + f(y)) = 2b + f(y) \implies a = b.$$

So f is injective.

Let's now use this property to derive the value of $f(x)$. If we constrain the FE to $P(0,y)$ it gives $f(2f(0) + f(y)) = f(y)$. Since $f(x)$ is injective, this equation gives $2f(0) + f(y) = y \implies f(y) = y - 2f(0)$. Therefore $f(x) = x - 2f(0)$ for all $x \in \mathbb{R}$.

In order to find the value of $f(0)$, we put $x = 0$ into the relation $f(x) = x - 2f(0)$ and obtain $f(0) = -2f(0)$ or $f(0) = 0$.

Hence the solution is: $f(x) = x$ for all $x \in \mathbb{R}$.

Example 4. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f(f(n)) = n, f(f(n+2)+2) = n$$

for all $n \in \mathbb{Z}$ and $f(0) = 1$.

Solution 4. We are given $f(0) = 1$, let's try to use this information by using it in the equations. Let $P(n)$ be the assertion that $f(f(n)) = n$ and $Q(n)$ be the assertion that $f(f(n+2)+2) = n$. $P(0)$ gives $f(f(0)) = 0$ or $f(1) = 0$.

Observing the equations, we notice that we have a relation between $f(f(n))$, n and $f(f(n+2)+2)$ and n. We look to combine these conditions.

We have

$$f(f(n+2)+2) = n \implies f(f(f(n+2)+2)) = f(n) \implies f(n+2)+2 = f(n)$$

where the first equality is from $Q(n)$ and last one from $P(f(n+2)+2)$.

Let's summarise what we have got till now:

$$f(0) = 1, f(1) = 0, f(n+2)+2 = f(n).$$

It's easy to guess that $f(n) = 1 - n$ is a solution. We are working on integers, so induction is a good way to prove our claim. This part is left for the reader to prove.

Example 5. Find the value of $f(486)$ where $f\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function such that

$$f(f(n)) = 3n$$

for all $n \in \mathbb{N}$.

Solution 5. It is obvious that $f(n)$ is injective but it does not help us make any further progress about the value of $f(n)$. Therefore we need a different approach to solve this problem.

Let's see the values of the function for some initial values of the domain. Note this is quite useful to get a sense of how the function is behaving. Since the domain is $\mathbb{N}$, we will plug in values like $1, 2$, etc.

Let $P(n)$ be the assertion of the problem statement, i.e $f(f(n)) = 3n$. We have an additional condition that f is a strictly increasing function (recall definition 2.4), we'll try to use it.

$$P(1) \to f(f(1)) = 3.$$

We wish to find $f(1)$. If $f(1) = 1$, then $f(f(1)) = f(1) = 1 \neq 3$. If $f(1) \geq 3$, then from the strictly increasing condition,

$$3 = f(f(1)) > f(2) > f(1) \geq 3,$$

because $f(1) > 2$ (remember that we are assuming $f(1) \geq 3$). This means that $3 > 3$ which is contradictory. Therefore $f(1) = 2$ and $3 = f(f(1)) = f(2)$.

Now that we have $f(1)$ and $f(2)$, we look to utilise these facts. $P(2)$ gives

$$6 = f(f(2)) = f(3), 9 = f(f(3)) = f(6), 18 = f(f(6)) = f(9) \cdots.$$

Can we guess the pattern? It looks like $f(3^n) = 2 \cdot 3^n$ and $f(2 \cdot 3^n) = 3^{n+1}$ but we need a mathematical proof for it. Induction seems a plausible way to proceed since we are dealing with natural numbers.

Base case: $n = 1 \to f(3) = 6$ which we have already found out earlier and $f(6) = 9$ which we have also seen before. Inductive case: Hypothesis: $f(3^n) = 2 \cdot 3^n$ and $f(2 \cdot 3^n) = 3^{n+1}$.

$$2 \cdot 3^{n+1} = f(f(2 \cdot 3^n)) = f(3^{n+1})$$

and

$$3^{n+2} = f(f(3^{n+1})) = f(2 \cdot 3^{n+1}).$$

Hence $f(3^n) = 2 \cdot 3^n, f(2 \cdot 3^n) = 3^{n+1}$ for all natural n. Note that $486 = 2 \cdot 3^5$ so $f(486) = f(2 \cdot 3^5) = 3^6 = 729$.

The answer is: $f(486) = 729$.[1]

Example 6. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that

$$f(-x) = -f(x), \quad f(x+1) = f(x) + 1, \quad f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$$

for all $x \in \mathbb{R}$ and $x \neq 0$.

Solution 6. There are no isolated terms which we could use to prove injectivity or surjectivity of $f(x)$. Also plugging values does not provide any idea about the behaviour of the function. Clearly we need to use a different approach to solve this problem.

We notice that there are some relations between $f(x)$ and $f(x+1), f(-x)$ and $f\left(\frac{1}{x}\right)$. Can we come up with a series starting from x and finally again reaching x using these possible transformations? In other words, we need a sequence of $x \to \ldots \to x$ using the moves $x \to x+1, x \to \frac{1}{x}, x \to -x$ for any x not equal to 0. Please see if you are able to find the cycle before looking at the cycle given below. Note that there could be many possible cycles.

$$x \to x+1 \to \frac{1}{x+1} \to \frac{-1}{x+1} \to 1 - \frac{1}{x+1} = \frac{x}{x+1} \to \frac{x+1}{x} = 1 + \frac{1}{x} \to \frac{1}{x} \to x.$$

Using the cycle, we get a relation in $f(x)$ which on simplifying gives $f(x) = 2x - f(x)$ for $x \neq 0, -1$. $\Rightarrow$ $f(x) = x$ for $x \neq 0, -1$. Lets' try to find $f(0), f(-1)$. We know $f(-1) = -f(1) = -1$ and $f(0) = -f(0) \implies f(0) = 0$.

The answer is: $f(x) = x$ for all $x \in \mathbb{R}$.

Conclusion

We have tried to show in this article how by repeatedly applying a few simple principles, we can make significant progress in understanding and solving functional equations. We will continue this theme in Part II of the article which will appear in the next issue. For now, here are a few sample problems which the reader could attempt on his/her own.

[1] $486 = 2 \cdot 3^5$, but what about 2001 which is neither a power of 3 nor twice a power of 3. Try to find the value of $f(2001)$. For the solution, refer my blog [3] at AoPS.

Example Problems

(1) (Korea 2000) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = (x-y)(f(x) + f(y))$$

for all $x, y \in \mathbb{R}$.

(2) Find all monotone functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(4x) - f(3x) = 2x$$

for all $x \in \mathbb{R}$.

(3) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y)$$

for all $x, y, z \in \mathbb{R}$.

(4) Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\left(\frac{f(x)}{y}\right) = yf(y)f(f(x))$$

for all $x, y \in \mathbb{R}^+$.

(5) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(x + y^2)f(yf(x)) = xyf(y^2 + f(x))$$

for all $x, y \in \mathbb{R}$.

(6) Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(3x) - f(x) \leq 8x^2 + 2x, f(2x) - f(x) \geq 3x^2 + x$$

for all $x \in \mathbb{R}$.

References

1. https://en.wikipedia.org/wiki/Continuous_function
2. www.matematikaria.com/unit/injective-surjective-bijective.html
3. www.artofproblemsolving.com/community/c453393

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The Constants of Mathematics

Part 1

SHAILESH SHIRALI

Science is full of constants. Probably the best known such constant is the velocity of light (c), made famous by Einstein's Special Theory of Relativity. (He postulated that all observers measuring the velocity of light in vacuum would obtain the same figure, regardless of their own velocity.) Other such constants, slightly less famous, are Planck's constant (h), the gravitational constant (G) which occurs in Newton's law of universal gravitation, the charge of the electron (e), the mass of the electron (m_e) and the mass of the proton (m_p). All these constants have *units* (so their values depend on the system of measurement), but there are also constants which are 'dimensionless'. For example, we have the 'fine-structure constant' α (also known as Sommerfeld's constant; it concerns the strength of the electromagnetic interaction between elementary charged particles) and constants like 3 (the number of independent dimensions of space) and 2 (which occurs as the exponent in so many force laws, e.g., Newton's universal law of gravitation).

In mathematics too, there are many constants. In one sense, of course, every number is a constant! But as in human society, in which all men are equal under the Constitution, yet some are "more equal than others" (apologies to George Orwell for this usage which is far removed from its original usage in *Animal Farm*), so too with numbers. Nature seems to have a particular love for some numbers, for they occur repeatedly in mathematical results, often in the most unexpected ways; numbers like π, e, γ and so on, and also numbers like 1 and 2.

Keywords: Constant, variable, irrational, commensurable

In this series, we make a whimsical journey visiting some well-known constants of mathematics; along the way we learn about their personalities, their peculiarities. In each case, we attempt to justify why mathematicians consider the number to be mathematically significant and hence is worthy of being called a 'constant'.

Pythagoras' constant: the square root of 2

We start with the number which has the honour of being the first one ever to be proved irrational: the square root of 2. It has the dubious honour of being the chief participant in the first great crisis in mathematics.

In what sense is $\sqrt{2}$ a mathematical constant? That is easy to see: the number is linked inextricably to the square, which is a fundamental geometric object. All squares are similar to one another, and the ratio of the diagonal to the side of any square is $\sqrt{2}$ (Figure 1). This follows from the theorem of Pythagoras, which explains the name given to the constant.

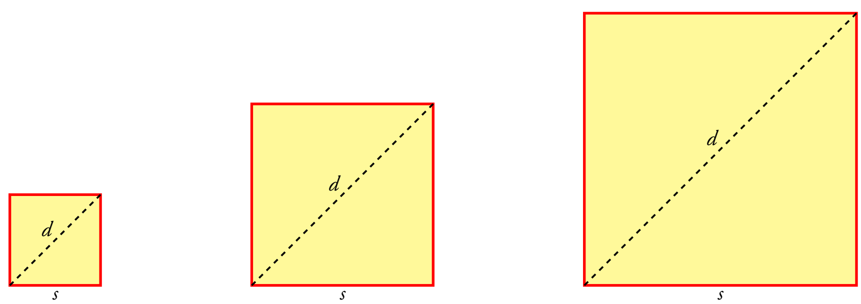


Figure 1. The ratio of diagonal to side, d/s, is the same for all squares: $d/s = \sqrt{2}$

As already stated, the constant $\sqrt{2}$ is notorious in being the first number shown to be irrational. This discovery was made by the school of Pythagoras (probably in the fifth century BCE), but it was expressed differently, thus: "The side and the diagonal of a square are not commensurable." (This means that no matter what unit of length we choose, it cannot fit a whole number of times into *both* the side *and* the diagonal.) A discovery of this kind if made today would be the source of much excitement and pleasure. But it appears that the discovery was not welcome to its discoverers! This will seem strange to us, but it has to be understood with reference to the Pythagorean world view, in which the role of the counting numbers was central. The word 'rational' may give us a clue to why this was so: nowadays, it is used to describe numbers that can be written as the ratio of two integers, but additionally it has the connotation of 'sane', 'orderly', 'logical', and so on. The fact that the same word is used to describe these two different attributes tells us that the Pythagorean view is still very much with us, so we continue to be Pythagoreans! This perspective may help us appreciate why the discovery that $\sqrt{2}$ is not commensurable provoked such a philosophic crisis, the first such in mathematics.

The Greeks were not the first to study the square root of 2. Earlier, the Babylonians studied it and had found some remarkable approximation schemes which are of interest to us even today.

Irrational nature of the square root of 2

We give several different proofs; each is (naturally) a "proof by contradiction". (Why 'naturally'? Because irrationality is essentially a negative concept; it asserts the *lack* of some characteristic, so there cannot be a direct proof of irrationality.)

Euclid's proof. This is the proof given in Euclid's *Elements*. It is perhaps the oldest formally articulated proof of any proposition in mathematics. It rests on two simple observations: (i) The square of an even integer is even. (ii) The square of an odd integer is odd. Here is the proof, expressed in modern algebraic language.

Suppose that $\sqrt{2} = a/b$ where a, b are positive integers. We may suppose that a, b are coprime, for if they do share a common factor, it can be 'canceled' from both the numbers, leaving the ratio a/b unchanged. But this means, in particular: a and b cannot *both* be even. By squaring the relation $\sqrt{2} = a/b$ we get:

$$2 = \frac{a^2}{b^2}, \quad \therefore \quad a^2 = 2b^2, \tag{1}$$

from which follow these statements, in sequence: a^2 is even, hence a is even, hence $a = 2c$ for some positive integer c. These in turn lead to the following:

$$a^2 = 4c^2, \quad \therefore \quad 2b^2 = 4c^2, \quad \therefore \quad b^2 = 2c^2, \tag{2}$$

from which follow these statements, in sequence: b^2 is even, hence b is even. It thus transpires that both a and b are even. But this contradicts what we said above: that a and b cannot both be even. We conclude that the supposition made at the start has to be invalid, and hence that $\sqrt{2}$ is not rational. □

A proof by descent. Since the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of positive integers is bounded below by 1, the following deduction is valid: *It is not possible to have an infinitely long, strictly decreasing sequence of positive integers.* (The two phrases 'infinitely long' and 'strictly decreasing' are crucial parts of this sentence.)

This may seem to be another of those 'obviously true' and trivial statements which cannot possibly yield anything of significance; but in fact many beautiful proofs are based on it. They are known collectively as **proofs by descent**. The proof we now offer, to show the irrationality of $\sqrt{2}$, is one such.

Suppose that a and b are positive integers such that $a/b = \sqrt{2}$. Then $b\sqrt{2} = a$. Using this property we define a set S as follows:

$$S = \text{the set of all positive integers } x \text{ such that } x\sqrt{2} \text{ is an integer.} \tag{3}$$

By definition, b lies in S; so S is non-empty. We shall now produce another positive integer which is smaller than b and lies in S.

The number we have in mind is $a - b$. First, we show that it has the desired property. It is certainly positive (for we have $a/b > 1$, hence $a > b$ and $a - b > 0$), and it is an integer, since a and b are integers. Now note that:

$$(a-b)\sqrt{2} = a\sqrt{2} - b\sqrt{2} = (b\sqrt{2}) \cdot \sqrt{2} - a = 2b - a. \tag{4}$$

Hence $a - b$ belongs to S. How can we be sure that $a - b$ is smaller than b? Let $b' = a - b$ and $a' = 2b - a$; then $a'/b' = \sqrt{2}$. Since a and b are integers, so are b' and a'. Since $a/b \approx 1.4$, it follows that $a > b$ but $a < 2b$, implying that $b' < b$. Hence $0 < b' < b$. We have thus found a positive integer b' which is smaller than b and lies in S.

This construction works with any integer in S. Thus we can find a positive integer b'' which is smaller than b' and lies in S. And so on.

We thus obtain an infinitely long, strictly decreasing sequence of positive integers. But such a sequence cannot exist!

Thus we have arrived at a contradiction. We conclude that the supposition made at the start (about the existence of the integers a and b) is invalid, and hence that $\sqrt{2}$ is not rational. □

Another route to the above proof is the following. Observe that if $x^2 = 2$, then

$$2 - x = x^2 - x, \quad \therefore\ 2 - x = x(x-1). \tag{5}$$

Therefore, if $x = \sqrt{2}$ then:

$$x = \frac{2-x}{x-1}. \tag{6}$$

Now suppose that $\sqrt{2}$ is a rational number. Let $\sqrt{2} = a/b$ where a, b are positive integers. Substituting $x = a/b$ in (**??**) we see that

$$\frac{a}{b} = \frac{2 - a/b}{a/b - 1} = \frac{2b - a}{a - b}.$$

We have arrived at the same expression and the same numbers ($a - b$ and $2b - a$) as earlier.

Pictorial proof. This puts into an attractive, pictorial form the argument just presented. It starts with the supposition that $\sqrt{2} = a/b$ where a and b are positive integers.

- $AB = b$
- $AC = a$
- $AP = b$
- $PC = a - b$
- $BQ = a - b$
- $CQ = 2b - a$

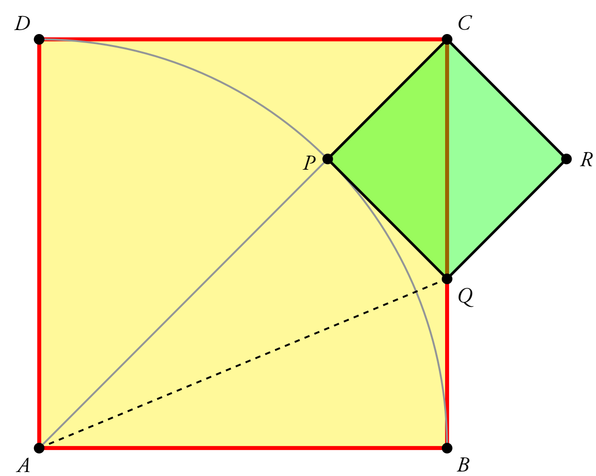


Figure 2.

Figure 2 displays a square $ABCD$ with side $AB = b$. Its diagonal AC has length $b\sqrt{2} = a$. By drawing an arc of a circle with radius b, centred at A, locate a point P on AC such that $AP = b$, and by drawing PQ perpendicular to PC, construct a square $CPQR$ on side CP, with Q on side CB. Join AQ. Since $AP = b$, we have $PC = a - b = PQ$.

Now consider $\triangle APQ$ and $\triangle ABQ$. They are RHS-congruent to each other, so $BQ = PQ$. It follows that $BQ = a - b$, and hence that $CQ = b - (a - b) = 2b - a$. Since a, b are integers, so are $a - b$ and $2b - a$. So the lengths of the side and diagonal of square $CPQR$ are positive integers.

Note what we have accomplished: starting with a square $ABCD$ whose side and diagonal have integer length, we have produced another square $CPQR$ whose side and diagonal also have integer length. Moreover, $CPQR$ is *strictly smaller* than $ABCD$. (Compare their diagonals: $CQ < CB$ and $CB < CA$, therefore $CQ < CA$.)

The same construction starting with square $CPQR$ will produce yet another square with integer side and diagonal, even smaller than square $CPQR$. The logic of the construction is such that we can continue this process forever. We thus get a shrinking sequence of integer-sided squares. This is clearly not possible — we cannot have indefinitely small, integer-sided squares! So we reach a contradiction, like earlier, and we conclude that we cannot construct such a configuration at all. Hence $\sqrt{2}$ is irrational. $\square$

Origami proof. The idea described above can be put in a pictorially attractive form in another way, using ideas from origami. Figure 3 (i) shows an isosceles right-angled $\triangle PQR$, right-angled at R. The bisector PS of $\measuredangle QPR$ has been marked. In Figure 3 (ii), the triangle has been folded along the angle bisector PS; what was originally $\triangle PSR$ has been folded upon $\triangle PST$, with side PR lying upon side PT.

Now suppose that $\sqrt{2}$ is a rational number, say $\sqrt{2} = a/b$ where a and b are positive integers. In Figure 3, choose the scale of the figure in such a manner that $PR = b$; then $PQ = a$. The sides of $\triangle PQR$ are b, b, a; these are all integers. Therefore, *$\triangle PQR$ is integer-sided, isosceles, and right-angled.* In Figure 3 (ii), $PT = PR$, hence $PT = b$ and $TQ = a - b$. In $\triangle TQS$, $\measuredangle TQS = 45^\circ = \measuredangle TSQ$, hence $TS = TQ$, i.e., $TS = a - b$. Since $\triangle PSR \cong \triangle PST$, we get $SR = ST$, i.e., $SR = a - b$; therefore $QS = b - (a - b) = 2b - a$. So the sides of $\triangle TQS$ are $a - b, a - b, 2b - a$; these too are all integers. Therefore, *$\triangle TQS$ is integer-sided, isosceles, and right-angled.*

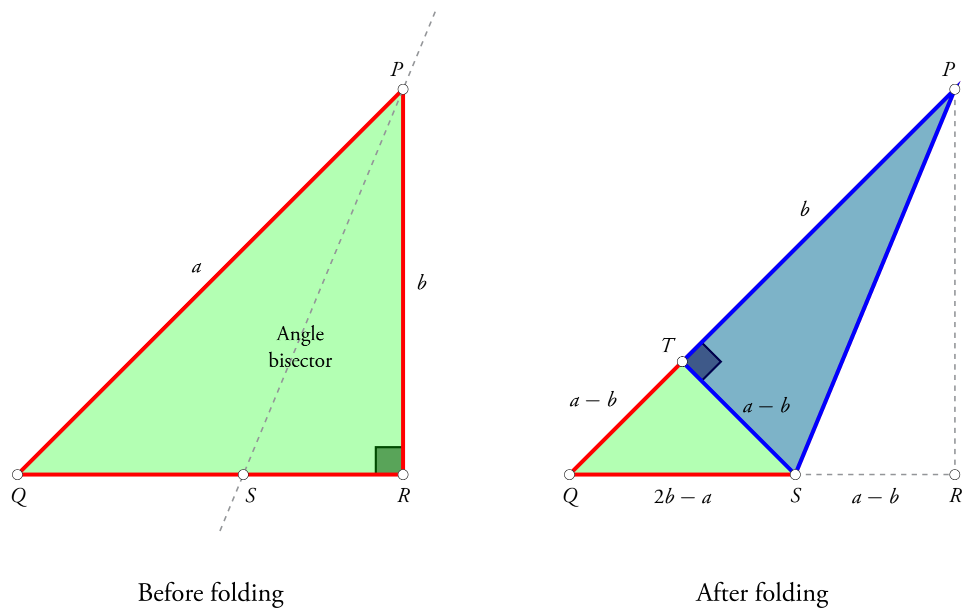


Figure 3.

Since $\triangle TQS$ lies within $\triangle PQR$, the sides of TQS are strictly smaller than the corresponding sides of PQR. Hence the existence of an integer-sided, isosceles, right-angled triangle has led to the existence of another such triangle but with strictly smaller sides. The very same construction applied to this smaller triangle will lead to the existence of yet another integer-sided, isosceles right-angled triangle.

This iterative step can be applied indefinitely, and we are forced to confront an infinite sequence of shrinking integer-sided triangles. This is clearly not possible; the condition that the sides are positive integers acts as an impassable barrier. Hence the initial assumption must be invalid; in other words, $\sqrt{2}$ cannot be a rational number. □

Computing the square root of 2

To find good decimal approximations for $\sqrt{2}$ we may use the well-known "long-division method". But rather than traverse this well-trodden path, we shall use a different approach for approximating the square root of 2. It is very 'low-tech' in its requirements: all it needs is the expansion formula for $(a - b)^2$.

We start with the easily verified fact that $\sqrt{2}$ lies between 1 and 2, and hence that:

$$0 < \sqrt{2} - 1 < 1. \tag{7}$$

Let $\alpha = \sqrt{2} - 1$. Since α lies between 0 and 1, the same is true for the quantities α^2, α^3, α^4, ...; that is, $0 < \alpha^n < 1$ for every positive integer n. Indeed, the larger the value of n, the closer the value of α^n to 0. This simple fact can be exploited to yield remarkably good approximations to α. Here's how we proceed. Squaring α using the binomial squaring formula, we get:

$$\alpha^2 = \left(\sqrt{2} - 1\right)^2 = 2 - 2\sqrt{2} + 1 = 3 - 2\sqrt{2}.$$

If we regard α^2 as a small quantity, i.e., $3 - 2\sqrt{2} \approx 0$, we get by division:

$$\sqrt{2} \approx \frac{3}{2}. \tag{8}$$

This is not a particularly good approximation, but it is noteworthy that we got it at all, and that too by such simple reasoning. We can improve it by continuing the squaring process. We get the following successively better approximations:

- $\alpha^4 = \left(3 - 2\sqrt{2}\right)^2 = 9 - 12\sqrt{2} + 8 = 17 - 12\sqrt{2}$, hence:

  $$\sqrt{2} \approx \frac{17}{12}. \tag{9}$$

  This is much better!

- $\alpha^8 = \left(17 - 12\sqrt{2}\right)^2 = 289 - 408\sqrt{2} + 288 = 577 - 408\sqrt{2}$, hence:

  $$\sqrt{2} \approx \frac{577}{408}. \tag{10}$$

  Even better ....

- $\alpha^{16} = \left(577 - 4082\sqrt{2}\right)^2 = 665857 - 470832\sqrt{2}$, hence:

  $$\sqrt{2} \approx \frac{665857}{470832}. \tag{11}$$

- $\alpha^{32} = \left(665857 - 470832\sqrt{2}\right)^2 = 886731088897 - 627013566048\sqrt{2}$, hence:

  $$\sqrt{2} \approx \frac{886731088897}{627013566048}. \tag{12}$$

It is worth examining how good these approximations are (each one necessarily yields an overestimate). Table 1 displays the results; each value may be compared with the actual value of $\sqrt{2}$ given in the last row. In just four steps, we have achieved close to twenty-five decimal place (d.p.) accuracy! That is indeed very impressive.

Remarks. Before closing this section we make two remarks.

- From any fraction a/b which is close to $\sqrt{2}$, in the sense that $|a - b\sqrt{2}|$ is a small quantity (close to 0; in any case, smaller than 1 in absolute value), we can obtain a better one by squaring, thus:

  $$\left(a - b\sqrt{2}\right)^2 = \left(a^2 + 2b^2\right) - 2ab\sqrt{2}.$$

  Hence the new approximation is $\left(a^2 + 2b^2\right) \div 2ab$, which may be written as:

  $$\frac{b}{a} + \frac{a}{2b}. \tag{13}$$

Number	Decimal expansion	Error
$\frac{17}{12}$	1.41666 ...	2×10^{-3}
$\frac{577}{408}$	1.41421 5686 ...	2×10^{-6}
$\frac{665857}{470832}$	1.41421 35623 7468 ...	3×10^{-12}
$\frac{886731088897}{627013566048}$	1.41421 35623 73095 04880 16896 ...	9×10^{-25}
$\sqrt{2}$	1.41421 35623 73095 04880 16887 ...	

Table 1. Rational approximations to $\sqrt{2}$

Example: From the approximation $7/5 = 1.4$ (accurate to one d.p.) we get:

$$\frac{5}{7} + \frac{7}{10} = \frac{99}{70} \approx 1.41428,$$

which is accurate to 4 d.p. And from this we get:

$$\frac{70}{99} + \frac{99}{140} = \frac{19601}{13860} \approx 1.414213564,$$

which is accurate to 8 d.p. One more application yields 17 d.p. accuracy!

- The same logic can be used to get good rational approximations to numbers like $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$; indeed, the square root of any rational number. But it will not work for cube roots, fifth roots, and so on. (Why not?)
- Some of you may recognise in this scheme a low-tech version of the well-known Newton-Raphson scheme for numerically solving arbitrary single variable equations.

Sightings of the square root of 2

A4 Paper. Did you know that the familiar A4-sized sheet of paper we use in printers and photocopiers incorporates the magic number $\sqrt{2}$? The number $\sqrt{2}$ has the following property: $\sqrt{2} : 2 = 1 : \sqrt{2}$. Hence, if we take a rectangular sheet of paper whose length to width ratio is $\sqrt{2} : 1$ and fold it in half along its longer side, the folded sheet will have the same shape as the original one (it has the same length-to-width ratio). This is just the property that defines A4-sized paper! (For, if a rectangular sheet whose length to width ratio is $x : 1$ has such a property, then we must have $x/2 : 1 = 1 : x$. This equation has only one solution, $x = \sqrt{2}$, as we must have $x > 0$. So there is only one such ratio which 'works.') If we fold such a sheet in two, along the longer side, we get a A5-sized sheet, and if we fold *that* in two, we get a A6-sized sheet. Similarly we have A3-sized paper which would yield A4-size if folded in half. The length-to-width ratios are the same for all these sheets; namely, $\sqrt{2} : 1$. See Figure 4.

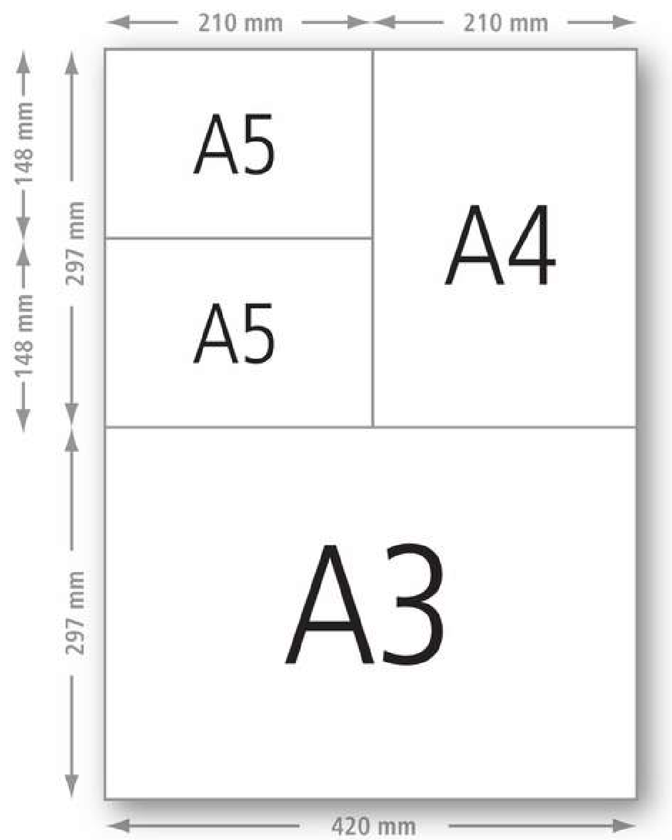


Figure 4. Paper sizes

…And a non-sighting of the square root of 2. The Boeing series of jet planes is well-known and their model numbers have become part of our everyday lexicon: Boeing 707, Boeing 747 and so on. Their very first model was the Boeing 707 and it has become part of folklore that it was so named because the angle between the wings and the body is 45° and, as is well-known, $\sin 45^\circ = 1/\sqrt{2} \approx 0.707$.

But this ancient wisdom has been debunked! In actual fact, the wingsweep angle of a Boeing 707 is 35°, not 45°. The actual reason behind the name is more pedestrian; see [1].

Figure 5. A Boeing 707; photo credit: https://www.boeing.com/history/products/707.page

To conclude, two beautiful formulas …

We conclude by displaying a couple of extremely beautiful expressions for the square root of 2.

A formula found by Euler (1707–1783). The first expression was found by the great Leonhard Euler. Try to prove it for yourself!

$$\sqrt{2} = \left(1 + \frac{1}{3}\right) \times \left(1 + \frac{1}{35}\right) \times \left(1 + \frac{1}{99}\right) \times \left(1 + \frac{1}{195}\right) \times \cdots . \tag{14}$$

The denominators in the fractions are

$$3 = 1 \times 3, \quad 35 = 5 \times 7, \quad 99 = 9 \times 11, \quad 195 = 13 \times 15, \quad \ldots .$$

A formula found by Francois Viète (1540–1603). The second expression is an amazing and beautiful formula connecting $\sqrt{2}$ and π:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdots \tag{15}$$

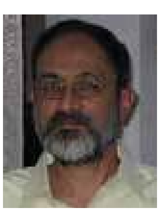
Try proving this for yourself. It is not too difficult! All you need is the following pair of results:

$$\sin 2x = 2 \cdot \sin x \, \cos x, \qquad \cos x = \sqrt{\frac{1+\cos 2x}{2}}.$$

Closing remark. We have seen a few occurrences of $\sqrt{2}$ in this brief article. There are many, many more such sightings of this number in the world of mathematics but we shall leave the task of uncovering them to you.

References

1. Mike Lombardi, "Why 7's been a lucky number", https://www.boeing.com/news/frontiers/archive/2004/february/i_history.html



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A NOTE OF APOLOGY

An important source from an article in Resonance, April 2009, titled Perfect Medians, Euler and Ramanujan by Dr A K Mallik, Department of Mechanical Engineering, Indian Institute of Technology, Kanpur [Email: akmallik@iitk.ac.in] which was used as a reference for the article Ramanujan And Some Elementary Mathematical Problems by Utpal Mukhopadhyay, published in the March 2018 issue of At Right Angles, was inadvertently omitted in the list of references cited.

We apologise to Dr Mallik for the omission.

Features

Addendum to Power Triangle

V G TIKEKAR

In Part II of this article [1], we had presented a unified approach by which, for any given positive integer k, the formula for the sum of the k-th powers of the first n natural numbers can be obtained. The method made use of a triangular arrangement of numbers called the *Power Triangle*. Its first few rows are given in Figure 1.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	3	2			
$n=3$	1	7	12	6		
$n=4$	1	15	50	60	24	
$n=5$	1	31	180	390	360	120

Figure 1. The first few rows of the Power Triangle

Here are the rules governing the formation of the Power Triangle. Denote the number in row n and column r by $T(n, r)$; here $n = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots, n + 1$. Then:

Rule 1: Row n has $n + 1$ numbers, $T(n, 1), T(n, 2), T(n, 3), \ldots, T(n, n + 1)$. We adopt the convention that $T(n, r) = 0$ if $r < 1$ or if $r > n + 1$. (In words: if the element at any position is absent, it is taken to be 0.)

Rule 2: The first number of every row is 1; so $T(n, 1) = 1$ for $n = 0,1,2, \ldots$.

Rule 3: The numbers in the successive rows of the power triangle are determined recursively as follows: for $n = 1,2,3\ldots$ and $r = 1,2,3, \ldots, n + 1$,

$$T(n, r) = (r-1) \cdot T(n-1, r-1) + r \cdot T(n-1, r). \qquad \text{(I)}$$

Keywords: Pascal triangle, power triangle, sums of powers

The Power Triangle with subscripts. In its original form, the Power Triangle had been presented using **subscripts**. See Figure 2; observe that each entry has a subscript which is identical to the r-value of its column. So this brief addendum is being offered for the sake of historical correctness.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=0$	1_1					
$n=1$	1_1	1_2				
$n=2$	1_1	3_2	2_3			
$n=3$	1_1	7_2	12_3	6_4		
$n=4$	1_1	15_2	50_3	60_4	24_5	
$n=5$	1_1	31_2	180_3	390_4	360_5	120_6

Figure 2. The first few rows of the Power Triangle, using subscripts

The law of formation of the Power Triangle when presented in this form is the following. The zeroth row has a single entry: 1, with subscript 1. For subsequent rows, the first entry (corresponding to $r = 1$) is 1, with subscript 1. Each subsequent entry (i.e., corresponding to $r > 1$) is given by the following sum, with the understanding that an empty space means that the corresponding entry is 0: *number in row n, column r is equal to the number in row n−1, column r times its subscript plus the number in row n−1, column r−1 times its subscript*. Note that this is simply the following rule expressed in words:

$$T(n, r) = (r-1)\cdot T(n-1, r-1) + r\cdot T(n-1, r).$$

We have already explained how the Power Triangle is used to get the desired formulas; so we do not elaborate on that now. All we need is this formula:

$$1^k + 2^k + \cdots + n^k = \sum_{r=1}^{k+1} \binom{n}{r}\cdot T\left(k, r\right).$$

References

1. V G Tikekar, "On the sums of powers of natural numbers, Part II", *At Right Angles*, March 2018, http://azimpremjiuniversity.edu.in/SitePages/ resources-ara-march-2018-sums-of-powers.aspx

PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1995. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.

Bisecting an Angle Using a Ruler

KASI RAO JAGATHAPU

Here is a simple way to bisect a given angle using the simplest and most familiar geometrical instrument: a ruler (see Figure 1). It works for any angle other than a straight angle (i.e., a 180º angle).

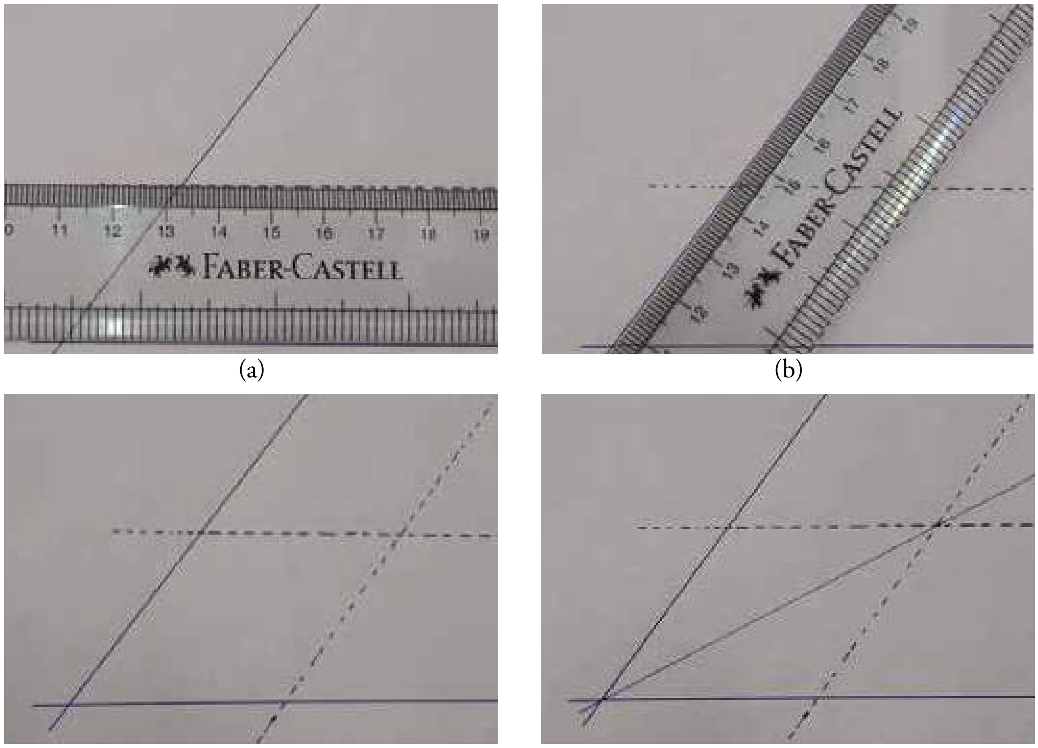
Figure 1.

While the procedure is simple to carry out, justifying it using the theorems of geometry may prove challenging to some students.

Here is the method. Given $\measuredangle ABC$ with vertex B, place the ruler so that one edge is aligned along ray BA and the opposite edge overlaps with the given angle. Using a pencil, draw a line m along the other edge; see Figure 2 (a). (For the vertex and line labels, please look at Figure 3; it depicts the construction schematically.) Next, place the ruler so that one edge is aligned along ray BC and the other edge overlaps with the given angle; draw a line n along

Keywords: Angle, bisection, ruler

the other edge; see Figure 2 (b). Let the lines m and n thus drawn intersect at point D. Then ray $\overrightarrow{BD}$ is the required bisector of $\measuredangle ABC$.



(a) (b) (c) (d)

Figure 2.

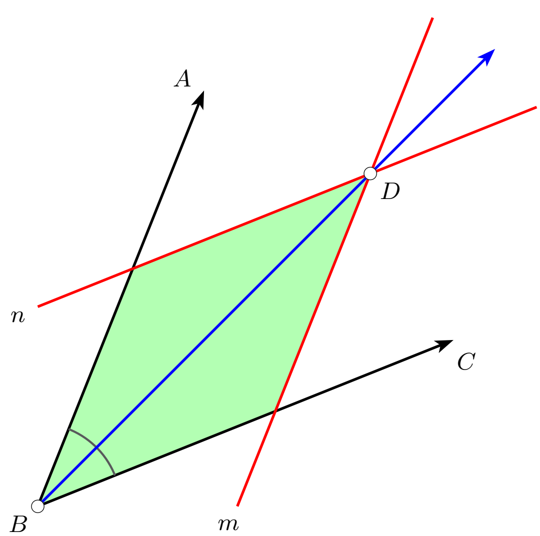
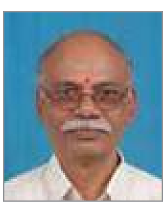


Figure 3. Schematic depiction of the procedure

The justification that the procedure works correctly is left as an exercise. A crucial geometrical fact that it draws on is this: *If two identical rulers are placed across each other, then the region overlapped by the two rulers is a rhombus* (see Figure 4).

Figure 4. A rhombus!



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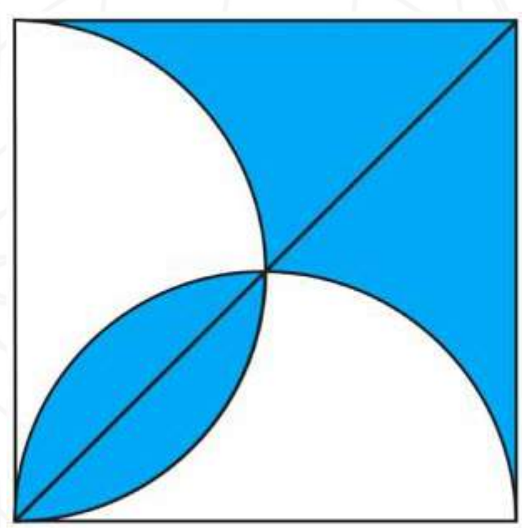
In the square below, two semicircles are overlapping in a symmetrical pattern. Which is greater: the area shaded blue or the area shaded white?



Extending the Definitions of GCD and LCM to Fractions

SHAILESH SHIRALI

With the LCM and GCD of natural numbers well-defined and an integral part of the middle school curriculum, one may wonder why this article embarks on a rather theoretical study of the LCM and GCD of rational numbers. But this article depicts exactly what a mathematician does – take a well-known concept and extend it to larger sets, testing the extended definition with backward compatibility with the original set. For the more able middle-schooler, this is an excellent opportunity to flex the muscles of conceptual understanding and constructive reasoning. Dive in!

It happens quite frequently in mathematics that we need to extend the definition of a mathematical concept to cover a larger domain than the one on which the concept was originally defined. Historically, such a progression is part of the very evolution of mathematics.

To give a simple example, consider the notion of sine and cosine of an angle. These notions arise from the consideration of right-angled triangles. If we stick strictly to the original definition, then it becomes absurd to talk of the sine and cosine of an obtuse angle. But one can easily extend the domains of definition of these functions to cover angles of arbitrary measure by considering, instead of a right-angled triangle, a circle of unit radius centred at the origin of a rectangular coordinate plane.

In such cases, it becomes imperative to check for *backward compatibility* (to borrow a term used more often in relation to software packages as they evolve and grow over time). That is, we must verify that the extended definition reduces to the original definition when considered over the original (reduced) domain. It is easy to verify that backward compatibility does hold in the case of the trigonometric functions.

And just as we have the notion of 'vibration testing' in engineering, in which we test the response of a newly engineered device in a stressful vibration environment to ascertain its points of weakness, so must we subject an extended definition to tests to ascertain *its* points of weakness.

Keywords: Definition, GCD, LCM, domain

We consider one such case here. We ask:

Is it possible to extend the definitions of GCD *and* LCM *to the rational numbers?*

In their current form, these functions are defined only for pairs of integers (not both zero). We recall their definitions here.

- Let a, b be integers, not both equal to 0 (note that $\text{GCD}(0,0)$ is not defined). Then:
  - → We define $\text{GCD}(a,0)$ to be $|a|$, provided that $a \neq 0$.
  - → If $a \neq 0$ and $b \neq 0$, then $\text{GCD}(a,b)$ is defined to be the largest positive integer c such that a/c and b/c are integers. (Note that the two conditions "$a \neq 0$ and $b \neq 0$" can be captured as a single condition by writing: "if $ab \neq 0$".)

  The definition will always yield a number satisfying the stated conditions since the set of positive integers c such that a/c and b/c are integers is nonempty (1 belongs to this set) and finite (since $c \leq |a|$ and $c \leq |b|$).
- Let $a \neq 0,\ b \neq 0$ be integers. We define $\text{LCM}(a,b)$ to be the smallest positive integer c such that c/a and c/b are integers. The definition will always yield a number satisfying the stated conditions since the set of positive integers c such that c/a and c/b are integers is nonempty (since ab belongs to this set).

The GCD can be efficiently computed using Euclid's division algorithm, and the LCM can then be computed using the relation

$$\text{GCD}(a,b) \times \text{LCM}(a,b) = ab, \tag{1}$$

which is true for all $a \neq 0,\ b \neq 0$.

Can we extend these definitions to cover rational numbers as well, keeping in mind the comments made earlier? Let us apply commonsense logic and see where it leads us.

Greatest common divisor of two positive rational numbers

We do not ordinarily use the term 'divisor' and 'multiple' in connection with non-integral rational numbers; so it is best to be clear at the start as to what these notions mean. We adopt the simplest approach here; the two terms are assumed to mean exactly the same thing as what they mean when used with reference to integers. So, if r and s are non-zero rational numbers, we say that r is a divisor of s if s/r is an integer; and in this situation we also say that s is a multiple of r. For example, $1/6$ is a divisor of $2/3$ (for $2/3 \div 1/6 = 4$, an integer) and $2/3$ is a multiple of $1/6$.

