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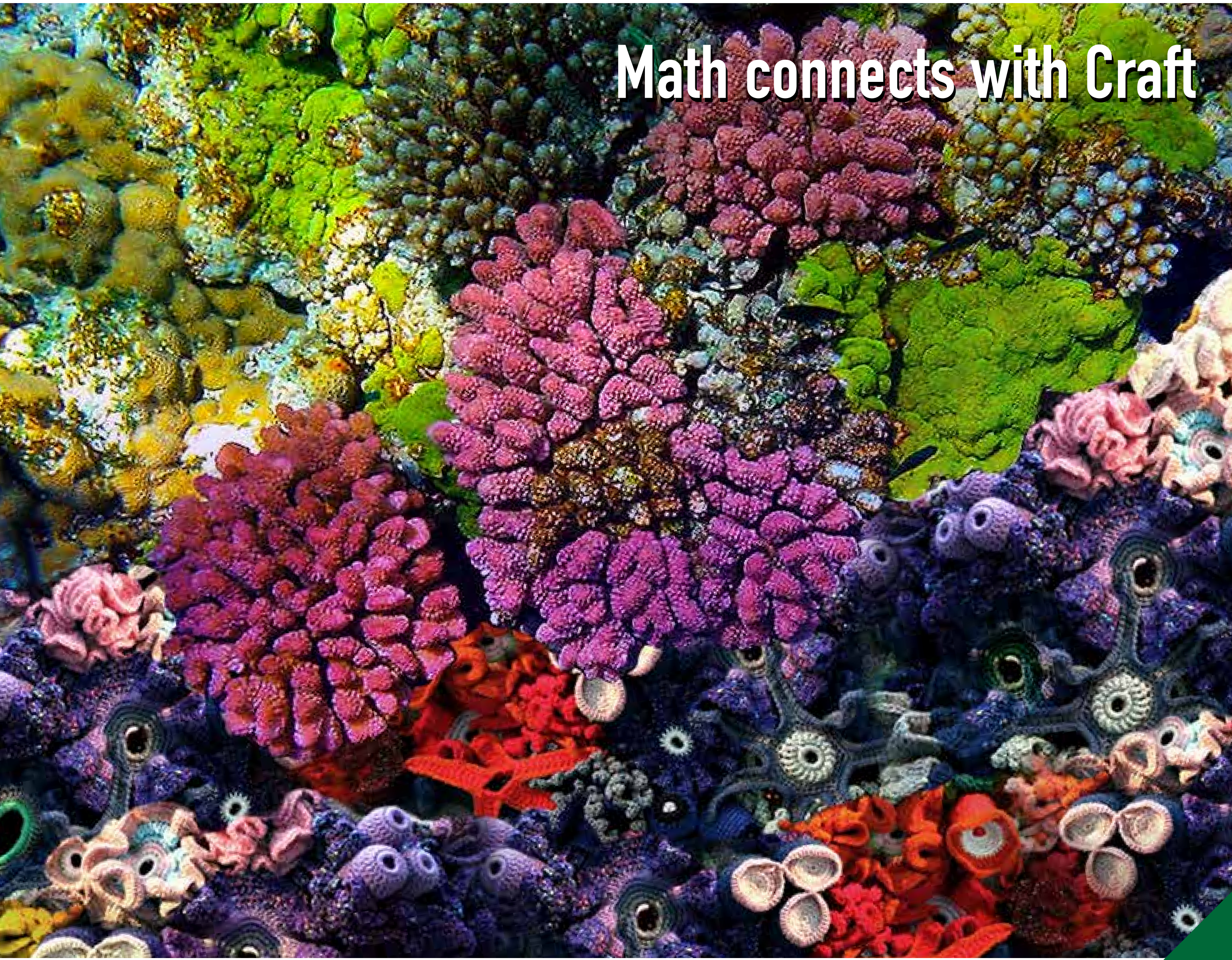
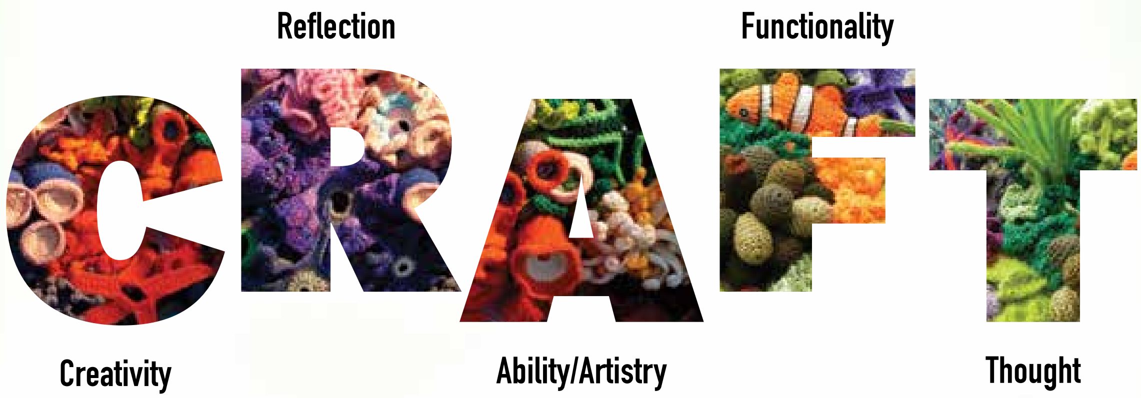
Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS



Math connects with Craft

PULLOUT
INTRODUCTION TO ALGEBRA - IV



Craft and Mathematics have long marched together but the partnership has been subtle and in many cases, almost ignored. In fact, many people who practise crafts such as knitting, crochet and quilting are self-confessed Math phobics. The very acronym of the word CRAFT that we have created made us realise that the projection of the stodgy mathematician hides these facets which are often called to play in a mathematical exploration.

Now you make well ask why a picture of a coral reef is on the cover of a math magazine which has decided to turn its focus on various crafts and the mathematics in them. Look carefully at the picture - among the natural corals are several crocheted replicas. We were stunned to see that others others who had investigated connections between craft and math had come up with very creative ideas. Check out the website https://ideas.ted.com/gallery-what-happens-when-you-mix-math-coral-and-crochet-its-mind-blowing/ which describes two sisters who used crochet to explore three dimensional shapes in a tactile way. Their project which eventually grew into a community effort called the Crocheted Coral Reef, captured their concern about the environment, their belief that mathematics had to be experienced to be learnt and their love of crochet.

Starting with a short article on quilting and the mathematics that the author used in it, we hope to periodically feature other such craft projects.

From the Editor's Desk . . .

At Right Angles is delighted to announce that we are now registered with the Registrar of Newspapers for India. You would notice a change in the numbering of our issues, this is in order to comply with RNI Guidelines.

The March 2019 issue is packed with articles for students, teachers and teacher educators of all classes. In Features, we carry thought provoking yet simple articles - Is it possible to have a lattice point equilateral triangle? Is there even more to know about 'e'? Who was W.W. Sawyer and what did he contribute to mathematics education? You are sure to find something to interest you here! In ClassRoom, you will find material on number patterns – noted by several student contributors: Bodhideep Joardar, Rahul Miraj and our youngest contributor so far - Adithya Rajesh. Striking results are not just reported, they are also explained and justified with proofs. And our sources vary from students to observations from colleagues in mathematics classrooms and mathematical tricks from websites- these are explored and explained. Magic triangles appear again - thanks to James Metz - and so does cryptography with an addendum by K G Mishra to the articles on the Hill Cipher. R Gomathy describes how she introduced her students to hands on mathematics by playing with tiles and creating tessellations - a creditable achievement in a government school classroom. And Low Floor High Ceiling is back with a dot sheet activity that pushes students to observe, experiment, conjecture and generalise.

TechSpace takes on a new arena this time – exploring probability with a graphics calculator. Barry Kissane describes just how to introduce students to the nuances of chance without having to resort to elaborate and endless activity. Problem Corner has its usual plethora of thought provoking and absorbing problems for all ages. And in Review, R. Mohan has an in-depth and personal encounter with W.W. Sawyer's *Mathematician's Delight*.

Padmapriya Shirali takes on a big problem in mathematics classrooms- Indices and Algebraic Identities in Part 4 of the Algebra PullOut series. And Swati Sircar has a take home gift for you all - a TearOut poster explaining Commutativity, Associativity and Distributivity of mathematical operations on Fractions.

We are happy to welcome Haneet Gandhi from the Department of Education (C.I.E), University of Delhi, to our Editorial Committee. Haneet has contributed several articles to previous issues, notably, a series on Tessellations, we are sure that her expertise will be invaluable in guiding novice contributors.

We hope that you will relish each page of this issue- do send in your feedback to AtRiA.editor@apu.edu.in Articles are welcome at the same address, please refer to the Call for Articles at the end of the magazine. Connect with us on our FaceBook page AtRiUM!

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an ‘academic’ and 'practitioner' oriented magazine.

Azim Premji University
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Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

LEGENDS - Primary School | Middle School | High School | Pre-University

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Contents contd.

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

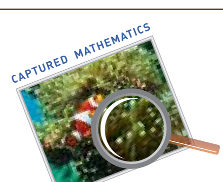
Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.



Captured Mathematics

Scattered across this issue, we present to you a collection of several interesting photographs which have captured some mathematical concepts.

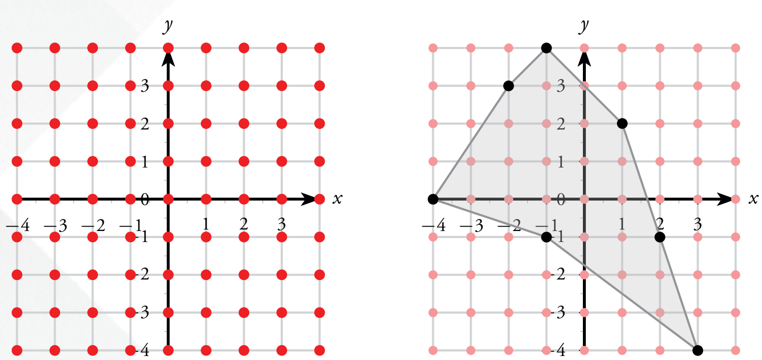
LATTICE POINT GEOMETRY Part I

Non-existence of a Lattice-point Equilateral Triangle

SHAILESH SHIRALI

In this article, Shailesh Shirali begins with a seemingly simple question but develops the answer into not one, but four different proofs! While the content focuses on mathematics that has many applications, some of which are mentioned here, the multiplicity of proofs is an added draw, helping the teacher to illustrate innovative ways of thinking and connections across approaches.

In the cartesian plane, coordinatized by a pair of rectangular axes, we say that a given point is a ***lattice point*** if its x- and y-coordinates are both integers. For example, the points with coordinates $(0, 1)$, $(1, 2)$, $(2, -3)$ and $(-3, 7)$ are lattice points, whereas $(3, 3.5)$ and $(2.3, 1)$ are not lattice points. The set of all lattice points in the coordinate plane is called a ***lattice***. A polygon in the coordinate plane all of whose vertices are lattice points is called a ***lattice polygon***. (See Figure 1.) There are many mathematical results of great interest pertaining to lattices and to lattice polygons, and we shall talk about some of them here.



Portion of a lattice
(i)

A lattice polygon
(ii)

Figure 1

Keywords: Lattice points, lattice-point triangle, irrational number, slope, congruence, descent

The notion of a lattice actually originated in the study of crystals and is a concept derived from crystallography. The definition of a lattice that we have adopted is a slightly restricted one, and crystallographers prefer a more general definition. But we shall not venture into that area for now.

The question we ask in Part I of this multi-part article is: *Using lattice points as vertices, can we find an equilateral triangle in the plane?* In short: ***Does there exist a lattice-point equilateral triangle?*** (To avoid needless complications arising from degenerate cases, we could specify 'lattice-point equilateral triangle with nonzero area'. But we shall assume this to be the case, implicitly.)

We can ask more generally about regular polygons in the lattice plane. Trivially, there exist squares in the lattice plane. How about regular pentagons? Or regular hexagons? Or regular heptagons or octagons? Clearly, there are infinitely many questions of this kind which can be posed.

For the moment we shall focus only on the equilateral triangle. We shall see that there is some elegant mathematics involved.

There does not exist a lattice-point equilateral triangle

In this section, we show that there does not exist an equilateral triangle whose vertices are distinct lattice points. We do so in four different ways. Why so many proofs of the same result? Isn't that an overkill? Possibly—but not to this author! For one thing, all the proofs are elegant, illustrating different mathematical themes; and they all lead in different directions. Precisely because of this, the basic result may be generalised in different ways.

First proof: Argument based on area. First we note the following fact: any lattice-point $\triangle ABC$ (i.e., a triangle whose vertices are lattice points) can be enclosed within a lattice-point rectangle in such a way that each vertex of the triangle either lies on a side of the rectangle or coincides with a vertex of the rectangle. Figure 2 illustrates how this may be done. Small variations occur depending on how the triangle is oriented, but the general idea is the same. It will always happen that at least one vertex of the triangle will coincide with a vertex of the rectangle. Observe that in this configuration, the sides of the rectangle necessarily have integer lengths. The same is true for the two shorter sides ('legs') of the three right-angled triangles that surround the given lattice triangle. It follows that the area of the rectangle is an integer, and the areas of the three right-angled triangles are all half-integers. (By 'half-integer' we mean a fraction whose denominator is at most 2.) This implies that the area of the given triangle is a half-integer as well. In particular, this means that the area of $\triangle ABC$ is a rational number.

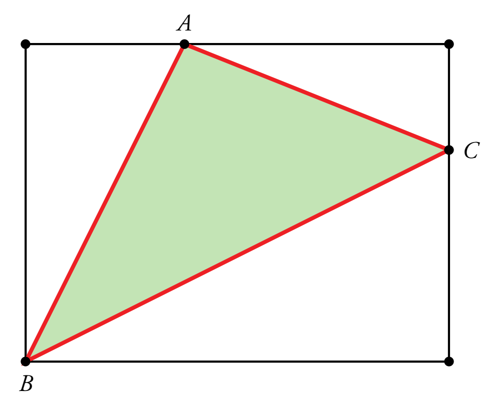


Figure 2

The argument sketched above is true for any lattice-point $\triangle ABC$. Now we consider the case when ABC is a lattice-point equilateral triangle with positive area. Let us apply the sine formula for the area of a triangle: "Area equals half the product of any two sides times the sine of the included angle." Since $\triangle ABC$ is equilateral by assumption, this yields:

$$\text{Area of } \triangle ABC = \frac{1}{2}AB^2 \cdot \sin 60^\circ = \frac{\sqrt{3}}{4}AB^2.$$

Let $A = (a, a')$ and $B = (b, b')$ where a, a', b, b' are integers. By the Pythagorean formula,

$$AB^2 = (a-b)^2 + (a'-b')^2 = \text{some nonzero integer},$$

implying that the area of the triangle is

$$\frac{\sqrt{3}}{4} \times \text{some nonzero integer}.$$

Hence the area of the triangle is an irrational number (since $\sqrt{3}$ is irrational). However, we had already concluded by a different line of argument that the area of the triangle is a rational number. So we arrive at a contradiction. That is, the assumption that there exists a lattice-point equilateral triangle leads to a contradiction and therefore cannot be true. Hence there does not exist a lattice-point equilateral triangle. □

Second proof: Argument based on slopes and angles. Our second proof is simpler than the first one and establishes a more general result. Thus, it can be said to be a stronger approach than the one used above.

The result we prove is easy to state: using only distinct lattice points as vertices, we cannot even construct a 60° angle! Clearly if this is the case, we cannot construct a lattice-point equilateral triangle either.

The proof uses the idea of slope. We claim the following: if A, B, C are three distinct lattice points such that AB is not perpendicular to BC, then $\tan \measuredangle ABC$ is a rational number. To see why, suppose that neither AB nor BC is parallel to the y-axis. Let m, n be the slopes of BA, BC respectively. Then, by a well-known formula from coordinate geometry, we have:

$$|\tan \measuredangle ABC| = \left| \frac{m-n}{1+mn} \right|.$$

By assumption, m, n are rational numbers and $mn \neq -1$. Hence $(m-n)/(1+mn)$ is a well-defined rational number; i.e., $\tan \measuredangle ABC$ is a rational number. If either AB or BC is parallel to the y-axis, then one of m, n is undefined; so we cannot use the above formula. However, the same conclusion holds. (Please fill in the details of the proof on your own.)

On the other hand, if $\measuredangle ABC = 60^\circ$, then its tangent equals $\sqrt{3}$, an irrational number.

So the assumption that a 60° angle can be formed using only lattice points as vertices leads to a contradiction. It follows that the phenomenon is not possible at all. □

Third proof: A number-theoretic argument. Assume that there exists a non-degenerate lattice-point equilateral $\triangle ABC$. By translating the triangle parallel to itself suitably, we can make one of its vertices coincide with the origin. This gives us a lattice-point equilateral triangle in which one vertex lies at the origin of the coordinate system. Assume that this vertex is B. Let $A = (a, b)$ and $C = (c, d)$, where a, b, c, d are integers. Since the triangle is equilateral, we must have, for some positive integer k,

$$\begin{aligned} a^2 + b^2 &= k, \\ c^2 + d^2 &= k, \\ (a-c)^2 + (b-d)^2 &= k. \end{aligned}$$

By adding the first two equations and subtracting the third one, we obtain the following:

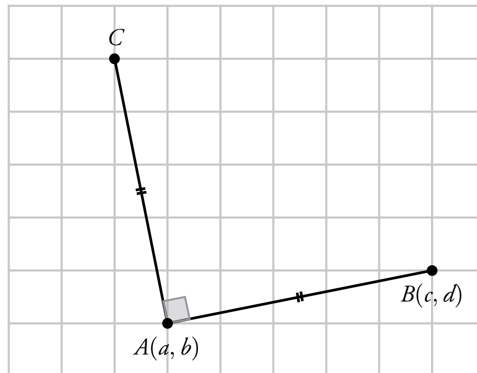
$$2ac + 2bd = k.$$

From this we deduce that k is even. This implies that a, b are both odd or both even, and, similarly, that c, d are both odd or both even. In short, a, b have the same parity and c, d have the same parity.

