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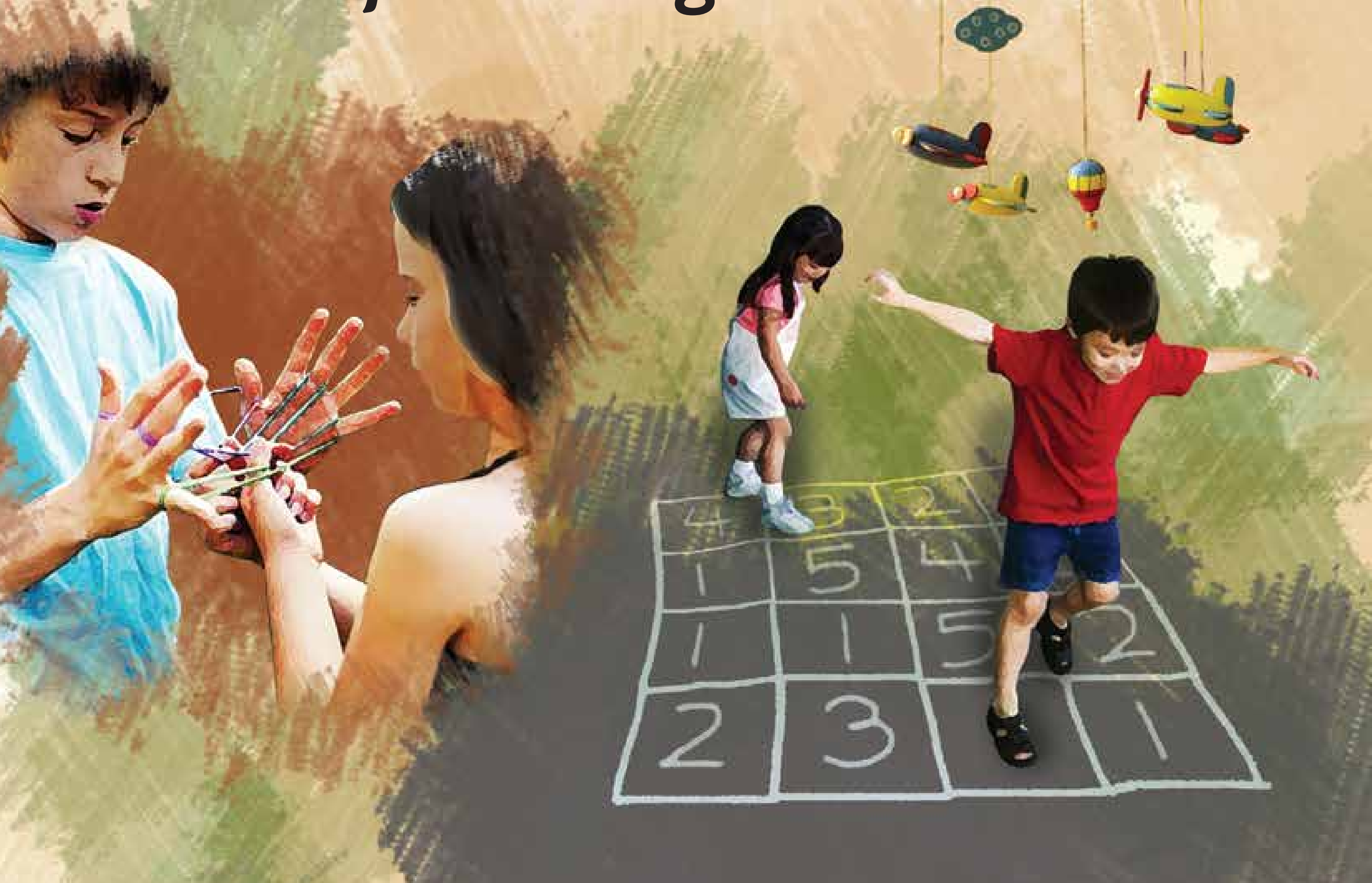
Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

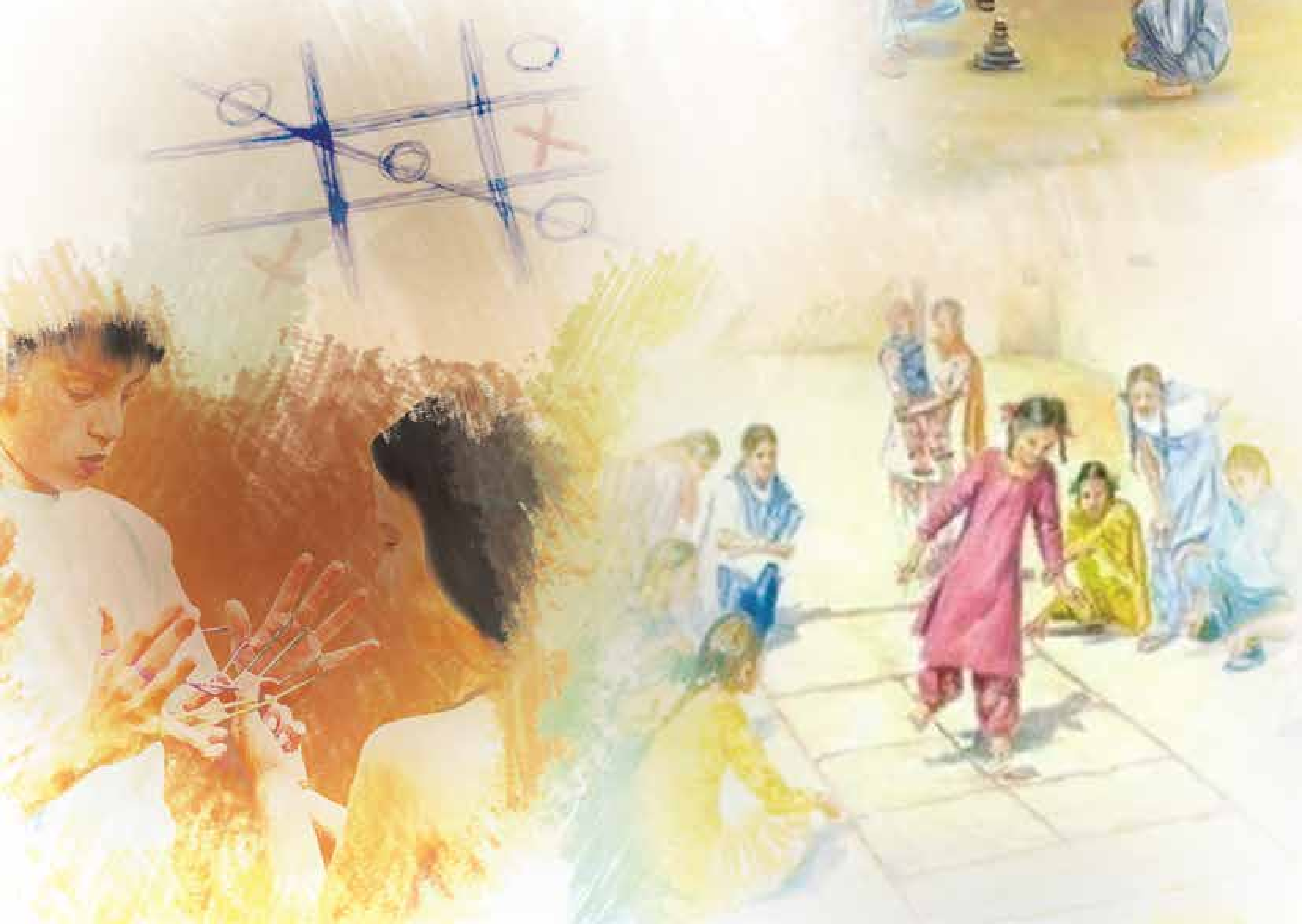
Let the Games Begin!!



Seeing Math in Games and Toys

PULLOUT
APPROACHES TO EQUATIONS

Lazy summer afternoons spent playing endless games of Sutli or hopscotch or any of those absorbing games have often become the topic of endless reminiscing about the good old days. Well, let's bring them back! With a math slant at that. Can the shapes made with knotted string lead to a better understanding of Euclidean geometry and 3 D space? Can the mobiles which hang over a baby's crib be used to understand how to solve equations? Can a teacher see how easy it is to reinvent a toy or a game in order to teach a mathematical concept or two? It's time to Deja View mathematics!



From the Editor's Desk . . .

And all of a sudden, we are at the last issue for 2018. It's tough to believe that we are now six years (and twenty issues) old. And what a journey it has been! We hope that you have enjoyed it as much as we did! Do remember that all past issues can be accessed at https://azimpremjiuniversity.edu.in/SitePages/resources-at-right-angles.aspx and individual articles as well as the whole magazine can be downloaded from here as well.

In Features, Shuborno Das wraps up his two-part article on Functional Equations and leaves you with a set of problems to practise what he preaches. Then we have the second part of Shailesh Shirali's article on Mathematical Constants, where we befriend that ubiquitous mathematical entity 'e'. Following which, Rahul Tikekar takes us on a whimsical 'what-if' journey through the various well-known incidents of the life of Ramanujan, that tragic hero of the Indian mathematics story.

In ClassRoom, young Satvik Kaushik describes an investigation on Armstrong Numbers. And his school mate Haren Sathvik shows us how mathematics is done – he pulls a theorem out of a discussion with his friend. Clearly, this young mathematician is not talking through his hat. A. Ramachandran delights us with Down To Earth Trigonometry – for all those seeking applications of mathematics, this article will be right up your street. The TearOut series with ready-to-use worksheets continues with Geometry explored using graph paper. You will love the innovative modification of Sutli- a traditional game played with string. As well as the insights into Interpretation of Errors in Arithmetic by Hriday Kant.

TechSpace is very with-it this time, we have an article on Mobile Puzzles, where the fine art of balancing equations is explored with software modelled on old-fashioned mobiles- the toys that hang over (and fascinate) a baby in the crib. The theme of Equations is viewed from the pedagogical angle in PullOut too, where Padmapriya Shirali weaves her usual magic and teaches us how to make the complex simple.

Problem Corner goes deep and wide, with problems ranging from Diophantine Equations to a narrative about the history of Mathematical Olympiads in India. Check out the Review for some wonderful readings on and about mathematics.

Remember our code given on the Contents Page. A discreet colour band at the top of each article indicates whether it is best suited for Primary (1-5), Middle School (6-8), High School (9-10) or Pre-University (11-12).

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

LEGENDS - Primary School | Middle School | High School | Pre-University

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Contents contd.

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Functional Equations

Part II

SHUBORNO DAS

In Part 1, I explained the different techniques to solve functional equations. In most cases, the steps to solve them are similar to those in solving algebraic equations. However there is one scenario where using the algebraic method to solve a functional equation may lead to an incomplete solution; i.e., only a subset of functions that solve the FE may be identified and not the complete list. This error is known as *pointwise trap*. In Part II of the two-part article on functional equations, we discuss the notion of pointwise trap and learn different ways to overcome it.

Pointwise Trap

What is a 'pointwise trap'? Consider the following example:

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x)^2 = xf(x)$.

Factorising the equation $f(x)^2 - xf(x) = 0$ gives $f(x)(f(x) - x) = 0$. Solving it like a normal equation would yield $f(x) = 0$ for all x **or** $f(x) = x$ for all x.

Is this the complete list of functions that solve the FE?

Consider a function $f(x)$ defined as follows: $f(x) = x$ for all x except $f(5) = 0$. If $x \neq 5$, then $f(x) - x = 0 \implies f(x)^2 = xf(x)$. If $x = 5$, then $f(x) = 0 \implies f(x)^2 = xf(x)$. So clearly this function also satisfies the FE. This proves that our earlier list of functions that satisfy the FE was incomplete.

It is not too difficult to deduce that there are infinitely many solutions to this FE. Refer Figure 1 as another example of such a function that satisfies the FE.

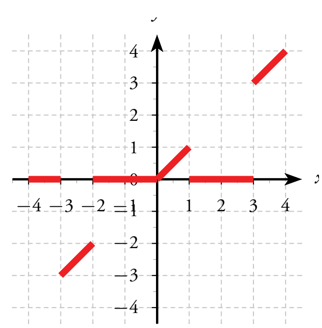

Figure 1. Example of a function solving the equation $f(x)^2 = xf(x)$

Keywords: Functional equation, function, domain, range, injective, surjective, even, odd, continuous, pointwise trap, solution

This mistake is known as a *pointwise trap*. In an equation, if $y(y-5)=0$, then $y=0$ or $y=5$ are the solutions and that is correct. However, if we blindly apply this approach to solve the FE, it leads to *pointwise trap*.

Properties of functions used to deal with the trap. In cases when an additional condition is imposed on the function, solving the problem becomes easier. We consider what happens when the following conditions are imposed separately: continuity, monotonicity, injectivity and surjectivity.

Continuity

Example 1. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the following condition: $f(x)^2 = xf(x)$ for all $x \in \mathbb{R}$.

Solution 1. After factorisation, we get $f(x)(f(x)-x)=0 \Rightarrow f(x)=0$ or $f(x)=x$ for all x. This means that for every real number x, either $f(x)=0$ or $f(x)=x$. In particular, we have $f(0)=0$. The trivial solution is $f(x)=0$ for all x.

In order to see if there is a non-trivial solution, assume that there is some real $r \neq 0$ such that $f(r)=r$. Suppose firstly that $r>0$. Then, continuity coupled with the condition $f(x)(f(x)-x)=0$ implies that $f(x)=x$ for all $x>0$. Next, suppose that $r<0$. Then, arguing exactly as earlier, we see that $f(x)=x$ for all $x<0$. This line of reasoning leads to four possible solutions:

- $f(x)=0$ for all x;
- $f(x)=x$ for all x;
- $f(x)=x$ for all $x \geq 0$ and $f(x)=0$ for all $x<0$;
- $f(x)=x$ for all $x \leq 0$ and $f(x)=0$ for all $x>0$.

So the continuity condition has brought down the number of solutions to just four (refer Figure 2).

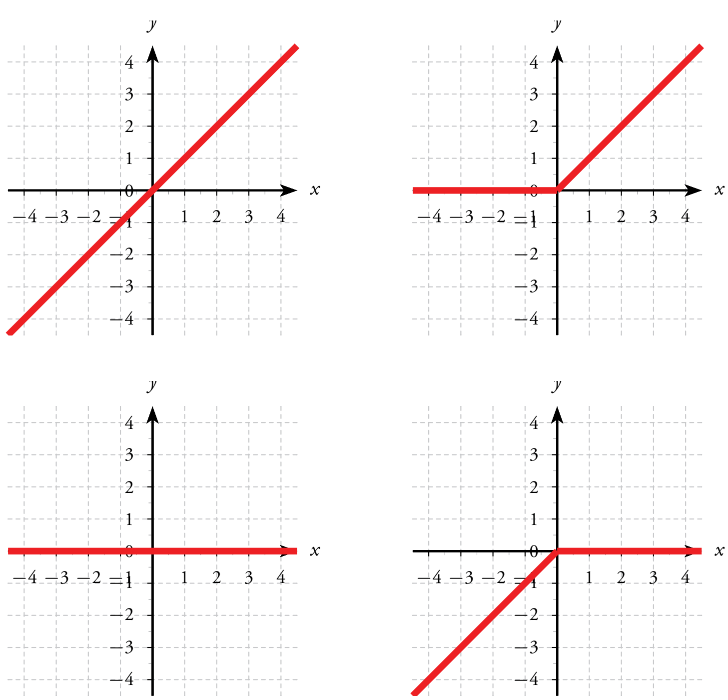

Figure 2. Solutions to example 1

Remark. A casual look at the equation may suggest that a possible solution is $f(x) = |x|$. Note that it satisfies the continuity requirement. However, it is easy to check that it does not satisfy the given condition for $x < 0$. (The quantity on the left side simplifies to x^2, whereas the quantity on the right side simplifies to $-x^2$.)

Monotonicity

Example 2. Find all strictly increasing functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the condition $f(x)^2 = xf(x)$ for all $x \in \mathbb{R}$.

Solution 2. After factorisation, we get $f(x)(f(x) - x) = 0 \Rightarrow f(x) = 0$ or $f(x) = x$ for all x. This means that for every real number x, either $f(x) = 0$ or $f(x) = x$. In particular, we have $f(0) = 0$. From the given condition, we have $f(x) > f(y)$ for all real $x > y$. Plugging $y = 0$, we have $f(x) > 0$ for $x > 0$ and plugging $x = 0$ gives $f(y) < 0$ for $y < 0$. Hence we have:

$$f(x) = x \text{ for all } x \in \mathbb{R}.$$

So the strict monotonicity condition has brought down the number of solutions to just one (refer Figure 3).

Injectivity

Example 3. Find all injective functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the condition $f(x)^2 = xf(x)$ for all $x \in \mathbb{R}$.

Solution 3. After factorisation, we get $f(x)(f(x) - x) = 0 \Rightarrow f(x) = 0$ or $f(x) = x$ for all x. This means that for every real number x, either $f(x) = 0$ or $f(x) = x$. In particular, we have $f(0) = 0$. Injectivity implies that there cannot be $a \neq 0$ such that $f(a) = 0$. Hence we have:

$$f(x) = x \text{ for all } x \in \mathbb{R}.$$

So the injectivity condition has brought down the number of solutions to just one (refer Figure 3).

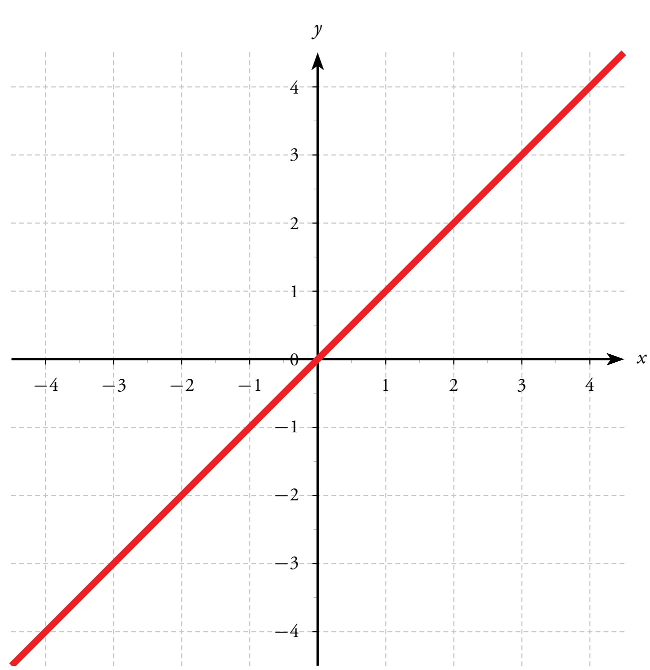

Figure 3. Solution to examples 2, 3, 4

Surjectivity

Example 4. Find all surjective functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the condition $f(x)^2 = xf(x)$ for all $x \in \mathbb{R}$.

Solution 4. After factorisation, we get $f(x)(f(x) - x) = 0 \Rightarrow f(x) = 0$ or $f(x) = x$ for all x. This means that for every real number x, either $f(x) = 0$ or $f(x) = x$. In particular, we have $f(0) = 0$.

Next, from surjectivity we realize that for every real b there exists a real a such that $f(a) = b$. Suppose $b \neq 0$; then, using the condition that $f(x) = 0$ or $f(x) = x$ for each x, we deduce that $a = b$ and hence that $f(x) = x$ is the only solution. Hence we have:

$$f(x) = x \text{ for all } x \in \mathbb{R}.$$

So the surjectivity condition has brought down the number of solutions to just one (refer Figure 3).

Use of contradiction in solving the pointwise trap. Another powerful technique to resolve pointwise trap is contradiction, particularly in multi-variable functional equations. Suppose in a multivariate FE, we end up getting $f(x) = g(x)$ or $f(x) = h(x)$ for each x (in the above examples, $g(x) = 0$ and $h(x) = x$). In such a case, we can use contradiction to prove that $f(x) = g(x)$ for all x or $f(x) = h(x)$ for all x are the only solutions, in the following manner. We assume for the sake of contradiction that there are distinct elements a, b in the domain such that $f(a) = g(a)$ and $f(b) = h(b)$. After that, we cleverly plug a, b either in the original equation or its variant to prove that $a = b$. This contradicts our assumption, implying that the only solutions are $f(x) = g(x)$ for all x and $f(x) = h(x)$ for all x.

Example Problems

Example 5. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2) + f(xy) = f(x)f(y) + yf(x) + xf(x+y)$$

for all $x, y \in \mathbb{R}$.

Solution 5. Let $P(x, y)$ be the assertion of the problem statement.

$P(0, 0)$ gives $2f(0) = f(0)^2 \Rightarrow f(0) = 0$ or $f(0) = 2$.

Case 1: $f(0) = 2$. $P(0, y)$ gives $2 + 2 = 2f(y) + 2y \Rightarrow f(y) = 2 - y$. On checking, we find that $f(y) = 2 - y$ does indeed solve the problem.

Case 2: $f(0) = 0$. $P(x, 0)$ gives $f(x^2) = xf(x)$. $P(-x, 0)$ gives $f(x^2) = -xf(-x)$. Thus

$$xf(x) = f(x^2) = -xf(-x) \Rightarrow x(f(x) + f(-x)) = 0.$$

For $x \neq 0$, $f(x) = -f(-x)$, therefore f is odd.

Now that we know that f is odd, it might be worth putting $y = -x$. We then get:

$$0 = f(x^2) + f(-x^2) = f(x)f(-x) - xf(x) = -f(x)^2 - xf(x)$$

$$\therefore\ 0 = f(x)^2 + xf(x) = f(x)(f(x) + x).$$

Hence $f(x) = 0$ or $f(x) = -x$ for each value of x.

This is the pointwise trap. We must avoid the trap using contradiction.

Clearly $f(x) = 0$ for all x is a trivial solution. Let $f(a) = 0$ and $f(b) = -b$ where $a, b \neq 0$. We plug these values in the original equation and try to work towards a contradiction.

We notice that there are a lot of $f(x)$ terms which may need to be eliminated to simplify the problem. We plug in $x = a$. Note that $f(a^2) = af(a) = 0$. Hence we have:

$$P(a, y): \quad f(ay) = af(a+y).$$

There are two f-terms here: $f(ay)$ and $f(a+y)$. We use the fact that $f(b) = -b$ and select a value of y such that the latter term becomes $f(b)$:

$$P(a, b-a): \quad f(a(b-a)) = af(a+(b-a)) = af(b) = -ab,$$

so

$$f\left(ab - a^2\right) = -ab.$$

But we know that $f(ab - a^2) = 0$ or $a^2 - ab$. This implies that either $0 = -ab$ or

$$a^2 - ab = -ab, \quad \therefore \ ab = 0.$$

So either $ab = 0$ or $a = 0$. But we had said at the start that both $a, b \neq 0$, so we have a contradiction. Hence it cannot be that $f(a) = 0$ and $f(b) = -b$ where $a, b \neq 0$. Therefore, the possible solutions are:

- $f(x) = 0$ for all x;
- $f(x) = -x$ for all x;
- $f(x) = 2 - x$ for all x.

It is easy to check by substitution that these are indeed solutions to the given problem.

Example 6. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f(n^2) = f(n+m)f(n-m) + m^2$$

for all $m, n \in \mathbb{Z}$.

Solution 6. Let $P(m, n)$ be the assertion of the problem statement. The problem looks like the identity $a^2 - b^2 = (a-b)(a+b)$. The solution to the problem seems to be $f(n) = n$ for all n. We proceed in a similar way.

$P(0, 0)$ gives $f(0) = f(0)^2 \Rightarrow f(0) = 0$ or 1.

Case 1: $f(0) = 1$. We need to eliminate the product term in the RHS. $P(n, n)$ gives $f(n^2) = f(2n) + n^2$. Putting $n = 2, f(4) = f(4) + 4$ which is not possible and we have no solution in this case. So we eliminate this possibility.

Case 2: $f(0) = 0$. $P(n, n)$ gives $f(n^2) = n^2$, hence

$$(n-m)(n+m) = n^2 - m^2 = f(n+m)f(n-m).$$

We cannot conclude from this that $pq = f(p)f(q)$ for all p, q as we are dealing with integers and $n + m$ and $n - m$ are not independent. Let $a = n - m$ and $b = n + m$. This means that a, b are of the same parity, hence $f(a)f(b) = ab$ for $a \equiv b \pmod 2$. We'll try using the original condition a little more. $P(n, 0)$ gives $f(n^2) = f(n)^2$. We also have $f(n^2) = n^2$. Hence $f(n)^2 = n^2$, so:

$$f(n) = \pm n.$$

This is the pointwise trap in this problem. For the sake of contradiction, assume that there exists an integer $k \neq 0$ (why can we assume this?) such that $f(k) = -k$. We know that $f(a)f(b) = ab$ for $a \equiv b \pmod 2$. Suppose m is any integer with the same parity as k. We have
$km = f(k)f(m) = -kf(m) \implies f(m) = -m$ for all m with the same parity as k, but we had $f(n^2) = n^2$.

Consider any perfect square $r^2 \neq 0$ with the same parity as k. For example, if k is odd, then we can choose $r^2 = 9$; if k is even, then we can choose $r^2 = 16$. Therefore

$$r^2 = f(r^2) = -r^2,$$

hence $r = 0$, which gives us a contradiction.

This means that there doesn't exist any k such that $f(k) = -k$.

Hence the answer is: $f(n) = n$ for all n which is indeed a solution to the FE.

Practice Problems

(1) (Japan MO Final 2004) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(xf(x) + f(y)) = f(x)^2 + y.$$

(2) (Iran 1999) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y.$$

(3) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$xf(y) + yf(x) = 2f(x)f(y).$$

(4) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x)f(x-y) + f(y)f(x+y) = x^2 + f(y)^2.$$

Note. Solutions to these problems will be given in the next issue (March 2019). Solutions to the practice problems posed in Part 1 of the article are given elsewhere in this issue.

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Features

Student Corner – Featuring articles written by students.

Solutions to Practice Problems in Functional Equations – Part I

SHUBORNO DAS

Problem 1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 - y^2) = (x - y)(f(x) + f(y))$$
for all $x, y \in \mathbb{R}$.

Solution 1. Let $P(x, y)$ be the stated property. Note that it implies that f is odd (this may be seen by interchanging x and y).

$$\begin{aligned}
P(0,0): &\quad f(0) = 0,\\
P(x,0): &\quad f(x^2) = xf(x),\\
P(0,x): &\quad f(-x^2) = -xf(x),\\
P(x,-y): &\quad f(x^2 - y^2) = (x+y)(f(x) - f(y)).
\end{aligned}$$

Comparing the last line with $P(x, y)$ gives

$$(x-y)(f(x)+f(y)) = (x+y)(f(x)-f(y)) \implies yf(x) = xf(y).$$

Hence for $x, y \neq 0$,

$$\frac{f(x)}{x} = \frac{f(y)}{y}.$$

Hence $\frac{f(x)}{x} = c$ for some constant $c \in \mathbb{R}$ and for all $x \neq 0$. Since $f(0) = 0$, $f(x) = cx$ for all x satisfies the original problem statement. Hence $f(x) = cx$ for all $x \in \mathbb{R}$.

Keywords: Functional equation, function, domain, range, injective function, surjective function

Problem 2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(4x) - f(3x) = 2x$$

for all $x \in \mathbb{R}$.

Solution 2. The problem may be restated as

$$f(x) - f\left(\frac{3x}{4}\right) = \frac{x}{2}, \quad \text{i.e.,}\ f(x) = f\left(\frac{3x}{4}\right) + \frac{x}{2}.$$

Iterating this relation, we get

$$\begin{aligned} f(x) &= f\left(\frac{3x}{4}\right) + \frac{x}{2} \\ &= f\left(\frac{9x}{16}\right) + \frac{x}{2} + \frac{3x}{8} \\ &= f\left(\frac{27x}{64}\right) + \frac{x}{2} + \frac{3x}{8} + \frac{9x}{32} + \dots. \end{aligned}$$

Let $a = \lim_{x \to 0^+} f(x)$. Thus we have for $x > 0$,

$$\begin{aligned} f(x) &= a + \frac{x}{2} + \frac{3x}{8} + \dots \\ &= a + \frac{x}{2}\left(1 + \frac{3}{4} + \frac{9}{16} + \dots\right) = a + 2x. \end{aligned}$$

Let $b = \lim_{x \to 0^-} f(x)$. By reasoning in the same way, we get $f(x) = b + 2x$ for $x < 0$.

Hence $f(x) = 2x + a$ for all $x > 0$, $f(x) = 2x + b$ for all $x < 0$ and $f(0) = c$ for some constants a, b, c. (Note that if f is to be continuous, then we must have $a = b = c$.)

Problem 3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y)$$

for all $x, y, z \in \mathbb{R}$.

Solution 3. Let $P(x, y, z)$ be the stated property.

$$\begin{aligned} P(0,0,0): &\quad f(0) = 0; \\ P(x,0,0): &\quad f(x^2) = xf(x); \\ P(0,y,z): &\quad f(yf(z)) = zf(y), \quad \therefore\ f(f(z)) = zf(1). \end{aligned}$$

Suppose $f(1) = 0$.

$$P(x,y,1): \quad xf(x) = f(x^2) = xf(x) + f(y), \quad \therefore\ f(y) = 0\ \text{ for all } y.$$

Next, suppose $f(1) \neq 0$. As $f(f(z)) = zf(1)$, f is injective. (For, if $f(a) = f(b)$, then $f(f(a)) = f(f(b))$, hence $af(1) = bf(1)$, hence $a = b$.) We also had $f(yf(z)) = zf(y)$. Plugging $y = z$, we get

$$f(yf(y)) = yf(y) = f(y^2), \quad \therefore\ yf(y) = y^2,$$

hence $f(y) = y$ for all $y \neq 0$. As $f(0) = 0$, $f(y) = y$ for all y.

Hence the solutions are $f(x) = x$ for all $x \in \mathbb{R}$ and $f(x) = 0$ for all $x \in \mathbb{R}$.

Problem 4. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\left(\frac{f(x)}{y}\right) = yf(y) \cdot f(f(x))$$

for all $x, y \in \mathbb{R}^+$.

Solution 4. Let $P(x, y)$ be the stated property. Since 0 is not an element of the domain, we first try to find $f(1)$.

$$P(1,1): \quad f(f(1)) = f(1) \cdot f(f(1)), \quad \therefore\ f(1) = 1.$$

Now LHS has a term $f\left(\frac{f(x)}{y}\right)$, we substitute in such a way that LHS becomes 1:

$$P(x, f(x)): \quad 1 = f(1) = f(x) \cdot f(f(x,))^2, \quad \therefore\ f(f(x)) = \frac{1}{\sqrt{f(x)}},$$

$$P(x, f(y)): \quad f\left(\frac{f(x)}{f(y)}\right) = f(y) \cdot f(f(y)) \cdot f(f(x)) = \sqrt{\frac{f(y)}{f(x)}}.$$

We cannot conclude from this that $f(x) = \frac{1}{\sqrt{x}}$ for all x; we first need to show that $\frac{f(x)}{f(y)}$ is surjective. We had

$$\frac{f\left(\frac{f(x)}{y}\right)}{f(y)} = y \cdot f(f(x)).$$

Clearly RHS is surjective (with an isolated y), hence so is LHS. Hence we can substitute $\frac{f(x)}{f(y)}$ as some positive real a. Therefore we have $f(a) = \sqrt{\frac{1}{a}}$.

Hence $f(x) = \frac{1}{\sqrt{x}}$ for all $x \in \mathbb{R}^+$.

Problem 5. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(x + y^2) \cdot f(y \cdot f(x)) = x \cdot y \cdot f(y^2 + f(x))$$

for all $x, y \in \mathbb{R}$.

Solution 5. Let $P(x, y)$ be the stated property. Clearly $f(x) = 0$ is a solution. We wish to explore whether there is a nontrivial solution, so let us assume that there exists some real x such that $f(x) \neq 0$. We now try to find $f(0)$:

$$P(x, 0): \quad x \cdot f(0) = 0 \text{ for all } x, \quad \therefore\ f(0) = 0.$$

Notice that there are no x terms inside a f-term except for $f(x)$. This might help us showing that f is injective. Let $f(a) = f(b)$. Let $y \neq 0$ and $a + y^2 \neq 0$, $b + y^2 \neq 0$. $P(a, y)$ and $P(b, y)$ give:

$$\frac{ayf(y^2 + f(a))}{a + y^2} = f(yf(a)) = f(yf(b)) = \frac{byf(y^2 + f(b))}{b + y^2}.$$

This yields

$$\frac{a}{a + y^2} = \frac{b}{b + y^2}.$$

Therefore $ab + by^2 = ab + ay^2 \Rightarrow a = b \implies f$ is injective.

The left hand side contains a term $(x + y^2)$, let us utilise this term effectively.

$$P(-y^2, y): \quad 0 = -y^3 f(y^2 + f(-y^2)).$$

Choose $y \neq 0$, we get $0 = y^2 + f(-y^2) \implies f(-y^2) = -y^2$. Now y is any real number, so $-y^2$ can assume any negative real value. Therefore $f(x) = x$ for all $x < 0$. Let $x, y > 0$.

$$P(-x, y): \quad (y^2 - x)f(-xy) = -xyf(y^2 - x),$$

$$\therefore (y^2 - x)(-xy) = -xyf(y^2 - x),$$

$$\therefore y^2 - x = f(y^2 - x) \text{ for all } x, y.$$

Therefore $f(x) = x$ for all $x \in \mathbb{R}$.

Hence $f(x) = 0$ for all $x \in \mathbb{R}$ and $f(x) = x$ for all $x \in \mathbb{R}$.

Problem 6. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$f(3x) - f(x) \leq 8x^2 + 2x, \quad f(2x) - f(x) \geq 3x^2 + x.$$

Solution 6. This is an unique problem since the problem is a functional inequality, which we have not encountered before. How do we proceed now? One of the approaches to solve a FE is to guess the solution. In this case $f(x) = x^2 + x$ is one of the solutions. (Note that for this function, equality holds in both the relations: $f(3x) - f(x) = 8x^2 + 2x$ and $f(2x) - f(x) = 3x^2 + x$.)

Therefore it makes sense to do the following substitution:

$$g(x) = f(x) - (x^2 + x).$$

Now how do the inequalities change?

$$g(3x) - g(x) = f(3x) - 9x^2 - 3x - f(x) + x^2 + x = (f(3x) - f(x)) - 8x^2 - 2x \leq 0$$

and

$$g(2x) - g(x) = f(2x) - 4x^2 - 2x - f(x) + x^2 + x = (f(2x) - f(x)) - 3x^2 - x \geq 0.$$

Hence we have:

$$g(3x) \leq g(x), \quad g(2x) \geq g(x).$$

We also have an additional condition of continuity. Hence:

$$g(x) \leq g\left(\frac{x}{3}\right) \leq g\left(\frac{x}{3 \cdot 3}\right) \leq \cdots \leq g\left(\frac{x}{3 \cdot 3 \cdot \cdots}\right) \leq g(0),$$

and

$$g(x) \geq g\left(\frac{x}{2}\right) \geq g\left(\frac{x}{2 \cdot 2}\right) \geq \cdots \geq g\left(\frac{x}{2 \cdot 2 \cdot \cdots}\right) \geq g(0).$$

Hence $g(x) = g(0)$. Let $g(0) = c$; then we get $g(x) = c$, hence $f(x) = x^2 + x + c$ which certainly does solve the functional inequality.

Hence $f(x) = x^2 + x + c$ for all $x \in \mathbb{R}$ and for some constant c.

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The Constants of Mathematics – Part II

The remarkable number *e*

SHAILESH SHIRALI

In this article, which is the second of our series on mathematical constants, we feature one of the most remarkable numbers in all of mathematics—the number e, a number that is bound to occupy centre place in any account of mathematics (along with its close cousin, the number π). It turns out that there is so much to say about e that we will need to devote two articles to this constant alone!

Probably the question that occurs to a reader who is meeting e for the first time is, why use the letter e for this number? Historically, it was the mathematician Leonhard Euler who first drew attention to the number and he gave it the symbol e. He probably chose this letter for its association with exponential functions and he may have used e to denote "***exponential number***." Today, we may like to think of it as "***Euler's number***." However, Euler was not the first mathematician to bump into the number; others had done so before him, but he was the first to realise its significance in mathematics. The web page [2] describes this beautifully:

> *The number e first comes into mathematics in a very minor way. This was in 1618 when, in an appendix to Napier's work on logarithms, a table appeared giving the natural logarithms of various numbers. However, that these were logarithms to base e was not recognised since the base to which logarithms are computed did not arise in the way that logarithms were thought about at this time. Although we now think of logarithms as the exponents to which one must raise the base to get the required number, this is a modern way of thinking. We will come back to this point later in this essay. This table in the appendix, although carrying no author's name, was almost certainly written by Oughtred. A few years later, in 1624, again e almost made it into the mathematical literature, but not quite. In that year Briggs gave a numerical approximation to the base* 10 *logarithm of e but did not mention e itself in his work.*

Keywords: Constant, variable, irrational, factorial, derangement

As can be seen from this quote, mathematicians had been holding e in their hands for many decades without realising it! The full story of the discovery (or should we say, the invention) of logarithms by John Napier makes for a fascinating study, but we shall not go into this for now. That requires a separate article all to itself!

After a series of hits and misses, it was in 1683 that the number e was first discovered. In that year, Jakob Bernoulli (one of the early members of the remarkable Bernoulli clan), while studying compound interest, considered the possibility of 'continuous compounding' and naturally encountered the limit of $\left(1+\frac{1}{n}\right)^n$ as n tends to infinity. He was able to show, using the binomial theorem, that the limit lies between 2 and 3. But it seems that he did not pursue this line of thinking beyond this point. So though we can say that e had finally been discovered, this is more a matter of historical hindsight; Bernoulli himself did not realise the significance of what he had discovered. In particular, he did not assign any name to the constant that he had discovered.

The big year when e finally made its appearance under the name we use today is 1731, in a letter that the mathematician Leonhard Euler wrote to Christian Goldbach. He mentioned the same limit that Bernoulli had given, showed that e is a sum of an infinite series (one with which we are very familiar these days), and calculated e to 18 decimal places. But he went well beyond this point; in this early work, he defined and explored the exponential function (defined not just over the real numbers but over the complex numbers as well) and brought out the connection between the exponential function and the sine and the cosine functions.

So though there can be dispute about the precise year of birth of e, there is no question that the year in which e becomes a true citizen of mathematics is 1731!

In the next few sections, we present a series of highlights of this remarkable number, showcasing its occurrence in numerous areas of mathematics.

Continuous compounding, à la Bernoulli.

The following formula is well-known to us: if a unit sum of money earns interest at the rate of r% per year, compounded n times a year at equal intervals (here n is a positive integer), then the amount A to which it grows at the end of one year is given by

$$A=\left(1+\frac{r/100}{n}\right)^n.$$

For the sake of exploring the underlying mathematics, we take $r=100$. The function to be explored then becomes:

$$A(n)=\left(1+\frac{1}{n}\right)^n. \tag{1}$$

We have now written $A(n)$ rather than just A, as the quantity depends on n. This is the function that Jakob Bernoulli had investigated. He noted that $A(n)$ increases with n, but the rate of increase comes down sharply with increasing n and $A(n)$ appears to reach a limiting value. Figure 1 illustrates this.

The graph of $A(n)$ indicates that as n increases without bound, the quantity $\left(1+\frac{1}{n}\right)^n$ tends to a limit which lies between 2.5 and 3. Convergence does not take place as rapidly as one may expect, as the following table illustrates:

n	10	25	50	100	1000	10000
$A(n)$	2.5937	2.6658	2.6915	2.7048	2.7168	2.7181

Even for $n=10000$, we have achieved an accuracy of only three decimal places.