Study of a particular case. To start with, let us try to find the GCD of a specific pair of fractions, say $15/4$ and $9/14$. Note that both the fractions have been given in their lowest terms. Suppose that the required GCD is $1/m$ times the first fraction and also $1/n$ times the second fraction, where m and n are positive integers. By the definition of greatest common divisor, m and n cannot have any divisors in common other than 1; i.e., $\text{GCD}(m,n) = 1$. So we have:

$$\frac{15}{4m} = \frac{9}{14n} \qquad \text{GCD}(m,n) = 1.$$

Cancelling common factors and simplifying, we get

$$\frac{m}{n} = \frac{35}{6}, \qquad \text{GCD}(m,n) = 1.$$

Since 35 and 6 have no factors in common other than 1, the relation $m/n = 35/6$ tells us that m is a multiple of 35 and n is a multiple of 6. And since $\text{GCD}(m, n) = 1$, it must be that $m = 35$ and $n = 6$. Hence the GCD of the two fractions is equal to

$$\frac{15}{4 \times 35} = \frac{3}{28} = \frac{9}{14 \times 6},$$

i.e., the GCD is equal to $3/28$. Now the numerator of $3/28$ is 3, which is equal to $\text{GCD}(15, 9)$, and the denominator of $3/28$ is 28, which is equal to $\text{LCM}(4, 14)$. So our reasoning has led us to the following:

$$\text{GCD}\left(\frac{15}{4}, \frac{9}{14}\right) = \frac{\text{GCD}(15, 9)}{\text{LCM}(4, 14)}.$$

The general case. Will this reasoning work in general? Let us apply the same reasoning to the pair of fractions a/b and c/d; here a, b, c, d are positive integers with $\text{GCD}(a, b) = 1$ and $\text{GCD}(c, d) = 1$. Suppose that the required GCD is $1/m$ times the first fraction and also $1/n$ times the second fraction, where m and n are positive integers with no divisors in common other than 1; i.e., $\text{GCD}(m, n) = 1$. So we have:

$$\frac{a}{bm} = \frac{c}{dn}, \qquad \text{GCD}(m, n) = 1,$$

$$\therefore \quad \frac{m}{n} = \frac{ad}{bc}, \qquad \text{GCD}(m, n) = 1.$$

By assumption, $\text{GCD}(a, b) = 1 = \text{GCD}(c, d)$. However, a and c may have common divisors other than 1; likewise for b and d. Let $\text{GCD}(a, c) = u$ and $\text{GCD}(b, d) = v$. We have, then:

$$\frac{ad}{bc} = \frac{a}{c} \times \frac{d}{b} = \frac{a/u}{c/u} \times \frac{d/v}{b/v} = \frac{(ad)/(uv)}{(bc)/(uv)}.$$

Hence:

$$\frac{m}{n} = \frac{(ad)/(uv)}{(bc)/(uv)},$$

and since $\dfrac{ad}{uv}$ and $\dfrac{bc}{uv}$ can have no factors in common, it must be that

$$m = \frac{ad}{uv}, \qquad n = \frac{bc}{uv}.$$

Hence the required GCD is

$$\frac{a}{b \times m} = \frac{uv}{bd} = \frac{c}{d \times n}.$$

Let us look more closely at the fraction in the middle; it can be written as:

$$\frac{uv}{bd} = \frac{u}{bd/v}.$$

The numerator of the fraction on the right side is $\text{GCD}(a, c)$. The denominator of the fraction is

$$\frac{bd}{v} = \frac{b \times d}{\text{GCD}(b, d)} = \text{LCM}(b, d).$$

We see, therefore, that

$$\text{GCD}\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{GCD}(a, c)}{\text{LCM}(b, d)}.$$

Least common multiple of two positive rational numbers

Study of a particular case. To start with, let us try to find the LCM of the same pair of fractions we studied earlier, 15/4 and 9/14. Suppose that the required GCD is m times the first fraction and also n times the second fraction, where m and n are positive integers. By the definition of least common multiple, m and n cannot have any divisors in common other than 1; i.e., GCD $(m, n) = 1$. So we have:

$$\frac{15m}{4} = \frac{9n}{14}, \quad \text{GCD}\,(m, n) = 1.$$

Cancelling common factors and simplifying, we get

$$\frac{m}{n} = \frac{6}{35}, \quad \text{GCD}\,(m, n) = 1.$$

Since 6 and 35 have no factors in common other than 1, the relation $m/n = 6/35$ tells us that m is a multiple of 6 and n is a multiple of 35. And since GCD $(m, n) = 1$, it must be that $m = 6$ and $n = 35$. Hence the LCM of the two fractions is equal to

$$\frac{15 \times 6}{4} = \frac{45}{2} = \frac{9 \times 35}{14},$$

i.e., the GCD is equal to 45/2. Now the numerator of 45/2 is 45, which is equal to LCM $(15, 9)$, and the denominator of 45/2 is 2, which is equal to GCD $(4, 14)$. So our reasoning has led us to the following:

$$\text{LCM}\left(\frac{15}{4}, \frac{9}{14}\right) = \frac{\text{LCM}\,(15, 9)}{\text{GCD}\,(4, 14)}.$$

The general case. Just as we did earlier, let us apply the same reasoning to the pair of fractions a/b and c/d; here a, b, c, d are positive integers with GCD $(a, b) = 1$ and GCD $(c, d) = 1$. Suppose that the required GCD is m times the first fraction and also n times the second fraction, where m and n are positive integers with no divisors in common other than 1; i.e., GCD $(m, n) = 1$. So we have:

$$\frac{am}{b} = \frac{cn}{d}, \quad \text{GCD}\,(m, n) = 1,$$

$$\therefore \quad \frac{m}{n} = \frac{bc}{ad}, \quad \text{GCD}\,(m, n) = 1.$$

Let GCD $(a, c) = u$ and GCD $(b, d) = v$. We have, then:

$$\frac{bc}{ad} = \frac{b}{d} \times \frac{c}{a} = \frac{b/v}{d/v} \times \frac{c/u}{a/u} = \frac{(bc)/(uv)}{(ad)/(uv)}.$$

Hence:

$$\frac{m}{n} = \frac{(bc)/(uv)}{(ad)/(uv)},$$

and since $\frac{bc}{uv}$ and $\frac{ad}{uv}$ have no factors in common, it must be that

$$m = \frac{bc}{uv}, \quad n = \frac{ad}{uv}.$$

Hence the required GCD is

$$\frac{a \times m}{b} = \frac{ac}{uv} = \frac{c \times n}{d}.$$

Let us look more closely at the fraction in the middle; it can be written as:

$$\frac{ac}{uv} = \frac{ac/u}{v}.$$

The numerator of the fraction on the right side is $ac/\text{GCD}(a, c) = \text{LCM}(a, c)$. The denominator of the fraction is $v = \text{GCD}(b, c)$. We see, therefore, that

$$\text{LCM}\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{LCM}(a, c)}{\text{GCD}(b, d)}.$$

Subjecting the extended definition to stress tests…

First test. Let us subject our formula to the simplest possible stress test, that of backward compatibility. That is, let us see if the formula reduces to the known formula for GCD if the rational numbers under consideration happen to be positive integers, i.e., with unit denominator.

In the statement,

$$\text{GCD}\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{GCD}(a, c)}{\text{LCM}(b, d)},$$

let $b = 1$, $d = 1$. The result then assumes the following form:

$$\text{GCD}(a, c) = \frac{\text{GCD}(a, c)}{\text{LCM}(1, 1)}.$$

But $\text{LCM}(1, 1) = 1$, so the statement assumes the form $\text{GCD}(a, b) = \text{GCD}(a, b)$, which is vacuously true. So the test of backward compatibility has been passed (though in a rather trivial manner).

Second test. We know that if a, b are positive integers, then

$$\text{GCD}(a, b) \times \text{LCM}(a, b) = ab.$$

Let us check whether such a relation holds for the GCD and the LCM of two positive rational numbers. Let the two positive rational numbers be a/b and c/d; here a, b, c, d are positive integers with $\text{GCD}(a, b) = 1$ and $\text{GCD}(c, d) = 1$. Then we have:

$$\begin{aligned}\text{GCD}\left(\frac{a}{b}, \frac{c}{d}\right) \times \text{LCM}\left(\frac{a}{b}, \frac{c}{d}\right) &= \frac{\text{GCD}(a, c)}{\text{LCM}(b, d)} \times \frac{\text{LCM}(a, c)}{\text{GCD}(b, d)} \\ &= \frac{\text{GCD}(a, c) \times \text{LCM}(a, c)}{\text{GCD}(b, d) \times \text{LCM}(b, d)} \\ &= \frac{ac}{bd} = \frac{a}{b} \times \frac{c}{d}.\end{aligned}$$

We see that the relation does hold in the domain of rational numbers. Our extended definition has just passed an important test!

Third test. We know that if a, b are positive integers, then there exist integers x and y such that

$$ax + by = \text{GCD}(a, b).$$

Let us check whether such a relation holds for the GCD of two positive rational numbers. Let the two positive rational numbers be a/b and c/d; here a, b, c, d are positive integers with $\text{GCD}(a, b) = 1$ and $\text{GCD}(c, d) = 1$. We must examine whether there exist integers x and y such that

$$\frac{ax}{b} + \frac{cy}{d} = \frac{\text{GCD}(a, c)}{\text{LCM}(b, d)}.$$

Multiplying both sides of the above relation by bd, we see that the question reduces to asking whether there exist integers x and y such that

$$ad \cdot x + bc \cdot y = \text{GCD}(a, c) \cdot \text{GCD}(b, d).$$

Since the following relation is clearly true,

$$\text{GCD}(ad, bc) = \text{GCD}(a, c) \cdot \text{GCD}(b, d),$$

it follows that there must indeed exist integers x and y such that

$$ad \cdot x + bc \cdot y = \text{GCD}(a, c) \cdot \text{GCD}(b, d),$$

and, therefore, that there must indeed exist integers x and y such that

$$\frac{ax}{b} + \frac{cy}{d} = \frac{\text{GCD}(a, c)}{\text{LCM}(b, d)}.$$

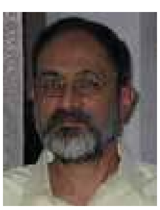
So the same kind of relation holds in the domain of rational numbers. Our extended definition has now passed a third important test!

It is impressive to see how far commonsense logic has led us. We were true to it and it has not let us down!

References

1. Wikipedia, "Greatest common divisor", https://en.wikipedia.org/wiki/Greatest_common_divisor.

2. Wikipedia, "Least common multiple", https://en.wikipedia.org/wiki/Least_common_multiple.



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Triangle Inequality - A Curious Counting Result

RAJAGOPALAN RVS

Counting and the study of mathematics go a long way back. In this article, geometry and counting come together to provide an interesting window to mathematical thinking and reasoning. Geometrical constructions provide a hands-on aspect and teachers of classes 6-10 can use this article to design GeoGebra investigations, mathematical discussions with trigger questions or even an unusual revision worksheet.

Geometric constructions using ruler-and-compass can be a rich learning experience in middle school and high school. Properties of parallel lines, triangles, various types of quadrilaterals and circles can be reinforced using constructions of geometrical figures, giving different combinations of parameters such as angles, lengths of sides, medians, altitudes, sums of sides, etc. Ideas from the locus concept can also be used to create interesting problems (e.g., [1]).

In the middle school, the triangle inequality can be appreciated by following a graded sequence of construction problems. Once they feel comfortable with the task of constructing triangles given the lengths of the sides or the measures of the angles (in different combinations), students can be asked whether it is always possible to construct a triangle given the lengths of the sides; e.g., "Is it possible to construct a triangle with sides of lengths 5 cm, 6 cm, 12 cm?" The students soon find that after drawing one side, say the one with length 5 cm, while trying to construct the remaining two sides, the relevant arcs do not intersect, so it is not possible to construct the triangle.

Keywords: Triangle, triangle inequality, side, length

From this, they can be guided to explore the triangle inequality. Let lengths a, b be given, with $b < a$; we wish to draw a triangle ABC in which BC has length a and AC has length b. Draw a segment BC with length a (Figure 1). With C as centre, draw a circle Γ with b as radius. Let U, V be the points where the circle intersects line BC, with U between B and C. Then $BU = a - b$ and $BV = a + b$. Vertex A clearly must lie somewhere on Γ (except points U and V). It should be clear that the third side of the triangle cannot be shorter than BU, nor can it be longer than BV. That is, the length c of the third side of the triangle must lie between $a - b$ and $a + b$, i.e., $c > a - b$ and $c < a + b$. Since there are infinitely many numbers between $a - b$ and $a + b$, the number of distinct triangles with two sides as a and b is infinite.

A remarkable fact emerges if we restrict the lengths of the sides of the triangle to integer values. For example, say we want to find the number of integer-sided triangles in which two sides have lengths 11 cm and 7 cm. The length of the third side must lie between 11 – 7 = 4 cm and 11 + 7 = 18 cm, and the lengths 4 and 18 are not possible, so the possibilities for the sides are (11, 7, 5), (11, 7, 6), (11, 7, 7), . . . , (11, 7, 16), (11, 7, 17). The total number of possibilities is 13 (= 17 – 5 + 1).

If the two sides have lengths 14 and 6, then the possibilities are (14, 6, 9), (14, 6, 10), (14, 6, 11), (14, 6, 12), . . . , (14, 6, 17), (14, 6, 18), (14, 6, 19); the number of possibilities is 11 (= 19 – 9 + 1).

Now consider the general case of an integer-sided triangle. Let two of the sides have specified integer lengths a, b, where $a > b$. Let c be the length of the third side (c too is an integer). Then the least possible value of c is $a - b + 1$, and the largest possible value of c is $a + b - 1$. Thus, c can take all integer values from $a - b + 1$ to $a + b - 1$. Now the number of integers from $a - b + 1$ to $a + b - 1$ is

$$(a + b - 1) - (a - b + 1) + 1 = 2b - 1.$$

So it is possible to construct precisely $2b - 1$ different integer-sided triangles with a and b as the lengths of two of its sides ($a > b$).

The striking fact here is that ***the number of different triangles depends only on the length of the smaller side***. So the number of integer-sided triangles with 11, 7 as the lengths of two sides is exactly the same as the number of integer-sided triangles with 1234,7 as the lengths of two sides, or with 157869,7 as the lengths of two sides; the number in each case is 13. All the possible integer-sided triangles with two sides having lengths 11 cm and 7 cm are drawn in Figure 2 (all with base BC). This yields a nice construction pattern!

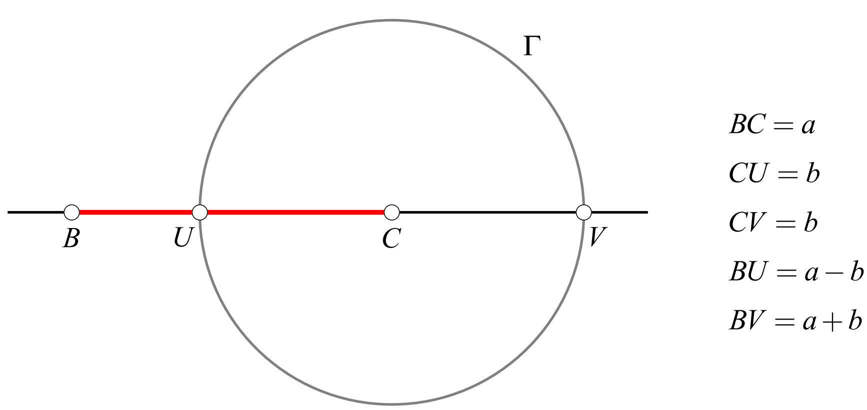

Figure 1. Constructing a triangle given the lengths of two of its sides

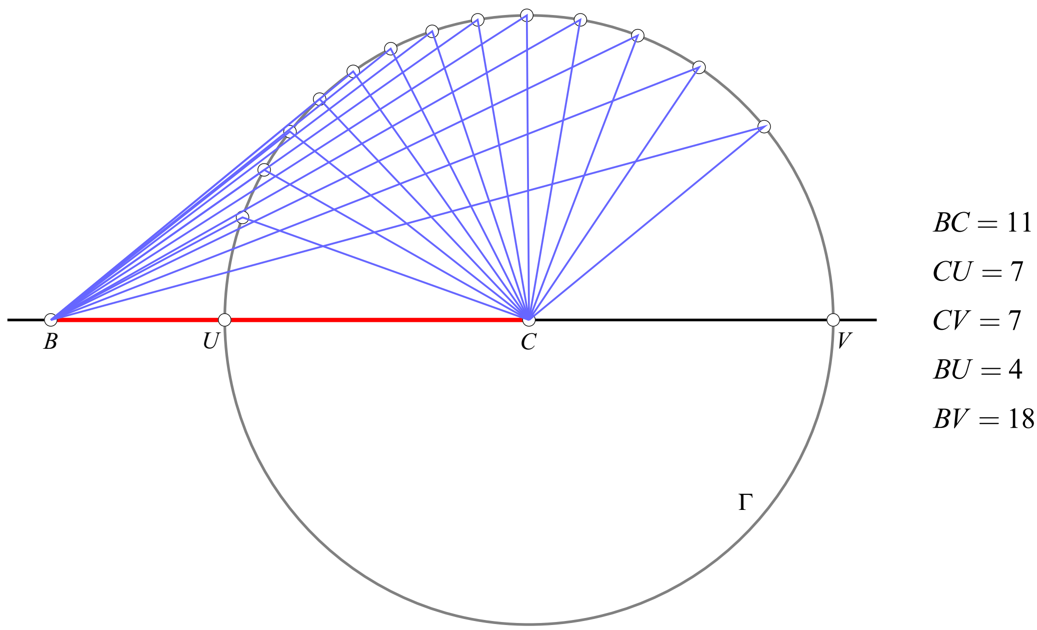

Figure 2. 13 different integer-sided triangles with 11 and 7 as the lengths of two of its sides

References

1. Sneha Titus & R Athmaraman, "Problems of the Middle School", http://azimpremjiuniversity.edu.in/SitePages/ resources-ara-march-2018-problems-of-middle-school.aspx.

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Low Floor High Ceiling

Four-telling Quadrilaterals

SWATI SIRCAR &
SNEHA TITUS

A mathematical investigation is an exciting way for students to learn to develop systematic reasoning and mathematical rigor in a relaxed and easy going manner. Setting up an investigation which achieves this objective is, however, not always easy. In addition, the teacher needs to facilitate the proceedings in such a manner that the student takes ownership of the investigation and develops these skills in a natural manner.

As a natural part of their everyday mathematics work, investigations help students:

- Explore problems in depth
- Find more than one way to solve many of the problems they encounter
- Reason mathematically and develop problem-solving strategies
- Examine and explain mathematical thinking and reasoning
- Communicate their ideas orally and on paper, using "clear and concise" notation
- Represent their thinking using models, diagrams and graphs
- Make connections between mathematical ideas
- Prove their ideas to others
- Develop computational skills – efficiency, accuracy and flexibility
- Choose from a variety of tools and appropriate technology
- Work in a variety of groupings – whole class, individually, in pairs, in small groups

Source: http://www.canalwinchesterschools.org/WTIMP.aspx

Keywords: Quadrilaterals, side lengths, angles, isosceles, trapeziums, cyclic, major segment, minor segment.

Clearly, these are objectives worth pursuing. Here is an investigation on types of quadrilaterals which, while starting with some simple hands-on activities and documentation of findings, ramps up to a conjecture and finally a proof about isosceles trapeziums. At several points, there are potential **I**nvestigation **Q**uestions which students can diverge to; however, in order to keep a somewhat linear flow, we have indicated these with a ***IQ#**. The interested teacher or student can design several new investigations based on these suggestions.

Part I: Classification of Quadrilaterals

Just as there is a triangle rule, according to which the length of any side of a triangle is greater than the difference of the lengths of the other two sides and less than the sum of the lengths of the other two sides, we have an equivalent quadrilateral rule. Here, the length of any side of a quadrilateral is less than the sum of the lengths of the other three sides.

1. Keeping this rule in mind, how many kinds of quadrilaterals can you make

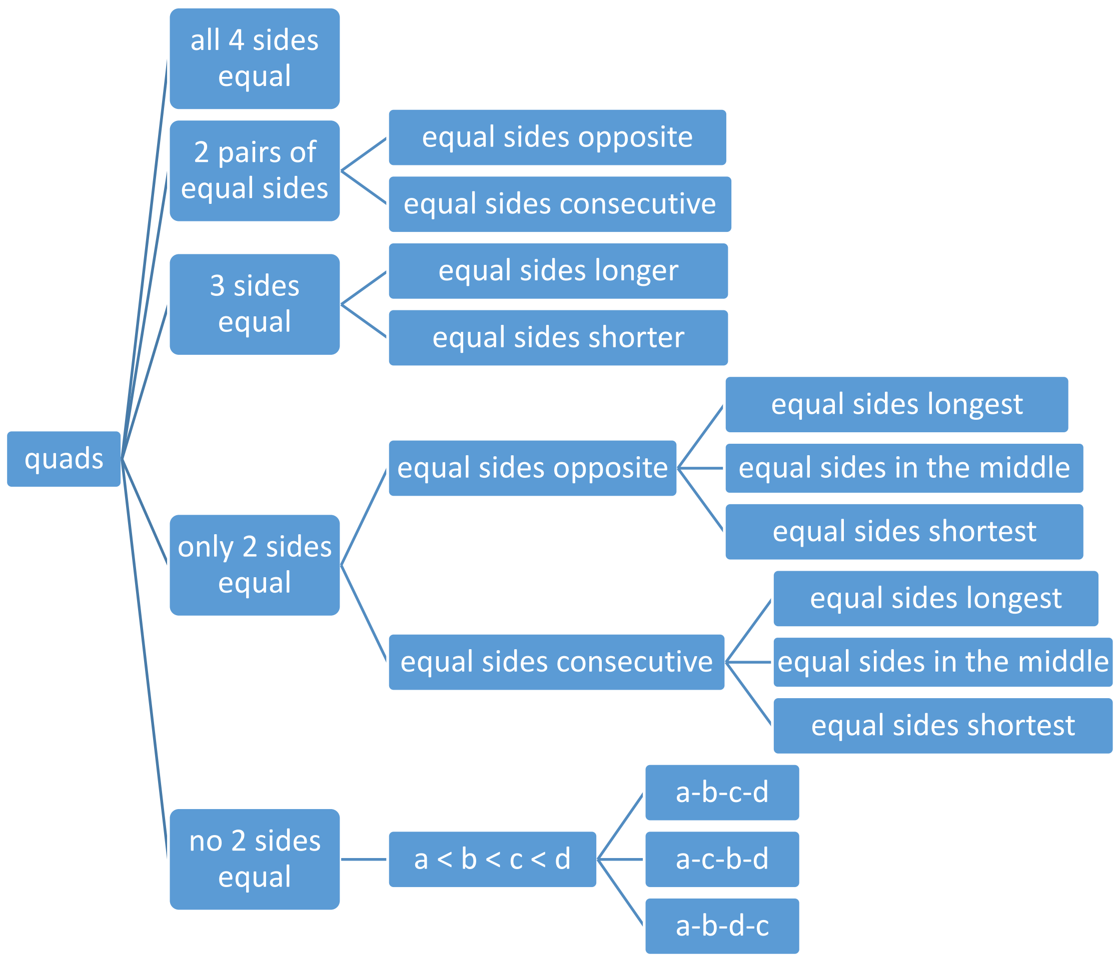
a. With 4 matchsticks **b.** With 8 matchsticks **c.** With 12 matchsticks **d.** With 16 matchsticks

Compile your results in a table as follows (some results have been given, a selection may be provided for students depending on their proficiency in this activity).

***IQ1:** While we have given only feasible quadrilaterals, it may be worthwhile to do a whole class investigation of all the possible side lengths which add up to 4, 8 ,12 or 16 and select those combinations which represent sides of a quadrilateral; do encourage students to write up and share their reasoning process.

	Side combinations	Possible quads
4 matchsticks	1-1-1-1	Rhombi
8 matchsticks (investigate why 1-2-4-1 is not possible)	2-2-2-2	Rhombi
	1-3-1-3	Parallelograms
	1-1-3-3	Kites and darts
12 matchsticks	3-3-3-3	
		Parallelograms
	2-2-4-4	
	2-2-3-5	Isosceles trapezium and general quadrilaterals
		General quadrilaterals with a pair of consecutive equal sides
		General quads with 4 different sides
16 matchsticks		
	⋮	⋮

2. **Try to generalize the pattern and capture it in a tree diagram as follows:**



***IQ2:** Many investigation questions can be designed at this point, using each branch of this tree diagram. Students who know the basic definition of the different kinds of quadrilaterals can see which branches can generate feasible (and not just theoretical) quadrilaterals of a particular kind.

***IQ3:** This is also an opportunity for students to study line symmetry and to classify all these quadrilaterals as those with or without line symmetry.

***IQ4:** For those who want to play with concave and convex quadrilaterals, we suggest straw models to investigate which of these quadrilaterals can be concave and which convex. Instructions for making straw models are given in the box.

To make a straw model for any of the above quads, take two straws of roughly equal length and cut them into four pieces as per specification.

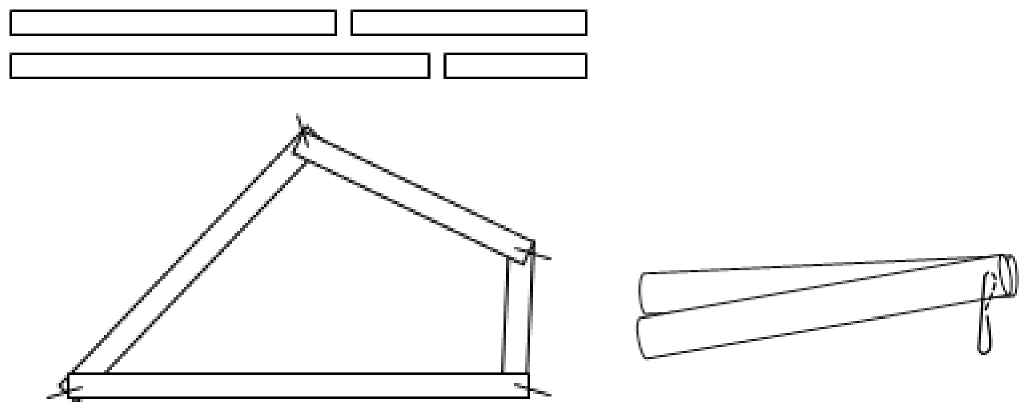


Figure 1.

For example, to make any of the quads with four different sides: cut one straw in roughly equal halves but not exactly so that one part is a bit longer than the other – these are b < c. Cut the second straw so that one part is much longer than the other– these are a < d.

The above cuts and the roughly equal length of the straws ensure that a < b < c < d. Now join them, with a stapler, in any of the given sequences to get the corresponding quad. While stapling, only one tooth of the pin should pass through the straw while the other will remain completely outside. After stapling, the straws should move freely forming a range of angles between them.

3. Which of the above branches can generate isosceles trapeziums?

Teacher's Note:
Students may need to refresh their definition of an isosceles trapezium: In Euclidean geometry, an **isosceles trapezoid** (**isosceles trapezium** in British English) is a convex quadrilateral with a line of symmetry bisecting one pair of opposite sides. It is a special case of a trapezoid. In any isosceles trapezoid two opposite sides (the bases) are parallel, and the two other sides (the legs) are of equal length (properties shared with the parallelogram). The diagonals are also of equal length. The base angles of an isosceles trapezoid are equal in measure (there are in fact two pairs of equal base angles, where one base angle is the supplementary angle of a base angle at the other base). https://en.wikipedia.org/wiki/Isosceles_trapezoid

***IQ5:** Mathematics, and geometry in particular, is rich in equivalent conditions as seen in the above, rather detailed, description. Isosceles trapeziums can provide an opportunity for students to investigate such equivalent conditions. Here is a sample question: 'If a quadrilateral has a line of symmetry bisecting a pair of opposite sides, what other properties does it have?'

***IQ6:** The converse can be more interesting. What is the converse of the above question? What are the minimum conditions to be specified to define an isosceles trapezium?

Apart from the quadrilaterals with four equal sides (in general, rhombi) and the ones with two pairs of equal and opposite sides (in general, parallelograms) in the tree diagram, only 5 types can yield isosceles trapeziums. These are:

i. 3 sides equal with equal sides longer, represented by b-b-b-a

ii. 3 sides equal with equal sides shorter, represented by a-a-a-b

iii. Only 2 sides equal and opposite and shortest, represented by a-b-a-c

iv. Only 2 sides equal and opposite and medium sized, b-a-b-c

v. Only 2 sides equal and opposite and longest, represented by c-a-c-b
Throughout we will use the notation that a < b < c

Part II: Moving to trapeziums via parallelograms.

1. **Suppose you are given both the lengths of the adjacent sides of a parallelogram. How many parallelograms can you construct?**

2. **Suppose you are given the lengths of all four sides of a trapezium, how many trapeziums can you draw? Investigate with the following example.**

Construct a trapezium ABCD with AB = 3cm, BC = 4cm, CD = 5cm, AD = 7cm. Assume AD || BC.

Hint:
Construct ΔCDE with CD = 5cm, CE = 3cm and DE = 3cm and extend DE to DA = 7 cm. Through C, draw CB = 4 cm. parallel to DA. How many such trapeziums can there be?

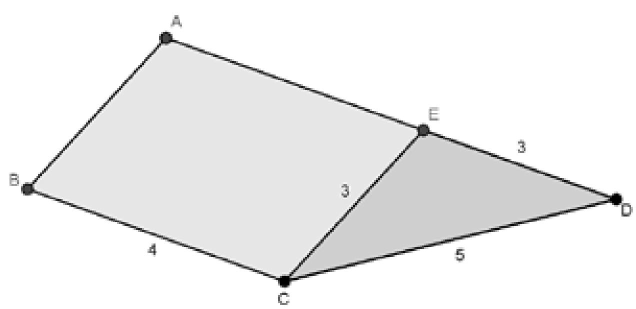


Figure 2. Trapezium with all four side lengths given

Teacher's Note:

Infinitely many parallelograms can be constructed since the angle between the given sides can be anything between 0° and 90°.

However, if the lengths of all 4 sides are given and the sides which are parallel are specified, then only one trapezium satisfies these constraints.

3. **Can you draw a quadrilateral ABCD with the same sides as given in 2 (AB = 3, BC = 4, CD =5, DA = 7) but with AB || CD? Justify.**

Teacher's Note:

Suppose it is possible. Then there is a point E on CD such that CE = AB = 3cm and ED = 2cm. Join AE. Now AB || CD ⇒ AB || CE and AB = CE ⇒ ABCE is a parallelogram ⇒ BC = AE = 4cm. Now in ΔADE, AD = 7cm < AE + ED = 4cm + 2cm = 6cm which is a contradiction! So it is impossible to draw the trapezium in this case.

4. **Now to generalise this: Consider trapezium ABCD such that BC || AD and BC < AD without loss of generality (Note that BC = AD implies ABCD is a parallelogram). Use the above split of trapezium ABCD into parallelogram ABCE and ΔCDE as mentioned in 2.**

Figure 3.

a. If BA ≤ CD without loss of generality, is BA + AD less than, equal to or greater than BC + CD?

b. If BC < AD, BA ≤ CD and BA + AD > BC + CD, is it possible to draw trapezium ABCD with BC || AD?

c. Given, BC < AD, BA ≤ CD and BA + AD > BC + CD, can you draw the trapezium with AB || CD?

d. What do you think happens if AB + BC = CD + DA?

Teacher's Note:

Using triangle inequality for the sides of ΔADE, where AE = BC and using the fact that ABCE is a parallelogram, it can be shown that CD < BA + AD – BC ⇒ BA + AD > BC + CD. Note that this is non-trivial since AD > BC but BA ≤ CD.

The converse is also true. The trapezium can be constructed for BC || AD (follow the hint given in 2) and not for AB || CD (check the Teacher's Note for 3). So for BC < AD and BA ≤ CD, BC || AD if and only if BA + AD > BC + CD.

When AB + BC = CD + DA, the triangle inequality collapses ΔADE into the line segment CD resulting in a collapse of parallelogram ABCE to the line segment BC + CD and we get a degenerate trapezium flattened to a line segment BD = BC + CD = BA + AD.

Part III: Focusing on the isosceles trapeziums.

1. **Consider any isosceles trapezium. Is it cyclic?**
2. **Consider any cyclic trapezium. Is it isosceles?**

Teacher's Note:

Suppose ABCD is an isosceles trapezium with BC || AD and BA = CD. Then ∠D = ∠A. [Hint: Split the trapezium into a parallelogram and a triangle to prove this.]

Assume BC < AD without loss of generality (BC = AD implies that ABCD is a rectangle which is cyclic). So ∠A + ∠C = ∠D + ∠C = 180° since BC || AD. So ∠B + ∠D = 360° – (∠A + ∠C) = 180° ⇒ ABCD is cyclic.

Alternatively, suppose ABCD is a cyclic trapezium i.e. BC || AD and ∠A + ∠C = ∠B + ∠D = 180°. ∠A + ∠C = 180° = ∠C + ∠D since BC || AD. So ∠A = ∠D ⇒ AB = CE = CD i.e. ABCD is isosceles. [The last step can be proved using the same hint as above.]

Therefore, a trapezium is isosceles if and only if it is cyclic.

***IQ7:** If I want to construct an isosceles trapezium what is the minimum amount of data that I need to have? Note: there are multiple possibilities e.g. (i) parallel sides and the distance between them, (ii) parallel sides and a base angle, etc. List as many possibilities as you can.

3. **Consider the isosceles trapezium a-c-b-c (remember that a < b < c). We have just seen that it will be cyclic. Is it possible to inscribe such a trapezium in the minor segment of a circle?**

4. **Determine if the isosceles trapezium a-b-b-b will always be in the major segment.**

Teacher's Note:

In ΔSQR (Figure 4), SR = b < QR = c ⇒ ∠SQR < ∠QSR ⇒ ∠SQR is acute ⇒ SPQR is a major arc

A similar proof can be devised for the trapezium a-b-b-b (Figure 5).

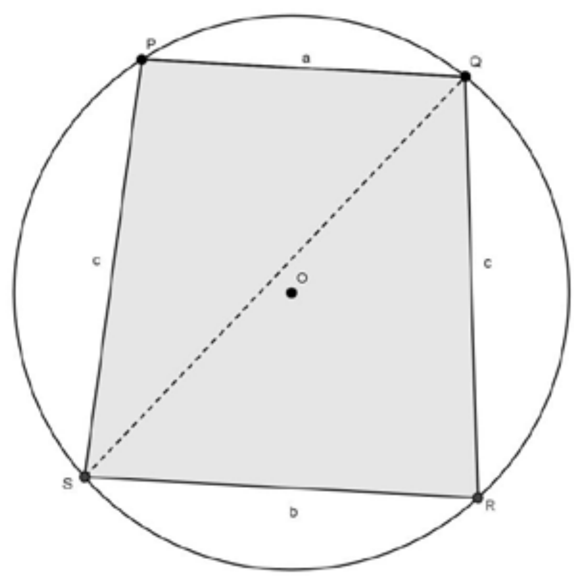


Figure 4. Isosceles trapezium a-c-b-c

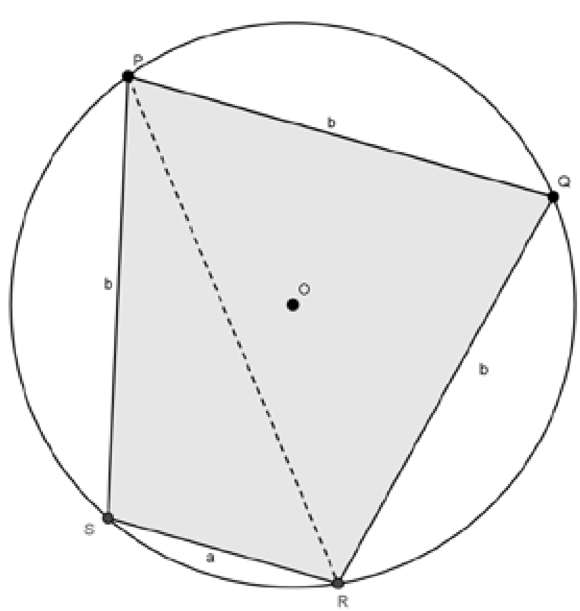


Figure 5. Isosceles trapezium a-b-b-b

5. Consider the isosceles trapezium a-a-a-b (remember $a < b$).

Note that $b < 3a$ to satisfy the quadrilateral inequality mentioned at the beginning of the article.

a. Suppose this trapezium has its longest side on the diameter. Prove that $b = 2a$.

b. Find the relation between a and b if this quadrilateral fits

   i. in a minor segment

   ii. in a major segment

Teacher's Note:

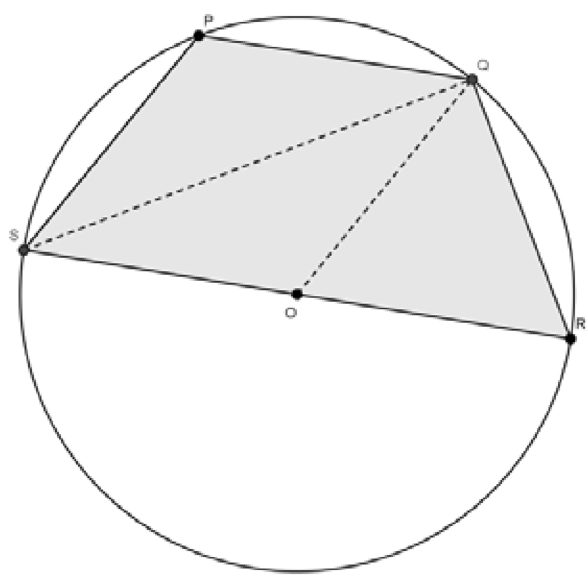


Figure 6. Isosceles trapezium in a semi-circle

$SP = PQ = QR = a$

$SR = b$

$\angle PQS = \angle QSR = \theta$ (alternate angles)

$\angle PQS = \angle PSQ = \theta$ (isosceles triangle)

$\therefore \angle PSR = 2\theta$

$\angle SQR = 90°$ (angle in semi-circle)

$\therefore \angle PQR = 90° + \theta = \angle QPS$

And $\angle PSR = \angle QRS =$

$90° - \theta = 2\theta$ (supplementary angles)

$\therefore 90° - \theta = 2\theta \Rightarrow \theta = 30°$

In ΔQOR, $\angle OQR = \angle QRO = 90° - \theta = 60°$

$\angle QOR = 2\theta = 60^0$

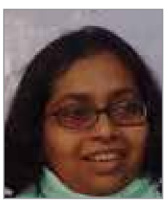
So ΔOQR is equilateral and $\therefore QR = a = OR = b/2$

If PQRS fits in the minor segment, then $\angle SQR > 90°$ $\Rightarrow$ with lengths SP = PQ = QR = a fixed, S and R should be further apart (resulting in a larger circumcircle) $\Rightarrow$ b > 2a and similarly if it fits in the major segment, then b < 2a.

Similar exploration can be done for the remaining two isosceles trapeziums. We will discuss those in a subsequent article.

Conclusion

Playing with matchsticks and straws can generate a lot of heat. Of the safe variety, thankfully. We hope that you will have fun investigating and writing up your discoveries. Do try these activities on dynamic geometry software such as GeoGebra. You are sure to make some exciting discoveries. And be sure to write them up!



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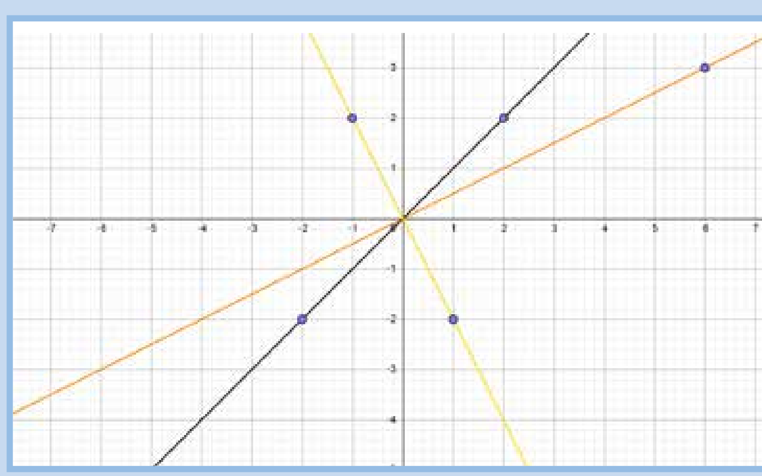
Sandeep Diwakar, from the Bhopal District Institute, Azim Premji Foundation, sent in the following question:

Can you draw a line through the origin (0, 0) such that it doesn't pass through any other lattice points? [lattice points = (m, n) such that m, n are integers]

To illustrate, in the diagram below, all the lines shown pass through at least one lattice point other than (0,0). Sandeep's question asks if it is possible to find a line which does not pass through any lattice point other than (0, 0).



Exploring Properties of Addition with Whole Numbers and Fractions

SWATI SIRCAR

This is inspired by the Pullout on Multiplication in At Right Angles, Vol 3 Issue 1 (http://teachersofindia.org/en/article/pullout-section-march-2014-teaching-multiplication) where Padmapriya Shirali mentioned how the commutative, associative and distributive properties of multiplication can be verified visually. What appealed to us was that the methods were free from computation and could be used or imagined for any combination of whole numbers no matter how large. We got interested in exploring the properties of addition in a similar manner. The basic processes for the commutative and associative properties of addition remain the same across the first three number sets, i.e., whole numbers, fractions and integers. In this article, we will discuss whole numbers and fractions. Usually these are not discussed in textbooks or classrooms. Even if they are mentioned at upper primary level, rarely any justification is given. Moreover they are often assumed. We felt that it was necessary for students to make sense of these properties and that visualization would be the ideal tool for this.

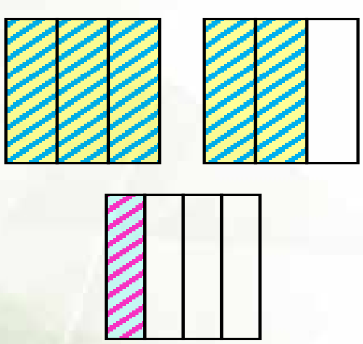


Figure 1.

We will be using two models: counters for whole numbers and unit square as the whole for fractions. So six will be represented by six counters and zero by the absence of any. $\frac{1}{4}$ will be represented by slicing a square into 4 equal vertical strips and shading 1 of them, and $\frac{5}{3}$ by slicing two squares in 3 strips each and shading 5 strips (Figure 1). The sums will be the total number of counters (for whole numbers) or the total shaded area (for fractions). Also for any sum $x + y$, x is shown on the left and y on the right.

Keywords: Whole numbers, fractions, Commutative Property, Associative Property

Commutative Property

The basic idea behind making sense of the commutative property is looking at a sum from two vantage points.

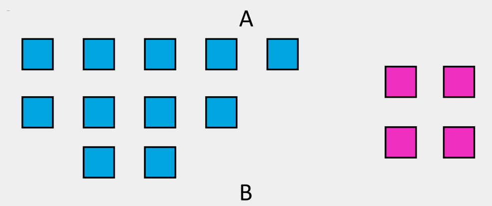


Figure 2.

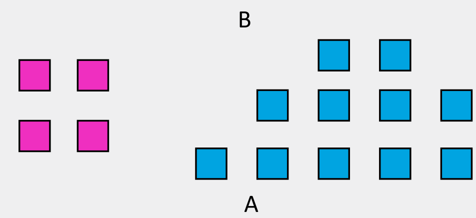


Figure 3.

So consider 11 + 4 as shown in Figure 2 (from point B) and observe how it looks from the point A. That is shown in Figure 3 and is nothing but a 180° rotated version of Figure 2. So from point B it is 11 + 4 and from point A it is 4 + 11.

With two groups of students situated at A and B recording their sums, it will make sense for them to conclude that 11 + 4 = 4 + 11, since view point does not change any of the concerned quantities. Note that this is true for any two whole numbers. It is easy to see that this holds even if one or both of the numbers are zero. For large numbers, say 100 or even 50, it may be cumbersome to arrange so many counters. So children should be encouraged to imagine the same for numbers as large as they can visualize.

When it comes to fractions, there are three possibilities:
1. Proper + proper **2.** Improper + proper (and ∴ proper + improper) **3.** Improper + improper

We will show an example of 2 and the other possibilities can be visualized in a similar way. Let us consider the sum as shown in Figures 4 and 5.

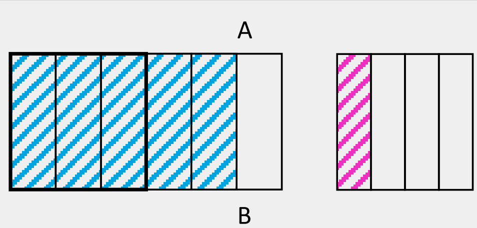


Figure 4.

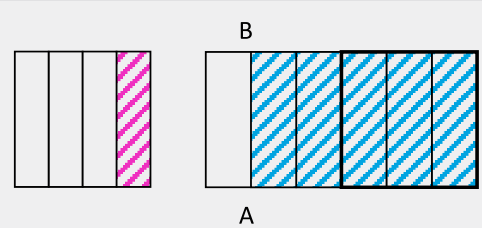


Figure 5.

Figure 4 is from B's perspective and it shows $\frac{5}{3}+\frac{1}{4}$ while Figure 5 shows the same sum from A's view point and it is $\frac{1}{4}+\frac{5}{3}$. The whole (unit square) is highlighted within the improper fraction. Again, since only the perspective changes, students may observe that the sums remain unchanged, i.e., $\frac{5}{3}+\frac{1}{4}=\frac{1}{4}+\frac{5}{3}$. Note that this holds for any two fractions. In fact, since any whole number can be represented by as many unit squares, this can also be extended to any sum involving a whole number and a fraction.

Associative Property

For this one, the basic idea is that if we have to add $x + y + z$, then it doesn't matter whether we combine first the x and y or y and z.