It may be the case that a, b, c, d are all even. In this case, $\triangle A'BC'$ where $A' = (a/2, b/2)$ and $C' = (c/2, d/2)$ is a lattice-point equilateral triangle as well (it has half the scale of the original triangle). The whole argument can be framed in terms of this triangle rather than the original one. By repeating this step as many times as needed, we eventually reach a stage where either A or C has at least one coordinate which is an odd number, i.e., at least one of a, b, c, d is odd. So there is no loss of generality in assuming that at least one of a, b, c, d is odd.

We now recall an important fact about square numbers: an even square is of the form $0 \pmod 4$, and an odd square is of the form $1 \pmod 4$.

We had observed above that a, b have the same parity and c, d have the same parity. Suppose that a, b are both odd and c, d are both even; then the relation $a^2 + b^2 = k$ shows that $k \equiv 2 \pmod 4$, while the relation $c^2 + d^2 = k$ shows that $k \equiv 0 \pmod 4$. We see a contradiction here. The same contradiction arises if we suppose that a, b are both even and c, d are both odd. We are forced to conclude that a, b, c, d all have the same parity. As we have assumed that at least one of them is an



$C = (a - d + b, b + c - a)$

Figure 3

odd number, it must be that a, b, c, d are all odd. Hence $a - c$ and $b - d$ are both even.

The relation $k = a^2 + b^2$ now tells us that k must be of the form 2 (mod 4). On the other hand, the relation $k = (a - c)^2 + (b - d)^2$ tells us that k must be of the form 0 (mod 4).

We have thus arrived at a contradiction, and this shows that it is not possible to find a lattice-point equilateral triangle. □

Fourth proof: An argument based on descent. Our fourth (and last) proof is subtler than the earlier ones. However, it uses important mathematical ideas and is worth studying deeply. Its basis lies in a fundamental symmetry of the lattice points of the coordinate plane: if about any lattice point as centre we perform a 90° rotation (either clockwise or anticlockwise), then lattice points get mapped to lattice points, and non-lattice points get mapped to non-lattice points. This may also be checked using simple computations: a 90° anticlockwise rotation about point $A(a, b)$ will take point $B(c, d)$ to point C where $C = (a - d + b, b + c - a)$. If a, b, c, d are integers, then obviously $a - d + b$ and $b + c - a$ are integers. So if A and B are lattice points, then C too is a lattice point. (See Figure 3.)

Now let us suppose that there exists a lattice-point equilateral triangle ABC. Figure 4 depicts the situation. Consider the effect of a 90° rotation (anticlockwise) about point A. Let the rotation take point B to point D. As per what we said above, D must be a lattice point.

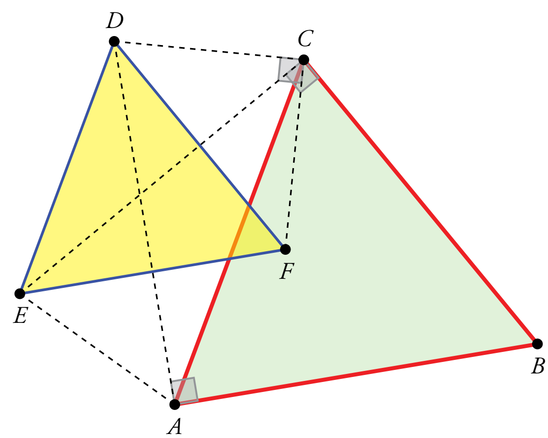


Figure 4

In the same way, let a 90° rotation (clockwise) be performed about point C as centre. Let this take point B to point E. Then E too is a lattice point. Finally, consider the effect of a 90° rotation about point C (anticlockwise). Let it take point D to point F. Then F too is a lattice point. Hence $\triangle DEF$ is a lattice-point triangle. We shall show that DEF is an equilateral triangle.

We start by noting that $\triangle CAD$ and $\triangle ACE$ are congruent isosceles triangles, each with apex angle 30°. Hence $AE = CD$, and $AEDC$ is an isosceles trapezium, with $ED \parallel AC$ and $\measuredangle ACD = \measuredangle CAE$. Since $\measuredangle ACD = 75°$, it follows that $\measuredangle CDE = 105°$.

Again, $CD = CF$ and $\measuredangle DCF = 90°$, so $\measuredangle CDF = 45°$. Hence $\measuredangle EDF = 60°$.

Next, observe that $\measuredangle DCE = 75° - 30° = 45°$. Since $\measuredangle CDF = 45°$ as well, it follows that

$CE \perp DF$. Since $CD = CF$, this means that CE bisects DF at right angles. Hence E is equidistant from D and F, i.e., $ED = EF$. Hence $\measuredangle EFD = \measuredangle EDF$, i.e., $\measuredangle EFD = 60°$. It follows that $\triangle DEF$ is equilateral. So $\triangle DEF$ is a lattice-point equilateral triangle.

Now let us compare the sizes of these two equilateral triangles. We have, from $\triangle CDF$:

$$\frac{DF}{CD} = \frac{1}{\sin 45°}.$$

Next, from $\triangle ACD$:

$$\frac{CD}{AC} = \frac{\sin 30°}{\sin 75°}.$$

Hence:

$$\frac{DF}{AC} = \frac{1}{\sin 45°} \times \frac{\sin 30°}{\sin 75°} = \frac{\sin 45°}{\sin 75°},$$

since $\sin^2 45° = \sin 30°$. Since $\sin 45°$ is smaller than $\sin 75°$, it follows that $DF < AC$. (In fact, $\sin 45°/\sin 75° = \sqrt{3} - 1 \approx 0.732 < 0.75 < 1$.)

Hence the equilateral triangle DEF is *strictly smaller* than the equilateral triangle ABC: its sides are shorter than $3/4$ of the sides of the original triangle. Thus, by following the geometrical procedure described above, we have generated a new lattice-point equilateral triangle whose sides are shorter than $3/4$ of the sides of the original triangle.

By applying the same procedure to $\triangle DEF$, we generate another lattice-point equilateral triangle GHI (say), whose sides are shorter than $3/4$ of the sides of $\triangle DEF$. And we can continue in this way, generating a sequence of lattice-point equilateral triangles whose sides are decreasing in a geometrical ratio which is strictly less than 1. But this is clearly impossible, because after a while we will obtain lattice-point triangles whose sides are less than 1 in length! However, the distance between two lattice points obviously cannot be less than 1. Hence such a sequence of triangles cannot exist.

It follows that a lattice-point equilateral triangle does not exist. $\square$

Remarks. As noted earlier, four proofs for the same result may seem like an overkill; but not if they illustrate important mathematical ideas, and that is certainly true of the proofs we have described. Each one is distinctive in its own way, though the first two have the common feature that they both depend on the irrationality of $\sqrt{3}$. The first three arguments are number theoretic, while the fourth proof is of a completely different nature.

How close can we come to finding a lattice-point equilateral triangle?

Having shown the impossibility of some phenomenon (in this case, the existence of a lattice-point equilateral triangle), we naturally want to know how close we can get to it. First we need a measure to assess how close is 'close'. We could choose a measure based on side-lengths, or one based on angles. Here is a possible measure of the discrepancy between a given triangle and an equilateral triangle, based on side-lengths: if the triangle has side-lengths a, b, c, then we compute the quantity q given by

$$q = \frac{ab + bc + ca}{a^2 + b^2 + c^2}.$$

It is a nice exercise (please try it) to show that if a, b, c are any three real numbers, then

$$ab + bc + ca \leq a^2 + b^2 + c^2,$$

with equality precisely when $a = b = c$. It follows that the computed quantity satisfies the inequality $0 < q \leq 1$, and equality holds precisely when the triangle is equilateral. Hence the gap between q and 1 is a measure of how far the triangle is from being equilateral.

We can now embark on a search for lattice-point triangles for which the q-value is very close to 1. How do we conduct such a search? Purely empirically, using a computer to check through millions of possibilities? Or is there a nicer way than that? Let us defer this question to the next part of this article. In the meantime, why don't we pass on the question to you? Please see how close you can come to finding a lattice-point equilateral triangle.

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CAPTURED MATHEMATICS

Image 1: The shadows of trees on either sides of the path intersect and form a quadrilateral

VC Lawn, University of Delhi — *Photo and Ideation: Kumar Gandharv Mishra*

Mathematical Relevance: Lines are formed by the intersection of planes. When the shadows (planes) of oppositely standing trees fall on the ground and intersect, four lines appear as borders of the quadrilateral formed in the centre of path. The quadrilateral appears to be a square.

What would happen if the light source was shifted closer to one of the trees?

The Constants of Mathematics

Part III

More on the remarkable number *e*

SHAILESH SHIRALI

In this article, which is the third of our series on mathematical constants, we continue our exploration of Euler's constant e. (Yes, we have had to devote more than one 'episode' to e, as there is so much to say about this number.)

More infinite series for e

In the previous part of this article, we pointed out that a simple consequence of the definition $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ is the following infinite series for e:

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}. \qquad (1)$$

We mentioned at the time that this infinite series converges quite rapidly. Let $B(n)$ denote the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$; then we have the following data:

n	$e - B(n)$
10	2.73×10^{-8}
20	2.05×10^{-20}
30	1.26×10^{-34}

Now here's the surprise: by making an apparently minor tweak to the series (1), we can increase the rate of convergence quite dramatically! Here is how we do it.

Keywords: Irrational, simple continued fraction, convergence, factorial

A tweaked series. The idea is very simple indeed: all we do is to combine adjacent pairs of terms. Thus we have:

$$\begin{aligned} e &= \left(\frac{1}{0!}+\frac{1}{1!}\right)+\left(\frac{1}{2!}+\frac{1}{3!}\right)+\left(\frac{1}{4!}+\frac{1}{5!}\right)+\cdots \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{(2n)!}+\frac{1}{(2n+1)!}\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{2n+1}{(2n+1)!}+\frac{1}{(2n+1)!}\right) \\ &= \sum_{n=0}^{\infty}\frac{2n+2}{(2n+1)!}. \end{aligned}$$

So we have the following result:

$$e=\frac{2}{1!}+\frac{4}{3!}+\frac{6}{5!}+\frac{8}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{2n+2}{(2n+1)!}. \tag{2}$$

Let $C(n)$ denote the sum $\frac{2}{1!}+\frac{4}{3!}+\frac{6}{5!}+\cdots+\frac{2n+2}{(2n+1)!}$; then we have the following data:

n	$e-C(n)$
10	9.30×10^{-22}
20	7.29×10^{-52}
30	3.23×10^{-86}

A hugely faster rate of convergence! Similarly, we have the following result:

$$e=1+\frac{3}{2!}+\frac{5}{4!}+\frac{7}{6!}+\frac{9}{8!}+\cdots=\sum_{n=0}^{\infty}\frac{2n+1}{(2n)!}. \tag{3}$$

This series too converges more rapidly than (1), but not as rapidly as (2). We leave the proof of (3) for you to find.

Yet more such tweaks are possible, by bracketing more terms together. Thus we have:

$$e=\sum_{n=0}^{\infty}\frac{(3n)^2+1}{(3n)!}=\frac{1}{0!}+\frac{3^2+1}{3!}+\frac{6^2+1}{6!}+\frac{9^2+1}{9!}+\frac{12^2+1}{12!}+\cdots. \tag{4}$$

This series converges even more rapidly than (2). Numerous such results are possible.

Simple continued fraction for *e*

In the previous part of this article, we had pointed out that the following fraction is a very good rational approximation for *e*:

$$\frac{23225}{8544},$$

which differs from *e* by roughly 6.7×10^{-9}. We had asked how such rational approximations can be found and noted that they come from the 'simple continued fraction' for *e*. We now elaborate on this comment.

First we explain what is meant by a *simple continued fraction* (SCF). This is best done by means of a few examples. Below, we express the fractions $\frac{7}{3}$, $\frac{7}{5}$ and $\frac{11}{7}$ as SCFs:

$$\frac{7}{3} = 2 + \frac{1}{3},$$

$$\frac{7}{5} = 1 + \cfrac{1}{2 + \cfrac{1}{2}},$$

$$\frac{11}{7} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}.$$

Observe that they are cumbersome to write! For this reason, short forms are used which consume less space and also are easier to typeset. Here is one short form which is used:

$$\frac{7}{3} = [2; 3], \qquad \frac{7}{5} = [1; 2, 2], \qquad \frac{11}{7} = [1; 1, 1, 3].$$

The examples shown above are *finite* SCFs. It is easy to show that every rational number can be expressed as a finite SCF. (There are precisely two finite SCFs corresponding to each irrational number. However, they differ in a rather inconsequential way.)

It follows immediately that *an infinite SCF must correspond to an irrational number*. One of the simplest and most elegant examples of this is the SCF for the golden ratio ϕ:

$$\phi = \frac{1}{2}\left(\sqrt{5} + 1\right) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}} = [1; 1, 1, 1, 1, 1, \ldots].$$

Here the SCF is made up solely of 1's.

It was Euler who found the SCF corresponding to e. It is a result of great beauty:

$$2 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}}}}}, \tag{5}$$

i.e.,

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, \ldots].$$

Observe the sequence of denominators:

$$1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\ 1,\ 1,\ 8,\ 1,\ 1,\ 10,\ 1,\ \ldots.$$

However, it will not be possible for us to give the proof of this result here; it is *way* beyond the scope of this article. Interested readers can refer to [1] or [6] for the proof.

Using the SCF to find a good rational approximation for *e*. The standard theory behind infinite SCFs tells us that if we truncate the SCF at any high denominator and compute the resulting finite SCF, the answer will be very close to the value of the infinite SCF. Here, let us compute the values of the SCFs obtained by truncating the infinite SCF at 6, 8 and 10, respectively; we get:

$$[2;1,2,1,1,4,1,1,6] = \frac{1264}{465},$$

$$[2;1,2,1,1,4,1,1,6,1,1,8] = \frac{23225}{8544},$$

$$[2;1,2,1,1,4,1,1,6,1,1,8,1,1,10] = \frac{517656}{190435}.$$

It follows that the fractions

$$\frac{1264}{465},\quad \frac{23225}{8544} \quad \text{and} \quad \frac{517656}{190435}$$

are progressively better rational approximations for e. (The fraction in the middle is the one we had exhibited earlier.) This answers the question we had raised in the earlier part of the article.

For more general information about continued fractions, the reader could refer to [7].

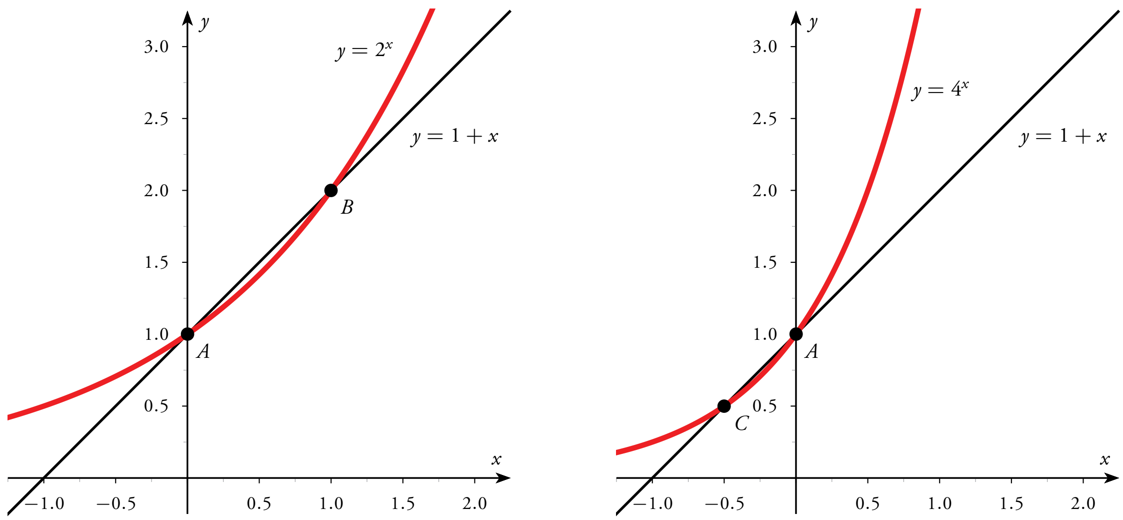
Difference between a SCF and a GCF. The significance attached to the word 'simple' in 'SCF' is that the numerators in the continued fraction are all 1's. If we relax this condition, we get constructs which are called *general continued fractions* or GCFs. For example:

$$\frac{268}{113} = 2 + \cfrac{3}{7 + \cfrac{5}{4 + \cfrac{2}{3}}}.$$

We have the following astonishing result—an infinite GCF for e:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cfrac{5}{6 + \cfrac{6}{7 + \cfrac{7}{8 + \cfrac{8}{9 + \cfrac{9}{\cdots}}}}}}}}}}. \tag{6}$$

How beautiful this result looks!