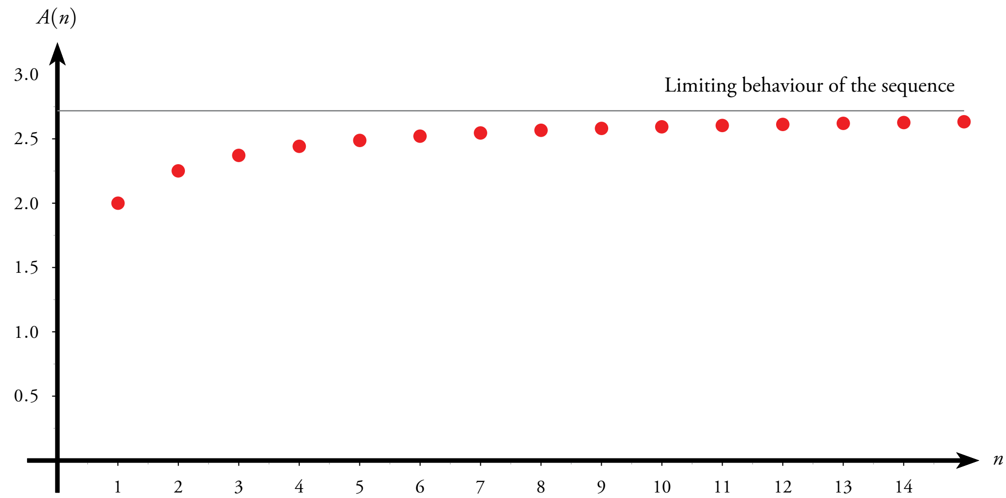


Figure 1.

To prove rigorously that $\left(1+\frac{1}{n}\right)^n$ tends to a limit takes a few steps. Generally, the proof takes the following form:

Establish that the sequence $\{A(n)\}_{n\geq 1}$ is *monotonic increasing*, i.e.,

$$A(1) < A(2) < A(3) < A(4) < \cdots.$$

In other words, we need to prove the following:

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} \qquad \text{for all } n \in \mathbb{N}. \tag{2}$$

Proving this inequality presents a nice challenge. Here is one way of proving it. For any positive integer n, we consider the following $n+1$ positive quantities:

$$\underbrace{1+\frac{1}{n},\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ \ldots,\ 1+\frac{1}{n},}_{n \text{ of these quantities}}\ 1.$$

Their geometric mean is

$$\left(1+\frac{1}{n}\right)^{n/(n+1)}.$$

The sum of the quantities is equal to

$$n\left(1+\frac{1}{n}\right)+1 = n+2,$$

hence their arithmetic mean is equal to

$$\frac{n+2}{n+1} = 1+\frac{1}{n+1}.$$

Invoking the result that the geometric mean of a list of positive numbers which are not all equal to one another is always strictly less than the arithmetic mean of the same list of numbers (this is the celebrated "AM-GM inequality"), we get

$$\left(1+\frac{1}{n}\right)^{n/(n+1)} < 1+\frac{1}{n+1},$$

i.e.,

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}.$$

Establish that the sequence $\{A(n)\}_{n\geq 1}$ is *bounded above*. In this particular case, we only need to establish that $A(n) < 3$ for all $n \in \mathbb{N}$. One way of proving this is to use the binomial theorem. First we verify computationally that $A(3) < 3$. Next we have, for all $n \geq 4$,

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \cdots + \frac{1}{n^n} \\
&< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\
&< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \qquad (3) \\
&< 3. \qquad (4)
\end{aligned}$$

A line of explanation is needed for the last two results. Inequality (3) follows from the claim

$$n! > 2^n \quad \text{for all } n \in \mathbb{N},\ n \geq 4,$$

which is best proved using induction (starting with $4! > 2^4$ as the base or 'anchor' of the induction), but we leave the details to the reader.

In the case of inequality (4), we simply use the well-known fact that

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 2.$$

With these two results established, it remains only to invoke a standard result from analysis ("a monotonically increasing sequence which is bounded above possesses a limit") which ensures that the sequence under study possesses a limit. We call the limit e. So the following may be considered to be a *definition* of e:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n. \qquad (5)$$

The following is now a straightforward consequence of (5):

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}. \qquad (6)$$

It is interesting to check the rate of convergence of the above infinite series for e. Let $B(n)$ denote the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$; then we have the following data:

n	10	25	50	100
$B(n)$	1.718281801	1.718281828	1.718281828	1.718281828

We see that the rate of convergence is quite rapid: $B(25)$, $B(50)$ and $B(100)$ agree to all the decimal places shown. Here is the value of e to 50 decimal places:

$$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ 77572\ 47093\ 69995\ 9\ldots.$$

Rational approximations for e. The double occurrence of the string '1828' in the above decimal expansion can prove misleading; a calculation to 9 decimal places might suggest that the value of e is the recurring decimal 2.7 1828 1828 1828 . . .. This is, of course, not so. However, this line of thinking yields the following rational approximation for e:

$$\frac{271801}{99990}.$$

The difference between this and e is roughly 2.8×10^{-10}. An approximation using smaller numbers is $\frac{27180}{9999}$; this differs from e by roughly 10^{-5}. But this is by no means the best possible rational approximation for e using relatively smaller numbers. A very much better approximation is

$$\frac{23225}{8544},$$

which differs from e by roughly 6.7×10^{-9}.

You may wonder how we hit upon the approximation $\frac{23225}{8544}$. The answer is that it comes from the simple continued fraction for e. However, we leave that discussion to the second part of this article.

Another limit for e. Only a couple of decades back, it was noticed that e can be expressed as a limit in another way which converges more rapidly than the expression used earlier. Namely, we have:

$$e = \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}} \right). \tag{7}$$

Writing $g(n)$ for the expression on the right side of (7), we have the following data:

n	10	50	100	500
$g(n)$	2.7194	2.718327	2.718293	2.7182823

It is not at all difficult to see intuitively why (7) is true. We urge you to find an intuitive justification for yourself. However, to devise an analytically rigorous proof takes more effort. We shall not go into the details in this article.

When the slope is 1

One of the many remarkable aspects of the number e is that it can be defined in several different ways, and these different definitions turn out to be all equivalent to each other. Here is one such way, in which we use the notion of slope of a curve.

We consider curves of the form $y = a^x$, where $a > 1$ is any real number. Figure 2 shows a few such curves. Since $a^0 = 1$ for all such a, all these curves pass through the point $P(0, 1)$. It is clear that the slope of the curve at P depends on a; let this slope be denoted by $h(a)$. The larger the value of a, the greater will be the slope. For $a = 1$, this slope is 0 (trivially so), and the slope can be made as large as we may please by taking a to be large enough.

It seems reasonable to suppose that there is a critical value of a such that $h(a) = 1$; i.e., such that the slope of the curve $y = a^x$ at the point $P(0, 1)$ is equal to 1. (This supposition can be justified rigorously, using standard methods of analysis, but we shall not go into the details here.) ***We define this critical number to be*** e.

It is relatively easy to show that this definition leads to the same limit definition that we had adopted earlier. To show how, let n be any large positive integer, and consider the point Q on the curve $y = e^x$

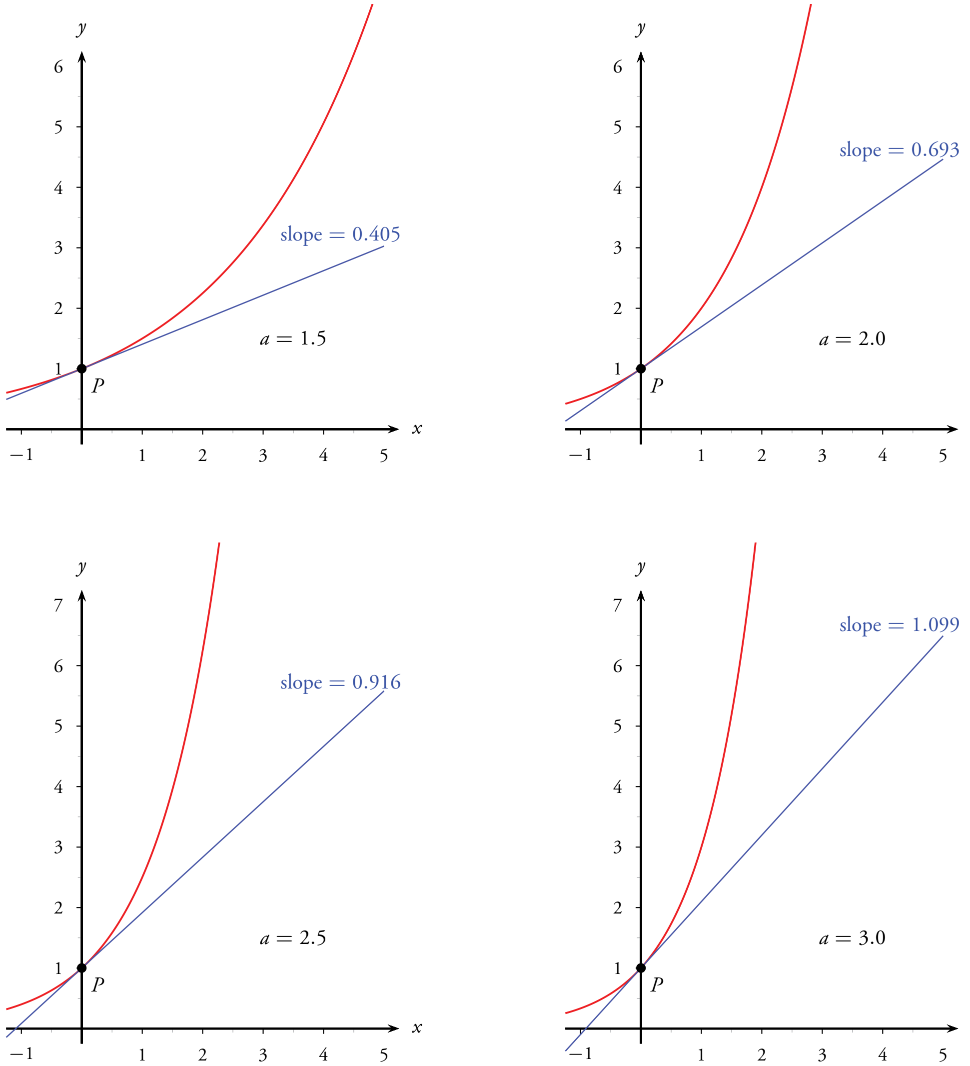


Figure 2. The curve $y = a^x$ for different values of a

(with e as just defined), with x-coordinate equal to $\frac{1}{n}$. That is,

$$Q = \left(\frac{1}{n},\, e^{1/n}\right).$$

The slope of chord PQ is equal to

$$\frac{e^{1/n} - 1}{1/n - 0} = n\left(e^{1/n} - 1\right).$$

If we let $n \to \infty$, then in the limit, the slope of chord PQ tends to the slope of the curve $y = e^x$ at the point P. By definition of the value of e, this slope is equal to 1. It follows that for large values of n, we must have

$$n\left(e^{1/n} - 1\right) \approx 1, \quad \therefore\ e^{1/n} - 1 \approx \frac{1}{n}, \quad \therefore\ e^{1/n} \approx 1 + \frac{1}{n}.$$

This implies that for large values of n, we must have

$$e \approx \left(1 + \frac{1}{n}\right)^n.$$

From this, it only takes a moment to conclude that

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

When the area is 1

Remarkably, there is also an approach using areas to hit upon the number e. This time, we consider the curve $y = \frac{1}{x}$. Figure 3 shows a sketch of the curve in the first quadrant. Let us define a function $f(t)$ for positive numbers t as follows: $f(t)$ = the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = t$. More precisely, it is the area of region $PQRS$ when it is traversed in an anticlockwise direction (see Figure 3 for the definitions of these points). Then f is a continuous function, and $f(1) = 0$.

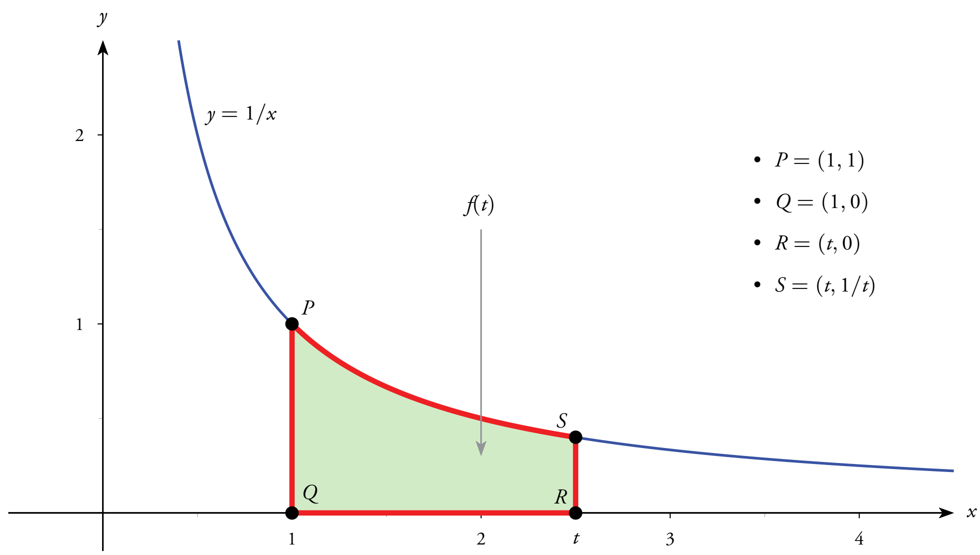


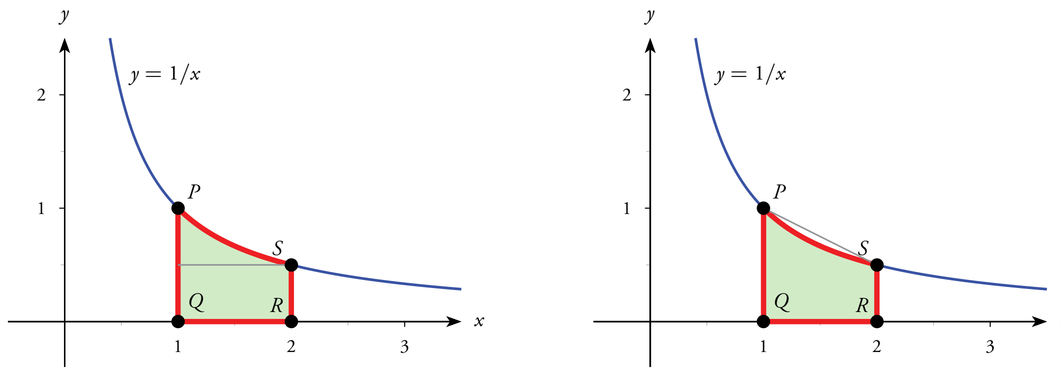
Figure 3.

The region $PQRS$ does not belong to any of the classes of regions studied up to class 10, as one of its sides is a curve which is not an arc of a circle. So it may not be obvious how we can find its area. (Remember that as far as this article is concerned, we have not yet started studying integration!) We therefore use a method of approximation, by inscribing rectangles within the region and by circumscribing rectangles about the region. By making these rectangles narrower and narrower, we get increasingly better approximations to the desired area.

For example, consider $f(2)$. In Figure 4 (a), we have drawn a rectangle inside $PQRS$. Its area is $1 \times 0.5 = 0.5$; hence $f(2) > 0.5$. In Figure 4 (b), we have drawn a trapezoid circumscribing $PQRS$. Its area is $\frac{1}{2}(1 + 0.5) \times 1 = 0.75$; hence $f(2) < 0.75$. It follows that $0.5 < f(2) < 0.75$. By drawing rectangles and trapezoids of width 0.5 (inscribed within the region and circumscribed about the region, respectively), we find that $0.58 < f(2) < 0.71$. These bounds show that $f(2) < 1$.

We may similarly consider $f(3)$. By inscribing narrower and narrower rectangles within the region and circumscribing narrower and narrower trapezoids about the region, we find that $1.04 < f(3) < 1.1$. These bounds show that $f(3) > 1$.

So we observe that $f(2) < 1$ and $f(3) > 1$. By continuity, we expect that there exists a critical value of t lying between 2 and 3 such that $f(t) = 1$. ***This critical value is called e.***



(a) (b)

Figure 4.

Having defined e this way, the onus is now on us to show that this definition implies that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Perhaps the reader could look for a proof of this claim on his or her own. However, as this article has already become somewhat long, and the proof is clearly not one that will fit in the margin of this page, we shall omit the proof for now and give it in the next part of this article, which will appear in the March 2019 issue of *At Right Angles*.

A mnemonic for the exponential number

A popular aspect of mathematical culture is to find easy-to-remember mnemonics for the decimal expansions of well-known numbers such as π and e. In [1], a mnemonic is presented for the first 40 digits of e. Since 0 presents an obvious difficulty in the design of such a mnemonic, the author uses an exclamation mark to represent 0. Commas, colons, semi-colons, quote signs and full-stop signs are to be ignored, and the position of the decimal point is assumed to be known. Here is his mnemonic:

> We present a mnemonic to memorize a constant so exciting that Euler exclaimed '!' when first it was found, yes, loudly '!.' My students perhaps will compute e, use power or Taylor series, an easy summation formula, obvious, clear, elegant!

Counting out the letters, we get the digits of e: 2.718281828 . . . .

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RAMANUJAN: *What might have been, if only…*

RAHUL TIKEKAR

A few weeks ago, I watched the film "The Man Who Knew Infinity" based on the book with the same title by Robert Kanigel. It tells the story of the prodigy Srinivasa Ramanujan whose contributions to the field of Mathematics were so profound that scholars are still trying to make sense of his work. I remember reading the book several years ago and being gripped by the story of his life. It is a very well-written book.

The movie, however, made a stronger impact on me in several ways: I realized how little, if anything, someone like me has contributed to this world; I realized what a waste of time it is to spend time perusing Facebook and WhatsApp posts; I appreciated the hard work and dedication of some of the finest researchers in the world; and I recognized the importance of Mathematics in solving many of the challenges we face in today's world, especially in the fields of science and technology. However, what was probably more revealing to me was that the story of Ramanujan is a story of misfortune and woe, worthy of a place among the best Greek tragedies. I couldn't but contemplate what might have been, had it not been for the series of unfortunate events, a perfect storm, that enveloped Ramanujan's life.

Let's start with the fact that he was born in the India of 1887, a country steeped in poverty, religious orthodoxy and illiteracy, burdened by centuries of foreign rule – whose sole purpose was to rob India of her riches and make her citizens work as peons while the masters enjoyed a life of pleasure. In such conditions, it was no wonder that very few Indians were competent enough to perceive Ramanujan's extraordinary mathematical ability. Wasn't it ironical that the only way to further his proficiency was to seek help from the very empire that was subjugating him? What if Ramanujan had

Keywords: Ramanujan, life, opportunities, prejudice, professional

been born in Britain, France, Germany, or even the USA? At the risk of losing bragging rights for Indians, but for the sake of furthering knowledge, couldn't it have helped Ramanujan to have been in an environment where his talent and gifts were not just recognized but cultivated and enriched?

Ramanujan's own religious beliefs prohibited a person from crossing the seas. As a Hindu, I don't believe that any such clause exists in the religion. Were Ramanujan and his family of the same mindset as some of the great explorers like Columbus, Francis Drake, Magellan and Vasco da Gama, the fear of distant lands would not have stopped him from seeking to travel where he could find opportunities to excel and shine. As with Galileo, religious beliefs nearly snuffed out the amazing results that he did eventually publish.

Then came Godfrey Hardy, a man so British and so married to Mathematics that he could not or would not or did not acknowledge that Ramanujan, while doing extraordinary work, was personally miserable. In the US, there is a popular slogan used by the state of California to promote its dairy products: *Great Cheese Comes from Happy Cows and Happy Cows Come from California*. While this may be only an advertising gimmick, there is merit in the statement that can and does apply to humans: a happy researcher will produce good results. If only Hardy could have just put aside his stiff British upper lip and opened up to Ramanujan; if only he could have enquired a little about Ramanujan's well-being; if only he had been more understanding and appreciative of Ramanujan's upbringing and Indian culture; if only he had been more involved, emotionally, with Ramanujan, as a friend. Could that bonding have established conditions where Ramanujan may have been happier and thus accomplished a whole lot more?

If only Hardy could have applied his remarkable mind to recognize that his genius ward was struggling to survive without his wife. Ramanujan was also unable to consume the food that was offered because he was a vegetarian. I can't but help wonder what could have been if Hardy would have offered to invite Ramanujan's wife to join him, or, at the very least, have offered to arrange for a cook to help with Ramanujan's dietary requirements. Could this have helped Ramanujan feel more accepted and appreciated? If only Hardy's first few statements to Ramanujan could have been: "What can we do to make your stay here comfortable?" Or: "If there is anything that is bothering you or keeping you sad, please let me know." Or: "Don't hesitate to come to me with any issues; think of me as your elder brother in this foreign land." Hardy assumed, incorrectly, that Ramanujan would just fit in like any other researcher of mathematics and enjoy everything that Cambridge offered.

And if these barriers by themselves were not sufficient to thwart progress, Ramanujan had to also deal with racial prejudice on the part of some of his Cambridge colleagues. I am always at a loss to explain how someone smart enough to work at an institution like Cambridge – where one is expected to be of a higher intellectual caliber than the rest of the society – can still harbour the notion that an entire race of humans can be inferior. Had these colleagues been more accepting of Ramanujan's strengths and encouraged and worked with him, instead of dismissing him, it may have boosted Ramanujan's creativity even more.

You would think that Ramanujan's cup of misery couldn't overflow any further. There was another villain lurking in his life, in the form of his very own mother. So possessive was she of her son's affections that she secretly intercepted letters his wife wrote and failed to mail them to Ramanujan. It is hardly a source of wonder that in an era when letters are the only means of communication, one would feel alone and depressed when one's own wife, the one person to be depended on for support, does not respond to your letters. One would be left with the feeling that perhaps she has moved on, increasing one's agony. If only Ramanujan's mother had stepped away from interfering in her son's life, he would have had at least one lifeline to use as a crutch, instead of attempting suicide. Perhaps Ramanujan's mother assumed that her son was in the best of company, living out his dream, and that once his wife joined him, he would forget all about his mother. History also has it that she was mean towards her daughter-in-law. Why blame the British for racial prejudice when one's own family can become a barrier in one's progress?

Finally, Ramanujan's own stubbornness is to blame for many of the issues he faced. He refused to get treatment for whatever was ailing him. He was in Britain, a place as good as any for good medical resources. If only he'd sought medical attention; if only he confided in Hardy his discomfort; if only he adapted to British food and acclimatized to the British weather; if only he found another friend in India to whom he could write and enquire about his wife, perhaps, just perhaps, things may have turned out better than they did. Another advertisement in the US has a tag line: *A Mind is a Terrible Thing to Waste*. In the case of Ramanujan, it was a great mind that was mostly wasted, and wasted for the wrong reasons. When Raj Reddy, the famous computer scientist and Turing award winner, was asked the secret of his success, he replied, modestly, that he was in the right place at the right time. In the case of Ramanujan, it appears that he was in the wrong place at the wrong time. I just want to raise my hands to the sky and shout, "Why?"

A closing comment

It can generally be argued that if a discovery is not made at a particular point in time by a certain person, it would be made sooner or later by someone else. Had Newton and/or Leibniz not invented Calculus when they did, probably someone else would have (indeed, calculus-like thoughts were very much in the air at the time). Lobachevsky and Bolyai independently worked on non-Euclidean geometry at around the same time. But Ramanujan appears to be an exception to this phenomenon. Keith Devlin articulates my thoughts beautifully in his book *Finding Fibonacci* (Princeton University Press, 2017). He argues that "had he [Ramanujan] not lived, it is likely that no one would have discovered many of the things he did." Extending this thought, had Ramanujan lived longer and in happier times, mathematics could have been so much the richer.

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Shapes with Sutli: The String Game

SANDHYA GUPTA

This article describes an activity where students created different geometrical shapes using a closed-loop string and developed conceptual understanding by engaging with properties of the shapes. The activity encouraged them to think deeply about the meaning of points, straight lines, edges, faces, and angles of geometrical shapes. Using standard models which are generally available, students only get to view geometrical shapes or build them by following a set of instructions. However, the activity described here provided students the opportunity to interact with each other, build various shapes and think creatively about how to prove their properties, thus, developing an in-depth understanding.

In many classrooms, Mathematics is taught as an abstract subject with emphasis on developing procedural knowledge. This renders Mathematics as a dry and difficult subject for most students. They develop a fear of the subject and prefer to drop it at the earliest possible opportunity. The way students engage with Mathematics is neither perceptibly useful nor interesting. However, if they are engaged in meaningful activities, which enable them to explore and visualize concepts, Mathematics can become a far more interesting subject.

Many students begin to dread the subject when they encounter theorems and proofs, the basic pillars of logic and reason. In the school curriculum, it is in the topic of geometry that the student is introduced to notions of argumentation and proof for the first time. Before working with proof, students need to visualize different shapes and their properties and also strengthen their understanding of the basic definition of shapes. This article describes a classroom activity where students of grade 9 constructed various two-dimensional and three-dimensional shapes while justifying and proving their properties at the same time. It is based on the old childhood game of "Sutli" which many of us may recall, and converts it into a Math game. [For the benefit of those who haven't heard of this game, it is played with a sutli or string, looped around the players' fingers to make patterns. It can be played by a single player and also in pairs by making patterns with their fingers. One partner makes a pattern,

Keywords: Geometrical shapes, classroom engagement, proof, visualization, 2D shapes, 3D shapes, square, cube, tetrahedron, octahedron, dodecahedron, games, kinesthetic activities.

the other builds on it, and the sutli keeps changing hands from one partner to the next till they get stuck, and can't make any more, or someone makes a mistake and the game has to be started all over again.]

In a traditional mathematics classroom, shapes are usually introduced through drawings on the blackboard. In some schools, students are shown 2D and 3D models of different geometrical shapes. A common hands-on activity entails cutting out and building a shape using a net. Such tasks, although interesting, allow students to create shapes by following a set of instructions, but do not help to develop geometrical reasoning related to the shapes.

The Activity

The objective of this activity is to build several 2D and 3D shapes and justify their properties through logical arguments [1]. The framework proposed by Cathy Humphreys [2] of *convince yourself, convince a friend, and then convince a sceptic* was used for justification. The activity was tried out with a class of 30 students where they worked in groups of about four. They were given 90 minutes to complete the task. Each group was given an 8 feet long closed-loop string and was required to build a given shape with the loop. The conditions posed were as follows:

a. The string knot cannot be opened at any time,

b. every person in the group must have at least one hand on the string,

c. the group must use the entire string,

d. the group must be able to justify their claim, and

e. no instrument from the geometry box may be used to build/measure the shape.

The shapes chosen for the building exercise were square, pentagon, cube, tetrahedron, square-based pyramid, and octahedron.

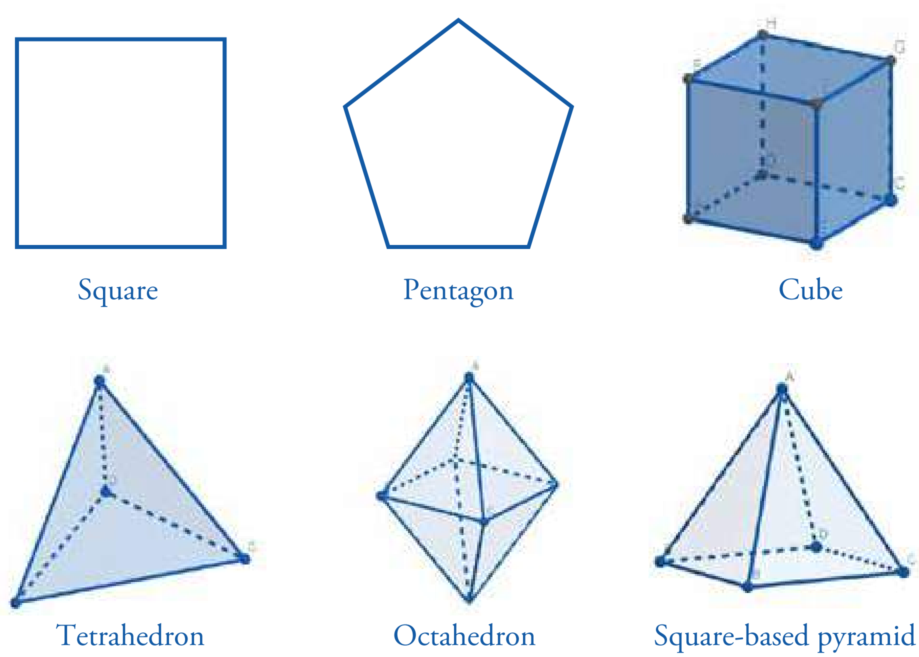


Figure 1: Shapes to be created by the closed loop string

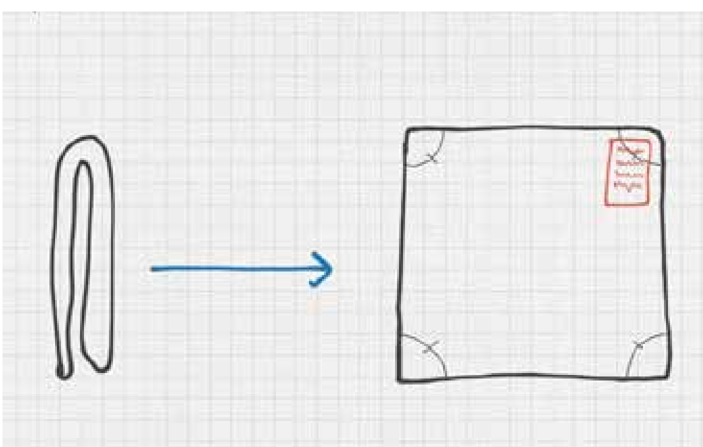
Discussion

Several methods were devised by students to build each shape. In this article we will discuss the methods for the square and cube. While the students attempted the task, the teacher facilitated the process by asking thought provoking questions. At the start of the activity, the teacher and support staff demonstrated the making of an equilateral triangle using the closed loop string and played the role of a friend and a sceptic. Some of the questions posed during this process were: how do you know it's a triangle? How can you prove all sides to be equal? What can you say about the angles? Why? In general, the teacher acted as the sceptic and 'pushed' the students to see and formulate multiple paths of thinking and justify their arguments regarding the shapes.

A. Building a Square

Different methods for building a square were implemented. Some of them are discussed below:

a. The loop was folded to get four equal sides and was opened up to form a quadrilateral (Figure 1). The interior angles were adjusted so that all were of equal measure (right angles). Some groups tried to prove that all angles are 90 degrees, while the others tried to prove that all angles are equal. To prove that a given angle was a right angle, students used the corner of a book or corner of wall and floor as a reference. The challenge in this approach was – how does one prove the corner of a book to be a right angle? The justification used was – if all the interior angles match the corner of the book, this would mean they are all equal, hence the quadrilateral would be square.


a


b

Figure 2: (a) Equally folded loop converted to square, angle measured with a book corner. (b) Students working on building a square

b. The loop was folded in such a way that a quadrilateral was formed with both its diagonals (Figure 3). All sides were compared to ensure their equality and both diagonals were also shown to be equal in order to justify that the quadrilateral was indeed a square. Some groups did not make the diagonal, but measured it with a stick.

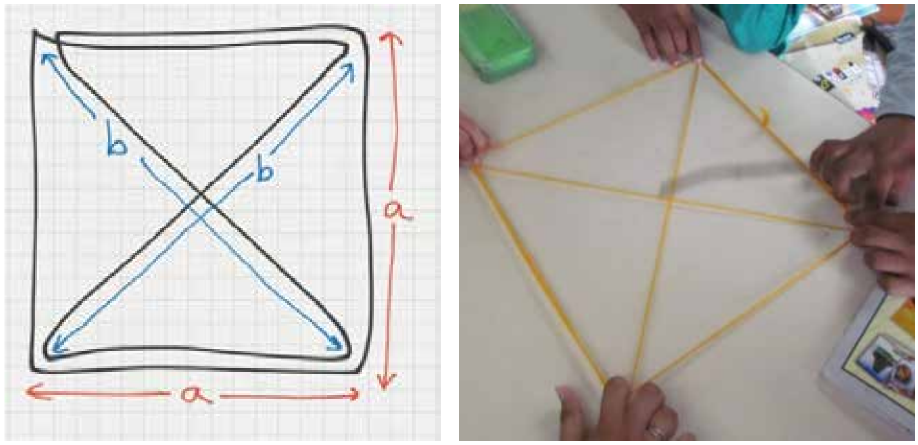


Figure 3a. Folding strategy for case (b)

Figure 3b. Students measuring the diagonal with a stick

c. In this method the loop was folded equally twice to get eight equal sides as shown in figure 4 below. Students held the centre point (mid-point of w shape in figure 4), and extended the edges to form a plus sign. All four angles of the plus sign were shown to be equal using the corner of a book. Finally the four central points of the thread were flipped to form the square.

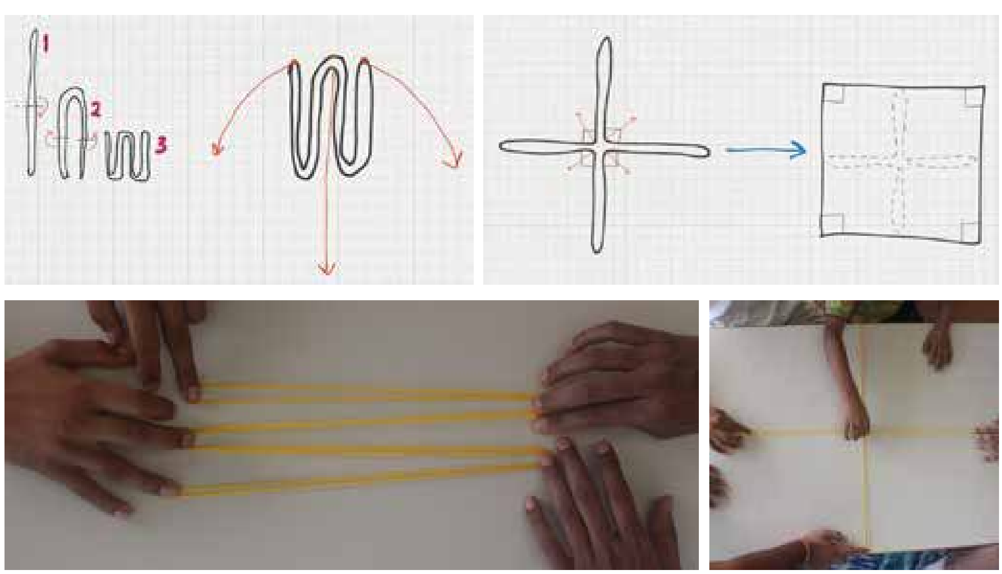


Figure 4: Folding strategy for case (c)

A critical point arose in the discussion - it is important to prove that all the four points are in the same plane. In general the students found it easy to prove the sides to be equal, but proving the angles to be equal was a bit challenging for most groups. The teacher guided the students by posing the following questions: What does it mean to have a right angle? Where do we see it around us? If all angles are 90 degrees, how do they compare with each other? If the student responds that it means all are equal, then how can you prove they are equal? What strategy can be used for this comparison?

One group also discussed the idea of making a separate knotted loop of Pythagorean triplet (3, 4, 5) to prove the corners to be right angles (as shown in Figure 5).

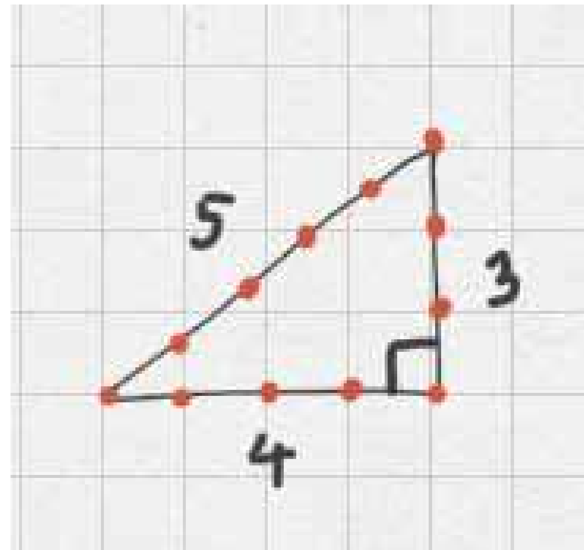


Figure 5: A closed loop with 12 equidistant knots being held as a right angled triangle

By the end of the exercise, the groups had developed the basic definition of a square. They articulated that 'A square is a quadrilateral in which all sides are equal and all angles are equal.' One key understanding that developed was, proving all interior angles to be equal is different from proving that an angle is 90°. To prove that a shape is a square, the following conditions must be met:

a. All the points are in the same plane, and all four edges are straight lines,

b. All sides are equal, and

c. One of the following:

    i. All interior corner angles are equal,

    ii. All interior corner angles are right angles,

    iii. The two diagonals are equal.

The groups were intentionally not probed about defining every shape before the building exercise. Initially, the gap in the definition of a square did not come out when they convinced each other in their groups. But when the facilitators came and questioned as sceptics, they realized what they were missing in their proof of a square. By the time the cube was presented to them, they were already asking the sceptic questions.

B. Building a Cube

There was an initial sense of disbelief when students were asked to build a cube with the loop string, but they quickly got to the task. One key element of the construction was having a clear understanding that for any face to be a square, all the points must lie on the same plane. It helped to draw the perspective view of the cube (as shown in Figure 1) on paper before starting to build the cube. It was important to understand the definition of a cube, and what was needed to prove that the structure was a cube. The importance of precise mathematical terms was apparent for a productive argument.

Some of the methods for building a cube were as follows:

a. **Loop method:** Open up the cube faces and visualize the cube net. Follow steps 1 to 3 as shown in Figure 6. Adjust each loop in the square shape as shown in step 4. Same coloured dots represent the corners of the squares meeting at the same vertex. Once all the vertices are connected a cuboid will be formed. Sides need to be adjusted for square faces.

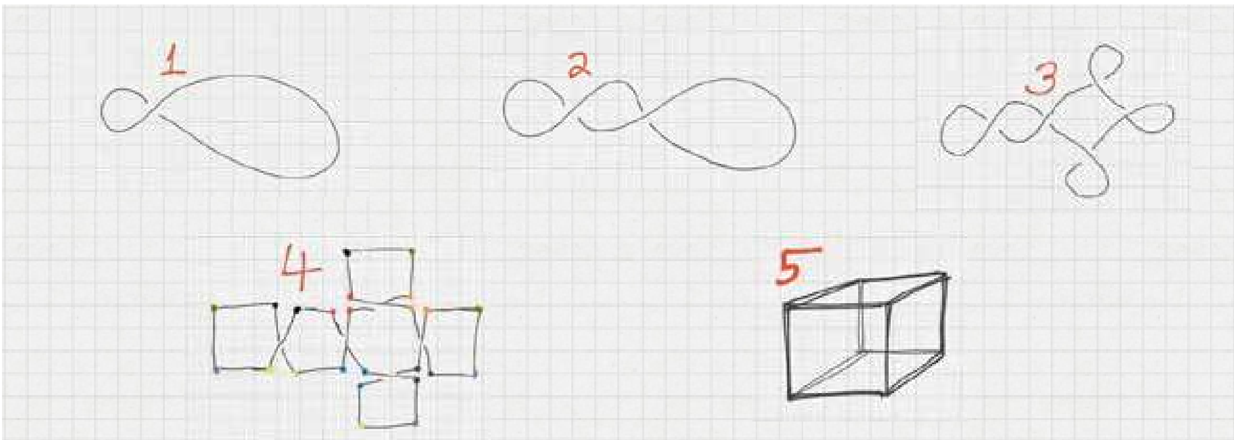


Figure 6. Loop method

b. **Pinch-and-Extend Method:** Make a square face on a flat surface, pinch a corner to make a small loop. Pull the small loop away from the square to make one edge. Keep the original square shape, but let it shrink. Do this to all four corners of the original square, and pull the new four edges vertically up from the square surface. There are still four edges of the cube remaining. Further extend and bend each loop edge at right angle filling in these four missing edges.