Figure 6.

For whole numbers, it may be best done as an activity. Pick any three whole numbers, say 7, 4 and 10 and represent them as piles of counters, i.e., the first pile with 7 counters, second pile with 4 counters, etc. (Figure 6). Now the sum 7 + 4 + 10 will be all the three piles combined into one. If only two piles can be combined at each step, then step 1 can be combining the first and second pile, i.e., the addition 7 + 4 and step 2 can be combining the third pile with this, i.e., adding 10 to the sum 7 + 4 or (7 + 4) + 10. On the other hand, step 1 can also be combining the second and third pile or 4 + 10 with step 2 as combining the first pile with this, i.e., 7 + (4 + 10). Both lead to the combination of all three piles into one with no extra counter coming in or going out. So the two sums can be justified to be equal, i.e., (7 + 4) + 10 = 7 + (4 + 10). Observe that the three whole numbers were chosen arbitrarily. For large numbers like 100, 1000, etc., if physical piles of counters are cumbersome to deal with, the process can be imagined. So this can be generalized for any three whole numbers.

For fractions, we can consider combining the shaded areas instead of piles of counters. There are eight possibilities:

1. All 3 proper

2. 2 proper and 1 improper

a. Proper + proper + improper
b. Proper + improper + proper
c. Improper + proper + proper

3. 1 proper and 2 improper

a. Proper + improper + improper
b. Improper + proper + improper
c. Improper + improper + proper

4. All 3 improper

We will show an example of 2c and leave the rest for the reader to explore.

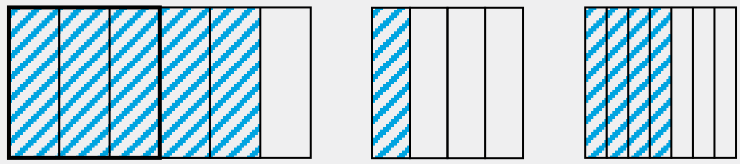
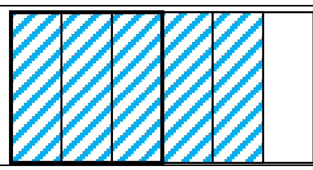
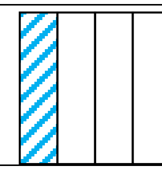
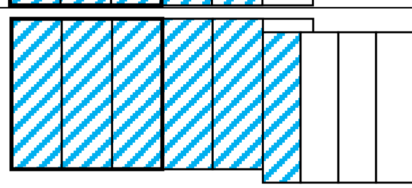
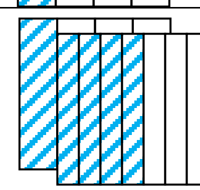
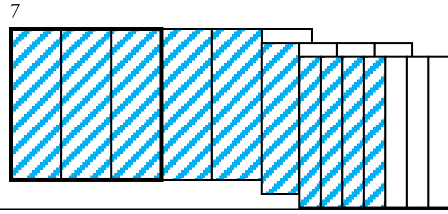
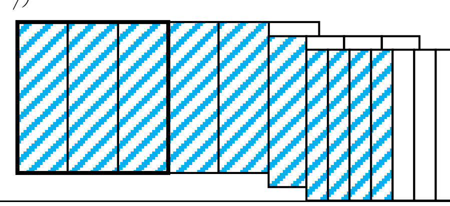


Figure 7.

Figure 7 represents the starting point of the example $\frac{5}{3}+\frac{1}{4}+\frac{4}{7}$.

We show step by step how the areas can be combined to show $\left(\frac{5}{3}+\frac{1}{4}\right)+\frac{4}{7}$ and $\frac{5}{3}+\left(\frac{1}{4}+\frac{4}{7}\right)$ respectively and that the end result is the same for both. So, $\left(\frac{5}{3}+\frac{1}{4}\right)+\frac{4}{7}=\frac{5}{3}+\left(\frac{1}{4}+\frac{4}{7}\right)$

Step 1	$\frac{5}{3}$ 	$\frac{1}{4}$ 
Step 2	$\frac{5}{3}+\frac{1}{4}$ 	$\frac{1}{4}+\frac{4}{7}$ 
Step 3	$\left(\frac{5}{3}+\frac{1}{4}\right)+\frac{4}{7}$ 	$\frac{5}{3}+\left(\frac{1}{4}+\frac{4}{7}\right)$ 

Note that this can be used for any three fractions and even a combination of fractions and whole numbers. Therefore this can be generalized for any combination of fractions and whole numbers. Also note that the total horizontal length of the shaded area is proportionate to the sum it represents. This can be used to show the sum on the number line. It is also a crucial step towards verifying these properties for rational and real numbers. *These will be discussed in a later article.*

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Visual Method for Fraction Multiplication

A generalised visual model for multiplication of fractions, based on the paper folding method, is presented.

SHAILAJA D SHARMA

Paper folding techniques have been successfully used to demonstrate multiplication of proper fractions in the classroom. This article may be used to make sense of the same techniques when applied to improper fractions. The problem at hand is to investigate how a product such as 3/2 x 4/3 may be demonstrated by paper folding.

A fraction is a ratio of two whole numbers. For the moment we consider only positive fractions; then they are formed as a ratio of two positive whole numbers and written in the form $\frac{a}{b}, b \neq 0$. Such a fraction is interpreted as follows:

Postulate 1: A collection of a equal-sized objects each of size $\frac{1}{b}$ units has a combined magnitude or size of $\frac{a}{b}$ units.

For example, a collection of 10 objects each of size $\frac{1}{3}$ metres has a total size of $\frac{10}{3}$ metres. A collection of 2 objects each of size $\frac{1}{5}$ square centimetres has size $\frac{2}{5}$ square centimetres. $\frac{22}{7}$ is a collection of 22 parts, each of which is equal to $\frac{1}{7}$ part of a defined object. If the defined object is a cord of length 1 metre, then we have 22 pieces of cord each of length $\frac{1}{7}$ metres, giving us $\frac{22}{7}$ metres in all.

What is $\frac{3}{4}$ of an apple? Here the defined object is the apple and we are talking about cutting the apple into 4 equal parts and taking 3 of those 4 parts. What is $\frac{5}{4}$ of an apple? Here, we start with 1 apple, divide it into 4 equal parts and add one more part equal to these 4 equal parts. Thus, in total we have 5 parts, each equal to $\frac{1}{4}$ of the original apple. We now have $\frac{5}{4}$ of an apple.

Keywords: fraction, proper, improper, multiplication, representation, visualisation

These concepts can be readily applied to extend fraction multiplication in the visual form by paper folding [1] to improper fractions. Consider a fraction multiplication of the form $\frac{a}{b} \times \frac{c}{d}$. For uniformity of treatment we shall always start with the second multiplicand, i.e., $\frac{c}{d}$. We shall depict $\frac{c}{d}$ visually, then formulate a procedure for finding $\frac{a \times c}{b \times d}$ visually and establish its equivalence with $\frac{a \times c}{b \times d}$ using Postulate 1.

Case 1: a < b, c < d

This case is already demonstrated [1]. In this case $\frac{a}{b}$ and $\frac{c}{d}$ are both proper fractions. A unit square is drawn and divided into d horizontal sections, of which c are selected. This selection represents the fraction $\frac{c}{d}$. The selected c horizontal sections are further divided into b vertical sections of which a are selected. The result of the cumulative selection process gives $a \times c$ cells, and the $b \times d$ partitions of the original unit square provide the size of each cell, viz. $\frac{1}{b \times d}$ units. Thus, the resulting magnitude, according to Postulate 1, is $\frac{a \times c}{b \times d}$ units $= \frac{\textit{no. of selected cells}}{\textit{no. of partitions of the unit square}}$ and represents the result of multiplication (Figure 1(i)).

$$Case\ 1 : \frac{a}{b} \times \frac{c}{d},\ a < b,\ c < d$$

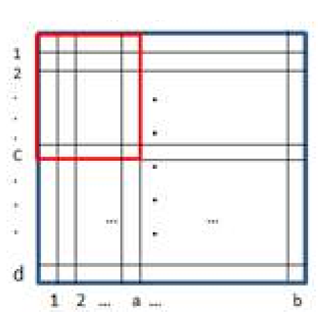

Figure 1(i). Multiplication of two proper fractions

Illustration 1: Consider the multiplication $\frac{1}{2} \times \frac{3}{4}$

Illustration 1: $\frac{1}{2} \times \frac{3}{4}$

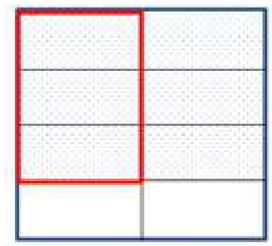

Figure 1(ii). The result is the ratio of the red to the blue areas, i.e. $\frac{3}{8}$

Take the unit square and divide it (or fold it) horizontally into 4 equal sections, thereafter select 3 adjacent sections. The selection is designated by the dotted portion in Figure 2(i). Next, divide the square vertically in two equal parts, and select one. The overlap of the selected areas is designated by the red outline Figure 1(ii). The original unit square is designated by the blue outline. The product is designated by the ratio of the red to the blue area. Each area is proportional to the number of sub-sections or tiles since they are equal-sized, and therefore the result is $\frac{3}{8}$. Note that the reference object shrinks at each step of this multiplication.

Case 2: a < b, c > d

In this case, the unit square has to be extended by as many sections as are required to obtain a total of c sections of size $\frac{1}{d}$ units each. That is to say, we divide the unit square into d equal horizontal sections, and then add $c - d$ sections of the same dimensions to the square, as shown in Figure 2(i). This enlarged rectangle now represents the improper fraction $\frac{c}{d}, c > d$. For taking $\frac{a}{b}, a < b$ part of this enlarged rectangle, the same is divided into b equal vertical sections, of which a are selected.

$$Case\ 2 : \frac{a}{b} \times \frac{c}{d}, a < b, c > d$$

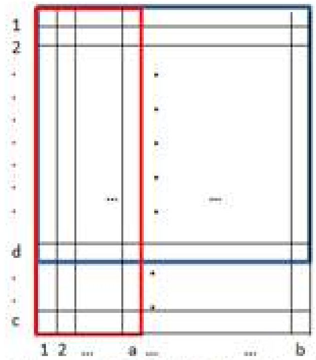

Figure 2(i). Multiplication of a proper and an improper fraction

Now, we have a total of $a \times c$ cells, all of which we have selected. The size of each cell is $\frac{1}{b \times d}$ units, since the unit square itself is now split into exactly $b \times d$ equal cells.

The result of multiplication is, as before

$$\frac{a \times c}{b \times d} \text{ units} = \frac{no.\ of\ selected\ cells}{no.\ of\ partitions\ of\ the\ unit\ square}$$

Illustration 2: Consider the multiplication $\frac{1}{2} \times \frac{4}{3}$

Illustration 2: $\frac{1}{2} \times \frac{4}{3}$

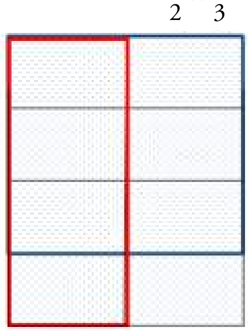

Figure 2(ii). The result is the ratio of the red to the blue areas, i.e. $\frac{4}{6}$

Take the unit square and divide it horizontally into 3 equal sections, thereafter append one more section to it as in Figure 2(ii). All 4 sections are selected at this stage (dotted portion in Figure 2(ii)). Next, divide the object vertically in two equal parts, and select one. The overlap of the selected areas is designated by the red outline Figure 2(ii). The original unit square is designated by the blue outline. The product is designated by the ratio of the red to the blue area and the result is $\frac{4}{6}$, which may be simplified algebraically into $\frac{2}{3}$. Note that the reference object enlarges in the first step and then contracts in the second step of this multiplication.

Case 3: a > b, c > d

$$Case\ 3 : \frac{a}{b} \times \frac{c}{d}, a > b, c > d$$

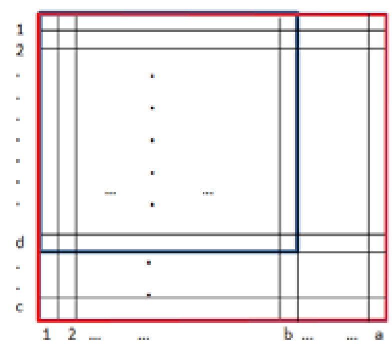

Figure 3(i). Multiplication of two improper fractions

This is the case where two improper fractions are being multiplied. The unit square is partitioned into d horizontal sections, and as before, $c - d$ sections of the same size are appended to it. This enlarged object is now sub-divided into b equal vertical sections, but they are inadequate for the purposes of selection, since we require a such vertical sections, $a > b$. Therefore $a - b$ additional vertical sections are appended to the object, as depicted in Figure 3(i). The object is further enlarged by this appendage, which is intuitively understandable, since the fractions are each larger than unity. All the cells resulting are required for the computation, viz. $a \times c$ cells. However, for the size of the cell, we revert to an examination of the original unit square, which we now find to be divided into exactly $b \times d$

parts. Therefore, the result of our computation is: $a \times c$ parts, each of size $\frac{1}{b \times d}$ units, or

$$\frac{a \times c}{b \times d} \text{ units} = \frac{\textit{no. of selected cells}}{\textit{no. of partitions of the unit square}}$$

Illustration 3: Consider the multiplication $\frac{3}{2} \times \frac{4}{3}$

Illustration 3: $\frac{3}{2} \times \frac{4}{3}$

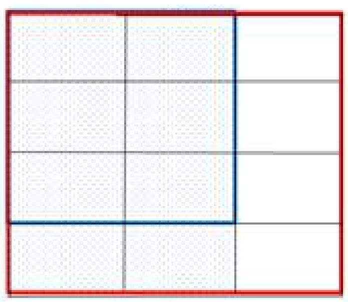

Figure 3(ii). The result is the ratio of the red to the blue areas, i.e. $\frac{12}{6}$

Take the unit square and divide it horizontally into 3 equal sections, thereafter append one more section to it as in Figure 3(ii). The entire 4 sections are selected at this stage (dotted portion in Figure 3(ii)). Next, divide the object vertically in two equal sections, and append another vertical section equal to each. The entire area is selected, designated by the red outline in Figure 3(ii). The original unit square is designated by the blue outline. The product is designated by the ratio of the red to the blue area and therefore the result is $\frac{12}{6}$, which may be simplified algebraically into 2. The reference object doubles.

Case 4: $b = 1$ or $d = 1$

In the case that either of the multiplicands is a whole number, it can be treated as an improper fraction with unit denominator and the above procedure can be applied.

By commutativity of multiplication of rational numbers, the remaining possible cases are trivially covered.

Note that, geometrically speaking, in every case, the result of multiplication of the two fractions is the ratio of the area of the red figure to that of the blue figure. This result, when extended to say, 3-dimensions, suggests that the product of 3 fractions may be visualized as the ratio of the volumes of two cuboids. The result may be extended to n dimensions.

Conclusion

We have shown with rationale that the paper folding method for fraction multiplication can be extended to improper fractions. This expands the scope for using visual methods to demonstrate how fractions interact with one another and can be used for pedagogical as well as creative exercises. Further, the result may be extended to n fractional factors.

Acknowledgements

The author thanks Proteep Mallik (APU) for the inspiration to write up this note and Swati Sircar and Sneha Titus for many helpful comments.

References

1. Shirali, P. (2012, Jun). Fractions - A Paper-Folding Approach. *At Right Angles*, 81-86. Also available on: http://teachersofindia.org/en/article/atria-pullout-section-june-2012

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Making Sense of Adding Unlike Fractions

RUPESH GESOTA

Nothing makes as much sense to a student as his or her own reasoning. And that is why a math class should give students the time and careful facilitation that enables this.

The problem at hand was 4/3 + 5/2.

Nothing makes as much sense to a student as his or her own reasoning. And that is why a math class should give students the time and careful facilitation that enables this.

The problem at hand was 4/3 + 5/2.

Here is an account of a class in which this problem was tackled by students who had understood the need/reason for fractions to be of the same size i.e., to have the same denominators so as to be able to add them easily. However, they had not yet arrived at any particular method to achieve this. This account is written by Rupesh Gesota, an engineer-turned-school-maths teacher. Check the 'Teacher's Blog' sub-page of the website www.supportmentor.weebly.com - in which this account. was first published- to know more about his adventures in teaching math. Given below is the description of the class in Rupesh's words.

Keywords: fraction, numerator, denominator, addition, sense-making

Looking at the problem 4/3 + 5/2, one of the students said that each of the unit fractions above i.e., 1/3 and 1/2 should be split into quarters. Most probably, the reason for this could be that the pictorial representations of both the quantities (that were drawn on the board) looked bigger than a quarter. (Quarters and halves are fractions that students are extremely familiar with.) All the students agreed with this suggestion... So I simply went ahead without showing any hesitation. This is what the picture looked like:

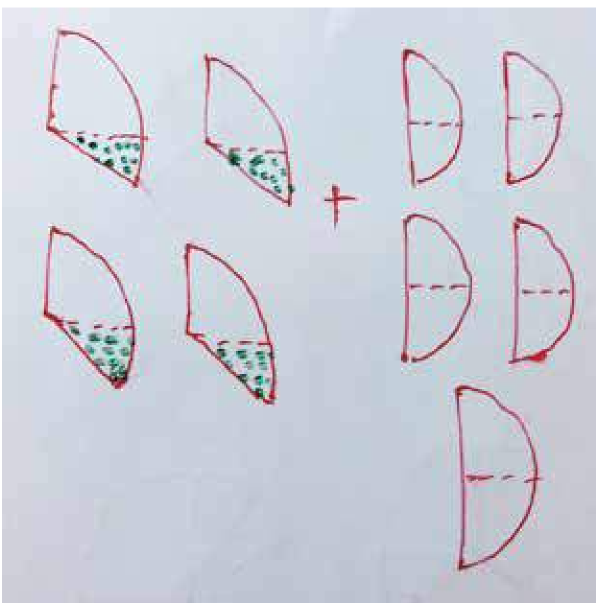


Figure 1.

Seeing it, the students said that we have 14 quarters in all plus 4 smaller pieces. When I asked them how to add the smaller (green) pieces to these 14 quarters, one of them argued –

The green piece is half of a quarter. So, 2 green pieces would make up 1 quarter.

This is not the first time I have witnessed a student giving this specific argument, i.e., misinterpreting this left over piece as half of one-fourth (Do you see why so many students would be saying/seeing it this way?).

I chose to ask the class about this viewpoint. And, unsurprisingly, the whole class completely agreed with this, except for one student.

She said - If 2 smaller (green) pieces sum up to 1 quarter, then 2 pieces of one-third should sum up to 3 quarters! That isn't true. So the green piece is not half of a quarter.

Isn't this a beautiful argument?

I looked at the class. Not everyone understood this. So a picture was drawn where a whole was first divided into three thirds, and then one third was erased. This visual instantly enabled them to see the difference between two-thirds and three-quarters.

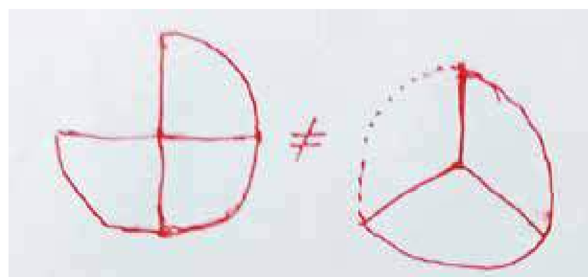


Figure 2.

So, now the problem was – What is the size/name of this smaller piece??

It did not take much time for one of them to shout – So then, THREE green pieces would make one quarter!

I must confess that when I heard this claim at first, I thought that it was just a random guess and hence would get eliminated through another line of argument. I did not pay attention to this and did not evaluate this new claim, probably because of the tone in which it was broadcasted and also probably because of its nature (since TWO didn't work, it must be THREE)!

However, I am glad that a couple of them took it seriously and they not just agreed with this claim

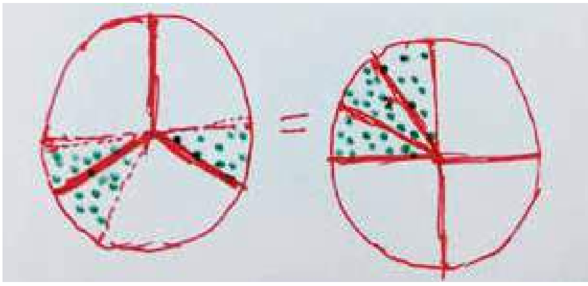


Figure 3.

but even proved it correct with the help of this diagram.

Now, isn't this too beautiful? :-)

Finally, when I probed, they could also give me the name of this green piece.

"Because 3 pieces make one quarter, 12 such pieces would make one whole, hence it's 1/12"

So now, we knew that 3 thirds is same as 4 quarters and the remaining one-third also had one quarter. That left us with a single green piece.

To this, one of them proposed - So let's represent each quarter in terms of this green piece now, because we know that 3 greens make one quarter.

I looked at the class again for their approval. Some required one more round of explanation but soon everyone was on the boat.

Finally, they transformed the original problem 4/3 + 5/2 i.e.,

4 thirds + 5 halves ---> 16 twelfths + 30 twelfths = 46 twelfths.

You might have noted that they did not multiply the Numerator and Denominator by the same number to get a common denominator.... Neither did they take the LCM, nor did they do any cross multiplication.

So what is your view about this approach?

PS: These students study in Marathi medium municipal schools and hail from disadvantaged backgrounds. To know more about and support this maths enrichment program, check the website www.supportmentor.weebly.com

RUPESH GESOTA is an engineer-turned-school-maths teacher. He loves to see the sparkles of understanding in the eyes of his students and he finds it inspiring to realise that he was part of this enlightenment process. He also loves working with their parents and teachers to make the process of Math-education meaningful as well as joyful. To read more of his experiences check his blog www.rupeshgesota.blogspot.com. He can be contacted at rupesh.gesota@gmail.com.

ISN'T IT WONDERFUL

$2^n - 1$ is prime $\Longrightarrow$ n is prime

$2^1 - 1 = 1$
$2^2 - 1 = 3$ Prime
$2^3 - 1 = 7$ Prime
$2^4 - 1 = 15$
$2^5 - 1 = 31$ Prime
$2^6 - 1 = 63$
$2^7 - 1 = 127$ Prime
$2^8 - 1 = 255$
$2^9 - 1 = 511$
$2^{10} - 1 = 1023$
$2^{11} - 1 = 2047$
$2^{12} - 1 = 4095$
$2^{13} - 1 = 8191$ Prime
$2^{14} - 1 = 16383$
$2^{15} - 1 = 32767$
$2^{16} - 1 = 65535$
$2^{17} - 1 = 131071$ Prime
$2^{18} - 1 = 262143$
$2^{19} - 1 = 524287$ Prime
$2^{20} - 1 = 1048575$

ART OF MATHEMATICS

This was posted on our FB page AtRiUM. We have a few questions for our readers. Is there sufficient evidence for us to conclude that if $2^n - 1$ is prime, then n is prime, for all n? If we cannot conclude this, then can we conjecture? ……..And then prove that our conjecture is true? We have done this, using the technique of proving the contrapositive of the statement. [If the statement is $p \Longrightarrow q$, then the contrapositive of this statement is $\sim q \Longrightarrow \sim p$. A statement and its contrapositive are logically equivalent and proving the contrapositive is equivalent to proving the statement.]

Here p is 'if $2^n - 1$ is prime' and q is 'n is prime'. Assume that n is not a prime, i.e. n is composite, then $n = mk$ where $1 < m, k < n$. Then, $2^n - 1 = 2^{mk} - 1 = (2^m)^k - 1 = (2^m - 1)(1 + 2^m + \ldots + (2^m)^{(k-1)})$ using the binomial theorem. So, $2^n - 1$ has factors other than itself and 1. Hence, it is composite. If n is composite, then, $2^n - 1$ is composite. So, if , $2^n - 1$ is prime, then n is prime. We leave you with a question: If n is prime, then is $2^n - 1$ always a prime? [This is called the converse of the given statement.]

ClassRoom

TearOut

Fun with Dot Sheets

Beginning with this issue, we start the TearOut series. In this article, we focus on investigations with dot sheets. Pages 1 and 2 are a worksheet for students, pages 3 and 4 give guidelines for the facilitator

Remember: Whenever a line has to be drawn, two grid points must be identified through which this line passes.

1. Angles

- On the square grid: Pick two adjacent dots and draw the line segment connecting them. Without using a protractor, draw the following angles at any end of the line segment: 45°, 135°, 225°
- On the isometric grid: Pick two adjacent dots and connect them with a line segment. Draw these angles at any end of the line segment: 30°, 60°, 90°, 120°, 150°, 210°

2. Collinear points

- Pick two points at random. Find a 3rd point that is collinear with them
- Verify collinearity with a scale. Can you prove collinearity? How?
- Repeat with other pairs of points on both the square and the isometric grids

3. Complete the rectangles and squares (check Figure A on Page 2)

- You are given two sides of a rectangle. Can you complete it?
- You are given one side of a square. Can you complete it? Is there only one square that you can draw using this line?
- Can you draw a square on the isometric grid if you are given one side? Why?

4. Parallel lines

- Pick any two points a bit far apart, draw the line joining them and pick a 3rd point not on the drawn line
- Draw a 2nd line through the 3rd point and parallel to the 1st line
- Justify that they are parallel

5. Perpendicular lines

Version 1	Version 2
Pick any two points a bit far apart, draw the line joining them and pick a 3rd point on the drawn line Draw a 2nd line through the 3rd point and perpendicular to the 1st line Verify that they are perpendicular. How?	Pick any two points a bit far apart, draw the line joining them and pick a 3rd point not on the drawn line Draw a 2nd line through the 3rd point and perpendicular to the 1st line Verify that they are perpendicular. How?

6. Reflections

Version 1	Version 2
Draw a horizontal or vertical line (mirror) and a scalene triangle on one side Reflect the triangle on the line	Use a mirror that is neither horizontal nor vertical

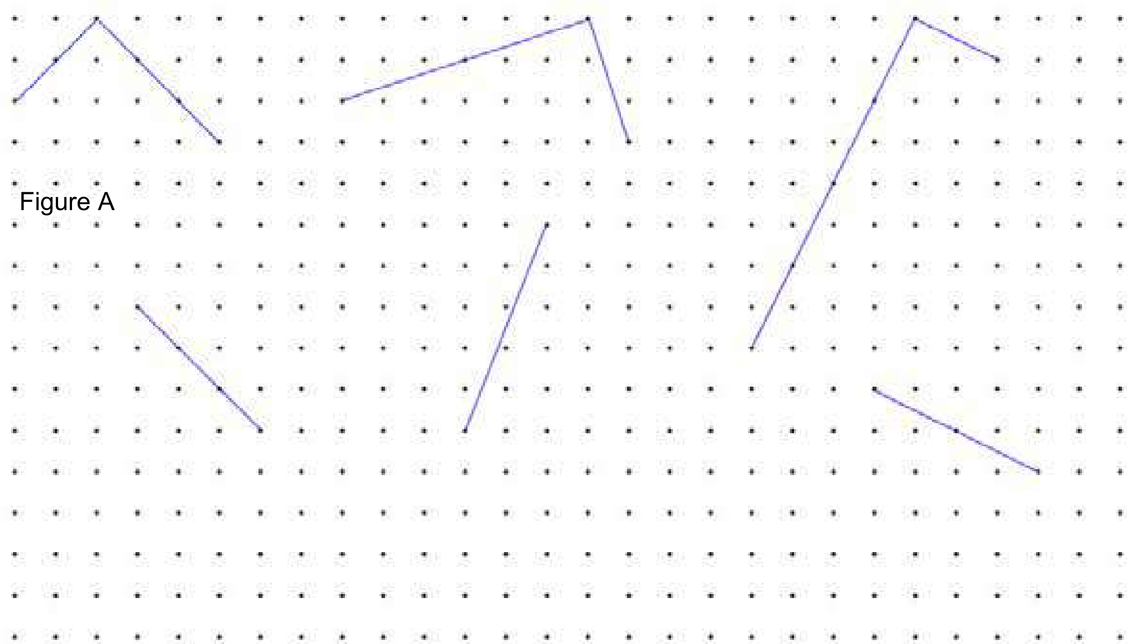
7. Rotations: Pick a point in the middle of the sheet and draw a scalene triangle

Square Grid	Isometric Grid
Rotate the triangle counter-clockwise by 90° and then by 180°	Rotate the triangle counter-clockwise by 60° and then by 120°

8. Double reflections (check Figures on Page 2)

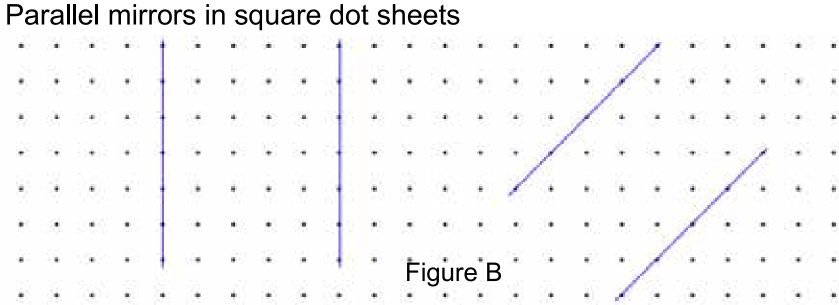
On parallel lines	On intersecting lines
Mirrors are a pair of horizontal or vertical grid lines (Figure B) Mirrors are a pair of (45°) slant parallel lines (Figure B)	On square grid: mirrors intersecting at 45° or 90° (Figure C) On isometric grid: mirrors intersecting at 30° or 60° (Figure D)

Complete the rectangles (2 sides given) and the squares (1 side given)

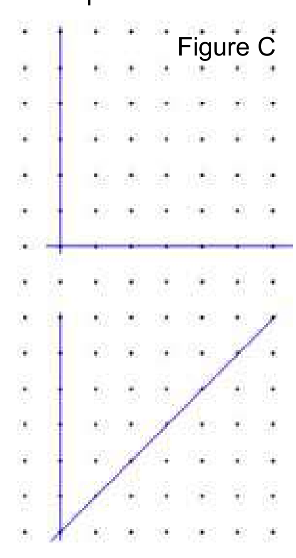


Use the following examples for pairs of mirrors for double reflection

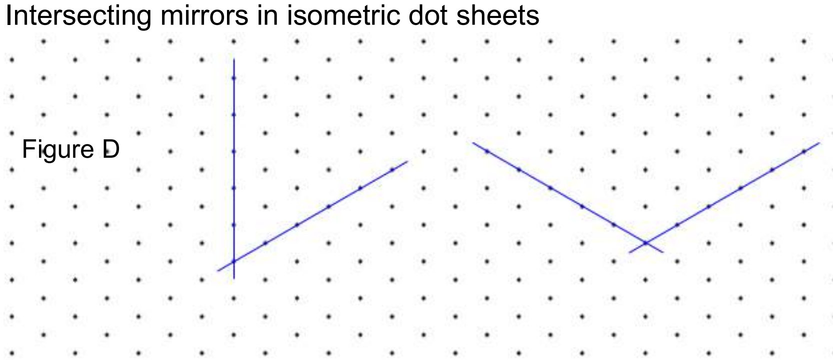
Parallel mirrors in square dot sheets



Intersecting mirrors in square dot sheets



Intersecting mirrors in isometric dot sheets



Exploring Spatial Understanding and Geometry on Square-Grids and Dot Sheets

The following activities can be done on square grids (Figure 1) and on rectangular dot sheets (Figure 2). These pave the way for more rigorous navigation of the Cartesian plane in the higher classes. Henceforth square grid will refer to both the actual square grid with lines as well as the rectangular dot sheets. [The advantage of rectangular dot sheets over the square grid from notebooks is that they do not have any lines.]

In general, one can start any of the activities on the square grid. Later, they should be tried on the isometric grid (Figure 3) as a challenge. Some activities should be done only on the isometric grid as indicated below.

Figure 1

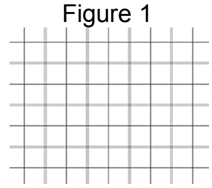


Figure 2

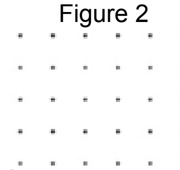
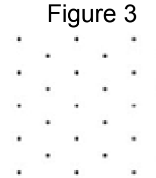


Figure 3



Materials required (other than dot sheets) will be scale, pencil, eraser and sharpener. It might help to have a protractor but that is necessary only for verification. Whenever a line has to be drawn, two grid points must be identified through which this line passes.

The activities are broadly in two categories: **A.** Drawing lines parallel to or inclined at a given angle to a given line **B.** Reflecting and rotating shapes

The topics which can be introduced or practised with these activities are Understanding Elementary Shapes (Class 6 NCERT Curriculum) and Symmetry (Class 7 NCERT Curriculum). However, with skillful facilitation, students can go far beyond these topics. These activities can be done with classes 5-8; the level of responses will of course depend on the topics that the students are familiar with. In many cases, students are asked to justify their answers and this will help them to develop their mathematical reasoning. This is also an opportunity for the teacher to facilitate their appreciation of mathematical rigor.

Throughout we have used *blue* for what is given and *pink* for what a child is supposed to do.

1. Angles - The features of the dot sheets should be utilized for drawing these angles. Children should be able to identify the square and equilateral triangle tiling in the respective dot sheets. Using the angles of these regular polygons they should be able to justify the presence of 90° and 60° angles. Halving these would generate 45° and 30° respectively which can be combined with the earlier angles to get the rest.

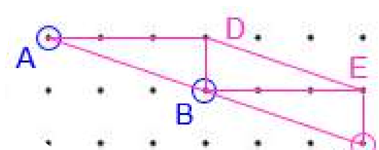


Figure 4

2. Collinear points If two consecutive grid points are selected, then finding a 3rd point would be too simple. So care must be taken to pick points which are further apart and not on the same grid line. The easiest way to find a 3rd collinear point is to mimic the path from the 1st point to the 2nd one to go to the 3rd from the 2nd. E.g. in Figure 4, A and B are the given points. The path from A to B and from B to C is '3 right and 1 down'. To prove that A, B and C are collinear, we can show ∠ABC = 180° by observing that ΔABD, ΔEDB and ΔBCE are congruent (Why?) and using the angle sum property of a triangle. This '3 right 1 down' provides a beginning into 'run and rise' whose quotient (i.e., rise over run) is slope. Children should be able to eye-estimate and do double, i.e., 6 right and 2 down, triple, etc. If the given run and rise are not coprime, they should be able to identify grid point(s) within the line segment AB.

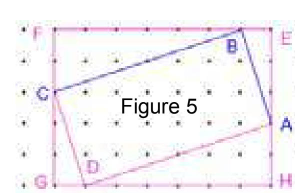


Figure 5

3. Complete the rectangles and squares - Whenever the given sides are at 45° slant, it is easier since mirror reflection can be used. However for the rest, justification can be provided with the help of congruent right triangles. E.g. in Figure 5, the same path is followed from C to D as in B to A resulting in ΔABE ≅ ΔCDG. So AB = CD and similarly BC = AD i.e. ABCD is a parallelogram with ∠B = 90° (given) making it a rectangle. For square, the right angle needs to be constructed and that can be done on either side of the given line. E.g. in Figure 6 – the path A to B: 2 down 4 right changes to 4 up 2 right for A to C. This leads to rotation of ΔABE to ΔCAF and contributes to the m · m' = -1 for slopes of perpendicular lines.

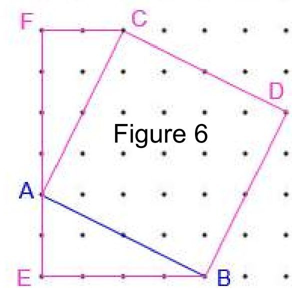


Figure 6

4. Parallel lines - Once again, the easiest is to follow the path from A to B and mimic that to go from C to D. That results in AB = CD and AB || CD. This can also be achieved by mimicking A to C i.e. 2 down 1 right for B to D as shown in Figure 7. Both can be thought of as a translation resulting in parallel lines.

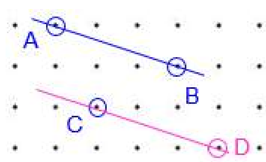


Figure 7

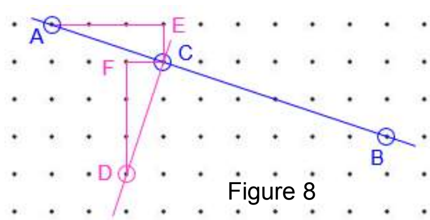


Figure 8

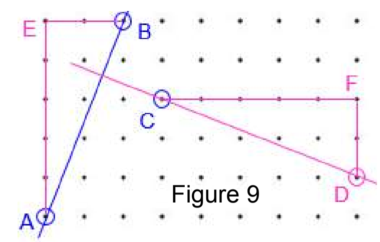


Figure 9

5. Perpendicular lines - It boils down to rotating a right triangle by 90°. E.g. in Figure 8, ΔACE is rotated to get ΔDCF while in Figure 9, ΔABE has been rotated to get ΔCDF. Both involves an exchange of run and rise and interchange of up and down.

6. Reflections - As a precursor to reflecting triangles (or any other shape), children should first reflect points on a line. Eye-estimation should suffice if they understand the properties of reflection. In particular, that if A' is the reflected image of A on the line PQ, then PQ is the perpendicular bisector of AA'. So to reflect a triangle, each of the vertices has to be reflected. Scalene triangles help in identifying which vertex got reflected to which one.

Various properties of reflection can and should be discussed after this activity. This includes the change in orientation, congruency of image and pre-image as well as image and pre-image being equidistant from the mirror. For version 2: the mirror can be at 45° with grid lines on square grid and at 30° on isometric grid.

7. Rotations - Similarly, rotating a point about another point should be tried first. Children should be able to eye-estimate and understand that if A' is the rotated image of a point A rotated by θ about another point O then ∠AOA' = θ.   Various properties of rotation can and should be discussed, especially that orientation remains the same.

8. Double reflection - This is an interesting exercise to observe that double reflection on parallel lines results in translation while that in intersecting lines generates a rotation. In addition, it is worth noting that the distance between image and translated pre-image is double the perpendicular gap between the parallel mirrors (Figure 10). This can be easily justified with the help of the image in between. Similarly it can be observed that the angle of rotation is double the angle between the intersecting mirrors. Proof again utilizes the in between image (Figure 11).

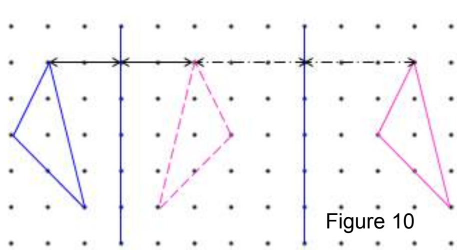


Figure 10

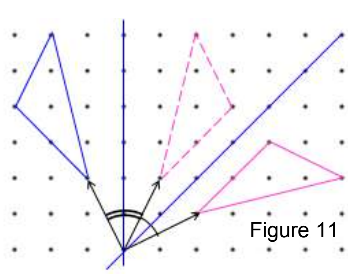


Figure 11

Mapping Triangle Shapes

A RAMACHANDRAN

The triangle inequality has a familiar cadence to it and most students can recite it spontaneously. In this article, we mathematise our understanding of possible triangle shapes, using the limits of values which the angles first, and then the sides, take. It's a great way for students to explore different ways of expressing their conceptual understanding.

Triangles are of different shapes. The shape of a triangle is determined by its angles, or, alternatively, by the ratios of its sides. We shall focus on the angle aspect now. To fix its shape, it is enough if two angles of a triangle are specified. So we could have a 2-dimensional 'map' where every point stands for a possible triangle shape and every possible shape is represented by a point in the map. For convenience, we could take the greatest and least angles of the triangle to be the variables. Let us denote the angles of the triangle as α, β and γ, satisfying the relation $\alpha \geq \beta \geq \gamma$. We could represent α and γ on the X-axis and Y-axis, respectively, of a plane graph.

Now α cannot be less than 60°, and γ cannot be greater than 60° (can you see why?), i.e.,

$$60° \leq \alpha < 180° \text{ and } 0° < \gamma \leq 60°.$$

(The intermediate angle β has the limits $0° < \beta < 90°$.)

Keywords: Triangles, angles, acute, right, obtuse, sides, inequalities, limits, maps

Thus only points in the α,γ plane lying within these limits can represent possible triangle shapes. Refer Rectangle ABCD in Figure 1.

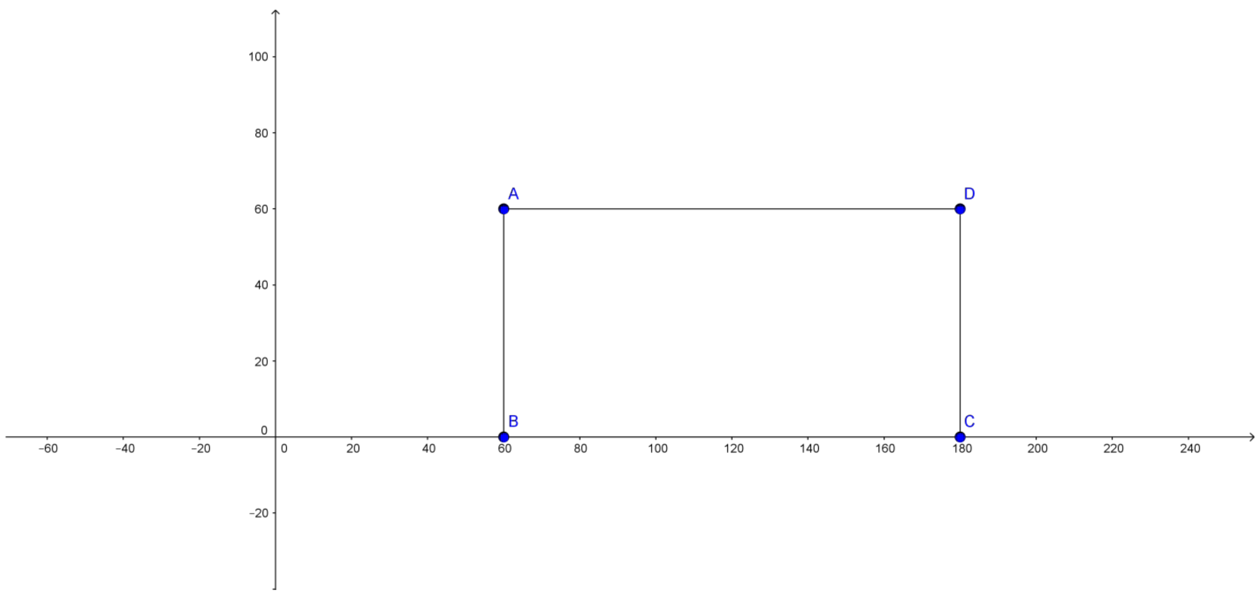


Figure 1.

Actually we have the following additional restrictions on γ for given α. The maximum value of γ for a particular value of α is given by the relation γ = (180° – α)/2 = 90° – α/2, while the minimum value is given by γ = 180° - 2α. These two relations define two straight lines in the α, γ plane, *AC* and *AE*, respectively, intersecting at p*A*nt A (see Figure 2).

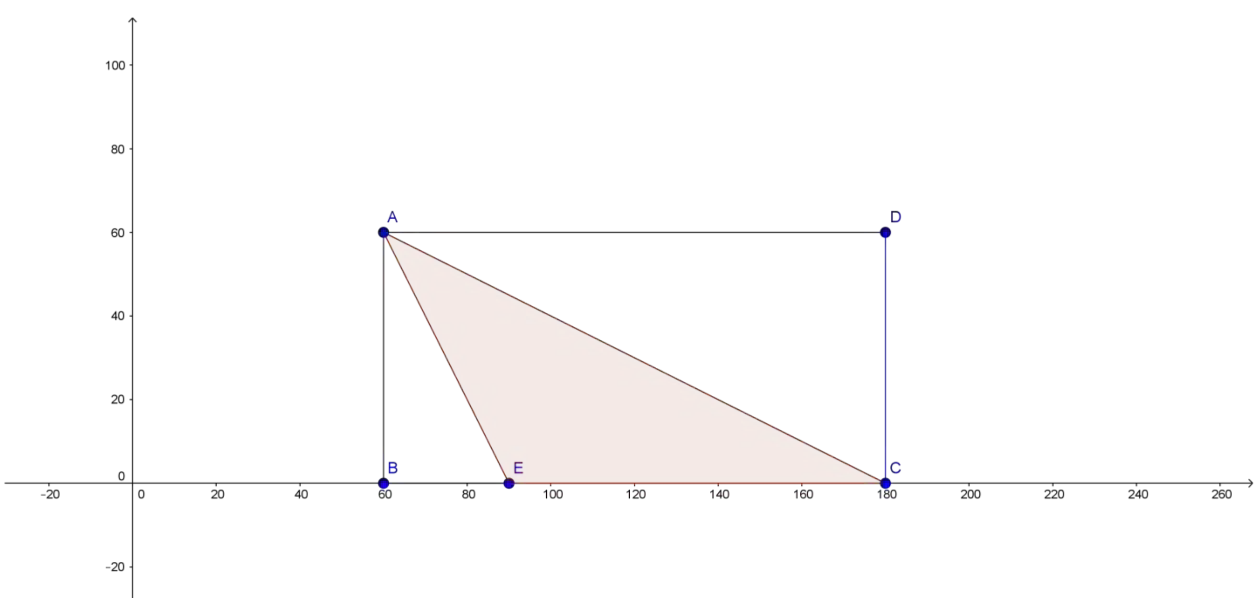


Figure 2.