(a) (b)

Graphs of $y = 1 + x$, $y = 2^x$ and $y = 4^x$

The only number satisfying a certain inequality

Consider the inequality $2^x \geq 1 + x$. Is this true for all real values of x? If we draw the graphs corresponding to $y = 2^x$ and $y = 1 + x$, we find that they intersect at $x = 0$ and $x = 1$. We observe from the graph that the inequality $2^x > 1 + x$ holds (strictly) for $x < 0$ and for $x > 1$. But for $0 < x < 1$, the inequality is reversed; we have $2^x < 1 + x$. So it is *not true* that $2^x \geq 1 + x$ for all real values of x; the inequality is falsified over the interval $0 < x < 1$. See Figure 1 (a).

Next, consider the inequality $4^x \geq 1 + x$. Is this true for all real values of x? Probing as we did earlier, we find that the graphs of $y = 4^x$ and $y = 1 + x$ intersect at $x = -0.5$ and $x = 0$; so the inequality is falsified over the interval $-0.5 < x < 0$. Outside of this interval, the inequality $4^x \geq 1 + x$ is valid. See Figure 1 (b).

How about the inequality $3^x \geq 1 + x$? Is this true for all real values of x? Finding the points of intersection of these two graphs involves more computation, but after some effort we find that there are intersection points at $x = 0$ and $x = -0.174$ (approximately); so the inequality is falsified over the interval $-0.174 < x < 0$. Outside of this interval, the inequality $3^x \geq 1 + x$ is valid.

How about the inequality $2.5^x \geq 1 + x$. Is this true for all real values of x? Once again, finding the points of intersection of the two graphs involves a fair bit of computation, but we find that there are intersection points at $x = 0$ and $x = 0.188$ (approximately); so the inequality is falsified over the interval $0 < x < 0.188$. Outside of this interval, the inequality $2.5^x \geq 1 + x$ is valid.

It is tempting to draw a conjecture from this pattern: namely, that for each real number $a > 1$, the inequality $a^x \geq 1 + x$ is never valid for the entire set of all real numbers x; that there is always some interval where the inequality is falsified. *But this conjecture is wrong!* It turns out that there is one (and precisely one) real number $a > 1$ for which it is true that $a^x \geq 1 + x$ for all real values of x, and that number is $a = e$.

In other words, the following claims are true:

(i) If $a > 1$ and $a \neq e$, then there exist real values of x for which $a^x < 1 + x$.

(ii) The inequality $e^x \geq 1 + x$ holds for all real values of x.

Proof of (i). We use the fact that *the only positive number a for which the graph of $y = a^x$ has slope 1 at $x = 0$ is $a = e$.* (This was proved in Part 2 of this series of articles. As noted there, this property can be used to *define* e.) Moreover, the following is true: if $a > e$, then the slope of $y = a^x$ at $x = 0$ is greater than 1, and if $0 < a < e$, then the slope of $y = a^x$ at $x = 0$ is less than 1. Let us see how these two facts imply claim (ii).

- Suppose that $a > e$. Then the slope of $y = a^x$ at $x = 0$ is greater than 1. This means that at the point $(0, 1)$, *the curve crosses the line $y = 1 + x$ from below to above*; i.e., the curve lies *below* the line in the region immediately to the left of $x = 0$, and it lies *above* the line in the region immediately to the right of $x = 0$. *This implies that $a^x < 1 + x$ in some region immediately to the left of $x = 0$.* In other words, there exists some negative number c, whose value naturally will depend on a, such that for $c < x < 0$, we have $a^x < 1 + x$. (Figure 1 (b) may make this clearer.)
- Suppose that $0 < a < e$. Then the slope of $y = a^x$ at $x = 0$ is less than 1. This means that at the point $(0, 1)$, *the curve crosses the line $y = 1 + x$ from above to below*; i.e., the curve lies *above* the line in the region immediately to the left of $x = 0$, and it lies *below* the line in the region immediately to the right of $x = 0$. *This implies that $a^x < 1 + x$ in some region immediately to the right of $x = 0$.* In other words, there exists some positive number d, whose value naturally will depend on a, such that for $0 < x < d$, we have $a^x < 1 + x$. (Figure 1 (a) may make this clearer.)

So statement (i) has been proved.

Proof of (ii). We first note that the fact that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

implies that the derivative of e^x is e^x. (This is a well-known fact, and it is studied in the +2 mathematics course. But for the sake of completeness, we include the proof here.) To see why, note that (by definition) the slope of the curve $y = e^x$ at the point $P\ (a, e^a)$ is

$$\lim_{h \to 0} \frac{e^{a+h} - e^a}{h} = e^a \cdot \lim_{h \to 0} \frac{e^h - 1}{h} = e^a \cdot 1 = e^a.$$

We now show how this result can be used to prove that $e^x \geq 1 + x$ for all real values of x.

Define a function g on the set of real numbers $\mathbb{R}$ as follows:

$$g(x) = e^x - 1 - x.$$

The derivative of g is $g'(x) = e^x - 1$. Since $e > 1$, the following statements are true:

- If $x < 0$, then $e^x < 1$, hence $g'(x) < 0$.
- If $x > 0$, then $e^x > 1$, hence $g'(x) > 0$.

It follows that $g(x)$ *is strictly decreasing when $x < 0$, and $g(x)$ is strictly increasing when $x > 0$.* Hence $g(x)$ achieves its global minimum at $x = 0$, i.e., $g(x) \geq g(0)$ for all x.

Since $g(0) = 0$, it follows that $g(x) \geq 0$ for all x, i.e., $e^x \geq 1 + x$ for all x.

Appendix

In the previous part of the article, we considered the curve $y = \frac{1}{x}$ and defined a function $f(t)$ for $t > 0$ thus: $f(t)$ = the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = t$. Then f is a continuous function, and $f(1) = 0$. We found, using simple computation, that $f(2) < 1$ and $f(3) > 1$. By continuity, there exists a value of t between 2 and 3 such that $f(t) = 1$. We claimed that this critical value is e. Let us give here the steps needed to prove this claim. (We shall leave the individual steps as problems for the reader.)

Step 1: Show that if $a > 0$ and $b > 0$, then $f(ab) = f(a) + f(b)$.

Step 2: Deduce that for any $a > 0$ and any positive integer n, $f(a^n) = n \cdot f(a)$.

Step 3: Show that for any positive integer n,

$$\frac{1}{n+1} < f\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$

Step 4: Deduce that for any positive integer n,

$$\frac{n}{n+1} < n \cdot f\left(1 + \frac{1}{n}\right) < 1,$$

and hence that

$$\frac{n}{n+1} < f\left(\left(1 + \frac{1}{n}\right)^n\right) < 1.$$

Step 5: In the above relation, let $n \to \infty$; the quantity on the extreme left then tends to 1, and the quantity on the extreme right is equal to 1. The 'sandwich principle' now applies, and we deduce that

$$\lim_{n\to\infty} f\left(\left(1 + \frac{1}{n}\right)^n\right) = 1.$$

Step 6: Deduce from the above that the value of t such that $f(t) = 1$ is equal to

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

But this number is e, by definition. This proves the claim that had been made.

Remark. We have yet to exhaust the list of remarkable features that e possesses!

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W.W. Sawyer
The Universal Math Teacher

VIVEK MONTEIRO

As a corollary of Section 8 of the Right to Education Act, every young Indian citizen in elementary school today has the legal right to get mathematics education of good quality. Perhaps India is the only country in the world, where this is legally mandatory. Two questions are now squarely on the agenda of Indian math education: "What is math education of good quality?" and, "Is it possible to ensure this for every child?"

Prof. W.W. Sawyer, who passed away in the year 2008 at the age of 97, was perhaps the first person to articulate math universalization in the sense of the RTE Act and to engage in depth with both these questions.

In a 1958 article, titled 'The Possibility of Universal Mathematical Literacy' (TPUML), he wrote: *"We are facing a great crisis and a great opportunity. Changes in science and technology are tending to make semi-skilled labour obsolete. The demand for highly educated people, teachers, mathematicians, technicians is rapidly increasing. Social and political dislocation can only be avoided if we are able to educate to a high standard far more people than has previously been considered possible. The ability to think mathematically will have to be taken for granted much as the ability to read a newspaper is at present. Such a change will seem fantastic to many people. So would universal literacy have seemed absurd a few centuries ago.*

Two possible viewpoints on twentieth century education.

a. It represents a close approach to the best that is humanly possible.

b. It represents the first gropings of a new society. Universal schooling is hardly a century old. Its standard of efficiency may be compared to the standards of industrial efficiency of 1750.

Keywords: mathematics education, history, reasoning, context

To enter a new stage of history is always difficult…

The population today divides sharply into those who hate and fear mathematics and a minority of mathematicians.

The remarkable thing is that such an outcome is accepted as normal. It is as if physical education cripples 90% of the children taking it."

The conviction that universalization of mathematics is possible, in the way that every citizen becomes comfortable with her mother tongue, is the foundation of Sawyer's work on math education that spanned almost 8 decades, all levels from elementary to university math and across several continents. His definition of success in mathematics, like all his writing, is simple but profound: *Complete success would mean that every individual felt, "I enjoyed the mathematics that I had time to learn. If I ever need or want to learn some more, I shall not be afraid to do so."*

Mathematics is commonly thought to be a difficult and esoteric subject, accessible only to a select and privileged few. Sawyer's entire lifework was dedicated to proving the converse – that ordinary people like you and me can be taught to understand, learn and enjoy mathematics.

In the following, we survey briefly some of his insights that are as relevant and pertinent today as when they were written many decades ago.

Sawyer points out that *rigour in the teaching of mathematics* should never be confused with the problem of *rigour in mathematics*. Though always gentle, he is unsparing in his criticism of the almost universal absence of rigour in math teaching. In 1946, Sawyer visited India and delivered a lecture to the Indian Mathematical Society titled "The Teaching of Mathematics." *Probably there is no subject which offers such possibilities for misunderstanding between teacher and pupil as mathematics does. The teacher stands at the blackboard. It is perfectly clear to him what the symbols mean, and what conclusion can be drawn from them. It is completely otherwise with many of the pupils. What the symbols are meant to represent, how the teacher knows what is right and what is wrong, what is the object of the whole business anyway – all this is wrapped in mystery. The great majority of students say to themselves,* ***"We shall never learn this stuff, but we want to get through the exam. We'll have to learn it by heart."***

This is not a satisfactory state of affairs. This learning by heart not only imposes a quite unnecessary strain on the student; it is also quite useless. It gives neither an understanding of the subject, nor the power to apply mathematics in ordinary life.

The more we can see things from the pupil's point of view, the better teachers we shall be. And the first question in the pupil's mind is, ***"Why do we have to do this at all?"*** *When I was at school, the boys were always asking this – and they never got a satisfactory answer. The teachers made up all kinds of answers, but they were none of them very convincing. The fact is, I think, that mathematics is taught because it is* ***the custom*** *to teach it.*

The same point is made, more sharply, in his article 'From Abstract to Concrete' (1962): *The depressing thing about arithmetic, badly taught, is that it destroys a child's intellect, and to some extent his integrity. Before they're taught arithmetic children won't give their assent to utter nonsense: afterwards, they will. Instead of looking at things and thinking about them, they make wild guesses in the hope of pleasing the teacher or an examiner.*

In his book *A Concrete Approach to Abstract Algebra* (1959) he writes about how not to teach: *"In planning such a course, a professor must make a choice. His aim may be to produce a perfect mathematical work of art, having every axiom stated, every conclusion drawn with flawless logic, the whole syllabus covered. This sounds excellent, but in practice the result is often that the class does not have the faintest idea of what is going on. Certain axioms are stated. How are these axioms chosen? Why do we consider these axioms rather than others? What is the subject about? What is its purpose? If these questions are left unanswered,* *students feel frustrated. Even though they follow every individual deduction, they cannot think effectively about the subject. The framework is lacking; students do not know where the subject fits in, and this has a paralyzing effect on the mind."*

But there is an alternative:

"On the other hand, the professor may choose familiar topics as a starting point. The students collect material, work problems, observe regularities, frame hypotheses, discover and prove theorems for themselves. The work may not proceed so quickly; all topics may not be covered; the final outline may be jagged. But the student knows what he is doing and where he is going; he is secure in his mastery of the subject, strengthened in confidence of himself."

And what constitutes good math teaching?

The most important thing in the early teaching of mathematics is that the student should form the habit of weighing evidence, of deciding for himself. (Vision in Elementary Mathematics 1964)

The essential thing is to arouse interest in the subject:

I am convinced that any attempt to teach a topic to uninterested pupils both puts a strain on the teacher and is without benefit to the learners.

*Being interested in something is a feeling, an emotion. Our emotions are not at our beck and call. It is no use me saying, **"I will tell you a story and you must try hard to be amused." Interest, like laughter and falling in love, is something that happens to us. Education results when adults are able to find the approach that will unlock the energy within a child and steer it into useful, or at least harmless, channels. This diversion of energy into acceptable channels is one of the most important aspects of teaching; it is a civilizing influence, and such influences have never been more needed than to-day.** (From Mathematics, Emotions, Things)*

Education is essentially the direction of mental energy. Children have abundant energy looking for an outlet…

At the various stages of development, a person's energies are concentrated on various objects which acquire the hue of romance – e.g., riding a bicycle, getting into a basketball team, love and courtship, …

Beauty is in the eye of the beholder. Any subject, any activity can acquire the halo of romance.

Thinking is extremely unsatisfactory and inefficient if the concentration of the mind by romance has not taken place.

(TPUML)

Therefore, the essence of good math teaching is 'motivation' and morale: *In a university lecture you can be sure all the appropriate results will be stated and proved. But the students are not always put in a position to see what the whole course is trying to do, where it came from and where it is going. I remember when I was at Cambridge I heard of only two lecturers who discussed the history of the subject.*

I would like to illustrate this point by discussing analysis. I believe that there was a survey which showed that the part of mathematics pupils enjoyed the most at school was calculus and at university what students enjoyed least was analysis. In school calculus is intuitive; you accept something if it sounds reasonable and it looks right. In universities it is exactly the opposite; everything must be proved with the utmost legalistic precision.

The schools approach corresponds to the way mathematicians worked in the 17th and 18th centuries; the universities to the way mathematicians thought in the 19th century.

Now in fact there is a very interesting explanation why, at a certain stage of history, mathematics changed from the first approach to the second. It arose from the interaction of music and mathematics. (From- Talk That Was Not Given)

Sawyer then goes on to narrate the origins of Fourier analysis and why and how this necessitates different concepts of convergence of functions. Everything he writes is in plain language, delightful prose, behind which is a solid and deep understanding of mathematics.

Born in 1911, WW Sawyer graduated from Cambridge University with specialization in the applied mathematics of quantum mechanics and relativity. Immediately thereafter, he began his long career dedicated to teaching and learning math. Initially, he taught for several years in Britain. His first book *Mathematician's Delight* (1943) (MD), was written with the aim "to dispel the fear of mathematics."

Sawyer's second book, edited by him, and with six out of ten chapters written by him, *Mathematics in Theory and Practice* (1948) deals with the importance of making and constructing with materials for introducing concepts of school mathematics.

His third book, *Prelude to Mathematics* (1955) conceived while he was Head of the Department of Mathematics at the University College (Gold Coast, now Ghana) (1948-50) is about 'How to grow mathematicians.'

On a personal note: I was introduced to Sawyer's *Prelude* by an older friend and college mentor Nitant (now Prof. V.M. Kenkre, Emeritus Professor of Physics at UNM, USA) during my first year in college. The book had a profound impact – I felt exhilarated and elated after reading it, though I did not understand all of it at that time. (Many other writers who have written about this book describe experiencing the same feeling of elation after reading it.) Professors at Princeton, after reading *Prelude*, invited Sawyer to the USA to work on curriculum. Sawyer, at the time was at Canterbury College, New Zealand (1951-56).

Sawyer's writings cover a broad spectrum from primary math to 'advanced' math. In *Vision in Elementary Mathematics*, he shows how a simple game "Think of a number," can be translated into an introduction to algebra – how unknown numbers can also be represented by things, and can be added and subtracted much like pebbles. This writer has used this approach to introduce algebra to hundreds of primary school teachers. The universal response is "I never realized algebra is this simple."

Mathematician's Delight covers school geometry, including an introduction to Calculus. Mathematics at the secondary level is also covered fairly thoroughly in three other of his books: *Designing and Making* (1957, co-authored with Srawley), *What is Calculus About (*1961) and *The Search for Pattern* (1970).

College and University level mathematics is covered in four other books *A Concrete Approach to Abstract Algebra* (1959), *A Path to Modern Mathematics* (APMM) (1966), *An Engineering Approach to Linear Algebra* (1972) and *A Numerical Approach to Functional Analysis* (1978).

Each of Sawyer's books is a perennial classic. Before getting into the details and procedures of any topic in mathematics he gives the reader an idea of 'where it came from and where it is going.' There is one sentence that describes everything that Sawyer wrote – he was dedicated to conveying to the reader 'What is this all about' in the simplest possible way.

High priests of the subject may find his style infuriating – for example, in his chapter on Affine spaces (*The Arithmetic of Space*, APMM), he introduces vector addition by adding and subtracting cats and dogs, later going on to fractional and negative cats and dogs – but the ordinary reader will find herself exclaiming, "Oh, I never realized it was so simple!"

A few more examples from his writing will illustrate the above point.