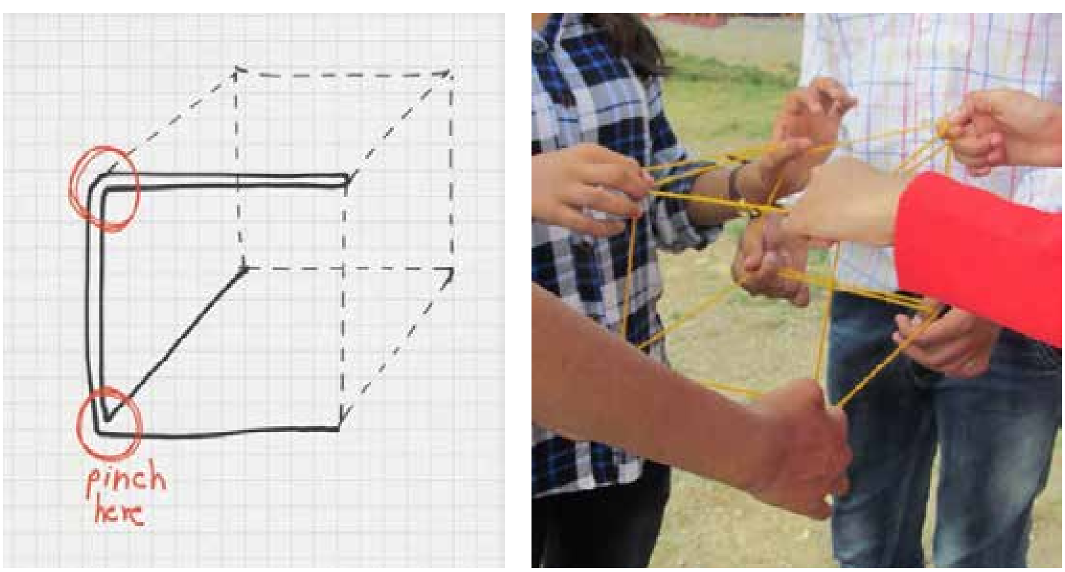


Figure 7: Pinch-and-Extend method

c. **Trace-the-Edge method:** Keep tracing all the edges of the virtual cube and come back to the starting point. Hold all the vertices and adjust to make all faces square-shaped to build the cube.

Proving the 3D design to be a cube shape was a challenge for many. The conceptual clarity regarding the properties of a cube was the key. Many groups struggled with the thought that they needed to prove each face to be a square, and that all adjacent planes were perpendicular to each other. After much deliberation, the groups reached a definition: "A cube is a closed 3-dimensional structure with six square faces, with three faces meeting at each vertex." The minimum conditions needed to prove a cube shape using a string are:

a. The 3-dimensional shape must have an outline of six faces,

b. All faces outlined must be square shapes that are congruent to each other,

c. Three faces meet at each vertex.

There was further deliberation on what it meant to be a 'face.' Since a thread was used, outline of the face was constructed and not the face per se.

Students also created a tetrahedron and an octahedron with the loops (Figure 8). A similar process of building the shape and developing a proof along the way was followed. Some groups went further and started working on other shapes such as a dodecahedron at the end of the 90-minute session.

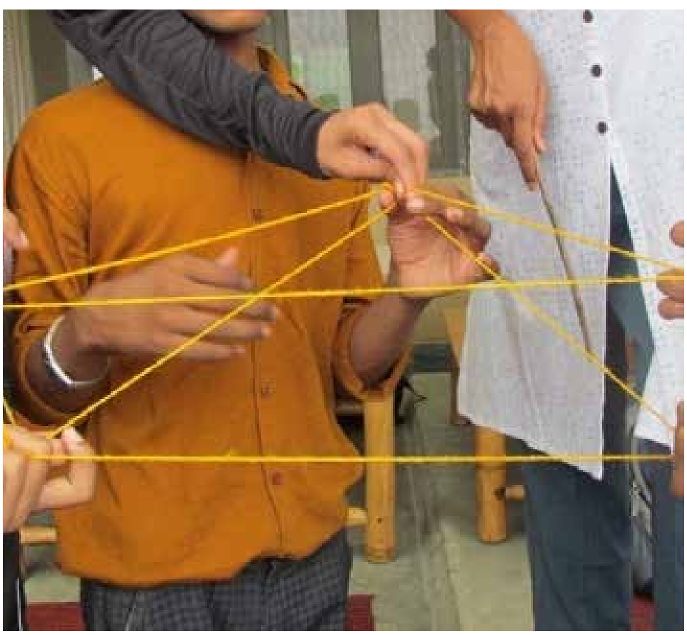


Figure 8: Students building square based prism and octahedron

The session facilitators played the role of the sceptic, encouraged students to think deeply about the meaning of a point, a straight line, an edge, a face, or an angle in the geometrical sense. Some students even argued about the meaning of a vertex while holding it with their fingers. The definitions of many geometrical terms were revisited, questioned and discussed. The importance of correct terminology and its mathematical meaning in a conversation became apparent. For example, it wasn't 'side' of a cube anymore, it was side of a square and edge of a cube; it wasn't being called 'corner' of a cube, it was a cube's 'vertex'; that in a pentagon all sides and interior angles need not be equal, only a regular pentagon will have that and so on. It was very satisfying to see that every group member engaged with the activity without any fear of failure. Students approached the task in multiple ways, listened to each other's ideas and collaborated in their work. Many students felt that they had developed a clear understanding of these shapes for the first time. Students who had initially defined square as a quadrilateral with equal sides, added 'all interior angles equal' to their definition. In the attempt to figure out the elements needed to prove a cube, the students thought about the angular relationship between not only two edges, but also between two planes. Students also realized that they didn't need to memorize the properties of the shapes, such as number of faces, edges, vertices, interior angles, etc., as they could now visualize the shape and work out their properties whenever required. The exploration of shapes did not stop with the session, but became a point of discussion even days after the session. This was indeed a high point of the activity.

Acknowledgement

The author is thankful to members of Aavishkaar team including Sarit Sharma, Bharat Shresth and Apoorwa Hooda for their support. She would also like to thank Shamli Manasvi, a participant in this activity for her ideas, illustrations and constructive feedback.

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Student Corner – Featuring articles written by students.

Arc - centre Theorem

HAREN SATHVIK

Student Haren Sathvik uses mathematics as a power tool to win an argument and prove his point. Exactly what the Position Paper of the National Focus Group on Mathematics says: Children use abstractions to perceive relationships, to see structure, to reason about things, to argue the truth or falsity of statements. Logical thinking is a great gift mathematics can offer us, and inculcating such habits of thought and communication in children is a principal goal of teaching mathematics. Read on to find out more.

In this short note, we discuss a theorem (arc-centre theorem) which states that an arc can have only one centre of curvature.

Centre of curvature: The centre of a circle of which the given arc is a part.

Introduction

When my classmates and I were preparing for the Regional Mathematical Olympiad, we were given the following problem:

A circle of radius 2 is centred at O. Suppose square OABC has side length 1; sides AB and CB are extended past B to meet the circle at D and E respectively (refer Figure 1). What is the area of the shaded region in the figure which is bounded by BD, BE and the minor arc connecting D and E?

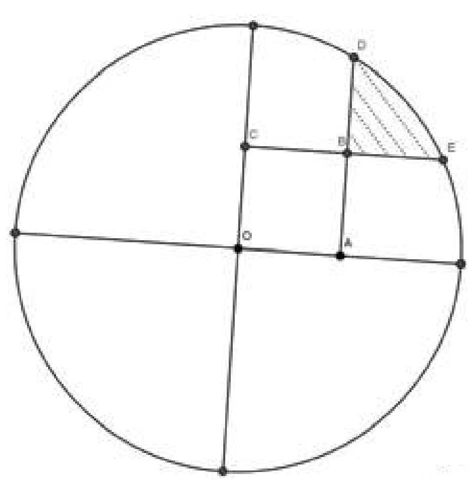

Figure 1.

Keywords: Centre of curvature, arc

I solved the problem using trigonometry. My friend Mahesh came up with a different argument. His approach was to construct another circle with centre B and then argue that arc DE must be its quarter circumference (since angle DBE = 90°). I felt this was wrong, as you cannot have two different circles (with two different centres) sharing an arc (i.e., arc DE). When we asked for help from our math mentor, he challenged us to prove the statement. I came up with a proof which I give in this note. That is, I prove the "arc centre theorem" – the statement that "A given arc has a unique centre of curvature."

Before proving the theorem, let us show how to construct a centre for a given arc using straightedge and compass.

Construction. Draw any two non-parallel chords with endpoints on the arc. For each of them, draw a second chord parallel to the first, also with endpoints on the arc. Construct the midpoint of each chord. For each pair of parallel chords, draw the line through their midpoints. The two lines then intersect at the centre of the circle [1].

Now let us prove our main theorem.

Theorem (Arc-centre theorem)

An arc of a circle can have only one centre of curvature.

Proof. Consider arc $\overline{ABC}$ where A and C are the endpoints and B is the midpoint of AC. Let L be a line through B, perpendicular to the tangent to arc $\overline{ABC}$ at B; that is, L is the normal to arc $\overline{ABC}$ at B (refer Figure 2). Suppose that it has more than one centre of curvature (in contradiction to the statement of the theorem). Let the two centres be O_1 and O_2 (refer Figure 3).

Assume $O_1O_2 = h$. Then we have:

$$AO_1 = BO_1 = CO_1 = r_1$$
$$AO_2 = BO_2 = CO_2 = r_2$$

Now we use the triangle inequality which states that in a triangle, the sum of any two sides is greater than the third side. So from triangle AO_1O_2,

$$AO_1 + O_1O_2 > AO_2, \text{i.e.,}\ r_1 + h > r_2 \qquad (1)$$

But since B, O_1, O_2 lie on a straight line,

$$BO_1 + O_1O_2 = BO_2, \text{i.e.,}\ r_1 + h = r_2 \qquad (2)$$

It is clear that (2) contradicts (1). Hence the initial supposition must be wrong. Therefore, we can say two different centres of curvatures do not exist for a single arc. Hence proved.

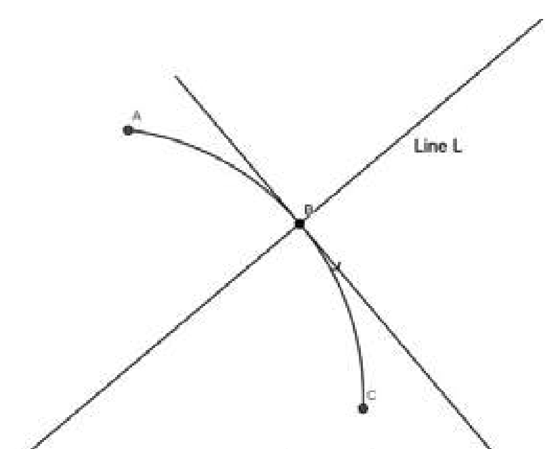

Figure 2.

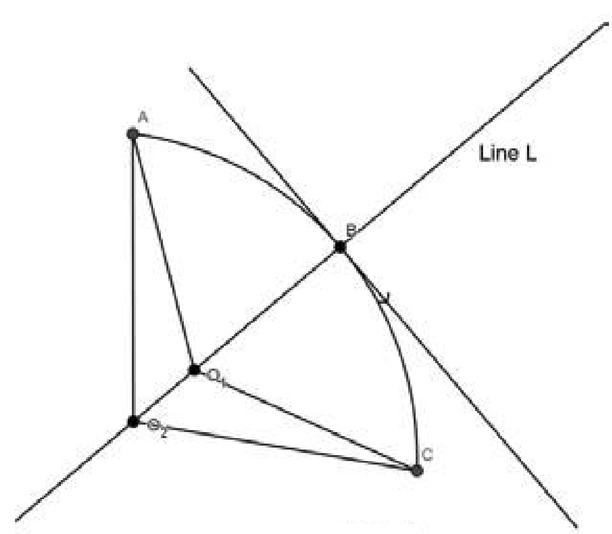

Figure 3.

Acknowledgement

The author would like to thank his math mentor for his support, motivation and cooperation. He would also like to thank an anonymous referee for his/her kind comments and suggestions, which led to a better presentation of this paper.

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1. https://math.stackexchange.com/questions/1113276/how-to-find-centre-of-a-circle-from-only-an-arbitary-arc-of-that-circle

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Vijay bought three used text books. He later sold them all at the same price. On the first book he made a profit of 50%. On the second book he took a loss of 50%. If the total amount of money he received from the sale of the three books is the same as the total amount of money he paid for the three books, find the percentage of profit or loss that he incurred on the third book. Also identify if Vijay made a profit or took a loss on the sale of the third book.

A Note on Armstrong Numbers

SATVIK U. KAUSHIK

In this article, student Satvik Kaushik investigates Armstrong numbers. Mathematical investigation is a powerful way for students to learn more about concepts that interest them. In mathematical investigations, students are expected to pose their own problems after initial exploration of the mathematical situation. The exploration of the situation, the formulation of problems and their solution give opportunity for the development of independent mathematical thinking and in engaging in mathematical processes such as organizing and recording data, pattern searching, conjecturing, inferring, justifying and explaining conjectures and generalizations. It is these thinking processes which enable an individual to learn more mathematics, apply mathematics in other disciplines and in everyday situations and to solve mathematical (and non-mathematical) problems.

Source: https://www.coursehero.com/file/26670868/What-is-mathematical-investigation-Mathematics-for-Teachingpdf/

In this note we provide a way for identifying Armstrong numbers. We also discuss their generalizations.

Introduction

In recreational number theory, an Armstrong number (named after Michael F. Armstrong) of the first kind is a number that is the sum of its own digits each raised to the power of the number of digits. For example, $153 = 1^3 + 5^3 + 3^3$. In this note we study three-digit Armstrong numbers of the first kind and their patterns in the general case (Armstrong numbers of the second kind) and we also list all the n-digit Armstrong numbers of the first kind.

Theorem 1. **There are only four three-digit Armstrong numbers of the first kind. They are 153, 370, 371 and 407.**

Proof. Consider a three-digit number $N = 100A + 10B + C$ where $A, B, C \in \{0, 1, 2, \ldots .9\}$ and $A \neq 0$. Assuming that it is an Armstrong number, we shall find the possible values of A, B, C.

From the definition of an Armstrong number, it follows that $A^3 + B^3 + C^3 = 100A + 10B + C$. We consider the different possible values of A.

Suppose that $A = 1$. Then $B^3 + C^3 = 10B + C + 99$. The number on the right side is at least 99 and at most 198. Hence $B < 6$ and $C < 6$. We can check that either $B = C = 4$, or one of B or C is 5. (The other possibilities clearly do not work out.)

Keywords: Armstrong number, exponents

Clearly $(B, C) = (4, 4)$ does not satisfy $B^3 + C^3 = 10B + C + 99$.

Hence one of B or C is 5. If $B = 5$, then $C^3 - C = 24$, so $C = 3$, hence the number is 153.

If $C = 5$, then $B^3 + 125 = 10B + 104$, so $B^3 - 10B = -21$, but this does not yield any positive integer solution for B.

Hence the first three-digit Armstrong number is 153.

That is, $\mathbf{1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153.}$

Next, suppose that $A = 2$. Then $B^3 + C^3 = 10B + C + 192$. The number on the right side is at least 192 and at most 291, hence either $B = C = 5$, or one of B or C is 6.

In neither case do we find any such B and C which fits the equation. So there is no three-digit Armstrong number starting with 2.

Next, suppose that $A = 3$; then $B^3 + C^3 = 10B + C + 273$. The number on the right side is at least 273 and at most 372, which implies $(B, C) = (5, 6)$ or $(6, 5)$ or B or C is 7. It is easy to check that the possibilities $(B, C) = (5, 6)$ and $(B, C) = (6, 5)$ do not work. Now consider the possibility $C = 7$; in this case $B^3 = 10B - 63$, but this gives no such B. Next, consider $B = 7$; in this case we have $C^3 = C$, which gives $C = 0$ or $C = 1$. Hence the possible numbers are $370, 371$. Equality works out for both these numbers.

That is, $\mathbf{3^3 + 7^3 + 0^3 = 27 + 343 + 0 = 370}$ and $\mathbf{3^3 + 7^3 + 1^3 = 27 + 343 + 1 = 371.}$

Now let us proceed with the case $A = 4$. In this case we have $B^3 + C^3 = 10B + C + 336$. The number on the right side is at least 336 and at most 435, which implies $(B, C) = (5, 6)$ or $(6, 5)$ or B or C is 7. It is easy to check that no solution exists with either $B = 7$ or $(B, C) = (5, 6)$ or $(6, 5)$. Hence we take $C = 7$, which gives $B^3 = 10B$. This yields $B = 0$. Hence the number is 407. This fits the requirement.

That is, $\mathbf{4^3 + 0^3 + 7^3 = 64 + 0 + 343 = 407.}$

If we experiment with the possibilities $A \in \{5, 6, 7, \ldots, 9\}$, we do not find any solutions for B and C.

Hence there are only four three-digit Armstrong numbers of the first kind; they are **153, 370, 371, 407.**

Note. In [1], the author Dr. MOLOY DE gives a brief description on Armstrong numbers as follows:

"Armstrong numbers of first kind are base dependent and they are certainly rare. They cannot have more than 60 digits in base 10, because for $n > 60$, $n9^n < 10^{n-1}$. Since there is an upper limit to their size, it is theoretically possible to find all of them, given sufficient computer time. However, 10^{60} is an unimaginably huge number, so such a 'brute force' approach would be unwise. Luckily, D. Winter proved in 1985 that there are exactly 88 base-10 Armstrong numbers of first kind, and they must have 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 16, 17, 19, 20, 21, 23, 24, 25, 27, 29, 31, 32, 33, 34, 35, 37, 38 or 39 digits. Of course, the one-digit Armstrong numbers of first kind are somewhat trivial since clearly $1^1 = 1, 2^1 = 2$ etc. The Armstrong numbers of first kind up to 10 digits are 1, 2, 3, 4, 5, 6, 7, 8, 9, 153, 370, 371, 407, 1634, 8208, 9474, 54748, 92727, 93084, 548834, 1741725, 4210818, 9800817, 9926315, 24678050, 24678051, 88593477, 146511208, 472335975, 534494836, 912985153, and 4679307774. The largest Armstrong number of first kind (in base 10) is the 39-digit beast: 115132219018763992565095597973971522401".

The following table gives a list of all n-digit Armstrong numbers of the first kind [2], [3].

No.	Armstrong Numbers of first kind
1	0, 1, 2, 3, 4, 5, 6, 7, 8, 9
3	153, 370, 371, 407
4	1634, 8208, 9474
5	54748, 92727, 93084
6	548834
7	1741725, 4210818, 9800817, 9926315
8	24678050, 24678051, 88593477
9	146511208, 472335975, 534494836, 912985153
10	4679307774, 32164049650, 32164049651, 40028394225, 42678290603, 44708635679, 49388550606, 82693916578,
11	94204591914
14	28116440335967
16	4338281769391370, 4338281769391371
17	21897142587612075, 35641594208964132, 35875699062250035
19	1517841543307505039, 3289582984443187032, 4498128791164624869, 4929273885928088826
20	63105425988599693916
21	128468643043731391252, 449177399146038697307
23	21887696841122916288858, 27879694893054074471405, 27907865009977052567814, 28361281321319229463398, 35452590104031691935943
24	174088005938065293023722, 188451485447897896036875, 239313664430041569350093
25	1550475334214501539088894, 1553242162893771850669378, 3706907995955475988644380, 3706907995955475988644381, 4422095118095899619457938
27	121204998563613372405438066, 121270696006801314328439376, 128851796696487777842012787, 174650464499531377631639254, 177265453171792792366489765
29	14607640612971980372614873089, 19008174136254279995012734740, 19008174136254279995012734741, 23866716435523975980390369295
31	1145037275765491025924292050346, 1927890457142960697580636236639, 2309092682616190307509695338915
32	17333509997782249308725103962772
33	186709961001538790100634132976990, 186709961001538790100634132976991
34	1122763285329372541592822900204593
35	12639369517103790328947807201478392, 12679937780272278566303885594196922
37	1219167219625434121569735803609966019
38	12815792078366059955099770545296129367
39	115132219018763992565095597973971522400, 115132219018763992565095597973971522401

Generalisations of Armstrong numbers

Now we discuss some generalizations of Armstrong numbers. In Theorem 1 we discussed n-digit Armstrong numbers of the first kind. Now we discuss the Armstrong numbers of second kind. Such a number has the property that it "is equal to the sum of the cubes of numbers composed of two successive digits or three successive digits or four successive digits and so on of the number." For example:

$153 = 1^3 + 5^3 + 3^3$ (Armstrong number of first kind);

$165033 = 16^3 + 50^3 + 33^3$ (Armstrong number of second kind);

$166500333 = 166^3 + 500^3 + 333^3$ (Armstrong number of second kind);

$166650003333 = 1666^3 + 5000^3 + 3333^3$ (Armstrong number of second kind).

First we prove the example stated above, then we explore these numbers and their generalizations.

Theorem 2.

$$\mathbf{1666\ldots^3 + 5000\ldots^3 + 3333\ldots^3 = 1666\ldots6\;5000\ldots0\;3333\ldots3},$$

where the numbers of 6's, 3's and 0's are the same in the three numbers on the left side.

Proof. Here $1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153$, $16^3 + 50^3 + 33^3 = 165033$. Here we can see a pattern among the results. To prove the pattern for all A, B, C, take A, B, C in terms of variable "n" such that we can say

n	A	B	C	ABC
1	$1 = \frac{10^1 - 4}{6}$	$5 = \frac{10^1}{2}$	$3 = \frac{10^1 - 1}{3}$	$153 = 1(10^2) + 5(10^1) + 3(10^0)$ $= \left(\frac{10^1 - 4}{6}\right)(10^{2(1)}) + \left(\frac{10^1}{2}\right)(10^{1(1)}) + \left(\frac{10^1 - 1}{3}\right)(10^{0(1)})$
2	$16 = \frac{10^2 - 4}{6}$	$50 = \frac{10^2}{2}$	$33 = \frac{10^2 - 1}{3}$	$165033 = 16(10^4) + 50(10^2) + 33(10^0)$ $= \left(\frac{10^2 - 4}{6}\right)(10^{2(2)}) + \left(\frac{10^2}{2}\right)(10^{1(2)}) + \left(\frac{10^2 - 1}{3}\right)(10^{0(2)})$
3	$166 = \frac{10^3 - 4}{6}$	$500 = \frac{10^3}{2}$	$333 = \frac{10^3 - 1}{3}$	$166500333 = 166(10^6) + 500(10^3) + 333(10^0) = \left(\frac{10^3 - 4}{6}\right)(10^{2(3)}) + \left(\frac{10^3}{2}\right)(10^{1(3)}) + \left(\frac{10^3 - 1}{3}\right)(10^{0(3)})$
4	$1666 = \frac{10^4 - 4}{6}$	$5000 = \frac{10^4}{2}$	$3333 = \frac{10^4 - 1}{3}$	$166650003333 = 1666(10^8) + 5000(10^4) + 3333(10^0) = \left(\frac{10^4 - 4}{6}\right)(10^{2(4)}) + \left(\frac{10^4}{2}\right)(10^{1(4)}) + \left(\frac{10^4 - 1}{3}\right)(10^{0(4)})$

$A = \dfrac{10^n - 4}{6}$, $B = \dfrac{10^n}{2}$ and $C = \dfrac{10^n - 1}{3}$.

Hence

$$\text{LHS} = A^3 + B^3 + C^3 = \left(\frac{10^n - 4}{6}\right)^3 + \left(\frac{10^n}{2}\right)^3 + \left(\frac{10^n - 1}{3}\right)^3 = \frac{10^{3n} - 10^{2n} + 2.10^n - 2.10^0}{6},$$

$$\text{RHS} = ABC = \left(\frac{10^n - 4}{6}\right)(10^{2n}) + \left(\frac{10^n}{2}\right)(10^n) + \left(\frac{10^n - 1}{3}\right)(10^0) = \frac{10^{3n} - 10^{2n} + 2.10^n - 2.10^0}{6}.$$

So $A^3 + B^3 + C^3 = 100A + 10B + C$ where $A = \dfrac{10^n - 4}{6}$, $B = \dfrac{10^n}{2}$ and $C = \dfrac{10^n - 1}{3}$.

Hence

$$1666\ldots^3 + 5000\ldots^3 + 3333\ldots^3 = 1666\ldots6\ 5000\ldots0\ 3333\ldots3.$$

Theorem 3. **There exist Armstrong numbers of the second kind corresponding to all three-digit Armstrong numbers of the first kind.**

Proof. Let us start with an Armstrong number of the first kind, say $100A + 10B + C = 370$.

Take the six-digit number $\overline{A_0B_0C_0}$ where A_0, B_0, C_0 are two-digit numbers such that

$$A_0^3 + B_0^3 + C_0^3 = 10^4 A_0 + 10^2 B_0 + C_0,$$

where A_0 has the form $\overline{3a}$ or $\overline{a3}$, B_0 has the form $\overline{7b}$ or $\overline{b7}$, and C_0 has the form $\overline{0c}$ or $\overline{c0}$.

Let us start with the possibility $A_0 = \overline{3a}$, that is, $A_0^3 + B_0^3 + C_0^3$ is a six-digit number starting with 3 and followed by a. There are 4 possible cases to consider.

Let us take the case where $B_0 = \overline{7b}$ and $C_0 = \overline{0c}$.

It should be clear that $30^3 \le \left(\overline{3a}\right)^3 \le 39^3$. We get similar bounds for B_0^3 and C_0^3.

Now we make use of the following relations:

- minimum possible value of $B_0^3 + C_0^3$ is equal to minimum possible value of $\left(A_0^3 + B_0^3 + C_0^3\right)$ − maximum possible value of A_0^3;
- maximum possible value of $B_0^3 + C_0^3$ is equal to maximum possible value of $\left(A_0^3 + B_0^3 + C_0^3\right)$ − minimum possible value of A_0^3.

From this we deduce that $B_0^3 + C_0^3$ lies between 240681 and 372999.

Since C_0^3 does not exceed 729 and $72^3 > 372999$, B_0 can be only 70 or 71.

If $B_0 = 70$, then $A_0^3 + C_0^3 \le 56999$. Hence A_0 lies between 30 and 38.

Here first two digits are 34, hence $A_0 \in [34, 38]$.

Hence $A_0^3 + B_0^3 + C_0^3 = (3a)^3 + 343000 + c^3$. It is clear that no 'a' satisfies this equation. Similarly, B_0=71 also doesn't give any solution.

Now let us take the case where $B_0 = \overline{b7}$ and $C_0 = \overline{0c}$.

We know that $0 \le C_0^3 \le 729$. It follows that B_0^3 lies between 239952 and 372999, hence $B_0 = 67$. This yields $\left(\overline{3a}\right)^3 + 300763 + \left(\overline{0c}\right)^3 = \overline{3a670c}$; from this we get $a = 3$ and $c = 0$ or 1.

If we continue the same for $B_0 = \overline{7b}$, $C_0 = \overline{c0}$, we get no solution.

If we continue the same for $B_0 = \overline{b7}$, $C_0 = \overline{0c}$, we get $B_0 = 67$ and $C_0 = 0$ or 1.

That is, if $A_0 = \overline{a3}$, then we get $A_0 = 33$, $B_0 = 67$, $C_0 = 00$ or 01.

Now observe that $3^3 + 7^3 + 0^3 = 370$ and $33^3 + 67^3 + 00^3 = 336700$.

This seems to be in the form $A^3 + B^3 + C^3 = \overline{ABC}$ where $A = \dfrac{10^n - 1}{3}$, $B = \dfrac{2.10^n + 1}{3}$ and $C = 000\ldots$ (n times). Since

$$A^3 + B^3 + C^3 = \left(\frac{10^n - 1}{3}\right)^3 + \left(\frac{2.10^n + 1}{3}\right)^3 + 0^3 = \frac{10^{3n} + 10^{2n} + 10^n}{3},$$

$$ABC = \left(\frac{10^n - 1}{3}\right)(10^{2n}) + \left(\frac{2.10^n + 1}{3}\right)(10^n) + 0(10^0) = \frac{10^{3n} + 10^{2n} + 10^n}{3},$$

it is true.

Hence the generalization for 153:

$$\mathbf{1666\ldots^3 + 5000\ldots^3 + 3333\ldots^3 = 1666\ldots6\ 5000\ldots0\ 3333\ldots3.}$$

The same generalisation works for the numbers 370, 371 and 407. These yield the four Armstrong numbers of the second kind.

Hence there exist Armstrong numbers of the second kind for all three-digit Armstrong numbers of the first kind.

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TearOut

Fun with Dot Sheets

Beginning with the last issue, we started the TearOut series. In this article, we focus on investigations with graph paper. Pages 3 & 4 give guidelines for the facilitator, pages 1 & 2 are a worksheet for students. This time we explore quadrilaterals and triangles using lattice points.

SWATI SIRCAR

The following activities can be done on centimeter or inch graph paper, rectangular dot sheets or square grids. They start with the first quadrant where all coordinates are non-negative. The origin is kept as one of the starting points so that patterns can be seen easily. Later, the activities have been generalized to other quadrants and starting points.

Geometry on graph paper

Get some graph papers, a pencil and a scale… you are all set to go… ☺

Quadrilaterals

1. **Take the origin O and any 2 points P and Q in the 1st quadrant, draw OP and OQ and complete the parallelogram OPSQ**
   - 1.1. How are the coordinates of S related to those of P and Q?
   - 1.2. How are the coordinates of the intersection of OS and PQ related to those of S?
   - 1.3. When will OPSQ be (i) a rectangle? (ii) a rhombus? (iii) a square?
   - 1.4. When will O, P and Q not form three vertices of a parallelogram? In that case, can they be three vertices of any other quadrilateral? How?
2. **Take origin O and any point P in the 1st quadrant, draw OP and**
   - 2.1. Draw a square OPSQ
     - 2.1.1. What are the possible coordinates of Q? How are they related to the coordinates of P?
     - 2.1.2. What are the coordinates of S? How are they related to those of P and Q?
   - 2.2. Draw a rhombus OPSQ and explore 2.1.1 and 2.1.2
   - 2.3. Draw a rectangle OPSQ and explore 2.1.1 and 2.1.2
3. *[optional] Take origin O and any point P in any quadrant and repeat all of 2.*
4. **Take origin O and any 2 points P and Q. Draw OP and OQ. Draw the parallelogram OPS_1Q. Draw PQ and the parallelograms $OPQS_2$ and OS_3PQ**
   - 4.1. What are the coordinates of S_1, S_2 and S_3?
   - 4.2. How are they related to the coordinates of P and Q?
   - 4.3. Comment on ΔOPQ and $\Delta S_1S_2S_3$
5. **Take any 3 points T, P and Q, draw the parallelograms TPS_1Q, $TPQS_2$, TS_3PQ and repeat 4.**

Triangles

1. **Take origin O and any point P on the positive x-axis and draw a scalene ΔOPQ such that Q is in the 1st quadrant (See Figure 1)**

   1.1. Comment on the coordinates of Q when ∠OPQ is
   (i) acute (ii) right (iii) obtuse

   1.2. If Q is on the y-axis comment on ∠QOP.

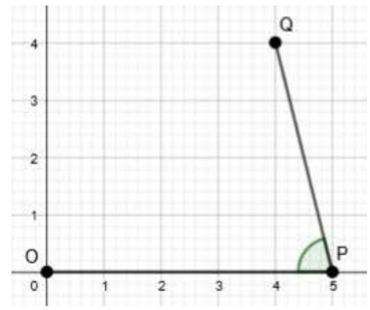


Figure 1

2. **Take origin O and any point P on the positive x-axis and draw an isosceles ΔOPQ such that Q is in the 1st quadrant and OQ = PQ (See Figure 2)**

   2.1. How are the coordinates of Q related to those of P?

   2.2. What is the height of ΔOPQ? How is that related to the coordinates of any of the points?

   2.3. Consider the following cases:

   2.3.1. If height = ½ × base i.e. ½ OP

   2.3.2. If height < ½ × base i.e. ½ OP

   2.3.3. If height > ½ × base i.e. ½ OP

   2.3.4. If height > base i.e. OP

   Comment on ∠OQP and check which is true: (i) OP > OQ or (ii) OP < OQ for each case

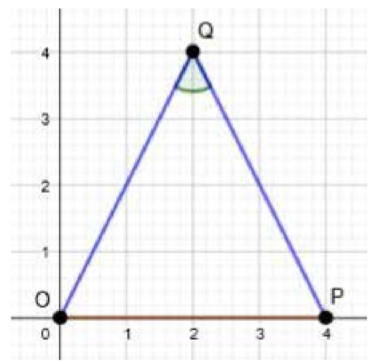


Figure 2

3. *[optional] Take origin O and any point P on the positive y-axis and repeat all of 1 and 2. (Q on x-axis in 1.2)*

4. *[optional] Take origin O and any point P on any of the axes and repeat all of 1 and 2*

5. **Take origin O and any point P in the 1st quadrant and draw a isosceles ΔOPQ such that Q is also in 1st quadrant and OP = OQ (See Figure 3)**

   5.1. How are the coordinates of Q related to those of P?

   5.2. If P is on the x-axis, where is Q? Comment on ∠POQ in that case.

   5.3. When will OP not form an isosceles ΔOPQ with OP = OQ? In that case,

   5.3.1. Can you get an isosceles triangle in the 1st quadrant with OP = PQ?

   5.3.2. How are the coordinates of Q related to those of P in that case?

   5.3.3. Comment on ∠OPQ.

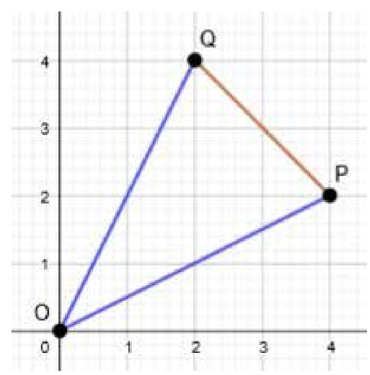


Figure 3

6. **Take origin O and any point P in any quadrant and repeat 5.1 and 5.2**

   6.1. If P is on the y-axis, where is Q? Comment on ∠OPQ in that case.

   6.2. If the x-coordinate and the y-coordinate of P are the same i.e. of the form (a, a)

   6.2.1. What are the coordinates of Q?

   6.2.2. Comment on ∠POQ in that case

7. **Take any 2 points T and P and draw triangles for all 3 angles in each case**

   7.1. a scalene ΔTPQ such that ∠TPQ (or ∠PTQ) is
   (i) acute angle (ii) right angle (iii) obtuse angle

   7.2. an isosceles ΔTPQ such that PT = PQ and ∠TPQ is
   (i) acute angle (ii) right angle (iii) obtuse angle

The only pre-requisite for these activities is familiarity with the origin, the axes – especially just the positive parts in most cases – and finding the coordinates of a given point on the plane. So this can be attempted at the upper primary level. It revisits some of the properties of parallelograms and triangles that students learn at this stage. It will also lay the foundation for some properties (or theorems) that they will learn at the secondary level.

These activities are designed to develop an intuitive sense of equal sides, right angles, etc., followed by observing the pattern in the coordinates of the points. As a starting point, we expect the students to use lattice points (i.e. points with integer coordinates) mostly. We also want students to use eye-estimate at this stage rather than construction so that they develop the intuitive sense as mentioned above. This would foster an understanding of the coordinate system and how it relates to the properties.

If done as a class, each student can record one set of values of the coordinates of P, Q and S. Later, these sets can be collated in a table. If done individually, each student will need to find different sets of values of coordinates of P, Q and S. From that tabulated data, the pattern of how these three points are related emerges and the relation can be generalized algebraically. Table 1 provides a set of values for the coordinates of P, Q, and S. Let us take a closer look activity by activity.

P	Q	S
(5, 2)	(2, 6)	(7, 8)
(3, 8)	(5, 7)	(8, 15)
(4, 2)	(7, 3)	(11, 5)
⋮	⋮	⋮

Table 1

Whenever we mention easy options, we mean lattice points. It may be possible to find other rational options and if a student finds those, then justification should be asked when needed.

Quadrilaterals

The key ideas include the vector addition (and subtraction) which is based on parallelograms (See Figure 4).

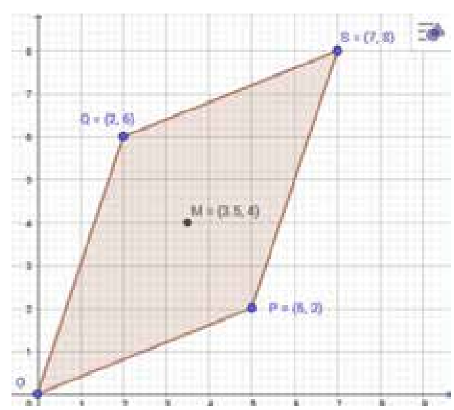


Figure 4

1. **Parallelograms**

   1.1. Vector addition: i.e. if P = (a, b) and Q = (c, d), then S = (a + c, b + d)

   1.2. Midpoint formula: Intersection of OS and PQ = $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$, in particular the coordinates are half of those of S

   1.3. Right angles ⇒ P and Q must be on the axes, equal sides ⇒ P = (a, b) and Q = (b, a) are easy options i.e.

   (i) P = (a, 0), Q = (0, b) or P = (0, a), Q = (b, 0)

   (ii) P = (a, b), Q = (b, a)

   (iii) P = (a, 0), Q = (0, a) or P = (0, a), Q = (a, 0)

   Note: If P = (a, b) where a, b are integers, it may be difficult to find another integer pair (c, d) ≠ (b, a) for Q such that OP = OQ or $a^2 + b^2 = c^2 + d^2$. One can draw circle with centre O and radius OP, but it may not go through any lattice point other than (b, a). It is possible that it may not pass through any other point that has rational coordinates.

   1.4. When O, P and Q are collinear

2. **Squares and more**

   2.1. If P = (a, b), then Q = (−b, a) or Q = (b, −a) — the coordinates are switched and one of them has the sign changed, coordinates of S = P + Q as before. The relation between coordinates of P and Q paves the way for understanding how slopes of perpendicular lines are related

2.2. Similar easy options like in 1.3 i.e. if P = (a, b) then Q = (b, a)

2.3. This is similar to finding a Q that will make ∠POQ a right angle like in 2.1, however since all sides of a rectangle are not necessarily equal there are more choices for Q like (–b, a), (–2b, 2a), (–3b, 3a) … or (b, –a), (2b, –2a), (2b, –3a) … i.e. (–nb, na) for any integer $n \neq 0$
Each one reinforces the vector addition

3. Generalizes the above with coordinates involving negative numbers

4. Creates the three possible parallelograms that are double of ΔOPQ.
$S_1 = P + Q = (a + c, b + d)$, $S_2 = Q - P = (c - a, d - b)$, $S_3 = P - Q = (a - c, b - d)$
Also O, P and Q are midpoints of S_2S_3, S_3S_1 and S_1S_2 respectively
∴ $\Delta S_1S_2S_3$ is made of four triangles congruent to ΔOPQ

5. Generalizes the vector addition and the rest of 4 away from the origin

Triangles

While construction of triangles with given specification is included in most upper primary syllabus and textbooks, students often don't develop the ability to draw or visualize different kinds of triangles on the square grid. These activities foster that intuitive sense and brings attention to the coordinates and relations among them. The formulas encountered later will become more meaningful with this experience.

1. **Scalene triangles**
   1.1. If P = (a, 0) and Q = (c, d), this fosters the understanding that ∠OPQ is greater, equal to or less than 90° if and only if c > a, c = a and c < a respectively, that ∠OPQ increases with "c" and and its being acute, right or obtuse is independent of "d"
   1.2. Reinforcing the angle between the axes

2. **Isosceles triangles**
   2.1. If P = (a, 0) then Q = (a/2, d) where d > 0
   2.2. That the height of ΔOPQ = d i.e. y-coordinate of Q
   2.3. Explores all possible isosceles triangles especially how the relation between the height and the base changes with the angle opposite to the base OP

   In particular, this can capture two types of acute isosceles in 2.3.3 and 2.3.4. OP > OQ for 2.3.1 i.e. right isosceles and 2.3.2 i.e. obtuse isosceles. OP < OQ for 2.3.4. The remaining case 2.3.3 is interesting since both cases are possible. It can be interesting to push students to understand what is the cut-off for OP = OQ, what kind of triangle that forms, what is the height of that triangle etc.