So we now have the triangular region *AEC* which is the required 'map' of triangular shapes. Point *A* represents the equilateral triangle shape. Points on line segment *AC*, excluding the endpoints *A* and *C*, stand for isosceles triangles of the form $\alpha > \beta = \gamma$. Points on line segment *AE*, excluding the endpoints *A* and *E*, stand for isosceles triangles of the form $\alpha = \beta > \gamma$. Points in the interior of the triangular region *AEC* represent scalene triangle shapes.

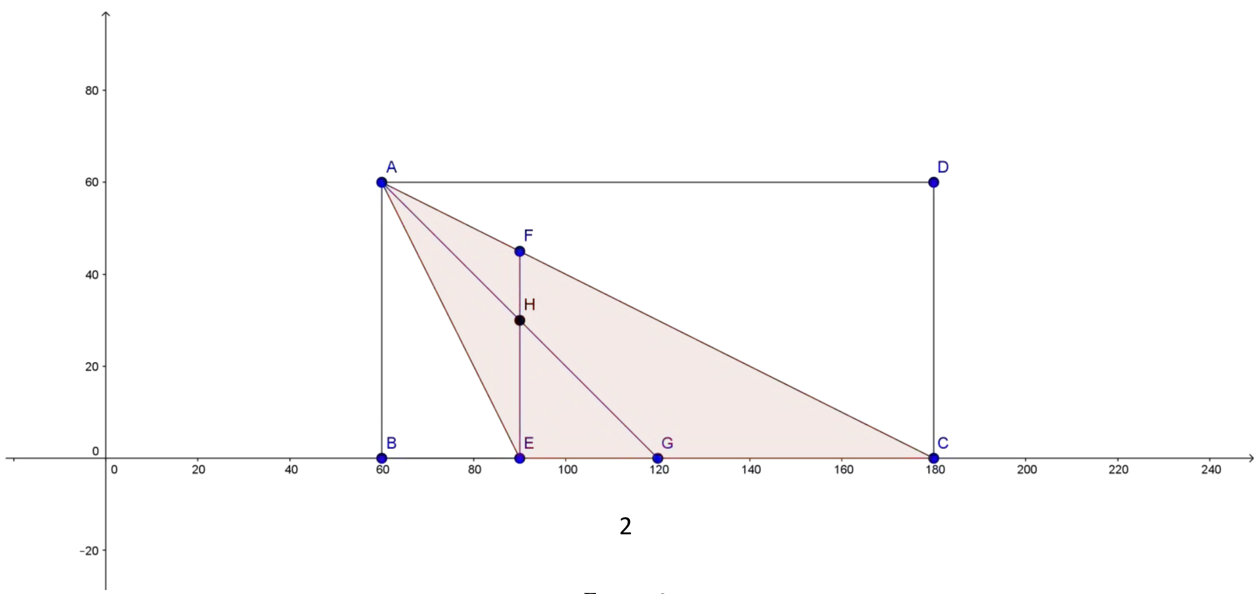


Figure 3.

In Figure 3 we see another line segment marked *EF*. Points on this line segment, excepting *E* itself, represent right-angled triangles. Point *F* itself represents the right-angled isosceles triangle (with angles 45°,45°,90°). Points in the interior of ΔAEF stand for acute-angled scalene triangles, while points within ΔFEC represent obtuse-angled scalene triangles. Also shown in Figure 3 is the line segment *AG* with a slope of -1. If we move along this line, starting from *A*, α increases while γ decreases to the same extent, leaving β unchanged. Hence points on this line segment, except *G* itself, represent triangles with angles in arithmetic progression. Point *H*, where this line intersects line *EF*, represents the 30°, 60°, 90° triangle, the only right-angled triangle with angles in arithmetic progression.

Let us now try a similar exercise taking the sides into consideration. We can take the side of intermediate length to be of unit length, the shortest of length φ and the longest of length ψ, with the proviso $\varphi \leq 1 \leq \psi$.

Since we have two variables, we can again think of a 2-D map, taking ψ on the *X*-axis and φ on the *Y*-axis. Now what are the limits on the values these can take? Clearly

$$0 < \varphi \leq 1 \quad \text{and} \quad 1 \leq \psi < 2.$$

(If $\psi \geq 2$, it would be longer than the sum of the other two sides.) So our 'map' is confined to the square area *PQRS* (Figure 4).

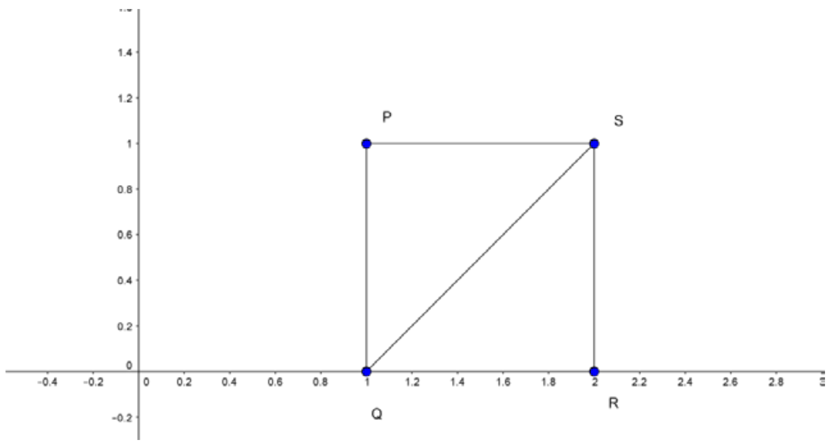


Figure 4.

Now there is a further constraint in the values ψ can take for a given φ value: ψ cannot equal or exceed $\varphi + 1$ at any point. So the line given by the equation $\psi = \varphi + 1$ or $\varphi = \psi - 1$ is a limiting line for the map (line *QS* in Figure 4). Our map is now confined to the triangular area *PQS*, excluding points on line *QS* itself.

Clearly, point *P* represents the equilateral triangle as its coordinates are $\psi = 1$, $\varphi = 1$. Points on the line segment *PQ*, except points *P* and *Q*, represent isosceles triangles where the unequal side is shorter than either of the equal sides. Points on the line segment *PS*, except *P* and *S*, represent isosceles triangles where the unequal side is longer than either of the equal sides. Points in the interior of Δ *PQS* stand for scalene triangles, since their ψ and φ values would be different, neither being equal to unity.

Now the question naturally arises: What about right-angled triangles? Now a right-angled triangle in our scheme would have to satisfy the condition $\psi^2 = \varphi^2 + 1$, or $\psi^2 - \varphi^2 = 1$. Now this is the equation for a hyperbola, one arm of which passes through the p *Q*ıt Q ($\psi = 1$, $\varphi = 0$) and intersects line *PS* at the p *T*nt T ($\psi = \sqrt{2}$, $\varphi = 1$); see Figure 5.

Needless to say, this point represents the isosceles right triangle. Points of the hyperbolic arc lying within Δ *PQS* represent other right triangle shapes.

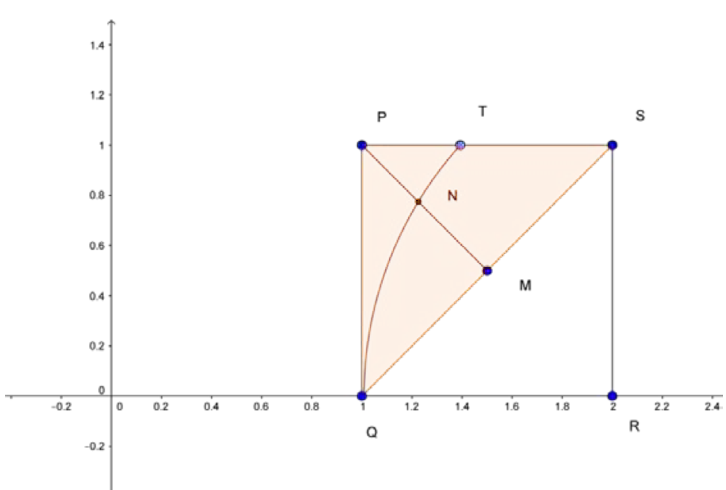


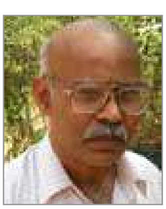
Figure 5.

Points in the interior of region *PQT* have lower ψ value and/or higher φ value compared to points on the arc *QT*. That is, they satisfy the inequality $\psi^2 < \varphi^2 + 1$, which means that the longest side faces an acute angle. So, such points stand for acute-angled scalene triangles. Similar arguments show that points in the region *TQS* stand for obtuse-angled scalene triangles.

Also shown in Figure 3 is the line segment *PM* with a slope of -1. If we move along this line starting from *P*, ψ increases, while φ decreases to the same extent, leaving the perimeter constant. Hence points on this line segment, except *M* itself, stand for triangles with same perimeter as the equilateral triangle represented by point *P*. In other words, the sides of such triangles would be in arithmetic progression, while maintaining an intermediate side length of one unit. Point *N*, where this line intersects arc *QT*, represents a right triangle with sides in A.P. As discussed in earlier articles in AtRiA, such a triangle must be a 3-4-5 triangle.

It is satisfying to see that these two approaches have resulted in 'Maps of triangle shapes' of similar structure. These maps of triangular shapes are themselves triangular. Points close to point *A* in the first case and close to point *P* in the second case represent shapes close to the equilateral triangle shape. Points close to *E* in the first case and close to *Q* in the second case represent triangles where one angle is much smaller than the other two, which are comparable, resulting in a dagger-like shape. Points close to vertex *C* in the first case and close to vertex *S* in the second case represent triangles where one angle is much larger than the other two, resulting in a bow-like shape. Points on line segment *EC* in the first case and *QS* in the second case represent triangles which have collapsed into line segments. R.I.P.

Addendum: The last observation relating an equilateral triangle to a 3-4-5 triangle can be contextualised differently. Let us say we set out to draw an ellipse, choosing as foci two points unit distance apart, and a string of length two units with ends secured at the foci. In the symmetrical position the string and base line together form an equilateral triangle. As we move the string aside, keeping it stretched, we reach a point where the string and base line form a right triangle. This triangle is a 3-4-5 triangle.



A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

Formulas for Special Segments in a Triangle

In many programs of study, the material on the formulas relating the sides and special segments in a triangle does not appear as part of the study of mathematics in high school. On the other hand, in many programs of study the background required to understand this subject is studied already by the ages of 13-15. This situation gives us the opportunity to teach the relationship formulas at an early stage, even before the studies of geometry have begun in the precise manner at the higher level of difficulty.

In this paper we propose a structure and a method for teaching the relationship formulas that has been tried with a group of students. Teaching the relationship formulas by this manner will present the students with many uses for material that has already been studied, and will expose them to new methods for solving problems in geometry and algebra.

This paper presents material that is suitable for students aged 13-15. The material includes: (1) Obtaining three formulas that relate special segments in the triangle to the sides of the triangle; (2) Using these formulas for proving three geometrical theorems; (3) Examples of problems in geometry that can be solved algebraically using these formulas; (4) Didactic recommendations for teaching this material.

DR. DAVID FRAIVERT, PROF. MOSHE STUPEL

Suitable for students in classes 9 and 10, this article uses formulas relating the sides of a triangle with its special segments such as the median and altitude. Students have an opportunity to derive new geometric relationships using familiar algebraic identities. The pedagogy strategy used is of worksheets with scaffolded questions which enable students to derive these relations for themselves. Answers to all questions are explained in the first part of the article. The article provides an interesting way to devise extension activities which enable students to both practise and build on learnt concepts.

The teaching of geometry at school is comprised of several stages. In the last stage (usually starting from the ages of 15-16), the study of geometry focuses on the logical structure of the topics and on proofs of theorems, followed by application of the knowledge gained to proving and solving problems and tasks.

In the preceding stage (usually at the ages of 13-14), some isolated subjects are studied, theorems are presented without proof or with a partial proof only. During this stage, the emphasis is laid on solving geometrical problems of calculation. Usually during this stage, the following topics are studied: segments, angles, angles between parallel lines, the sum of the angles in the triangle, congruence and similarity of triangles, the Pythagorean Theorem, etc. (Note: the topics of the similarity of triangles and the Pythagorean Theorem are studied at the ages of 13-14, as an extension of the topic of ratios and proportions.) In parallel, as part of the

Keywords: Formula development, Relationship formulas in the triangle, Algebraic Identities, Characteristics of isosceles triangles, Heron's formula, Nonlinear systems.

studies of algebra, the following topics are studied: solution of equations of the first degree, algebraic identities, operations with algebraic fractions, simplification of algebraic expressions.

From the history of mathematics we know that great mathematicians discovered by chance the famous theorems in geometry by performing mathematical manipulations of different formulas.

Solving a task using different methods

Some mathematical tasks can be solved using different methods – by using mathematical tools from the same field, by using tools from a different field or by combining tools from several fields. The larger the toolbox available to the student, the higher is his/her chance of successfully dealing with the mathematical tasks, and the more capable is he or she of finding the solution by the shortest and simplest method. Using a wide variety of mathematical tools, one can discover unorthodox solutions or proofs which accentuate the beauty of mathematics, increase motivation and the joy of both teaching and learning the subject [1], [2], [3], [4].

Use of formulas

As early as possible, in their primary education in mathematics students learn to use formulas, such as: calculation of the area of a triangle, calculation of the volume of a box, velocity calculations, etc. As students progress in their studies, formulas are added, such as: the Pythagorean Theorem, the sum of an arithmetic progression, calculation of the weighted average and standard deviation, the Laws of Sines and Cosines, etc. The question is whether the use of a formula to calculate a particular value constitutes knowledge and a technical skill of substituting values in a formula – as expected from low-achieving and intermediate-achieving students, or a tool that can be used to develop new formulas, to find proofs to theorems and to solve unique problems, as can be expected from the advanced and excelling students. These students are able and deserve to rise to a higher mathematical level, on which they have the ability to develop new formulas, and subsequently to know how to use them as a tool that allows them to deal with different tasks (see [5]). The use of formulas has a significant importance in the age of computerized technology, since it allows the student to investigate and deal with various tasks throughout all the fields of mathematics in a dynamic manner, as well as in other areas of daily life.

In this paper we shall present use of material that is acquired usually before the ages of 14-15, as a sufficient basis for the development of formulas that relate sides and special segments in a triangle, which allow one to prove new properties in shapes and to perform different calculations.

Studying the relationship formulas at the ages of 14-15 shall give the students the following advantages:

a) Turning learned material into a useful tool both at the present stage and later during their studies.

b) Deepening knowledge in various fields in geometry and algebra and the ability to implement this knowledge.

c) Acquaintance with the method for solving geometrical problems related to the triangle by the algebraic method.

In order to develop the formulas relating sides and special segments in the triangle, one requires knowledge in the following topics: Similarity of triangles (definition and condition of similarity by two angles), the Pythagorean Theorem, the expansion of the square of a sum or difference, the difference of squares formula, operations with algebraic fractions, and simplification of algebraic expressions.

The proposed program for developing and using the relation formulas is based on this knowledge only and is composed of the following parts:

1. Obtaining the formulas of relations between special segments in the triangle and the sides of the triangle.
2. Proving two tests that permit one to determine if the triangle is an isosceles triangle:
   a. The test of two medians of equal length.
   b. The test of two angle bisectors of equal length.
3. Obtaining Heron's formula for calculating the area of a triangle.
4. Solving problems of calculation in a triangle by an algebraic method.

Obtaining the formulas of relation between special segments in the triangle and the sides of the triangle.

Let ABC be some triangle, the lengths of whose sides are

$BC = a$, $AC = b$ and $AB = c$.

$AH = h_a$ is the altitude to the side a.

$AL = l_a$ is the bisector of the angle $\measuredangle BAC$.

$AM = m_a$ is the median to the side a (see Figure 1).

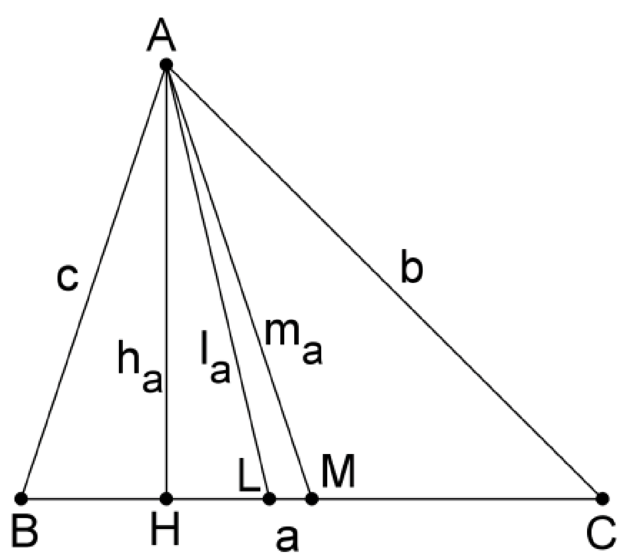


Figure 1.

The following well-known formulas relate the segments h_a, l_a and m_a to the segments a, b and c:

$$h_a^2 = \frac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{4a^2}; \quad (1)$$

$$m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4}; \quad (2)$$

$$l_a^2 = bc\frac{(b+c)^2 - a^2}{(b+c)^2}. \quad (3)$$

We hereby present the methods for obtaining these formulas based on the material studied by students at the ages of 13-14.

Obtaining the first formula for the altitude in the triangle, which is the starting formula for proving the formulas 1 to 3.

Given is the triangle $\triangle ABC$, in which AH is the altitude to the side BC (see Figure 2).

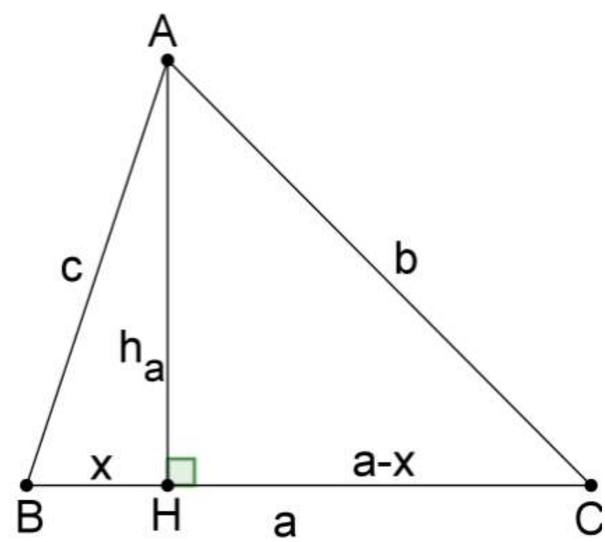


Figure 2.

We denote: $BC = a$, $AC = b$, $AB = c$, $AH = h_a$, $BH = x$, $HC = a - x$. *By the Pythagorean Theorem*, in the right-angled triangle $\triangle ABH$ there holds: $h_a^2 = c^2 - x^2$, in the right-angled triangle $\triangle ACH$ there holds: $h_a^2 = b^2 - (a-x)^2$.

Therefore, $c^2 - x^2 = b^2 - (a-x)^2$, and *from the square of the difference formula,* we have $c^2 - x^2 = b^2 - a^2 + 2ax - x^2$, from where we have for x: $x = \dfrac{a^2 + c^2 - b^2}{2a}$.

We substitute the obtained expression for x in the formula for h_a^2, to obtain:

$$h_a^2 = c^2 - \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 \tag{4}$$

The formula (4) is the first formula that expresses the square of the altitude of the triangle by the lengths of its sides and is also the basic formula for proving the formulas (1) to (3) .

Obtaining a second formula (Formula (1)) for the altitude in a triangle.

By using the difference of squares formula $a^2 - b^2 = (a-b)(a+b)$ on the right-hand side of (4), we obtain:

$$h_a^2 = \left(c - \frac{a^2 + c^2 - b^2}{2a}\right)\left(c + \frac{a^2 + c^2 - b^2}{2a}\right),$$

from which, *by adding the fractions* in each pair of parentheses, and *by using the abridged multiplication formula* $(a \pm c)^2 = a^2 \pm 2ac + c^2$, we obtain:

$$\begin{aligned} h_a^2 &= \frac{2ac - a^2 - c^2 + b^2}{2a} \cdot \frac{2ac + a^2 + c^2 - b^2}{2a} \\ &= \frac{b^2 - (a^2 - 2ac + c^2)}{2a} \cdot \frac{(a^2 + 2ac + c^2) - b^2}{2a} = \frac{b^2 - (a-c)^2}{2a} \cdot \frac{(a+c)^2 - b^2}{2a} \\ &= \frac{(b - a + c)(b + a - c)}{2a} \cdot \frac{(a + c - b)(a + c + b)}{2a} \\ &= \frac{(a + b + c)(a + b - c)(a + c - b)(b + c - a)}{4a^2}. \end{aligned}$$

Obtaining the formula for the median (Formula (2)).
In Figure 3 it is given that

$AM = m_a$ is the median to the side BC, $AH = h_a$ is the altitude to the side BC, $BC = a$, $AC = b$ and $AB = c$, and hence:

$$BM = MC = \frac{a}{2}$$

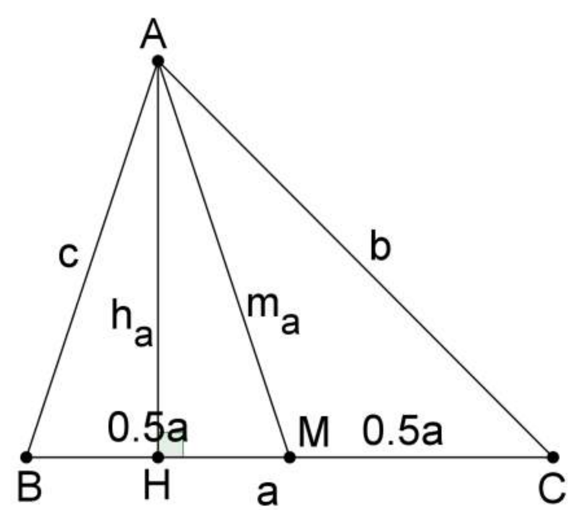


Figure 3.

We consider the triangle $\triangle ABM$, whose sides are $AB = c$, $BM = \dfrac{a}{2}$ and $AM = m_a$; the segment AH is an altitude in this triangle, and therefore from (4) for $AH^2 = h_a^2$ there holds:

$h_a^2 = AB^2 - \left(\dfrac{BM^2 + AB^2 - AM^2}{2BM}\right)^2$. After substitution we obtain:

$$h_a^2 = c^2 - \left(\frac{\left(\frac{a}{2}\right)^2 + c^2 - m_a^2}{2 \cdot \frac{a}{2}}\right)^2 \quad or: \quad h_a^2 = c^2 - \left(\frac{\frac{a^2}{4} + c^2 - m_a^2}{a}\right)^2. \qquad (5)$$

By comparing (4) and (5), it follows that

$$\left(\frac{a^2+c^2-b^2}{2a}\right)^2 = \left(\frac{\frac{a^2}{4}+c^2-m_a^2}{a}\right)^2.$$

Now observe that $a^2+c^2-b^2$ and $\frac{a^2}{4}+c^2-m_a^2$ have the same sign; for if $\measuredangle B$ is acute, then both $a^2+c^2-b^2$ and $\frac{a^2}{4}+c^2-m_a^2$ are positive (from $\triangle ABC$ and $\triangle ABM$, respectively); and if $\measuredangle B$ is obtuse, then both $a^2+c^2-b^2$ and $\frac{a^2}{4}+c^2-m_a^2$ are negative. Hence the equality sign is preserved if we take square roots of both sides in the above equality. On doing so and cancelling common factors, we obtain the desired formula, $m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4}$.

Obtaining the formula for the angle bisector (Formula (3)).
In Figure 4 it is given that:

$AL = l_a$ is the bisector of the angle $\measuredangle BAC$, $AH = h_a$ is the altitude to the side BC, $BC = a$, $AC = b$ and $AB = c$. Note that AH is also the altitude to the side BL in the triangle $\triangle ABL$.

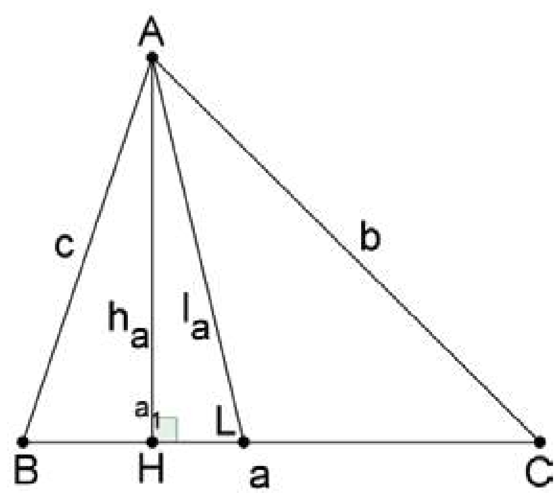


Figure 4.

The lengths of the sides of the triangle $\triangle ABL$ are: $AB = c$, $AL = l_a$ and $BL = a_1$.

From equation (4) for the altitude h_a in the triangle $\triangle ABL$ there holds:

$h_a^2 = AB^2 - \left(\frac{BL^2+AB^2-AL^2}{2BL}\right)^2$ and after substitution we obtain:

$$h_a^2 = c^2 - \left(\frac{a_1^2+c^2-l_a^2}{2a_1}\right)^2. \tag{6}$$

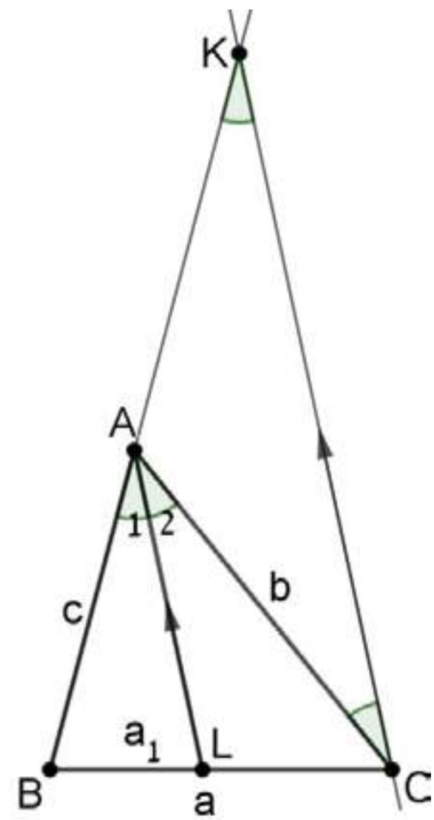


Figure 5.

We now express the length of the segment $BL = a_1$ through the side lengths of the triangle $\triangle ABC$. We carry out the following auxiliary construction (as shown in Figure 5):

Through the vertex C we draw the straight line m that is parallel to the line AL, and which intersects the continuation of the side BA at the point K. This creates two pairs of equal angles: $\measuredangle A_1 = \measuredangle K$ (*corresponding angles in these parallel lines and their secant* BK) and $\measuredangle A_2 = \measuredangle ACK$ (*alternate angles in the same parallel lines and their secant* AC). In addition we have $\measuredangle A_1 = \measuredangle A_2$, and therefore $\measuredangle K = \measuredangle ACK$. Hence it follows that $\triangle ACK$ is an isosceles triangle in which $AK = AC = b$. In the triangles $\triangle ABL$ and $\triangle KBC$ there are two pairs of equal angles: $\measuredangle A_1 = \measuredangle K$, where $\measuredangle B$ is a common angle. Thus, *from the similarity theorem (angle, angle)*, the triangles are similar, $\triangle BAL \sim \triangle BKC$. Hence, *from the definition of similar triangles*, there holds $\frac{BA}{BK} = \frac{BL}{BC}$. After the lengths of the obtained segments are substituted in the proportion, we obtain: $\frac{c}{b+c} = \frac{a_1}{a} \Rightarrow a_1 = \frac{ac}{b+c}$.

We substitute the expression for a_1 in (6), and obtain

$$h_a^2 = c^2 - \left(\frac{\left(\frac{ac}{b+c}\right)^2 + c^2 - l_a^2}{2\frac{ac}{b+c}} \right)^2. \tag{7}$$

By comparing (4) and (7) we obtain: $\dfrac{a^2+c^2-b^2}{2a} = \dfrac{\left(\dfrac{ac}{b+c}\right)^2 + c^2 - l_a^2}{2\dfrac{ac}{b+c}}$ from which follows:

$\dfrac{a^2+c^2-b^2}{2a} \cdot \dfrac{2ac}{b+c} = \dfrac{a^2c^2}{(b+c)^2} + c^2 - l_a^2$, and hence we have for l_a^2 :

$$l_a^2 = \frac{a^2c^2 + (b+c)^2c^2}{(b+c)^2} - \frac{c(a^2+c^2-b^2)}{b+c} \Rightarrow l_a^2 = c\frac{a^2c + (b+c)^2c - (a^2+c^2-b^2)(b+c)}{(b+c)^2}.$$

After opening the parentheses (multiplication of polynomials, use of the abridged multiplication formula and collecting similar terms in the denominator of the fraction), we obtain:

$$l_a^2 = c\frac{2b^2c + bc^2 + b^3 - a^2b}{(b+c)^2} = bc\frac{2bc + c^2 + b^2 - a^2}{(b+c)^2} = bc\frac{(b+c)^2 - a^2}{(b+c)^2}$$

Thus we obtained formula (3).

Proving the signs (tests) of an isosceles triangle

Sign 1 (test of an isosceles triangle based on two medians of equal length)

A triangle in which there are two medians of equal lengths is an isosceles triangle.

In $\triangle ABC$ it is given that:

$AD = m_a$ is the median to the side $BC = a$,

$BE = m_b$ is the median to the side $AC = b$,

and also $AD = BE$ (as shown in Figure 6).

Prove that $\triangle ABC$ is an isosceles triangle.

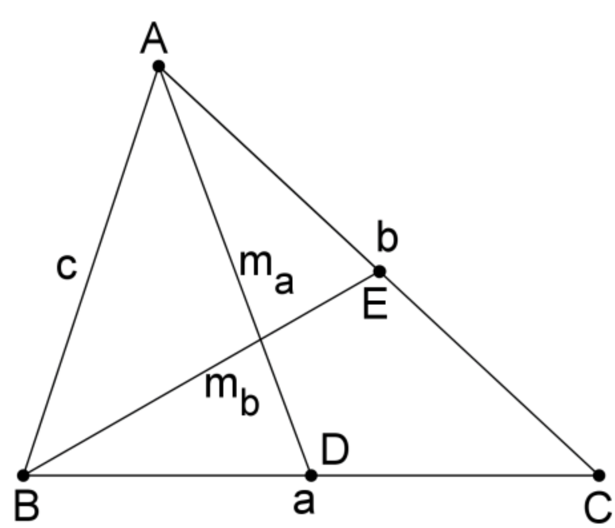


Figure 6.

Proof

We use the relation (2) between the length of the median and the lengths of the triangle's sides. From this formula:

$$m_a^2 = \frac{b^2 + c^2}{2} - \frac{a^2}{4} \text{ and } m_b^2 = \frac{a^2 + c^2}{2} - \frac{b^2}{4}.$$

From the data it follows that $\frac{a^2 + c^2}{2} - \frac{b^2}{4} = \frac{b^2 + c^2}{2} - \frac{a^2}{4}$, and hence:

$2a^2 + 2c^2 - b^2 = 2b^2 + 2c^2 - a^2 \Rightarrow 3a^2 = 3b^2$, which means that $a = b$, and therefore the triangle is an isosceles one.

Note: of course, the first test can also be proven using other ways, without using the relation formula; for example, using the properties of a parallelogram. However, this material is studied at a later stage, when we focus on the logical structure of the material and on proofs.

Sign 2 (test of an isosceles triangle by two equal angle bisectors)

A triangle in which there are two angle bisectors of equal lengths is an isosceles triangle.

In $\triangle ABC$ it is given that:

$AD = l_a$ is the bisector of the angle $\measuredangle BAC$,

$BE = l_b$ is the bisector of the angle $\measuredangle ABC$,

and also $AD = BE$ (as shown in Figure 7).

Prove that $\triangle ABC$ is an isosceles triangle.

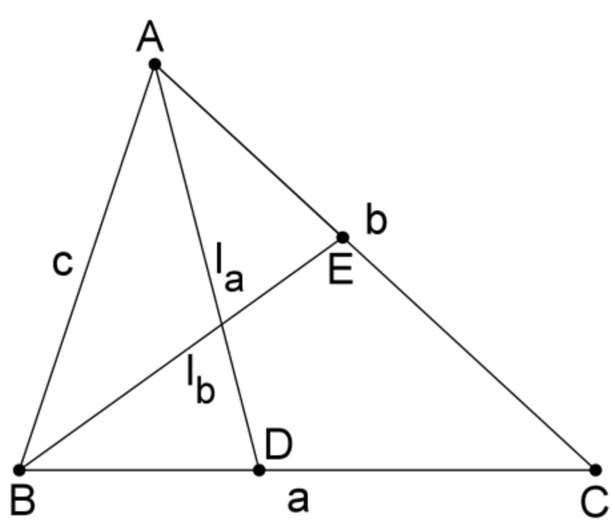


Figure 7.

Proof

From formula (3) that relates the angle bisector to the side lengths of the triangle, there holds: $l_a^2 = bc\frac{(b+c)^2 - a^2}{(b+c)^2}$ and also $l_b^2 = ac\frac{(a+c)^2 - b^2}{(a+c)^2}$.

From the data we have that: $ac\frac{(a+c)^2 - b^2}{(a+c)^2} = bc\frac{(b+c)^2 - a^2}{(b+c)^2}$, and hence:

$$a(b+c)^2\left[(a+c)^2 - b^2\right] = b(a+c)^2\left[(b+c)^2 - a^2\right].$$

By using the formulas for the difference of squares and the square of a sum, we have:

$$a\left(b^2 + 2bc + c^2\right)(a+c-b)(a+c+b) = b\left(a^2 + 2ac + c^2\right)(b+c-a)(b+c+a)$$
$$\left(b^2 + 2bc + c^2\right)\left(a^2 + ac - ab\right) = \left(a^2 + 2ac + c^2\right)\left(b^2 + bc - ab\right),$$

and after multiplying polynomials and collecting similar terms we obtain:

$$3a^2bc - 3ab^2c + a^2c^2 - b^2c^2 + ac^3 - bc^3 + a^3b - ab^3 = 0,$$

grouping together

$$3abc(a-b) + c^2\left(a^2 - b^2\right) + c^3(a-b) + ab\left(a^2 - b^2\right) = 0,$$
$$(a-b)\left[3abc + c^2(a+b) + c^3 + ab(a+b)\right] = 0.$$

The factor in the square brackets is always positive, therefore there holds $a - b = 0$, or $a = b$, and the triangle is an isosceles one.

Note: the theorem stating that **a triangle in which the lengths of two angle bisectors are equal is an isosceles triangle** is known in literature as the **Steiner-Lehmus Theorem**. This theorem is also called the **"Internal bisectors problem"** and **"Lehmus's Theorem"** [6-8].

Since the original proof by Steiner and Lehmus, dozens of different proofs have been suggested for this theorem both using geometrical tools and using tools from other fields in mathematics, or by a combination of different tools [9-11]. This is an example of a case in which both a theorem and its converse are true, however the proof in one direction is easy and immediate, and the proof in the other direction is much more difficult.

Obtaining Heron's formula for the area of a triangle

Heron's formula for the area of a triangle whose side lengths are a, b and c is $S_{\triangle ABC} = \sqrt{p(p-a)(p-b)(p-c)}$, where $p = \frac{a+b+c}{2}$ is half the perimeter of the triangle.

Proof

We use formula (1) for the altitude of a triangle in the following form:

$$h_a^2 = \frac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{4a^2},$$

we multiply both sides of the formula by $\frac{a^2}{4}$ and obtain:

$$\frac{a^2h_a^2}{4} = \frac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{16}.$$

Hence:

$$\left(\frac{ah_a}{2}\right)^2 = \frac{a+b+c}{2} \cdot \frac{a+c-b}{2} \cdot \frac{a+b-c}{2} \cdot \frac{b+c-a}{2}. \tag{8}$$

Since:

$$p-a = \frac{a+b+c}{2} - a = \frac{b+c-a}{2}, p-b = \frac{a+b+c}{2} - b = \frac{a+c-b}{2}, p-c = \frac{a+b+c}{2} - c = \frac{a+b-c}{2}$$

and

$$\frac{ah_a}{2} = S_{\triangle ABC}.$$

After substituting in (8), we obtain the equality: $S^2_{\triangle ABC} = p(p-a)(p-b)(p-c)$, and hence:

$$S_{\triangle ABC} = \sqrt{p(p-a)(p-b)(p-c)}.$$

Didactic recommendations: examples for applying the material during lessons.

Activities for obtaining the auxiliary formula (4)

1) In each of the two right-angled triangles shown in Figure 2, express the square of the side AH (h_a^2) using the lengths of the other sides. Use the Pythagorean theorem.
2) Equate the expressions for h_a^2 that you obtained in order to solve the equation and obtain an expression for x.
3) Obtain the formula (4).

Activity for obtaining formula (1)

1) Factor the right-hand side of formula (4) using the formula
   $a^2 - b^2 = (a-b)(a+b)$.
2) Using the formulas
   $(a-b)^2 = a^2 - 2ab + b^2$ and
   $a^2 - b^2 = (a-b)(a+b)$, factor each of the factors you obtained into two additional factors.

Activity for obtaining formula (2)

1) Observe Figure 3 and the data given beside it, and determine the lengths of the sides of the triangle ABM.
2) Write down the formula (4) using the sides of the triangle ABM and its altitude AH.
3) Equate the expression for h_a^2 that you obtained in the previous section with the expression for h_a^2 that appears in formula (4) and solve the equation obtained for m_a^2 .

Activity for obtaining formula (3)

1) Observe Figure 4 and the data given beside it, and determine the lengths of the sides of the triangle ABL.
2) Write down the formula (4) using the sides c, a_1 and l_a of the triangle ABL and its altitude AH.
3) In Figure 5 it is given: the triangle ABC, whose sides are a, b, c; the angle bisector AL of the angle $\measuredangle BAC$, whose length is l_a, the straight line CK that is parallel to AL (K belongs to the continuation of the side BA). Find the following in the figure (show your work): (a) Two similar triangles; (b) An isosceles triangle.
4) Write down a proportion that contains the four sides with one end at the point B of the similar triangles found in the previous section. Using this proportion, express a_1 using the lengths of the other three sides.
5) Using the obtained formula for a_1, and the formula from Section 2, express h_a^2 using a, b, c and l_a .
6) Equate the expression for h_a^2 that you obtained in the previous section with the expression for h_a^2 that appears in formula (4) and solve the equation obtained for l_a^2 .

Activity for discovering and obtaining a proof for the test (sign) of an isosceles triangle based on two equal medians.

1) Use Formula (2) and write down an expression for the square of the median to the side AC (expression for m_b^2).
2) Equate the expressions for m_a^2 and m_b^2, and simplify the equality obtained.
3) From the result you obtained draw a conclusion concerning the triangle ABC.

Activity for discovering and obtaining a proof for the test (sign) of an isosceles triangle based on two equal angle bisectors.

1) Use Formula (3) and write down an expression for the square of the angle bisector of $\measuredangle ABC$ (for l_b^2).
2) Equate the expressions for l_a^2 and l_b^2, and simplify the equality obtained based on the following instructions:
   a) Divide the two sides of the equality by a common factor.
   b) Transform the proportion to an equality of products.
   c) Open all the parentheses, group all the terms on the left-hand side of the equality, and simplify it.
   d) Factor the left-hand side of the equality obtained into two factors, where one factor is $a - b$ (use the method "Factoring Trinomials by Grouping).
   e) Explain why the second factor (The expression in the large parentheses) is always positive.
   f) Draw conclusion, what is the condition on $a - b$, which assures the existence of the equality. Draw a conclusion concerning the triangle ABC.

Activity for obtaining Heron's formula

1) Multiply both sides of the formula (1) by the expression $\frac{a^2}{4}$.
2) Determine the geometrical meaning of the expression you obtained on the left-hand side of the new equality.
3) Write down the right-hand side of the equality as the product of four fractions, each of which has the denominator 2.
4) Denote the fraction $\frac{a+b+c}{2}$ by p.
5) What is the geometrical meaning of p?
6) Write down each of the expressions $p - a$, $p - b$ and $p - c$ as a fraction which only has the lengths of the sides of the triangle ABC.
7) Write down the right-hand side of the equality as the product of factors that contain p.
8) Express the area of the triangle ABC (S_{ABC}) using expressions that contain p.

Solving calculation problems in a triangle using an algebraic method (using the relation formulas (1) to (3)).

The use of the developed formulas for the calculation of the side lengths of a triangle, as presented in the three examples below is based on the fact that students have the technical skills for solving algebraic equations of the first and the second degree.

Example 1

In the $\triangle ABC$ it is given that:

$AB = 6$, $m_{BC} = 6$ is the median to the side BC, $m_{AC} = 4$ is the median to the side AC.

Calculate the lengths of the sides AC and BC.

$$\left[\text{answer}: AC = \sqrt{\frac{146}{3}}, BC = \sqrt{\frac{56}{3}}\right]$$

Example 2

In the $\triangle ABC$ it is given that:

$BC = 5$, $m_{BC} = \frac{1}{2}\sqrt{209}$ is a median to the side BC, $l_{BC} = 4\sqrt{3}$ is the bisector of the angle $\measuredangle BAC$.

Calculate the lengths of the sides AB and AC.

[answer: $AC = 9$, $AB = 6$]

Example 3

In the $\triangle ABC$ it is given that:

$BC = 10$, $l_{BC} = 6\sqrt{2}$ is the bisector of the angle $\measuredangle BAC$, $h_{BC} = 3\sqrt{7}$ is the altitude to the side BC.

Calculate the lengths of the sides AB and AC.

[answer: $AC = 12$, $AB = 8$]

Notes

Problem 1 is solved by means of double use of the formula (2).

Problem 2 is solved by using the formulas (2) and (3).

Problem 3 is solved by using the formulas (1) and (3).

Summary

Known algebraic formulas for calculating the lengths of certain segments in a triangle were developed, as well as a formula for calculating the area of a triangle – using algebraic manipulations which are within the skill set of students aged 14-15. One should expect that after presenting this method to students aged 14 or more, who had acquired the skill of using simple algebraic formulas, they would be able to develop more complex formulas that may allow mathematical tasks to be solved and proofs for theorems on a high level of difficulty to be found.

References

1. Tall, D.(2007) 'Teachers as Mentors to encourage both power and simplicity in active material learning', Plenary Lecture at the *Third Annual Conference for Middle East Teachers of Science, Mathematics and Computing, 17-19, March 2007*, Abu-Dhabi.
2. Liekin, R.: 2009, 'Multiple proof tasks: Teacher practice and teacher education', *ICME Study 19*, Vol. 2, pp. 31-36.
3. Levav-Waynberg, A. and Leikin, R.(2009). 'Multiple solutions for a problem: A tool for evaluation of mathematical thinking in geometry', *In Proceedings of CERME 6, January 28th - February 1st 2009*, Lyon, France, pp. 776-785.

4. Stupel, M, & Ben-Chaim, D. (2013). One problem, multiple solutions: How multiple proofs can connect several areas of mathematics. Far East Journal of Mathematical Education, Vol.11(2), pp. 129-161.

5. Fraivert, D. & Stupel, M. (2014). Development and implementation of algebraic formulas for calculation in trapezoids. In the book of the international conference "Mathematical education: current state and future perspectives". Dedicated to the 95th anniversary of Prof. A. Stolyar. A. Kuleshov University' Mogilev, Beloruse State

6. Coxeter, H. S. M. and Greitzer, S. L. "The Steiner-Lehmus Theorem." §1.5 http://www.amazon.com/exec/obidos/ASIN/0883856190/ref=nosim/ericstreasuretroGeometry Revisited. Washington, DC: Math. Assoc. Amer., pp. 14-16, 1967.

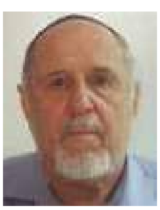
7. Coxeter, H. S. M. http://www.amazon.com/exec/obidos/ASIN/0471504580/ref=nosim/ericstreasuretroIntroduction to Geometry, 2nd ed. New York: Wiley, p. 9, 1969.

8. Hunter, J. A. H. and Madachy, J. S. http://www.amazon.com/exec/obidos/ASIN/0486231100/ref=nosim/ericstreasuretroMathematical Diversions. New York: Dover, pp. 72-73, 1975.

9. Oxman, V. (2004). "On the Existence of Triangles with Given Lengths of One Side and Two Adjacent Angle Bisectors." *Forum Geom.*Vol. **4**, 215-218.

10. Abu-Saymeh, S.; Hajja, M., and ShahAli, H. A.(2008) "Another Variation on the Steiner-Lehmus Theorem." *Forum. Geom.* Vol. 8, 131-140.

11. Hajja, M. (2008). A short trigonometric proof of the Steiner-Lehmus Theorem. *Forum Geom.* Vol. 8, 39-42.



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ClassRoom

Theorem Concerning A Magic Triangle

SHAILESH A SHIRALI

Magic Triangles and Squares are often used as a 'fun activity' in the math class, but the magic of the mathematics behind such constructs is seldom explained and often left as an esoteric mystery for students. An article that can be used by teachers in the middle school (6-8) to justify to students that everything in mathematics has a reason and a solid explanation behind it. Plus a good way to practise some simple algebra.