The best way to learn geometry is to follow the road which the human race originally followed: Do things. make things, notice things, arrange things, and only then reason about things. (Mathematician's Delight. MD)

At the secondary level, Sawyer emphasised elementary approaches to calculus, again based on work with 12 to 15-year-old children. He starts the discussion in ***MD*** with – *"The Basic Problem": The basic problem of differential calculus is the following: we are given a rule for finding where an object is at any time, and are asked to find out how fast it is moving.*

He summarises a beautiful discussion on complex function theory in the following words

There is a blanket theorem, which I have never seen stated in quite this way in any textbook:

Within a circle centred at the origin, not containing any singularity, you can safely carry out any operation on the power series that might occur to a sane mathematics student. ('The importance of the unbelievable')

This writer and the team at Navnirmiti were in touch with Prof. Sawyer through his daughter and son-in law during the final five years of his life. After Sawyer's demise, we were greatly honoured to receive a collection of his books, translations and handwritten notes, which are archived at the Sawyer Memorial in Pune. The articles referred to here are available at two websites www.marco-learningsystems.com, set up by Marc Alder, and www.wwsawyer.org, set up by Navnirmiti. VEM has been translated into Marathi. It is not easy to translate Sawyer. However, it will greatly benefit math education in India, if we can collectively translate his writings into Indian languages as soon as possible.

VIVEK MONTEIRO completed his doctorate in theoretical physics from the State University of New York, Stony Brook, in 1974. He returned to India and worked for two years at TIFR before switching fields to take up full time trade union work in 1977. He is currently Secretary, CITU Maharashtra State Committee. Through Navnirmiti, a self-reliant organization, of which he is a founder-advisor, he has continued working actively in math and science education. He is a trustee of the Sawyer Memorial trust.

CAPTURED MATHEMATICS

Image 2: The tamarind shaped fruit is known by different names in India. *Jalebi* is one of the popular names because of its twisted surface and structure similar to the famous Indian sweet Jalebi.

Mathematical Relevance:
The fruit has a unique design and a twisted shape of the kind one encounters in the study of topology.

Photo and Ideation:
Kumar Gandharv Mishra

Pithecellobium Dulce or Jalebi

Addendum to Hill Ciphers

KUMAR GANDHARV MISHRA

The Hill Cipher method in cryptography has been described in detail in earlier issues of *At Right Angles* [1] [2]. Through those articles, the reader will be familiar with the art of encryption and decryption of messages using matrices and modular arithmetic. The Hill Cipher method requires a substitution table comprising letters and numerals, and a fixed key invertible matrix that is used to encrypt the plain text into the cipher text. Further, the inverse of the key matrix is used to decrypt the cipher text back to the plain text. A substitution table must comprise of the characters and their corresponding numerical values. Table 1 with 29 characters (numbered from 0 to 28) is a particular choice, which we will use in this article.

The process of encryption and decryption requires the plaintext to be converted to a matrix using the numerals from the substitution table. Further, the plaintext matrix is to be pre multiplied or post multiplied by the key matrix to obtain a matrix that leads to the cipher text. When the values of numerals in this product matrix exceed 28, modular arithmetic is used, i.e., the product matrix is reduced modulo 29. This requires knowledge and understanding of modular arithmetic and Ghosh has shown how modular arithmetic can also be performed with technical tools like MS-Excel.

Blank Space	**.**	**?**	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	
L	**M**	**N**	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
14	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**

Table 1. Substitution Table[a]

[a] '0' represents a blank space

Keywords: Cryptography, Hill cipher, code, substitution, matrix

However, lack of familiarity with technology and absence of topics like 'modular arithmetic' at school level pose a challenge in introducing 'Cryptography' to students. In this article we introduce an alternative technique by *faking* numerical values which are either negative or greater than 28. The reader is requested to refer to the previous articles on Hill Ciphers to recall the original method. In this article, we illustrate the cases when the matrix of ciphertext has negative elements or elements greater than 28, i.e., $c_{ij} > 28$ or $c_{ij} < 0$.

Let's try this:

Suppose you wish to send the message '**HELLO HOW R U?**' to your friend.

Choose an invertible 3 by 3 matrix K as the key and share it with your friend. The sender and the receiver are required to know the key but it is to be kept secret from others. Let us choose K as follows

$$K = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 1 \\ 3 & -2 & 1 \end{bmatrix}$$

The message - **HELLO HOW R U?** has 14 characters including spaces. We would form our message matrix based on the key matrix. As the key matrix is of order 3 by 3, the message matrix must be either a 3 by 5 or 5 by 3 matrix.

Note: The order of the message matrix M will depend on the order of the key matrix K. To get a new matrix C (for ciphertext), we will need to pre-multiply or post-multiply M by K (depending on the order of the message matrix M created by the sender of the message). For example, if M is of order 3 by 4 then we need to compute KM = C whereas if M is of order 4 by 3 we will compute MK = C.

Part I – ENCRYPTION

The process of encryption is as follows:

Step 1: In **HELLO HOW R U?** each character is replaced with its corresponding number from Table 1. We obtain the following string of numbers.

10_7_14_14_17_0_10_17_25_0_20_0_23_2

Step 2: Since the key matrix K is of order 3 by 3, we may create a message matrix M of order 3 by 5. The elements in the string will be positioned in M row wise and then the remaining empty positions can be filled with '0'.

$$M = \begin{bmatrix} 10 & 7 & 14 & 14 & 17 \\ 0 & 10 & 17 & 25 & 0 \\ 20 & 0 & 23 & 2 & 0 \end{bmatrix}$$

Step 3: We need to pre-multiply M by K to get the matrix for obtaining ciphertext.

Thus C = KM

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 1 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 10 & 7 & 14 & 14 & 17 \\ 0 & 10 & 17 & 25 & 0 \\ 20 & 0 & 23 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -30 & 7 & -32 & 10 & 17 \\ 40 & 4 & 34 & 5 & 34 \\ 50 & 1 & 31 & -6 & 51 \end{bmatrix}$$

Step 4: The values obtained can be arranged as a new string:

$$-30\_7\_-32\_10\_17\_40\_4\_34\_5\_34\_50\_1\_31\_-6\_51$$

Note that some of the elements in the string are greater than 28 and some are even negative integers. These values are not in Table 1. How will you proceed? Well, the idea is to convert (fake) these values in terms of a numeral in Table 1 and substitute with corresponding letters but your friend should be aware of the method of making these conversions and be able to retrieve the characters from Table 1.

Let's see how:

Rule of conversion while sending ciphertext:

i. **If $C_{ij} > 28$** • Divide C_{ij} by 28, and then find the quotient and remainder For example, in the numeral string take '50' $50 = 1.28 + 22, q = 1, r = 22$ • Represent the new element as $r^{q+} = 22^{1+}$ Similarly $31 = 3^{1+}, 34 = 6^{1+}, 40 = 12^{1+}, 51 = 23^{1+}$	ii. **If $C_{ij} < 0$** • Find difference between 28 and C_{ij} For example, in the numeral string take '–32' Difference: $28 - (-32) = 60$ Divide the difference obtained by 28 $60 = 2.28 + 4, q = 2, r = 4$ Represent the new element for '–32' as $r^{q-} = 4^{2-}$ Similarly $-30 = 2^{2-}, -6 = 6^{1-}$

Table 2

The sender can now use the following new string of numerals:

$$2^{2-}\_7\_4^{2-}\_10\_17\_12^{1+}\_4\_6^{1+}\_5\_6^{1+}\_22^{1+}\_1\_3^{1+}\_6^{1-}\_23^{1+}$$
$$\textit{for}\ -30\_7\_-32\_10\_17\_40\_4\_34\_5\_34\_50\_1\_31\_-6\_51$$

Using Table 1, the obtained letter string would be:

$$?^{2-}\_E\_B^{2-}\_H\_O\_J^{1+}\_B\_D^{1+}\_C\_D^{1+}\_T^{1+}\_.\_A^{1+}\_D^{1-}\_U^{1+}$$

You will send this modified letter (the cipher text) string to your friend

$$?^{2-}\_E\_B^{2-}\_H\_O\_J^{1+}\_B\_D^{1+}\_C\_D^{1+}\_T^{1+}\_.\_A^{1+}\_D^{1-}\_U^{1+}$$

Part II – DECRYPTION

Step 1: When your friend receives the cipher text, he/she will be required to convert the letter string to a numeral string using Table 1 as follows

$$2^{2-}\_7\_4^{2-}\_10\_17\_12^{1+}\_4\_6^{1+}\_5\_6^{1+}\_22^{1+}\_1\_3^{1+}\_6^{1-}\_23^{1+}$$

Step 2: Note that within this string of characters, some elements are different from others as these have superscripts. These are converted (fake) numerals which need to be reconverted to their original values.

Rule of conversion after cipher text has been received

r^{q+}	r^{q-}
• When the superscript is q+, he/she understands that the original numerals are greater than these converted numerals. • To retrieve these, he/she simply adds q times 28 to r. For example: $23^{1+} = 1.28 + 23 = 51$ Similarly, $22^{1+} = 50, 12^{1+} = 40, 6^{1+} = 34, 3^{1+} = 31$	• When the superscript is q^-, he/she understands that the original numerals are negative integers. • To retrieve these he/she adds q times 28 to r and further subtracts this result from 28. For example: $6^{1-} = 28 - (1.28 + 6) = 28 - 34 = -6$ Similarly, $4^{2-} = -32, 2^{2-} = -30$

Table 3

In this way, your friend converts the received letter string

$$?^{2-}\_E\_B^{2-}\_H\_O\_J^{1+}\_B\_D^{1+}\_C\_D^{1+}\_T^{1+}\_.\_A^{1+}\_D^{1-}\_U^{1+}$$

$$-30\_7\_-32\_10\_17\_40\_4\_34\_5\_34\_50\_1\_31\_-6\_51$$

Step 3: It should be kept in mind that Matrix C was generated by pre-multiplication of M by K. For obtaining M, C needs to be pre multiplied by K^{-1}. In other words, C must be of order 3 by *m*. The elements of this string fit into a 3 by 5 matrix C and Matrix M is obtained.

$$M = (K)^{-1}C = \begin{bmatrix} 1/3 & 4/3 & -2/3 \\ 1/3 & 7/3 & -5/3 \\ -1/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} -30 & 7 & -32 & 10 & 17 \\ 40 & 4 & 34 & 5 & 34 \\ 50 & 1 & 31 & -6 & 51 \end{bmatrix}$$

$$M = \begin{bmatrix} 10 & 7 & 14 & 14 & 17 \\ 0 & 10 & 17 & 25 & 0 \\ 20 & 0 & 23 & 2 & 0 \end{bmatrix}$$

Step 4: The final numeral string obtained from this matrix is

*10_7_14_14_17_0_10_17_25_0_20_0_23_2_*0

Using Table 1, the message is decrypted as:

HELLO HOW R U?

In the above example, we have pre-multiplied M by K. We can also generate the new matrix for ciphertext by post-multiplying M by K, i.e., C = MK. In that case, M will be of order 5 by 3.The reader is urged to try this as an exercise.

As Table1 has 29 values, we can also use 29 in place of 28 during the conversion process (Table 2 and Table 3). For convenience, we chose 28 as it is the largest numerical value which appears here, but it is not necessary. Any number can be chosen for this conversion. It will only change the ciphertext, not the final result. The number chosen for conversion (28, 29 or any other number) must be pre-known and private

PROPERTIES OF MULTIPLICATION

Visual justification for fractions (and whole numbers)

After looking at visual justifications for properties of addition for whole numbers and fractions, this poster considers the properties of multiplication for the same number sets.

The model: A unit square represents the whole, ¼ is represented by slicing a square into 4 equal vertical strips and shading 1 of them, and 5/3 by slicing two squares in 3 strips each and shading 5 strips (Figure 0).

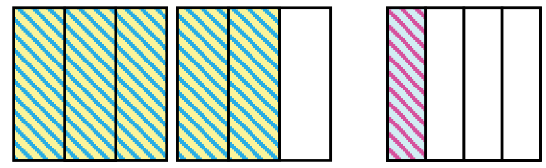


Figure 0

The commutative and distributive properties are easy extensions of the whole numbers case considering the array model of multiplication. In these, the product is represented by the rectangular area whose vertical side represents the 1st multiplicand and the horizontal side represents the 2nd one. So $\frac{2}{3} \times \frac{4}{7}$ is shown by Figure 1 below where the shaded area is 2×4 out of 3×7 parts i.e. $\frac{2\times4}{3\times7}$. Figure 4 illustrates $\frac{4}{3} \times \frac{2}{7}$ where the shaded area is 4×2 parts of the unit square or 3×7 i.e. $\frac{4\times2}{3\times7}$. Note that since $\frac{4}{3}$ is an improper fraction shown along the vertical length, it spills over the unit square (shown with a thick border).

Commutative Property of Multiplication

There are 3 possibilities:

1. Proper × proper
2. Improper × proper (and ∴ proper × improper)
3. Improper × improper

We show one example of 1 and 2 each and 3 can be done in a similar way.

Proper × proper:

$\frac{2}{3} \times \frac{4}{7}$

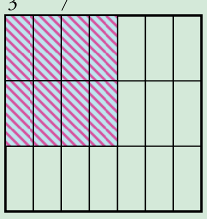


Figure 1

$\frac{4}{7} \times \frac{2}{3}$

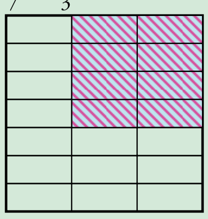


Figure 2

Figure 1 is rotated by 90° to get Figure 2. In Figure 1, the shaded area is 2×4 parts out of 3×7 parts i.e. $\frac{2\times4}{3\times7}$ whereas in Figure 2, it is 4×2 parts out of 7×3 parts i.e. $\frac{4\times2}{7\times3}$.

Note that rotation is a rigid transformation. So the lengths and hence the areas remain unchanged. So $\frac{2}{3} \times \frac{4}{7} = \frac{4}{7} \times \frac{2}{3}$. Therefore this can be generalized for any two proper fractions.

Proper × improper:

$\frac{2}{7} \times \frac{4}{3}$

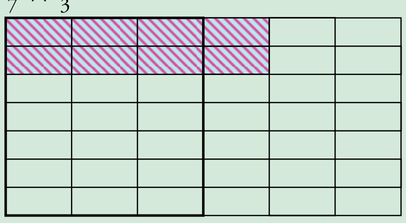


Figure 3

$\frac{4}{3} \times \frac{2}{7}$

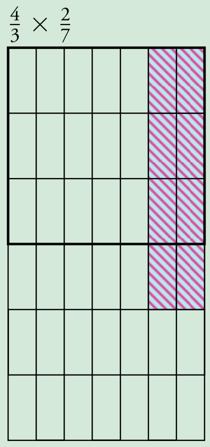


Figure 4

Note:

Figure 3 is rotated by 90° to get Figure 4 as in the previous case. So the areas remain unchanged i.e. $\frac{2}{7} \times \frac{4}{3} = \frac{4}{3} \times \frac{2}{7}$.

In Figure 3, the 2nd multiplicand $\frac{4}{3}$ is an improper fraction. So the shaded area spills over the unit square along the horizontal dimension. In Figure 4, the 1st multiplicand is an improper fraction making the shaded area spill over the unit square along the vertical dimension.

This model can be used to represent whole number × fraction since any improper fraction is a sum of a natural number and a proper fraction. Multiplications involving 0 are by definition 0. So the commutative property is vacuously true when zero is involved.

Distributive Property of Multiplication

There are 6 possibilities:

1. Proper × the sum
   a. Proper × (proper + proper)
   b. Proper × (proper + improper)
   c. Proper × (improper + improper)
2. Improper × the sum
   a. Improper × (proper + proper)
   b. Improper × (proper + improper)
   c. Improper × (improper + improper)

Note that proper + improper = improper + proper by commutativity of addition, which has been discussed in the previous article. We show an example of 1a and 2b each. The reader can explore the remaining.

Proper × (proper + proper)

$\frac{3}{4} \times \left(\frac{4}{7} + \frac{2}{3}\right) = \frac{3}{4} \times \frac{4}{7} + \frac{3}{4} \times \frac{2}{3}$

The total horizontal length of both shaded areas represents the sum $\frac{4}{7} + \frac{2}{3}$ while the vertical length is $\frac{3}{4}$. So the total shaded area is $\frac{3}{4} \times \left(\frac{4}{7} + \frac{2}{3}\right)$. On the other hand, pink area represents $\frac{3}{4} \times \frac{4}{7}$ while the green region depicts $\frac{3}{4} \times \frac{2}{3}$. So the total shaded area is $\frac{3}{4} \times \frac{4}{7} + \frac{3}{4} \times \frac{2}{3}$. Therefore $\frac{3}{4} \times \left(\frac{4}{7} + \frac{2}{3}\right) = \frac{3}{4} \times \frac{4}{7} + \frac{3}{4} \times \frac{2}{3}$

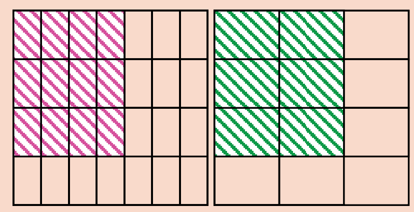


Figure 11

Improper × (proper + improper):

$\frac{5}{3} \times \left(\frac{2}{7} + \frac{7}{4}\right) = \frac{5}{3} \times \frac{2}{7} + \frac{5}{3} \times \frac{7}{4}$

Figure 12 can be understood along the same lines as Figure 11. Note that the shaded area spills over the unit square whenever an improper fraction is involved and along that dimension.

This model works for visually justifying the distributive property of multiplication for any combination of three fractions (and natural numbers including zero).