3. Reverses the axes for 1 and 2

4. Generalizes 1 and 2 to the negative parts of the axes

5. **More isosceles**
   5.1. Similar easy option as Quad 1.3 (ii)
   5.2. Similar to Quad 1.3 (iii)
   5.3. Similar to Quad 1.4 If P = (a, a) and connecting back to 2

6. Generalizes to coordinates with negative numbers
   6.1. If P = (0, b) then Q = (±b, 0) and ∠OPQ = 45°. Why?
   6.2. Q = (a, –a) or Q = (–a, a) i.e. reflection of P on x- or y-axis respectively forming a right angle ∠POQ

7. Generalizes away from origin, fostering the sense of drawing almost all possible scalene and isosceles triangles. It also exposes students to consider all the possibilities when side-wise and angle-wise classifications of triangles are combined. Note that one type of triangle is not included. This one can't be drawn with only lattice points as its vertices. Look out for the proof in the next issue!

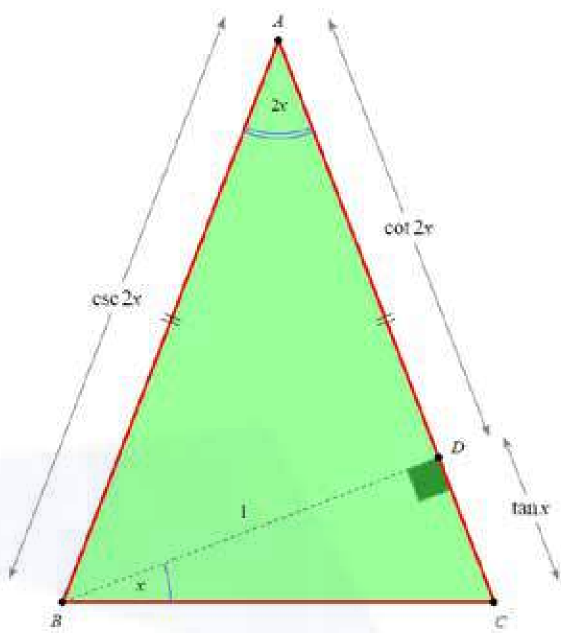
ClassRoom

Geometric Proof of a Trigonometric Identity

SHAILESH SHIRALI

The following diagram is a "proof without words" (also known as a 'visual proof') for the trigonometric identity

$$\csc 2x = \cot 2x + \tan x.$$



Keywords: Identity, trigonometry, visual proof

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Interpretation of ERRORS IN ARITHMETIC

HRIDAY KANT DEWAN

Errors should not be viewed as a setback but as an opportunity to learn more about the student's thought processes.

Correcting errors to get the right product instead of analysing the learning trajectory is a quick fix that does not nurture deep learning

Very often, errors are caused by over-generalisation of rules which are transmitted to the student as short-cuts to getting the required answer.

Certain repetitive drill and practice tasks set by teachers can contribute to students developing these quick fixes which in fact, can divert them from the concept that they are meant to understand and practise.

Teachers must plan tasks that help children understand the basic concept being taught. They must also observe the child doing the task and, together with the child, examine the child's procedural thinking.

Errors are often referred to as windows into the minds of learners and also as part of the ladder of learning. The effort to look at the work of the learner and attempting to construct the conceptual and procedural underpinning of the response-expressions of the child provides many insights. This analysis can also be useful in mathematics as it points out the care the teacher may take in conversing with learners as it may offer many learnings and takeaways for the teacher. It has been spelt out sufficiently well that the work of children has to be considered far more carefully than merely sorting into the categories of right and wrong. The manner in which the learner has reached the answer provides insights about the way she thinks and approaches the question. Many times, the answer being correct does not imply that the problem has been tackled appropriately. The correct answer could well be a consequence of fortuitous mistakes and coincidences. In an article 'Errors or ladders of learning' many years ago, Agnihotri[1] suggested that errors could be the steps in the ladders of learning and indicate the path through which the journey of learning could take place. This has indeed been stated and argued before and after by many persons. In the learning of language there are many examples of this and some of them are relatively better understood. For example, there is the phenomenon of over generalisation in responding to exceptions to the rules. The simplest example is of 'go' and 'goed' or the example in Hindi of saying टूटाना for तोडना. The formulation presented to the child, 'इस गिलास को मत तोडना' became 'इस गिलास को मत टूटाना. The implied

[1] Agnihotri, Rama Kant. 1988. 'Errors as learning strategies'. In Indian Journal of Applied Linguistics 14.1.: 1-14. Bahri Publications Delhi

Keywords: Errors, correction, remediation, column addition, drill and practice

meaning in both is 'breaking' as a part of the imperative, 'Do not break the glass'. Many more examples can be added to this from language and from other areas as well.

While this happens in the natural learning process of the child, as she tries to engage with situations and attempts to create expressions that communicate her ideas, the patterns and rules that she has over-generalised start entering her conversations and then, based on the responses received, she herself slowly corrects them. Yet, there is a tendency of teachers to forcibly correct the rules children need to use and not give the children the opportunity to recognise them and correct them. By setting up a formal process to correct the error, the teacher loses the chance to help the student understand the underlying pattern and the reason for the error.

If we ask ourselves the question as to how this, or something else that is akin to an over-generalisation of rules happens in the context of mathematics, then we can try to analyse the work of children and connect it to the kind of processes they have gone through and the possible generalisations that they can make. Given the nature of mathematical objects and the manner in which relationships between them are constructed, there are no or very few exceptions to the definitions and the rules. Where there are exceptions, they are indicated in the beginning itself. For example, in the defining relationships for a rational number, where p/q, where $q \neq 0$ and p, q are within the set of integers or in calculus finding limits with the denominator tending to zero. In doing operations and in solving problems, there are however, created algorithms. And these may be inappropriately spelt out and wrongly used. These algorithms and short-cuts are created to find quick answers and are sometimes provided by teachers and sometimes shared through exchanges. The effort is focused towards finding a response to the task without fully understanding the expectation or the task, leading to inappropriate use of the technique. It is these phenomena that we would explore in this article. We do this through the piecing together of response patterns to three distinct kinds of mathematical situations. In all these, some form of shortcut strategy to reach the expected answer is seen. We will then try to link it to some phenomena that may be fairly widely used by children to respond to the tasks they have been given so as to complete them with minimum effort. All the examples presented in this paper have at their base a widely used task that is common across schools. This task continues to influence the development of mathematical understanding and impacts the subsequent mathematical ideas developed by the learner. The strategy or the method adopted for the task, produces the desired answer and is efficient in providing the response the teacher wants but the manner in which it constructs understanding has wider implications for the conceptual structures of the child. The purpose of the analysis of these examples is to illustrate the often stated point that short-cuts and special algorithms to arrive at the answer can lead to such conceptual confusions and generalisations that fail to give the user the ability to appreciate the procedures being used and use them in appropriate situations in an appropriate manner only. The tasks that are given to children must be well thought out and must expect her to use her conceptual, procedural and cognitive knowledge instead of just reproducing information. It also suggests that examining how children do the task is important for the teacher to observe and she must also talk to the child about the way she is doing the task and the logic she has behind it.

Considering a few response patterns

It is generally noticed that: i) children have difficulties considering a fractional representation as one number; ii) they also have difficulties in dealing with two-digit numbers and on operations with them. The nature of responses has been reported on at many places and if a teacher considers the responses of her students to such problems beyond marking them as right or wrong, she will discover many patterns in the responses. One such pattern likely to be present in the responses even till class 8 and beyond is typified by the following example

$$\frac{1}{3}+\frac{1}{2}=43 \quad \text{and} \quad \frac{1}{3}+\frac{1}{2}=\frac{2}{5}$$

and similar responses in subtraction and multiplication sums.

The second example is of addition of two or more numbers with carry over and also to some extent addition of numbers with an unequal number of digits.

The examples of these are

$$\begin{array}{rrr} & 2 & 7 \\ + & 3 & 8 \\ \hline 5 & 1 & 5 \end{array}$$

and other such equivalent examples.

When the number of digits is unequal or a problem of algorithmic addition with numbers presented in the same manner as above with digits displaced by a small amount from a strict column format, interesting responses may be seen. For example

128 + 64 can become

$$\begin{array}{lccc} & 1 & 2 & 8 \\ + & 6 & 4 & \\ \hline & 7 & 6 & 8 \end{array}$$

Or

$$\begin{array}{lcccc} & & 1 & 7 & 9 \\ + & 2 & 6 & 1 & \\ \hline & 2 & 7 & 8 & 9 \end{array}$$

Similar responses are seen in subtraction where the larger digit in the numerals is the one from which the other smaller digit is subtracted. For example, 64 – 38 gives the answer 34

$$\begin{array}{rrr} & 6 & 4 \\ - & 3 & 8 \\ \hline & 3 & 4 \end{array}$$

Another task that illustrates this rather well is the belief that once the child knows the rule of carry over, then she can be given numbers of any size to add. So generally, when we think children in class 4 or 5 (if not earlier) know addition of 2- and 3-digit numbers we start giving numbers like this-

$$\begin{array}{r} 74345212 \\ 52136128 \\ +\ 214321 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{r} 28750 \\ 13250 \\ +\ 8950 \\ \hline \end{array}$$

Clearly students are unable to read the numbers and know what they are and what the sum should be and so they are doing column additions without reading the numbers as one number. When given subtraction problems, they are at a loss when they have to borrow. So even if they get the correct answer in these, it is not as if they are learning any mathematical concepts or developing any ability in mathematics. It would seem that the rules of addition have been over-generalised without comprehension.

Then there is division as well. Consider these for example:

$$\begin{array}{r} 11 \\ 3\overline{)44} \\ \underline{3} \\ 14 \\ \underline{3} \\ 11 \end{array} \quad \text{or} \quad \begin{array}{r} 11 \\ 3\overline{)44} \\ \underline{44} \\ 00 \end{array}$$

$$\begin{array}{r} 11 \\ 6\overline{)85} \\ \underline{66} \\ 21 \end{array} \quad \text{or} \quad \begin{array}{r} 11 \\ 5\overline{)\ 75} \\ \underline{-55} \\ 20 \end{array}$$

All these children do problems like these correctly:

$$\begin{array}{r} 21 \\ 4\overline{)\ 84} \\ \underline{-8} \\ 4 \\ \underline{4} \\ 0 \end{array} \quad \text{and} \quad \begin{array}{r} 7 \\ 8\overline{)58} \\ \underline{56} \\ 02 \end{array} \quad \begin{array}{r} 20 \\ 2\overline{)\ 40} \\ \underline{-40} \\ 00 \end{array}$$

It is when they have to deal with numbers that have to be seen as 2-digit numbers and a clear understanding of place value is required that they slip into errors/short cuts. These are all examples from Class 3 of a school.

All teachers of mathematics have seen these examples. They become a path over which many travel and slowly overcome but many others get stuck with it and as they are faced with more and more mathematics they become more entrapped in them.

The responses below are from adults who have enrolled in an open course on teaching-learning of mathematics. In response to what is wrong with the solution to $\frac{8}{15} + \frac{3}{15} = \frac{11}{30}$ the reasons were interesting as we can decipher a pattern underlying them.

One response was that she has added the numerators and the denominators. What would have been the correct thing to do is to add the numbers above and below each other and then take the sum of the two numbers.

$$\text{So, } 23 + 18 = 41$$

This is a simple response emerging from the habit of column addition to which we would return.

In response to the solution to $\frac{3}{4} + \frac{3}{5}$, the answer given was even more complex; it goes

$$\frac{3}{4} + \frac{3}{5} = \frac{4}{3} + \frac{5}{3} = \frac{9}{3}$$

The numbers are reversed to give a common denominator and then the numbers on top are added.

The emphasis and importance given to column writing and dealing with numbers (or rather digits) in the same column independent of the place in the whole number seems to suggest there is either a cognitive inability or there is something that is done in early classes that forms this pattern. The fact that all children display more elaborate cognitive and conceptual abilities in their standard or routine tasks leaves us to examine the manner in which they are expected to engage with numbers and the opportunities that are created for them. One example of a strategy observed by a person closely observing children while they are attempting to quickly do the tasks being given by their teachers in rural schools of central India illustrates this and offers an interesting insight. It is usual practice in beginning classes to have children write numbers from 1 to 100 in a column or a row format.

1	2	3	4	5	.	.	.	.	.	10
11	12	13	14	15	.	.	.	.	.	20
21	.	.	.	.	.	.	.	.	.	30

Or in the form

1	11	21	.	.	.	.	.	.	91
2	12	22	.	.	.	.	.	.	92
9	19	.	.	.	.	.	.	.	99

Children are asked to do this and it is supposed that they would write these as is required, through a systematic following of the order. What was observed was that children follow a strategy that makes the task easier for them; they write one row or column as required and then repeat the same up to 9. The matrix you get is

1	2	3	4	5	.	.	.	.	.	10
11	12	13	14	15	.	.	.	.	.	20
21	_2	_3	_4	.	.	.	.	.	.	30
31	_2	_3	_4	.	.	.	.	.	.	40

Then two small steps lead to listing all the numbers from 1 to 100.

The step is to write 2, 3, 4, 5, 6, 7, 8, 9 in front of each number of the respective column giving this on the slate:

1	2	3	4	5	.	.	.	.	.	10
11	12	13	14	15	.	.	.	.	.	20
21	22	23	24	.	.	.	.	.	.	30
31	32	33	34	.	.	.	.	.	.	40

and then you can write 10, 20, ..., etc. in the last column or write 1, 2 ....., 9 and then add a zero at the end of each. They repeat this pattern when numbers above 100 are to be written or numbers are to be written in horizontal rows.

Very often, children using a slate have these part columns already written. And these can be quickly filled in the way required. This gives them time to do other things that they want to do. It is fascinating to see how they manage to do this when asked to write counting numbers in reverse sequence. They only need to write digits 9 to 1 and manage the second column appropriately in the same 9 to 1 pattern. In other tasks also, similar patterns are evoked to produce the expected answer. There is no need to understand why in the 2 digit numbers, the digits at different locations are different in the sense they represent different quantities, or in the order of the numbers, what is the logic of the number names, etc. What is enough is to know 1 to 9 and the strategy to write the answer.

So what is the learning from all this for teachers of mathematics?

a. Giving tasks that are mechanical, particularly in the absence of conversation about the task, what they did and why, children can use strategies that change the task to one that only requires the production of the answers without engaging with any concept that should have been required to do the task with understanding. It does not also make them think about what, how or even the procedures involved. The tasks become amenable to alternative short-cuts and do not even help with procedural knowledge or memorisation of facts. Attempts to make tasks complex and challenging has to go along with ensuring that children understand the question and its relevance, be able to read out the questions and estimate the answer.

b. Avoid giving children short-cuts in the form of alternative simple strategies or routes to reach the answer. The strategy of column addition, just looking at the digits and without reading and understanding the numbers is another such example. The problem of short-cuts does not end here and carries on to later classes as well. Some of the examples are short-cuts in word problems, namely hints like all together means addition, spent and left mean subtraction, sharing means division or following BODMAS without thinking. All these lead to confusions that complicate. Similarly, in trying to simplify the meaning of letter numbers, terms like 4a, calling it (say) 4 apples, and so 4a cannot be added to 3b as apples and bananas cannot be added. This suggests, however, that 4a + 3b can be added to give 7f (fruits). And also the confusion that the other letters in the book represent other objects and the consequent confusions and road blocks in learning. These are just a few types of confusions but each of these confusions manifest themselves in multiple ways.

c. Practice is useful but only when it is not mechanical and repetitive and requires the need to develop an understanding of the problem and about how to proceed to the solution.

d. In situations where the number of children is large and the time available for the teacher is at a premium, resorting to techniques of occupying children with tasks like writing numerals can lead to the false sense that an understanding has developed in children. In addition, it gives children generalisations and rules that are grossly inappropriate and wrong

In considering the work of children and while constructing tasks for them, it is important to ensure that they are able to form and articulate their understanding and get reasonable feedback on that. The fact that the teacher may not be able to look at all of them necessitates that they work with each other not in an 'expert' and novice relationship but in a peer relationship. It is of course true that these small group interactions would yet have hierarchies of knowledge but those are what they have constructed as peers. The groups and the relationships cannot be designed and constructed by the teacher such that the 'smart' student guides the weaker one, learning should be independent and supported by a discussion of errors.

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ClassRoom

Some Down-to-Earth Trigonometry

A RAMACHANDRAN

This article aims to present some applications of trigonometry to earth sciences. We assume that the earth is a perfect sphere.

Latitudes and longitudes are imaginary circles that run East-West and North-South respectively, on the earth's surface. Unlike longitudes (or meridians), latitudes (or parallels) vary in length. The Equator is the longest. The others decrease in size till the poles, which are just points. The length of the other latitudes can be expressed as fractions of the length of the Equator.

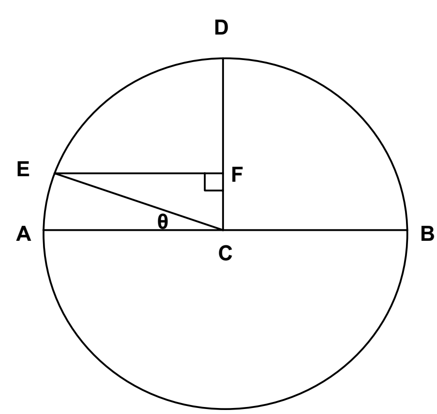

Figure 1

In Figure 1, which shows a section of the earth through the poles, AB is an equatorial diameter. C is the centre of the earth and D the North Pole. E is a point on latitude θ°. The radius of the latitude

$$EF = EC \cos\theta = R\cos\theta,$$

R being the radius of the earth. So we see that the radii of the latitudes and thereby the latitudes themselves, vary as the cosine function as we move from 0° (Equator) to 90° (the poles). The latitudes that are half the length of the Equator are the 60° latitudes.

Keywords: Trigonometry, latitude, longitude, sphere, curved surface area, horizon.

Now we consider the earth's surface area enclosed by a given latitude and the Equator. We invoke the celebrated theorem of Archimedes which states that the surface area of a sphere enclosed between two parallel planes equals a similar part of the (curved) surface area of an enveloping cylinder with axis perpendicular to the planes (Refer Figure 2).

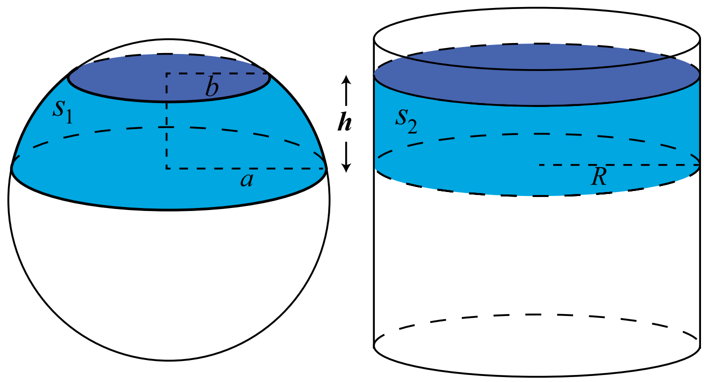

Figure 2. Source: http://mathworld.wolfram.com/

So we need to obtain the perpendicular distance between the equatorial plane and the plane of the given latitude. This is represented in Figure 1 by the distance

$$FC = R \sin \theta.$$

So the extent of the earth's surface enclosed between the equator and a given latitude θ is

$$2\pi R\,(R \sin \theta) = 2\pi R^2 \sin \theta.$$

This reaches the hemispherical curved surface area of $2\pi R^2$ at the poles. Half the earth's surface area lies between the 30° N and S latitudes. It may be of geographical interest to note that about 40% of the earth's surface lies in the tropical zone, about 52% in the temperate zone and about 8% in the frigid zone.

....

The shortest route between two places on earth over the earth's surface lies along the great circle passing through them. Great circles are the largest circles that can be drawn on the surface of the globe. There are an infinite number of them. The equator is one. Two opposite longitudes together make one. There is a unique great circle passing through a pair of points, unless they happen to be antipodal (diametrically opposite) points, in which case there are an infinite number of them.

To find the distance along a great circle route we need to know the angular separation of the points, i.e., the angle between the radii connecting the given points to the centre of the earth. This can be obtained from the latitude and longitude of these points. We take the following observations from 3-D coordinate geometry:

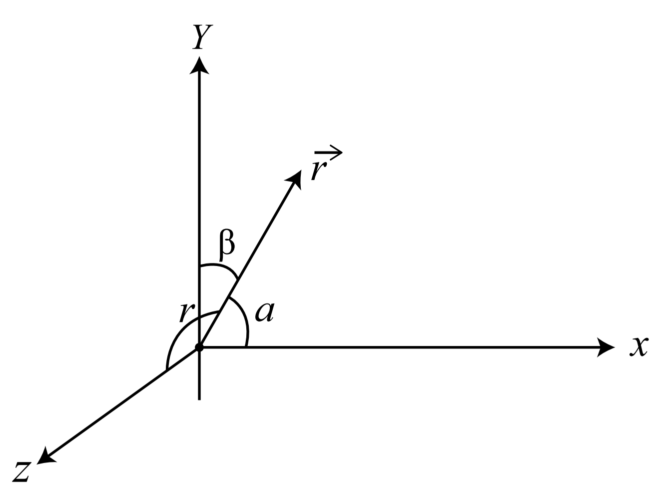

Figure 3

Any line in 3-space makes angles α, β, γ with the X, Y, Z axes respectively (Figure 3). If α_1, β_1, γ_1 and α_2, β_2, γ_2 are the angles made by two lines with the coordinate axes respectively, then the angle θ between the lines can be obtained from the equation

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2.$$

In the context of the earth, we can choose the lines from the centre of the earth to the North pole, the point (0°, 0°) and the point (0°, 90° E) respectively as the coordinate axes. Then, if the latitudes of the two places are ϕ_1 and ϕ_2 and longitudes λ_1, λ_2 we can say that the cosine of the central angle θ

$\cos\theta = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\lambda_1 \cos\phi_2 \cos\lambda_2 + \cos\phi_1 \sin\lambda_1 \cos\phi_2 \sin\lambda_2$, which simplifies to

$\cos\theta = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 (\cos\lambda_1 \cos\lambda_2 + \sin\lambda_1 \sin\lambda_2)$ or

$\cos\theta = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda.$

We follow the convention that North, East are positive, while South, West are negative. See Boxed item at the end of the article for a fuller explanation.

The product of the central angle (in radians) and the radius of the earth gives the distance between the given places 'as the crow flies.'

The following could be checked out by the reader:

If both points lie on the Equator the central angle is the difference in longitude between the places; if the points lie on the same longitude the central angle is the difference in latitude; if the points are antipodal {in such a case the points can be taken to be [x° N, y° E] and [x° S, (180° – y°)W]} the central angle is 180°.

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There is a limit to the extent of the earth's surface visible from any point above ground level. The limit of vision is the horizon. How far is the horizon? Obviously it is a function of the height of the observer/ detector above ground level. (Again we assume a perfectly spherical earth lacking atmosphere.)

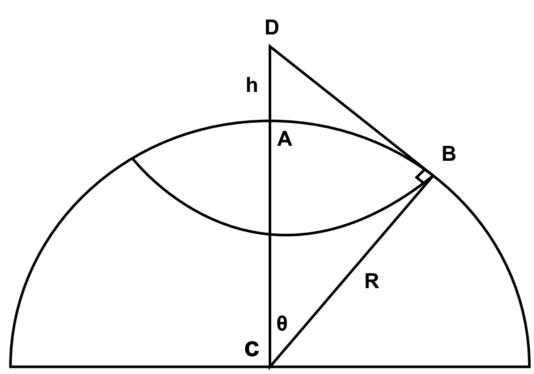

Figure 4

As can be seen from Figure 4, the horizon is the locus of the points of contact of tangents drawn from the observer to the earth's surface. The required distance d is then the arc length $AB = R\theta$ (θ in radians).

$$\text{But } \cos\theta = \frac{R}{R+h}.$$

From the above we get

$$d = R\cos^{-1}\frac{R}{R+h}.$$

This equation can be rearranged to give

$$h = R[\sec(d/R) - 1],$$

enabling one to calculate the height required to see to a given distance.

An alternative approach is to use the Pythagoras theorem to get

$$DB^2 = DC^2 - CB^2 = (h+R)^2 - R^2 = h^2 + 2hR$$

$$\text{So} \quad DB = \sqrt{(h^2 + 2hR)}$$

Now, if h << R, we can say

$$DB = \sqrt{2hR}.$$

And we could then say $d = \sqrt{2hR}$, approximating the arc length AB to the line of sight distance DB.

Substituting 6370 km for R, and converting to metres we have $d = 3570\sqrt{h}$ metres.

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Now we shall derive an expression for the extent of the earth's surface visible from a point.

Again taking a cue from Archimedes we need to find the vertical extension of the part of the sphere enclosed by the horizon. If BE is perpendicular to CD (Figure 5), then the required distance is AE, which we shall denote by 'k.' Now

$$BC^2 = CE.CD, \text{ or}$$

$$R^2 = (R - k)(R + h), \text{ which gives}$$

$$k = Rh/(R + h).$$

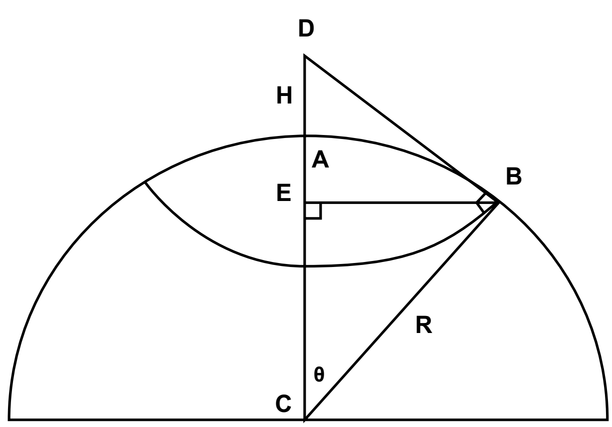

Figure 5

So the extent of the earth's surface visible from point D is

$$2\pi R[Rh/(R + h)] = 2\pi R^2[h/(R + h)] = 2\pi R^2 (1 - \cos\theta).$$

As a fraction of the curved surface area of the hemisphere, this becomes $h/(R + h) = 1 - \cos\theta$.

It can be seen that when h = 0, the visible area = 0. When h is very large the area approaches $2\pi R^2$. When h = R, visible area $= \pi R^2$, half the curved surface of the hemisphere.

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The above expression could also be considered as an expression for the extent of the earth's surface from any point of which an object located at the given height above the earth would be in line of sight. This view of the matter has more practical implications. The Global Positioning System, the American satellite-based navigation system, initially depended on a set of 24 satellites placed in orbits around the earth at an approximate height of 20200 km above the earth's surface, which is more than 3 times the earth's radius. So, substituting 3R for h in the equation derived above we find that each satellite has 3/4 of the earth's hemispherical surface area or 3/8 of earth's total surface in line of sight. So 24 satellites judiciously located would cover the earth's surface $\frac{3}{8} \times 24 = 9$ times over. This fully bears out the claim made that any place on earth would at any instant be in line of sight of at least 6 satellites. The system was later augmented to 32 satellites which by similar arguments can be said to cover earth's surface 12 times over, justifying the claim that at least 9 satellites would always be in line of sight from any point on earth.

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In the course of the earth's revolution around the sun, different latitudes in the tropical zone get vertical rays of the sun at midday on different days of the year. The Equator experiences it on the equinoxes (21, March and 23, September), the Tropic of Cancer on the summer solstice (21, June) and the Tropic of Capricorn on the winter solstice (21, December). The movement of the 'overhead midday sun' between the Tropics follows a sinusoidal path. We can calculate the date(s) of direct midday sun at a particular latitude or conversely the latitude that experiences it (declination, δ) on a given date.

For the earth-sun system we could designate the following coordinate axes:

- The line of centres of earth and sun at spring equinox as the X axis
- The line perpendicular to the above and lying in the earth's orbital plane (ecliptic) as the Y axis
- The line perpendicular to the above lines (perpendicular to ecliptic) as the Z axis.

Now assuming the earth's orbit to be circular with the sun at the centre, and a uniform orbital speed for the earth, it can be shown that

$$\sin\delta = \sin 23.45^\circ \sin(360^\circ N/365.25),$$

where N is the number of days elapsed after the spring equinox. A positive value indicates a latitude in the northern hemisphere while a negative value indicates a latitude in the southern hemisphere.

The above formula can be approximated to

$$\delta = 23.5^\circ \sin(360^\circ N/365).$$

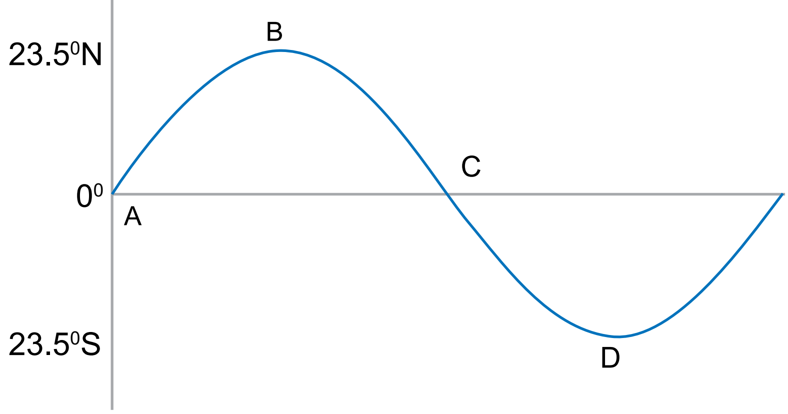
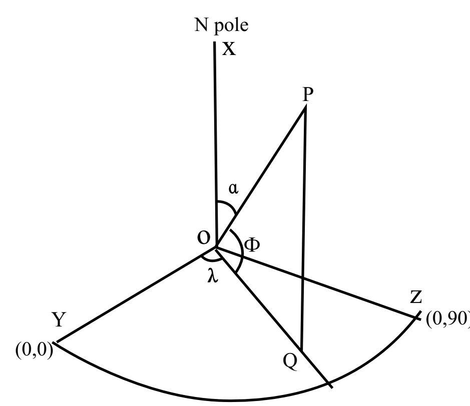


Figure 6

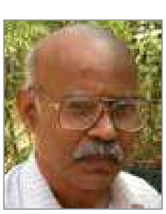
As can be seen (Figure 6) the graph of this relation is steep around the equinoxes and has low values of slope around the solstices, where the sun seems to linger, justifying the term 'solstice' (= stationary sun).

A, B, C, D stand for spring equinox, summer solstice, autumn equinox and winter solstice respectively.



The cosines of the angles made by a straight line in 3-space with the X, Y and Z axes are generally denoted by α, β and γ. In the context of the earth, we can take the centre of the earth to be the origin and the ray towards the North pole as the X axis. Then the ray towards the point on the earth's surface with latitude 0° and longitude 0° could be the Y axis, while the ray towards the point with latitude 0° and longitude 90° E could be the Z axis. Now if P is a point on the earth's surface its α value is the complement of Φ, the latitude of P. So cos α = sin Φ. To obtain cos β we multiply cos Φ with cos λ, the longitude of point P. That is, we project P on the equatorial plane to Q, and then project Q onto the Y axis. To obtain cos γ we multiply cos Φ with sin λ as the angle between OQ and the Z axis is the complement of the longitude λ. If $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are the angles made by two radii of earth with the coordinate axes respectively, and θ is the angle between the radii, we have

$\cos\theta = \sin\Phi_1 \sin\Phi_2 + \cos\Phi_1 \cos\lambda_1 \cos\Phi_2 \cos\lambda_2 + \cos\Phi_1 \sin\lambda_1 \cos\Phi_2 \sin\lambda_2$, as mentioned earlier in this article.



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Area covered by Two Intersecting Circles

$\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$

Websites and focus interest groups are a good source for interesting problems. But it's rarely that one gets down to solving these; more often they go into a to-do list. We hope that the solution presented here will encourage you to try more of these.

Look at the steps of the process: Visualisation, definition of the problem, connection to known formulas and then good old mathematical processing. Problem solved!

Here is a challenging problem in mensuration. It was posed in the LinkedIn group "Math, Math Education, Math Culture" by one of the members, Robert Lewis.

Figure 1 shows portions of two circular disks: ω, with centre $A\,(0,0)$ and radius 4, and Γ, with centre $B\,(8,8)$ and radius 8. Their boundaries intersect at points C, D.

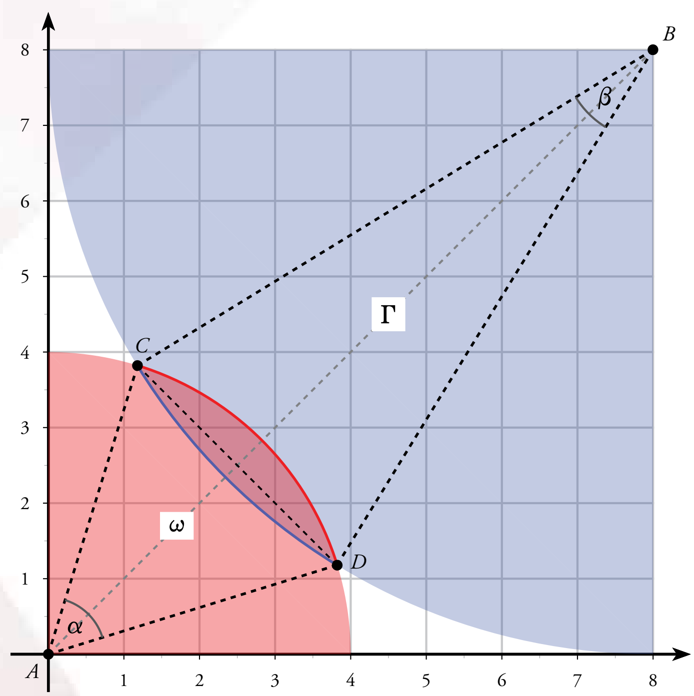

Figure 1.

Keywords: Circle, disk, lune, kite, area, cosine rule

The problem is to find the area of the region enclosed within both the disks, i.e., the area of the lune enclosed by the two circular arcs CD. (A 'lune' is any region bounded by two circular arcs, or by a line and a circular arc.)

Let $\alpha = \measuredangle CAD$ and $\beta = \measuredangle CBD$. The area of a sector with central angle θ in a circle of radius r is $\frac{1}{2}r^2\theta$ (here θ is measured in radians), and the area of an isosceles triangle with equal sides r enclosing an angle θ is $\frac{1}{2}r^2 \sin\theta$. Hence the required area is

$$\frac{4^2}{2}(\alpha - \sin\alpha) + \frac{8^2}{2}(\beta - \sin\beta)\,. \tag{1}$$

It remains to find the angles α and β and then make the necessary substitutions. Consider $\triangle BAD$. Since $AD = 4$, $BD = 8$ and $AB = 8\sqrt{2}$, the cosine rule gives:

$$\cos\frac{\alpha}{2} = \frac{4^2 + (8\sqrt{2})^2 - 8^2}{2\cdot 4\cdot 8\sqrt{2}} = \frac{5\sqrt{2}}{8},$$

$$\cos\frac{\beta}{2} = \frac{8^2 + (8\sqrt{2})^2 - 4^2}{2\cdot 8\cdot 8\sqrt{2}} = \frac{11\sqrt{2}}{16}.$$

Hence:

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = 2\left(\frac{5\sqrt{2}}{8}\right)^2 - 1 = \frac{9}{16},$$

$$\cos\beta = 2\cos^2\frac{\beta}{2} - 1 = 2\left(\frac{11\sqrt{2}}{16}\right)^2 - 1 = \frac{57}{64},$$

and:

$$\sin\alpha = \sqrt{1 - \frac{9^2}{16^2}} = \sqrt{\frac{16^2 - 9^2}{16^2}} = \sqrt{\frac{7\times 25}{16^2}} = \frac{5\sqrt{7}}{16},$$

$$\sin\beta = \sqrt{1 - \frac{57^2}{64^2}} = \sqrt{\frac{64^2 - 57^2}{64^2}} = \sqrt{\frac{7\times 121}{64^2}} = \frac{11\sqrt{7}}{64}.$$

So the area of the lune is:

$$\frac{4^2}{2}\left(\cos^{-1}\frac{9}{16} - \frac{5\sqrt{7}}{16}\right) + \frac{8^2}{2}\left(\cos^{-1}\frac{57}{64} - \frac{11\sqrt{7}}{64}\right). \tag{2}$$

To get the numerical value we need a calculator (or 'tables'). We find the area to be $1.173 + 0.555 \approx 1.728$ square units. Note that $\cos^{-1} 9/16$ and $\cos^{-1} 57/64$ must be evaluated in radians.

A variation

The proposer (Robert Lewis) next modified the problem to make it more challenging, as follows. Leaving disk ω unchanged, he replaces Γ by a disk with radius r and centre (r, r). For the two disks to intersect and the lune to have positive area, r must lie between $4(\sqrt{2}-1)$ and $4(\sqrt{2}+1)$, as can be shown by solving an appropriate pair of equations. This is a nice problem in itself, but we leave it to you to solve.

Here is his challenge:

Find some values of r for which the area of the lune is a nice quantity.

As 'nice' is not a well defined mathematical attribute, we must give it a suitable interpretation. We shall take it to mean: "expressible in terms of familiar quantities". ('Familiar' is not a mathematical attribute either! — but we mean: frequently met numbers like the integers, the rationals, numbers like $\sqrt{2}$, $\sqrt{3}$, π, e, $\ln 2$, $\ln 3$, and so on. Let us keep the notion loose, deliberately, and not attempt to make it more precise than this.) With this interpretation, we plunge in and look for suitable values of r.

We use the same notation and the same figure, except that now $B = (r, r)$ and the radius of Γ is r. Let $\alpha = \measuredangle CAD$ and $\beta = \measuredangle CBD$. The area of the lune is, as earlier:

$$\frac{4^2}{2}(\alpha - \sin\alpha) + \frac{r^2}{2}(\beta - \sin\beta). \tag{3}$$

We must find α and β. Consider $\triangle BAD$. Since $AD = 4$, $BD = r$ and $AB = r\sqrt{2}$, the cosine rule gives:

$$\begin{aligned} \cos\frac{\alpha}{2} &= \frac{4^2 + (r\sqrt{2})^2 - r^2}{2\cdot 4\cdot r\sqrt{2}} = \frac{(r^2+16)\sqrt{2}}{16r}, \\ \cos\frac{\beta}{2} &= \frac{r^2 + (r\sqrt{2})^2 - 4^2}{2\cdot r\cdot r\sqrt{2}} = \frac{(3r^2-16)\sqrt{2}}{4r^2}. \end{aligned} \tag{4}$$

From these we get expressions for $\cos\alpha$, $\cos\beta$, $\sin\alpha$, $\sin\beta$. Substituting these in turn into (3) and doing a substantial amount of simplification (but we will spare you the details), we arrive at the following expression $f(r)$ for the area of the lune:

$$\begin{aligned} &\frac{4^2}{2}\cos^{-1}\frac{r^4 - 32r^2 + 256}{64r^2} + \frac{r^2}{2}\cos^{-1}\frac{5r^4 - 96r^2 + 256}{4r^4} \\ &\quad - \frac{\sqrt{-(r^4 - 96r^2 + 256)}}{2}. \end{aligned} \tag{5}$$

For what values of r will $f(r)$ assume a form expressible in terms of known quantities? Surely, those for which the $\cos^{-1}$ term simplifies to a recognizable form. This means that we want the expressions

$$P = \frac{r^4 - 32r^2 + 256}{64r^2}, \qquad Q = \frac{5r^4 - 96r^2 + 256}{4r^4}$$

to take values like 0, $1/2$, $1/\sqrt{2}$, $\sqrt{3}/2$, $(\sqrt{5}-1)/4$ and 1, for which the inverse cosine has a recognizable form. (This list can be extended, but we have not attempted to do so.)