According to the Wikipedia entry [2], "***A magic triangle** … is an arrangement of the integers from 1 to n on the sides of a triangle with the same number of integers on each side, … so that the sum of integers on each side is a constant, the **magic sum** of the triangle.*" It then adds: *Unlike magic squares, there are different magic sums for magic triangles of the same order.* They are also known as ***perimeter magic triangles*** [1].

The number of integers on each side is called the ***order*** of the magic triangle. The order is clearly equal to $(n + 3)/3 = n/3 + 1$. (As a corollary, we see that n must be a multiple of 3.) Figure 1 displays a third-order magic triangle with $n = 6$ and magic sum 9.

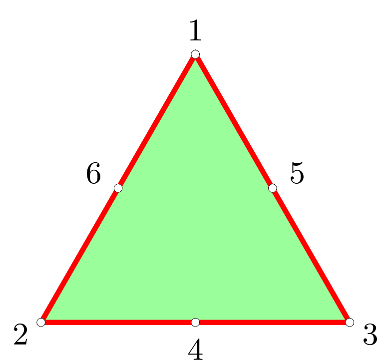


Figure 1. Third-order magic triangle with $n = 6$

Keywords: Magic triangle, perimeter magic triangle, magic sum, arithmetic progression

In this short note, we study magic triangles with $n = 9$ (which means that they are of order four). That is, we arrange the integers 1 to 9 on the sides of a triangle, with four integers on each side, in such a way that the sum of the integers on each side is the same. We discover, quite by chance, a striking result concerning the three numbers placed at the vertices. Specifically, we show the following:

Theorem. *The vertex numbers of a fourth-order magic triangle, when arranged in order, form an arithmetic progression.*

In fact, this is also true of third-order magic triangles, as Figure 1 illustrates. (We can see that the property holds for the magic triangle shown in the figure. But it is true for all third-order magic triangles. The proof of this is left as an exercise.)

Note that x, y, z are sums of pairs of numbers.

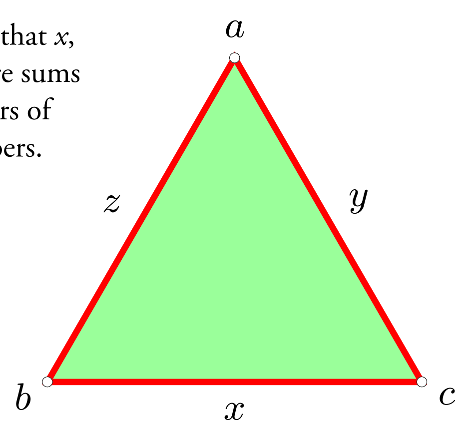


Figure 2. General relationships for fourth-order magic triangles ($n = 9$)

Proof. Let a, b, c be the numbers at the three vertices (see Figure 2). Let x, y, z be the sums of the other two numbers on the three edges, respectively (x on edge b—c; y on edge c—a; z on edge a—b). Let s be the magic sum of this triangle. Then we have the following relations:

$$\left.\begin{aligned} b + x + c &= s, \\ c + y + a &= s, \\ a + z + b &= s. \end{aligned}\right\} \qquad (1)$$

By adding the three relations we get:

$$2(a + b + c) + (x + y + z) = 3s. \qquad (2)$$

We also have:

$$a + b + c + x + y + z = 1 + 2 + 3 + \cdots + 9 = 45. \qquad (3)$$

Hence:

$$a + b + c = 3s - 45. \qquad (4)$$

So the sum of the numbers at the vertices is $3s - 45$. Note that this is a multiple of 3. So the sum of the vertex numbers is necessarily a multiple of 3.

Since $a + b + c \geq 1 + 2 + 3 = 6$ and $a + b + c \leq 9 + 8 + 7 = 24$, we get $6 \leq 3s - 45 \leq 24$, and therefore:

$$17 \leq s \leq 23. \qquad (5)$$

It follows that $s \in \{17, 18, 19, 20, 21, 22, 23\}$. We look at each possibility in turn.

The case s = 17: This possibility implies that $a + b + c = 6$ and can take place if and only if $\{a, b, c\} = \{1, 2, 3\}$. But in that case, the vertex numbers form an AP, as required. Figure 3 displays one of the magic triangles corresponding to this situation (there is one other such triangle which we will leave for the reader to find).

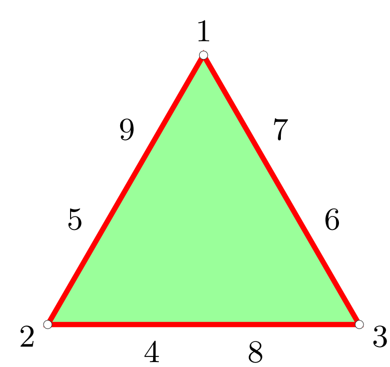


Figure 3. Fourth-order magic triangle with magic sum 17

The case s = 18: Rather to our surprise, we find that this possibility cannot occur at all! (So the assertion that the vertex numbers form an AP in this case is vacuously true, in the sense that it cannot be falsified.) But to see why takes a few steps which we now describe.

If $s = 18$, then we must have $a + b + c = 9$. The only sets of three distinct integers whose sum is 9, the integers all lying between 1 and 9 (inclusive), are the following: {1, 2, 6}, {1, 3, 5} and {2, 3, 4}. (Please verify for yourself that these are the only possibilities.) In the latter two cases, the vertex numbers form APs; there is nothing more to show. So we focus on the first possibility, where the vertex numbers are 1, 2, 6. The situation is as depicted in Figure 4.

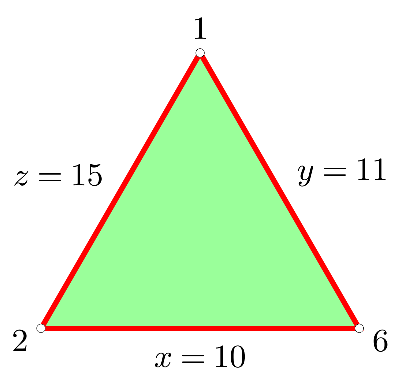


Figure 4. Analysis of a fourth-order magic triangle with magic sum 18

Consider the placement of the number 9. Can 9 be part of the pair whose sum is x? If so, then the other number of that pair must be 1. However, 1 has already been 'used up' (at a vertex). It follows that 9 cannot be part of the pair whose sum is x. Can 9 be part of the pair whose sum is y? If so, then the other number of that pair must be 2. However, 2 has already been 'used up' (at another vertex). It follows that 9 cannot be part of the pair whose sum is y. Can 9 be part of the pair whose sum is z? If so, then the other number of that pair must be 6. However, 6 has already been 'used up' (at yet another vertex). It follows that 9 cannot be part of the pair whose sum is z. All the possibilities have now been eliminated, which means that 9 has no place at all! But this means that the vertex numbers cannot be 1, 2, 6. Note the crucial role played by the number 9 in the above argument. Let us describe this role by saying that 9 *is a **witness** to showing the impossibility of having 1, 2, 6 as the vertex numbers.*

It turns out that the other two possibilities listed—with vertex numbers 1,3,5 and 2,3,4 respectively—also do not 'work'; in both cases, we are unable to construct the relevant magic triangle. And in both cases, the number 9 acts as a witness to show their impossibility. The relevant diagrams are shown in Figure 5. However, we omit the argument as it goes along exactly the same lines as the argument made above.

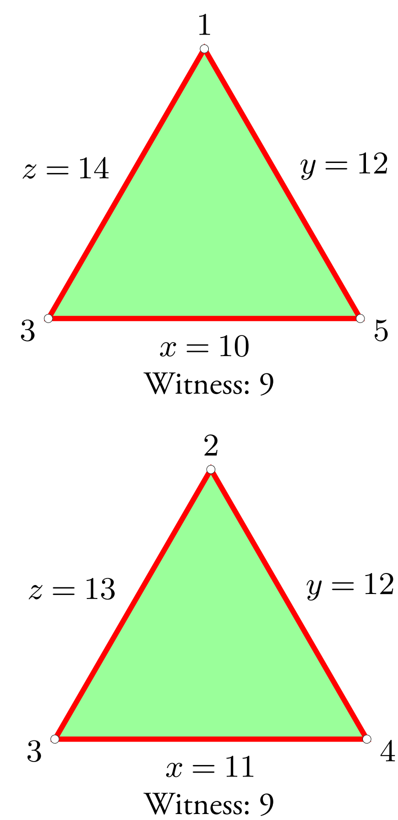


Figure 5. Analysis of fourth-order magic triangles with magic sum 18

It follows that if $s = 18$, the statement that the vertex numbers form an AP is vacuously true.

The case s = 19: This possibility implies that $a + b + c = 12$. The only sets of three distinct integers, all between 1 and 9 (inclusive), who sum is 12, are the following: {1, 2, 9}, {1, 3, 8}, {1, 4, 7}, {1, 5, 6}, {2, 3, 7}, {2, 4, 6} and {3, 4, 5}. Of these, the ones that need closer examination are the following:

$$\{1, 2, 9\}, \{1, 3, 8\}, \{1, 5, 6\}, \{2, 3, 7\}. \qquad (6)$$

The first two cases are studied in Figure 6(a) and Figure 6 (b). In each case we need a witness that will play the role played by 9 in the earlier analysis. The relevant witnesses are listed alongside the captions. We leave it to the reader to verify that the witness plays its expected role in each case.

The remaining two cases are depicted in Figure 6 (c) and Figure 6 (d). As earlier, the relevant witnesses are listed alongside the captions. Once again, we leave the missing steps in the argument to be filled in by the reader. It follows that if $s = 19$, the vertex numbers form an AP in all the cases.

The case s = 20: This possibility implies that $a + b + c = 15$. We tackle the problem differently in this case. We must show that *a*, *b*, *c* (in some order) form an AP. This is equivalent to showing that one of the numbers *a*, *b*, *c* is 5. (The possible values for (*a*, *b*, *c*) then become (1, 5, 9), (2, 5, 8), (3, 5, 7), (4, 5, 6), all of which are APs.) Suppose not; that is, suppose that 5 occurs as an interior number on one of the edges, say on edge *b*—*c*. Let the remaining number (i.e., the fourth number) on edge *b*—*c* be k. Then we have the following relation:

$$b + c + k + 5 = 20, \ \therefore \ b + c + k = 15. \qquad (7)$$

We also have $a + b + c = 15$. Comparing the two relations, we see that $a = k$. But this means that the same number has been used twice (it has acted as a witness, to use the former term). This is contrary to the stated requirement that no number can be used more than once. It follows

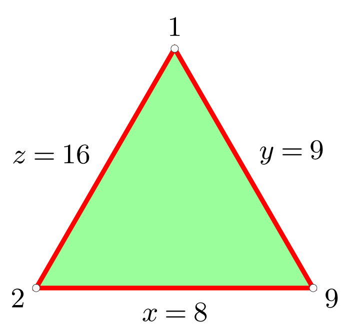


Figure 6 (a). Witness: 7

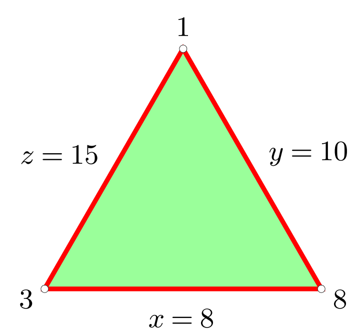


Figure 6 (b). Witness: 7

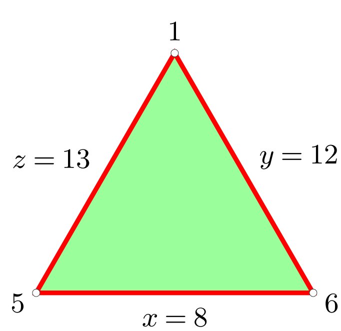


Figure 6 (c). Witness: 7

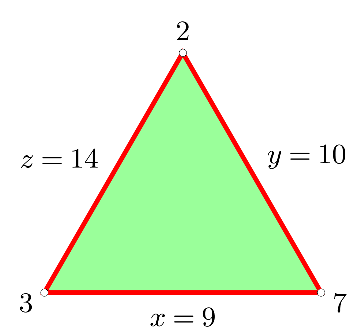


Figure 6 (d). Witness: 7

Figure 6. Analysis of fourth-order magic triangles with magic sum 19

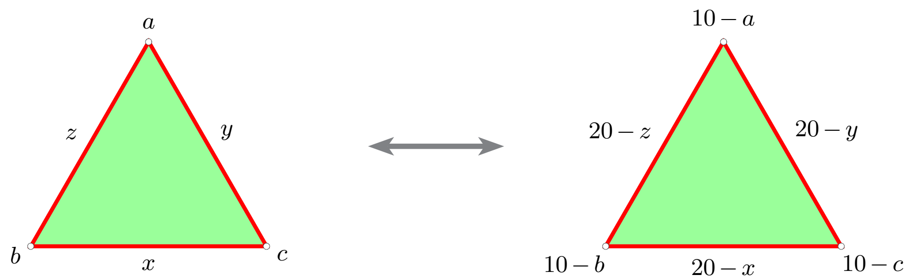


Figure 7. Fourth-order magic triangles with magic sums s and 40−s

that one of the numbers a, b, c is 5, and therefore that a, b, c (in some order) form an AP.

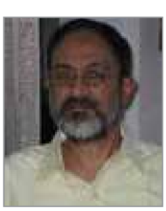
The cases s = 21, 22, 23: The remaining cases (s = 21, 22, 23) are best handled by appealing to symmetry. Given a fourth-order magic triangle with magic sum s, if we replace every entry by its tens-complement, i.e., we replace a, b, c, . . . by $10 - a$, $10 - b$, $10 - c$, . . ., respectively (this means that we replace x, y, z by $20 - x$, $20 - y$, $20 - z$, respectively), we get a fourth-order magic triangle whose magic sum is $40 - s$ (see Figure 7).

If the magic sum s is one of the numbers 21, 22, 23, then $40 - s$ is one of the numbers 19, 18, 17, which means that the earlier analysis applies. Since the vertex numbers for the modified magic triangles do form an AP (proved above), the same must be true for the vertex numbers of the original magic triangles.

We remark in closing that the nonexistence of a magic triangle with $s = 18$ implies, in the light of the above remark, the nonexistence of a magic triangle with $s = 22$. Here too, the assertion that "the vertex numbers form an AP" is vacuously true.

References

1. Harvey Heinz. "Perimeter Magic Triangles." http://www.magic-squares.net/perimeter.htm#Order4PerimeterMagicTriangles
2. Wikipedia. "Magic triangle (mathematics)." https://en.wikipedia.org/wiki/Magic_triangle_(mathematics)



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Characterisation of a Right Triangle

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

Let us start by making a few remarks on the notion of ***characterisation*** in mathematics, a theme that is central to the subject. The notion has relevance in other settings as well, but we will restrict ourselves to its meaning in mathematics. Given any set S which has been specified in some well-defined manner, we may want a test by which we can decide membership of this set. That is, we want to fill in the blanks in the following sentence in some appropriate and meaningful way:

$$\text{Entity } x \text{ belongs to } S \iff \_\_\_\_ \ \_\_\_\_ \ \_\_\_\_.$$

To be of any interest, the test must not be a mere restatement of the defining property of the set. If this requirement is met, we call this a *non-trivial characterisation* of the set. Some of the most interesting and nicest results of mathematics are non-trivial characterisations of one kind or another.

Here are two simple examples which illustrate the theme.

Right triangles: A non-trivial characterisation of the set of right triangles is Pythagoras's theorem: *A triangle is right-angled if and only if the square of one of the sides is equal to the sum of the squares of the other two sides*. The beauty of this result is its compactness and its surprise value: there is no obvious reason whatever why the result should be true. (But the surprise is spoiled to some extent by the great fame of this result!)

Keywords: Right triangle, characterisation

Prime numbers: Do there exist non-trivial characterisations of the set of prime numbers? This is an enormously interesting and deep question which has occupied the attention of mathematicians for over two millennia. Ancient Chinese mathematicians came close when they stated that a positive integer $n > 1$ is prime if and only if $2^n - 2$ is divisible by n. This turns out to be *almost* correct! (The correct statement is: If a positive integer $n > 1$ is prime, then $2^n - 2$ is divisible by n. The converse statement is false.) The answer to our question is: Yes, there do exist such characterisations, but to understand them requires a substantial buildup of concepts and we do not dwell on them for now.

Against this background, we offer the following surprising characterisation of right-angled triangles. It is adapted from Problem 1 of the European Girls' Mathematical Olympiad (EGMO), 2013 [1].

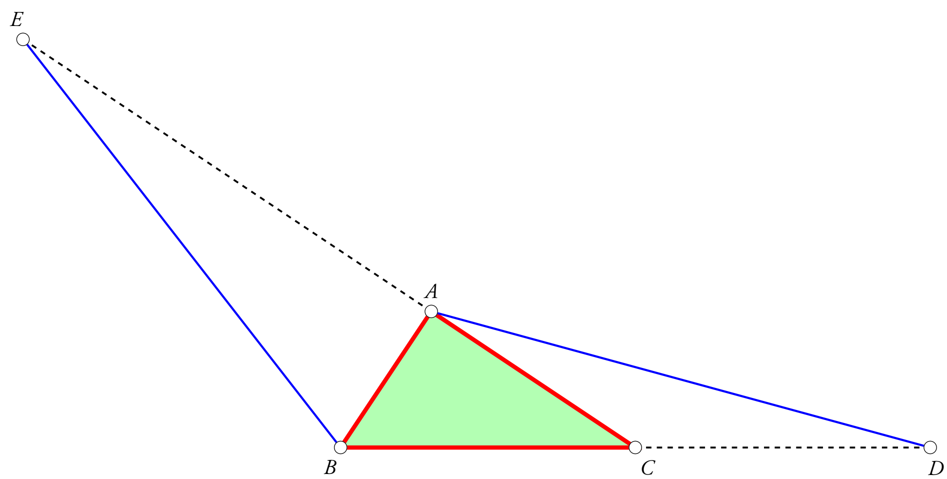


Figure 1.

Theorem 1 (EGMO-2013-1)**.** *Given any* $\triangle ABC$*, extend* BC *to* D *and* CA *to* E *so that* $\overrightarrow{BD} = 2 \cdot \overrightarrow{BC}$ *and* $\overrightarrow{CE} = 3 \cdot \overrightarrow{CA}$*. (See Figure 1.) Join* AD *and* BE*. Then we have the following result:*
$\measuredangle BAC = 90^\circ \iff AD = BE.$

Remark. Before plunging into the proof, we note that there appears to be a basic lack of symmetry about the result; given that it is $\measuredangle A$ which is ultimately going to be the right angle, the condition surely should not discriminate between vertices B and C. (For example, in the same setup, Pythagoras's theorem asserts that $a^2 = b^2 + c^2$; note that the condition is symmetric in b and c, i.e., it does not discriminate between the vertices B and C.) But it appears to do just that. The resolution of this is the following. We find that if in Theorem 1, we completely swap the roles of B and C, we get (as anticipated) another correct statement. That is, the following is true.

Theorem 2 (EGMO-2013-1)**.** *Given any* $\triangle ABC$*, extend* CB *to* F *and* BA *to* G *so that* $\overrightarrow{CF} = 2 \cdot \overrightarrow{CB}$ *and* $\overrightarrow{BG} = 3 \cdot \overrightarrow{BA}$*. (See Figure 2.) Join* AF *and* CG*. Then we have the following result:*
$\measuredangle BAC = 90^\circ \iff AF = CG.$

Proof. Given the remarks made earlier, it suffices to prove either Theorem 1 or Theorem 2. We choose to prove Theorem 2 and we do so using vector algebra.

Let A be the origin, and let $\overrightarrow{u} = \overrightarrow{AB}$, $\overrightarrow{v} = \overrightarrow{AC}$. Then we have $\overrightarrow{BF} = \overrightarrow{CB} = \overrightarrow{u} - \overrightarrow{v}$ and $\overrightarrow{GA} = 2\overrightarrow{AB} = 2\overrightarrow{u}$, so:

$$\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF} = \overrightarrow{u} + (\overrightarrow{u} - \overrightarrow{v}) = 2\overrightarrow{u} - \overrightarrow{v},$$

$$\overrightarrow{GC} = \overrightarrow{GA} + \overrightarrow{AC} = 2\overrightarrow{u} + \overrightarrow{v}.$$

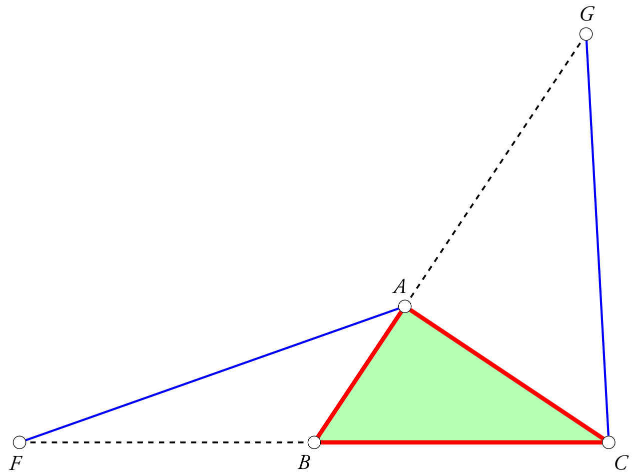


Figure 2.

Hence

$$AF^2 = 4\,\overrightarrow{u}\cdot\overrightarrow{u} - 4\,\overrightarrow{u}\cdot\overrightarrow{v} + \overrightarrow{v}\cdot\overrightarrow{v},$$
$$CG^2 = 4\,\overrightarrow{u}\cdot\overrightarrow{u} + 4\,\overrightarrow{u}\cdot\overrightarrow{v} + \overrightarrow{v}\cdot\overrightarrow{v}.$$

It follows that

$$AF = CG \iff \overrightarrow{u}\cdot\overrightarrow{v} = 0 \iff \overrightarrow{u}\perp\overrightarrow{v},$$

i.e.,

$$AF = CG \iff \measuredangle BAC = 90^\circ.$$

This proves the desired result. However, the proof yields a bit more. For, we have:

$$AF^2 - CG^2 = -8\,\overrightarrow{u}\cdot\overrightarrow{v}.$$

Since $\overrightarrow{u}\cdot\overrightarrow{v}$ is positive when $\measuredangle A$ is acute and negative when $\measuredangle A$ is obtuse, we are able to make a more complete statement:

$$AF < CG \iff \measuredangle A < 90^\circ,$$
$$AF = CG \iff \measuredangle A = 90^\circ,$$
$$AF > CG \iff \measuredangle A > 90^\circ$$

A geometric proof. You may notice that there is something odd about the above proof. Though it is simple in terms of the algebra involved, it does not tell us *why* the result is true; it does not yield any understanding of the result. At the end of the proof, one submits to the force of its logic but is left with no understanding of "what is going on." (It is, surely, reasonable to expect that of a proof.) So it seems worthwhile to be on the lookout for a proof that yields some geometric insight into the configuration.

We respond to the challenge by presenting the following proof. But we cast it in a different way: we present it as a theorem about a *parallelogram*. In the wording below, we have tried to label the vertices in such a way that the analysis conducted earlier is compatible with the new diagram. (See Figure 3.)

Theorem 3. *Let EBDG be a parallelogram. Let C be the midpoint of BD, and let EC meet the diagonal BG at A. Join AD. Then we have the following equivalence:*

$$\measuredangle BAC = 90^\circ \iff DA = DG.$$

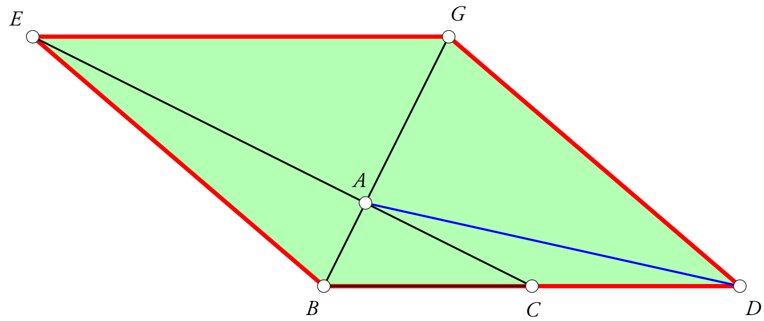


Figure 3. $\measuredangle BAC = 90^\circ \iff DA = DG$

Proof. The proof can be accomplished using vector algebra, along the same lines as above. We leave the details to the reader. For now, we present a geometric proof. The proof is anchored on the rotational symmetry of a parallelogram, namely, the half-turn symmetry about the common midpoint of its two diagonals.(See Figure 4.)

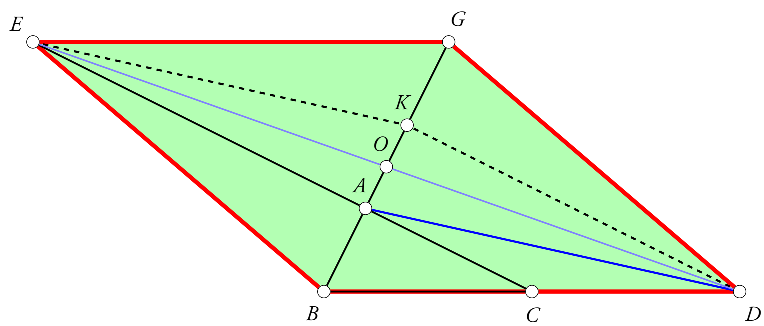


Figure 4. Proving that $\measuredangle BAC = 90^\circ \iff DA = DG$

We subject the parallelogram $BDGE$ to a half-turn about its centre O and let K be the image of A under this transformation. Note that D and E swap places under the same transformation (since O is the midpoint of DE). Join EK and KD. Observe the following: (i) O is the midpoint of AK; (ii) $EADK$ is a parallelogram, since AK and DE bisect one another; (iii) $BA = AK = KG$ (i.e., A and K are points of trisection of diagonal BG).

The proof now rolls on its own! Suppose $\measuredangle BAC = 90^\circ$. Then $EA \perp BK$, from which it follows by rotational symmetry of the parallelogram that $DK \perp AG$. It follows that $\triangle DKA \cong \triangle DKG$, and hence that $DA = DG$.

Conversely, if $DA = DG$, then $\triangle DAG$ is isosceles. We also have $KA = KG$, hence $\triangle DKA \cong \triangle DKG$ ('SSS' congruence), from which it follows that $DK \perp AG$ and hence that $EA \perp BA$, i.e., $\measuredangle BAC = 90^\circ$.

This proof seems much more satisfying!

References

1. European Girls' Mathematical Olympiad 2013, https://www.egmo.org/egmos/egmo2/

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

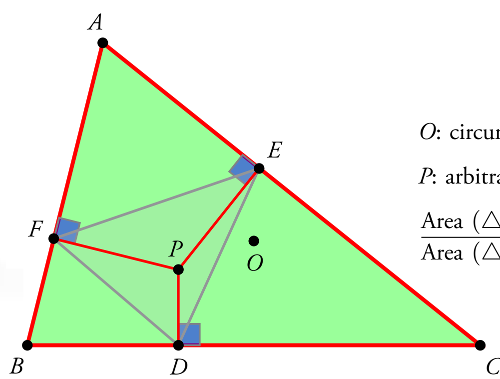
How to Prove it

SHAILESH SHIRALI

In this episode of "How To Prove It", we prove a beautiful and striking formula first found by Leonhard Euler; it gives the area of the pedal triangle of a point with reference to another triangle.

Euler's formula for the area of a pedal triangle

Given a triangle ABC and a point P in the plane of ABC (note that P does not have to lie within the triangle), the **pedal triangle** of P with respect to $\triangle ABC$ is the triangle whose vertices are the feet of the perpendiculars drawn from P to the sides of ABC. See Figure 1. The pedal triangle relates in a natural way to the parent triangle, and we may wonder whether there is a convenient formula giving the area of the pedal triangle in terms of the parameters of the parent triangle. The great 18th-century mathematician Euler found just such a formula (given in Box 1). It is a compact and pleasing result, and it expresses the area of the pedal triangle in terms of the radius R of the circumcircle of $\triangle ABC$ and the distance between P and the centre O of the circumcircle.



$$\frac{\text{Area }(\triangle DEF)}{\text{Area }(\triangle ABC)} = \frac{1}{4}\left(1 - \frac{OP^2}{R^2}\right)$$

Figure 1. Euler's formula for the area of the pedal triangle of an arbitrary point

Keywords: Circle theorem, pedal triangle, power of a point, Euler, sine rule, extended sine rule, Wallace-Simson theorem

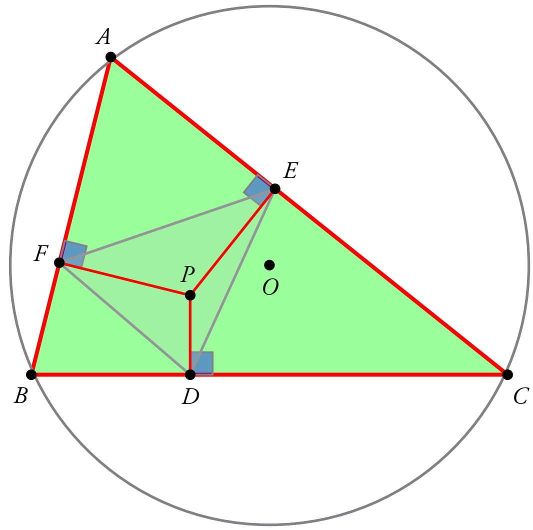


Figure 2. Proof of Euler's formula – step I

Euler's formula for the area of a pedal triangle

Theorem 1 (Euler)**.** Given a reference triangle ABC and a point P in the plane of the triangle, the ratio of the area of the pedal triangle of P to the area of $\triangle ABC$ is given by:

$$\frac{\text{Area of } \triangle DEF}{\text{Area of } \triangle ABC} = \frac{1}{4}\left(1 - \frac{OP^2}{R^2}\right), \tag{1}$$

where O is the circumcentre and R is the radius of the circumcircle of $\triangle ABC$.

Box 1

The occurrence of the distance OP in this formula comes as a major surprise. The reader is invited to look for a proof before reading on. It is a pretty challenge!

Proof of Euler's formula. The proof will unfold in several stages. The sine formula for area (see Figure 2) tells us that

$$\text{Area of } \triangle DEF = \frac{1}{2}\left(DE \cdot DF \cdot \sin \measuredangle EDF\right). \tag{2}$$

We now obtain simplified expressions for each term on the RHS of this formula: DE, DF and $\sin \measuredangle EDF$. This will yield the desired result.

First, consider DE. Note that $DCEP$ is a cyclic quadrilateral, CP being a diameter of its circumcircle (to see why, note that $\measuredangle PDC = 90^\circ = \measuredangle PEC$). The 'extended sine rule' (i.e., the statement that in any triangle, the ratio of each side to the sine of the opposite angle equals the diameter of the circumcircle of the triangle) applied to $\triangle CDE$ tells us that

$$\frac{DE}{\sin \measuredangle ECD} = CP, \tag{3}$$

and so:

$$DE = CP \cdot \sin C. \tag{4}$$

In exactly the same way, we get:

$$DF = BP \cdot \sin B. \tag{5}$$

Next, we seek an expression for $\sin \measuredangle EDF$. For this, we extend BP till it meets the circumcircle again at point K; draw segments CK, AK (see Figure 3).

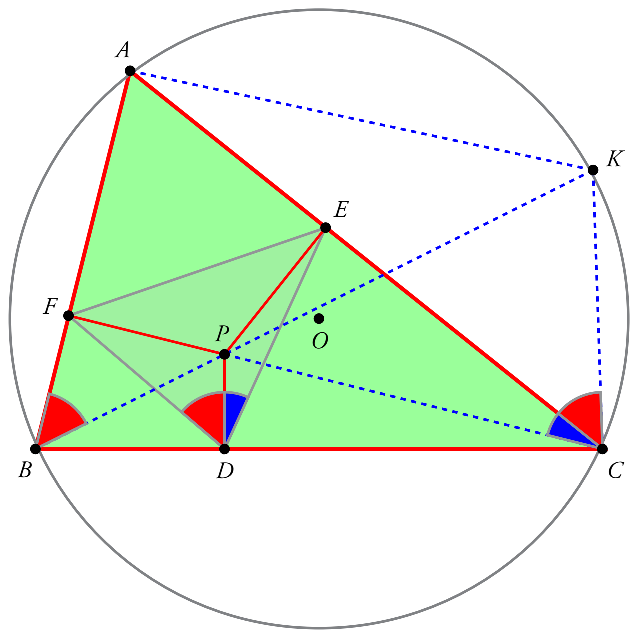


Figure 3. Proof of Euler's formula – step II

First we note that $\measuredangle EDF = \measuredangle KCP$ (as marked in the figure); for: $\measuredangle PDE = \measuredangle PCE$ and $\measuredangle PDF = \measuredangle PBF = \measuredangle KBA = \measuredangle KCA$. Hence we get by addition: $\measuredangle EDF = \measuredangle KCP$, and so: $\sin \measuredangle EDF = \sin \measuredangle KCP$. Next:

$$\frac{PK}{\sin \measuredangle KCP} = \frac{CP}{\sin \measuredangle PKC} = \frac{CP}{\sin A}. \tag{6}$$

This yields:

$$\sin \measuredangle KCP = \frac{PK \cdot \sin A}{CP} = \sin \measuredangle EDF. \tag{7}$$

Hence:

$$\begin{aligned} \text{Area of } \triangle DEF &= \frac{1}{2} CP \cdot \sin C \cdot BP \cdot \sin B \cdot \frac{PK \cdot \sin A}{CP} \\ &= \frac{1}{2} BP \cdot PK \cdot \sin A \, \sin B \, \sin C. \end{aligned} \tag{8}$$

Now we invoke another well-known formula for the area of a triangle which follows from the extended sine rule and the sine formula for area:

$$\begin{aligned}\text{Area of } \triangle ABC &= \frac{1}{2} bc \, \sin A \\ &= \frac{1}{2} 2R \sin B \cdot 2R \sin C \cdot \sin A \\ &= 2R^2 \, \sin A \, \sin B \, \sin C.\end{aligned}$$

Hence we obtain:

$$\text{Area of } \triangle DEF = \frac{1}{2} BP \cdot PK \cdot \frac{\text{Area of } \triangle ABC}{2R^2}. \tag{9}$$

Next, note that $BP \cdot PK$ is simply the **power** of the point P with respect to the circumcircle, and this is equal to $R^2 - OP^2$. (Some of you may not recall the definition of "power of a point with respect to a given circle". For your convenience, we have assembled all the relevant concepts and formulas in the appendix at the end of this article.) Hence:

$$\frac{\text{Area of } \triangle DEF}{\text{Area of } \triangle ABC} = \frac{R^2 - OP^2}{4R^2} = \frac{1}{4}\left(1 - \frac{OP^2}{R^2}\right), \tag{10}$$

as claimed.

A corollary to Euler's formula: the Wallace-Simson theorem. If P lies on the circumcircle, then $OP = R$, so the formula tells us that the area of the pedal triangle is 0. This is equivalent to asserting that the vertices of the pedal triangle lie in a straight line. In this form, the statement is well-known as the Wallace-Simson theorem (see Figure 4):

Theorem 2 (Wallace & Simson)**.** The feet of the perpendiculars dropped from a point on the circumcircle of a triangle to the sides of the triangle lie in a straight line.

The Wallace-Simson theorem can be proved directly, by old-fashioned "angle-chasing". (Try to find a proof on your own! Or see Appendix 2.) The line on which points D, E, F lie is called the **pedal line** of P (or, in older texts, the **Simson line** of P).

Appendix 1: Power of a Point

Let Γ be a circle with centre O, and let P be any point in the plane of the circle. To start with, consider the case when P lies outside the circle. Let ℓ be any line through P, and let ℓ cut the circle at points A, B. (See Figure 5.) Let PT be a tangent from P to the circle. Then $PA \cdot PB = PT^2$. To see why, we only need to see that $\triangle PAT \sim \triangle PTB$ (compare the angles of the triangles to see why). By the theorem of Pythagoras, $PT^2 = OP^2 - r^2$. Hence $PA \cdot PB = OP^2 - r^2$. Since the value of $OP^2 - r^2$ does not depend on the choice of line ℓ, it follows from this that the product $PA \cdot PB$ is independent of ℓ; it is constant over all such lines.

If P lies within the circle, a different figure needs to be drawn; the tangent PT now does not exist, but the conclusion remains the same: the product $PA \cdot PB$ is independent of the choice of line ℓ, and the constant value is equal to $OP^2 - r^2$. The reader is invited to draw the relevant figure in this case and to verify the stated conclusion.

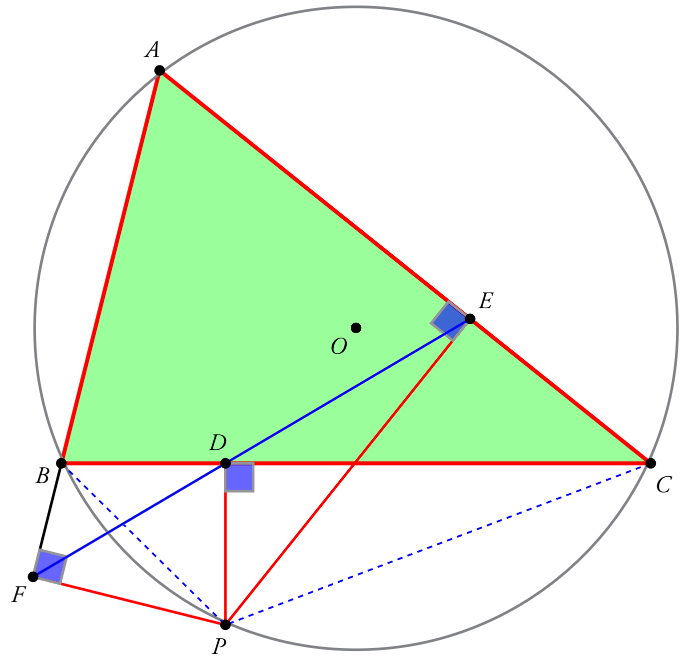


Figure 4. The Wallace-Simson theorem: the feet of the perpendiculars from P to the sidelines of the triangle lie in a straight line

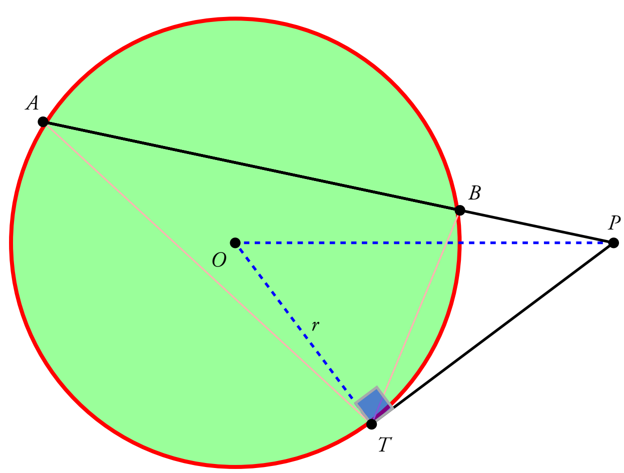


Figure 5.

If P lies outside the circle (as in Figure 5), then $OP > r$, hence $PA \cdot PB > 0$. If P lies inside the circle, then $OP < r$, hence $PA \cdot PB < 0$. And if P lies on the circle, then $OP = r$, hence $PA \cdot PB = 0$.

The quantity $OP^2 - r^2$ is called the **power of point P with respect to the circle**. Thus the power is positive for points P lying outside the circle, 0 for points P on the circle, and negative for points P lying inside the circle.

Appendix 2: Proof of the Wallace-Simson theorem

In order to prove that points F, D, E lie in a straight line, we must prove that $\measuredangle PDF + \measuredangle PDE = 180^\circ$. (See Figure 6.) Now $\measuredangle PDF = \measuredangle PBF$ (from the cyclic quadrilateral $PDBF$); and $\measuredangle PDE + \measuredangle PCE = 180^\circ$ (from the cyclic quadrilateral $PCED$; to see why it is cyclic, note that $\measuredangle CDP$ and $\measuredangle CEP$ are both right angles). So the task comes down to proving that $\measuredangle PBF = \measuredangle PCE$. But this readily follows from the fact that quadrilateral $ABPC$ is cyclic: $\measuredangle PBF$ is an exterior angle of this quadrilateral, and it is equal to the interior opposite angle which is $\measuredangle PCA$.

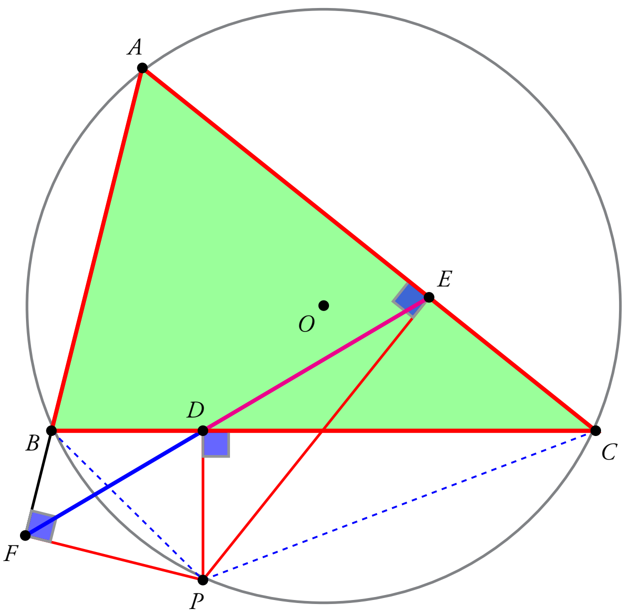


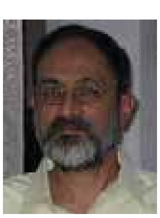
Figure 6.

References

1. A. Bogomolny, "Sides and Area of Pedal Triangle" from *Interactive Mathematics Miscellany and Puzzles*, http://www.cut-the-knot.org/triangle/PedalTriangle.shtml, Accessed 03 January 2016

2. R. A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, 1960, 135–141

3. Weisstein, Eric W. "Pedal Triangle." From *MathWorld*–A Wolfram Web Resource. http://mathworld.wolfram.com/PedalTriangle.html



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An Angle-in-a-Quadrilateral Problem

$\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$

Shown in Figure 1 is quadrilateral *ABCD* with $DA = AB = BC$ and $\measuredangle DAB = 74°$, $\measuredangle ABC = 166°$. The problem is to find the measure of $\measuredangle BCD$.

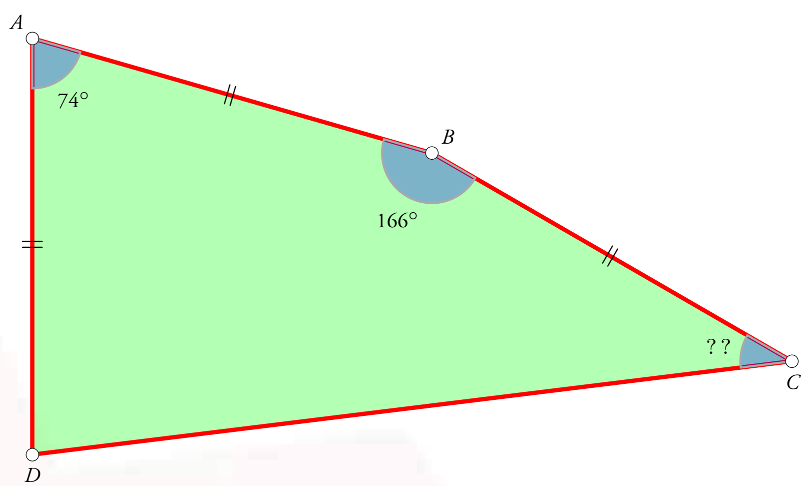

Figure 1.

The problem (taken from [1]) is challenging but admits many different solutions, some of which are very elegant (which is what makes this problem so interesting). We present some of these solutions here.

A solution using trigonometry.
Since $AD = AB$, $\measuredangle ABD = 53°$. Let $AB = a$. From the isosceles $\triangle ABD$ (Figure 2) we get $DB = 2a \sin 37°$. (To see why, imagine dropping a perpendicular from *A* to base *BD*.)

Keywords: Quadrilateral, angle

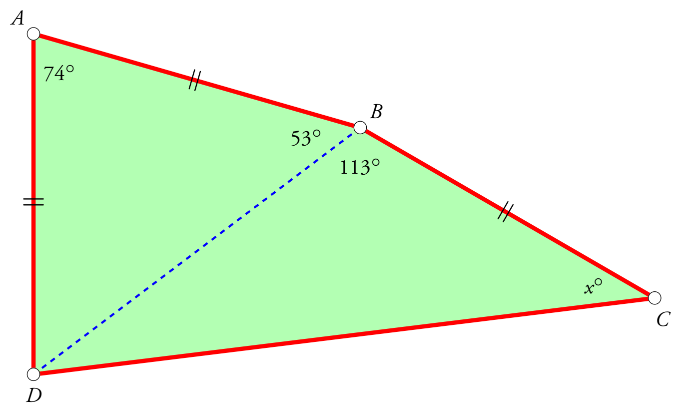

Figure 2.