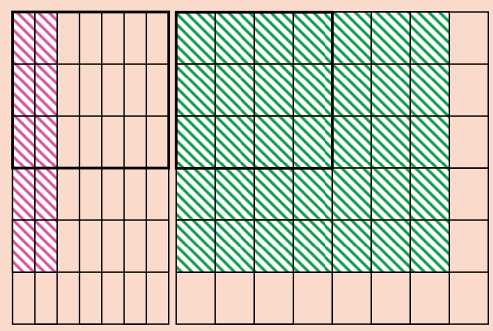


Figure 12

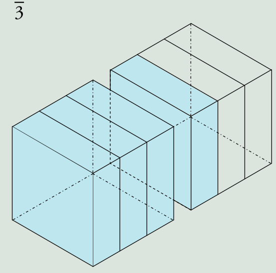
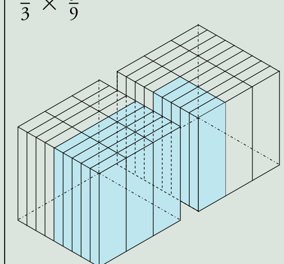
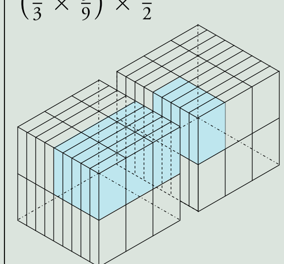
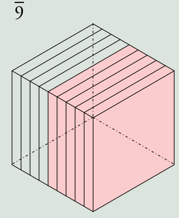
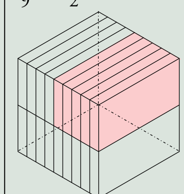
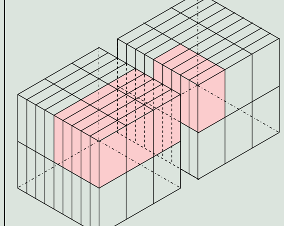
Associative Property of Multiplication

Associative property of multiplication involves the product of three fractions. So we use volume and 3D i.e. the unit cube as the whole and represent the three fractions along the y, x and z axes respectively.

There are 8 possibilities:

1. All three proper
2. Two proper and one improper
   a. Proper × proper × improper
   b. Proper × improper × proper
   c. Improper × proper × proper
3. One proper and two improper
   a. Proper × improper × improper
   b. Improper × proper × improper
   c. Improper × improper × proper
4. All three improper

We show an example of 2c and the rest can be done in a similar manner. Given the 3D representation we show it step-wise. We consider the example $\left(\frac{4}{3} \times \frac{5}{9}\right) \times \frac{1}{2} = \frac{4}{3} \times \left(\frac{5}{9} \times \frac{1}{2}\right)$

	Step 1	Step 2	Step 3
Left hand side: $\left(\frac{4}{3} \times \frac{5}{9}\right) \times \frac{1}{2}$	$\frac{4}{3}$  Figure 5	$\frac{4}{3} \times \frac{5}{9}$  Figure 6	$\left(\frac{4}{3} \times \frac{5}{9}\right) \times \frac{1}{2}$  Figure 7
Right hand side: $\frac{4}{3} \times \left(\frac{5}{9} \times \frac{1}{2}\right)$	$\frac{5}{9}$  Figure 8	$\frac{5}{9} \times \frac{1}{2}$  Figure 9	$\frac{4}{3} \times \left(\frac{5}{9} \times \frac{1}{2}\right)$  Figure 10

For visual clarity, we have kept a gap between the two unit cubes. But that need not be there.
Note that the end volume is the same for both and that this can be generalized for any three fractions (and natural numbers).

This poster, created by SWATI SIRCAR, provides visual justifications which are not the same as proofs. It is based on the PullOuts by Padmapriya Shirali – Multiplication (Vol. 3, Issue 1) and Fraction (Vol. 1, Issue 1), and Visual Method for Fraction Multiplication by Shailaja D Sharma (Vol. 7, Issue 2)

between the sender and receiver. Here, apart from the key matrix (private) we have also involved a 'number' as our private element.

Points to be kept in mind:

i. The key matrix should be a non- singular matrix.

ii. The order of multiplying message matrix by key matrix, i.e., pre multiplication or post multiplication by key matrix must be pre decided between the sender and the receiver.

iii. If the elements (letters) of message are fewer than the number of elements of matrix M, then fill the vacant positions with '0'.

iv. Symbols of $(+)$ and $(-)$ in r^{q+} and r^{q-} respectively can be replaced by new ones which both sender and receiver can decide (denoting addition and subtraction respectively).

v. Larger sentences may be fragmented into groups of three letters such as **HEL, LOH, OWR, U?**. Each group may be treated as a 1 by 3 or 3 by 1 matrix, which may be multiplied with key matrix and then cipher matrix for each group may be obtained separately. In the same way inverse of each cipher matrix may also be obtained separately.

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Student Corner – Featuring articles written by students.

Finding the number of ordered tuples having a given LCM

RAHIL MIRAJ

Counting is one of the first skills that a student of mathematics learns. Here we feature an article that observes patterns while counting and generalises this pattern. While the actual combinatorics and use of the binomial theorem may be appreciated only by students of classes 11 and 12, students of High School will certainly be able to follow the reasoning in the two examples given. It is important for students to have such gentle introductions to mathematical notation and theorems used at a more senior level.

In this article, we derive a formula for the number of ordered n-tuples of positive integers whose LCM is a given integer. More specifically, we prove the following theorem.

Theorem. *The number of ordered n-tuples whose LCM is $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, where $p_1, p_2, \cdots, p_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are non-negative integers, is equal to*

$$\prod_{i=1}^{m} \left[(a_i + 1)^n - a_i^n\right]. \qquad (1)$$

Before proceeding with the proof of this claim, we illustrate the use of the formula with some examples.

Example 1. To find the number of ordered pairs (a, b) of positive integers whose LCM is $p^2 q^4 r^2$, where p, q, r are prime numbers.

Solution. No prime number other than p, q, r can divide either a or b. Let us write

$$a = p^u q^v r^w, \quad b = p^x q^y r^z,$$

where u, v, w, x, y, z are non-negative integers. Since the LCM of a, b is $p^2 q^4 r^2$, it must be true that

$$u, x \in \{0, 1, 2\}, \quad v, y \in \{0, 1, 2, 3, 4\}, \quad w, z \in \{0, 1, 2\},$$

and

$$\max(u, x) = 2, \quad \max(v, y) = 4, \quad \max(w, z) = 2.$$

Keywords: LCM, primes, multiplication principle

The ordered pair (u, x) can be any of the following (5 possibilities):

$$(0,2),\ (2,0),\ (1,2),\ (2,1),\ (2,2).$$

The ordered pair (v, y) can be any of the following (9 possibilities):

$$(0,4),\ (4,0),\ (1,4),\ (4,1),\ (2,4),\ (4,2),\ (3,4),\ (4,3),\ (4,4).$$

The ordered pair (w, z) can be any of the following (5 possibilities):

$$(0,2),\ (2,0),\ (1,2),\ (2,1),\ (2,2).$$

Therefore, using multiplication principle, the number of ordered pairs (a, b) is

$$5 \times 9 \times 5 = 225.$$

Now we solve the same problem by using the theorem. Here,

$$m = 3,\ n = 2,\ a_1 = 2,\ a_2 = 4,\ a_3 = 2.$$

Hence the required number is

$$\begin{aligned} &\prod_{i=1}^{3} \left[(a_i + 1)^2 - a_i^2\right] \\ &= \left[3^2 - 2^2\right] \cdot \left[5^2 - 4^2\right] \cdot \left[3^2 - 2^2\right] \\ &= 5 \times 9 \times 5 = 225. \end{aligned}$$

Example 2. To find the number of ordered triples (a, b, c) of positive integers whose LCM is $p^2q^4r^3s^2$, where p, q, r, s are prime numbers.

Solution. No prime number other than p, q, r, s can divide a, b, c. Let us write

$$a = p^u q^v r^w s^x, \quad b = p^{u'} q^{v'} r^{w'} s^{x'}, \quad c = p^{u''} q^{v''} r^{w''} s^{x''},$$

where $u, v, w, x, u', v', w', x', u'', v'', w'', x''$ are non-negative integers. Since the LCM of a, b, c is $p^2q^4r^3s^2$, it must be true that

$$u, u', u'' \in \{0,1,2\}, \quad v, v', v'' \in \{0,1,2,3,4\}, \quad w, w', w'' \in \{0,1,2,3\}, \quad x, x', x'' \in \{0,1,2\},$$

and

$$\max(u, u', u'') = 2, \quad \max(v, v', v'') = 4, \quad \max(w, w', w'') = 3, \quad \max(x, x', x'') = 2.$$

The ordered triple (u, u', u'') can be any of the following (19 possibilities):

$$\begin{array}{lllllll} (2,0,0), & (2,1,0), & (2,0,1), & (2,1,1), & (2,2,0), & (2,0,2), & (2,1,2), \\ (2,2,1), & (2,2,2), & (0,2,0), & (0,2,1), & (1,2,0), & (1,2,1), & (0,2,2), \\ (1,2,2), & (0,0,2), & (1,0,2), & (0,1,2), & (1,1,2). & & \end{array}$$

The ordered triple (v, v', v'') can be any of the following (61 possibilities):

$$\begin{array}{lllllll} (0,0,4), & (0,4,0), & (4,0,0), & (1,1,4), & (1,4,1), & (4,1,1), & (2,2,4), \\ (2,4,2), & (4,2,2), & (3,3,4), & (3,4,3), & (4,3,3), & (4,4,4), & (0,1,4), \\ (1,0,4), & (0,2,4), & (2,0,4), & (0,3,4), & (3,0,4), & (1,2,4), & (2,1,4), \\ (1,3,4), & (3,1,4), & (2,3,4), & (3,2,4), & (0,4,1), & (1,4,0), & (0,4,2), \\ (2,4,0), & (0,4,3), & (3,4,0), & (1,4,2), & (2,4,1), & (1,4,3), & (3,4,1), \\ (2,4,3), & (3,4,2), & (4,0,1), & (4,1,0), & (4,0,2), & (4,2,0), & (4,0,3), \\ (4,3,0), & (4,1,2), & (4,2,1), & (4,1,3), & (4,3,1), & (4,2,3), & (4,3,2), \\ (4,4,0), & (4,4,1), & (4,4,2), & (4,4,3), & (4,0,4), & (4,1,4), & (4,2,4), \\ (4,3,4), & (0,4,4), & (1,4,4), & (2,4,4), & (3,4,4). & & \end{array}$$

The ordered triple (w, w', w'') can be any of the following (37 possibilities):

$$\begin{array}{lllllll} (0,0,3), & (0,3,0), & (3,0,0), & (1,1,3), & (1,3,1), & (3,1,1), & (2,2,3), \\ (2,3,2), & (3,2,2), & (3,3,3), & (0,1,3), & (1,0,3), & (0,2,3), & (2,0,3), \\ (1,2,3), & (2,1,3), & (0,3,1), & (1,3,0), & (0,3,2), & (2,3,0), & (1,3,2), \\ (2,3,1), & (3,0,1), & (3,1,0), & (3,0,2), & (3,2,0), & (3,1,2), & (3,2,1), \\ (3,3,0), & (3,3,1), & (3,3,2), & (3,0,3), & (3,1,3), & (3,2,3), & (0,3,3), \\ (1,3,3), & (2,3,3). & & & & & \end{array}$$

The ordered triple (x, x', x'') can be any of the following (19 possibilities):

$$\begin{array}{lllllll} (2,0,0), & (2,1,0), & (2,0,1), & (2,1,1), & (2,2,0), & (2,0,2), & (2,1,2), \\ (2,2,1), & (2,2,2), & (0,2,0), & (0,2,1), & (1,2,0), & (1,2,1), & (0,2,2), \\ (1,2,2), & (0,0,2), & (1,0,2), & (0,1,2), & (1,1,2). & & \end{array}$$

Therefore, using multiplication principle, the number of ordered triples (a, b, c) is

$$19 \times 61 \times 37 \times 19 = 814777.$$

Now we solve the same problem by using the theorem. Here,

$$m = 4, \;\; n = 3, \;\; a_1 = 2, \;\; a_2 = 4, \;\; a_3 = 3, \;\; a_4 = 2.$$

Hence the required number is

$$\begin{aligned} &\prod_{i=1}^{4} \left[(a_i + 1)^3 - a_i^3\right] \\ &= \left[3^3 - 2^3\right] \cdot \left[5^3 - 4^3\right] \cdot \left[4^3 - 3^3\right] \cdot \left[3^3 - 2^3\right] \\ &= 19 \times 61 \times 37 \times 19 = 814777. \end{aligned}$$

Proof of the theorem

For convenience, we repeat the statement of the theorem:

Theorem. *The number of ordered n-tuples whose LCM is $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, where $p_1, p_2, \cdots, p_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are non-negative integers, is equal to*

$$\prod_{i=1}^{m} \left[(a_i + 1)^n - a_i^n\right]. \tag{2}$$

Let $(b_1, b_2, \ldots, b_n)$ be an ordered n-tuple of positive integers whose LCM is

$$p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}.$$

We must count the number of such n-tuples. Let

$$b_j = p_1^{c_{1j}} \cdot p_2^{c_{2j}} \cdots p_m^{c_{mj}} \qquad (1 \le j \le n). \tag{3}$$

We must have:

$$c_{11}, c_{12}, \ldots c_{1n} \in \{0, 1, 2, \ldots, a_1\}, \quad \max\left(c_{11}, c_{12}, \ldots c_{1n}\right) = a_1. \tag{4}$$

How many possibilities are there for the n-tuple $(c_{11}, c_{12}, \ldots c_{1n})$, subject to (4)? At least one c_{1j} must be at its maximum possible value (a_1). Let us fix these c_{1j}'s at the start; let k of the c_{1j}'s be at their maximum possible value ($1 \le k \le n$). These c_{1j}'s can be chosen in $\binom{n}{k}$ possible ways. Each of the remaining c_{1j}'s can take a_1 possible values (i.e., all the values from 0 to $a_1 - 1$). Hence the number of choices for the remaining c_{1j}'s is a_1^{n-k}. Hence the number of possibilities for the n-tuple $(c_{11}, c_{12}, \ldots c_{1n})$, subject to (4), is

$$\sum_{k=1}^{n} \binom{n}{k} \cdot a_1^{n-k}.$$

By the binomial theorem, this quantity is equal to

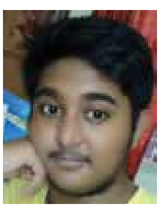
$$(a_1 + 1)^n - a_1^n.$$

The same reasoning holds for the other tuples. Hence, by the multiplication principle, the number of possible ordered n-tuples $(b_1, b_2, \ldots, b_n)$ is equal to

$$\left[(a_1+1)^n - a_1^n\right] \cdot \left[(a_2+1)^n - a_2^n\right] \cdot \cdots \cdot \left[(a_m+1)^n - a_m^n\right], \tag{5}$$

i.e., $\prod_{i=1}^{m} \left[(a_i+1)^n - a_i^n\right]$.

Acknowledgments
I thank my father Prof. Dr. Farook Rahaman for several illuminating discussions.



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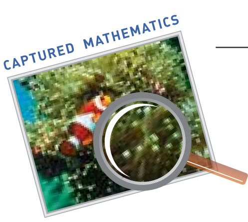

Image 3: Light Shows are a major attraction in cultural and musical fests. Fun with these lights reveals different patterns and shapes.

Lighting during Music Fest, Hindu College, DU

Photograph and Ideation: Kumar Gandharv Mishra

ClassRoom

Cutting a Square into Equal Parts

VINAY NAIR

Pre-requisites

The article talks about a simple activity which can be performed with students of primary, middle and high school. The shape that is used to discuss here is a square and hence it is expected that students know the basic properties of a square. The article also talks about using lines. Even if students don't have a Euclidean notion of definition of a line, that idea can be instilled as the teacher executes this activity. Similarly, students might have a notion of the words *equal, area, congruence, similar*, etc., and this activity is very useful in clarifying the subtle difference between terminologies as well as the rigour of mathematical language.

Another important aspect is that the activity allows the teacher to introduce concepts like countable, uncountable and infinite (not in the notion of different types of infinity, but in the sense of a layman's usage of the words). This is a very fruitful exercise because many a time all these words are mixed up in the usage by children.

Towards the end of the article, inside a table, the reader will find the question ***Why*** asked quite a few times. This is *why* the activity is useful for high school students as well because they will be in a better position to reason as to why they believe the answer is so for a particular number of lines. If they have to give reasons, they might have to take a dip into the axioms and definitions of certain concepts, which they don't do otherwise while solving school problems. At primary and middle school level also the reasoning can (and should) be asked, but at that age, they might not be introduced to the idea of axioms and building theories in Mathematics by changing axioms.

Last, but not the least, the activity can be explored using other shapes as well and a comparative study made between different shapes.

Keywords: Square, symmetry, equality, congruency, area, generalisation

"In how many ways can you divide a square into equal parts using a line?" is a question that I have often asked middle school students. Interestingly, I get new answers every time I ask this question and the questions that follow. The most common answer is 'Four,' the four ways being:

Category - I

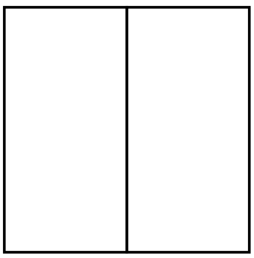

Figure 1.1

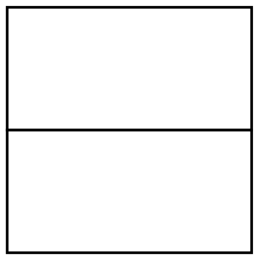

Figure 1.2

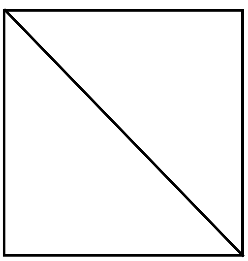

Figure 1.3

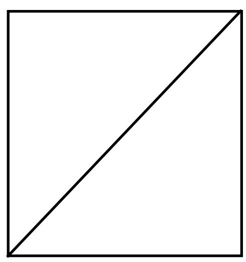

Figure 1.4

I imagine the reason why most people think of these four ways is because they would have done something similar in their own lives through paper folding (cutting out a square from a rectangular sheet of paper), opening and closing books, cutting cakes or chocolates, folding bedsheets/mats, etc. One can think of so many situations in which such folding must have been done.