So now we equate each of P and Q to 0, $1/2$, $1/\sqrt{2}$, $\sqrt{3}/2$, $(\sqrt{5}-1)/4$ and 1, solve for r, and examine the outcome. At the end of this exercise (for which we omit the details; it is rather tedious!), we find only two suitable candidate values of r, both of which are obtained by solving the equation $P = 1/2$:

$$r = 2\sqrt{6} - 2\sqrt{2}, \qquad r = 2\sqrt{6} + 2\sqrt{2}.$$

Substituting these values into the expression for $f(r)$ and simplifying the expressions, here is what we get:

- If $r = 2\sqrt{6} - 2\sqrt{2}$, the area of the lune is
$$\pi\left(16 - \frac{20}{\sqrt{3}}\right) - 8.$$
- If $r = 2\sqrt{6} + 2\sqrt{3}$, the area of the lune is
$$\frac{4\left(\pi(4+\sqrt{3}) - 6\sqrt{3} - 6\right)}{3}.$$

Inasmuch as 'nice' is a term which can be interpreted in different ways, the question posed at the start is open-ended and will doubtless permit many more such answers.

Closing remark

What exactly is 'nice'? As already noted, this notion is not well defined. And yet, we do have a vague sense of its meaning. When we evaluate an integral (say), we are happier with 2 as an answer than with 2.1; we are happier with π than with 3.140; we are happier with $\sqrt{2}$ than with $\sqrt{2.1}$. Probably you will agree with the author that the above answers are nicer than the answer we got for the original problem, $\frac{4^2}{2}\left(\cos^{-1}\frac{9}{16} - \frac{5\sqrt{7}}{16}\right) + \frac{8^2}{2}\left(\cos^{-1}\frac{57}{64} - \frac{11\sqrt{7}}{64}\right)$. So, implicitly, we do seem to operate with a sense of what is nice and what is not so nice! But let us leave it at that. The point was to have a bit of fun and nothing more!

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Mobile Puzzles – Making Sense of Variables and Equations

SANGEETA GULATI

For the middle school student, the transition from arithmetic to algebra is often quite daunting. In grade VI, the concept of a 'variable' is encountered for the first time. This is the stage where either a child embraces the newly introduced 'Algebra' or gets overwhelmed with the idea of numbers being replaced by letters of the alphabet. This is also the stage where the students learn to solve equations and find the value(s) of the unknown(s). Quite often students get lost in the working of equations and memorize the rules without understanding the logic.

The equality sign "=" poses a challenge in the learning of algebra. While learning arithmetic in the primary years, children think of the equality sign as a prompt to find the answer to a mathematical sentence such as 15 + 8 = ? It is perhaps viewed as a symbol which connects the problem to the answer or an operation to be performed. However, while learning algebra, the student is baffled when she encounters a statement such as $a + b = c$, or $2x + 1 = 5$, where the "=" symbol represents an equivalence rather than a prompt to produce an answer. At this stage, children need scaffolding to move from an *operational view* of the equality sign to a *relational view*. A popular strategy to help the child develop a relational view of the equality sign and solve equations is to use the metaphor of a balance scale. In this model, equations are thought of as consisting of two sides of equal weight. Thus performing operations on each side must maintain the balance. Figure 1 represents this idea.

Keywords: Mobile, online manipulative, algebra, equation, solution

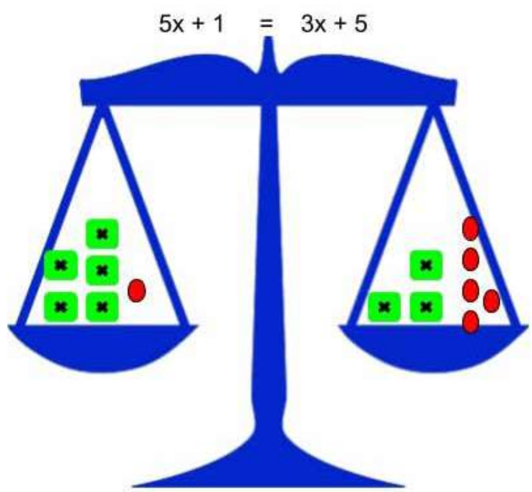

Figure 1

The website SolveMe Puzzles (http://solveme.edc.org/) uses the balance approach to enable the learner to visualize and explore the solutions of equations. It has an excellent collection of puzzles namely, Mobiles, Who Am I? and Mystery Grid. In this article we shall help the reader navigate through Mobiles ,which are colourful, interactive sculpture puzzles, designed to support algebraic reasoning in a fun and interactive manner.

The first screen or the main menu (http://solveme.edc.org/mobiles/#) provides the user with the option to '**Play**' or '**Build**' a puzzle. By clicking on '**Play**', we are taken to a screen (Figure 2), which offers three options – Explorer, Puzzler, Master; each is a collection of puzzles with increasing level of difficulty.

A mobile puzzle presents multiple balanced collections of objects. The horizontal beams are always suspended at the middle by strings and for that reason the two ends of each beam have the same amount of weight on them. Beams and strings weigh nothing and identical shapes represent the same weight whereas different shapes may have the same or different weights. The puzzler is asked to determine the unknown weight. Actually, the mobile puzzle presents a

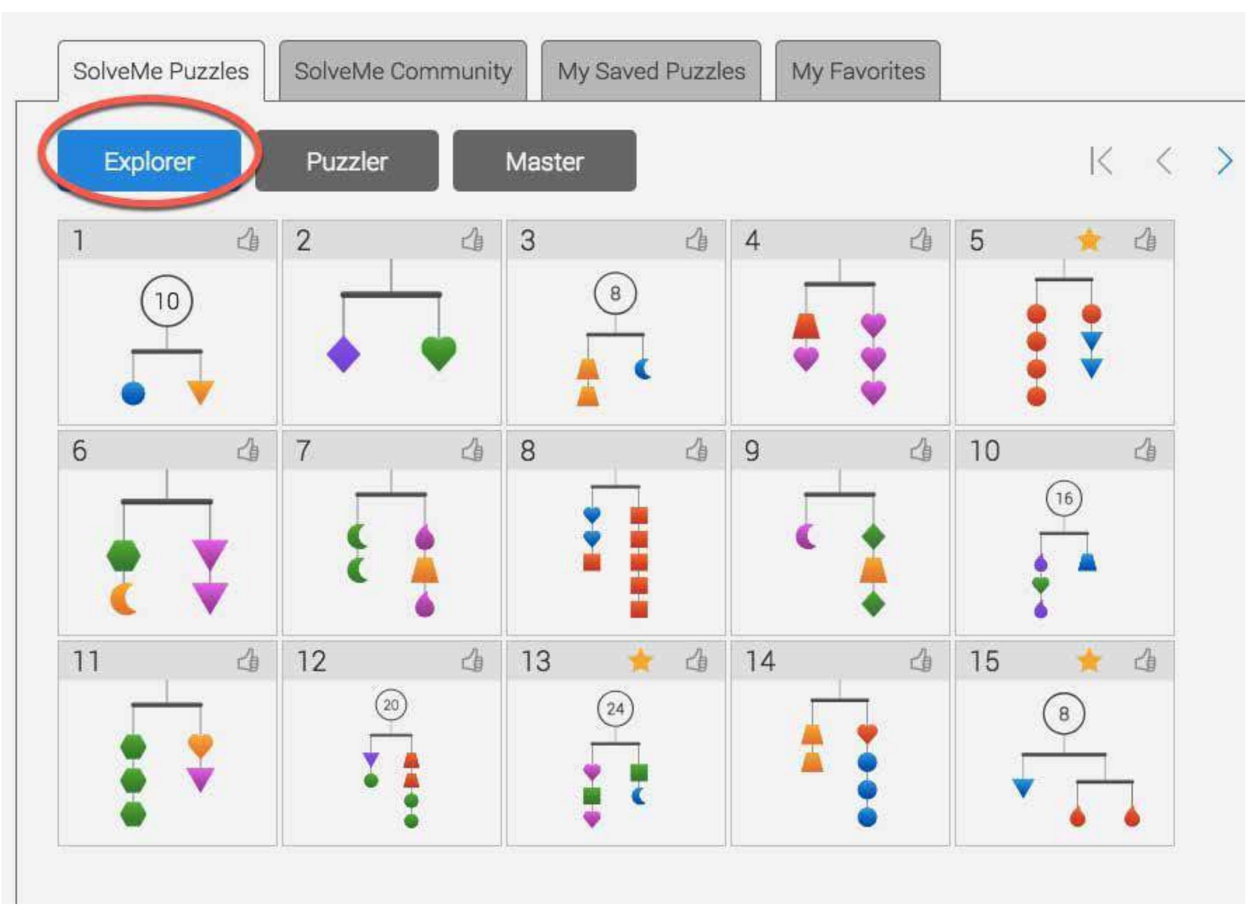

Figure 2

system of equations in the form of a picture, which highlights the underlying structure.

In 'Explorer' the initial puzzles are very simple and can serve as a 'warm-up' for the first time user. For example, Puzzle #5 (see Figure 3) shows four orange circles on the left side of the balance, each having a value equal to 4. The right side has two blue triangles and one orange circle. The learner needs to observe that removing one orange circle from each side will not disturb the balance. Once this is done, the two blue triangles on the right are equivalent to three orange circles on the left and both represent a value equal to 12. Thus, each blue triangle must represent 6. This would encourage learners to use proportional thinking to find the answer.

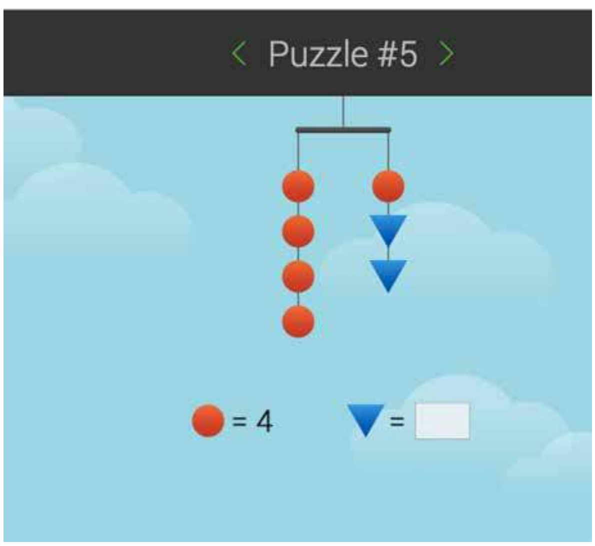

Figure 3

Other puzzles such as Puzzle #12 (Figure 4) have a circle with a number placed on top of the beam, which represents the total weight. This indicates that the values of the weights on both sides of the balance must add up to this number. Here, the weights on each side must add up to 10. Further the green circle is equivalent to 3, so this value may be substituted on both sides wherever there is a green circle. The idea of substitution, that is, replacing a shape by a number may be reinforced through such examples.

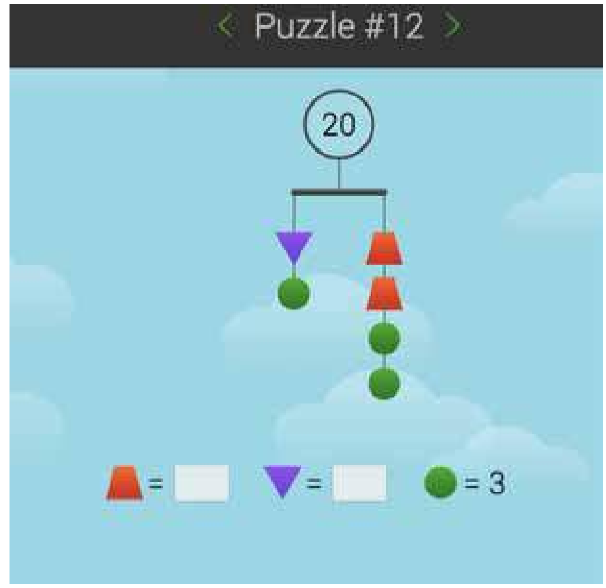

Figure 4

Some puzzles have two beams, for example, Puzzle #28(Figure 5). It has four blue triangles on the lower beam. Since this equals 8 (each blue triangle being equal to 2), the moon and the square on the right side of the upper beam must also add up to 8. Using the Pen tool (at the bottom right of screen), the user may scribble their observations (Figure 6) and calculate the values of the variables. The eraser may be used to make corrections.

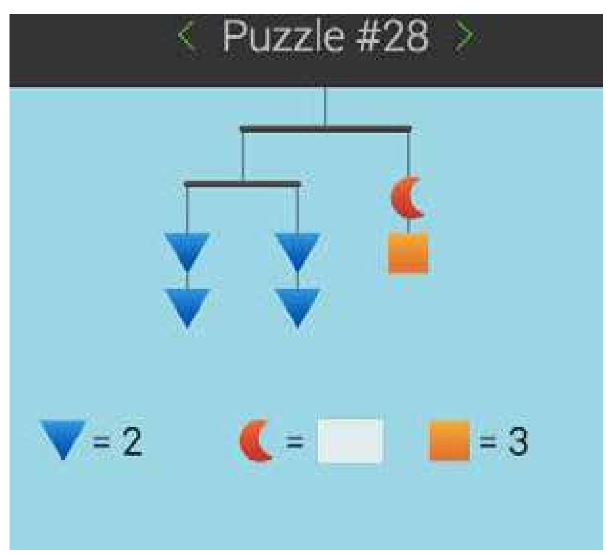

Figure 5

[1] Mobile: A mobile is essentially a hanging structure that supports baby toys and objects that stimulate and entertain the baby

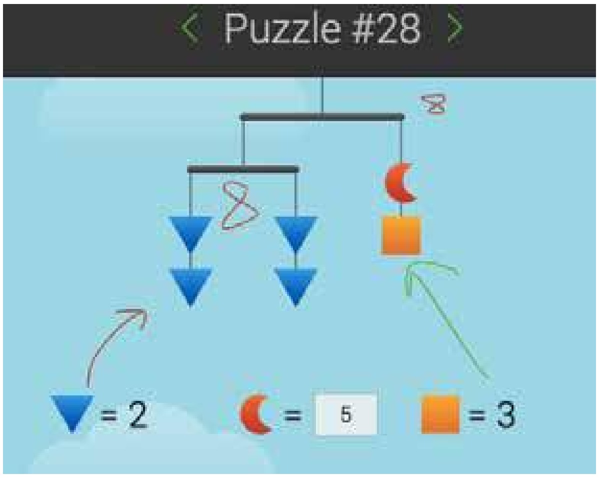

Figure 6

The sheer variety of puzzles seems to provide ample scope for practice and to build up the reasoning and strategy to solve for unknowns.

The Puzzle#28 may be used to introduce the concept of equations. Puzzle#28 may be represented by the equation: $4t = m + s$ ('t' for triangle, '*m*' for moon and '*s*' for square). We may substitute 2 for '*t*' and 3 for '*s*' to find the value of '*m*'.

The puzzles increase in complexity and become more interesting. In Puzzle#58(Figure 7), the user is required to find the values of all three types of weights, namely the '*d*' (the drop), '*c*' (the circle) and '*t*' (the triangle).

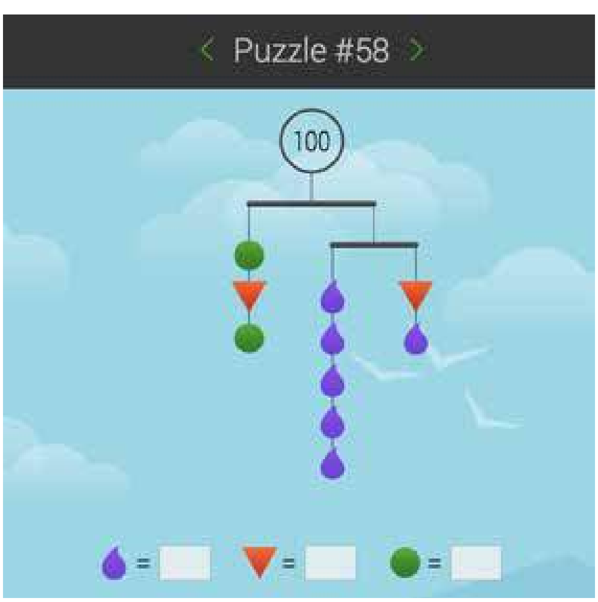

Figure 7

The only numerical value given is that of 100 in the circle. This figure translates to the following three equations

$5d = t + d$ (lower beam)

$2c + t = 6d + t$ (balancing both sides of the upper beam)

$2c + t + (6d + t) = 100$

All three equations may be used to solve the puzzle.

A middle school (or younger) student may work out such a puzzle by observing the weight of 100 equally divided into 50 each across the upper beam and further into 25 each across the lower beam. Five drops equaling 25 gives the first value; one drop (*d*) equals 5, which gives one triangle weight to be 20 leading us back to the upper beam left end giving the value of each green circle to be 15. The observation may vary from student to student and so the reasoning suggested here is one of the many possible ways to get the solution.

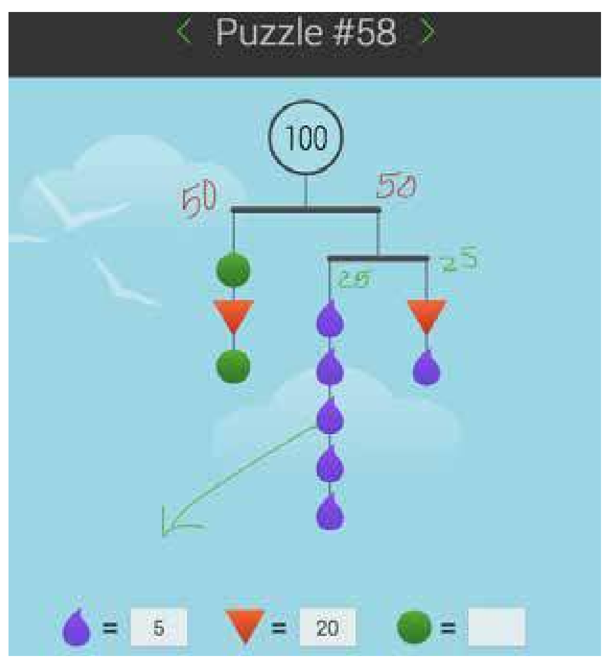

Figure 8

The logic of these puzzles reinforces the kind of reasoning required for solving systems of equations. The visual format makes these puzzles very appealing to a wide range of learners starting from middle grades to adults. Interactive features including instant response to an incorrect value (Figure 9) and a celebratory message on submission of correct values (Figure 10) keep the user engaged and motivated.

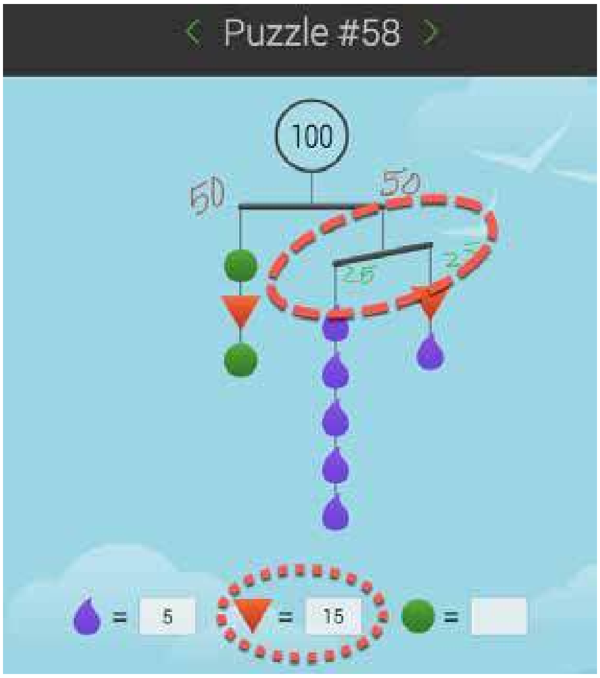

Figure 9

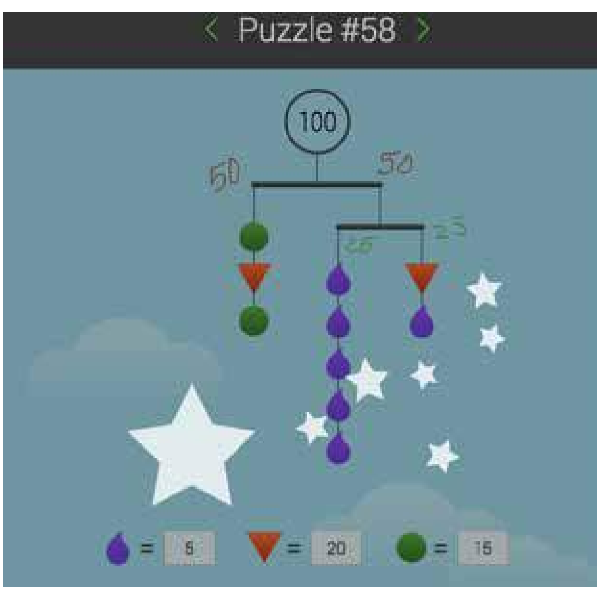

Figure 10

Now that you have experienced the 'Play' feature of SolveMe Mobiles, let us explore the "build" option which allows you and your students to create puzzles; building mobiles is a great exercise in reverse thinking. It is recommended that you create an account so that the puzzles created by you are saved and can be used at any given time. Students under 13 years of age can also create the account without using any email id.

To build your own mobile, first choose a shape (Figure 11) and then choose its colour (Figure 12).

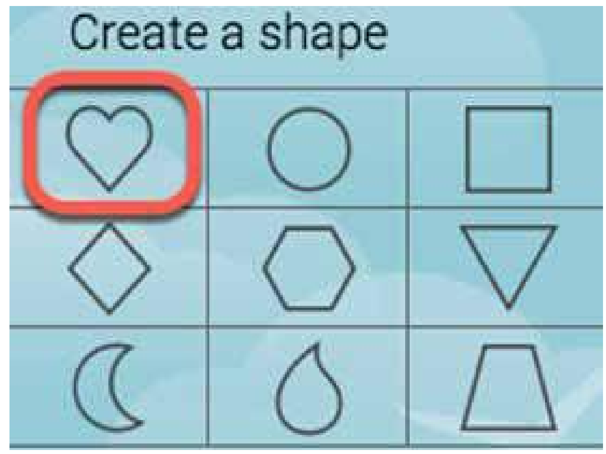

Figure 11

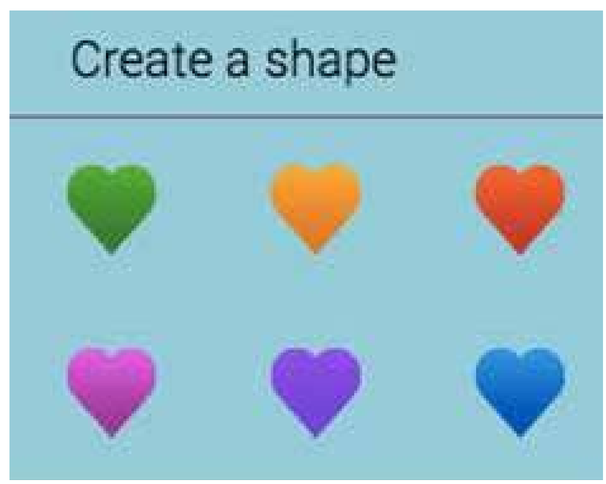

Figure 12

Then, set the weights of your shapes and drag the shapes and any other beams that you want from the Spare Parts bin onto the main screen. You can drag a total weight circle if you want to use it as a clue (Figure 13).

Now decide which clues to give the player. Which of the weights are you going to hide? (You must hide at least one to make a puzzle.) Use the toggle button to hide weights and the little "x" to remove unwanted shapes.

Figure 13

For example, the puzzle in Figure 13 shows the total weight of the mobile (26) and the weight of the orange circle (3). The weights of the purple heart and the blue drop are hidden for the puzzle-player to figure out.

Finally, you can save & play your puzzle (Figure 14) and you can share it!

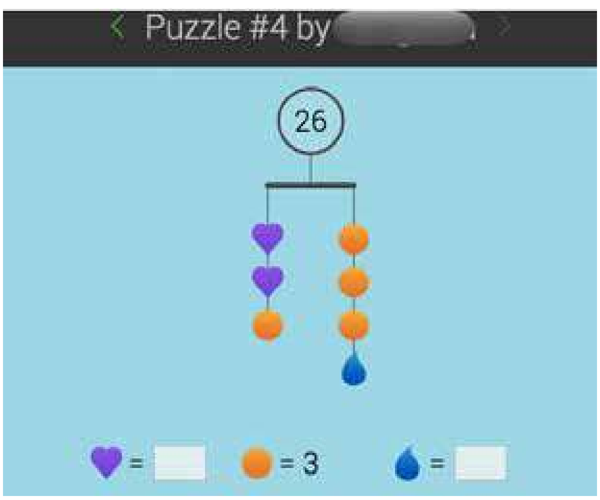

Figure 14

You can find the puzzles you have created under "My Saved Puzzles" in the Puzzle Menu.

In the Classroom: Teacher can give a certain number, which is the total weight, and students can be asked to create their own mobiles. They may work in groups or individually drawing out the mobiles on paper before recreating on the website, and they can challenge others to solve their mobiles.

We hope this article will help the teacher to design some interesting tasks on solving equations for her students. Although this method of mobile puzzles is not meant to replace the actual procedural method for solving equations, it can certainly be used as a precursor to motivate the student. Students may be asked to sketch the mobiles and then designate values to the weights/shapes leading to formation of equations. Class competitions can be organized wherein students make mobiles with multi-levels and varied number of weights/shapes and challenge the class to solve their creations.

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Problems for the MIDDLE SCHOOL

Problems Related to Divisibility

Problem Editor: A. Ramachandran

Factors and multiples, tables and long division – students who are relieved at mastering these in numbers are confounded when the same topics rear their head in algebra. Here is a nice collection of problems that allow students to play with algebraic expressions and study them through the lens of divisibility. Throughout this article, the letter n stands for a natural number. The problems are arranged in two sets; those in Set 1 can be easily attempted by students in middle school who have a reasonable comfort level with algebra. Those in Set 2 are advanced problems for which knowledge of the factor theorem is also required.

Set 1

Problem VII-3-M.1.1

Show that the expression $n(n+1)(2n+1)$ is always divisible by 6. (You may have come across this expression in a certain context, but let us not invoke that now.)

Problem VII-3-M.1.2

Show that the expression $n(n+1)(2n+1)(3n^2+3n-1)$ is always divisible by 30.

Keywords: Factors, multiples, divisibility, remainders, variables, factor theorem

Problem VII-3-M.1.3

You must have come across divisibility tests for the numbers from 2 to 10 or 12, except 7. Here is a 'divisibility test' for 7.

Consider a number of d digits. Remove the last (units) digit to get a number with $d - 1$ digits. Diminish this number by twice the digit that was removed. Now the original number is divisible by 7 if and only if the number so obtained is divisible by 7. If it is not easy to decide on the divisibility of the new number by 7, the procedure could be repeated until a convenient number is arrived at. The question is: prove that this works, that this is a reliable test.

Solutions

Problem VII-3-M.1.1

Show that the expression $n(n+1)(2n+1)$ is always divisible by 6.

Solution

Let us check this expression for a few values of n.

n	$n(n+1)(2n+1)$	Value	Multiple of 6?
1	$1 \times 2 \times 3$	6	Yes $6 = 6 \times 1$
3	$3 \times 4 \times 7$	84	Yes $84 = 6 \times 14$
13	$13 \times 14 \times 27$	4914	Yes $4914 = 6 \times 819$
20	$20 \times 21 \times 41$	17220	Yes $17220 = 6 \times 2870$

So for the values of n that we have considered, $n(n+1)(2n+1)$ is a multiple of 6. Of course, this is not a proof; for that, we need to consider the general case. For a number/expression to be divisible by 6, it must be divisible by 2 as well as by 3.

Regarding divisibility by 2, there are two possibilities:

(i) If n is even, then $n(n+1)(2n+1)$ is a multiple of 2.

(ii) If n is odd, then $(n+1)$ is even, so $n(n+1)(2n+1)$ is a multiple of 2.

Regarding divisibility by 3, there are three possibilities:

(i) n is a multiple of 3. Then clearly, $n(n+1)(2n+1)$ is a multiple of 3.

(ii) $n+1$ is a multiple of 3. Then clearly, $n(n+1)(2n+1)$ is a multiple of 3.

(iii) Neither of these is a multiple of 3. Now, any number when divided by 3 leaves a remainder of 0, 1 or 2. No other remainder is possible. (Do you see why?) If neither n nor $n+1$ is a multiple of 3, then n has to leave a remainder of 1 when divided by 3, because if it leaves a remainder of 2 then $(n+1)$ will be divisible by 3. (Consider any number which leaves a remainder of 2 when divided by 3 such as $8, 17, 23$ and you will see that the next number will then be a multiple of 3.) So n has to be of the form $3k+1$, where k is a whole number. Then $2n+1$ would equal $6k+3$, a multiple of 3.

So we see that in any case one of the three factors turns out to be a multiple of 3, ensuring that the product is a multiple of 6.

Please note that using this technique of checking the divisibility by 2 and 3 and concluding that the number is divisible by 6 if it is divisible by both 2 and 3, is possible only because 2 and 3 are relatively prime, i.e., they have no common factors. For example, checking the divisibility of a number by 12 by checking the divisibility by 6 and 2 will not work, though we can check the divisibility by 3 and 4.

Problem VII-3-M.1.2

Show that the expression $n(n+1)(2n+1)(3n^2+3n-1)$ is always divisible by 30.

Solution

This problem can build on our learning from the previous one. Since 6 and 5 are relatively prime, we check if the expression is divisible by 6 as well as by 5.

The first three factors of the given expression are common to the earlier one, thus assuring us of divisibility by 6. Now we need to ensure divisibility by 5. There are five possibilities based on the 5 possible remainders $(0, 1, 2, 3$ and $4)$ when a number is divided by 5.

(i) n is a multiple of 5. Then we have nothing more to do.

(ii) n is of the form $5k+1$, k being a whole number; substituting $5k+1$ for n in the last factor yields $75k^2+45k+5$, clearly a multiple of 5.

(iii) n is of the form $5k+2$; now the third factor turns out to be $10k+5$, a multiple of 5.

(iv) n is of the form $5k+3$; now the last factor turns out to be $75k^2+105k+35$, again a multiple of 5.

(v) n is of the form $5k+4$; now the second factor turns out to be $5k+5$.

So we see that in each case, one of the four factors is a multiple of 5; thereby the given expression is divisible by 30 for all n.

Problem VII-3-M.1.3

A test for divisibility by 7. Consider a number of d digits. Remove the last (units) digit to get a number with $d-1$ digits. Diminish this number by twice the digit that was removed.

Now the original number is divisible by 7 if and only if the number so obtained is divisible by 7. If it is not easy to decide on the divisibility of the new number by 7 the procedure could be repeated until a convenient number is arrived at.

A few worked examples to clarify matters.

Example 1. Take the number 259. Take the units digit '9'. Subtract $2 \cdot 9 = 18$ from 25 to get 7. Since this is a multiple of 7, so is 259.

Example 2. Take the number 8883. Subtract $3 \cdot 2 = 6$ from 888 to get 882. Now subtract 4 from 88 to get 84. You could also go ahead and subtract 8 from 8 to get 0, a multiple of 7. So 8883 is a multiple of 7.

Example 3. Take the number 98. Subtract $8 \cdot 2 = 16$ from 9 to get -7, a multiple of 7. So 98 is a multiple of 7.

Example 4. Take the number 1234. Subtract 8 from 123 to get 115. Now subtract 10 from 11 to get 1, not a multiple of 7. So 1234 is not a multiple of 7.

Solution
Let us denote by y the units digit of the d digit number (that we want to test for divisibility by 7) and by x the rest of the number (which would be of $d - 1$ digits) taken en bloc. As an illustration, in example 1, y is 9 and x is 25 and d (the number of digits) is 3.

Then the value of the number is $10x + y$ and the number obtained after carrying out the divisibility procedure is $x - 2y$.

$10x + y$ can be written as $10x - 20y + 21y$ or $(10x - 20y) + 21y$. Since $21y$ is divisible by 7, this number $10x - 20y + 21y$ will be divisible by 7 if and only if $10x - 20y$ is divisible by 7.

$10x - 20y = 10(x - 2y)$ and so if $(x - 2y)$ is a multiple of 7, then $10x + y$ is a multiple of 7 and vice-versa. This is the required proof.

On similar lines, here is a test of divisibility by 13. Add four times the units digit of the number under test to the rest of the number. The original number is divisible by 13 if and only if the number obtained is so. You may find it interesting to prove the same!

Set 2

Statement of Factor Theorem

If a polynomial in x reduces to zero when we substitute $x = a$, then $x - a$ is a factor of the polynomial.

For example, when we substitute $x = 1$ in $x^2 - 2x + 1$, we get $1 - 2 + 1$ which is 0. Therefore, $x - 1$ is a factor of $x^2 - 2x + 1$.

Problem VII-3-M.2.1
For what values of n is the expression

(i) $a^n + b^n$ divisible by $a + b$

(ii) $a^n + b^n$ divisible by $a - b$

(iii) $a^n - b^n$ divisible by $a + b$

(iv) $a^n - b^n$ divisible by $a - b$?

(Or, taking b to be equal to 1, for what values of n is $a^n \pm 1$ divisible by $a \pm 1$?)

Problem VII-3-M.2.2
Show that if the expression $2^k + 1$ were to be prime, k being a natural number, we should have $k = 1$, i.e., 2^0 or $k = 2^n$.

Problem VII-3-M.2.3
Show that if $2^n - 1$ were to be prime, n must be prime.

Solution to Problem VII-3-M.2.1
We invoke the factor theorem for the solution. To check for divisibility of $a^n + b^n$ by $a + b$, we need to substitute $a = -b$ in $a^n + b^n$. This yields $(-b)^n + b^n$, which can equal 0 only when n is odd.

To check for divisibility of $a^n + b^n$ by $a - b$, we need to substitute $a = b$ in $a^n + b^n$. The resulting expression can never equal 0.

To check for divisibility of $a^n - b^n$ by $a + b$, we need to substitute $a = -b$ in $a^n - b^n$, which can equal 0 only if n is even.

To check for divisibility of $a^n - b^n$ by $a - b$, we need to substitute $a = b$ in $a^n - b^n$. The resulting expression equals 0 for all values of n.

As corollaries to these we could say.

(i) $a^n + 1$ is divisible by $a + 1$ only for odd n.

(ii) $a^n + 1$ is not divisible by $a - 1$ for any n.

(iii) $a^n - 1$ is divisible by $a + 1$ only for even n.

(iv) $a^n - 1$ is divisible by $a - 1$ for all n.

Problem VII-3-M.2.2
Show that if the expression $2^k + 1$ were to be prime, k being a natural number, we should have $k = 1 (= 2^0)$ or $k = 2^n$.

Solution
If $k = 1$, the given expression takes the value 3, a prime number. So we have proved the result in this case.

If k were not of the form 2^n, then it would have at least one odd factor.

For example, if $k = 17$, then it has the odd factors 1 and 17.

If $k = 15$, then it has the odd factors 3 and 5.

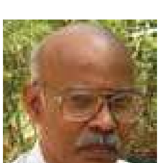
If $k = 36$, it has the odd factors 3 and 9.

So the given expression could be written as $2^k + 1 = (2^p)^q + 1$, with $k = p \times q$, and q being odd. Then by the observation of the previous problem, the given expression is divisible by $2^p + 1$. So it cannot be prime. Note that k being of the form 2^n does not imply that $2^k + 1$ is prime. In other words the converse of the problem statement is not true. For more on this theme look up 'Fermat number/prime.'

Problem VII-3-M.2.3
Show that if $2^n - 1$ were to be prime, n must be prime.

Solution
If n is not prime then it can be factored in this form:

$n = u \times v$, with $u, v \neq n, 1$, but with the possibility of $u = v$. So the given expression can be written as $2^{u \cdot v} - 1 = (2^u)^v - 1 = (2^v)^u - 1$. Then by the observation of the previous problem, this has $2^u - 1$ and $2^v - 1$ as factors. So $2^n - 1$ cannot be prime unless n itself is prime. Again note that n being prime is no guarantee that $2^n - 1$ is prime. That is, the converse of the problem statement is not true. For more on this theme look up 'Mersenne number/prime.'



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Pigeonhole Principle: Some Applications

PRITHWIJIT DE

The Pigeonhole Principle (PHP) or the Dirichlet Principle is perhaps the easiest theorem that exists in all of Mathematics. It states that if $n+1$ pigeons are put into n pigeonholes, then there is at least one pigeonhole with more than one pigeon. The proof is as easy as the statement. Assume the contrary. Then every pigeonhole has at most one pigeon and therefore the total number of pigeons is at most n. A contradiction.

It is natural to ask what is so special about something so trivial. The answer lies in the applications. The goal of this article is to serve up a delectable collection of examples of applications of PHP to the reader.

We start with a couple of simple examples.

(1) *Among any* 13 *people, there are two who have their birthdays in the same month.*

(2) *There are n married couples. How many of the* $2n$ *people must be selected in order to guarantee that one has selected a married couple?*

Think of each couple as a pigeonhole. There are n pigeonholes. If we select $n+1$ people and put each one in the pigeonhole corresponding to the couple to which they belong, then some box will contain two people; that is, we will have selected a married couple. To see that $n+1$ is the smallest, notice that n people may be selected without choosing a married couple, if only the wives or only the husbands are selected.

Keywords: Pigeonhole principle, pigeons, pigeonhole

(3) *If $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$, then there is at least one pair that differ by* 1.

The idea is to pair up the consecutive integers to form n pairs: $(2k-1, 2k)$ for $1 \leq k \leq n$ and in so doing, we reduce it to the previous example.

Try this one:

If $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$, then there is at least one pair whose greatest common divisor is 1.

Interestingly, if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$, then there always is a pair (a, b) such that either a divides b or b divides a. How do we show this? Note that any positive integer m can be written as $2^k t$ where k is a non-negative integer and t is an odd positive integer. In particular if m is odd, then $k = 0$ and $t = m$. As a consequence of this observation, we see that each positive integer $m \in \{1, 2, 3, \ldots, 2n\}$ can be written as $2^k t$ where k is a non-negative integer and $t \in \{1, 3, 5, \ldots, 2n-1\}$.

Thus there are n different admissible values of t. As $n+1$ different numbers are chosen, there are two among these with the same value of t, i.e., the same odd part. If the numbers are $a = 2^r t$ and $b = 2^s t$, then it is obvious that a divides b if $r < s$ and b divides a otherwise.

The next example may surprise you:

Given a set of m integers $\{a_1, a_2, \ldots, a_m\}$, there exists a subset of this set with the property that the sum of all its elements is divisible by m.

As a set with m elements has 2^m subsets and therefore $2^m - 1$ nonempty subsets, there are $2^m - 1$ possibilities for the subset we are looking for. For a small enough m (say $m \leq 6$), it is possible to manually enumerate all the possibilities and find the desired subset. But this method fails badly for $m \geq 7$ as the number of possibilities increases rapidly. What do we do then? Let us think a bit. What do we want? We want a subset of $\{a_1, a_2, \ldots, a_m\}$ with the desired property. Consider the m integers

$$a_1, \quad a_1 + a_2, \quad \ldots, \quad a_1 + a_2 + \cdots + a_m.$$

If any one of these sums is divisible by m, then we have our required subset. If none is divisible by m, then upon division by m, each one leaves a remainder that lies between 1 and $m-1$, both inclusive. But then we obtain m remainders, each between 1 and $m-1$. Hence by the PHP, there exist two identical remainders. If the corresponding sums are

$$a_1 + a_2 + \cdots + a_p \quad \text{and} \quad a_1 + a_2 + \cdots + a_q,$$

with $q > p$, then it follows by subtraction that the sum $a_{p+1} + a_{p+2} + \cdots + a_q$ is divisible by m; hence the desired subset is $\{a_{p+1}, a_{p+2}, \ldots, a_q\}$. Observe that the proof gives us more than what was asked: it gives us an algorithm to find the subset and quite remarkably the subset consists of some consecutive numbers in the list.

The reader may want to try the following problem. It is similar to the one discussed above.

Prove that, for any $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$, there exist two among them whose difference is divisible by n.