Next, invoking the sine rule in $\triangle BCD$ and using the fact that supplementary angles have equal sines, we get:

$$\frac{DB}{\sin x^\circ} = \frac{BC}{\sin(x^\circ + 113^\circ)},$$

$$\therefore \quad \frac{2a\,\sin 37^\circ}{\sin x^\circ} = \frac{a}{\sin(x^\circ + 113^\circ)}.$$

This yields: $2\,\sin 37^\circ \cdot \sin(x^\circ + 113^\circ) = \sin x^\circ$, and so:

$$2\,\sin(67^\circ - x^\circ) = \frac{\sin x^\circ}{\sin 37^\circ}.$$

Now we resort to a clever argument based on the monotonic nature of the sine function over the interval from 0° to 90°. To start with, note that we obviously have $0 < x < 67$. (Else the quantities on the two sides of the above supposed equality have opposite signs.)

Now suppose that $x < 37$. Then the quantity on the right side is less than 1. On the other hand, the supposition that $x < 37$ leads to the following: $67 - x > 30$, hence

$$\sin(67^\circ - x^\circ) > \sin 30^\circ,$$

$$\therefore \quad 2\,\sin(67^\circ - x^\circ) > 2\sin 30^\circ,$$

$$\therefore \quad 2\sin(67^\circ - x^\circ) > 1.$$

So if $x < 37$, the quantity on the left side is greater than 1, while the quantity on the right side is less than 1. We have arrived at a contradiction. Hence it cannot be that $x < 37$. The same reasoning works if we assume that $x > 37$; now we find that the quantity on the left side is less than 1, while the quantity on the right side is greater than 1. So this possibility does not work out either. Since x can neither be less than 37 nor greater than 37, it follows that $x = 37$. Hence $\measuredangle BCD = 37^\circ$.

A pretty solution combining trigonometry and geometry.

Here is an elegant and pleasing solution that combines geometry and trigonometry and makes effective use of the identity $\sin\theta = \sin(180^\circ - \theta)$. Let the diagonals AC, BD of the quadrilateral intersect at E (Figure 3). An easy angle computation shows that $\measuredangle DEC = 120^\circ$.

We have now, applying the sine rule to $\triangle ADC$ and $\triangle BDC$ respectively:

$$\frac{AD}{\sin\measuredangle ACD} = \frac{CD}{\sin 67^\circ},$$

$$\frac{BC}{\sin\measuredangle BDC} = \frac{CD}{\sin 113^\circ}.$$

Since $\sin 67^\circ = \sin 113^\circ$, the quantities on the right-hand sides of the two equalities are equal. We also have $AD = BC$. It follows that $\sin\measuredangle ACD = \sin\measuredangle BDC$, and therefore that $\measuredangle ACD = \measuredangle BDC$, as both angles are acute (indeed, $\measuredangle BDC + \measuredangle ACD = 60^\circ$).

Hence $\measuredangle ECD = 30^\circ$ and, therefore, $\measuredangle BCD = 37^\circ$.

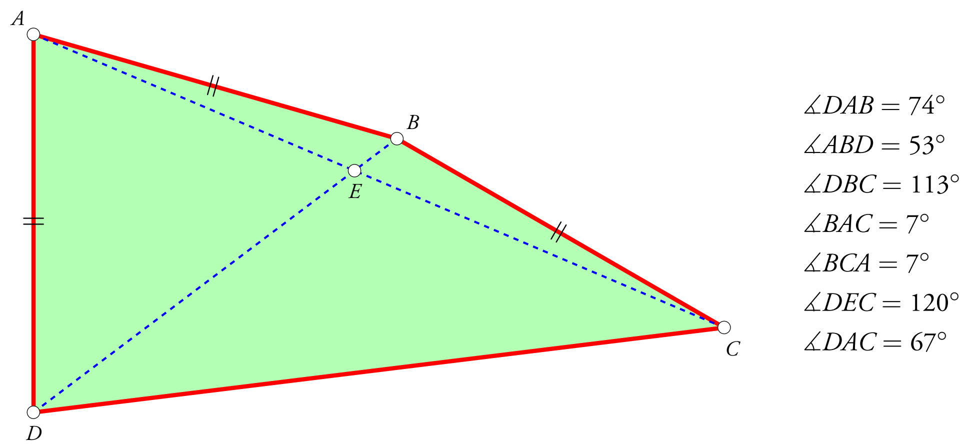

Figure 3.

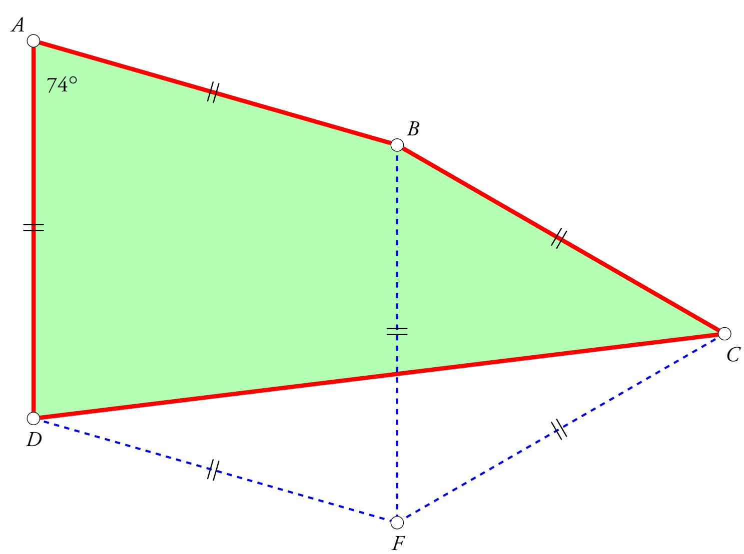

Figure 4.

An elegant pure geometry solution.
Next, we present an extremely elegant solution that draws on basic geometrical ideas about parallelograms (Figure 4). Draw $\overrightarrow{BF} = \overrightarrow{AD}$; then $ABFD$ is a parallelogram, and since $AB = AD$, it is a rhombus (as shown).

From this we deduce that $\measuredangle ABF = 106°$, and therefore that $\measuredangle FBC = 60°$. Since $BF = BC$, this makes BFC an equilateral triangle, so $\measuredangle BCF = 60°$. Again, in the isosceles $\triangle FCD$, $\measuredangle CFD = 74° + 60° = 134°$, hence $\measuredangle FCD = 23°$. It follows that $\measuredangle BCD = 60° - 23° = 37°$.

Another elegant pure geometry solution.
We conclude by presenting yet one more extremely elegant pure geometry solution. Locate point K such that $\triangle AKB$ is equilateral (Figure 5). Then we also have $AK = AD$ and $BK = BC$. And since $\measuredangle KAB = \measuredangle KBA = 60°$, we have $\measuredangle KAD = 14°$ and $\measuredangle KBC = 106°$. These in turn imply that $\measuredangle AKD = 83°$ and $\measuredangle BKC = 37°$. Also, obviously, $\measuredangle AKB = 60°$.

But now note that $83° + 60° + 37° = 180°$. This means that points D, K, C lie in a straight line! So the picture shown is not accurate (it was deliberately shown that way; note that we chose to

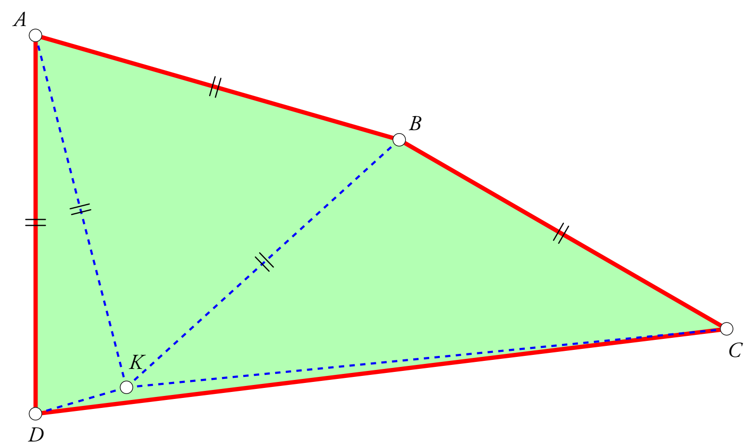

Figure 5.

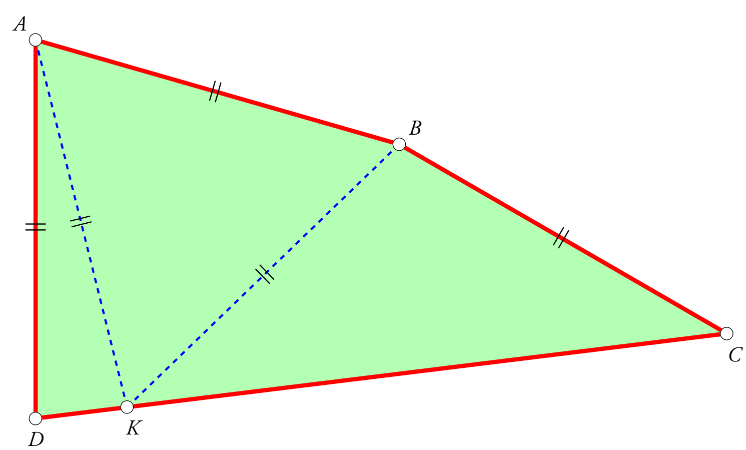

Figure 6.

locate K inside the quadrilateral, but we could as well have shown K outside the quadrilateral); the actual picture is as shown in Figure 6.

It follows immediately from the above that $\measuredangle BCK = 37^\circ$, i.e., $\measuredangle BCD = 37^\circ$.

Remark. It is noteworthy that the pure geometry solutions (the last two solutions presented above) featured the use of an equilateral triangle. This is a theme which occurs very often in the solutions of such problems.

References

1. Mathematics Stack Exchange, "Is there a way to solve for the missing angle?" https://math.stackexchange.com/questions/2564493/is-there-a-way-to-solve-for-the-missing-angle?newsletter=1&nlcode=838029%7c198e

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TechSpace

One Dimensional Cellular Automata:

The Totalistic Approach

JONAKI B GHOSH & ROHIT ADSULE

The topic of cellular automata has many interesting and wide ranging applications to real life problems emerging from areas such as image processing, cryptography, neural networks, developing electronic devices and modelling biological systems. In fact cellular automata can be a powerful tool for modelling many kinds of systems. In the March 2018 issue of At Right Angles we had introduced the basic ideas which form the foundation of the *Elementary Cellular Automata* (ECA) as defined by Stephen Wolfram. The reader is urged to go through the article before reading this.

The topic of Cellular Automata lends itself to interesting investigations which are well within the reach of high school students. We had illustrated the simple and yet powerful ideas in the previous article where we had described and analysed the behaviour of the 256 ECAs. In this article we shall provide a brief recap for the first time reader before moving on to the concept of Totalistic Cellular Automata.

Briefly defined, a *cellular automaton* is a collection of cells on a grid of a specified shape that evolves through discrete time steps according to a set of rules based on the state (or color) of the neighbouring cells. Cellular Automata may be one, two or three – dimensional. In this article and in the earlier one we have limited ourselves to exploring the one dimensional cellular automata on a grid of square cells where each row of the grid represents a generation or an iteration of the automata.

Keywords: cellular automata, grids, colour, state, neighbouring, rule, pattern, generation, iteration

The defining characteristics of a cellular automaton are

i. A grid of cells

ii. Each cell has a *state* – dead or alive. Cells which are alive may be coloured black or numbered 1 and cells which are dead may be numbered 0 and are white.

iii. Each cell in the grid has a *neighbourhood*. A neighbourhood of a given cell is a set of cells which are adjacent to it. This may be chosen in various ways. E.g., if we consider a linear grid of square cells, then the neighbourhood of each cell will be the two adjacent cells – one to its left and the other to its right.

iv. Finally every cellular automaton must have a *defining rule* based on which it grows and evolves in discrete time steps. For example, in a square grid, each row of cells may be considered as a separate generation of cells. Thus the first row is the initial generation (or generation 0) where each cell has a state (0 or 1). The state of each cell in the second row must be a function of its neighbouring cells in the row above it (that is the initial row). This may be written as

$$(\text{Cell state}_t) = f(\text{Neighbouring Cell state}_{t-1})$$

To begin with let us consider a linear grid of 8 cells where every cell has state 0 except the 5^{th} cell which has a state 1.

Figure 1. A linear grid of 8 cells where the 5^{th} cell is a live cell.

This linear grid of square cells will be referred to as **generation 0** (or row 0). The states of cells in **generation 1**(that is row 1) will be determined by the neighbourhood of each cell in the row 0 which comprises of the three cells just above it. Clearly the states of these three cells may be any one of the following

000 001 010 100 011 101 110 111

The fact that in any three – cell neighbourhood, there are three cells each with state value either 0 or 1 implies that there are $2^3 = 8$ ways of colouring these cells. Thus there are 8 neighbourhood configurations described by the triples of 0's and 1's as shown above. However conventionally, while defining an ECA, these neighbourhoods are taken in the following specific order.

111 110 101 100 011 010 001 000

Each of these configurations will determine the state of the middle cell of the three cells just below it in the next row, which may again be either 0 or 1. However the state of the leftmost corner cell in row 1 will be determined by the state of the cell just above it in row 0, its right neighbour and the last cell in row 0. Similarly, the state of the rightmost corner cell in row 1 will be determined by the state of the cell just above it in row 0, its left neighbour and the first cell in row 0.

Let us now arbitrarily assign 0's and 1's to all 8 neighbourhood configurations as follows

111	110	101	100	011	010	001	000
0	0	0	1	1	1	1	0

Pictorially this may be represented as

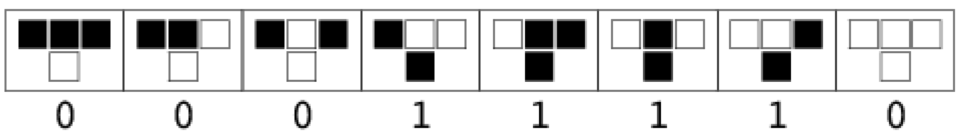

Figure 2. A rule set for a one-dimensional cellular automaton

This arbitrary assignment (also known as the *rule set*) will be the *defining rule* which will determine how this particular automaton will evolve. Note that this defining rule 00011110 may be treated as a binary number whose decimal representation may be obtained as follows

$$0 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 30$$

This kind of a rule set generates an *elementary cellular automaton*. The ECA which evolves from this rule set is referred to as **Rule 30**. However a different assignment of 0's and 1's would lead to a different rule set and a different ECA. Since each of the 8 groups of three cells may be assigned 0 or 1, this leads to $2^8 = 256$ possible assignments. Thus, in all, there are 256 ECA rules.

In our previous article we had used Mathematica, a powerful Computer Algebra System and NICO an open source software to explore the 256 ECAs. Both Mathematica and NICO were used to obtain the graphic (pictorial) representations of all the 256 ECAs. We observed the evolutionary pattern of each ECA through the first 100 iterations and categorised the ECAs into specific classes based on the patterns manifested by them.

A Totalistic Cellular Automaton

In this article we shall focus on the notion of generating a cellular automaton based on the totalistic approach. A *totalistic cellular automaton* is a cellular automaton in which the rules depend only on the total (or equivalently, the average) of the values of the cells in a neighbourhood. Wolfram introduced this idea in 1983. Like an ECA, the evolution of a one-dimensional totalistic cellular automaton can be completely described by a rule specifying the state a given cell will have in the next generation based on the *sum* of the values of the three cells consisting of the cell just above it in the grid, the one to its left and the one to its right.

Let us consider the case of a three cell neighbourhood (as described in the earlier section) where each cell has a value of 0 or 1.

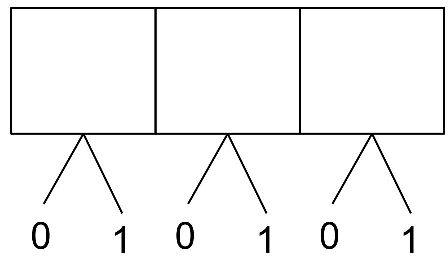

Figure 3. A three- cell neighbourhood

Note that each neighbourhood of three cells will have a total value of 0 when all the three cells have value 0 as shown.

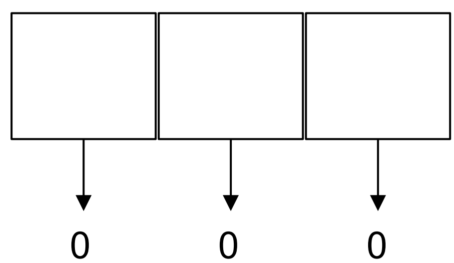

Figure 4. A three cell neighbourhood where all cells have state 0.

The total value will be 1 when one of the three cells has value 1 and the other two have value 0. This can happen in $3_{C_1} = 3$ ways.

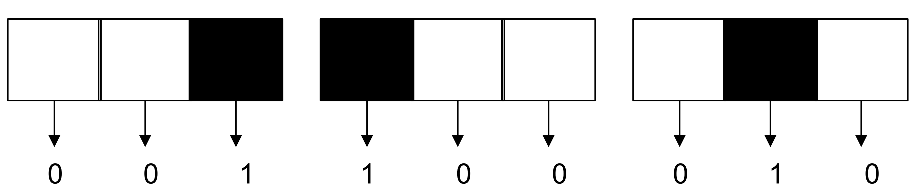

Figure 5. Three – cell neighbourhoods with total value equal to 1.

Similarly for a total value of 2, two out of the three cells must have value 1 which can happen in $3_{C_2} = 3$ ways.

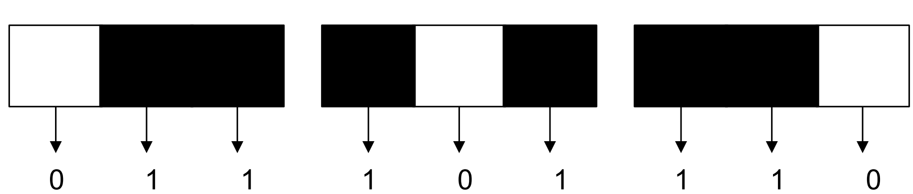

Figure 6. Three – cell neighbourhoods with total value equal to 2.

Finally the total value of 3 occurs when all cells have value 1 and this can occur in only 1 way.

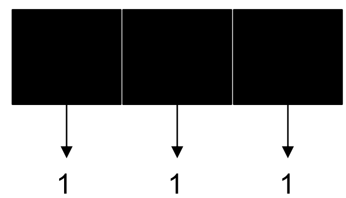

Figure 7. A three - cell neighbourhood with total value equal to 3.

Thus while we had $2^3 = 8$ neighbourhood possibilities in the ECAs, here we have only 4 neighbourhood possibilities – those with values 0, 1, 2 and 3. Further, a neighbourhood total of 0 may be assigned to a cell numbered 0 or 1. Similarly neighbourhood totals of 1, 2 and 3 can be also assigned to either 0 or 1. Thus there are only $2^4 = 16$ possible totalistic cellular automata (with three cell neighbourhoods) whereas there were $2^8 = 256$ ECAs! The totalistic approach considerably reduces the number of possible automata.

A 5 – cell totalistic cellular automaton

In our project we decided to explore the case of an automaton with five cell neighbourhoods. A five cell neighbourhood can have $2^5 = 32$ possible colourings if each cell is assigned values 0 or 1. Some examples are shown in Figure 8. Further, all these 32 neighbourhoods can lead to a cell with state 0 or 1. Hence there will be $2^{32} = 4294967296$ cellular automata with five cell neighbourhoods. We realised that it would be too difficult to explore such a large number of cases.

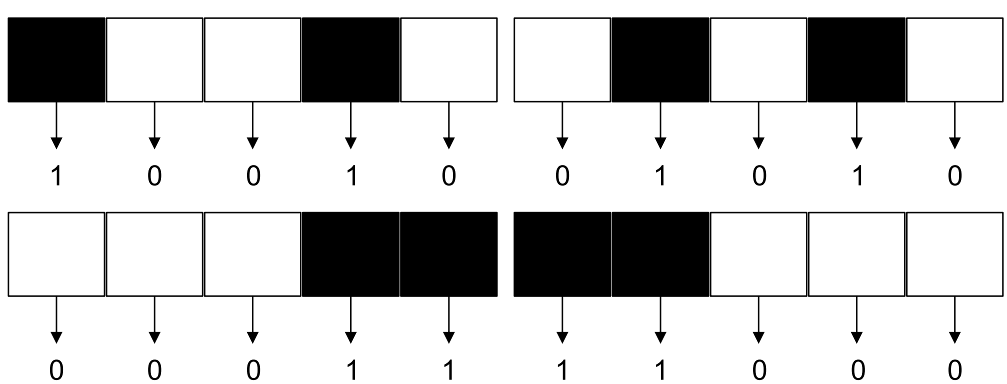

Figure 8. Five - cell neighbourhoods for a Totalistic Cellular Automaton.

However if we go by the totalistic approach, the sum of the values of a 5 – cell neighbourhood can be 0, 1, 2, 3, 4 or 5, thus reducing the neighbourhood possibilities to 6 only! Each of these neighbourhoods can be assigned to 0 or 1 in the next generation of cells thus leading to $2^6 = 64$ possible automata rules. We have chosen to number the rules from 0 to 63. Thus **Rule 0** will represent the case when all total neighbourhood values (from 0 to 5) are assigned to 0. Similarly, **Rule 63** will represent the case when all total neighbourhood values (from 0 to 5) are assigned to 1. Since these two rules lead to all white cells or all black cells in subsequent generations, they may be referred to as trivial cases.

As a non-trivial case, let us consider **rule 53** which has a binary representation 110101 (as it can be expressed in powers of 2 as $1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$).

In order to define rule 53, we shall place the binary digits of 53 in correspondence with the sum values starting with 5 till 0. This leads us to the following rule

Sum	5	4	3	2	1	0
Assignment of 0 or 1	1	1	0	1	0	1

Table 1. Rule 53 totalistic cellular automata rule

When applied on a grid of square cells with a single live cell in the centre of row 1, this rule leads to the following intricate and symmetric triangular pattern.

Figure 9: Rule 53

Table 2 lists out the defining rules of all 64 totalistic automata. Note that in any row of the table we will find the binary digits of the corresponding rule number in the columns numbered 5,4,3,2,1 and 0.

Sum	**5**	**4**	**3**	**2**	**1**	**0**
Rule 0	0	0	0	0	0	0
Rule 1	0	0	0	0	0	1
Rule 2	0	0	0	0	1	0
Rule 3	0	0	0	0	1	1
Rule 4	0	0	0	1	0	0
Rule 5	0	0	0	1	0	1
Rule 6	0	0	0	1	1	0
Rule 7	0	0	0	1	1	1
Rule 8	0	0	1	0	0	0
Rule 9	0	0	1	0	0	1
Rule 10	0	0	1	0	1	0
Rule 11	0	0	1	0	1	1
Rule 12	0	0	1	1	0	0
Rule 13	0	0	1	1	0	1
Rule 14	0	0	1	1	1	0
Rule 15	0	0	1	1	1	1
Rule 16	0	1	0	0	0	0
Rule 17	0	1	0	0	0	1

Table 2 Defining rules for a 5 - cell totalistic cellular automaton

Sum	5	4	3	2	1	0
Rule 18	0	1	0	0	1	0
Rule 19	0	1	0	0	1	1
Rule 20	0	1	0	1	0	0
Rule 21	0	1	0	1	0	1
Rule 22	0	1	0	1	1	0
Rule 23	0	1	0	1	1	1
Rule 24	0	1	1	0	0	0
Rule 25	0	1	1	0	0	1
Rule 26	0	1	1	0	1	0
Rule 27	0	1	1	0	1	1
Rule 28	0	1	1	1	0	0
Rule 29	0	1	1	1	0	1
Rule 30	0	1	1	1	1	0
Rule 31	0	1	1	1	1	1
Rule 32	1	0	0	0	0	0
Rule 33	1	0	0	0	0	1
Rule 34	1	0	0	0	1	0
Rule 35	1	0	0	0	1	1
Rule 36	1	0	0	1	0	0
Rule 37	1	0	0	1	0	1
Rule 38	1	0	0	1	1	0
Rule 39	1	0	0	1	1	1
Rule 40	1	0	1	0	0	0
Rule 41	1	0	1	0	0	1
Rule 42	1	0	1	0	1	0
Rule 43	1	0	1	0	1	1
Rule 44	1	0	1	1	0	0
Rule 45	1	0	1	1	0	1
Rule 46	1	0	1	1	1	0
Rule 47	1	0	1	1	1	1
Rule 48	1	1	0	0	0	0
Rule 49	1	1	0	0	0	1
Rule 50	1	1	0	0	1	0
Rule 51	1	1	0	0	1	1
Rule 52	1	1	0	1	0	0
Rule 53	1	1	0	1	0	1
Rule 54	1	1	0	1	1	0
Rule 55	1	1	0	1	1	1
Rule 56	1	1	1	0	0	0
Rule 57	1	1	1	0	0	1
Rule 58	1	1	1	0	1	0
Rule 59	1	1	1	0	1	1
Rule 60	1	1	1	1	0	0
Rule 61	1	1	1	1	0	1
Rule 62	1	1	1	1	1	0
Rule 63	1	1	1	1	1	1

Table 2

Classification of the 5 – cell totalistic cellular automata (TCA)

We decided to explore all 64 TCAs by writing a program in Java (see Appendix I). The program produces the graphic image of the first 100 iterations of a specified rule number. After observing all 64 TCAs we tried to classify them based on their evolutionary behaviour. Interestingly, we found that the TCAs may be classified into four major categories which are very similar to the case of the ECAs. These categories are also mentioned in the research literature associated with cellular automata.

1. **Uniform:** All cells in the grid are either black or white.
2. **Repetitive:** These automata reveal regular alternating pattern or a block of cells which repeat themselves throughout the grid.
3. **Nested or Fractal-like:** These automata lead to Sierpinski triangle like fractal patterns exhibiting clear self- similarity or other nested patterns.
4. **Random or chaotic:** These are patterns which cannot be placed in any of the above three categories. There is no fixed pattern in these automata and their evolution is highly unpredictable.

The rule numbers which fit into the above categories are listed in the following table.

Category	TCA rule number	Characteristics
Uniform	0,4,8,12,16 (multiples of 4)	All cells are white (trivial case)
	63	All cells are black (trivial case)
	54, 57, 58, 59, 61, 62	Black cells with a border of black and white cells
Repetitive	1,3,5,7,11,13,15,31	Repetitive pattern of rows of black and white cells. 7,11,13,15 and 31 have a different configuration at the border.
	19, 23	Alternating rows of black and white with a central band pattern.
Nested	2, 6, 14, 30, 33, 34	Sierpinski triangle like structure
	21, 25, 38, 42, 49	Nested but not Sierpinski like
Random	17, 29, 35, 47	A generally random pattern with a border which gets wider as the iterations increase.
	9, 27, 41, 45, 50, 51	A generally random pattern with a uniform border which remains the same as the iterations increase.
Chaotic	10, 18, 22, 26, 33, 43, 46	A seemingly chaotic behaviour (although symmetrical)
All black with a fern-like repetitive pattern in the border	53, 54	It is difficult to place this in any of the above categories.

Table 3. Classification of the 64 Totalistic Cellular Automata.

Here are some examples of TCAs which evolve from one live cell in the centre of the top row of the grid. One may note that all TCAs are symmetrical about the height of the triangular pattern. While describing them we will focus on the evolution of the cells in one half of the triangular pattern.

Figure 10. Rule 57: Uniform - a black triangle is formed with a border of white cells.

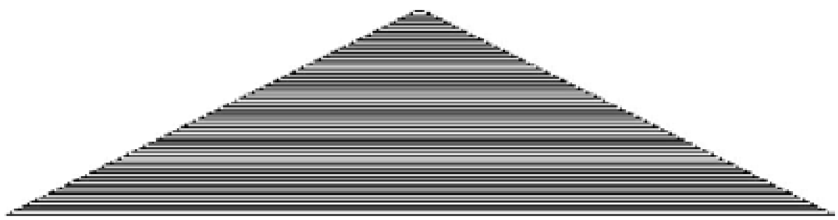

Figure 11. Rule 3: Repetitive pattern comprising rows of black and white cells

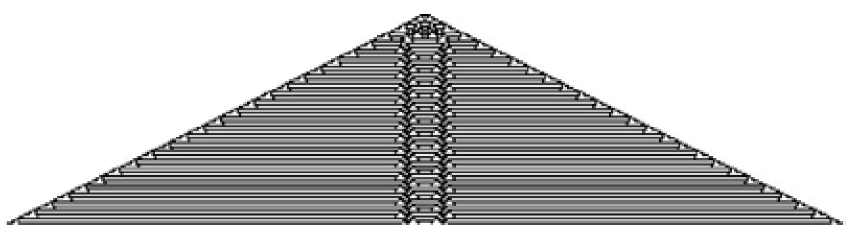

Figure 12. Rule 23: Alternating rows of black and white with a central band pattern.

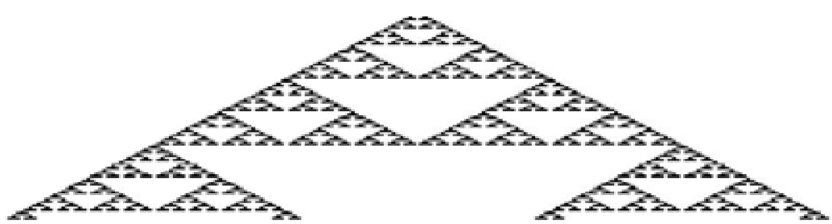

Figure 13. Rule 6: Sierpinski triangle like structure

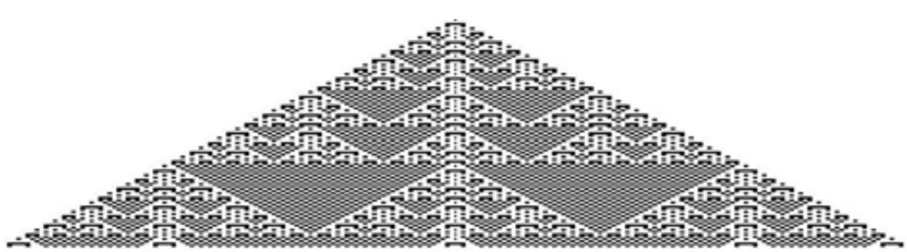

Figure 14. Rule 25: Nested Structure

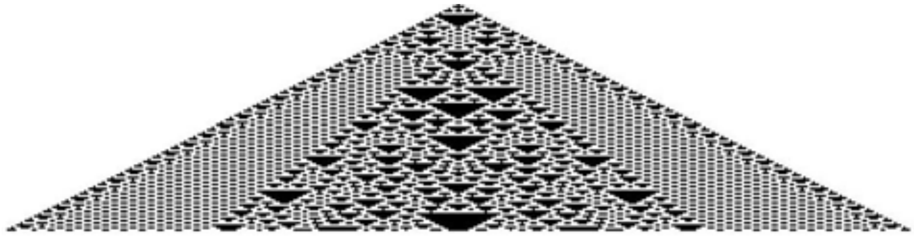

Figure 15. Rule 35: A generally random pattern with a border which gets wider as the iterations increase.

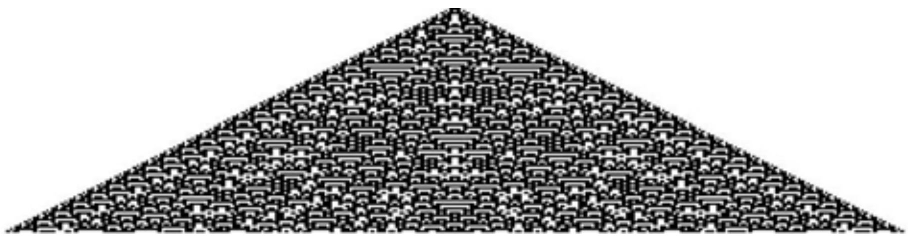

Figure 16. Rule 27: A generally random pattern with a uniform repetitive border.

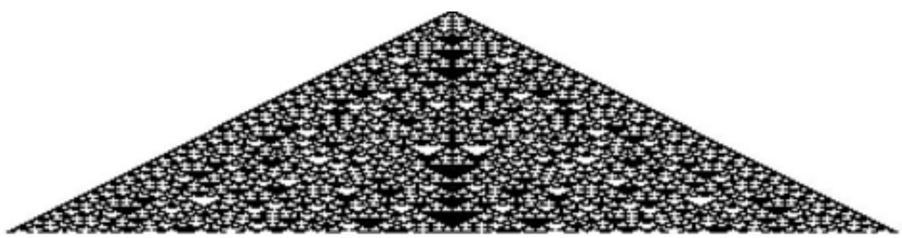

Figure 17. Rule 46: A seemingly chaotic pattern

Concluding discussion

In this article we have described the second phase of our exploratory study on one - dimensional Cellular Automata. In order to define our own cellular automata we adopted the totalistic approach where the state of a given cell is dependent on the sum of values of its five neighbouring cells (the cell just above it, two to the left and two to the right) in the previous iteration. These led to neighbourhood values ranging from 0 to 5, each of which can give rise to a cell with value 0 or 1. This defining rule led to 64 cellular automata with some interesting patterns. We developed a code in Java to study their evolutionary patterns. The results were interesting as we were able to classify the 64 automata into four major categories – Uniform, Repetitive, Nested and Random. However an additional category with a repetitive fern-like pattern in the border was also identified which did not fit into the other four categories. The results of our investigations have been compiled in Table 3.

In this project we have explored two colour (two – state) ECAs and TCAs. It would be interesting to explore other kinds of automata which emerge when there are more than two states. A treatment may be found in http://mathworld.wolfram.com/TotalisticCellularAutomaton.html.

We hope we have succeeded in taking the reader on an exciting journey into the computational world of the one - dimensional cellular automata!

APPENDIX I

JAVA Code for the Totalistic Cellular Automaton

```
import java.awt.*;
import java.awt.Color;
import java.awt.Graphics;
import javax.swing.JFrame;
import javax.swing.JPanel;
import javax.imageio.ImageIO;
import java.awt.image.BufferedImage;
import java.io.*;
public class Processor extends JPanel
{
  static int line=0;
  static int max_lines=120;
  static BufferedImage[]images=new BufferedImage[max_lines];
  public Processor()
  {
    setSize(1000,1000);
  }

  @Override
  public void paintComponent(Graphics g)
  {
    System.out.println("paintComponent called from "
+ Thread.currentThread().getName());
    //new Exception().printStackTrace(System.out);
    super.paintComponent(g);
    if(line==0)
    {
      g.setColor(Color.WHITE);
      g.fillRect(0,0,1000,1000);
      g.setColor(Color.BLACK);
      g.fillRect(500,0,2,2);
    }
    else if(line!=0)
    {
      System.out.println("Line-1="+(line-1));
      g.drawImage(images[line-1],0,0,null);
      int x_cor=500-line*4;
      int y_cor=0+line*2;//The x and y coordinates of the points where the new squares are to be added from
      for(int lv=1;lv<=4*(line+1)-3;lv++)
      {
        BufferedImage img=images[line-1];
        if(lv>1)
          x_cor=x_cor+2;//Value adjusted for each square to be drawn in the same line
        int test_x=x_cor-3;
        int test_y=y_cor-1;
        int sum=0;
        for(int lv2=1;lv2<=5;lv2++)
        {
          System.out.println("(x,y):"+test_x+","+test_y);
          Color c=new Color(img.getRGB(test_x,test_y));
          if(c.getRed()==0&&c.getGreen()==0&&c.getBlue()==0)
          {
            sum+=1;
          }
          test_x=test_x+2;
        }
```

```
            if(sum==5||sum==4||sum==2||sum==0)//Set rule number here
            {
               g.fillRect(x_cor,y_cor,2,2);
            }
         }
      }
   }

   public static BufferedImage toBufferedImage(Component component)
   {
      BufferedImage image = new BufferedImage(component.getWidth(), component.getHeight(), BufferedImage.
TYPE_INT_RGB);
      Graphics g = image.getGraphics();
      component.paint(g);
      return image;
   }

   public static void main(String[]args)throws IOException
   {
      JFrame frame=new JFrame("Automaton");
      frame.setDefaultCloseOperation(JFrame.EXIT_ON_CLOSE);
      frame.setSize(1000,1000);
      BufferedImage image;
      Processor t=new Processor();
      frame.add(t);
      image=new BufferedImage(1,1,BufferedImage.TYPE_INT_RGB);
      image=toBufferedImage(t);
      images[0]=image;
      line++;
      while(line<max_lines)
      {
         image=toBufferedImage(t);
         images[line]=image;

         line++;
      }
      //Fie I/O operation
      File f=new File("Rule 100.png");
      ImageIO.write(images[line-1],"PNG",f);
      frame.setVisible(true);
   }

}
```

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Middle School Problems
Understanding Circular Motion

A. RAMACHANDRAN

Here are some problems related to circular motion, along with their solutions and some extension activities.

Problem VII-2-M.1: **On Coins and Plates**

A. Coin Rolling Outside

(i) A coin (or disc) rolls around another similar coin held down firmly, maintaining constant contact and without slipping. After it has executed one turn around the fixed coin, how many times has it turned around itself?

(ii) What if the fixed coin or disc was twice the size of the rolling coin/disc?

(iii) Or n times the size? (See Figure 1.)

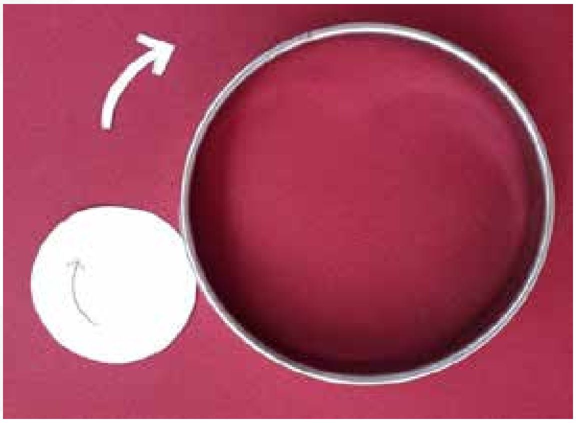

Figure 1.

Keywords: circular, rolling, angle, clock, orbit

B. Coin Rolling Inside

(iv) A coin/disc could also roll in a similar fashion along the edge of a circular depression (see Figure 2). Assuming the depression to be twice the size of the disc, if the disc executes one turn around the rim, how many times has it turned around itself?

(v) What if the depression were *n* times the size of the disc? (If the disc and depression were of the same size no movement would be possible.)

Figure 2.

Problem VII-2-M.2: **On Analog Clocks**

(i) How many times do the hour hand and minute hand of an analog clock/timepiece overlap in a 12 hour period? The two hands overlap at 12:00. What is the exact time when they do so again?

(ii) How many times do the two hands occupy diametrically opposite positions?

(iii) How many times do the two hands find themselves at right angles to each other

a. With hour hand 'downstream'

b. With minute hand 'downstream' (all in a 12 hour period)?

Problem VII-2-M.3: **On Planetary Motion**

Two planets circle the same sun in concentric circular orbits located in the same plane, with the sun at the centre, both moving in an anticlockwise direction. Planet A, occupying the inner orbit, takes x days to circle the sun once, while planet B, occupying the outer orbit, takes y days, with $y > x$. (Let us consider the 'days' to be 'earth days.')

(i) If they are in a straight line with the sun today, both being on the same side of the sun, after how many days will they again be in a line with the sun (again being on the same side of the sun)? Note that they need not occupy the same initial positions in their orbits.

(ii) Would the answer to the above be the same if the requirement is that they are in a line but on opposite sides of the sun on both occasions?

(iii) Or if they are to be at right angles with planet A 'downstream' on both occasions?

(iv) As above, but with planet B 'downstream'?

Solutions

Problem VII-2-M.1: **On Coins and Plates**

A. Coin rolling outside: When a circular disc rolls on a path, the number of turns it makes around itself equals the path length divided by the circumference of the disc. In the case of a disc (of radius R_{roll}) rolling around another stationary circular disc (of radius R_{stat}), this turns out to be the ratio of their circumferences or, equivalently, of their radii. However, in the process of making a circuit around the stationary disc, the moving disc executes one extra turn around itself. So the number of turns is given by the expression $\frac{R_{stat}}{R_{roll}} + 1$. Applying this formula, we get the following answers:

(i) Twice

(ii) Thrice

(iii) $n + 1$ times.

B. Coin rolling inside: The argument is similar to the one above. In this case the movement along the rim is in a sense/direction opposite to the movement around itself. Therefore the required expression is $\frac{\text{Rstat}}{\text{Rroll}} - 1$. Applying this formula, we get the answers (iv) Once (v) n – 1 times. Do try it out with actual materials.

Problem VII-2-M.2: **On Analog Clocks**

After 12:00, the minute hand races ahead of the hour hand, completing one full turn in 1 hour. By this time the hour hand has moved an angular distance of 30°. Now the minute hand needs to catch up with the hour hand. The time taken for this is obtained by dividing 30° by the angular velocity of the minute hand relative to the hour hand, which is 360°/hour -30°/hour = 330°/hour. The result is $\frac{1}{11}$ hour. So the exact time when the overlap happens is $1\frac{1}{11}$ hours past 12:00. Similar arguments show that subsequent overlaps happen at times $2\frac{2}{11}$ hours, $3\frac{3}{11}$ hours, etc. The last of the series is $11\frac{11}{11}$ hours which is actually 12 hours again. So the overlap happens 11 times in 12 hours at equally spaced intervals of time and positions on the dial.

Or one could reason like this: Let the angle between the hour hand and the 12 'o'clock position be x and the angle between the minute hand and the 12 'o'clock position be y. Now we have $y = 12x - 360\,N$ degrees, where N is an integer with $0 \le N \le 11$. When the hands overlap, we have $x = y = 12x - 360N$, which leads to $x = \frac{360N}{11}$. If $N = 0$, $x = y = 0$, which stands for noon/midnight. As N takes values 1, 2, 3,……10, x takes values $\frac{360}{11}, \frac{360 \times 2}{11}, \frac{360 \times 3}{11}, \ldots\ldots \frac{360 \times 10}{11}$.

If $N = 11$, then $x = \frac{360 \times 11}{11}$ which again stands for noon/midnight. So there are 11 overlapping situations equally spaced in time and position on the dial, as can be practically demonstrated with an analog clock/timepiece.

The answer to all sub-questions (ii) and (iii) is the same as above: 11 times in 12 hours. That is, any specified relative position of the hands occurs on 11 equally spaced occasions in 12 hours.

Addendum 1: Take an analog clock/timepiece where the hours are marked by dots or bars and no numerals are shown. Let the device be set to Indian Standard Time. Now hold the device upside down. Mentally advance the hour hand by 15°, i.e., the angle corresponding to a half hour. The device now shows Greenwich Mean Time/Universal Time. This is exact except that a.m./p.m. needs to be assigned. Check it out for a few positions and then try to prove that it always works.

Problem VII-2-M.3: **On Planetary Motion**

The situation is similar to the above except that the units are different. The angular velocities of planets A and B are $360/x$ degrees per day and $360/y$ degrees per day, respectively. Initially planet A races ahead and completes one full circuit in x days. By this time planet B has moved an angular distance of $\frac{360x}{y}$ degrees. Planet A now has to catch up with this at a relative angular

velocity of $\frac{360}{x}$ degrees per day $-\frac{360}{y}$ degrees per day $= \frac{360(y-x)}{xy}$ degrees per day.

The time required for this is

$$\frac{360x}{y}\text{degrees} \div \frac{360(y-x)}{xy}\text{degrees per day} = \frac{x^2}{y-x}\text{days.}$$

We now need to add the time x to this to get the total time that has to elapse to have the two planets in a line with the sun again (being on the same side of the sun), which turns out to be

$$\left(x + \frac{x^2}{y-x}\right)\text{ days} = \frac{xy}{y-x}\text{ days.}$$

i) – (iv) The same holds true for the other cases mentioned.

Addendum 2: The problem about the planets can be related to our own solar system. The time interval between two occurrences of the same relative position is termed the Synodic period of either planet with respect to the other. Considering the Earth there can be two situations. Earth could be planet *B* as in the above problem. Then planet *A* would be a planet whose orbit lies inside that of the Earth (an 'inferior' planet). So if we know the orbital periods of Earth and the inferior planet we can calculate the synodic period. Alternatively and more practically, if we know Earth's orbital period and the synodic period of the other planet (this is observable from Earth), we can calculate the orbital period of the inferior planet. Denoting the synodic period by *S*, we have $S = \frac{xy}{y-x}$. This can be reformulated to give $x = \frac{Sy}{S+y}$.

If we take planet *A* to be Earth then planet *B* would be a planet moving in an orbit outside Earth's orbit (a 'superior' planet). As earlier, we could, from a knowledge of Earth's orbital period and the synodic period of the other planet (again observable from Earth), calculate the orbital period of the superior planet. The formula for *S* should now be reformulated to give $y = \frac{Sx}{(S-x)}$. It may be useful in this case to express the time periods in 'earth years' rather than 'earth days,' to simplify the computations.

The orbital and synodic periods of the planets in our neighbourhood can be obtained from the internet. The reader is invited to verify the statements made above. Despite the many simplifying assumptions made (circular and coplanar orbits and uniform orbital speed) there is good agreement.

APOLOGY FOR AN ERROR

We apologise for an error in the Middle School Problems of the March 2018 issue of At Right Angles. The solution that was presented for Problem VII-1-M.6 was actually the solution to Problem VII-1-M.7.