Some students come up with non-routine solutions, like:

Category - II

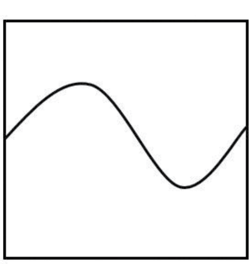

Figure 2.1

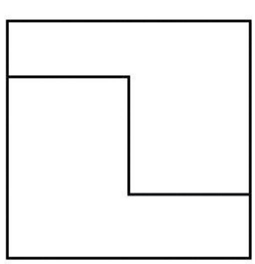

Figure 2.2

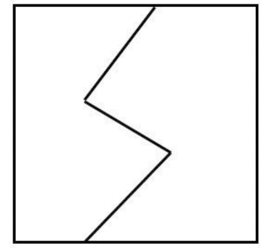

Figure 2.3

Surprisingly, such solutions usually don't come from the older kids who have been conditioned by thinking patterns; and we rarely see adults making such designs. When students come up with such designs, as a teacher one might be tempted to say that they are *wrong* as the question clearly tells us that we must use ***a line***, and that line should be always **straight**. But this is a good way to kill the creativity of a student. Instead, when we ask them to explain what they mean by *lines*, the students who have drawn this understand these dividers also as a line probably because they might have seen queues outside a ticketing office that go in these shapes which we call colloquially as *lines!*

Most of the students in the class don't accept the divisions in category II. This is an opportunity for the teacher to discuss in the class - *What is a line?* I've heard a nice definition from many students for a *straight line* - A straight line is a line that doesn't bend. If we can avoid the mathematical rigour of defining a line, we can probably accept this definition. Usually, at this point, the majority of the class agrees that there are four ways of dividing a square into equal parts using a line if we consider divisions under Category - I, and infinite[1] ways if we consider Category - II. At this juncture, I tell them that even in Category - I, there are more than four ways of dividing it and challenge them to find more ways. Subsequently, I have seen a few students drawing the image below:

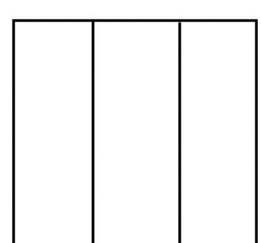

Figure 2.4

When I have asked them to read the question again, I have found out that they have taken the phrase *'…a line'* in the question to be *'…one line.'* When this is clarified, they take back their answer. However, there is always some kid who thinks of the idea sketched below:

Category - III

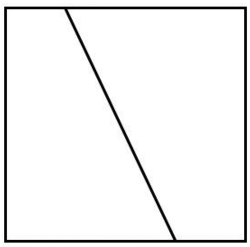

Figure 3.1

There is a general confusion amongst students when you ask them if the parts shown are equal parts. Many say, *'Yes'*, some say, *'No'*. When the latter are asked the reason for their answer, most of them say that they don't seem to be equal. One of the interesting answers that I got from a student recently was - *'How can they be equal if they are not symmetric?,* (By *symmetry*, the student was referring to reflective symmetry.) So I asked the class if they thought it is true that if two things have to be equal, they should have reflective symmetry as well. After some discussion, a few of them came up with counter examples where two things were equal even when they did not have reflective symmetry. Thus, this new category of dividing a square into equal parts is added to the answer. But there are still a few who want to be convinced that the two parts are equal. So they are asked to construct a square and draw a line from a point on one edge in such a way that when the line meets the opposite edge, the length from corner to starting point is the same as that from the opposite corner of the square to the end point. And then, measure all the sides and angles of the two shapes formed; or cut out the shapes and see if they fit exactly one on top of each other. Again, the question still remains - how many such possibilities exist? Most say, *'Two'* and then say *'Four'*.

Category - III

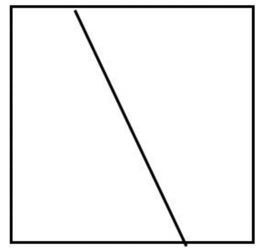

Figure 3.1

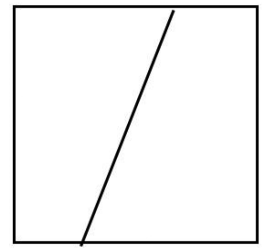

Figure 3.2

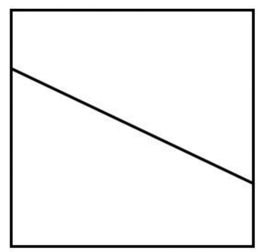

Figure 3.3

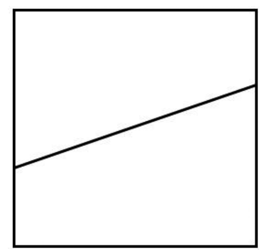

Figure 3.4

But in no time, someone else comes up saying that more ways are possible because you can alter the distance where the divider meets the sides of the square and still cut it into equal parts. Thus, the class agrees that there are **many, many** ways to cut a square into equal parts using a line.

Cutting a square into equal parts using two lines

The next question that follows is –*"In how many ways can you divide a square into equal parts using two lines?"* The following two answers come up:

Category - IV

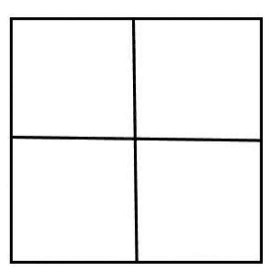

Figure 4.1

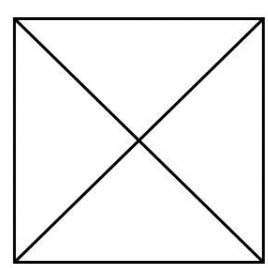

Figure 4.2

Category - V

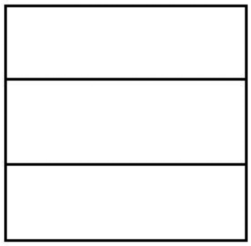

Figure 5.1

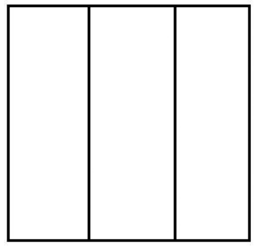

Figure 5.2

In most cases, students come up with either Category IV or V and not both. This is an interesting observation. Naturally, the next question is - *How many ways are there to cut a square into four equal parts and three equal parts using two lines?* Students start working on it immediately.

If you would have noticed, until the last question, we have never mentioned the number of equal parts into which they are supposed to divide the square. Everywhere it says, 'divide the square into equal parts.' Why? When we don't give them the number of equal parts, we are not conditioning them or putting any constraint on them in thinking. This frees their imagination which is very important to bring about creativity through mathematics.

Let us come back to the question. Most students who would have explored say that there are only two ways in Category IV and two ways in Category V. However, there have always been two or three students out of a class of 30-35, who draw the image below and say that they have found a new way.

Category - VI (a)

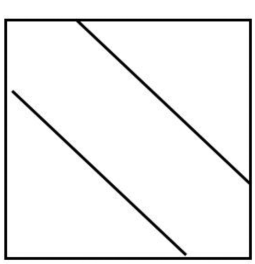

Figure 6.1

Category - VI (b)

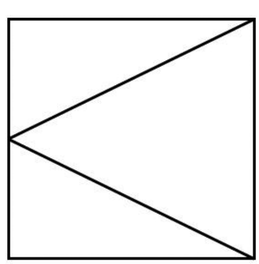

Figure 6.2

(Before you read ahead, please try to think why the students call them equal parts. Please note that in Fig 6.1 and 6.2, even though the students did not specify the lengths of the two line segments, they are assuming the line segments in each square to be of equal lengths. Also the triangles formed in Fig 6.1 are isosceles triangles, and the line segments in Fig 6.2 meet each other at the midpoint of the side of the square.)

Yes, in the case of Category VI (a), they think so because they have considered *equal* to be of *equal quantities or equal area*. They are certainly not wrong because the word 'equal' is very ambiguous. This is a great opportunity to talk about a few important points, starting with - What do you mean by 'equal'? Those who consider their answer of Category VI is to be correct say that 'equal' means 'equal quantity'. Others say that 'equal' means 'same shape and same size'. The class arrives at a conclusion that depending on how we define *equal*, we can accept or reject Category VI. We move ahead with the meaning of *equal* as *same shape and same size*[2].

The students who see Category VI (b) as equal parts see them as equal because all the three parts are triangles and they aren't able to see the difference between the shapes of the triangles. It usually takes a little time for those students to realise that the right angled triangles are half the size of the isosceles triangle.

Next, we ask the students to find out whether there could be more than two ways of dividing the square in Category IV. Most of them don't see more ways but there is always one or two in the class who think of this:

Category - VII

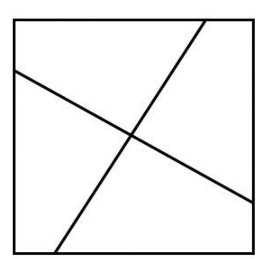

Figure 7.1

To see that all the four parts could be of the same shape and size is not easy for many. Those who might have spent a considerable amount of time in observing shapes and patterns closely while they were younger, are able to spot that the four parts could be identical. For others, they can be asked to try to cut out the shapes (after being instructed how one needs to draw the lines so that the four parts can be equal) and verify. For students in high school, this could be a good question to *prove*. Some may also observe that the lines have to be perpendicular to each other if the four parts have to be equal (Why?[3]). Again, they are asked to find out how many possibilities exist and they come up with the answer - **many many** (**Why?**). Some students also start feeling that the answer is always **many many** and say that there are **many many** ways to divide a square into three equal parts using two lines. While some students object to this, they are unable to reason it out. We will leave it to the reader to explore the question - Prove or disprove that there are infinite ways of dividing a square into four congruent parts using two lines.

Some students draw the picture below also.

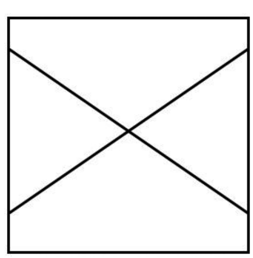

Figure 7.2

In such instances, we need to reiterate the meaning of *equal as same shape and size*. Still, some see the four parts as being of equal shape and size. If there are students who see it that way, they need to be asked to see the four parts and see what kinds of shapes are formed. Such students are usually very few in number but the fact is that they see the four shapes as equal parts which is something that the teacher might not have imagined.

Cutting a square into equal parts using three lines

The next question that the students explore is - *In how many ways can a square be cut into equal parts using three lines?* Most students find out the two ways shown below.

Fig 8.1

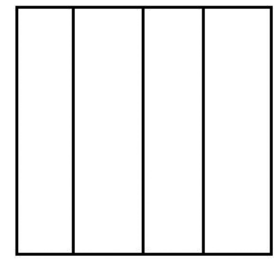

Fig 8.2

Some observe that using three lines, we could divide the square into six equal parts.

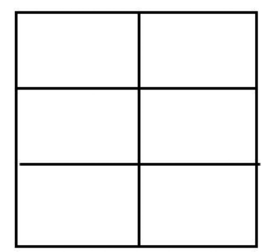

Fig 8.3

By this time, when we ask the students - 'What are the questions that you would like to explore now?' they say, 'How many ways are there to divide a square into six equal parts using three lines and four equal parts using three lines?' Without any further instruction from the teacher, they start pursuing both the problems. Once they come up with their answers, the teacher again asks - 'Why do you feel so?' This is a very important question, both for the teacher and for the student. It is important for the teacher because it is an opportunity to understand how young minds think, and it is important for the students because they start reasoning in response to this question *Why*.

If you happen to try this question in a classroom, there would be more solutions that students might give and many of them would be incorrect. Instead of saying whether it is right or wrong, try asking them *Why do they feel it is correct?*

This exploration can continue and one can ask students to make a table as below and explore by changing the number of lines and the number of equal parts.

Shape	No. of lines	No. of equal parts	How many ways?
Square	1	2	Many many (Why?)
Square	2	3	Two (Why?)
Square	2	4	Many many (Why?)
Square	3	4	Many many (Yes! It is true. But how?)
Square	3	5	How many ways? Why? (Can you prove it?)
Square	3	6	How many ways? Why?
Square	4	8	How many ways? Why?
Square	5	8	How many ways? Why? (There exists a solution)
Square	6	8	How many ways? Why? (The answer is more than 1)
Square	7	8	How many ways? Why?

The next interesting exercise could be to change the shape from a square into some other shape, say a circle or a rectangle or an equilateral triangle and make the same table. Observe if the number of ways will still remain the same for a certain number of lines and equal parts. Can we arrive at some observational generalisations followed by some logical generalisation if possible? Can we try to find (and prove) that for x number of lines in a shape, y number of equal parts are NOT possible? The exploration is endless!

Learning Outcomes

1. Students start observing shapes more closely.
2. A deeper understanding of 'congruence' can be developed before the topic of congruence is introduced in high school geometry.
3. Improving reasoning and observation skills.
4. Attempting observational generalisations and the ability to come up with claims.
5. Importance of *defining* and understanding terms (like *equal*) before engaging in problem-solving.

Questions for further exploration for high school students

1. Try writing down proofs for all the *WHYs*.
2. In a square, we can draw three lines and divide the square into at most seven parts. Is it possible to divide the square into seven parts of equal area? Can we justify our answer?
3. If there are n number of lines drawn, what are the possible number of equal parts that can be made from a square and other shapes?

Teacher Notes

1. I have observed that most students in middle school who use the word 'infinite' confuse the term with 'uncountable' or 'a very large number' or 'the largest number'. Hence, I use a new term with them which is **many many**. In this article, I have used the term **many many** for 'infinite' as the term 'infinite' won't be clear to most middle school students.
2. We don't use the word 'congruence' because it is not a vocabulary that they are familiar with. But 'same shape and same size' is something that early middle school students can comprehend.
3. Wherever there is a 'Why' put up, it is a question for the reader to see if they can prove it with some mathematical rigour.

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Playing with Tiles
Beginning to Tessellate

R.GOMATHY

Many people are of the opinion that mathematics is only about numbers and number operations, and thus myths related to who can do mathematics and who cannot, abound. It is possible that children may struggle with numbers, but it is hard to believe that there could be a child who doesn't recognize patterns. We see children creating patterns all the time using stones, sticks, leaves, flowers, finger prints, vegetable carvings, rubber stamp impressions and also mathematical shapes. Often they create patterns unknowingly as part of their games and activities. Children should look for patterns as a means of understanding and learning mathematics. "Looking for patterns trains the mind to search out and discover the similarities that bind seemingly unrelated information together as a whole…. A child, who expects things to make sense, looks for the sense in things and from this sense develops understanding. A child who does not see patterns often does not expect things to make sense and sees all events as discrete, separate and unrelated." (Baratta, as cited in Burns, 112)[1]



The word 'Tessellation' comes from the Latin word *'tessella'* which means a small cube or a tile. When we say tessellation, we mean filling a surface or a plane with flat shapes without any gaps or overlap. So it may also be called as a pattern of shapes that fit perfectly together.

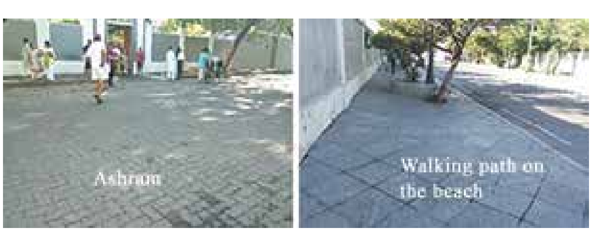
Keywords: tessellation, pattern, play, tiles, colour, shape

In this article I (a primary school teacher) wish to share the experiences when my Grade III students engaged with a two-dimensional patterning exercise, called Tessellation or Tiling.

Let me introduce my students to you. We belong to a small school in Puducherry and I teach them Language (English), Mathematics and EVS. The students mentioned in this article are of age group seven- eight years. Ours is a small classroom (22 by 22 feet) but I manage to maintain a mathematics corner, equipped with lots of manipulatives, in our small room. To state some, we have material for learning counting, place value, four basic arithmetic operations, and fractions. Both teacher and students take joint responsibility of arranging and maintaining these materials.

The beginning

Many good examples of tessellations can be located in historical monuments but as my students could not be taken to these sites, I planned to show them the examples of tessellations from our easily available resources, such as the pathway of Bharathi Park in Puducherry, veranda grills of our houses, window designs in the church and the tiles in kitchens and restrooms of houses and schools, and of course, tiling patterns in the children's dress materials.



In our syllabus, tiling patterns are introduced from Class II onwards. Since last year, I give tessellation activities to Grade II children. However, this year while doing the activities on tessellations, I noticed a huge difference in the approach of the children.