These were examples based on properties of numbers. Let us look at some examples from Geometry.

Five points are chosen inside a square of side length 2 units. Prove that there are two points which are at most $\sqrt{2}$ units apart.

One thing has to be made precise before we tackle this problem. By the phrase 'inside a square' we mean the interior as well as the boundary of the square. Observe that the diagonals are $2\sqrt{2}$ units long and half this length is $\sqrt{2}$. Partition the square into four identical squares by drawing two mutually perpendicular lines through the midpoints of the two pairs of opposite sides. The maximum distance between two points inside each of these squares is $\sqrt{2}$. There are 4 squares and 5 points. By PHP one of these squares will have at least two of the chosen points inside it. Here is a similar problem for the reader:

Determine an integer m such that if m points are chosen within an equilateral triangle of side length 1 *unit, there are two which are at most* $1/n$ *units apart.* (Naturally, m will depend on n, so we can write m_n or $m = f(n)$ instead of just m.)

For some applications, a slightly stronger form of PHP is used.

If n objects are placed in k boxes ,where $n = qk + r$, q and r are positive integers and $0 < r < k$, then at least one box contains more than q objects.

The truth of this statement is entirely obvious and the reader may see at once that setting $q = 1$ gives us the form of PHP that was introduced in the beginning. Let us see some applications of the strong form of PHP.

Each of the vertices of a regular pentagon is coloured either black or white. Both colours are used. Prove that there are three vertices of the pentagon which receive the same colour, and that these form an isosceles triangle.

One quickly observes that $n = 5$, $k = 2$ and as $5 = 2 \times 2 + 1$, $q = 2$, and concludes that some three vertices must receive the same colour. Let the vertices be named A, B, and C. Now two cases arise.

Case 1: *The vertices A, B and C are adjacent.*

In this case, it is easy to see that $AB = BC$ and hence the triangle ABC is isosceles.

Case 2: *The vertices A, B are adjacent and C is opposite AB.*

In this case $CA = CB$, so the triangle is isosceles.

Another colouring problem in the same spirit.

Consider six points in the plane such that no three of them are collinear. Join every pair of points and colour every edge thus obtained either red or blue. If both colours are used, then there must be a triangle whose sides are either all red or all blue.

Six points joined pairwise give rise to $\binom{6}{2} = 15$ edges. There are two colours. By the strong form of PHP there are at least 8 edges of one colour. But it doesn't lead anywhere. There exist colourings of 8 edges with the same colour without forming a triangle. In fact one can colour 9 of these 15 edges without obtaining a triangle with sides of one colour. What do we do now? The argument is slightly tricky. Consider one of the six points and name it A. It is connected to the remaining five points, B, C, D, E and F (say). Each of these five edges receives one of the two colours: red or blue. Thus $n = 5$ and $k = 2$ giving $q = 2$ and by PHP there are at least three edges among these five with the same colour. Without loss of generality assume that the edges AB, AC and AD are coloured red. Look at the triangle BCD. If all its edges are blue, we have found a 'blue triangle'. If one of the sides of triangle BCD is red, say BC is red, then we have found a 'red triangle' in ABC. The reader may verify that the smallest number of points required to ensure the existence of a monochromatic triangle is six by conjuring up counterexamples for $n = 3, 4, 5$. The next example involves a pyramid.

The base of a pyramid is a convex polygon with 9 sides. Each of the diagonals of the base and each of the edges on the lateral surface of the pyramid is coloured either black or white. Both colours are used. (Note that the sides of the base are not coloured.) Prove that there is a monochromatic triangle.

The reader may like to have a go at this. Here is another interesting example. In this example, we use upper case letters to denote vertices of a polygon and lower case letters to denote the numbers associated with the vertices. Thus, A is a vertex and a is the number associated with it, etc.

Consider a regular polygon with 100 vertices. To each vertex a natural number from the set $\{1, 2, 3, \ldots, 49\}$ is assigned. Prove that there are four vertices A, B, C and D which form a parallelogram ABCD and for which $a + b = c + d$.

A direct application of the strong form of PHP tells us that there is some number that is assigned to at least 3 vertices. But unfortunately this does not lead us anywhere closer to the solution. The beauty of this problem lies in the following simple geometric fact: **The chord joining a given vertex to its farthest neighbour is a diameter of the circumscribing circle.** But how many such diameters are there? 50. If PQ is a diameter, then $0 \leq |p - q| \leq 48$. Thus the difference can take 49 different values. Therefore by PHP there exist at least two diameters for which the absolute values of the difference between the numbers at the endpoints are same. If AC and BD are two such diameters, then $|a - c| = |b - d|$. Without loss of generality we may assume $a \geq c$ and $d \geq b$ to obtain $a + b = c + d$. Note that $ABCD$ is in fact a rectangle.

We now turn to a problem from Algebra which involves PHP. This was asked in the Regional Mathematical Olympiad 2017 conducted in the Maharashtra and Goa region.

Let $P(x)$ and $Q(x)$ be polynomials of degree 6 and degree 3 respectively, such that

$$P(x) > Q(x)^2 + Q(x) + x^2 - 6$$

for all real values of x.

If all the roots of $P(x)$ are real numbers, then prove that there exist two roots of $P(x)$, say α, β, such that $|\alpha - \beta| < 1$.

It is not clear at the outset how this problem is related to PHP. More often than not, Olympiad problems are like that! One wouldn't know right away what will work and what won't. There is an element of surprise in most of the Olympiad problems and this problem is no exception. Let us see how to solve it. However, please bear in mind that the solution presented here need not be the only possible way of solving the problem.

First observe that if $P(x)$ has two identical roots, then the conclusion is obvious. Therefore assume that the roots are distinct. Next a bit of algebraic manipulation is carried out to write the right hand side of the inequality as

$$(Q(x) + 1/2)^2 + x^2 - (5/2)^2.$$

Now observe that the given inequality holds for all real values of x. Thus, in particular, it holds for the roots of $P(x)$. Therefore if u is a root of $P(x)$, then

$$0 = P(u) > (Q(u) + 1/2)^2 + u^2 - (5/2)^2,$$

and it follows that $|u| \leq 5/2$. In other words $u \in [-5/2, 5/2]$. Thus all six roots lie in an interval of length 5 units. That's it. Now its over to PHP. Divide the interval into five subintervals of unit length. These are the pigeonholes and the roots are the pigeons. It is evident that there are two or more pigeons in one hole. Thus there exist roots α, β of $P(x)$ such that $|\alpha - \beta| < 1$.

We end this article with a gem of an example. This can be 'experimentally verified' by the reader in real life.

Suppose that $n^2 + 1$ *people are lined up shoulder to shoulder in a straight line. Then it is always possible to choose* $n + 1$ *people from the line to take one step forward so that going from left to right their heights are either non-decreasing or non-increasing.*

The reader may verify this assertion for small values of n and then build up an argument leading to the proof of the statement.

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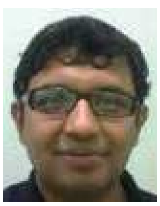

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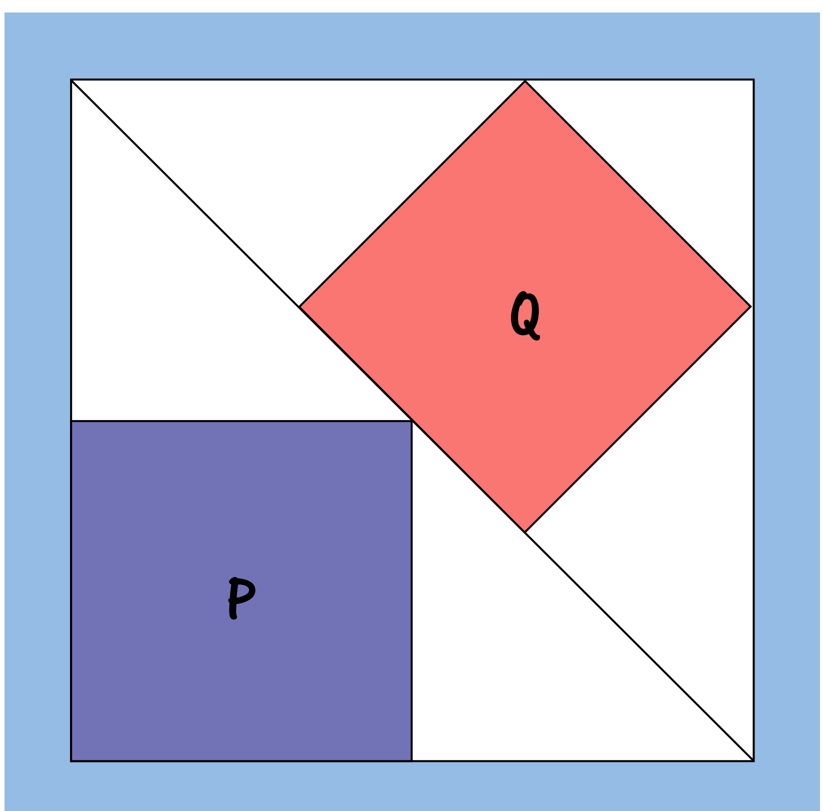
Find the ratio of the areas P : Q



Send in your solutions to AtRiA.editor@apu.edu.in

Problem Corner

Adventures in PROBLEM SOLVING

SHAILESH SHIRALI

In this edition of 'Adventures' we study a few miscellaneous problems, mostly from the Pre-Regional Mathematics Olympiad (PRMO; this year's PRMO was conducted on August 19 in centres all over the country). As usual, we pose the problems first and present solutions later.

Miscellaneous problems

Problem 1. Consider all 6-digit numbers of the form $abccba$ where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7. (Problem 3 of PRMO 2018)

Problem 2. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$. (Problem 10 of PRMO 2018)

Problem 3. If $a, b, c \geq 4$ are integers, not all equal, and $4abc = (a+3)(b+3)(c+3)$, then what is the value of $a+b+c$? (Problem 18 of PRMO 2018)

Problem 4. A positive integer k is said to be 'good' if there exists a partition of the set $\{1, 2, 3, \ldots, 20\}$ into disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there? (Problem 22 of PRMO 2018)

Problem 5. Find all prime numbers p such that $\frac{1}{p}\left(2^{p-1} - 1\right)$ is a perfect square. (Problem posed on the Math Stack Exchange website)

Keywords: Place value, digits, divisibility, triangles, medians, subsets, partitions

Solutions to the problems

Solution to problem 1

Let n be a 6-digit number of the form *abccba*; then

$$n = 10^5 a + 10^4 b + 10^3 c + 10^2 c + 10b + a = 100001a + 10010b + 1100c.$$

Take remainders modulo 7. We get $n \equiv 6a + c \pmod 7 \equiv c - a \pmod 7$. So, for n to be a multiple of 7, we must have $a \equiv c \pmod 7$. Note that the value of b does not affect divisibility of n by 7. Moreover, we must have $a > 0$, as it is the leading digit of the number. We now list the possibilities.

- $a = 1$. Since $a \equiv c \pmod 7$, the possible values of c are $1, 8$ (2 choices). There are 5 choices for b (namely, $1, 3, 5, 7, 9$). This yields 10 possibilities.
- $a = 2$. The possible values of c are $2, 9$ (2 choices). With 5 choices for b, this yields 10 possibilities.
- $a = 3$. The only possible value of c is 3. This yields 5 possibilities.
- $a = 4$. The only possible value of c is 4. This yields 5 possibilities.
- $a = 5$. The only possible value of c is 5. This yields 5 possibilities.
- $a = 6$. The only possible value of c is 6. This yields 5 possibilities.
- $a = 7$. The possible values of c are $0, 7$ (2 choices). This yields 10 possibilities.
- $a = 8$. The possible values of c are $1, 8$ (2 choices). This yields 10 possibilities.
- $a = 9$. The possible values of c are $2, 9$ (2 choices). This yields 10 possibilities.

Hence the total number of possibilities is $5 \times 10 + 4 \times 5 = 70$.

Solution to problem 2

Here we make use of (i) the theorem of Pythagoras; (ii) the fact that the circumcentre of the right-angled triangle lies at the midpoint of its hypotenuse; (iii) the fact that the point of intersection of two medians of a triangle is a point of trisection of each median.

Let $BG = 2x$, $CG = 2y$; then $GE = x$, $GF = y$. Also

$$BC^2 = BG^2 + CG^2 = 4\left(x^2 + y^2\right).$$

We are told that $AD = 30$. Hence $GD = 10$. Since triangle BGC is right-angled at G, its circumcentre lies at the midpoint of its hypotenuse, i.e., at D. It follows that $DB = 10$, and therefore that $BC = 20$. Combining this fact with what we deduced above, we see that

$$x^2 + y^2 = 100.$$

From the right-angled triangles BGF and CGE, we obtain

$$BF^2 = BG^2 + FG^2 = 4x^2 + y^2,$$

$$CE^2 = CG^2 + EG^2 = x^2 + 4y^2.$$

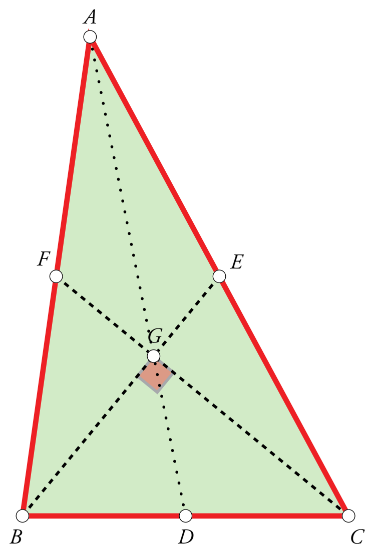


Figure 1.

Hence

$$AB^2 + AC^2 = 4\left(5x^2 + 5y^2\right) = 20\left(x^2 + y^2\right) = 2000,$$

and so

$$AB^2 + AC^2 + BC^2 = 2400.$$

Therefore the required answer is $2400/100 = 24$.

Solution to problem 3

Given that $a, b, c \geq 4$ are integers and

$$4abc = (a+3)(b+3)(c+3),$$

to find the value of $a + b + c$. The problem additionally requires that a, b, c are not all equal. However, this seems an unnecessary requirement; for if $a = b = c$, then we would obtain $4a^3 = (a+3)^3$, implying that the cube root of 4 is a rational number, which is not the case. Hence a, b, c cannot all be equal. Without any loss of generality, we may assume that $a \leq b \leq c$. From the given equation we obtain:

$$4 = \left(1 + \frac{3}{a}\right)\left(1 + \frac{3}{b}\right)\left(1 + \frac{3}{c}\right),$$

therefore

$$\left(1 + \frac{3}{c}\right)^3 \leq 4 \leq \left(1 + \frac{3}{a}\right)^3.$$

Solving these inequalities individually for a and c, we obtain

$$a \leq 5.1 \leq c,$$

so $a \leq 5$ and $c \geq 6$. Therefore $a \in \{1, 2, 3, 4, 5\}$. As we have also been told that $a \geq 4$, it follows that $a \in \{4, 5\}$. We consider both these possibilities.

- If $a = 4$, the given equation leads to $16bc = 7(b+3)(c+3)$. This may be rewritten as $9bc - 21(b+c) = 63$, which yields $(3b-7)(3c-7) = 112$. It follows that the pair $(3b-7, 3c-7)$ is one of the following possibilities:

$$(1, 112), \quad (2, 56), \quad (4, 28), \quad (7, 16), \quad (8, 14).$$

For b, c to assume integer values, both factors must be of the form 2 (mod 3), hence the pair $(3b-7, 3c-7)$ must be either $(2, 56)$ or $(8, 14)$. Hence the triple (a, b, c) is one of the following:

$$(4, 3, 21), \quad (4, 5, 7).$$

Of these, the first possibility need not be listed as we had supposed that $a \leq b$.

- If $a = 5$, the given equation leads to $bc - 2b - 2c - 6 = 0$. This may be rewritten as $(b-2)(c-2) = 10$. It follows that the pair $(b-2, c-2)$ is one of the following possibilities:

$$(1, 10), \quad (2, 5),$$

and therefore that the triple (a, b, c) is one of the following:

$$(5, 3, 12), \quad (5, 4, 7).$$

Neither of these possibilities needs to be listed as we had supposed that $a \leq b \leq 4$.

Hence the only triple (a, b, c) which satisfies the relations

$$4 \leq a \leq b \leq c, \quad 4abc = (a+3)(b+3)(c+3)$$

is $(4, 5, 7)$. This yields $a + b + c = 16$.

Solution to problem 4

A positive integer k is said to be 'good' if there exists a partition of the set $\{1, 2, 3, \ldots, 20\}$ into disjoint proper subsets such that the sum of the numbers in each subset of the partition is k; to find all the good numbers. Clearly, any such k must be a proper divisor of 210, and since 20 itself must belong to some subset, we must also have $k \geq 20$. These two requirements yield six possible values of k:

$$21, \ 30, \ 35, \ 42, \ 70, \ 105.$$

We examine each of these six possibilities for feasibility.

- $k = 21$ is feasible as we can form the following 10 two-element subsets: $\{1, 20\}$, $\{2, 19\}$, $\{3, 18\}$, $\{4, 17\}$, …, $\{8, 13\}$, $\{9, 12\}$ and $\{10, 11\}$, each with sum 21.
- $k = 30$ is feasible as we can form the following 7 subsets (note that they are not all of the same size): $\{10, 20\}$, $\{11, 19\}$, $\{12, 18\}$, $\{13, 17\}$, $\{14, 16\}$, $\{4, 5, 6, 15\}$ and $\{1, 2, 3, 7, 8, 9\}$, each with sum 30.
- $k = 35$ is feasible as we can form the following 6 subsets: $\{15, 20\}$, $\{16, 19\}$, $\{17, 18\}$, $\{1, 2, 3, 4, 5, 6, 14\}$, $\{7, 8, 9, 11\}$ and $\{10, 12, 13\}$, each with sum 35.
- $k = 42$ is feasible as we can form the following 5 subsets: $\{20, 19, 3\}$, $\{18, 17, 7\}$, $\{16, 15, 11\}$, $\{14, 13, 12, 2, 1\}$ and $\{10, 9, 8, 6, 5, 4\}$, each with sum 42.
- $k = 70$ is feasible as we can form the following: $\{20, 19, 18, 13\}$, $\{17, 16, 15, 14, 8\}$ and $\{12, 11, 10, 9, 7, 6, 5, 4, 3, 2, 1\}$, i.e., 3 subsets, each with sum 70.
- $k = 105$ is feasible: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ and the complement set $\{15, 16, 17, 18, 19, 20\}$ each have sum 105.

We see that all six possibilities yield partitions of the required type. Note that we had to resort to ad hoc methods to find these partitions. In general, the problem of finding such partitions has no easy solution, requiring a great amount of computational work.

Solution to problem 5

To find all prime numbers p such that $\frac{1}{p}\left(2^{p-1}-1\right)$ is a perfect square. See [1].

The 'little theorem' of Fermat assures us that if p is any odd prime number, then the quantity

$$\frac{1}{p}\left(2^{p-1}-1\right)$$

is an integer. Here are the values taken by this expression for the first few odd primes p:

p	3	5	7	11	13	17	...
$(2^{p-1}-1)/p$	1	3	9	93	315	3855	...

The only square numbers we spot in the second row are 1 and 9. We shall show that these are in fact the only square numbers possible, i.e., the only primes p for which $\frac{1}{p}\left(2^{p-1}-1\right)$ is a perfect square are $p=3$ and $p=7$.

The stated condition implies that p is odd. Let $p=2k+1$, where k is a positive integer. Then $2^{p-1}-1=2^{2k}-1=\left(2^k-1\right)\left(2^k+1\right)$, so

$$\frac{2^{p-1}-1}{p}=\frac{\left(2^k-1\right)\left(2^k+1\right)}{p}.$$

Now p can be a divisor of only one of 2^k-1, 2^k+1, as these two quantities cannot share any common factor greater than 1. We consider both the possibilities.

- Suppose that p is a divisor of 2^k-1. Then the quantities $\left(2^k-1\right)/p$ and 2^k+1 are coprime (indeed, the quantities 2^k-1 and 2^k+1 themselves are coprime, being consecutive odd numbers), and as their product is a perfect square, each of them must be a perfect square. That is, we must have for some integers a, b,

  $$\frac{2^k-1}{p}=a^2, \quad 2^k+1=b^2.$$

  The second equality yields $b^2-1=2^k$, hence $(b-1)(b+1)=2^k$. This implies that $b-1$ and $b+1$ are both powers of 2. Moreover, we also have $(b+1)-(b-1)=2$. But the only two powers of 2 that differ by 2 are $2^2=4$ and $2^1=2$. Hence it must be that $b+1=4$, i.e., $b=3$, which yields $k=3$. Hence $2^k-1=7$, which tells us that $p=7$. So this possibility yields just one prime number, namely $p=7$.

- Suppose that p is a divisor of 2^k+1. Then the quantities $\left(2^k+1\right)/p$ and 2^k-1 are coprime, and as their product is a perfect square, each of them must be a perfect square. That is, we must have for some integers a, b,

  $$\frac{2^k+1}{p}=a^2, \quad 2^k-1=b^2.$$

  The second equality yields $2^k=b^2+1$. As the quantity on the left side is even, b must be odd, hence $b^2\equiv 1 \pmod 4$, therefore $b^2+1\equiv 2 \pmod 4$. This implies that $2^k\equiv 2 \pmod 4$. The only positive integer k for which this is true is $k=1$. (If $k\geq 2$, then $2^k\equiv 0 \pmod 4$.) Hence $k=1$, so p is a divisor of $2^1+1=3$. Hence $p=3$. So this possibility too yields just one prime number, namely $p=3$.

So there are just two prime numbers for which the stated condition is true: 3 and 7.

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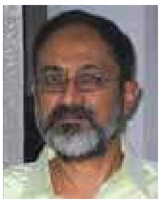

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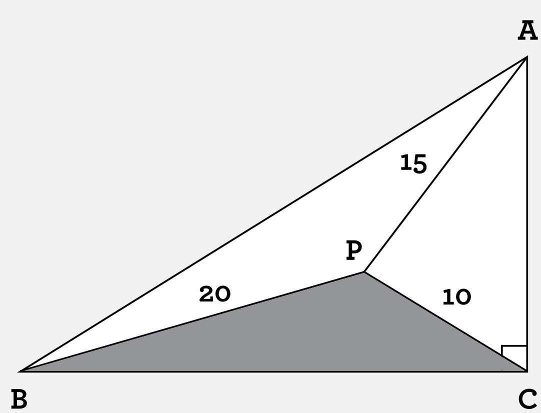
Find the Area of **the Shaded Region** A(PBC)

$AC \perp BC$, $|BC| = 2|AC|$, $|BP| = 20$ cm,

$|AP| = 15$ cm, $|CP| = 10$ cm



Mathematics Olympiads in India

PHOOLAN PRASAD

The history of Mathematical Olympiad (MO) activity in India is not available anywhere. Hence, while recording this history, we also mention the people who initiated and nurtured this activity. However, before we talk about it, we first highlight a few aspects of the International Mathematical Olympiad (IMO).

The International Mathematical Olympiad – an overview

The International Mathematical Olympiad (IMO for short) is a major event in the world mathematical scene today, featuring close to 100 different countries, but it began on a very small note. The first IMO was held in 1959 when Romania invited a small number of countries from what was then known as the Eastern Bloc to participate in the event. The following seven countries participated in this IMO: Romania, Hungary, Czechoslovakia, Bulgaria, Poland, Union of Soviet Socialist Republics (USSR, more commonly known as Soviet Union; the Union ceased to exist in 1991) and the German Democratic Republic (more commonly known as East Germany; note that West Germany and East Germany united in 1989).

Here is a quote from the official IMO website, [5]: "The International Mathematical Olympiad (IMO) is the World Championship Mathematics Competition for High School students and is held annually in a different country. The first IMO was held in 1959 in Romania, with 7 countries participating. It has gradually expanded to over 100 countries from 5 continents. The IMO Board ensures that the competition takes place each year and that each host country observes the regulations and traditions of the IMO."

Keywords: International Mathematical Olympiad, IMO, NBHM, HBCSE, AMTI, DAE

And here is a quote from the Wikipedia website, [9]: "The IMO examination consists of six problems. Each problem is worth seven points, so the maximum total score is 42 points. No calculators are allowed. The examination is held over two consecutive days; each day the contestants have four-and-a-half hours to solve three problems. The problems chosen are from various areas of secondary school mathematics, broadly classifiable as geometry, number theory, algebra, and combinatorics. They require no knowledge of higher mathematics such as calculus and analysis, and solutions are often short and elementary. However, they are usually disguised so as to make the solutions difficult. Prominently featured are algebraic inequalities, complex numbers, and construction-oriented geometrical problems, though in recent years the latter has not been as popular as before."

One of the extremely attractive traditions of the IMO is the procedure for selection of the examination problems. Participating countries are requested to send problem proposals to the host country a few months in advance of the IMO. It is expected that these proposals are original and have been kept confidential; in particular, it is expected that they have not been seen by the students participating from that country. Generally, 100 or more proposals are received by the host country, which then prepares from this large collection a shortlist of 30 problems. This is done by the Problem Committee set up by the host country. Needless to say, the task of preparing the shortlist is complex and requires the problem committee to intensively study the proposals that have been received (and to find their own solutions to these problems – no mean task).

Each participating team consists of a Team Leader, a Deputy Team Leader and up to 6 student participants. The team leaders arrive at the IMO venue a few days in advance of the IMO and all leaders together form the Jury, which selects six problems from the shortlist after a long discussion spread over two full days. After the problems have been selected, the Jury further subdivides them into two groups of three problems each (each set of three problems to be done at one sitting; two sittings are required). The wordings of the problems are examined carefully to avoid ambiguities and misinterpretations. Translations into numerous languages are also done at this point in time. By tradition, the three problems on each day include one that is regarded as 'fairly easy', one that is 'moderately difficult', and one that is 'extremely difficult.' (Of course, these terms are relative! It has happened at times that the Jury has completely underestimated the level of difficulty of a problem.)

The Deputy Leader arrives at the IMO venue later with the students and he/she and the participants stay separately from the Jury, without any communication with the leaders. This is important from the standpoint of security and confidentiality.

There is an elaborate but well-worked out procedure for evaluation of the scripts and award of grades. Needless to say, this is an exhausting period of time for the Team Leaders and the Deputy Team Leaders. By tradition, each problem is created out of 7 points, so the maximum score possible for any individual participant is 42 points.

Officially, the IMO is not a team event (though unofficial tallies of team scores are maintained; in this article, we shall not refer to these unofficial team rankings at all), and participants are ranked based on their individual scores. Medals are awarded to the highest ranked participants; slightly fewer than half of them receive a medal. The cutoffs (minimum scores required to receive a gold, silver or bronze medal respectively) are chosen so that the numbers of gold, silver and bronze medals awarded are approximately in the ratio 1 : 2 : 3. Participants who do not win a medal but who score 7 points on at least one problem receive an 'Honourable Mention.' Special prizes may be awarded for solutions of outstanding elegance or involving good generalisations of a problem. The details are available at [2].

Concerning IMO medalists, [10] notes the following: "A number of IMO medalists have gone on to become notable mathematicians. Some IMO participants have either received a Fields Medal, a Wolf Prize or a Clay Research Award, awards which recognise groundbreaking research in mathematics; a European Mathematical Society Prize, an award which recognizes young researchers; or one of the American Mathematical Society's awards (a Blumenthal Award in Pure Mathematics, Bôcher Memorial Prize in Analysis, Cole Prize in Algebra, Cole Prize in Number Theory or Veblen Prize in Geometry and Topology) recognizing research in specific mathematical fields."

A few IMO medalists have also gone on to become notable computer scientists, receiving prizes such as the Nevanlinna Prize, the Knuth Prize and so on (awards which recognise outstanding research in theoretical computer science).

Occasionally, the IMO features some exceptionally young participants. Just to mention two such individuals: Terence Tao (Australia), who won a gold medal in 1988, at age 13 years (he also won a bronze medal in 1986 and a silver medal in 1987, at age 11 years); Akshay Venkatesh (Australia) who won a bronze medal in 1994, at age 12 years. Both these mathematicians were awarded Fields medals (in 2006 and 2018 respectively).

Mention may be made of Ciprian Manolescu, the only person to achieve three perfect scores—at IMO 1995 (Canada), IMO 1996 (India) and IMO 1997 (Argentina). He is now a Professor of Mathematics at the University of California, Los Angeles.

To see how various countries have performed over the years, please see [6]. To see how India has performed over the years, please see [3].

IMO does not give much importance to the names of the leaders and deputy leaders at its site. However, the names of leaders and deputy leaders are available for any particular year; see [4]. By changing the year in this link, we can get information for other years.

Mathematical Olympiad activity in India

Now we come to Mathematics Olympiad activity in India. In 1968, only 12 countries participated in IMO, mostly from eastern Europe and just three other countries: Italy, Sweden and UK. But there was a visionary in India, P. L. Bhatnagar (at the Indian Institute of Science (IISc), Bangalore) or PLB as he was known, who organised the first ever Mathematical Olympiad (MO) in India for the students of Bangalore in January 1968 under the auspices of the Bangalore Mathematical Association. In the first two such MOs (the second was held in December 1968 at Bangalore, Mysore, Dharwar, Gulbarga and Mangalore), only one candidate qualified for an Olympiad prize. PLB left IISc in 1969 but as the President of the Association of the Mathematics Teachers of India (AMTI), he continued organising MO activities in various cities in India and AMTI continues to hold MO activities at the national level even today.

Some members (led by Phoolan Prasad, a Ph.D. student of PLB) of the Department of Applied Mathematics of IISc again started MO in 1978 for students of Bangalore city and then continued this activity. It became an official activity of IISc through its Centre of Continuing Education. A highlight of this activity was a one-day lecture program on the culture of mathematics for those who qualified in the MO examination. Many distinguished mathematicians and scientists from IISc and outside Bangalore (including S. Ramaseshan, Director of IISc, a physicist and T. Desiraju, a neuroscientist at NIMHANS) were invited to talk on mathematics and on mathematical sciences. Meanwhile, the first Ph.D. student of PLB, J. N. Kapur (then a member of the National Board of Higher Mathematics (NBHM), DAE, Govt. of India) proposed that NBHM should organise MO for participation in IMO. The MO lecture program of IISc had caught the attention of NBHM and NBHM requested IISc to conduct training program for the IMO, which started at IISc in 1986. Many enthusiastic mathematicians at IISc and some invitees like S. A. Shirali took part in the training program. M. S. Raghunathan (Chairman of NBHM), Phoolan Prasad (Coordinator, School Committee, NBHM), Izhar Husain, A. M. Vaidya and R. Subramanian played important roles in organising MO in India and IMO training program.

The IMO training camp in India started at IISc from 1986 and continued till 1993. In 1994 the training camp shifted to the Bhabha Atomic Research Centre (BARC) campus in Trombay, Mumbai. At present, all MO activities are organized by the Homi Bhabha Centre for Science Education (HBCSE) on behalf of NBHM; see [1].

India first participated in an IMO in 1989 at Braunschweig in West Germany (unification with East Germany had not yet happened) and the Indian team earned 4 silver medals and 1 Honourable Mention (HM). Till this time, IMO training camps at IISc were organised by mathematicians at IISc and a few others from outside (including C. R. Pranesachar, B. J. Venkatachala, both former students of IISc, and C. S. Yogananda) who were deeply interested in mathematical problem-solving. S. A. Shirali continued participating in the training camps. In 1990, NBHM felt a need to appoint permanent faculty for the work of organising MO activity and teaching at the IMO training camps. On request from NBHM, IISc set up a Mathematics Olympiad Cell in 1991 at the Department of Mathematics, which continues to function at IISc. Three NBHM Teacher Fellows were appointed in the Olympiad cell during 1991 and 1993.

Under involvement from NBHM and faculty of IISc, MO activity in India and the IMO training program (with about 10 mathematicians joining as resource persons from various parts of India) grew in intensity and enthusiasm since 1989. Several medalists from India have gone on to become mathematicians and computer scientists across the world. Mention should be made here of K Soundararajan, who won a silver medal in IMO 1991 (Sweden) and who has gone on to do outstanding work in analytic number theory (see [7]); and Subhash Khot, who won silver medals in IMO 1994 (Hong Kong) and IMO 1995 (Canada) and who was awarded the Rolf Nevanlinna Prize in 2014 for outstanding work in theoretical computer science; see [8].

A notable event in the history of Olympiad activity in India was the hosting of IMO 1996 in Mumbai, under the chairmanship of A M Vaidya. It is important to point out that for the mathematicians who worked with dedication for the IMO activity in India, winning medals at the IMO was never the primary aim; rather, the aim was to draw attention to the need for quality mathematics education at the school level. This remains an urgent need today.

A look at how IMO problems have evolved over the years

As mentioned earlier, the first IMO was held in 1959. Looking back, the level of the problems posed in those early years seems impossibly low in comparison with the level of the problems posed these days. We list some problems for comparison.

IMO 1959, Problem 1. Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number n.

IMO 1959, Problem 4. Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.

IMO 1970, Problem 1. Let M be a point on the side AB of $\triangle ABC$. Let r_1, r_2 and r be the radii of the inscribed circles of triangles AMC, BMC and ABC. Let q_1, q_2 and q be the radii of the escribed circles of the same triangles that lie in the angle ACB. Prove that

$$\frac{r_1}{q_1}\cdot\frac{r_2}{q_2}=\frac{r}{q}.$$

IMO 1970, Problem 4. Find the set of all positive integers n with the property that the set $\{n \,:\, n,\ n+1,\ n+2,\ n+3,\ n+4,\ n+5\}$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

IMO 2010, Problem 1. Determine all functions $f\colon \mathbb{R}\to\mathbb{R}$ such that the equality

$$f(\lfloor x\rfloor\cdot y)=f(x)\cdot\lfloor f(y)\rfloor$$

holds for all $x, y \in \mathbb{R}$. (Here $\lfloor z\rfloor$ denotes the greatest integer less than or equal to z.)

IMO 2010, Problem 4. Let P be a point inside the triangle ABC. The lines AP, BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M respectively. The tangent to Γ at C intersects the line AB at S. Suppose that $SC = SP$. Prove that $MK = ML$.

Remark. Contrast Problems 1 and 4 in IMO 1959 with Problems 1 and 4 in IMO 2010.

A few problems proposed for the IMO by India

Listed below are a few problem proposals from India that were shortlisted for consideration by the problem committee of the host country. In cases where the problem was selected for the IMO, we have made a note of this fact. (Please note that the list is not a complete one, we have only given a sampling of the problems.)

Shortlisted for IMO 1989; proposed by Shailesh Shirali. A bicentric quadrilateral is one that is both inscribable in and circumscribable about a circle. Show that for such a quadrilateral, the centers of the two associated circles are collinear with the point of intersection of the diagonals.

IMO 1990, Problem 1; proposed by C R Pranesachar. Given a circle with two chords AB, CD that meet at E, let M be a point of chord AB other than E. Draw the circle through D, E, and M. The tangent line to the circle DEM at E meets the lines BC, AC at F, G, respectively. Given $AM/AB = \lambda$, find GE/EF.

Shortlisted for IMO 1992; proposed by Shailesh Shirali. Two circles $G1$ and $G2$ are inscribed in a segment of a circle G and touch each other externally at a point W. Let A be a point of intersection of a common internal tangent to $G1$ and $G2$ with the arc of the segment, and let B and C be the endpoints of the chord. Prove that W is the incentre of the triangle ABC.

Shortlisted for IMO 1992; proposed by C R Pranesachar. Show that in the plane there exists a convex polygon of 1992 sides satisfying the following conditions:

(i) its side lengths are $1, 2, 3, \ldots, 1992$ in some order;

(ii) the polygon is circumscribable about a circle.

IMO 1992, Problem 2; proposed by B J Venkatachala. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f\left(x^2 + f(y)\right) = y + \left(f(x)\right)^2.$$

IMO 2002, Problem 5; proposed by B J Venkatachala. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y, u, v \in \mathbb{R}$,

$$\left(f(x) + f(y)\right)\cdot\left(f(u) + f(v)\right) = f(xu-yv)+f(xv+yu).$$

IMO 1998, Problem 2; proposed by R B Bapat. In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

Shortlisted for IMO 2014; proposed by N V Tejaswi. There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e., from clockwise to anticlockwise or *vice versa*.

Suppose that Turbo's path entirely covers all circles. Prove that n must be odd.

Some memorable problems from the IMOs

The mechanism by which problems are posed for the IMOs is a unique and remarkable one, and over the years some truly beautiful and memorable problems have been created through this tradition. We mention a few here.

IMO 1988, Problem 6. Let a and b be two positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab + 1}$$

is a perfect square.

IMO 1990, Problem 3. Find all positive integers n having the property that

$$\frac{2^n + 1}{n^2}$$

is an integer.

IMO 1991, Problem 2. Let $n > 6$ and let $a_1 < a_2 < \cdots < a_k$ be all the natural numbers that are less than n and relatively prime to n. Show that if $a_1, a_2, \ldots, a_k$ is an arithmetic progression, then n is a prime number or a natural power of 2.

IMO 1993, Problem 5. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine whether there exists a strictly increasing function $f\colon \mathbb{N} \to \mathbb{N}$ with the following properties:

(a) $f(1) = 2$;

(b) $f(f(n)) = f(n) + n \quad (n \in \mathbb{N})$.

IMO 1993, Problem 6. Let n be an integer greater than 1. In a circular arrangement of n lamps $L_0, L_1, \ldots, L_{n-1}$, each one can be either ON or OFF. We start with the situation where all lamps are ON, and then carry out a sequence of steps, $\text{Step}_0, \text{Step}_1, \ldots$. If L_{j-1} (j is taken mod n) is ON, then Step_j changes the status of L_j (it goes from ON to OFF or from OFF to ON) but does not change the status of any of the other lamps. If L_{j-1} is OFF, then Step_j does not change anything at all. Show that:

(a) There is a positive integer $M(n)$ such that after $M(n)$ steps, all lamps are ON again.

(b) If n has the form 2^k, then all lamps are ON after $n^2 - 1$ steps.

(c) If n has the form $2^k + 1$, then all lamps are ON after $n^2 - n + 1$ steps.

IMO 1996, Problem 5. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to AF. Let R_A, R_C, R_E be the circumradii of triangles FAB, BCD, DEF respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

Remark. Problem 6 of IMO 1988 was for many years regarded as "the most difficult problem ever posed in an IMO." But it seems likely that problem 5 of IMO 1996 now has that label.

Acknowledgement. The author expresses his sincere thanks to Dr Shailesh Shirali who first suggested the writing of an article on the history of Mathematical Olympiad in India.

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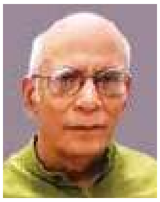

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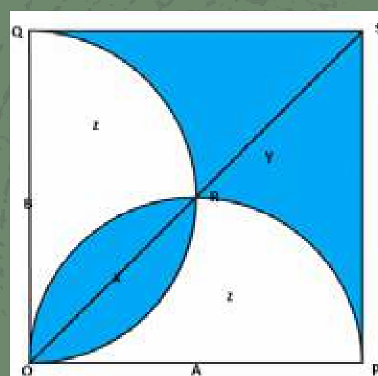
Question: *In the square below two semicircles are overlapping in a symmetrical pattern. Which is greater: the area shaded blue or the area shaded white ?*



Reader Tejash Patel sent us this solution

Solution: *As shown in the figure, Let OP = OQ = 2a.*
And OB = OA = BR = AR = a.

Now x = 2[Area of sector OBR – Area of ΔOBR] = $2[\frac{\Pi a^2 \times 90}{360} - \frac{1}{2}a^2] = 2[\frac{\Pi a^2}{4} - \frac{1}{2}a^2]$

$$\therefore x = \frac{\pi a^2}{2} - a^2 \;\ldots\ldots\ldots\ldots\ldots(1)$$

Now $x + z = \frac{1}{2}\pi a^2$

$$\therefore z = \frac{1}{2}\pi a^2 - x = \frac{1}{2}\pi a^2 - (\frac{\pi a^2}{2} - a^2) = a^2 \;\ldots\ldots\ldots(2)$$

Now y = $4a^2 - [2z + x] = 4a^2 - [2a^2 + \frac{\pi a^2}{2} - a^2] = 3a^2 - \frac{\pi a^2}{2}$ …………(3)

Area shaded blue = $x + y = \frac{\pi a^2}{2} - a^2 + 3a^2 - \frac{\pi a^2}{2} = 2a^2$

Area shaded white = $2z = 2a^2$.