Also, there was an error in the statement of Problem VII-1-M.6. The corrected statement and the solution to this problem are given below.

Problem VII-1-M.6

Construct the locus of a point which moves so that it remains at equal distance from two given parallel straight lines l and m. Describe the locus in words.

Construct the locus of a point which moves so that it is a vertex of a trapezium of area 25 cm^2 one of whose parallel sides AB = 4 cm is at equal distance from two given parallel straight lines l and m which are at a distance of 5 cm from each other.

The first part of this question is extremely simple: the locus is a line parallel to the two given lines and situated midway between the two of them.

In the second part, side AB is situated on the above line. The parallel side is on either *l* or *m*. The base of the required trapezium is therefore 2.5 cm. Using the formula Area = 1/2 × base × sum of parallel sides, we find that the sum of the parallel sides is 20 cm. Since one of the parallel sides is 4 cm, the other side has length 16 cm. A point D can be chosen on either *l* or *m*, this fixes the remaining vertex C on this line at a distance of 16 cm on either side of D. So there are two possible trapeziums for each position of D on *l* and again on *m*.

Problem Corner

Alternative Solution to 'A Triangle Problem'

SANJIB RUDRA

Here is an alternative solution to the following problem which was studied in the November 2016 issue of *AtRiA*:

Two sides of a triangle have lengths 6 *and* 10*, and the radius of the circumcircle of the triangle is* 12*. Find the length of the third side.*

Let the triangle be ABC, with sides $a = BC = 6$ and $b = CA = 10$ (Figure 1). The radius of the circumcircle is 12. We must find c, the length of side AB. Let $\measuredangle ACB = x$; then reflex $\measuredangle AOB = 2x$, so $\measuredangle AOB = 360^\circ - 2x$. From $\triangle ABC$ we get, using the cosine rule,

$$\cos x = \frac{6^2 + 10^2 - c^2}{2 \times 6 \times 10} = \frac{136 - c^2}{120}.$$

From $\triangle AOB$ we get, again using the cosine rule,

$$\cos(360^\circ - 2x) = \frac{12^2 + 12^2 - c^2}{2 \times 12 \times 12} = \frac{288 - c^2}{288}.$$

Hence:

$$\cos 2x = \frac{288 - c^2}{288}.$$

Since $\cos 2x = 2\cos^2 x - 1$, we have:

$$\frac{288 - c^2}{288} = 2\left(\frac{136 - c^2}{120}\right)^2 - 1,$$

$$\therefore\ c^4 - 247c^2 + 4096 = 0, \qquad \text{(on simplification)}.$$

Keywords: Triangle, circumradius, cosine rule

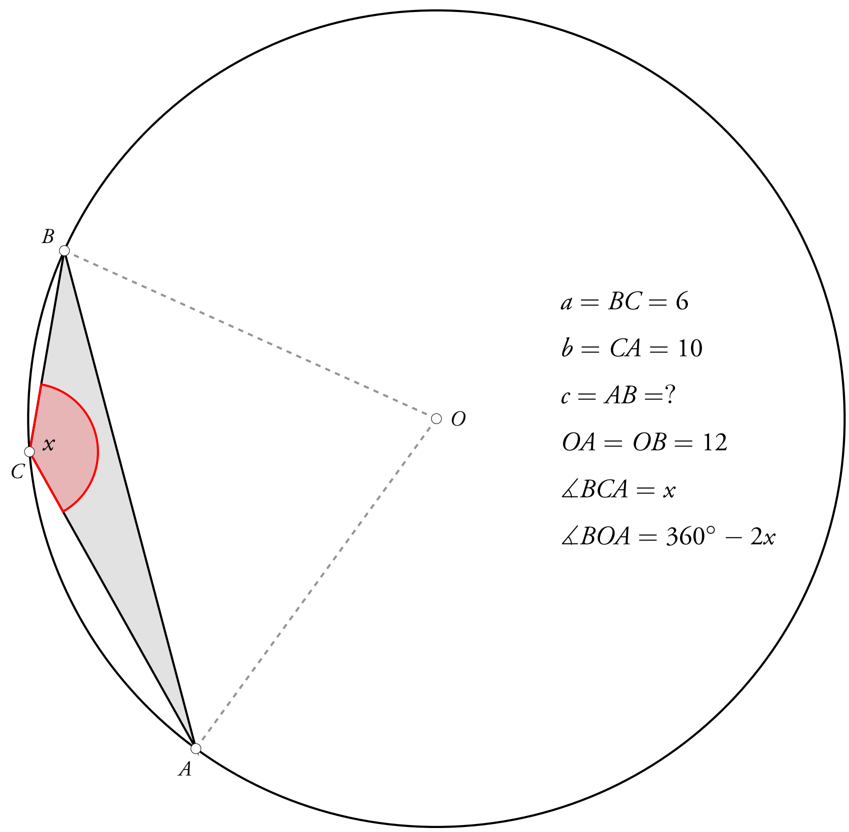

Figure 1.

Let $d = c^2$; then $d^2 - 247d + 4096 = 0$. The solution of this quadratic equation is

$$d = \frac{247 \pm 5\sqrt{1785}}{2},$$

giving $d \approx 229.123$ and $d \approx 17.877$. Hence, taking square roots,

$$c \approx 15.137, \qquad c \approx 4.228.$$

These are the two possible lengths of AB.

Problem Corner

Problems for the
Senior School

PRITHWIJIT DE &
SHAILESH SHIRALI

Problems for Solution

Problem VII-2-S.1 Let *AB* be a fixed line segment in the plane. Let *O* and *P* be two points in the plane and on the same side of *AB*. If $\measuredangle AOB = 2\measuredangle APB$, does it necessarily follow that *P* lies on the circle with centre *O* and passing through *A* and *B*?

Problem VII-2-S.2 Let *ABC* be an equilateral triangle with centre *O*. A line through *C* meets the circumcircle of triangle *AOB* at points *D* and *E*. Prove that the points *A, O* and the midpoints of segments *BD*, *BE* are concyclic. [Tournament of Towns]

Problem VII-2-S.3 Three non-zero real numbers are given. If they are written in any order as coefficients of a quadratic trinomial, then each of these trinomials has a real root. Does it follow that each of these trinomials has a positive root? [Tournament of Towns]

Problem VII-2-S.4 *D* is the midpoint of the side *BC* of triangle *ABC*. *E* and *F* are points on *CA* and *AB* respectively, such that *BE* is perpendicular to *CA* and *CF* is perpendicular to *AB*. If *DEF* is an equilateral triangle, does it follow that *ABC* is equilateral? [Tournament of Towns]

Problem VII-2-S.5 A boy computed the product of the first n positive integers and his sister computed the product of the first m even positive integers where $m \geq 2$. Is it possible for them to get the same result?

Solution to problem VII-1-S.1 Two hundred students are positioned in 10 rows, each containing 20 students. From each of the 20 columns thus formed, the shortest student is selected, and the tallest of these 20 (short) students is labelled *A*. These students return to their initial places. Next, the tallest student in each row is selected, and from these 10 (tall) students, the shortest is labelled *B*. Who is taller, *A* or *B*?

Keywords: tournament of the towns, coin problems

If A and B stand in the same row, then B is taller than A, since B is the tallest student in that row. If A and B stand in the same column, then again B is taller than A, since A is the shortest student in that column. Finally, if A and B do not stand in either the same column or the same row, let C be that student standing in the same column as A and in the same row as B. Then B is taller than C (since B is the tallest in that row), and C is taller than A (since A is the shortest in that column). Hence, in all cases, B is taller than A.

Solution to problem VII-1-S.2 Given 13 coins, each weighing an integral number of grams. It is known that if any coin is removed, then the remaining 12 coins can be divided into two groups of 6 with equal total weight. Prove that all the coins are of the same weight.

First, it follows from the conditions of the problem that each coin weighs either an even number of grams or an odd number of grams. Since any set of twelve coins can be divided into two groups of equal weight, a set of twelve coins weighs an even number of grams. This total weight remains an even number if one of the twelve coins is exchanged with the thirteenth coin. But this is possible only if the weights of the coins interchanged are of the same parity, and this holds for any of the twelve coins initially weighed. Hence, either each coin weighs an even number of grams or each coin weighs an odd number of grams.

Now subtract from the weight of each coin, the weight of the lightest coin (possibly two or more coins may have the same weight, but this is unimportant). This may be thought of as producing a *new* set of coins, and this new set clearly satisfies the conditions of the problem. (One or more coins may be thought of as having *zero weight*.) It is clear that each coin in the *new* set weighs an even number of grams. If now we divide each weight by 2 and think of this as providing a *new* set of weights, this new set again satisfies the conditions of the problem.

Assume now that **not** all the coins are of the same weight. In this case, not all the weights of the second set (obtained by subtracting the weight of the lightest coin from the original weights of each of the coins) will be zero. If we continue to divide by 2, thus obtaining *new* sets satisfying the conditions of the problem, we finally arrive at a set of coins of which some are of even weight (at least one is of zero weight) and some are of odd weight (continued division of an even number by 2 finally produces an odd number). But such a set satisfying the conditions of the problem has been shown to be impossible. This contradiction proves the assertion of the problem statement.

Solution to problem VII-1-S.3 Show that there are infinitely many positive integers A such that $2A$ is a square, $3A$ is a cube and $5A$ is a fifth power.

First, observe that 2,3,5 divide A. So we may take $A = 2^{\alpha}3^{\beta}5^{\gamma}$. Considering $2A$, $3A$ and $5A$, we observe that $\alpha + 1$, β, γ are divisible by 2; α, $\beta + 1$, γ are divisible by 3, and α, β, $\gamma + 1$ are divisible by 5. We can choose $\alpha = 15 + 30n$; $\beta = 20 + 30n$; $\gamma = 24 + 30n$. As n varies over the set of natural numbers, we get an infinite set of numbers of required type.

Solution to problem VII-1-S.4 An infinite sequence of positive integers $a_1, a_2, \ldots, a_n, \ldots$ satisfies the condition $\sum_{k=1}^{m} a_k^3 = \left(\sum_{k=1}^{m} a_k\right)^2$, *i.e.*,

$$a_1^3 + a_2^3 + a_3^3 + \cdots a_m^3 = \left(a_1 + a_2 + a_3 + \cdots + a_m\right)^2$$ for each positive integer m. Determine the sequence.

Putting $m = 1$ we get $a_1 = 1$. If $m=2$, then $1 + a_2^3 = (1 + a_2)^2$ and this implies $a_2 = 2$. If we assume that we have shown $a_j = j$ for $1 \le j \le k$ for some $k \ge 3$ then

$$1^3 + 2^3 + \cdots + k^3 + a_{k+1}^3 = (1 + 2 + \cdots + k + a_{k+1})^2,$$

and noting that $1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^2 = \left(\frac{1}{2}k(k+1)\right)^2$, we get after simplification,

$$(a_{k+1} - k - 1)(a_{k+1} + k) = 0,$$

whence $a_{k+1} = k + 1$. Thus by induction it follows that $a_n = n$ for every positive integer n.

Solution to problem VII-1-S.5 The function $f(n) = an+b$, where a and b are integers, is such that for every integer n, the numbers $f(3n+1)$, $f(3n)+1$ and $3f(n)+1$ are three consecutive integers in some order. Determine all such functions $f(n)$.

We have

$$f(3n + 1) = 3an + a + b,$$

$$f(3n) + 1 = 3an + b + 1,$$

$$3f(n) + 1 = 3an + 3b + 1.$$

Observe that $b \neq 0$, for if $b = 0$, then $f(3n)+1 = 3f(n) + 1$, which contradicts the fact that they are consecutive integers. Also, $3f(n) + 1 - f(3n) - 1 = 2b$, an even non-zero integer. Thus these two numbers cannot be consecutive. Therefore $f(3n+1)$ is the middle number and hence

$$2f(3n + 1) = f(3n) + 1 + 3f(n) + 1,$$

whence $a = b + 1$ and $f(n) = (b + 1)\, n + b$. Also,

$$|f(3n + 1) - f(3n) - 1| = |b| = 1.$$

Thus $b = \pm 1$ and hence $f(n) = 2n + 1$, and $f(n) = -1$ are the only solutions. It is easy to check that both satisfy the condition of the problem.

The Odd-Even Tale

PRITHWIJIT DE

We are introduced to the concept of an even number and an odd number in primary school or even earlier. Any natural number divisible by 2 is even; if it is not, it is odd. The definition is extended to integers once we learn the arithmetic of negative whole numbers. Then we make simple observations such as: the sum of two even numbers is even, as is the sum of two odd numbers, and sum of an even number and an odd number is odd. Carrying on, we deduce that the sum of an even number of even numbers or odd numbers is even, and so is the sum of an odd number of even numbers, whereas the sum of an odd number of odd numbers is odd. This article aims at discussing some problems where these simple observations come into play.

Problem 1. At a party, each guest shakes hands with a certain number of guests. Is it true that the number of guests who have shaken hands with an odd number of guests is even? (It is taken for granted that each handshake is between precisely two persons; there are no handshakes featuring three or more hands!)

Solution to Problem 1. Let there be N guests present in the party. Suppose they are numbered $G_1, G_2, \ldots, G_N$. Let h_k, $1 \le k \le N$, be the number of handshakes performed by the guest G_k. Let

$$T = h_1 + h_2 + \cdots + h_N.$$

Keywords: Odd, even, parity

First we show that T is even. Here is a nice way of proving it. Imagine that there is a counter placed in the party hall and initially it is set at zero. Whenever there is a handshake, the counter counts the number of hands involved in it. Thus at every handshake, the count on the counter increases by 2. Since initially the count is an even number (0 is even), the final count has to be an even number. Therefore, T is even. Without loss of generality, if we assume that $h_1, h_2, \ldots, h_M$, $M < N$, are odd then it follows that

$$h_1 + h_2 + \ldots + h_M = T - (h_{M+1} + \cdots + h_N)$$

is even, which shows that M must be even, for if M is odd, then on the left side we would have an odd number of odd numbers which necessarily add up to an odd number; on the other hand, all the quantities on the right side are even, consequently the right side is even. Hence M is even.

Problem 2. The numbers 1 through 10 are written in a row. Can the signs '$+$' and '$-$' be placed between them, so that the value of the resulting expression is 0?

Solution to Problem 2. We know that

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$$

If some of the '$+$' signs are replaced by '$-$' then the sum changes by an even number. (More specifically, the sum changes by twice the sum of the numbers thus altered, i.e., it changes by an even number.) Since the original sum is odd, no matter how many sign changes are made, the resulting sum will remain an odd number. Therefore it will not be possible to reach zero at any stage.

Problem 3. The numbers $1, 2, 3, \ldots, 2016, 2017$ are written on a blackboard. We decide to erase from the board any two numbers, and replace them with their positive difference. This process is continued till a single number remains on the blackboard. Can this number be zero?

Solution to Problem 3. Consider what happens at each stage to the sum of all the numbers on the blackboard before and after the number replacement. At any stage, let the numbers erased be a and b with $a > b$. Then the sum of all the numbers changes by

$$(a + b) - (a - b) = 2b,$$

an even number. *Hence, at each stage, the sum of the numbers changes by an even number*. Therefore the parity of the final sum and the initial sum will be the same. As the sum $1 + 2 + \cdots + 2016 + 2017$ is odd, the final number written on the board is odd too, and hence cannot be 0.

Problem 4. Can one form a 'magic square' with the first 36 prime numbers?

For the benefit of the reader, a "magic square" here means a 6×6 array of boxes, with a number in each box, and such that the sum of the numbers along any row, column, or diagonal is constant. The answer is NO. Why? Perhaps the reader would like to figure it out.

Problem 5. Let $a_1, a_2, \ldots, a_{2017}$ be a permutation of $1, 2, \ldots, 2017$ such that $a_k \neq k$ for every $k \in \{1, 2, \ldots, 2017\}$. Is the product

$$(a_1 - 1)(a_2 - 2) \cdots (a_{2017} - 2017)$$

even or odd?

Solution to Problem 5. There is an odd number of terms in the product. Suppose the product is odd. What can we say about each term? Each term must be odd. Thus for every k with $1 \leq k \leq 2017$, $a_k - k$

is odd. Now comes the crucial observation. It is motivated by the fact that $a_1, a_2, \ldots, a_{2017}$ is a permutation of $1, 2, \ldots, 2017$, and a permutation keeps the sum of the numbers unchanged. Therefore

$$a_1 + a_2 + \cdots + a_{2017} = 1 + 2 + \cdots + 2017,$$

and this can be re-written as

$$(a_1 - 1) + (a_2 - 2) + \cdots + (a_{2017} - 2017) = 0.$$

Note that the right side of this equality is even whereas each summand on the left side is odd, by assumption. But this cannot happen, because an odd number of odd numbers cannot add up to an even number. Thus we arrive at a contradiction and it arose from our assumption that the product is odd. Therefore the product must be even.

Note that the proposition is not true if the number of numbers we start with is even. For instance, if we had 2018 numbers, the claim could not have been made; in fact, it would have been false. To prove the falsity of the claim for an even number of numbers, we have to exhibit a permutation of this set of numbers for which the product defined in the statement of the problem is odd. We let the reader find such a permutation.

Problem 6. Let a, b and c be odd integers. Prove that the polynomial $ax^2 + bx + c$ does not have a rational root.

Solution to Problem 6. This is a very nice problem. We will solve it in two different ways. There is a reason for doing so, but we will not divulge it right now and rather let the suspense hang in the air. The first method is the textbook method: extract the roots and analyse them. The roots are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Assume that $b^2 - 4ac \geq 0$, so that the roots are real. If we want the roots to be rational, then $\sqrt{b^2 - 4ac}$ must be a positive integer. Let there exist a positive integer x such that

$$\sqrt{b^2 - 4ac} = x.$$

Observe that x is odd, since b^2 is odd and $4ac$ is even. Upon simplification, we get

$$ac = \left(\frac{b-x}{2}\right)\left(\frac{b+x}{2}\right).$$

Observe that both $b - x$ and $b + x$ are even numbers. Thus $\frac{b-x}{2}$ and $\frac{b+x}{2}$ are integers. In fact, they are both odd integers, because ac is odd. But then $b = \frac{b-x}{2} + \frac{b+x}{2}$ is even, contrary to the stated fact that it is odd. This contradiction shows that such a positive integer x does not exist.

Now we are ready for the second method. This is also a proof by contradiction, but it does not require extraction of roots. Here is how it runs. Suppose that the given equation has a rational root $x = \dfrac{p}{q}$, where p and q are integers, $q \neq 0$ and the greatest common divisor of p and q is 1, i.e., p and q are coprime. Thus

$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0.$$

Clearing the denominators leads to

$$ap^2 + bpq + cq^2 = 0.$$

Since p divides ap^2 as well as bpq, it must be that p divides cq^2. Similarly, q divides ap^2. But since p and q are coprime, we conclude that p divides c and q divides a. Thus p and q are odd and so is $ap^2 + bpq + cq^2$. This contradicts the above statement that $ap^2 + bpq + cq^2 = 0$, an even number.

Now the time has come to reveal the reason for discussing the second method. If we study the proof closely, we see that the vital part of the argument is to prove that p and q are odd. Once this is done, the rest of the proof relies on the fact that the sum of an odd number of odd numbers cannot be even. Nowhere did we use the fact that we are dealing with a quadratic polynomial. All that mattered in the end was that an odd number of terms were present in the expression. This opens up the possibility of generalising the proposition to polynomials of arbitrary degree with odd coefficients and having an odd number of terms. To put it in precise mathematical terms, pick an even number k and consider a finite sequence of natural numbers $n_0 < n_1 < \cdots < n_k$ and odd integers $a_{n_0}, a_{n_1}, \cdots, a_{n_k}$. Construct the polynomial

$$a_{n_k}x^{n_k} + a_{n_{k-1}}x^{n_{k-1}} + \cdots + a_{n_1}x^{n_1} + a_{n_0}.$$

This polynomial has $k+1$ terms, which is odd because k is even. This polynomial does not have a rational root. Why? Let us emulate the argument that we used for the quadratic. It is evident that zero is not a root of the polynomial. If possible, let there be a non-zero rational root $x = \frac{p}{q}$, where p and q are coprime integers and $q \neq 0$. Then

$$a_{n_k}\left(\frac{p}{q}\right)^{n_k} + a_{n_{k-1}}\left(\frac{p}{q}\right)^{n_{k-1}} + \cdots + a_{n_1}\left(\frac{p}{q}\right)^{n_1} + a_{n_0} = 0.$$

Multiplying both sides of the equation by q^{n_k} leads to

$$a_{n_k}p^{n_k} + a_{n_{k-1}}p^{n_{k-1}}q^{n_k - n_{k-1}} + \cdots + a_{n_1}p^{n_1}q^{n_k - n_1} + a_{n_0}q^{n_k} = 0.$$

As before, we observe that p divides a_{n_0} and q divides a_{n_k}, hence both are odd. Thus every term on the left hand side of the preceding equation is odd and there are an odd number of them. Therefore the sum of these terms cannot be zero. This contradiction shows that the polynomial cannot have a rational root.

Note that we did not explicitly extract the roots of the polynomial equation in order to carry out the analysis and complete the argument. The reader may be aware that finding roots of a general polynomial is a herculean task. The second method not only overcomes this difficulty, it also reduces the complexity of the problem to a great extent.

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Adventures in Problem Solving A Problem About Divisors

SHAILESH SHIRALI

In this edition of 'Adventures' we study a curious problem concerning the divisors of a certain number. Partial information has been provided about the divisors and on that basis we are required to find the number. The information provided seems at first sight to be meagre in the extreme. But strangely, it suffices to make progress. Read on!

The problem

> *For an arbitrary positive integer n, list its divisors in increasing order, starting with 1 and ending with n. Let the divisors be $d_1, d_2, d_3, \ldots, n$ where $d_1 = 1$ and $d_1 < d_2 < d_3 < \cdots < n$. Find all possible values of n for which the following property is satisfied:*
>
> $$d_8 + d_{10} + d_{11} = n. \tag{1}$$

In other words, find all possible values of n for which the 8-th, 10-th and 11-th divisors of n add up to n itself.

The problem looks formidable (which Olympiad problem does not?), but, as we shall see, there are enough clues to solve it. In fact, we shall uncover a surprising conclusion.

Two observations

At the start, we make a simple yet easily missed observation which holds the key to the solution of this problem:

I: If d is a divisor of a positive integer n, then n/d too is a divisor of n.

Keywords: *Divisor, constraints, combinations, systematic reasoning*

The divisors d and n/d are called *complementary divisors* of n. For example, 2 and 5 are a pair of complementary divisors of 10.

Our second observation is a well-known theorem of elementary number theory:

II: Let the prime factorization of a positive integer n be $n = p^u \cdot q^v \cdot r^w \cdots$ where $p, q, r, \ldots$ are distinct prime numbers, and $u, v, w, \ldots$ are positive integers. Then the number of divisors of n is given by the product

$$(u+1)\cdot(v+1)\cdot(w+1)\cdots. \tag{2}$$

Example 1: Consider the integer 12; its factorization into primes is $12 = 2^2 \cdot 3$. We expect the number of divisors of 12 to be $(2+1)\cdot(1+1) = 6$. And 12 does indeed have 6 divisors (they are: $1, 2, 3, 4, 6, 12$).

Example 2: Consider the integer 30; its factorization into primes is $30 = 2 \cdot 3 \cdot 5$. We expect the number of divisors of 30 to be $(1+1)\cdot(1+1)\cdot(1+1) = 8$. And 30 does indeed have 8 divisors (they are: $1, 2, 3, 5, 6, 10, 15, 30$).

It is an easy exercise to prove the formula. We only need to use these facts: (i) a divisor of n must be made up of the same primes that divide n; (ii) the power to which a prime number divides the divisor cannot exceed the power to which that prime number divides n itself.

Solution to the problem

Let $a = n/d_8$, $b = n/d_{10}$ and $c = n/d_{11}$. Then $1 < c < b < a$, and the condition $d_8 + d_{10} + d_{11} = n$ translates to:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1. \tag{3}$$

The first task, clearly, is to identify all positive integer solutions of the above equation. We shall show that with the condition $c < b < a$, there is just one solution.

Suppose that $c \geq 3$. Then $a > b > c \geq 3$, hence:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{3} + \frac{1}{3} + \frac{1}{3},$$

$$\therefore \ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1. \tag{4}$$

So if $c \geq 3$, then relation (3) cannot be satisfied. Since $c > 1$, it follows that $c = 2$, i.e., $d_{11} = n/2$. *Hence, 2 is one of the prime divisors of n.*

Since $c = 2$, we get:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2}, \tag{5}$$

and $a > b > 2$, i.e., $a > b \geq 3$.

Suppose that $b \geq 4$. Then $a > b \geq 4$, hence:

$$\frac{1}{a} + \frac{1}{b} < \frac{1}{4} + \frac{1}{4},$$

$$\therefore \ \frac{1}{a} + \frac{1}{b} < \frac{1}{2}. \tag{6}$$

So if $b \geq 4$, then relation (5) cannot be satisfied. Since $b > 2$ (obtained above), it follows that $b = 3$, i.e., $d_{10} = n/3$. *Hence, 3 is one of the prime divisors of n.*

Having obtained $c = 2$ and $b = 3$, we get a by substitution:

$$\frac{1}{a} = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, \qquad \therefore\ a = 6.$$

Let us summarize what we have obtained till now: 2, 3 *and* 6 *are divisors of n, and*:

$$d_1 = 1, \quad d_2 = 2, \quad d_3 = 3, \quad \ldots, \quad d_8 = \frac{n}{6}, \quad d_{10} = \frac{n}{3}, \quad d_{11} = \frac{n}{2}. \tag{7}$$

Since 2 and 3 are prime divisors of n, it follows that n has the following form:

$$n = 2^u \cdot 3^v \cdot w, \tag{8}$$

where u, v are positive integers, and w is a positive integer not divisible by either 2 or 3, i.e., $\gcd(w, 6) = 1$.

Between $n/2$ and n, there can clearly be no further divisors of n; do you see why? Hence the next divisor after d_{11} must be n itself; that is, $d_{12} = n$. So we obtain another important property:

$$n \textit{ has precisely } 12 \textit{ divisors.} \tag{9}$$

Next we ask: Which integers have exactly 12 divisors? On listing the different ways in which 12 can be written as a product of integers exceeding 1,

$$12 = 6 \times 2 = 4 \times 3 = 3 \times 2 \times 2, \tag{10}$$

we deduce that the positive integers which have exactly 12 divisors are of the following four kinds:

$$p^{11}, \quad p^5 \cdot q, \quad p^3 \cdot q^2, \quad p^2 \cdot q \cdot r, \tag{11}$$

where p, q, r are unequal primes. For example, each of the following has exactly 12 divisors:

$$2^{11}, \quad 3^{11}, \quad 2^5 \times 3, \quad 3^5 \times 2, \quad 2^3 \times 3^2, \quad 3^3 \times 2^2, \quad 2^2 \times 3 \times 5, \quad \ldots.$$

Moreover, the forms listed in (11) cover all the possibilities.

We see that there are infinitely many integers having 12 divisors. From this infinite collection, we need to select those which satisfy the conditions in our problem.

There has been no mention of d_9 till now. But we can say this: Since $d_8 = n/6$ and $d_{10} = n/3$, it must be that $d_9 = n/4$ **or** $d_9 = n/5$. Hence n has either 4 or 5 as a divisor — but not both.

Combining this deduction with what we got earlier, we see that precisely one of the two possibilities listed below holds:

(a) $n = 2^u \cdot 3^v \cdot r$ where u, v are positive integers with $u \geq 2$, and r is a positive integer not divisible by 2, 3 or 5; **OR**:

(b) $n = 2 \cdot 3^v \cdot 5^w \cdot s$ where v, w are positive integers and s is a positive integer not divisible by 2, 3 or 5.

To this finding, we bring the fact that n must have one of the forms listed in (11). It follows that n must be one of the following:

- $n = 2^5 \times 3 = 96$;
- $n = 2^3 \times 3^2 = 72$;
- $n = 3^3 \times 2^2 = 108$;
- $n = 2^2 \times 3 \times w = 12w$ where w is a prime number greater than 5;

- $n = 3^2 \times 2 \times 5 = 90$;
- $n = 5^2 \times 2 \times 3 = 150$.

Hence the integers which satisfy the stated condition are precisely the following:

$$72, \quad 90, \quad 96, \quad 108, \quad 150, \quad 12w,$$

where w is any prime number greater than 5.

Here are three sample verifications. Observe that the relation $d_8 + d_{10} = d_{11}$ holds in each case, and this is true by virtue of the equality $1/6 + 1/3 = 1/2$; or, equivalently, $n/6 + n/3 = n/2$.

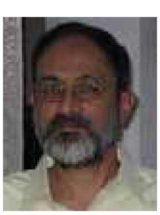
- $n = 90$: The divisors of n are
$$1,\ 2,\ 3,\ 5,\ 6,\ 9,\ 10,\ 15,\ 18,\ 30,\ 45,\ 90,$$
and we may check that $15 + 30 + 45 = 90$.
- $n = 108$: The divisors of n are
$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12, 18,\ 27,\ 36,\ 54,\ 108,$$
and we may check that $18 + 36 + 54 = 108$.
- $n = 12 \times 11 = 132$ (the form $12w$ with $w = 11$): The divisors of n are
$$1,\ 2,\ 3,\ 4,\ 6,\ 11,\ 12,\ 22,\ 33,\ 44,\ 66,\ 132,$$
and we may check that $22 + 44 + 66 = 132$.

We have discovered a surprising fact: There are infinitely many integers which satisfy the stated condition!

A problem for you to tackle …

Now we close with a problem for you; it is a small variation of the problem we have solved, but the outcome turns out to be quite different.

> For an arbitrary positive integer n, list its divisors in increasing order, starting with 1 and ending with n. Let the divisors be $d_1, d_2, d_3, \ldots, n$ where $d_1 = 1$ and $d_1 < d_2 < d_3 < \cdots < n$. Find all possible values of n for which the following property is satisfied: $d_7 + d_{10} + d_{11} = n$.



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Problem Corner

The Three Circles Problem

$\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$

In this article, we study the following problem. *Three circles of equal radius r are centred at the vertices of an equilateral triangle ABC with side 2a. Here we assume that $r > a$. Find the area of the three-sided region DEF enclosed by all three circles, in terms of r and a.* (See Figure 1.)

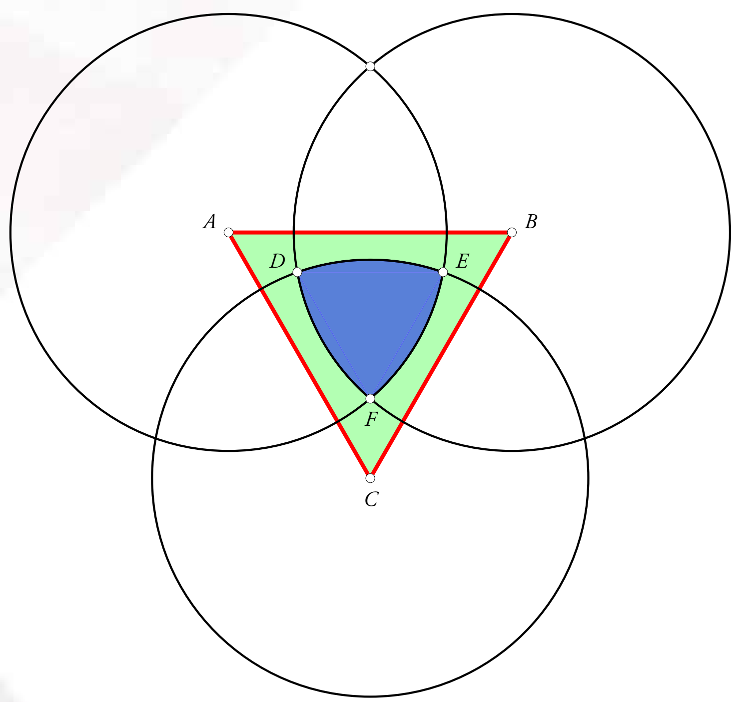

Figure 1.

Solution. We carry out the analysis as shown below.

Keywords: *Circles, intersection, area*

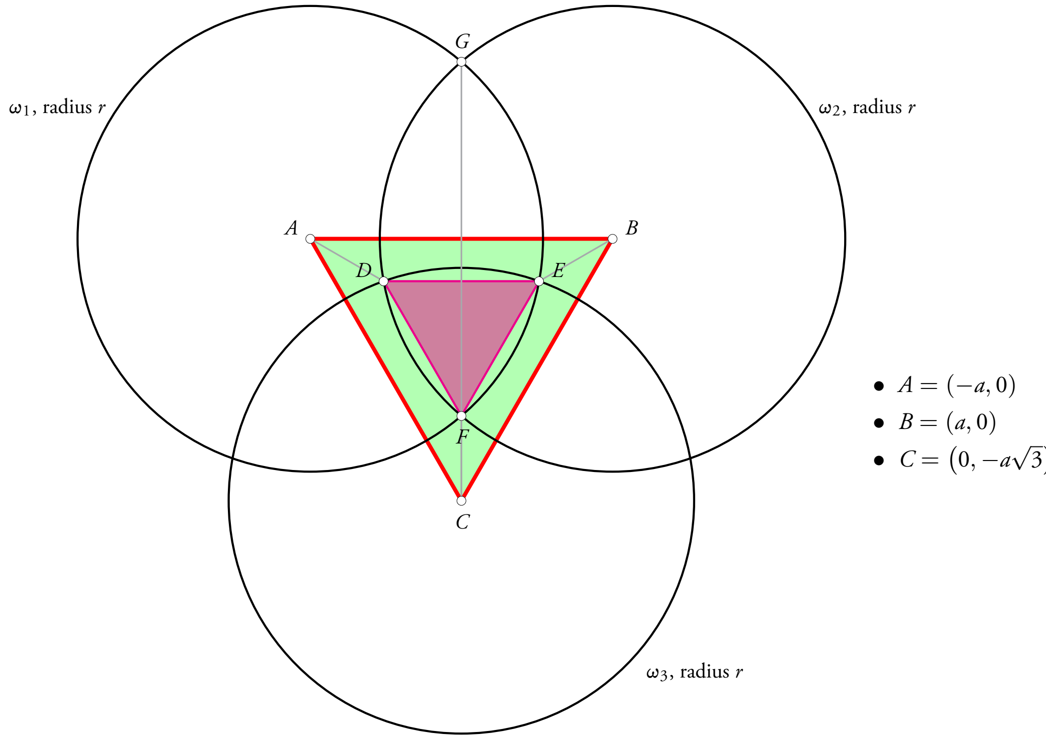

Figure 2.

(1) Mark points D, E, F, G as shown. Using Pythagoras's theorem, we obtain $FG = 2\sqrt{r^2 - a^2}$. Let $2d$ be the length of DE. We must first find d in terms of r and a.

(2) Assign coordinates as shown:

$$A = (-a, 0), \quad B = (a, 0), \quad C = \left(0, -a\sqrt{3}\right).$$

The equations of the three circles then are:

$$\begin{aligned} \omega_1 : \quad & (x+a)^2 + y^2 &= r^2, \\ \omega_2 : \quad & (x-a)^2 + y^2 &= r^2, \\ \omega_3 : \quad & x^2 + \left(y + a\sqrt{3}\right)^2 &= r^2. \end{aligned}$$

(3) The coordinates of points D, E, F, G can now be worked out by solving pairs of simultaneous equations. Here is what we get:

$$D = \left(\frac{a - \sqrt{3\left(r^2 - a^2\right)}}{2}, \frac{-a\sqrt{3} + \sqrt{r^2 - a^2}}{2}\right),$$

$$E = \left(\frac{-a + \sqrt{3\left(r^2 - a^2\right)}}{2}, \frac{-a\sqrt{3} + \sqrt{r^2 - a^2}}{2}\right),$$

$$F = \left(0, -\sqrt{r^2 - a^2}\right),$$

$$G = \left(0, \sqrt{r^2 - a^2}\right).$$

(4) The length of DE can now be worked out from the coordinates of D and E:

$$DE = \sqrt{3\left(r^2 - a^2\right)} - a.$$

(5) The area of triangle DEF can now be worked out using the above expression:

$$\text{Area of } \triangle DEF = \frac{\sqrt{3}}{4}\left(\sqrt{3\left(r^2 - a^2\right)} - a\right)^2$$

$$= \frac{3r^2\sqrt{3} - 2a^2\sqrt{3} - 6a\sqrt{r^2 - a^2}}{4}.$$

(6) Next, we find $\theta = \measuredangle DCE$, using the length of DE:

$$\sin\theta = \frac{DE/2}{r}$$

$$= \frac{\sqrt{3\left(r^2 - a^2\right)} - a}{2r}.$$

(7) This allows us to find the area of the minor segment bounded by segment DE and circle ω_3:

$$\text{Area of segment } D\omega_3 E = \frac{r^2(\theta - \sin\theta)}{2}.$$

(8) Finally, the area of the region DEF is given by:

$$\text{Area of region } DEF = \text{Area of } \triangle DEF + 3 \cdot \text{Area of segment } D\omega_3 E.$$

This simplifies, after a lot of work, to:

$$-\frac{3}{2}a\sqrt{r^2 - a^2} + \frac{3}{4}a\sqrt{2a\left(\sqrt{3\left(r^2 - a^2\right)} + a\right) + r^2}$$

$$-\frac{3}{4}\sqrt{3}\sqrt{\left(r^2 - a^2\right)\left(2a\left(\sqrt{3\left(r^2 - a^2\right)} + a\right) + r^2\right)}$$

$$+ 3r^2\sin^{-1}\left(\frac{\sqrt{3\left(r^2 - a^2\right)} - a}{2r}\right) - \frac{1}{2}\sqrt{3}a^2 + \frac{3\sqrt{3}r^2}{4}$$

This is the required area.

(9) For $r = 10, a = 6$, we get:

$$\text{Area of region } DEF = 36\left(\sqrt{3} - 4\right) + 300\sin^{-1}\left(\frac{1}{10}\left(4\sqrt{3} - 3\right)\right)$$

$$\approx 39.4628 \text{ square units.}$$

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Problem About a Finite Sequence

$\mathcal{C} \otimes \mathcal{M} \alpha \mathcal{C}$

The following problem appeared in the International Mathematical Olympiad (IMO) of 1977, held that year in (the former Republic of) Yugoslavia (see [1]; it is the second problem in the paper):

In a finite sequence of real numbers, the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

It is not difficult to show that such a sequence cannot have more than 16 terms. Suppose there is a 17-term sequence $a_1, a_2, a_3, \ldots, a_{16}, a_{17}$ having the stated property. We shall show that this leads to a contradiction. We use the terms of the sequence to construct the following 7×11 matrix:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}

As per the given information, the sum of the terms in each row is positive, hence the sum of all the terms in the entire matrix is positive. Also as per the given information, the sum of the terms

Keywords: Finite sequence, IMO

in each column is negative, hence the sum of all the terms in the entire matrix is negative. We have an obvious contradiction. Hence there cannot exist a 17-term sequence having the stated property.

The interesting question now arises: *Does there exist a 16-term sequence having such a property, and if so, how do we construct such a sequence?* Here is one line of reasoning which helps us find such a sequence. Suppose that $a_1, a_2, a_3, \ldots, a_{16}$ is a 16-term sequence with the stated property. Let us now see what can be gleaned from this information. Let b_k be the sum of the first k terms of the a-sequence, i.e.,

$$b_k = a_1 + a_2 + a_3 + \cdots + a_k.$$

Observe that there are 16 numbers $b_1, b_2, b_3, \ldots, b_{16}$. Can we state which of these is the smallest and which one is the largest? It turns out that the complete chain of order relations can be deduced using only the stated property. For we have:

$$0 < b_{11} < b_4 < b_{15} < b_8 < b_1 < b_{12} < b_5 < b_{16} < b_9 < b_2 < b_{13} < b_6.$$

For example, $b_{11} < b_4$ is true because $a_5 + a_6 + \cdots + a_{10} + a_{11} < 0$; similarly for the other relations. We also have:

$$b_{10} < b_3 < b_{14} < b_7.$$

Finally we have $b_7 < 0$. This means that

$$b_{10} < b_3 < b_{14} < b_7 < 0 < b_{11} < b_4 < b_{15} < b_8 < b_1 < b_{12} < b_5 < b_{16} < b_9 < b_2 < b_{13} < b_6.$$

So we have deduced the full chain of order relations! Quite remarkable. . .

These relations now suggest a way of constructing a sequence with the required property: *assign to the quantities $b_{10}, b_3, b_{14}, b_7, b_{11}, b_4, b_{15}, b_8, \ldots, b_{13}, b_6$* ***any*** *16 numbers in increasing order, the first four of them being negative, and then solve for the unknown quantities $a_1, a_2, a_3, \ldots, a_{16}$*. For example, we may put $b_{10} = -4$, $b_3 = -3$, $b_{14} = -2$, $b_7 = -1$, $b_{11} = 1$, $b_4 = 2$, $b_{15} = 3$, $b_8 = 4, \ldots, b_6 = 12$. Here are the values displayed in tabular form:

b_{10}	b_3	b_{14}	b_7	b_{11}	b_4	b_{15}	b_8	b_1	b_{12}	b_5	b_{16}	b_9	b_2	b_{13}	b_6
– 4	– 3	– 2	– 1	1	2	3	4	5	6	7	8	9	10	11	12

Or, rearranging the entries and displaying them in a more convenient form:

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
5	10	– 3	2	7	12	– 1	4	9	– 4	1	6	11	– 2	3	8

These yield the following, in turn:

$a_1 = b_1 = 5;$

$a_2 = b_2 - b_1 = 10 - 5 = 5;$

$a_3 = b_3 - b_2 = -3 - 10 = -13;$

$a_4 = b_4 - b_3 = 2 + 3 = 5;$

$a_5 = b_5 - b_4 = 7 - 2 = 5;$

$a_6 = b_6 - b_5 = 12 - 7 = 5;$

$$a_7 = b_7 - b_6 = -1 - 12 = -13;$$

$$a_8 = b_8 - b_7 = 4 - (-1) = 5;$$

and so on. Proceeding systematically in this manner, we obtain all the numbers. Here are the resulting values:

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
5	5	– 13	5	5	5	– 13	5	5	– 13	5	5	5	– 13	5	5

Isn't it curious that the sequence assumes only two different values? (Of course, the values are decided by the choice of the 16 numbers made earlier.)

It should be clear from the way we have found the values of the a_i that, regardless of what values we give to the quantities $b_{10}, b_3, b_{14}, b_7, b_{11}, b_4, b_{15}, b_8, \ldots, b_{13}, b_6$, we will always obtain a-values that fit the various equations.

References

[1] International Mathematical Olympiad, https://www.imo-official.org/year_info.aspx?year=1977

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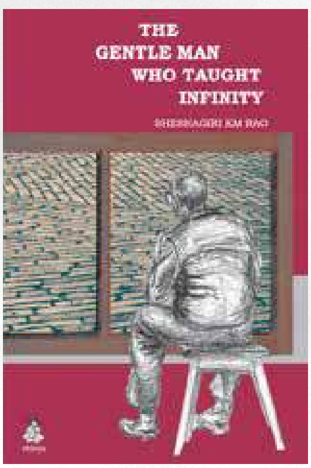
The Gentle Man Who Taught Infinity

by Sheshagiri KM Rao

Reviewed by Sneha Titus



This book review has appeared in the July 2018 issue (Volume 23 Number 7) of Resonance: journal of science education, published by the Indian Academy of Sciences. We are very grateful to the Indian Academy of Sciences for giving us permission to reprint this review.

Rarely has a book so perfectly matched its title. *The Gentle Man Who Taught Infinity* by Sheshagiri KM Rao is one such and what a gentle read it was! Written as a tribute to the mathematics teacher who influenced his life, Sheshagiri Rao has managed to show us with his account just how far reaching a teacher's influence can be and how this teacher did it, not commandingly or overtly or even intentionally but with his sheer love for the subject he taught and his innate respect for the students he taught.

The narrative is set in Bangalore, the pitch is set right from the start with a description of the city that the author grew up in, a far cry from the bustling metropolis it is now. Sheshagiri Rao's childhood memories will certainly strike a chord with readers who grew up in the sixties and seventies – of going by cycle rickshaw to school, of climbing trees at play during the long evenings at home, of booking 'trunk calls' and of the fascination with 'church-run schools' with their emphasis on education in English. Which eventually took the author to Baldwin Boys School where, in the eighth grade, he encountered the chief protagonist of this book, his mathematics teacher Mr. Channakeshava.

The teacher, his craft and his subject – these have been the recurring themes of many a teacher education program. In this book, which I would recommend as illustrative reading for anyone teaching or taking these courses, the author explains how, in the person and the practice of his math teacher, these

Keywords: mathematics, pedagogy, math-phobia, teacher, craft, subject

three were entwined. In almost every anecdote, these emerge inseparably and subtly. Clearly, they produced a deep and lasting impression on the student, strongly evident in the incidents he reports in his own teaching career.