Grade II generally regarded this activity as a 'fun activity,' mainly associating it with their prior experiences of making kolams and other floor designs. This year, I noticed that as these children moved to Grade III, their approach started becoming more mathematical. This was evident from the way they had started using mathematical terms while doing the tessellation activities. I observed them using words such as 'square', 'triangle', 'hexagons', 'sides (edges)', 'corners(vertices)', 'tiling', and 'gaps' more often. One could see an emergence of mathematical maturity as they used mathematically-oriented vocabulary. Using precise mathematical terms reflected their connection with the concepts.

Moving on… Engaging and learning

In this section, Grade III children's journey of learning tessellations has been shared. We began with 'completing the pattern' exercise and gradually moved to learning about tiling two-dimensional shapes.

Completing the pattern:

All pattern related exercises begin with the task of completing a given incomplete pattern. This exercise acquaints children with the intricacies of placing the units correctly. I knew that my Grade III children were already familiar with creating linear patterns; this time I gave them an activity to complete a pattern involving two-dimensional shapes. To give them a hint, an example of tessellation pattern was displayed in front of the class and the children were asked to first copy and then complete the pattern in the worksheet given to them.

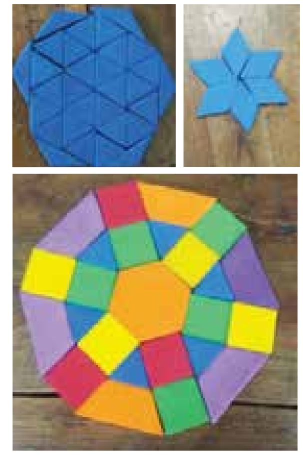
Based on their work, I could easily place the children under three categories.

1. Children who could complete the pattern as expected. Only 4 children were able to continue the tessellation pattern as expected.
2. Children who could complete the pattern partly. These children did make a pattern based on their own ideas but their work was different from what was expected. I could place 8 children in this category.
3. Children who did not work on expanding the pattern and instead took it as a 'fun activity' of colouring. Three children were seen taking it as a colouring activity.

After completing the pattern, the students were given space to explain their thoughts. This discussion helped children identify the mistakes they had made and to also reflect on how they had understood/misunderstood the activity.

Understanding the basics of Tessellation:

Next, we moved on to learning the basics behind tessellations. We used some geometrical shapes from the tessellation kit available in our mathematics corner. I divided the class into three groups with five children in each group and they were asked to use the shapes from the kit and fill up the top surface of the table in such a way that there were no gaps or overlaps between the shapes. Examples of the floor tiling patterns which we had identified earlier in and around our Puducherry city were shared. Each group worked to make their tiling patterns.



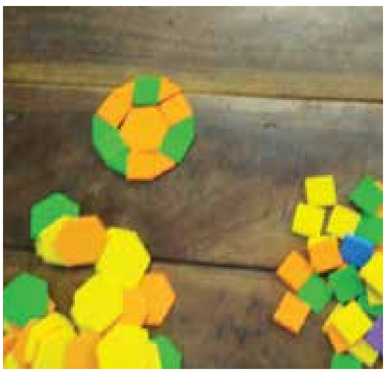
Preliminary engagement

Complex patterns

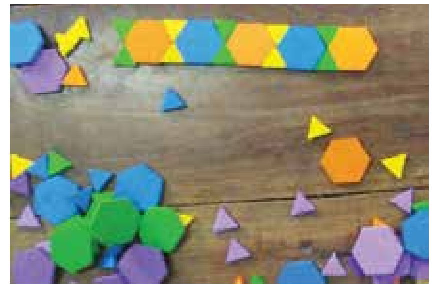
After this activity I wanted to categorise the students based on their level of understanding of the concept. Some children could work only to a preliminary level while some understood the idea and made complex tessellation patterns. As one can see, children at preliminary level were those who placed the shapes edge-to-edge with no gaps and no overlaps but were unable to proceed with the same pattern throughout.

While doing this activity, the children also learnt about the idea of rotational symmetry, particularly of square, equilateral triangle and regular hexagon. I asked the children to rotate these shapes and see if the orientation of these shapes changed after undergoing a rotation of 180 degrees. Children articulated that 'square remains the same, likewise hexagon also remains the same, but the triangle turns upside down.' They used their own informal language (in Tamil, which is their mother tongue) to express this.

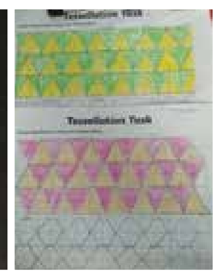
Afterwards, the children started their work on creating a tiling pattern but this time I had to facilitate their working. I gave triangles and squares to one group, hexagons and triangles to the second group, and hexagons, triangles and squares to the third group. Their task was to create a tiling pattern with the shapes given to them.



Initially, the children got attracted to the colours of the tiles and did not bother to place them in a tessellation network. One could see gaps between the tiles and there was no specific pattern in their work. I had to remind them of their earlier learnings and soon after the children refined their work and created some tiling patterns.



Then I gave them some square-grid sheets and asked them to draw their respective tessellations. I could see the children were ready to handle the complexities of creating tiling patterns on the square-grid sheets. They showed remarkable improvement.



Some reflections

At times, students find it difficult to imagine things. Often they need additional support in doing so and in such situations concrete material does the scaffolding. I am reminded of Cobb & Bowers's (1999)[2] thoughts, "Vygotsky took interest in the fact that human action is mediated by tools and signs. He says that tools could be physical or psychological. Researchers says that tools have profound influence on mathematical thinking and the way students develop mathematical understanding." Connecting with my work, I can say, the colourful tessellation kit helped my children understand the concept of tessellations. My children of class III were beginners to the tessellation activity; therefore I had to use the tessellation and 'rangometry' kits. In the Class III textbook, tiling activity is confined only to a colouring activity and children find it very difficult to imagine the symmetrical and rotational patterns from the textbook. When they handle the pieces of the kit, they feel motivated to do the activity. The physical touch of the shapes also familiarizes them to the properties of the shapes.

I also noticed that students find tessellations with hexagons and triangles easiest. Rarely did they try their hands on tessellating regular triangles and squares. This may be because of the influence of drawing *Kolam and Rangoli* designs in their houses. One of the girls, Rohini, said that she could make very attractive designs as she helps her mother in making *Kolam* designs. Many other girls shared similar experiences.

Another important observation is that the children in Class III were not interested in tiling the entire surface. They rather preferred making shapes of house, boat, flower pots and *Rangoli* designs. This shows that students keenly observe their surroundings and are very attached to their environment. Some of the girls, instead of making two dimensional tiles, began stacking the shapes. On asking, they said they liked to construct buildings using those tiles. Was this due to their prior experience of playing with the building blocks, I wonder.

According to the learning outcomes of the NCERT, by the time children reach Class III, they must be able to create and expand patterns. This tessellation activity was helpful as, through it, my Class III students were able to visualize and handle the shapes and deduce their properties. I am confident that this foundational activity will help my children understand geometry in a better way. I could see them recognizing and explaining the properties of shapes. They were making intricate monohedral tiling patterns with regular polygonal tiles. Although these concepts are not dealt with in detail in NCERT Class III textbook, my students were doing this intuitively. They could see rotational symmetry in shapes, recognize and describe the properties of regular shapes, and could appreciate the aesthetic aspect in mathematics.

The author is grateful to Dr. Haneet Gandhi, Delhi University, who contributed significantly to the presentation of this article.

References

1] Cobb, P., Bowers, J. (1999). Cognitive and Situated Learning Perspectives in Theory and Practice. Educational Researcher, Vol. 28, No. 2, pp. 4.

2] http://archive.dimacs.rutgers.edu/nj_math_coalition/fwfinal/ch11/ch11_k-02.html

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Low Floor High Ceiling Tasks

Squaring the Dots

Think into the Box

SWATI SIRCAR & SNEHA TITUS

A square dot sheet has equally spaced dots aligned vertically and horizontally. Many interesting investigations can be devised with this simple learning material. We began with a question that had been posted in the Thinking Skills PullOut (November 2015): *Can you draw a square with just 1 dot inside?* The resulting investigation has branched out in multiple directions; we will explore a few and leave you to try the rest. As usual, we have designed this investigation in the Low Floor High Ceiling style in which we welcome explorers to step easily into the low-floored classroom with fairly easy questions. As the investigations proceed, the questions get more challenging and the high ceiling is designed to keep even the most able students thinking, conjecturing, proving and, in short, being mathematicians.

Task 1: Squares with sides along vertical and horizontal axes.

Note: Squares are always drawn with dots at the vertices.

- How many dots are enclosed within a 1×1 square?
- As the size of the square increases, is there a pattern in the number of dots inside?
- Is it possible to generalise to the number of dots inside an $n \times n$ square?
- Is there more than one way of summing the dots inside the squares?
- Do all these ways of summing the dots give the same formula for the number of dots inside an $n \times n$ square?

Keywords: Dot sheets, squares, counting, slopes, pattern, generalising, algebra

Task 2: Squares with sides inclined to the vertical.

Note: Squares are always drawn with dots at the vertices.

- Draw the smallest square with its diagonals along vertical and horizontal axes. What angle are its sides inclined at?
- If the distance between two adjacent horizontal (or vertical) dots is 1 unit, what is the length of the side of this square?
- Is it possible to get a square with the sides inclined at other angles to the vertical?
- If the distance between two adjacent horizontal (or vertical) dots is 1 unit, find the length of the side of this new square.
- Can you find a general way of generating more and more such squares? Try to find the smallest square each time.
- Can you find a general representation for the sides of the squares thus generated?

Task 3: Counting the dots inside squares with sides inclined to the vertical.

- For each type of square that you have drawn, draw bigger and bigger squares and count the number of dots inside. Make a table for each type of square.
- Can you find a pattern to sum the dots inside?
- Can you find a pattern connecting the number of dots inside to the squares of increasing inclination?

Teacher Notes:

We have simply given **our** final outcomes - an exploration usually starts in an open-ended manner and it is important for students not to be constrained at this time. As they start observing, they will notice patterns and at this time, gentle facilitation will nudge them along the path to more systematic documentation. Do remember that there can be other discoveries that they make and other paths that they may want to follow as their interest deepens.

Task 1: The figures 1.1 and 1.2 and the following table show the results of our investigations.

Side length	Formula 1 for adding dots	Formula 2 for adding dots	Formula 3 for adding dots	No of dots inside
1				0
2				1
3	2 + 2	2 × 2	1 + 2 + 1	4
4	3 + 3 + 3	3 × 3	1 + 2 + 3 + 2 + 1	9
5	4 + 4 + 4 + 4	4 × 4	1 + 2 + 3 + 4 + 3 + 2 + 1	16
n	$(n-1)$ added $(n-1)$ times	$(n-1) \times (n-1)$	$2(1 + 2 + \ldots(n-2)) + (n-1)$ $= (n-2)(n-1) + (n-1)$	$(n-1)^2$

Table 1

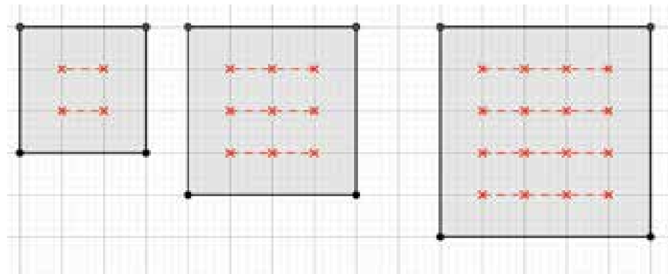

Figure 1.1

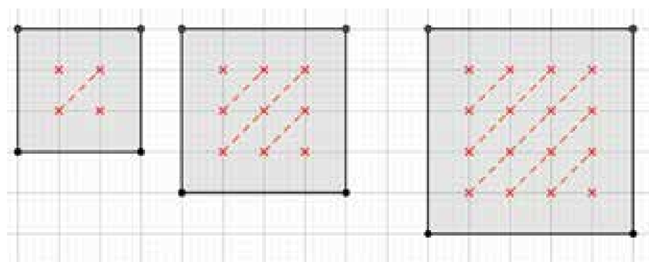

Figure 1.2

Task 2:

Squares with sides at increasing tilts. See Figure 2. Students who are not used to the idea of slope may find it difficult to go beyond the first tilt, but do give them time to explore. The idea of slope and the relationship between slopes of perpendicular lines will emerge very naturally- even without explicit statements.

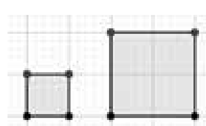

Figure 2.1

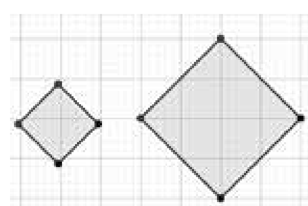

Figure 2.2

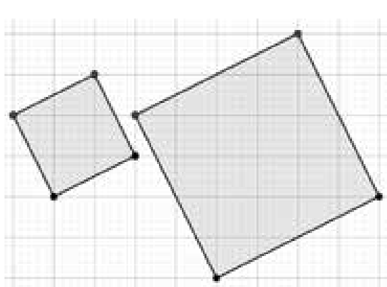

Figure 2.3

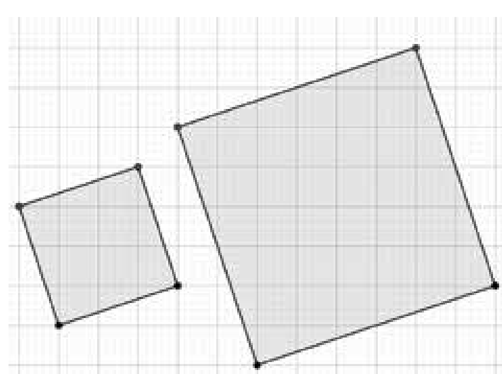

Figure 2.4

Relationship between tilt and length of the side of the square

Figure	Side of square	Tilt in words
2.1	1 unit	Vertical and horizontal sides
2.2	√2 units	From the left dot go right 1 and up 1 to get to the dot on the right. Reverse this to go to the dot below.
2.3	√5 units	From the left dot, go right 2 and up 1 to get to the dot on the right. Reverse this to go to the dot below.
2.4	√10 units	From the left dot, go right 3 and up 1 to get to the dot on the right. Reverse this to go to the dot below.

Table 2

Task 3:

Figure 3 shows the summing patterns obtained for squares with sides increasing in multiples of √2. Students can experiment with colour and see newer patterns emerging, which we may not have reported here.

Please note that the grid size had to be adjusted due to constraints of space. Consequently, some of the vertices of the squares may have moved off the dots. However the initial condition of the investigation was preserved, i.e., the vertices of the squares are on the dots.

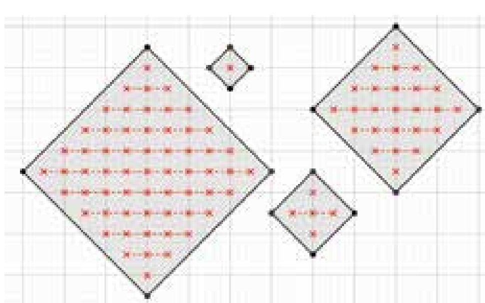

Figure 3.1

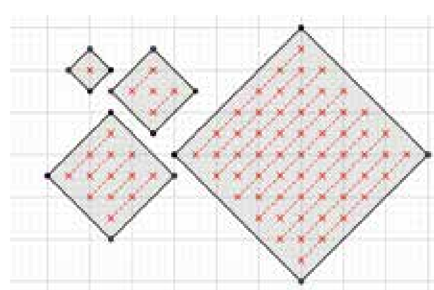

Figure 3.2

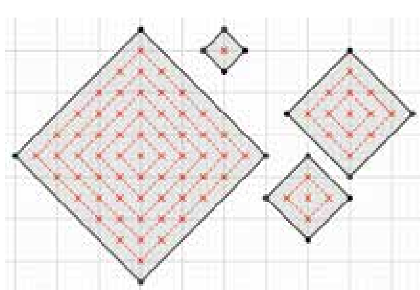

Figure 3.3

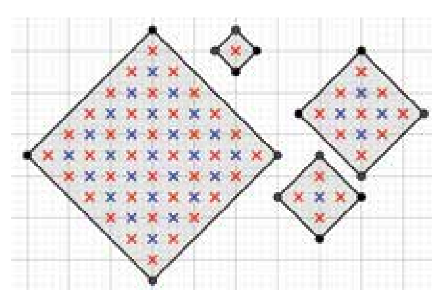

Figure 3.4

Side length	Formula 1 for adding dots	Formula 2 for adding dots	Formula 3 for adding dots	No of dots inside
$\sqrt{2}$	1	1	1	1
$2\sqrt{2}$	$1 + 3 + 1$	$2 + 1 + 2$	$1 + (2 \times 2)$	5
$3\sqrt{2}$	$= 1 + 3 + 5 + 3 + 1$ $= 2(1 + 3) + 5$	$3 + 2 + 3 + 2 + 3$	$1 + (2 \times 2) + (2 \times 3) + (2 \times 1)$ $= 1 + 4 + 8$	13
$6\sqrt{2}$	$= 1 + 3 + 5 + 7 + 9 + 11 + 9 + 7 + 5 + 3 + 1$ $= 2(1 + 3 + 5 + 7 + 9) + 11$	$6 + 5 + 6 + 5 + 6 + 5 + 6 + 5 + 6 + 5 + 6$	$= 1 + (2 \times 2) + (2 \times 3 + 2 \times 1) + (2 \times 4 + 2 \times 2) + (2 \times 5 + 2 \times 3) + (2 \times 6 + 2 \times 4)$ $= 1 + 4 + 8 + 12 + 16 + 20$	61
$n\sqrt{2}$	$= 2(1 + 3 + 5 + \ldots.. (2n - 3)) + 2n - 1$ $= 2(n - 1)2 + 2n - 1$ $= 2n^2 - 2n + 1$	$(n) \times (n) + (n - 1)(n - 1)$ $= 2n^2 - 2n + 1$	$= 1 + (2 \times 2) + (2 \times 3 + 2 \times 1) + \ldots\ldots.. (2n + 2(n - 2))$ $= 1 + 4 + 8 + \ldots.. + 4(n - 1)$ $= 1 + 2(n - 1)(n)$ $= 2n^2 - 2n + 1$	$2n^2 - 2n + 1$ $= n^2 + (n - 1)^2$

Table 3

Similarly, other patterns for summation can emerge and so do opportunities for beautiful pictures as students begin to notice sets of patterns. See Figure 4 and Table 4.