∴ Area shaded blue and Area shaded white are equal.

P.S. One of our very visual readers suggested that we simply join PQ for a justification of equality (based on the symmetries of the square and circle) to spring to mind.

Problem Corner

Arsalan's Amazing Area Problems

SHAILESH SHIRALI

On the Facebook page (AtRiUM: At Right Angles, Us and Math) linked to this magazine, one of our contributors, Arsalan Wares, has been astonishingly prolific in posting problems. A good many of these have had to do with regular hexagons; more specifically, with the areas of polygonal regions drawn within such hexagons. It is both astonishing and pleasing to see such a rich diversity of problems arising from this simple and familiar structure.

In this article, we study a few of these problems and demonstrate (if at all such a fact is in need of demonstration!) the great power and versatility of the vector method in a certain class of geometric problems. For the reader's convenience, we have listed the relevant formulas at the end of the article, in the appendix.

Problem 1: Quadrilateral within a hexagon
Shown in Figure 1 is a regular hexagon *ABCDEF*. The three diagonals emanating from vertex *E* are drawn. These give rise to four triangles *EFA*, *EAB*, *EBC* and *ECD*. The centroids of these four triangles are the points *P*, *Q*, *R* and *S*, respectively.

Problem: *Find the ratio of the area of the quadrilateral PQRS to that of the hexagon ABCDEF.*

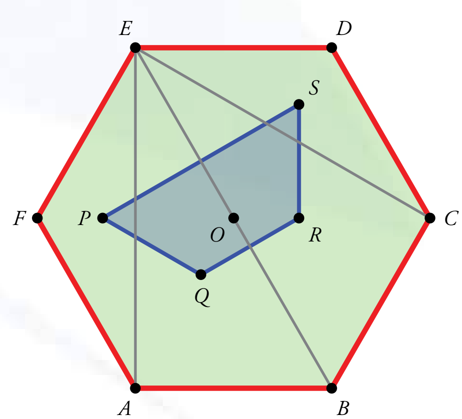


Figure 1.

Keywords: regular hexagon, triangle, area, ratio, vector, cross product

Let O be the circumcentre of hexagon $ABCDEF$. Referred to this point as origin, let the position vectors of A and B be $\mathbf{a}$ and $\mathbf{b}$, respectively. Then the position vectors of the other vertices of the hexagon are easily found in terms of $\mathbf{a}$ and $\mathbf{b}$. More specifically:

- the position vector of C is $\mathbf{c} = -\mathbf{a} + \mathbf{b}$;
- the position vector of D is $\mathbf{d} = -\mathbf{a}$;
- the position vector of E is $\mathbf{e} = -\mathbf{b}$;
- the position vector of F is $\mathbf{f} = \mathbf{a} - \mathbf{b}$.

From these, we easily deduce the position vectors of the centroids of the four triangles:

- the position vector of P is $\mathbf{p} = \frac{2}{3}(\mathbf{a} - \mathbf{b})$;
- the position vector of Q is $\mathbf{q} = \frac{1}{3}\mathbf{a}$;
- the position vector of R is $\mathbf{r} = \frac{1}{3}(-\mathbf{a} + \mathbf{b})$;
- the position vector of S is $\mathbf{s} = -\frac{2}{3}\mathbf{a}$.

From these relations, we deduce the following:

$$\mathbf{PR} = \mathbf{r} - \mathbf{p} = \mathbf{b} - \mathbf{a},$$
$$\mathbf{QS} = \mathbf{s} - \mathbf{q} = -\mathbf{a}.$$

From these relations in turn, we deduce the vector area of the quadrilateral $PQRS$:

$$\text{Vector area of } PQRS = \frac{1}{2}\left(\mathbf{PR} \times \mathbf{QS}\right) = \frac{1}{2}\left(\mathbf{a} \times \mathbf{b}\right).$$

The vector area of hexagon $ABCDEF$ is 6 times the vector area of a triangle OAB and hence is equal to

$$6 \times \frac{1}{2}\left(\mathbf{a} \times \mathbf{b}\right) = 3\left(\mathbf{a} \times \mathbf{b}\right).$$

It follows that the area of the quadrilateral is $\frac{1}{6}$ of the area of the hexagon; so the required ratio of areas is $1 : 6$.

Quadrilaterals and triangles within a triangle

Shown in Figure 2 is an arbitrary triangle ABC. The midpoints of sides BA and BC are R and S respectively, and the points of trisection of side AC are points P and Q respectively, with P being closer to A than to C. Two of the resulting regions are shaded green and another two regions are shaded red.

Problem: *Find the ratio of the total area coloured red to the total area coloured green.*

Let point A serve as the origin, and let the position vectors of points B and C be $6\mathbf{b}$ and $6\mathbf{c}$ respectively. Then the position vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ of points P, Q, R, S are as follows:

$$\mathbf{p} = 2\mathbf{c}, \qquad \mathbf{q} = 4\mathbf{c},$$
$$\mathbf{r} = 3\mathbf{b}, \qquad \mathbf{s} = 3\left(\mathbf{b} + \mathbf{c}\right)$$

Now we compute the position vectors $\mathbf{d}, \mathbf{e}, \mathbf{f}$ of points D, E, F respectively. Let $SD : DA = u : 1 - u$ and $BD : DQ = v : 1 - v$. Then we can write:

$$\mathbf{d} = 3(1-u)\left(\mathbf{b} + \mathbf{c}\right), \quad \mathbf{d} = 4v\mathbf{c} + 6(1-v)\mathbf{b}.$$

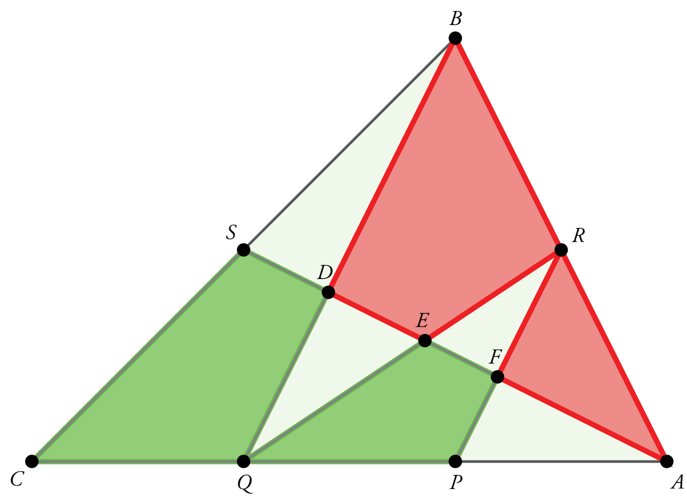


Figure 2.

Hence we have $3(1-u)\,(\mathbf{b}+\mathbf{c}) = 4v\mathbf{c} + 6(1-v)\mathbf{b}$. Since vectors **b** and **c** are linearly independent, we obtain the following inequalities:

$$3(1-u) = 6(1-v), \quad 3(1-u) = 4v.$$

Solving these equations for u and v in the usual way, we obtain: $u = 1/5$, $v = 3/5$. It follows that $AD : AS = 4 : 5$, $BD : DQ = 3 : 2$ and

$$\mathbf{d} = \frac{12}{5}\,(\mathbf{b}+\mathbf{c})\,.$$

In the same way we obtain the position vectors **e** and **f** of E and F respectively. We find that $AE : AS = 4 : 7$, $RE : EQ = 3 : 4$ and

$$\mathbf{e} = \frac{12}{7}\,(\mathbf{b}+\mathbf{c})\,;$$

and also $AF : AS = 2 : 5$, $RF : FP = 3 : 2$ and

$$\mathbf{f} = \frac{6}{5}\,(\mathbf{b}+\mathbf{c})\,.$$

These results allow us to get the ratio we want. We shall proceed using vector algebra. We have already obtained the position vectors of all the points in the figure. We now make use of these as follows.

- The vector area of triangle ARF is equal to

$$\frac{1}{2}\,(\mathbf{AR}\times\mathbf{AF}) = \frac{1}{2}\left(3\mathbf{b}\times\frac{6}{5}(\mathbf{b}+\mathbf{c})\right) = \frac{9}{5}(\mathbf{b}\times\mathbf{c}).$$

- The vector area of quadrilateral $RBDE$ is equal to $\frac{1}{2}\,(\mathbf{EB}\times\mathbf{RD})$. Now

$$\mathbf{EB} = 6\mathbf{b} - \frac{12}{7}(\mathbf{b}+\mathbf{c}) = \frac{30}{7}\mathbf{b} - \frac{12}{7}\mathbf{c},$$

$$\mathbf{RD} = \frac{12}{5}(\mathbf{b}+\mathbf{c}) - 3\mathbf{b} = -\frac{3}{5}\mathbf{b} + \frac{12}{5}\mathbf{c}.$$

Hence the vector area of quadrilateral $RBDE$ is

$$\frac{1}{2}\cdot\frac{18}{35}\,(5\mathbf{b}-2\mathbf{c})\times(-\mathbf{b}+4\mathbf{c}) = \frac{9\times 18}{35}(\mathbf{b}\times\mathbf{c})$$

- Hence the vector area of the region coloured red is equal to

$$\left(\frac{9}{5}+\frac{9\times 18}{35}\right)(\mathbf{b}\times\mathbf{c})=\frac{45}{7}(\mathbf{b}\times\mathbf{c}).$$

- Next we compute the vector area of quadrilateral *PFEQ*. Now

$$\mathbf{PE}=\frac{12}{7}(\mathbf{b}+\mathbf{c})-2\mathbf{c}=\frac{12}{7}\mathbf{b}-\frac{2}{7}\mathbf{c},$$

$$\mathbf{FQ}=4\mathbf{c}-\frac{6}{5}(\mathbf{b}+\mathbf{c})=-\frac{6}{5}\mathbf{b}+\frac{14}{5}\mathbf{c}.$$

  Hence the vector area of a quadrilateral *PFEQ* is equal to

$$\frac{1}{2}\cdot\frac{4}{35}(6\mathbf{b}-\mathbf{c})\times(-3\mathbf{b}+7\mathbf{c})=\frac{78}{35}(\mathbf{b}\times\mathbf{c}).$$

- Finally we compute the vector area of quadrilateral *QDSC*. We have:

$$\mathbf{QS}=3(\mathbf{b}+\mathbf{c})-4\mathbf{c}=3\mathbf{b}-\mathbf{c},$$

$$\mathbf{DC}=6\mathbf{c}-\frac{12}{5}(\mathbf{b}+\mathbf{c})=-\frac{12}{5}\mathbf{b}+\frac{18}{5}\mathbf{c}.$$

  Hence the vector area of quadrilateral *QDSC* is equal to

$$\frac{1}{2}\cdot\frac{6}{5}(3\mathbf{b}-\mathbf{c})\times(-2\mathbf{b}+3\mathbf{c})=\frac{21}{5}(\mathbf{b}\times\mathbf{c}).$$

- Hence the vector area of the region coloured green is equal to

$$\left(\frac{78}{35}+\frac{21}{5}\right)(\mathbf{b}\times\mathbf{c})=\frac{45}{7}(\mathbf{b}\times\mathbf{c}).$$

We see that the total area of the region coloured green is identical to the total area of the region coloured red. The desired ratio is 1 : 1.

Hexagon within a regular hexagon

The third problem we study is much more complex than the first two, but the same methods suffice to produce a solution.

Shown in Figure 3 is a regular hexagon *ABCDEF*. The points of trisection of the sides *AB*, *CD* and *EF* are located: *G* and *H* are the points of trisection of *AB*; *I* and *J* are the points of trisection of *CD*; and *K* and *L* are the points of trisection of *EF*. We now draw the following six segments: *AI*, *BL*, *CK*, *DH*, *EG*, *FJ*. (Observe the symmetries in the choice of the segments.) These six segments divide the hexagonal region into 19 different regions.

Problem: *Find the ratios of the areas of these regions to that of the hexagon.*

It is not hard to see that the 19 regions can be grouped into five subsets. Six of the regions have the smallest area (z), then another six regions have the next smaller area (y); then we have three regions with the next smaller area (x), another three regions with a larger area (w), and finally the central region with area v. The symmetries of the hexagon tell us that the six regions marked z are congruent to each other, as are the six regions marked y, and likewise for the three regions marked x and the three regions marked w. (*Comment.* In claiming that some regions are smaller in area than others, we are simply using visual observation. However, we do not make use of these observations in the solution presented below.)

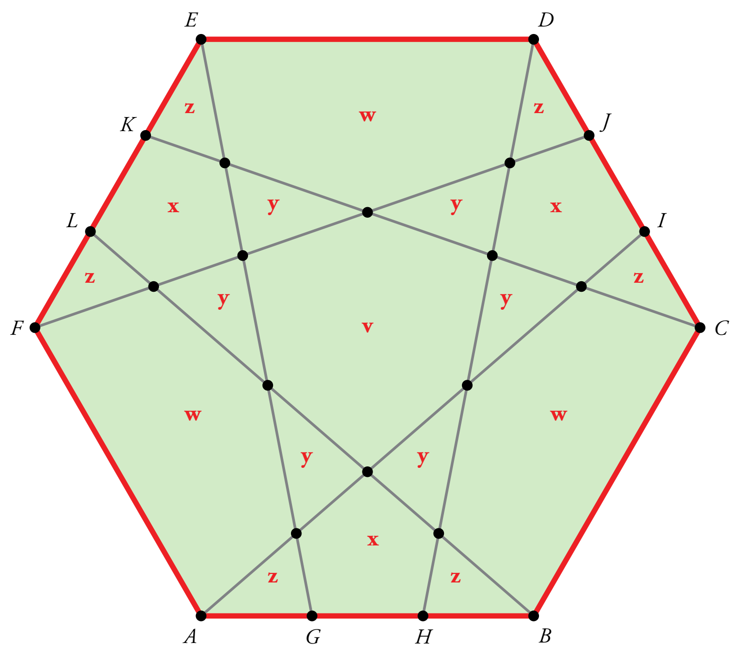


Figure 3.

We assign the same position vectors to the various points as in the first problem. That is, treating the centre of the hexagon to be the origin of the vector coordinate system, let the position vectors of A and B be $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the position vectors $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ of the remaining vertices C, D, E, F of the hexagon are as follows:

$$\mathbf{c} = -\mathbf{a} + \mathbf{b}, \quad \mathbf{d} = -\mathbf{a}, \quad \mathbf{e} = -\mathbf{b}, \quad \mathbf{f} = \mathbf{a} - \mathbf{b}.$$

We now obtain the position vectors $\mathbf{g}, \mathbf{h}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}$ of the six points of trisection G, H, I, J, K, L:

$$\mathbf{g} = \frac{1}{3}(2\mathbf{a} + \mathbf{b}), \qquad \mathbf{h} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b}),$$

$$\mathbf{i} = \frac{1}{3}(-3\mathbf{a} + 2\mathbf{b}), \qquad \mathbf{j} = \frac{1}{3}(-3\mathbf{a} + \mathbf{b}),$$

$$\mathbf{k} = \frac{1}{3}(\mathbf{a} - 3\mathbf{b}), \qquad \mathbf{l} = \frac{1}{3}(2\mathbf{a} - 3\mathbf{b}).$$

We now work out the position vectors of the four points of intersection lying on any one line segment, say segment FJ. This exercise will enable us to find the ratios into which the points divide that segment. By symmetry, we expect that all six line segments are divided by the points that lie on them in the same proportions.

- Let FJ and LB intersect at M (see Figure 4, which is the same as Figure 3 but has been reproduced here for convenience; note that additional points M, N, P, Q are shown in this figure), and let

  $$FM : MJ = r : 1 - r, \quad LM : MB = 1 - s : s.$$

  Then the position vector $\mathbf{m}$ of M is given by each of the following expressions:

  $$\mathbf{m} = \frac{r}{3}(-3\mathbf{a} + \mathbf{b}) + (1 - r)(\mathbf{a} - \mathbf{b}),$$

  $$\mathbf{m} = \frac{s}{3}(2\mathbf{a} - 3\mathbf{b}) + (1 - s)\mathbf{b}.$$

Equating the two expressions for $\mathbf{m}$ and making use of the fact that vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, which implies that the coefficients of $\mathbf{a}$ on the two sides of the equality sign are equal, and so also for the coefficients of $\mathbf{b}$, we obtain the following pair of simultaneous equations in the unknowns r and s:

$$2r + 3(s-1) = 0, \quad 6r + 2s - 3.$$

Solving these equations, we obtain the following values for r and s:

$$r = \frac{3}{14}, \quad s = \frac{6}{7}.$$

From this we obtain:

$$\mathbf{m} = \frac{1}{7}(4\mathbf{a} - 5\mathbf{b}).$$

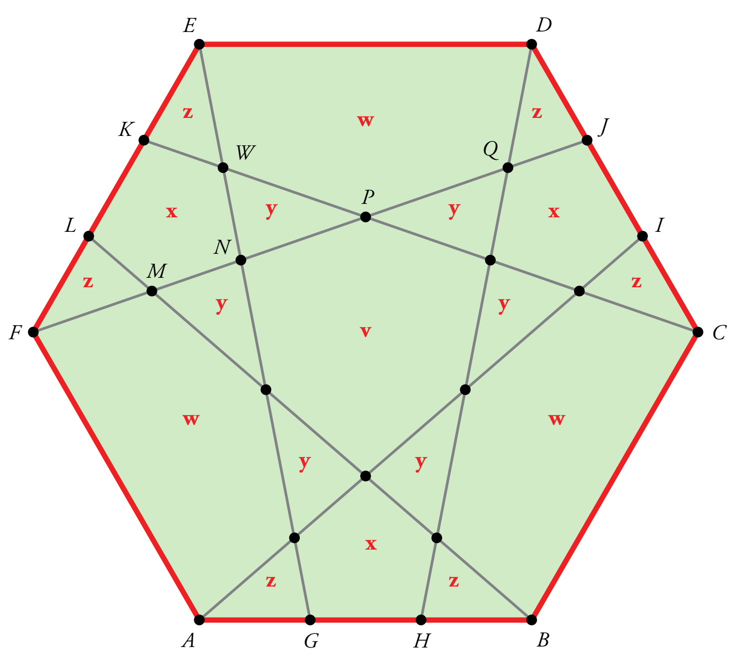


Figure 4.

We are now in a position to compute the area of $\triangle FML$, i.e., we can now find out the value of z. Namely, the vector area of this triangle is equal to

$$\begin{aligned}
&\frac{1}{2}(\mathbf{f}\times\mathbf{m} + \mathbf{m}\times\mathbf{l} + \mathbf{l}\times\mathbf{f}) \\
&= \frac{1}{2}\Big((\mathbf{a}-\mathbf{b})\times\frac{1}{7}(4\mathbf{a}-5\mathbf{b}) + \frac{1}{7}(4\mathbf{a}-5\mathbf{b})\times\frac{1}{3}(2\mathbf{a}-3\mathbf{b}) + \frac{1}{3}(2\mathbf{a}-3\mathbf{b})\times(\mathbf{a}-\mathbf{b})\Big) \\
&= \frac{1}{2}\Big(-\frac{1}{7} - \frac{2}{21} + \frac{1}{3}\Big)\mathbf{a}\times\mathbf{b} = \frac{1}{21}\mathbf{a}\times\mathbf{b}.
\end{aligned}$$

Since the vector area of hexagon $ABCDEF$ is $3\mathbf{a}\times\mathbf{b}$, we see that the area of $\triangle FML$ is $\frac{1}{63}$ times the area of the hexagon. So if the area of the hexagon is taken to be 1 square unit, then $z = \frac{1}{63}$.

- Let FJ and EG intersect at N, and let $FN : NJ = r : 1-r$. From the symmetry of the figure, it follows that we also have $EN : NG = r : 1-r$. Hence the position vector $\mathbf{n}$ of N is given by each of the following expressions:

$$\mathbf{n} = \frac{r}{3}(-3\mathbf{a}+\mathbf{b}) + (1-r)(\mathbf{a}-\mathbf{b}),$$
$$\mathbf{n} = \frac{r}{3}(2\mathbf{a}+\mathbf{b}) + (1-r)(-\mathbf{b}).$$

Equating the two expressions for $\mathbf{n}$, we obtain $r = 3/8$. This yields the position vector of N:

$$\mathbf{n} = \frac{1}{4}(\mathbf{a}-2\mathbf{b}).$$

- Let FJ and CK intersect at P, and let $FP : PJ = r : 1-r$. From the symmetry of the figure, it follows that we also have $CP : PK = r : 1-r$. Hence the position vector $\mathbf{p}$ of P is given by each of the following expressions:

$$\mathbf{p} = \frac{r}{3}(-3\mathbf{a}+\mathbf{b}) + (1-r)(\mathbf{a}-\mathbf{b}),$$
$$\mathbf{p} = \frac{r}{3}(\mathbf{a}-3\mathbf{b}) + (1-r)(-\mathbf{a}+\mathbf{b}).$$

Equating the two expressions for $\mathbf{p}$, we obtain $r = 3/5$. This yields the position vector of P:

$$\mathbf{p} = -\frac{1}{5}(\mathbf{a}+\mathbf{b}).$$

- Let KC and EG intersect at W. By symmetry, W divides KC in the same ratio as M divides LB. We have already computed and found that $LM : MB = 6 : 1$. Hence the position vector $\mathbf{w}$ of W is

$$\mathbf{w} = \frac{1}{7}(2(\mathbf{a}-3\mathbf{b}) + (-\mathbf{a}+\mathbf{b})) = \frac{1}{7}(\mathbf{a}-5\mathbf{b}).$$

- We are now in a position to compute the area of $\triangle WNP$, i.e., we can now find out the value of y. Namely, the vector area of this triangle is equal to

$$\begin{aligned}
&\frac{1}{2}(\mathbf{w}\times\mathbf{n} + \mathbf{n}\times\mathbf{p} + \mathbf{p}\times\mathbf{w}) \\
&= \frac{1}{2}\left(\frac{1}{7}(\mathbf{a}-5\mathbf{b})\times\frac{1}{4}(\mathbf{a}-2\mathbf{b}) - \frac{1}{4}(\mathbf{a}-2\mathbf{b})\times\frac{1}{5}(\mathbf{a}+\mathbf{b}) - \frac{1}{5}(\mathbf{a}+\mathbf{b})\times\frac{1}{7}(\mathbf{a}-5\mathbf{b})\right) \\
&= \frac{1}{2}\left(\frac{3}{28} - \frac{3}{20} + \frac{6}{35}\right)\mathbf{a}\times\mathbf{b} = \frac{9}{140}\mathbf{a}\times\mathbf{b}.
\end{aligned}$$

Since the vector area of hexagon $ABCDEF$ is $3\mathbf{a}\times\mathbf{b}$, we see that the area of $\triangle WNP$ is $\frac{3}{140}$ times the area of the hexagon. So if the area of the hexagon is taken to be 1 square unit, then $y = \frac{3}{140}$.

- Next we determine the value of x, but we shall use a different approach. From the ratios uncovered so far, we know that

$$FM : FN : FP : FQ = \frac{3}{14} : \frac{3}{8} : \frac{3}{5} : \frac{6}{7}.$$

Hence we have

$$FM : MN : NP : PQ : QJ = 60 : 45 : 63 : 72 : 40.$$

These ratios imply that

$$\frac{\text{Area of }\triangle PWN}{\text{Area of }\triangle PKF} = \frac{PW}{PK}\cdot\frac{PN}{PF} = \frac{PQ}{PJ}\cdot\frac{PN}{PF} = \frac{72}{112}\cdot\frac{63}{168} = \frac{27}{112}.$$

But we also have

$$\frac{\text{Area of }\triangle PWN}{\text{Area of }\triangle PKF} = \frac{y}{y+x+z} = \frac{3/140}{3/140+x+1/63}.$$

Hence

$$\frac{3/140}{3/140+x+1/63} = \frac{27}{112}, \quad \therefore\ x = \frac{13}{252}.$$

- Next we compute the value of v, using an approach similar to the one above. We have drawn yet another copy of the figure below (Figure 5), again for convenience. This time, we have also marked the point S where DH intersects KC and the point T where DH intersects LB.

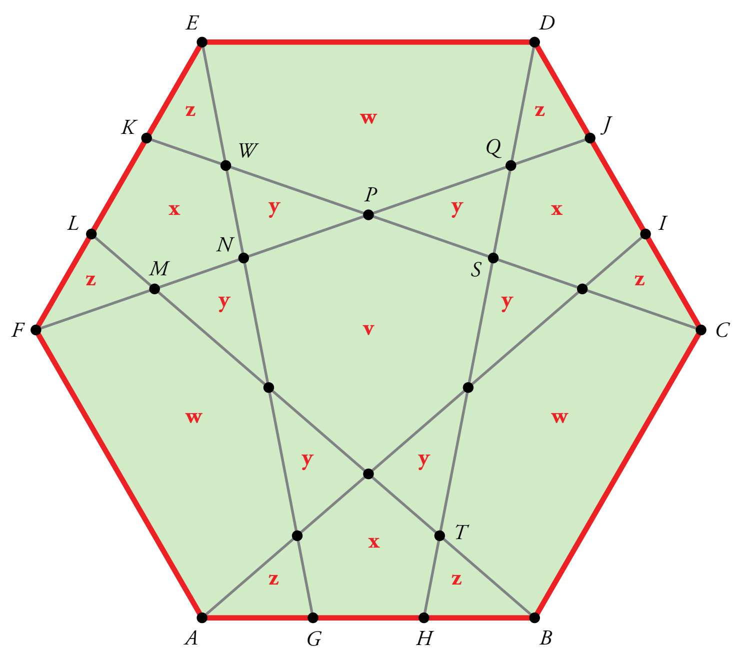


Figure 5.

We have:

$$\frac{\text{Area of }\triangle QPS}{\text{Area of }\triangle QMT} = \frac{QP}{QM}\cdot\frac{QS}{QT} = \frac{QP}{QM}\cdot\frac{MN}{MQ} = \frac{72}{180}\cdot\frac{45}{180} = \frac{1}{10}.$$

But we also have:

$$\frac{\text{Area of }\triangle QPS}{\text{Area of }\triangle QMT} = \frac{y}{3y+v} = \frac{3/140}{9/140+v}.$$

Hence:

$$\frac{3/140}{9/140+v} = \frac{1}{10}, \quad \therefore\ v = \frac{3}{20}.$$

- Having computed z, y, x, v, we find the value of w by subtraction:

$$3w = 1 - 3x - 6y - 6z - v = 1 - \frac{3\cdot 13}{252} - \frac{6\cdot 3}{140} - \frac{6\cdot 1}{63} - \frac{3}{20} = \frac{33}{70}, \quad \therefore\ w = \frac{11}{70}.$$

So we have, in conclusion:

$$z : y : x : w : v = \frac{1}{63} : \frac{3}{140} : \frac{1}{63} : \frac{3}{20} : \frac{11}{70} = 20 : 27 : 65 : 198 : 189.$$

Appendix: Two key formulas

We have made repeated use of the vector formulas for the area of a triangle and the area of a quadrilateral. We mention the formulas here for the reader's convenience.

Vector formula for area of a triangle

Given a triangle ABC (Figure 6 (a)), the vector area of the triangle is given by the following formula:

$$\text{Vector area of } \triangle ABC = \frac{1}{2}\left(\mathbf{AB} \times \mathbf{AC}\right),$$

where, by convention, the vectors are listed in such an order that the rotation which takes **AB** to **AC** is in the *anticlockwise* direction. (This is purely a convention. Naturally, as far as magnitudes of areas are concerned, the order does not matter.)

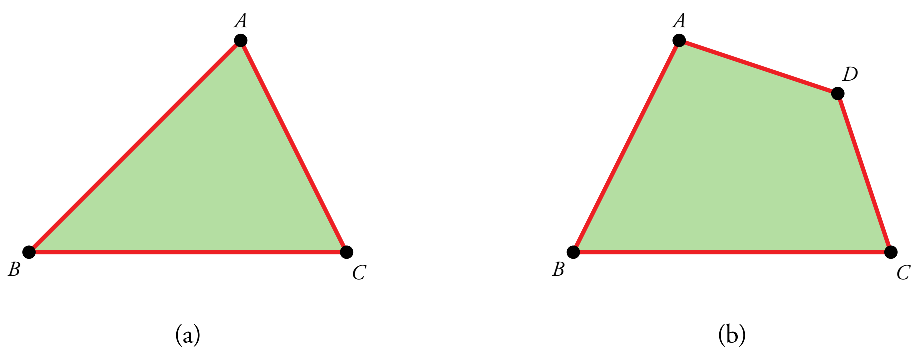


Figure 6.

Vector formula for area of a quadrilateral

Given a quadrilateral $ABCD$ (Figure 6 (b)), the vector area of the quadrilateral is given by the following formula:

$$\text{Vector area of quadrilateral } ABCD = \frac{1}{2}\left(\mathbf{AC} \times \mathbf{BD}\right).$$

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Problem Corner

A Pythagoras-style Diophantine Equation and its Solution

$\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$

The Pythagorean equation $x^2+y^2=z^2$ (to be solved over the positive integers $\mathbb{N}$) is a much-studied one; many articles have appeared in this magazine alone, devoted to this equation. A close relative to this is the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$ (which can be written as $x^{-1}+y^{-1}=z^{-1}$; in this form, its similarity to the Pythagorean equation is readily seen), and this too has been studied many times in *At Right Angles*.

In this note, we study another equation which visually resembles the Pythagorean equation and which too is required to be solved over the positive integers:

$$\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}=\frac{1}{\sqrt{z}}. \tag{1}$$

Write

$$x=m^2a, \quad y=n^2b, \quad z=k^2c, \tag{2}$$

where m, n, k are positive integers and a, b, c are 'square-free' positive integers, i.e., they are not divisible by any square number greater than 1. (So a, b, c are products of *distinct* prime numbers.) Any positive integer can be uniquely written in this form, i.e., as a product of a perfect square and a square-free positive integer. Making these substitutions, we get:

$$\frac{1}{m\sqrt{a}}+\frac{1}{n\sqrt{b}}=\frac{1}{k\sqrt{c}},$$

$$\therefore\ \ km\sqrt{ac}+kn\sqrt{bc}=mn\sqrt{ab}. \tag{3}$$

Squaring both sides of (3), we get:

$$k^2m^2ac+2mnck^2\sqrt{ab}+k^2n^2bc=m^2n^2ab.$$

Keywords: Diophantine equation, rational number, integer, prime number, square-free number

From this relation, we deduce that $2mnck^2\sqrt{ab}$ is an integer, and therefore that $\sqrt{ab}$ is a rational number. But if the square root of an integer is a rational number, then it is an integer. Hence $\sqrt{ab}$ is an integer. We know that in the prime factorisations of a and b, each prime occurs just once. If we combine this condition with the deduction that $\sqrt{ab}$ is an integer, we realize right away that $a = b$.

Again, (3) can be written as

$$mn\sqrt{ab} - km\sqrt{ac} = kn\sqrt{bc}. \tag{4}$$

Squaring both sides of (4), we get:

$$m^2n^2ab - 2knam^2\sqrt{bc} + k^2m^2ac = k^2n^2bc.$$

From this relation, we deduce (just as we did earlier) that $2knam^2\sqrt{bc}$ is an integer, therefore that $\sqrt{bc}$ is a rational number, therefore that $\sqrt{bc}$ is an integer, therefore that $b = c$. Hence $a = b = c$. A striking conclusion!

This means that $x = m^2a$, $y = n^2a$ and $z = k^2a$ for some positive integers m, n, k, a. Equation (1) now yields:

$$\frac{1}{m\sqrt{a}} + \frac{1}{n\sqrt{a}} = \frac{1}{k\sqrt{a}}, \qquad \therefore \ \frac{1}{m} + \frac{1}{n} = \frac{1}{k}. \tag{5}$$

It is remarkable that in attempting to solve the equation $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{z}}$ over $\mathbb{N}$, we have ended up (essentially) with the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$, also to be solved over $\mathbb{N}$! We know very well how to solve this equation; all we need to do now is to invoke what we had discovered earlier.

A corollary to what we discovered above is the following: *If coprime positive integers x, y, z satisfy the relation $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{z}}$, then each of x, y, z is a perfect square.* This is so because the condition that x, y, z are coprime forces $a = 1$, implying that $x = m^2$, $y = n^2$ and $z = k^2$. A neat result!

A specific example. Let us illustrate this by taking, say $z = 20$. So we seek all solutions (x, y) in positive integers to the equation $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}}$.

Since $20 = 2^2 \times 5$, what we showed earlier implies that $\sqrt{x}$ and $\sqrt{y}$ are integer multiples of $\sqrt{5}$. Let $x = 5m^2$ and $y = 5n^2$ where m and n are positive integers. Then the equation reduces to $\frac{1}{m} + \frac{1}{n} = \frac{1}{\sqrt{4}}$, i.e., $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$. We know very well how to solve this kind of equation! We have:

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2},$$

$$\therefore \ 2(m+n) = mn,$$

$$\therefore \ mn - 2(m+n) = 0,$$

$$\therefore \ (m-2)(n-2) = 4.$$

Since 4 can be written as a product of two positive integers in the following ways,

$$4 = 1 \times 4 = 2 \times 2 = 4 \times 1,$$

it follows that

$$(m-2, n-2) \in \{(1,4),\ (2,2),\ (4,1)\},$$

and hence that

$$(m, n) \in \{(3,6),\ (4,4),\ (6,3)\}.$$

Since $x = 5m^2$ and $y = 5n^2$, it follows that

$$(x, y) \in \{(45,180),\ (80,80),\ (180,45)\}.$$

Hence the equation $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}}$ has these positive integer solutions and no more.

Much the same approach can be followed for any given value of z.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Problems for the SENIOR SCHOOL

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem VII-3-S.1

Let $f(x) = x^2 + bx + c$ where b is a negative integer and c is a real number. Suppose the sum of the roots of $f(f(x))$ is a prime number. Prove that $f(f(x))$ has no real root in the interval $(0, 1)$.

Problem VII-3-S.2

Let k be a given positive integer. Determine all real x, y, z such that $xyz \neq 0$ and

$$x^k + y^{k+1} = z^{k+2}, \quad x^{k+1} + y^{k+2} = z^{k+3}, \quad x^{k+2} + y^{k+3} = z^{k+4}.$$

Problem VII-3-S.3

A quadratic polynomial $f(x) = ax^2 + bx + c$ has no real roots. It is given that b is a rational number, and exactly one of c and $f(c)$ is a rational number. Is it possible for the discriminant of $f(x)$ to be a rational number? [Russian Mathematical Olympiad]

Problem VII-3-S.4

The sequence $\{a_n\}_{n\geq 0}$ is defined as follows:

$$a_0 = 1, \quad a_1 = 3, \quad a_{n+1} = a_n + a_{n-1} \text{ for all } n \geq 1.$$

Find all integers $n \geq 1$ for which $na_{n+1} + a_n$ and $na_n + a_{n-1}$ share a common factor greater than 1.

Problem VII-3-S.5

Consider the sequence $\{10^n\}_{n\geq 1}$. Prove that the sum of no two terms of the sequence is a perfect square.

Keywords: Quadratics, roots, functions, circles, triangles, equilateral

Solutions of Problems in Issue VII-2 (July 2018)

Solution to problem VII-2-S.1

Let AB be a fixed line segment in the plane. Let O and P be two points in the plane, on the same side of AB. If $\measuredangle AOB = 2\measuredangle APB$, does it necessarily follow that P lies on the circle with centre O and passing through A and B?

Not necessarily. Consider the circle passing through A, O and B. The magnitude of $\measuredangle AOB$ does not change if O moves on arc AB on the same side of AB as P. Thus there are infinitely many positions of O for which $\measuredangle AOB = 2\measuredangle APB$ but $OA \neq OB$. If we have the additional hypothesis that $OA = OB$, then the claim can be proven to be true.

Solution to problem VII-2-S.2

Let ABC be an equilateral triangle with centre O. A line through C meets the circumcircle of triangle AOB at points D and E. Prove that the points A, O and the midpoints of segments BD, BE are concyclic. [Tournament of Towns]

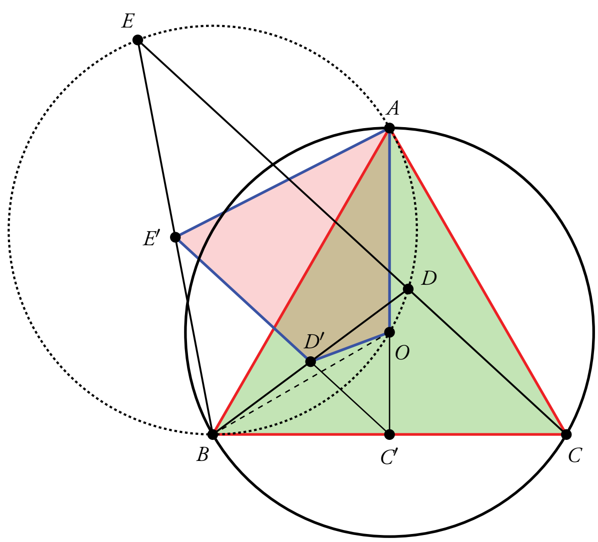


Figure 1.

Let D' and E' be the midpoints of BD and BE respectively. Extend $E'D'$ to meet BC at C'. Since $E'C'$ is parallel to EC and E' is the midpoint of EB, C' is the midpoint of BC. Therefore

$$C'D' \cdot C'E' = \left(\frac{1}{2}CD\right) \cdot \left(\frac{1}{2}CE\right) = \frac{1}{4}CD \cdot CE.$$

Observe that $\measuredangle OBC = \measuredangle OAB = 30^\circ$. Therefore, CB is tangent to the circumcircle of AOB and $CB^2 = CD \cdot CE$.

Thus $C'D' \cdot C'E' = \frac{1}{4}CD \cdot CE = \frac{1}{4}CB^2 = C'B^2 = C'O \cdot C'A$. This shows that A, O, D' and E' are concyclic.

Solution to problem VII-2-S.3

Three nonzero real numbers are given. It is given that if they are written in any order as the coefficients of a quadratic trinomial, then each of these trinomials has a real root. Does it follow that each of these trinomials has a positive root? [Tournament of Towns]

Let the three nonzero real numbers be a, b, c. Since a, b and c are all non-zero, 0 is not a root of any of the six trinomials under consideration. Suppose that $ax^2 + bx + c$ has two negative roots $-r$ and $-s$, where r and s are positive numbers. Then

$$ax^2 + bx + c = a(x + r)(x + s),$$

so $b = a(r + s)$ and $c = ars$ both have the same sign as a. Hence we may assume that they are all positive. Since one of the roots is real, both are real, so that we have $b^2 \geq 4ac$. Similarly, $c^2 \geq 4ab$ and $a^2 \geq 4bc$. Multiplication yields $(abc)^2 \geq (8abc)^2$, which is a contradiction. It follows that each of the six trinomials has a positive root.

Solution to problem VII-2-S.4

D is the midpoint of the side BC of triangle ABC. E and F are points on CA and AB respectively, such that BE is perpendicular to CA and CF is perpendicular to AB. If DEF is an equilateral triangle, does it follow that ABC is equilateral? [Tournament of Towns]

We shall show by actually constructing a counterexample that triangle ABC is not necessarily equilateral. Start with an equilateral triangle DEF. (See Figure 2.) Draw a segment BC through D, such that D is the midpoint of BC and BC is perpendicular to DE (at D), with F closer to B than to C. Construct a semicircle with centre D and radius DE. Since $DF = DE$, point F lies on this semicircle. Extend BF and CE to meet at A.