Much has been said about first impressions – often they are linked to meeting dashing and flamboyant characters. For a student to remember his first mathematics class in standard 7 for over thirty years, one would expect that his teacher was one such. And yet, the very gentleness of 'Channa' as his students affectionately called him, is what is emphasized in his very first class. That, and his lesson which though a simple exercise in multiplication, revealed beautiful patterns in mathematics and hooked his students to the subject in a way that revision, practice and even good marks couldn't do so far. *'I'd never seen beauty in mathematics until then, even in my wildest dreams. What happened in the period that day was quite astonishing even as it was fun.'* And almost immediately, the author talks about the roadblocks which he himself or others would have put in the way of such an approach – the inevitable chorus of lack of time and the need to complete the syllabus. Among his many arguments for such an approach, the following would be the most persuasive……… *'With Channa, the teaching was clear and cogent. Maths learning was almost effortless because we began to see things with more clarity.'*

The more sceptical reader may then question how such an unusual teacher could thrive in a conventional school. Remember that this book is set in the 1970s, a time when schools thrived on rules, discipline, and corporal punishment and when good marks were the single benchmark of excellence. (What has changed, the cynic may say, but let's be optimistic.) Rao makes it clear that the explorations that the class embarked on were sandwiched between more conventional classes. What remained, however, was the excitement that mathematics could bring.

At the beginning of the book, Rao makes the claim that he has tried to keep the maths simple. Having seen this in many books about math, I was a little cynical but he has managed to do this. Again, I see Channa's influence; he explains almost everything from first principles and in the Additional Notes arranged chapter-wise at the back of the book, there is detailed information on the mathematical topics mentioned in each chapter. Even better, the author has catered to the more serious reader by providing a list of books for those who want to delve deeper.

As a student, the author was exposed to many famous problems which are not in any school level syllabus. In the same gentle manner in which he was taught, he describes mathematical celebrities such as the Bridges of Konigsberg, the Four Colour Problem, the Barber's Paradox and Fermat's Last Theorem. Having encountered these only much after I left school, I can only envy the students whose teacher shared his own joy in the subject with mathematical story telling. Small wonder that he was remembered long after they graduated.

At the end of the book, the author describes Mr. Channakeshava's life and how his various struggles and responsibilities kept him anchored to his job at Baldwin Boys. He stresses on the point that nothing can stop the genuine learner – and here, we can learn by example. Not only was Channa constantly trying new and different problems, he was also an avid reader of a wide variety of books in a wide variety of languages. For those who want a slice of the problem pie, the section 'Whet thy appetite: Channa's 20' is a delightful appendix of twenty of Channa's chosen problems which he solved over the years and whose solutions he regularly submitted to a variety of math magazines and journals.

Skill building is in nowadays; from Rao's book, one begins to see that even in the seventies, Channa realised the importance of this. The author touches on an approach advocated by his own father when he found a topic difficult –

'gudipaatam', he calls it – we know it variously as 'mugging', 'learning by heart' and 'parroting'. Though not overtly stated, it is clear from various incidents that Channa was a teacher for whom content came second to skills. Using stories, sketches, arguments and even the drama of the QED which he appended to each successful proof, he taught his students the skills of visualization, representation, logical reasoning and mathematical communication.

There is a beautiful section on the need for proof and the difference between a proof and a demonstration. Clearly, the author has connected the nature of mathematics to his teacher's craft. Only one exposed to the pedagogy of mathematics could do so, but what stands out is that Channa made a lasting impression on many, many students. The author has taken the trouble to contact many of his school fellows, their memories of Channa are shared in the book. Including the opinions of a variety of students who went on to varied careers and who spoke from different perspectives certainly served to prove his point.

The book does tend to meander a bit and some themes are revisited through the lens of different incidents. But this is perhaps necessary to pencil in a more detailed picture of the beloved teacher.

In the Author's Note, Rao says that this book was written – among others – for parents who struggled with mathematics as students and who are now in the difficult position of prescribing a medicine which they themselves found difficult to swallow. Certainly, there are many readers who will identify with parental anxiety, expectations of the student and of the school and the reassurance extended by a benevolent teacher. *Mathematics is full of such curiosities, which can be studied by just about anyone* was one of Channa's quotes and the author goes on to say that everybody has the seeds of mathematical ability, they just need to be nurtured. Which is what Channa did for his students.

Rao says that this book was driven by both hope and anger at the way schools let down their students. Various aspects of Channa, the man, are described in the book. All of them factor in to Channa, the teacher – his immaculate brown suit, his gentlemanly ways, his sense of duty to his family, his steadfastness in protecting his values. Whether it is in describing how he used blackboard space or maintained discipline without resorting to the rod or remained both aloof and familiar outside the classroom, Rao manages to draw for us the picture of the Gentle Man who Taught Infinity, leaving us with the hope that as long as there are teachers like Mr. Channakeshava, schools would benefit and not harm students.

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Review

Mini Review
Numero Friendly Board Games

Designed by Pratima Patil

Reviewed by Sneha Titus

Just imagine, in a routine mathematics class a teacher enters the class room with a colorful board game. Instead of instructing students to take out their math textbooks/note books and setting work for them, he just opens the game board and allows students to play the game. The eyes of the students sparkle and they enjoy playing. Even the back benchers (who generally do not get involved in class room work) come forward to play and give a neck to neck fight to the scholars in the class. While playing the games, children get familiar with numbers and their interesting properties such as factors, multiples, square numbers, prime numbers and so on.

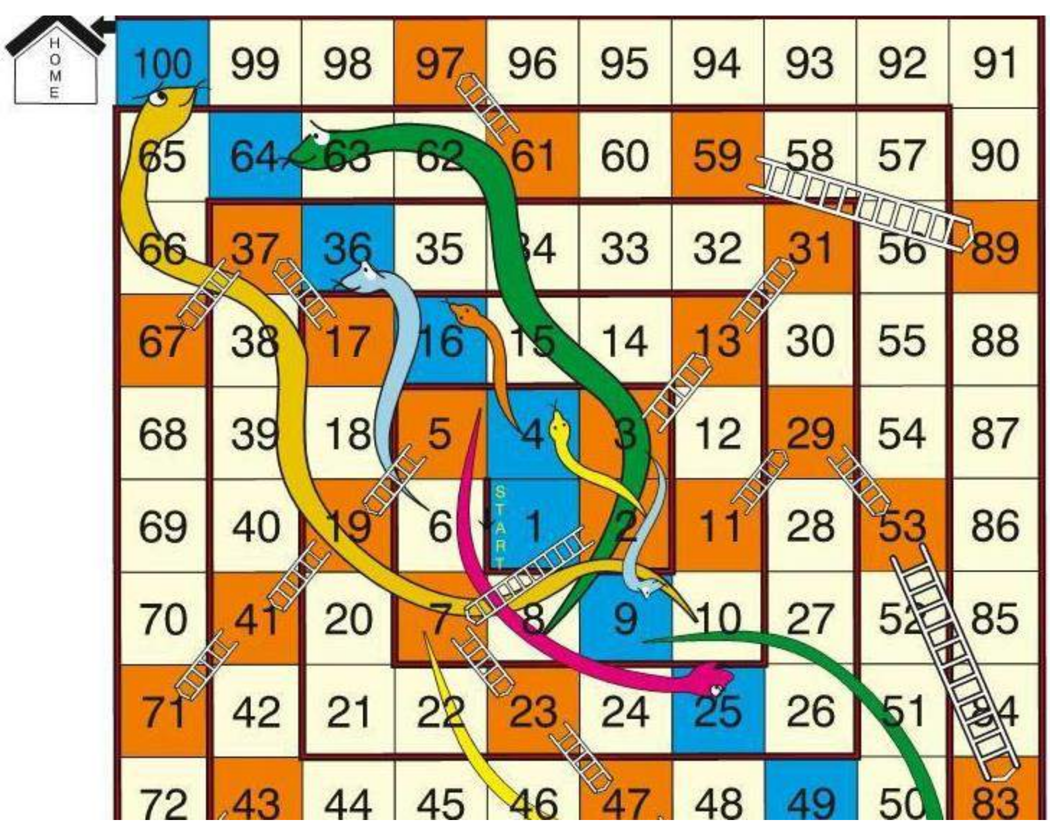
These are the words of the creator (Pratima Patil from Navi Mumbai) of Numero Friendly Board Games which I had the opportunity to review recently. I saw Squares & Primes and Dido – you may guess that the first is modeled on Snakes & Ladders and the second on that eternal favourite, Ludo. Pratima has used the fact that numbers can be arranged in different ways to exhibit interesting properties and sequences. For example, in Squares & Primes, the numbers from 1 to 100 are arranged spirally in a 10×10 grid - all the square numbers fall on either one of two diagonals and all the prime numbers on slanted lines. The climb up the ladders is from a smaller prime to a larger prime. And the slide down the snake is from a perfect square to its square root. The objective of course, is to start at 1 (at the centre of the board) and to end by going past 100 in the smallest number of moves. During the course of which, students encounter prime and composite numbers, perfect squares and their square roots, the concept of chance and fair play and of engaging with mathematics naturally and without fear. Many math games have some element

Keywords: Board game, chance, fair play, squares, primes, multiples, factors

of 'testing knowledge' but here, students play without needing to know the right answer to get ahead. I find that incredibly liberating. Of course, careful facilitation can help students absorb new definitions and relationships (what if you had landed on 1, could you climb a ladder, how many primes between 1 and 100, how many possible snakes, these are just a few questions that come to my mind). Even more interesting, students may be motivated to create their own boards with different arrangements and different rules- this would be constructivism in the math classroom!

Dido helps students practise Factors and Multiples with a board that has four separate tracks for four players and a dice that bears the first six primes on the 6 faces. Forward movement is allowed only if the number on the track in front of the player's piece is divisible completely by the number on the face of the dice just rolled. The destination at the centre of the board carries the number 30030, you can guess how it's related to the numbers on the dice. The game gives players an opportunity to document their moves on flashcards and I find that an extremely useful device for students to reflect on outcomes and their implication. Dido can be played at several levels – the creator has also created a colour code that relates factors to multiples but in my opinion, the first level becomes too simple with the colour coding and the second level may end up being confusing unless all rules are clearly spelt out. However, students will certainly be interested enough to try the game and set ground rules at their comfort level. One concern that I have is that the windmill shape of the Dido board may make it difficult to store or to use often without bending. Also, if the board is too small, the higher levels of the game may not be feasible. I've been told that these factors will be redressed soon.

But these are minor quibbles; if students engage joyously in innovative math games then more power to such creators!



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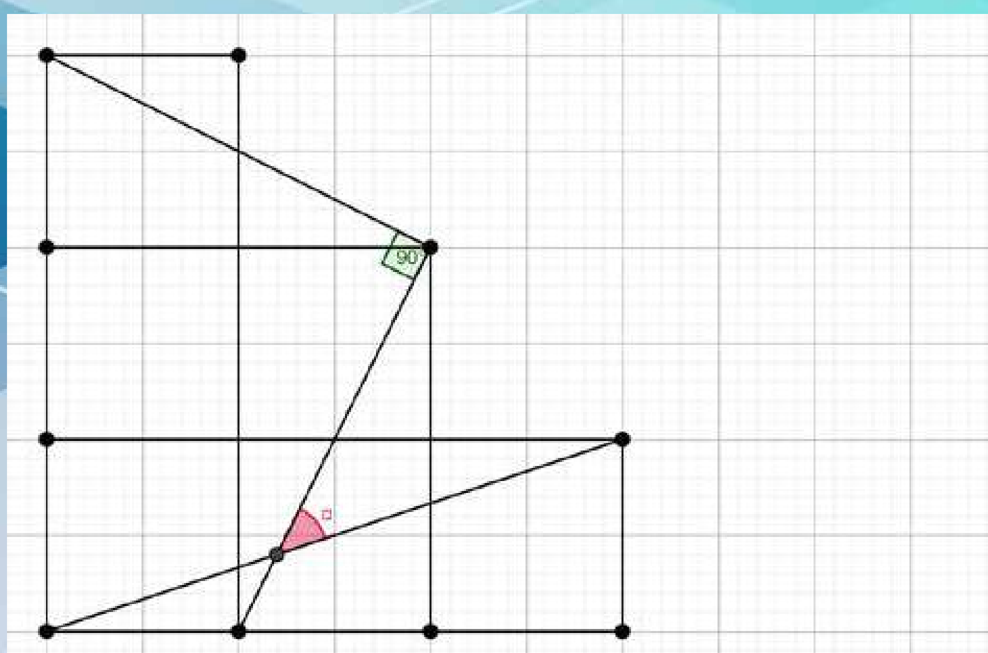
PRATIMA PATIL has just received copy right for the game boards as well as for the literary work based on these games. Also she has applied for the patent at the Patent Office. She has demonstrated the games with good response from schools and teacher training institutes. The game app is under process and all these products will soon be available under the name NUMERO FRIENDLY. I'm certainly looking forward to having these board games available commercially, meanwhile for those who can't wait, Pratima Patil may be contacted at pratimap0309@gmail.com or mobile +917715952612 / +91 9892351862. Pratima is very keen that she tests the games with children from different schools, so do contact her if you would like to run a trial.

Find angle α in as many different ways as possible



Filler contributed by Rupesh Gesota

The Closing Bracket . . .

Our country, at least as a matter of official policy, recognizes teaching to be a profession. Teacher education institutions are often separate from colleges of arts and science, and are regulated by a separate council. The latest National Curriculum Framework for Teacher Education announces this orientation in its subtitle: “Towards Preparing Professional and Humane Teacher” (NCTE 2010). But it is pertinent to ask, as many do, if teaching is really a full-fledged profession. After all, teachers are hardly comparable to lawyers or doctors in terms of their status and power. Nearly a half century ago, Etzioni in the U.S. amongst others, triggered controversy by declaring that teaching was only a semi-profession, one that did not deserve full a professional status like law and medicine (Etzioni 1969). According to Etzioni, along with teaching, nursing, social work, even engineering were only semi-professions. Today many educators would still say that teaching is yet to attain full professional status, while disagreeing with the view that it *ought to remain* a semi-profession. Indeed, policy documents, such as the position paper on Teacher Education of the National Curriculum Framework (NCERT 2006), emphasize the need to strengthen the professional standards of teaching.

What makes an occupation a profession? A variety of authors have laid down a number of essential characteristics of a profession. Those that figure in nearly every list are a shared foundational knowledge on which the professional practice is grounded, a code of ethics, autonomy and control in maintaining standards for entry and membership, and social (and in contemporary times, economic) status. Besides these general characteristics, some concrete markers of professional status are often mentioned as important: mobility and portability of benefits and independence of a particular employer, influence on policy, time set aside for professional development and self-improvement besides leave time, etc. It is clear that teaching is yet to acquire many of these characteristics in real terms.

Education has a formative role in shaping a person's beliefs, emotions and values, in the formation of dreams and aspirations, in building one's self-image and character – in short, in forging what makes us distinctly human. Whatever arrangement a society might put in place in order to educate its members, most educators would agree that the actual enterprise of education must be carried out through direct and caring interaction between human beings. Education is about the preservation and transmission across generations of what is distinctly human and hence human beings have to be central actors in this enterprise. Teachers are these central actors and it is in the fitness of things that society should accord them status and respect and recognize them as professionals.

Some measures needed to strengthen teaching as a profession are of a practical nature. For example, the need to form strong, well-knit professional organizations that embody the highest values and principles of the profession. The most common professional associations of teachers are those of subject teachers – associations of mathematics teachers or of science teachers. Most teachers have a dual identity, as teachers in general and as teachers of a subject. A teacher who teaches mathematics in school, especially beyond the elementary level, often has a strong identity as a mathematics teacher. While this is natural under the current organization of school curricula, one's identity as a mathematics teacher must not eclipse the larger mandate of teaching as a whole and its far-reaching goals and ultimate purpose.

The activities of a professional association must not become routine and mechanical exercises, but must centrally address the agenda of strengthening teaching as a profession. Meetings and conferences must be occasions to discuss fundamental issues that affect teaching as a profession. For e.g., what is the nature of knowledge that underpins the teaching of mathematics? What is the relationship between disciplinary mathematical knowledge and school mathematical knowledge? These questions are related to the very practical question of how much and what aspect of disciplinary mathematics a teacher should know in order to teach more effectively. Teachers must, for instance, balance the picture of mathematics as knowledge essentially about abstract objects and structures with the connections of mathematics to models and contexts, which are central to teaching and learning.

The task of building mathematics teaching as a profession is a large task, one that we cannot afford to postpone. I believe that AtRiA contributes to this task in a significant way and that teachers as professionals, find value in its pages. In the future, I hope that this contribution will become even stronger.

References

[1] Etzioni, A. (1969). *The semi-professions and their organization: Teachers, nurses, social workers*. Free Press.

[2] National Council for Teacher Education. (2010). *National curriculum framework for teacher education: Towards preparing professional and humane teacher*. New Delhi: NCTE.

[3] National Council of Educational Research and Training. (2006). *Position Paper of the National focus group on Teacher Education for Curriculum Renewal*. New Delhi: NCERT.

K. Subramaniam

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Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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Published at Azim Premji Foundation for Development, Azim Premji University Pixel B Block, PES College of Engineering Campus, Electronic City, Bangalore 560 100; **Editor:** Shailesh Shirali

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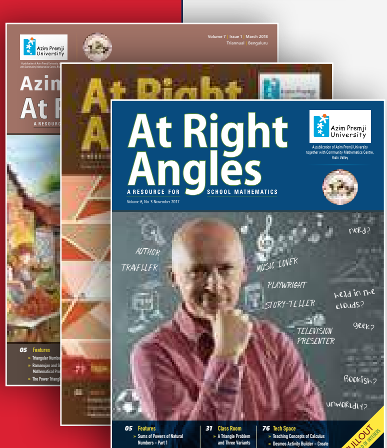

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Articles involving all aspects of mathematics are welcome. An article could feature: a new look at some topic; an interesting problem; an interesting piece of mathematics; a connection between topics or across subjects; a historical perspective, giving the background of a topic or some individuals; problem solving in general; teaching strategies; an interesting classroom experience; a project done by a student; an aspect of classroom pedagogy; a discussion on why students find certain topics difficult; a discussion on misconceptions in mathematics; a discussion on why mathematics among all subjects provokes so much fear; an applet written to illustrate a theme in mathematics; an application of mathematics in science, medicine or engineering; an algorithm based on a mathematical idea; etc.

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Publisher
Azim Premji University together with Community Mathematics Centre, Rishi Valley.

Editors
Currently drawn from Rishi Valley School, Azim Premji Foundation, Homi Bhabha Centre for Science Education, Lady Shri Ram College, Association of Math Teachers of India, Vidya Bhavan Society, Centre for Learning.

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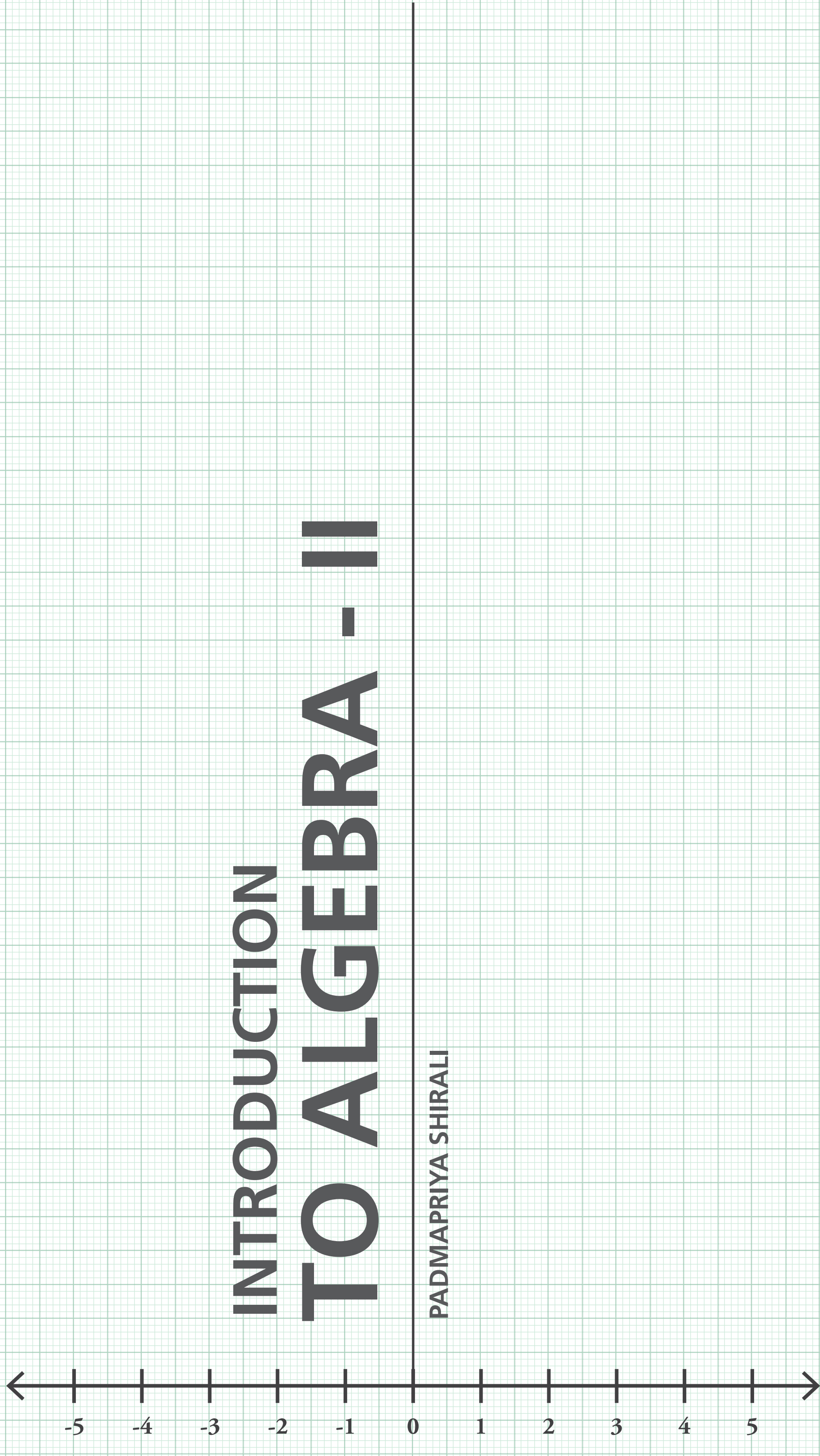
A Resource for School Mathematics

INTRODUCTION TO ALGEBRA - II

PADMAPRIYA SHIRALI



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INTRODUCTION

This article is Part II of the series 'Algebra – a language of patterns and designs.' The approach is based on the perception of algebra as a generalisation of relationships.

In Part I [http://teachersofindia.org/en/article/introduction-algebra-right-angles-pullout], we introduced the ideas of *variable*, *constant*, *term* and *expression* via numerical patterns. Various operations (addition, subtraction, multiplication) involving terms and expressions were also studied.

Now in Part II, we revisit the usage of words such as *variable*, *term* and *expression* and concepts and operations involving terms and expressions in the context of ***geometric designs*** (line designs, 2-D designs, 3-D designs).

Geometric designs are seen everywhere. Tile designs on floors, brick or stone work on walls, partitions of windows and doors, cardboard boxes (toothpaste boxes, soap boxes …) and many other everyday objects can be described in algebraic form.

As always, we begin with familiar concrete objects and use algebraic language to describe them before moving progressively to abstract algebraic expressions.

Ideally students should be exposed to algebra initially through the pattern approach followed by the design approach. However, the two approaches are independent of each other.

By approaching algebra through different routes, we will be able to make a robust link between informal algebra at the primary stage to the more formal algebra which students encounter later. Also it will facilitate students' fluency in the language of algebra, i.e., understanding variables and symbols and being able to use algebraic rules correctly.

Prior knowledge: Students need to be familiar with notions such as line segment, length, region and area of a rectangle, area of a square, capacity, volume of a cuboid and volume of a cube.

Keywords: Algebra, language, pattern, geometric design, variable, constant, term, expression, operation.

ACTIVITY 1

Objective: Introduction to design language (in the context of line designs) and the usage of variables for different lengths.

Materials: Sets of straws or straight sticks of different lengths. Dot paper

A few examples of real life situations:

How do we describe this ladder?

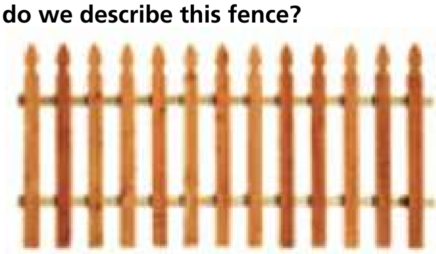
The ladder is made up of some long sticks and some short sticks.

If we take the ***length*** of the long stick as ***'l units'*** and the *length* of the short stick as ***'s units'***, we can describe the ladder design as 2*l* and 5*s* or **2*l* + 5*s***.

How do we describe this house?
We can express the house design as 5*l* and 3*s*.

We can also express it as **5*a*+3*b*** (where a stands for the length of the long stick and b stands for the length of the short stick).

How do we describe this fence?



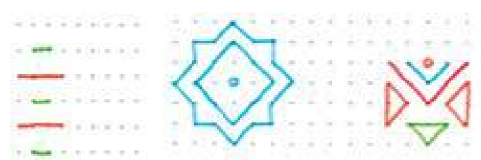
How many *l*'s? How many *s*'s? Here *l* is the length of the long stick and *s* is the length of the short stick.

Initially students can make designs using two different lengths and describe it as an expression.

Later on they can make designs using three or four different lengths.

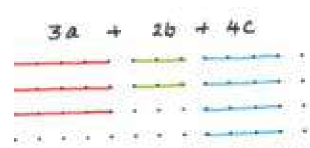
Here is a design made of three different lengths. It can be expressed as 3*a* + 2*b* + 4*c*.

Students can now be given designs of a similar kind to describe using appropriate design language.

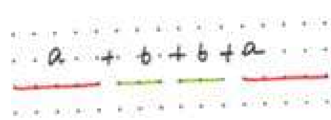


Set them exercises of the reverse type as well. Give students dot paper and ask them to create designs for some given expressions.

Example: 3*a* + 2*b* + 4*c*



Example: *a* + *b* + *b* + *a*



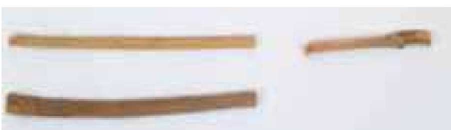
ACTIVITY 2

Objective: Addition and subtraction of expressions through line designs

Materials: Straws or sticks, Dot paper

Ask the first student to build a design to show a given expression, say, $4p + 3q$.

Ask the second student to show $2p + 1q$



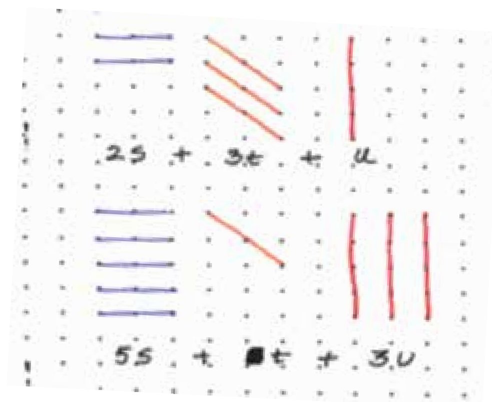
Verify that the second student has chosen the straws that correspond to the lengths p and q chosen by the first student. If not, it is an opportunity for discussing that p and q stand for specific lengths and line segments of the same length will be represented by the same letter. The students need to understand that different variables represent different numbers.

Now, how do we read these two designs together? They will be read as $6p + 4q$.

Ask students to record the design and expressions in a dot paper to show addition of expressions.

Give students a few more sets of expressions for which they can make corresponding drawings and sum them.

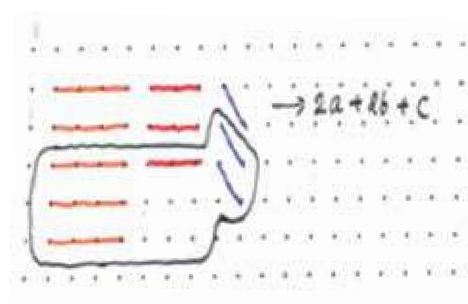
Example: $2s + 3t + u$, $5s + t + 3u$



In a similar manner one can demonstrate subtraction of expressions. Lay out the design for an expression, say, $5a + 3b + 3c$.

Circle the sticks to be removed. Say, $3a + b + 2c$.

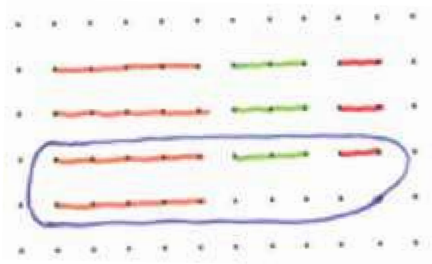
What is left?



$2a + 2b + c$.

This operation can be recorded on the dot paper as shown.

Give students some line segment designs as shown here to record the expressions and the subtraction of the given expressions.



They will give the expression for the initial design set up, then for the removed set (lines enclosed within loop are to be removed) and finally for the remaining set.

Note: Since these are concrete examples the teacher cannot, as yet, give examples of the following kind: Subtract $a + c$ from $2a + 3b$.

Give students a few more sets of expressions for which they can make corresponding drawings, subtract them and give the expression for what is left, like the following:

- Set up a line segment design to show subtraction of $4a + 2b + c$ from $7a + 5b + 3c$.
- Set up a line segment design to show subtraction of $5c + 8d + e$ from $10c + 8d + 3e + f$.

ACTIVITY 3

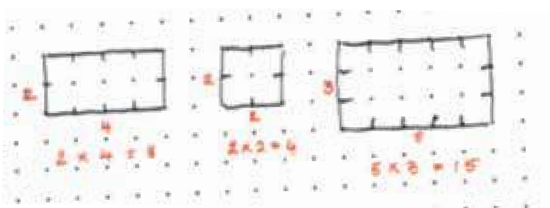
Objective: Design language for plane regions involving two variable terms.

Materials: Rectangles of different sizes (collection of visiting cards and greeting cards will help)

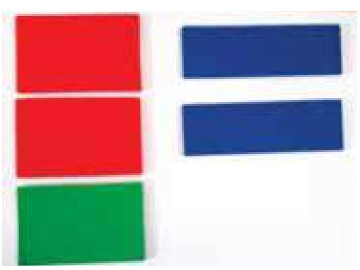
Prior knowledge: Familiarity with area formula for rectangles and squares.

Here, again, it is good to start using examples from real life situations before moving into abstract designs.

Revise the concept of area of rectangle and square using grid shapes as shown.



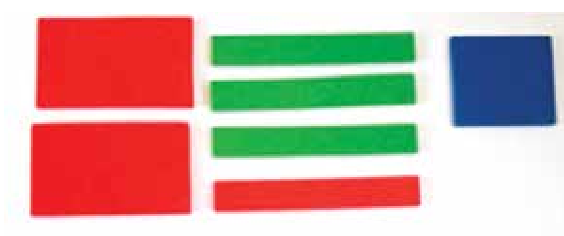
Set up a design with rectangles of two different sizes.



The length and breadth of the rectangles can be named using variables.

How do we describe this set up?
It would be $3ab + 2cd$.

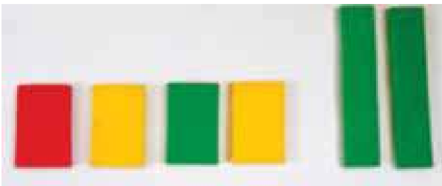
What will be the expression for this set up?



It would be $2ab + 4ef + gh$.

It is possible that different rectangles may have one common edge. Discuss the need for the usage of the same variable in such situations. This possibility is taken up in the next activity.

Give students some plane region designs as shown to draw in the dot paper and record the expressions.



Similarly give some expressions for which they need to make corresponding plane designs,

Example:

$3ab + 2pq + mn$,

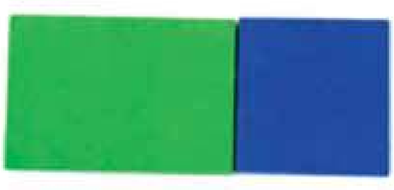
$2a^2 + 3b^2 + c^2$,

$ab + a^2 + b^2$.

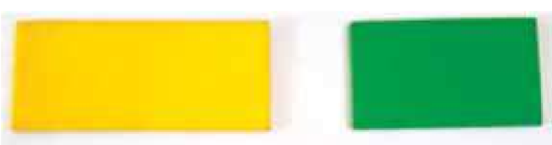
ACTIVITY 4

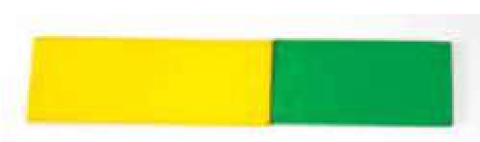
Objective: Design language for composite figures.

Materials: Rectangles and squares of different sizes which have one edge in common. Dot paper



When two rectangles or a rectangle and a square have one edge in common they can be combined to form a composite figure as shown.





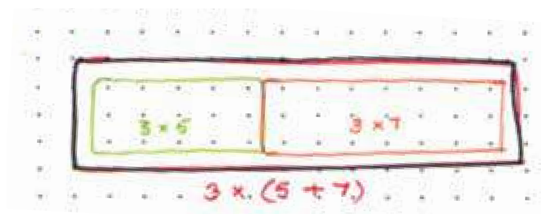
Note for the teacher: This will give rise to multiplication of a two term expression by a single term (binomial by a monomial).

Teacher can first place the shapes separately and state the design language as $ab + ac$.

Now the shapes can be brought together and stated as $a(b + c)$.

Through this, the law $ab + ac = a(b + c)$ (i.e., the distributive law) is established.

This can be further reinforced by assigning numerical values to the variable and demonstrated through dot arrays as shown.

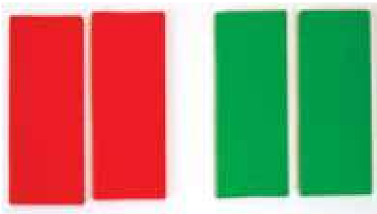


$$3 \times 5 + 3 \times 7 = 3 \times (5 + 7)$$

What will be the expression for this set up?

It would be $jk + jk + jk = j(k + k + k) = 3jk$.

What is the expression for this set up?



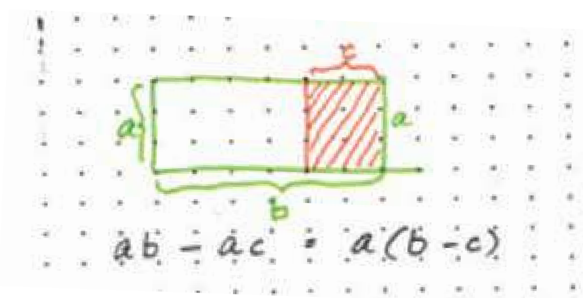
$2ab + 2ab = 4ab$.

Give students some plane region designs as shown above to draw in the dot paper and record the expressions.

Similarly give some expressions, as given here, for which they need to make corresponding plane designs and write the expression for the composite figure.

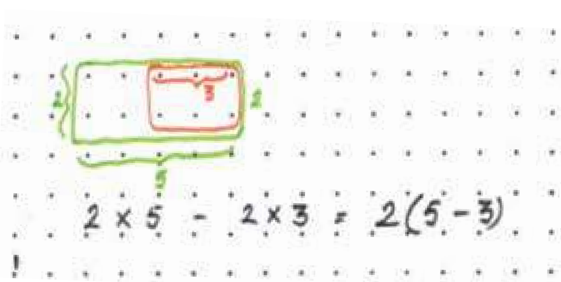
$$kl + l^2, \quad 3pq + 2pr + p^2$$

At this point teacher can also discuss the design for $ab - ac$. It would look like the following.

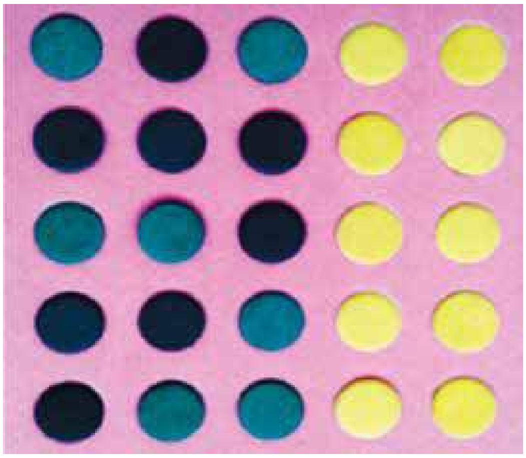


Again it can be established that $ab - ac = a(b - c)$.

This can be further reinforced by assigning numerical values to the variable and demonstrated through dot arrays as shown.



$2 \times 5 - 2 \times 3 = 2 \times (5 - 3)$



$5 \times 5 - 5 \times 2 = 5 \times (5 - 2)$

A few more such examples can be discussed.

$pq + pr + ps$

$ab + cb + db + fb$

Ask students to show using dot paper:

$a(b + c + d) = ab + ac + ad$.

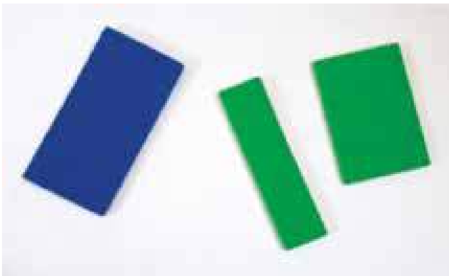
$p^2 + pq + pr = p(p + q + r)$.

ACTIVITY 5

Objective: Rules about addition, subtraction of like and unlike terms, Multiplication of expressions by a single variable

Materials: Rectangles and squares of different sizes. Dot paper

Vocabulary: Like rectangles, unlike rectangles, Like terms, unlike terms



Ask students to pick up any two rectangles. Pose the question 'are these rectangles alike?' or 'are these rectangles different?'

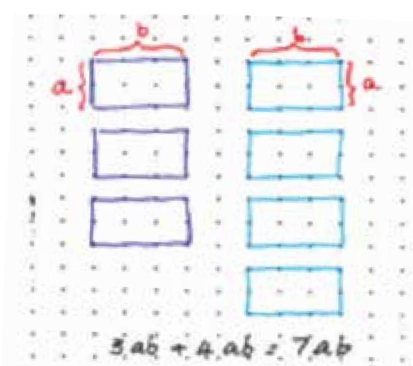
If the two rectangles have the same length and breadth they are like rectangles.

Students may pick up two rectangles which have the same length but a different breadth or which have the same breadth but different lengths or which have different lengths and different breadths.

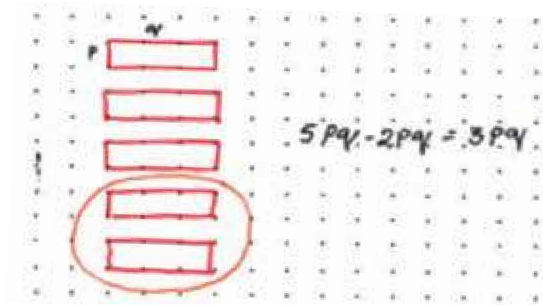
How do we describe them?

If two rectangles differ either in length or breadth or both they are unlike rectangles.

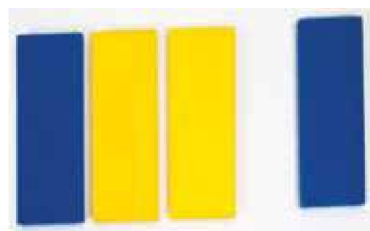
Show them that like rectangles are referred to using like terms and that like terms can be added only to like terms. Similarly like terms can be subtracted only from like terms.



Example: $3ab + 4ab = 7ab$

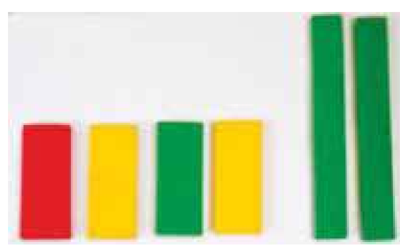


$5pq - 2pq = 3pq$



$4cd - cd = 3cd$

Also show them that unlike rectangles are referred to by using unlike terms and unlike terms cannot be added or subtracted.



$4xy + 2ab$

Let students draw figures in the dot paper to demonstrate the following.

$ab + ab + ab = 3ab$.
$3mn + 5rs - mn - 3rs = 2mn + 2rs$.

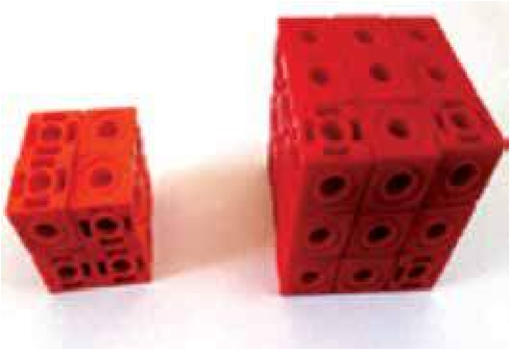
Students will now be in a position to make the transition to handle addition and subtraction of sets of abstract expressions of the following type:

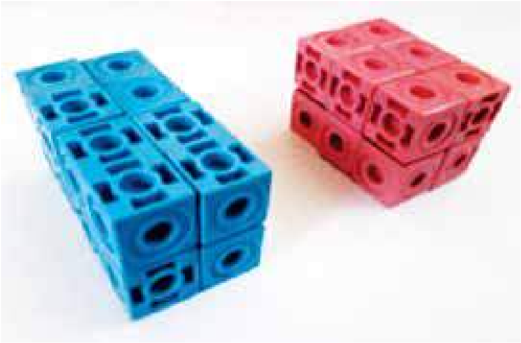
Add:	Subtract:	Multiply:
$2ab + 3cd + ef$ $ab + 2cd + ef$ $4a^2 + 6b^2 + c^2$ $2a^2 + b^2 + 3c^2$	$5ab + 4cd + fg$ $ab + 3cd + fg$	$a + 2b + c$ with d

ACTIVITY 6

Objective: Volume

Materials: unit cubes, triangular. Dot paper





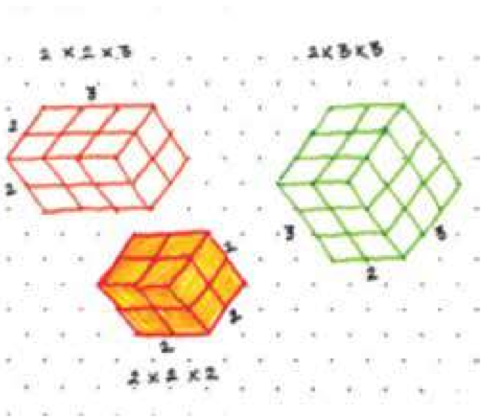
Initially let students build cubes and cuboids of different sizes.

Let them note down the dimensions in a table format (length, breadth, height, volume) to discover the formula for the volume of a cube or a cuboid.

They can fill the volume column by counting the number of cubes and noticing the relationship between the length, breadth and height.

Since the volume of 3D blocks is given in terms of length, breadth and height, the design language involves the use of the product of three variables.

Students can record these on triangular dot paper (also called isometric dot paper).



ACTIVITY 7

Objective: Design language for sets of blocks

Materials: Cuboids and cubes of different sizes and same size. Cardboard boxes of the same kind (soap boxes, toothpaste boxes)

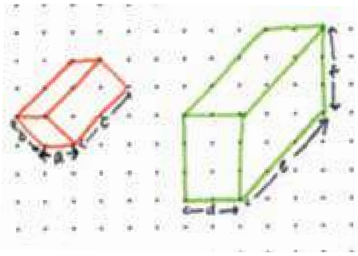
Prior knowledge: Volume of cube and cuboid
Vocabulary: Like terms, unlike terms

Show that cuboids having same length, breadth and height are referred to by like terms as shown.

Teacher can place different combinations of boxes and get the children to describe the set up using design language.

Example: $3abc + 2def$

Sides of unlike cuboids will be indicated by different letters (variables) and are referred to by unlike terms.

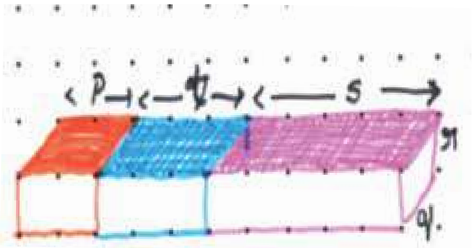


ACTIVITY 8

Objective: Design language for combined blocks. Multiplication and factorisation

Materials: Cuboids and cubes which have common faces can be brought together. Design language for them can be given initially considering the blocks separately and then as combined.

$abc + abc + abc = 3abc$



Show that $pqr + tqr + sqr = (p + t + s)\ qr$

At this point students should now be ready for addition and subtraction of abstract expressions of the regular kind. **For example:**

Add:	Subtract:	Multiply:	Factorise:
$abc + 2def$	$5pqr + 3uvw$	$p(p^2 + pq + pr + qr)$	$a^3 + a^2 b + a^2c$
$3abc + 4def$	$pqr + 2uvw$		
$2a^2b + b^2c + c^2a$	$7a^2b + a^2c + 3a^2d$		
$3a^2b + 4b^2c + c^2a$	$5a^2b + a^2c$		
$a^2b + 2b^2c + c^2a$			

CONCLUSION

By the time we complete the activities suggested in Part I and Part II in this series, students should feel comfortable in the use of algebraic concepts and words such as *variable*, *term* and *expression,* and in performing various operations using them.

They should be in a position to manipulate similar abstract expressions. Once the general principles of 'like' and 'unlike' terms have been understood and the rules of operations internalized, students should be ready for the take-off stage. They should be able to use the same laws and principles with higher powers and multi-variable terms and expressions.

Part III of this series will be on *approaches to equations.*

Padmapriya Shirali

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