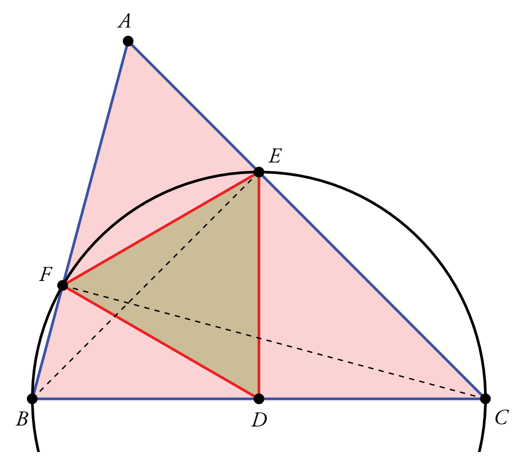


Figure 2.

Since $\measuredangle BEC = 90^\circ = \measuredangle BFC$, BE and CF are altitudes of triangle ABC. Since A lies on the extension of CE and DE is the perpendicular bisector of BC, $AB < AC$. Hence ABC is not equilateral.

Solution to problem VII-2-S.5

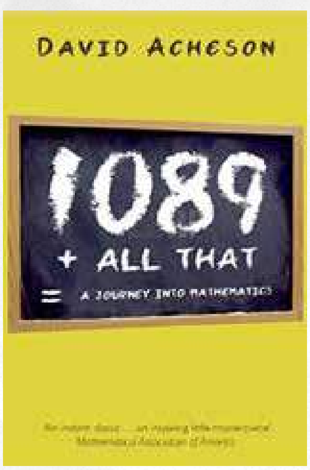
A boy computed the product of first n positive integers and his sister computed the product of the first m even positive integers where $m \geq 2$. Is it possible for them to get the same result?

The product of the first m even positive integers is $2^m \cdot m!$. Suppose that $n! = 2^m \cdot m!$ for some $m \geq 2$. Clearly $m \neq 2$; so $m > 2$. But then $n \geq 3$ in order for both $n!$ and $m!$ to be divisible by 3. In each product, every third factor is divisible by 3. For them to be divisible by the same power of 3, $n!$ can have at most two more terms than $m!$. Thus $n = m + 1$ or $n = m + 2$. If $n = m + 1$, then $m + 1 = 2^m$ which leads to $m = 1$. If $n = m + 2$, then $(m + 1)(m + 2) = 2^m$; but this has no solution in integers. (If $m > 2$, the product $(m + 1)(m + 2)$ has an odd factor > 3; but 2^m can have no odd factor > 1.) Thus we arrive at contradictions and we conclude that the two products cannot be the same.

1089 AND ALL THAT: A JOURNEY INTO MATHEMATICS

by David Acheson

Reviewed by Venkatesh Onkar



Let's say that you are a high school student, studying primarily the humanities and social sciences. Perhaps, in your middle and high school years, your experience of mathematics was something like that of the brave student Molesworth in a book called *Down with Skool* (as you can make out, Molesworth can't spell very well). Molesworth is at this moment approaching his teacher with his opinions about algebra:

> 'Sir sir please sir sir please?'
> 'Yes molesworth?'
> 'I simply haven't the fogiest about number six sir.'
> 'Indeed, molesworth?'
> 'It's just a jumble of letters sir i mean i kno i
> couldn't care less whether i
> get it right or not but what
> sort of an ass sir can hav written this book.'
> *(Maths master give below of rage and tear across*
> *room with dividers. He hurl me three times round*
> *head and then out of the window.)*
>
> *Down with Skool!,*
> *by Geoffrey Willans and Ronald Searle*

Keywords: math phobia, beauty, truth

David Acheson in his marvellous book *1089 and All That* quotes Molesworth (as above), adding that it strikes him as “rather sad" that there are students for whom algebra is just a “jumble of letters." Indeed, the whole book seems written to redress this feeling of sadness; there is an attempt to convey, through all the pages and the examples, the beauty and power of mathematics, in as simple a manner as possible. Even students with an elementary grasp of the discipline are shown some shining glimpses of the landscape, enough to evoke a desire to learn more. This is why, even though I teach mainly the humanities and the social sciences, I recommend this book highly to all you high school students out there (and indeed to anyone who has been somewhat intimidated by the subject).

Among the many rich chapters in the book, one in particular – *Great Mistakes* – stands out in terms of a potential attitude towards learning mathematics. As lay people, we tend to think of mathematics as a rather static body of knowledge, proven for all time. This chapter deals with subtle problems that have been misunderstood by even the great mathematicians, and it demonstrates something of the openness of mind necessary to look at problems (perhaps non-mathematical ones as well?) afresh. For instance, the author describes *Kakeya's problem:* “Find the smallest region in which a needle of unit length can be reversed, i.e., manoeuvred so that it rotates completely through 180 degrees." After walking us through the various figures through which the needle can be reversed (a circle with radius $\frac{1}{2}$, an equilateral triangle with height 1, for example), the book drops a bombshell by suggesting that the area of the figure in which the needle is to turn can be as *small as we like*, as long as we construct it given certain criteria. Obviously there is no proof offered, as the problem must be a complex one, but this example certainly grips us by suggesting that old problems have to be looked at afresh, and that there is certainly nothing to be taken for granted in learning mathematics. This attitude of wonder and freshness in exploration permeates the book as a whole.

A lot of the book deals with applied mathematics. In the chapter titled *The Heavens in Motion*, Acheson explains how the ancient Greeks constructed ellipses, and how they understood that an ellipse can be created by slicing through a cone; how Kepler discovered that the orbits of the planets are ellipses with the sun at one focus; and finally how Newton completed our understanding by proposing that the force F on a planet (moving in an elliptical orbit) is inversely proportional to the square of the distance from the sun (this also elegantly predicts that planets move faster when closer to the sun). For good measure, there is also a chapter that introduces the calculus in, the author suggests, “a concise and uncompromising way.”

In a more contemporary slant, the author discusses chaos theory: “the study of irregular erratic motion which is extremely sensitive to initial conditions.” The examples involving chaos theory are too subtle to summarize quickly (and, obviously, any summary here will spoil your pleasure in reading about them in the book itself!), but, needless to say, they are drawn from everyday life, the realm in which chaos theory operates, and they show us the mathematical power and beauty that underlie the perfectly ordinary world. And there is a stunning chapter on the transcendental number e, which explores the practical applications of “this strange number," as the author calls it, in all manner of problems from the spreading of disease to understanding how a milk droplet splashes on a smooth surface!

Apart from the applied angle, obviously the book discusses the realm of pure mathematics in lucid and elegant language. The classic example, Euclid's proof of the infinity of primes, is explained wonderfully clearly. Acheson also considers imaginary numbers (the mysterious notion of the square root of negative numbers) and shows, through an examination of history, how dealing with these counter-intuitive mathematical entities *as though they were real* led to perfectly correct mathematical solutions to problems.

And, in a final chapter that I have by no means fully grasped, the author leads us, with obvious relish, to what he calls "the most stunning result in the whole subject, so far" – the equation

$$\mathbf{e}^{\mathbf{i}\pi} = -\mathbf{1}.$$

While I will have to sit at some length with my mathematically talented friends to understand this result even in the very simplified form Acheson presents it, the author's awe at the process leading to this result is palpable:

> Firstly, we have obtained it by putting together a whole variety of relatively sophisticated mathematical ideas,including calculus, infinite series and imaginary numbers.
>
> Secondly, the formula is of great practical value; it is the sole reason, really, why virtually any engineering or physics book on oscillations has both e and i . . . all over the place, greatly simplifying many of the calculations.

Both at the beginning and at the end of the book, Acheson characterises mathematics in the following ways:

1. Wonderful theorems
2. Beautiful proofs
3. Great applications

1089 and All That, short and simple though it is, captures the spirit of the above points marvellously. Often I found myself wishing that the book explored a particular idea in more depth, and this is, I think, the sign of a successful and exciting introduction to an intellectually vast, deep and complex field.

VENKATESH ONKAR works at Centre For Learning, Bangalore, where he teaches sociology and history. He may be contacted at vonkar@gmail.com

Review

Introducing Steven Strogatz

Reviewed by Shashidhar Jagadeeshan

For those who don't know him, Steven Strogatz is the Jacob Gould Schurman Professor of Applied Mathematics at Cornell University in the US. He is a passionate educator who spends considerable time and energy trying to communicate the intricacies of mathematics to the lay audience.

Strogatz believes that *"Mathematicians are notoriously bad at [communication],"* and he goes on to comment jokingly about the mathematical community: *"we're writing on the board, with our back to the audience – it's like they don't even need to be there as far as we're concerned."*

He is the author of several best-selling books like *The Joy of X: A Guided Tour of Math, From One to Infinity*. He writes frequently for the New York Times, and appears regularly on National Public Radio in the US.

The purpose of this short review is to alert readers to Steven Strogatz's work (http://www.stevenstrogatz.com) in general, and to draw special attention to 15 pieces, under the broad heading of 'Elements of Maths,' that he wrote for the New York Times from January 2010 to May 2010 (http://www.stevenstrogatz.com/essays/?tag=Elements+of+Math).

Here is how the author describes the purpose of these articles:

"I'll be writing about the elements of mathematics, from pre-school to grad school, for anyone out there who'd like to have a second chance at the subject — but this time from an adult perspective.

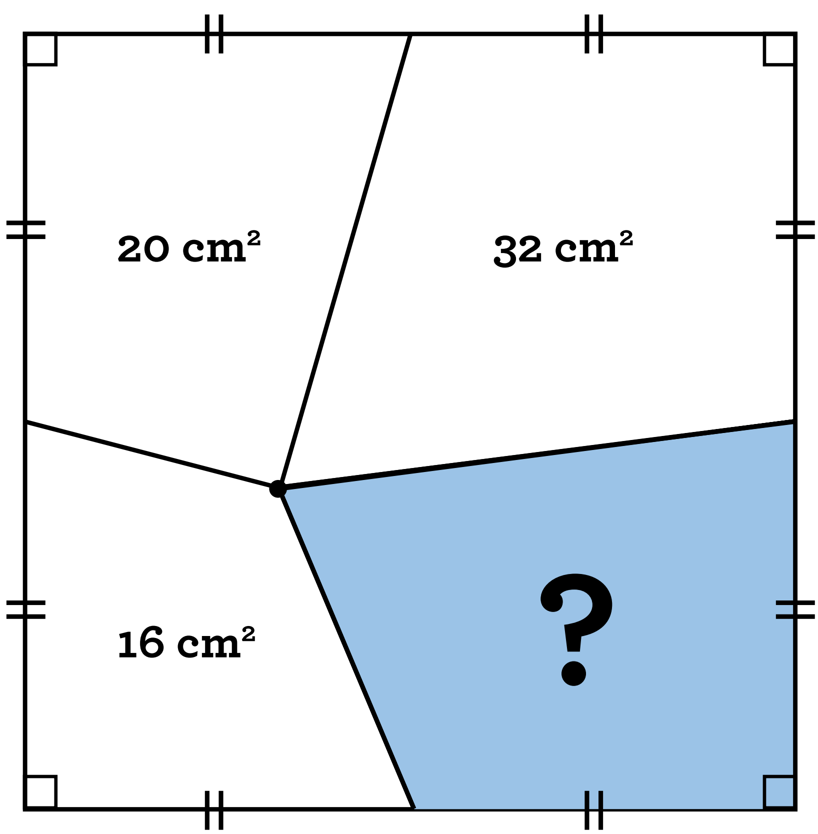
Keywords: communication, about math

It's not intended to be remedial. The goal is to give you a better feeling for what math is all about and why it's so enthralling to those who get it."

So the topics range from basic arithmetic to Group theory. The articles have catchy titles like 'From fish to infinity' and 'The enemy of my enemy' and investigate important areas in mathematics. Each article brings new perspectives to fundamental concepts, draws upon examples and illustrations from daily life, and ends with a list of suggested reading to deepen one's understanding.

For the lay person, these articles give a nuanced introduction to mathematics. For the student, they offer novel ways of thinking about the mathematics they are learning. And for the teacher, they are a great resource not only to enhance understanding of topics we take for granted, but also to enrich one's teaching.

SHASHIDHAR JAGADEESHAN received his PhD from Syracuse University in 1994. He has been teaching mathematics for over 25 years. He is a firm believer that mathematics is a human endeavour, and his interest lies in conveying the beauty of mathematics to students and demonstrating that it is possible to create learning environments where children enjoy learning mathematics. He is the author of Math Alive!, a resource book for teachers, and has written articles in education journals sharing his interests and insights. He may be contacted at jshashidhar@gmail.com.



Send in your solutions to AtRiA.editor@apu.edu.in

The Closing Bracket . . .

The Closing Bracket for this issue was prompted by Dan Meyer's post http://blog.mrmeyer.com/2018/learning-the-wrong-lessons-from-video-games/. I've been following this blog for more than 5 years now and find that Dan Meyer is fascinated equally by technology as well as by pedagogy. Though many of us have our doubts about the invasion of technology and my generation is for the most part, resolved to use it judiciously, its insidious presence has insinuated itself in our lives. For better or for worse. And just as in that knotty relationship, which this phrase comes from, we need, as teachers to, if not embrace, at least accept and try to understand, the effect which technology has on our classroom and how we can harness its psychology to improve our pedagogy.

In this post, Dan Meyer talks about video games and how they respond when a player makes a move. He quotes Karl Groos who claimed in *The Play of Man* that "the joy in being a cause" is fundamental to all forms of play. 'One hundred years later, Phil Daro would connect Groos's theory of play to video gaming: *Every time the player acts, the game responds [and] tells the player your action causes the game action: you are the cause.*'

And this is Dan Meyer's point. He says that designers of interactive math learning software, think that interactivity makes math learning a game. But in such games, we don't do what video games do. We don't let the students experience free fall. We cushion them with warnings, with guard rails, with expressions of dismay at the 'wrong' answer and with encouragement to try again to get the 'right' answer. If we look at a video game instead, it lets the player fall, it takes stock of the current position and expects the player to experience the consequences of the path that he or she has taken. There is no option to try again, no choice but to move on from where they have landed. Pretty much like real life.

A very useful pointer for designers of math learning software to keep in mind! But can it work in the current most commonly used scenario which presents endlessly repetitive problems for students to solve? I don't think so. Which means that math learning software has to design tasks which use mathematical concepts familiar to students of the age group to which the software is designed. But that is at a deep, yet ephemeral level. What the student should work with would be a situation which necessitates the use of these concepts, in tasks which are fairly open-ended and which can accept the student's input and move on to the next level, using the consequences of this input. That's when students will experience the 'joy in being a cause'.

Is the task too much of an ask? And can it be attempted by a teacher in a classroom without technology? As many classrooms in India are? It's certainly worth a try! Too many generations of students have found the math classroom boring and challenging in the dullest of ways. It's a different matter that these students as parents and teachers have continued to feel that drill and practice are the only way to learn mathematics. Though I would never decry the value of practice, it should not be the way for students to learn to hate mathematics. It is time to have students work on and get involved in tasks which are not just solved but engaged with.

Many of us teachers have tried to move on with the times and make our students feel a part of the classroom. But as Alan Wigley says in the Mathematics Teacher MT141 December 1992, teachers set up secure pathways for students to traverse. Questions, if asked, are intended to nudge the student along to the 'right answer'. The fascinating detours and the perplexing mazes, can also be fantastic learning opportunities for students. Why don't we allow them to stray off the beaten path and discover that mathematics is not always linear?

Sneha Titus
Associate Editor

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Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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Articles involving all aspects of mathematics are welcome. An article could feature: a new look at some topic; an interesting problem; an interesting piece of mathematics; a connection between topics or across subjects; a historical perspective, giving the background of a topic or some individuals; problem solving in general; teaching strategies; an interesting classroom experience; a project done by a student; an aspect of classroom pedagogy; a discussion on why students find certain topics difficult; a discussion on misconceptions in mathematics; a discussion on why mathematics among all subjects provokes so much fear; an applet written to illustrate a theme in mathematics; an application of mathematics in science, medicine or engineering; an algorithm based on a mathematical idea; etc.

Also welcome are short pieces featuring: reviews of books or math software or a YouTube clip about some theme in mathematics; proofs without words; mathematical paradoxes; 'false proofs'; poetry, cartoons or photographs with a mathematical theme; anecdotes about a mathematician; 'math from the movies'.

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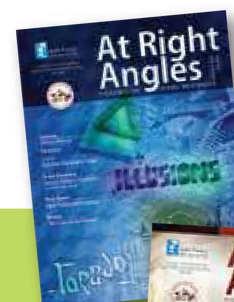

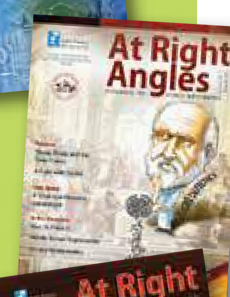

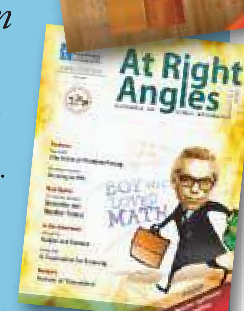

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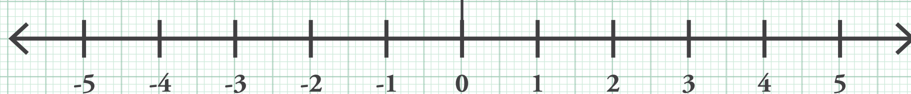
At Right Angles
A Resource for School Mathematics

APPROACHES TO EQUATIONS

PADMAPRIYA SHIRALI



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INTRODUCTION

Introduction of a complex topic is always a challenge. If one intends to root it in the student's daily experience, one is compelled to select a concrete model which serves the purpose but may have limited scope. At some point, the student will need to abstract out the general notion in order to build a broader sense of the concept.

The topic of 'Equations' can be approached in several ways. The choice of approach has a strong impact on the conceptual image which a student builds about a given concept. Hence, the choice is crucial in helping a student in understanding the concept as well as in developing the procedure for solving the problems.

However, every approach has its limitations and can be used only for solving certain types of problems. Its use is limited and it may become necessary to expose students to other approaches when the type or complexity of the problems alters.

For the teacher, there are crucial decisions to be made: when and how to introduce the concept, and how much emphasis should be placed on the corresponding skills.

Equations encompass varied types of problems. Here are some:

$$x + 5 = 7, \qquad 2\,(x + 10) - 3x = 16, \qquad 2x + 3 = 3x - 7,$$

$$2x + 3y = 11,\ 3x + 2y = 14, \qquad x^2 + 6x + 9 = 0, \qquad 2^x = x^2.$$

Obviously, procedures for solving these problems vary greatly.

How does one introduce the idea of equations (without going into a formal definition) to students? An algebraic equation is an equality involving variables.

In this article, I focus on two well-known approaches, the balance scale approach and the machine approach.

The **balance scale** approach essentially uses the analogy of a weighing balance.

For a weighing balance to be balanced, the weight on the left must equal the weight on the right.

Similarly, in an equation, the value of the expressions on the two sides are equal. The expressions on the two sides do not look the same, but when solved for a value they are equal.

We can model simple equations with a balance scale.

The **machine** approach treats the 'x' as an input on which one or more operations are performed in a definite sequence, in order to produce an output.

Both these approaches ultimately help the student to gain an understanding of what an equation is, to formulate an equation to represent a given problem, and to solve equations in one variable (one unknown), making proper use of symbols.

It is important that teachers spend sufficient amount of time helping students to build the capacity for framing equations, right from the beginning. In order to develop this capacity, it is good to use visuals wherever possible and to select daily life problems posed in simple language. The teacher needs to give equal emphasis to the forming of equations and to problem-solving procedures.

Another crucial point to note with regard to problem-solving is that students may use other methods to solve the problems; for example, trial-and-error. Some methods may be quite long-winded. At no point should the teacher discount these methods. The teacher can acknowledge that these methods are valid and follow it up with showing the standard and efficient methods.

Keywords: Algebra, language, balance, machine, equality, solution.

BALANCE APPROACH - ACTIVITY 1

Objective: To demonstrate how the same change performed on both sides maintains the level of a balance scale.

Materials: Balance with two pans; Different weights: 50gm, 100gm, 500gm

Let students place some weights to bring the balance to a level position.

Ask them what would happen if a 100gm weight is removed from the left side pan.

How would the balance look? (Which side will go down?)

What should one do to the right side pan to bring the balance to a level position?

Now try adding a 500gm weight to the right side pan.

How does the balance look now?

What should be done to the left side pan to bring the balance to a level position?

Now halve the weight on the left side pan of the balance.

What needs to be done to the right side of the pan to bring the balance to a level position?

Similarly, try to place three times the weight on one side and see what needs to be done on the other side.

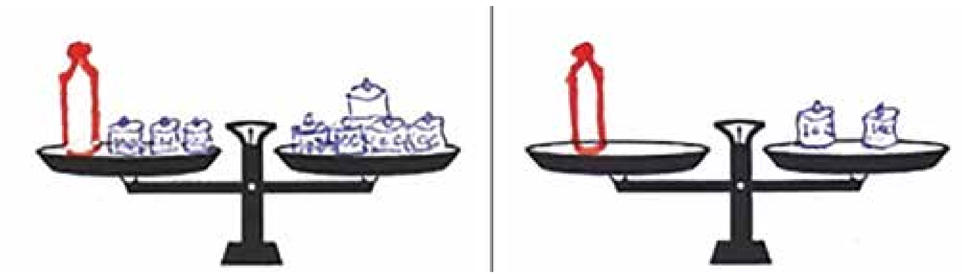
Note: The purpose of exposing students to this activity is to help them understand that an equilibrium situation gets affected by any changes that are made on one side and that it is necessary to compensate this by the same action on the other side.

ACTIVITY 2

Objective: To help children learn to build equations for a given situation with one operation and find the value for which the equation holds true.

Materials: Bottles or packets of similar kind and 100 gram weights

Note: Teacher can also make symbolic drawings of these on the board as shown in the picture.



The picture here shows 1 bottle and three 100gm weights on the left hand side and five 100gm weights on the right hand side.

Pose the question: What do you see on the left hand side?

Do we know the weight of the bottle? How shall we name its weight? Since the students have already been exposed to the idea of using the letter 'x' as a variable to represent an unknown quantity, they will have no difficulty in accepting its usage in this situation.

What do we see on the right hand side?

Is the balance in the level position?

How do we represent all this information as an equation?

$$x + 300 = 500.$$

What would be the weight of the bottle?

Students should be able to give the answer to this immediately.

However, the teacher needs to expose them to the procedure of inverse operation as well.

Note: Teacher should discuss and explain 'inverse operations' for all four basic operations at this point.

300 gm can be removed from both sides to maintain the balance in level position.

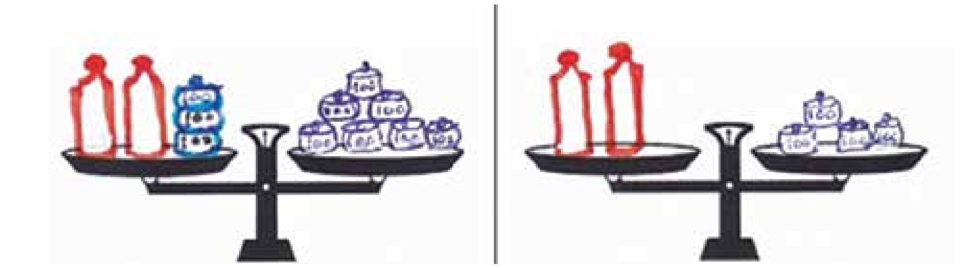
$$x + 300 - 300 = 500 - 300.$$

Hence, $x = 200$.

Teacher can do more problems of this kind involving other operations before moving on to the next level.

ACTIVITY 3

Objective: To help the children to learn to build equations for a given situation with two operations and find the value for which the equation holds true.



The picture here shows 2 bottles and three 100gm weights on the left hand side and seven 100gm weights on the right hand side.

How do we represent this information as an equation?

Again, talk about the weight of the bottle as the unknown 'x' and help the students to formulate the equation.

$$2x + 300 = 700.$$

What would be the weight of the bottle which is denoted here by x?

Students need to internalise that 'x' stands for some definite quantity in each situation.

Some students may be able to figure out an answer to this through mental calculations.

Help them verify their answer by following the procedure of inversion operations as well.

The visual aid helps students in thinking about what can be removed from both the sides.

$$2x + 300 - 300 = 700 - 300 \text{ (inverse of addition is subtraction)}$$

Point out that +300 and –300 cancel each other.

$$2x \div 2 = 400 \div 2 \text{ (inverse of multiplication is division)}$$

$$\text{Hence } x = 200.$$

At the introductory stage, students should use inverse operations as part of their working. At a later point they may see the equivalence of writing it only on one side as the other side will inevitably cancel out. That is, instead of writing

$$2x + 300 - 300 = 700 - 300$$

they will write

$$2x = 700 - 300.$$

The teacher can do more problems of this kind involving other operations before moving on to the next level.

Note: The teacher can show transposing variables and numbers from one side of the equation to the other after working through a few problems.

ACTIVITY 4

Objective: To expose the children to a variety of equations involving variables with negative coefficients, fractions and decimals.

How would we solve this? (Here the variable is on the right side.)

$$10 = x - 32.$$

Point out that an equation remains the same if the expressions are interchanged. (This is equivalent to seeing the balance scale from two sides; when the viewer changes sides, the left pan becomes the right one and vice versa.)

It can be written as $x - 32 = 10$ and solved in the normal way.

How would we solve this? (Here the variable comes with a negative sign.)

$$12 - x = 5.$$

Note: Students may not yet be ready to handle variables with negative coefficients.

Point out that $-x$ can be cancelled out by using $+x$.

$$12 - x + x = 5 + x,$$

$$12 = 5 + x.$$

This can be written as $5 + x = 12$ and solved in the normal way.

It is good to expose students to equations which come with fractions and decimals as well.

$$x + 1.5 = 4,$$

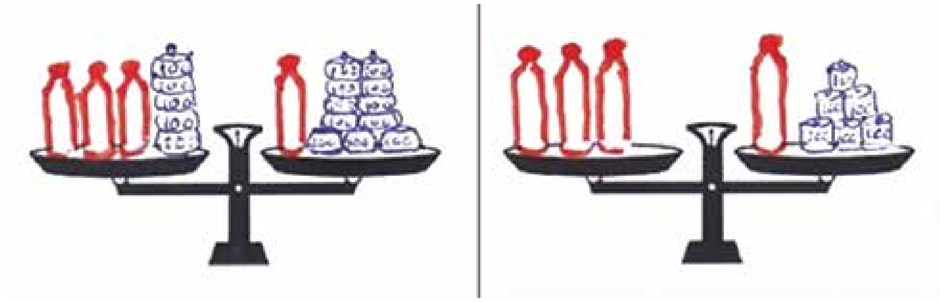
$$a - \frac{1}{2} = 7,$$

$$\frac{1}{2}\,b = 16,$$

$$\frac{5}{4}\,y = 10.$$

ACTIVITY 5

Objective: To help the children to learn to build equations for situations where there is an unknown on both sides.



The picture here shows 3 bottles and five 100gm weights on the left hand side and 1 bottle and eleven 100gm weights on the right hand side.

Do the students see that the bottles on the left hand side and right hand side are identical and hence will have the same weight?

How do we represent this information as an equation?

$3x + 500 = x + 1100.$

Ask the students 'where will you begin'?

This problem can be resolved by first doing an inverse operation for 500 or it can also be resolved by first removing an x (a bottle!) from both sides of the balance.

It would be good for the students to see that it does not make a difference which of the two ways they choose to begin with.

One way:

$$3x + 500 - 500 = x + 1100 - 500,$$
$$3x = x + 600,$$
$$3x - x = x + 600 - x,$$
$$2x = 600,$$
$$x = 300.$$

Other way:

$$3x + 500 - x = x + 1100 - x,$$
$$2x + 500 = 1100,$$
$$2x + 500 - 500 = 1100 - 500,$$
$$2x = 600,$$
$$x = 300.$$

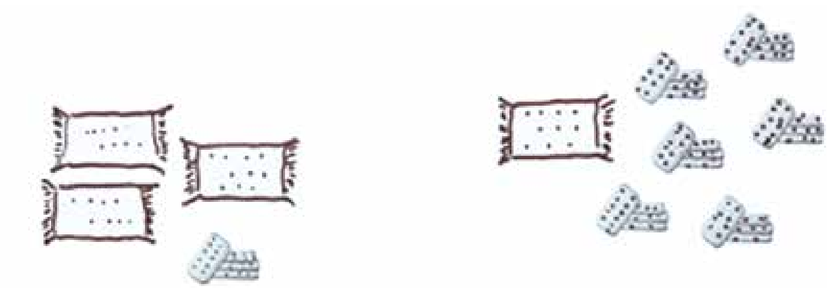
The teacher can do more problems of this kind involving other operations before moving on to the next level.

ACTIVITY 6

Objective: To help the children to learn to build equations for various other situations involving visuals and word problems

Each van holds the same number of people.

If the number of people on both sides is equal, how many people are in the van?



Each packet has the same number of biscuits.

If the number of biscuits on both sides is equal how many biscuits are in each packet?

ACTIVITY 7

Objective: To let the students demonstrate their understanding of equations by creating story problems

Give students a few equations of varied kind as below and ask them to write story problems for them.

Think of a story represented by the equation $4x = 8$.

Think of a story represented by the equation $(4x - 2) + 7 = 33$.

Think of a story represented by the equation $5\,(x - 3) = 20$.

ACTIVITY 8

Objective: To understand equivalence of equations, substitution as a way to check the correctness of the answer and the interchangeability of expressions.

The teacher needs to take care to see that students have grasped the following points.

Do they see that all these equations are the same?

$$3x = 6,$$

$$6 = 3x,$$

$$3x = 2 + 4.$$

How do the students check if the obtained value is correct?

The teacher will need to show substitution as a way of checking the answer.

Do they see that interchanging expressions from one side to another does not change the equation?

Ex. $2x + 7 = 3x - 2$ is the same as $3x - 2 = 2x + 7$.

Questions for equations are often posed in different types of wording.

Ex. Solve for x.

Do students see that to solve an equation means to find the value of the unknown?

What value of x will make this equation true?

MACHINE APPROACH

I will now take up the *machine approach* for introducing equations. The machine approach can be used independently as it helps the students visualise the way an equation is formed in a sequential manner and understand the procedure of undoing for solving them. However, the machine approach is best suited for problems with a single variable expression on one side and a number on the other side.

The teacher can play this game with the students. Let the students figure out what the teacher did after completing Activity 9.

Game 1: I will detect your number!

Teacher: Think of a number → Add 5 to the number → Multiply by 4 → Subtract 2

What is your answer?

Student: '38'.

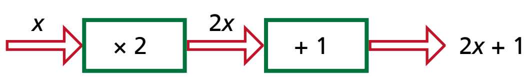
Teacher: Your number is 5.

ACTIVITY 9

Objective: To help students to learn to 'undo' what has been done in a two-stage machine.

The teacher can draw on the board an input output machine as shown in the picture.

Here is an input-output machine. It takes a number as an input and performs two operations on them and produces an output.



Look at the input output table of this machine.

5	11
3	7
12	25
7	15

What does this machine do? What are its two operations?

Students will quickly see that each number gets multiplied by 2 and 1 is added to the product to get the answer.

Now pose the question 'If the output of this machine is 29, what is the input?'

Again, students may be able to answer this by working it out mentally.

Encourage them to form an equation.

If the input is 'y' then

$$y \times 2 + 1 = 29.$$

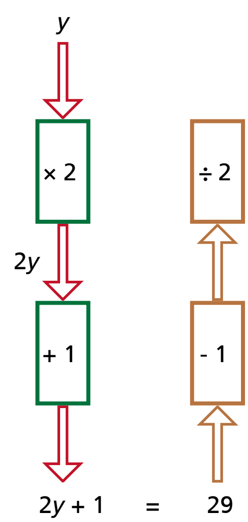
Some students prefer to draw machines going from top to bottom. They can practise the undoing process on the diagram.

$$2y + 1 = 29.$$

It is possible to solve this equation by going backwards or undoing what has been done.

The last operation performed by the machine was 'add 1'. To undo that, 1 must be subtracted from 29.

$$29 - 1 = 28$$



The previous operation was multiplication by 2. To undo that, 28 has to be divided by 2.

$$28 \div 2 = 14.$$

That gives 14.

More such examples with various operations can be worked out.

Game 2: Can the students now figure out how the teacher deduced their number in game 1?

Now ask the students to come up with 'Think of a number' instructions, (with four or five operations) which they can undo to deduce the starting number.

Let them make a diagram to show the series of operations.

Note: Certain combinations of operations will require the usage of brackets while writing an equation. Discuss the need and usage of brackets in such situations.

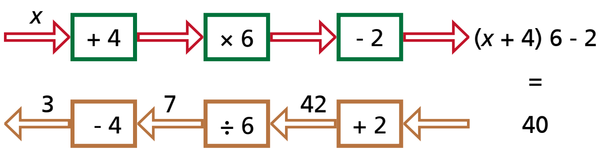
ACTIVITY 10

Objective: To help students to learn to 'undo' what has been done in a three stage machine.

Teacher can draw such a machine as the one given below.

$$+\,4 \times 6 - 2$$

Here is a chain of three machines.



Pose the question 'what comes out if you feed in 5?'

If the output of the machine is 40, how will you express this as an equation?

Point out that this equation requires the use of brackets as the sum of the two numbers is being multiplied by 6.

$$(x + 4) \times 6 - 2 = 40$$

How is this 'undone'?

Last operation is subtraction of 2 to get 40.

$$40 + 2 \text{ is } 42$$

The preceding operation is multiplication by 6 which gave 42.

$$42 \div 6 \text{ is } 7$$

The preceding operation is addition of 4 which gave 7.

$$7 - 4 \text{ is } 3$$

$$\text{Hence } x = 3$$

The teacher can draw different machines and specify an output for each machine. Students can find the corresponding input for them.

ACTIVITY 11

Objective: To help the students to tackle word problems (with usage of brackets) involving equations, through balance approach.

Ex. Yash has 15 stamps. Asif has 17 stamps.

Yash gives Asif some stamps. Now Asif has 3 times as many as Yash.

How many did Yash give Asif?

Here is a trial and error approach:

No. of stamps given by Yash	No. of stamps with Yash	No. of stamps with Asif
0	15	17
1	14	18
2	13	19
3	12	20

Is there a row where Asif has 3 times as much as Yash? Not as yet.

It is more efficient to use another approach.

Students could make drawings initially to aid in their understanding.

Let x stand for the number of stamps given by Yash to Asif.

Yash	**Asif**
15	17
$15 - x$	$17 + x$
$3(15 - x)$	$17 + x$

=

How many will be with Yash now? $15 - x$

How many will be with Asif? $17 + x$

Statement says that Asif has 3 times as much as Yash.

$$17 + x = 3(15 - x)$$

How do we solve this equation?

Let us first multiply out the brackets.

$$17 + x = 45 - 3x$$

After this step, students may work it out in different ways.

This is one way.

$$17 + x + 3x = 45 - 3x + 3x,$$

$$17 + 4x = 45,$$

$$17 + 4x - 17 = 45 - 17,$$

$$4x = 28,$$

$$x = 7$$

ACTIVITY 12

Objective: To help the students to tackle problems involving equations by undoing or the machine approach.

Example 1:

A bus picked up a certain number of passengers at the first stop. At the second stop it picked up five more. At the third stop it picked up the same number of passengers as were in the bus. At the fourth stop three passengers got out. There are 23 passengers left in the bus. How many passengers got into the bus at the first stop?

How many passengers were picked up at the first stop? Unknown. Call it x.

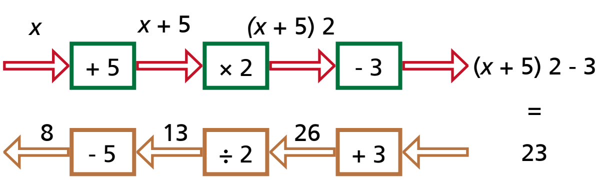
How many passengers were picked up at the second stop? The total number of passengers will now be $x + 5$.

What happened at the third stop? The passengers doubled in number. So the total number of passengers is now $2(x + 5)$.

What happened at the fourth stop? Three got out. The total number of passengers is now $2(x + 5) – 3$.

How many are in the bus now? 23

So $2\,(x + 5) – 3 = 23$.



Example 2:

A fruit seller marked each pomegranate with a certain price. When he found he could not sell them at that price, he reduced the price by Rs.4. He then managed to sell 15 of them for Rs.390. What was his marked price?

What was his marked price? Unknown. Call it x.

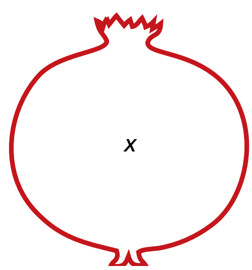
What was the price after the reduction? $x – 4$

How many pomegranates did he sell? 15

How much money did he get? $15\,(x – 4)$

What did he earn? Rs. 390

Hence $15(x – 4) = 390$.

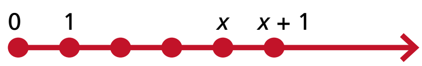


ACTIVITY 13

Objective: Expose students to a variety of word problems (variable on one side)

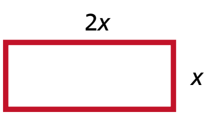
Teach students different ways of depicting information to help them comprehend problems.

The sum of two consecutive numbers is 83. Find the numbers.



$x + x + 1 = 83$

The length of a rectangle is twice its breadth. If the perimeter is 72 cm, find the length and breadth of the rectangle.



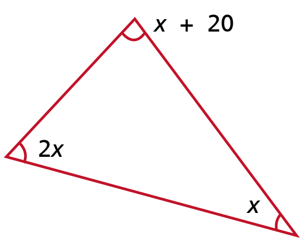
$2x + x + 2x + x = 72$

There are 650 students in a school. If the number of girls is 106 more than the boys, how many boys are there in the school?

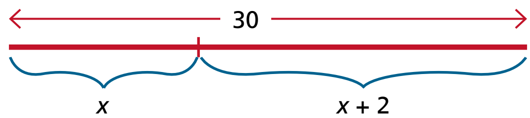
$x + 106$ x

$x + x + 106 = 650$

One angle, A, of a triangle is twice as large as another angle, B. The measure of the third angle is 20 degrees greater than the measure of angle B. Find the three angles.



An electrician cuts a 30ft length of wire into two pieces. One piece is 2 ft longer than the other. How long are the two pieces?

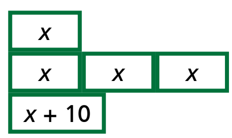


$x + x + 2 = 30$

In the school dining hall, there were some students on Monday. On Tuesday there were twice as many. When they were counted on Tuesday there were 230 students. How many were there on Monday?

$2x = 230$

Tanvi has some marbles. Sonia has three times as many as Tanvi. Ami has 10 more than Tanvi. Ami has 25 marbles. How many marbles do they have altogether?



$x + 10 = 25$

Class 5 has one more student than class 6. Class 4 has three fewer students than class 5.

Altogether there are 92 students in classes 4, 5 and 6. How many students are there in each class?

Class 4	**Class 5**	**Class 6**
$x + 1 - 3$	$x + 1$	x

$x + 1 - 3 + x + 1 + x = 92$

Arsh weighs 8 kg more than Yuga. Find their weights if the sum of their weights is 80kg.

On a farm there were some hens and sheep. Altogether there were 8 heads and 22 feet.

How many hens were there?

Challenge!

There are three buckets: one red, one blue and one yellow. Each holds a maximum of 5 litres. Liquid is measured carefully in whole number of litres and poured into the buckets, a different number of litres in each one. If the liquid in the red bucket was poured into the blue bucket, it would then contain the same amount of liquid as the yellow bucket. Half the content in the yellow bucket is the same as twice that in the red bucket. How much liquid is there in each bucket?

Sam's grandmother has an old recipe for cherry buns. To make them, she weighs two eggs. Then she takes the same weight in flour, and in sugar and in butter. She mixes all this together and then she adds half the weight of the 2 eggs in chopped glace cherries. She has enough mixture to put 45 grams in each of 12 paper cake cases. What was the weight of one egg?

Acknowledgement: Source for challenge problems: nrich (https://nrich.maths.org/).

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Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box' Padmapriya may be contacted at padmapriya.shirali@gmail.com