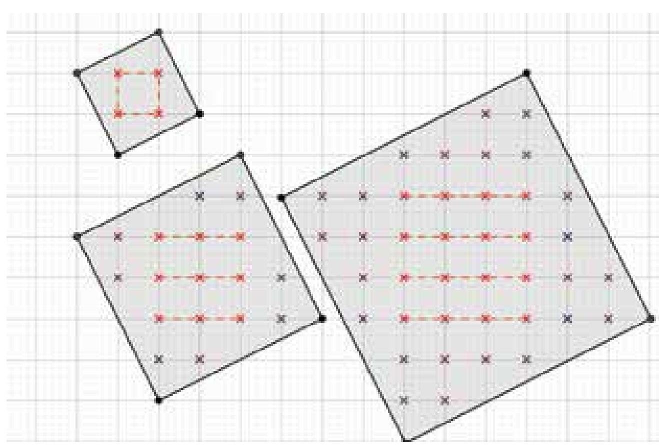

Figure 4

Side length	Formula 1 for adding dots	Formula 2 for adding dots	Formula 3 for adding dots	No of dots inside
$\sqrt{5}$	2×2	$2 + 2$		4
$2\sqrt{5}$	$3 \times 3 + 4 \times 2$	$2 + 4 + 5 + 4 + 2$		17
$3\sqrt{5}$	$= 4 \times 4 + 4 \times 4 + 4 \times 2$	$2 + 4 + 4 \times 7 + 4 + 2$		40

Table 4

Try and find a formula 3 and general representations for each formula.

Mathematisation has never been so much fun and we hope that you have enjoyed the possible avenues of exploration that we have opened up.

Another way to approach this exploration is to take the original question in the Thinking Skills PullOut (http://teachersofindia.org/en/ebook/thinking-skills-pullout), look at the number of dots enclosed by the squares and ask if it is possible to enclose any number of dots with a square.

Clearly 1, 4, 9, 16…. n^2 can be enclosed with squares.

The 45° tilted ones include 1, 5, 13, 25... dots – again a very clear pattern of $n^2 + (n - 1)^2$.

Up to 20, the numbers for which it was possible is either $4k$ or $4k + 1$. This can be explained to an extent if you revisit the squares in Figures 1.1, 1.2, 3.1-3.4 and 4. The squares fall in two groups: (i) with a dot at the centre e.g. the squares with 45° tilt and (ii) with no dot at the centre e.g. the biggest square in Figure 4. If you leave the dot at the centre, when it is there, you can split the remaining dots equally in four quadrants. [See Figure 6]. (Clearly, this is related to the four-fold rotational symmetry of the square about its central point.) So if each quadrant has k dots, then squares in (i) have $4k + 1$ dots in them while those in (ii) have $4k$ dots inside.

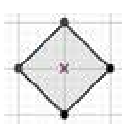

Figure 6.1

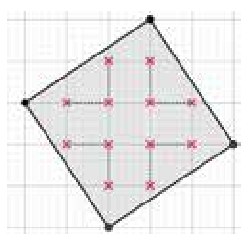

Figure 6.2

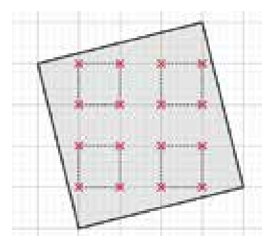

Figure 6.3

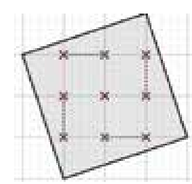

Figure 6.4

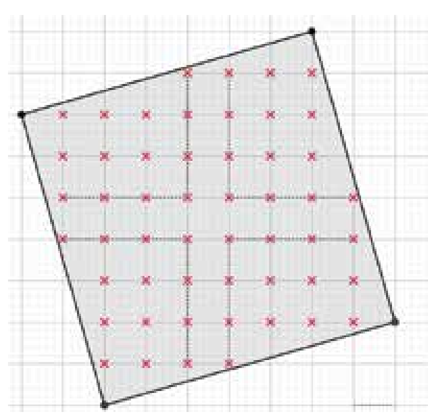

Figure 6.5

Is it possible to 'square' $4k$ or $4k + 1$ dots for each whole number k? Not sure... 20, 21, i.e., $k = 5$ appear to be impossible. Maybe you will find a square that has exactly 20 or 21 dots inside it!

We hope that we have left you with food for thought!

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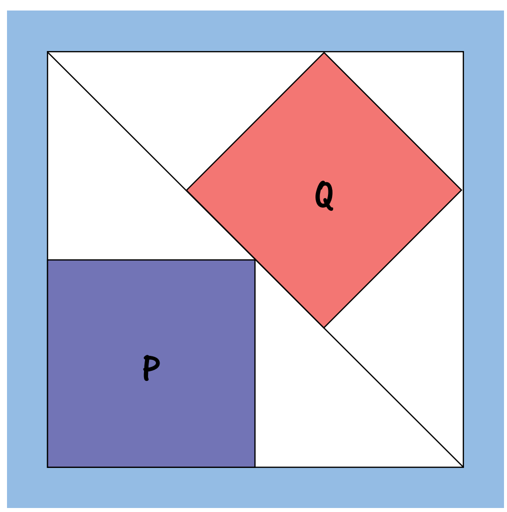
Solution to the puzzle on page 78 of the November 2018 issue

Find the ratio of the areas P : Q



Let side of P be x. Then area = x^2 *and*

length of the diagonal of the bigger swaure = $2\sqrt{2}\,x$.

Now side of Q = $\frac{1}{3} \times 2\sqrt{2}\,x$.

$\therefore$ *Area of Q* = $\frac{1}{9} \times 8x^2$.

Ratio of areas of P and Q = $1 : \frac{8}{9} = 9 : 8$

Correct solution sent in by: **Vincent D K**

Correct solutions were also sent in by A. Kumaran and Shri Krishna Anand

Addendum to Theorem concerning A MAGIC TRIANGLE

SHAILESH SHIRALI

In the July 2018 issue of *At Right Angles*, the topic of magic triangles was explored, a 'magic triangle' being "an arrangement of the integers from 1 to n on the sides of a triangle with the same number of integers on each side so that the sum of integers on each side is a constant, the 'magic sum' of the triangle." [1] The number of integers on each side is the 'order' of the magic triangle; it is equal to $(n + 3)/3 = n/3 + 1$.

The following result was stated in the article: *The vertex numbers of a fourth-order magic triangle, when arranged in order, form an arithmetic progression*. This is illustrated by the magic triangle in Figure 1, where the vertex numbers are 1,2,3.

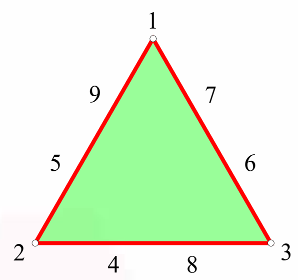
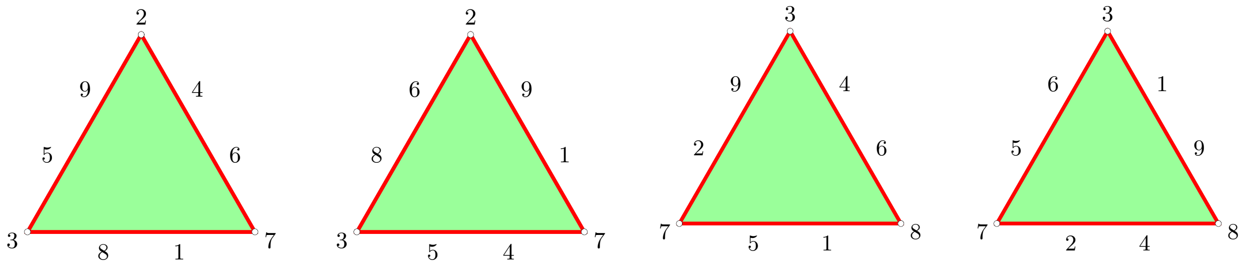


Figure 1. Fourth-order magic triangle with magic sum 17

A few weeks back, this author received an email from Mr James Metz of Hawaii, pointing out that this result is in error. (He also offered more observations on fourth order magic triangles. See his article later in this issue). As proof, he offered a few counterexamples (Figure 2):

Keywords: Magic triangle, magic sum, arithmetic progression



(a) Magic sum 19 (b) Magic sum 19 (c) Magic sum 21 (d) Magic sum 21

Figure 2. Counterexamples to the stated claim

A remarkable state of affairs: we 'proved' the result, yet here we find four different counterexamples to the claim! This challenges us to find out where we went wrong in the supposed proof. This article concerns itself with tracking down the error.

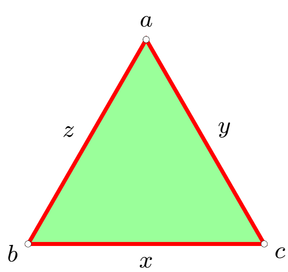
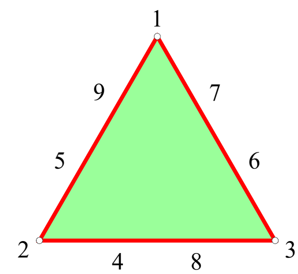


Figure 3. General relationships for fourth-order magic triangles (n = 9)

Proof. Let us recall our 'proof'. We denoted by a, b, c the numbers at the vertices (Figure 3); by x, y, z the sums of the other two numbers on the three edges respectively (x on edge bc; y on edge ca; z on edge ab); and by s the magic sum of the triangle. We deduced that $a + b + c = 3s - 45$ and noted that this means that the sum of the vertex numbers is a multiple of 3. Next we showed that $17 \le s \le 23$. (This follows from $a + b + c \ge 1 + 2 + 3 = 6$ and $a + b + c \le 9 + 8 + 7 = 24$.) We then looked at each possible value of s in turn. We go over these arguments in brief.

s = 17: This possibility implies that $a + b + c = 6$ and takes place if and only if $\{a, b, c\} = \{1, 2, 3\}$. In this case the vertex numbers form an AP, as required. Figure 4 displays one of the magic triangles corresponding to this situation.



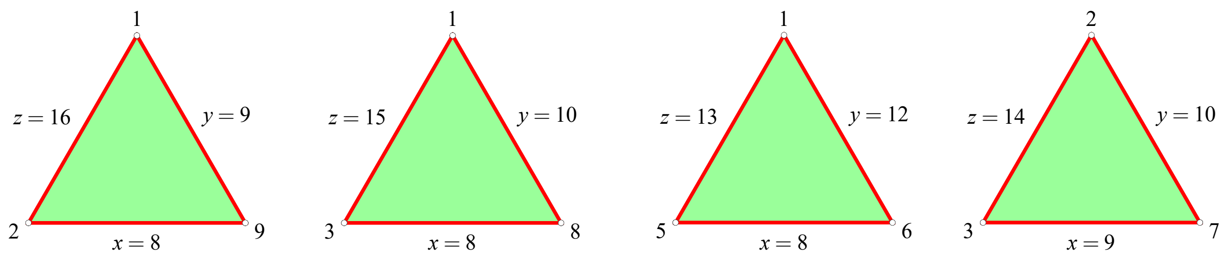
Note that x, y, z are sums of *pairs* of numbers.

Figure 4. Fourth-order magic triangle with magic sum 17

s = 18: This possibility cannot occur. For, if $s = 18$, then $a + b + c = 9$. The sets of three distinct integers between 1 and 9 (inclusive) whose sum is 9 are {1,2,6}, {1,3,5} and {2,3,4}. Consider the first possibility. By focusing on the possible position of 9, we discover that the magic triangle cannot be completed; in each case, some number is required in two different locations, i.e., two copies of that number are required. Hence there is no fourth order magic triangle with vertex numbers 1,2,6 and magic sum 18. Noting the role played by 9, we call it a *witness* to the impossibility of this configuration.

The other possibilities listed also do not work; once again, 9 acts as a witness to show their impossibility. Hence if $s = 18$, the statement that the vertex numbers form an AP is vacuously true.

s = 19: This implies that $a + b + c = 12$. The sets of three distinct integers between 1 and 9 (inclusive) whose sum is 12 are {1, 2, 9}, {1, 3, 8}, {1, 4, 7}, {1, 5, 6}, {2,3,7}, {2, 4, 6} and {3, 4, 5}. The sets that need examination are {1, 2, 9}, {1, 3, 8}, {1, 5, 6} and {2, 3, 7} (in the remaining three cases, the vertex numbers already



(a) Witness: 7 (b) Witness: 7 (c) Witness: 7 (d) Witness: 7 ???

Figure 5. Analysis of fourth-order magic triangles with magic sum 19

form an AP). The first three cases are studied in Figure 5 (a), 5 (b) and 5 (c). As earlier, in each case we need a witness that plays the role earlier played by 9. The relevant witnesses are listed alongside the captions. (Please check that they fulfill their duties faithfully.)

What about the fourth case, depicted in Figure 5 (d)? We had claimed earlier that 7 is again a witness, and we left the missing steps in the argument to be filled in by the reader. *But this is where we went wrong; 7 does not work out as a witness.* Indeed, no witness can be found for this configuration! And as the counter example provided by James Metz shows, there really is a magic triangle with vertex numbers 2, 3, 7 and magic sum 19 (Figures 2 (a) and 2 (b)).

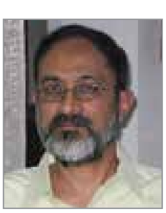
It follows that if $s = 19$, the claim that the vertex numbers form an AP is not true. We have found the error in our reasoning.

By virtue of the other result proved in the original article (that proof is valid, there is nothing wrong with it!), where we found a mapping between magic triangles with magic sum s and magic sum $40 - s$, we infer that the claim that the vertex numbers form an AP will be false for the case $s = 21$ as well. The counterexamples found by James Metz are consistent with this statement.

Acknowledgement. I thank James Metz most sincerely for writing to me and pointing out this error. It is always a humbling experience to an author when an error is spotted. It shows the extreme need for accuracy in one's reasoning and one's writing. It also shows the extreme need for care in not passing off to the reader the task of checking an argument!

References

[1] Wikipedia. "Magic triangle (mathematics)." https://en.wikipedia.org/wiki/Magic_triangle_(mathematics)



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Student Corner – Featuring articles written by students.

Two Striking Number Patterns

ADITHYA RAJESH

Sums of squares of the natural numbers from the Pascal triangle

The array below shows the first 12 rows of the Pascal triangle. (*Editor's note*. The triangle has been typeset in a left justified manner, different from the usual depiction.)

1											
1	1										
1	2	1									
1	3	3	**1**								
1	4	6	**4**	1							
1	5	10	**10**	5	1						
1	6	15	**20**	15	6	1					
1	7	21	**35**	35	21	7	1				
1	8	28	**56**	70	56	28	8	1			
1	9	36	**84**	126	126	84	36	9	1		
1	10	45	**120**	210	252	210	120	45	10	1	
1	11	55	**165**	330	462	462	330	165	55	11	1

Here are the entries of the *fourth column* of this triangular array (with a 0 included in the front), written in row form:

$$0,\ 1,\ 4,\ 10,\ 20,\ 35,\ 56,\ 84,\ 120,\ 165,\ 220,\ 286,\ 364,\ 455,\ 560,\ 680,\ \ldots.$$

Keywords: Number pattern, sum of squares, triangular number, cube

We add *pairs of consecutive members* of this sequence and get these numbers:

$$1,\ 5,\ 14,\ 30,\ 55,\ 91,\ 140,\ 204,\ 285,\ 385,\ 506,\ 650,\ 819,\ 1015,\ 1240,\ \ldots.$$

We have obtained the sums of squares of the consecutive natural numbers:

$$\begin{aligned}
1 &= 1^2,\\
5 &= 1^2+2^2,\\
14 &= 1^2+2^2+3^2,\\
30 &= 1^2+2^2+3^2+4^2,\\
55 &= 1^2+2^2+3^2+4^2+5^2,\\
91 &= 1^2+2^2+3^2+4^2+5^2+6^2,\\
140 &= 1^2+2^2+3^2+4^2+5^2+6^2+7^2,\\
204 &= 1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2,\\
285 &= 1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2,\\
385 &= 1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2,
\end{aligned}$$

and so on.

Relationship between the cubes and the triangular numbers

Starting with the sequence of cubes (1, 8, 27, 64, 125, ...), we compute the differences between pairs of consecutive cubes, then the differences between consecutive numbers of that sequence, and so on. Here is what we get:

1		8		27		64		125		216		...
	7		19		37		61		91		...	
		12		18		24		30		...		
			6		6		6		...			

We see that in the third row (12, 18, 24, ...), the numbers are consecutive multiples of 6, starting with 12. This means that the numbers in the second row (7, 19, 37, 61, ...) all leave the same remainder under division by 6. That common remainder is 1. Subtracting 1 from all the numbers in this row, we get these numbers which are all multiples of 6:

$$6,\ 18,\ 36,\ 60,\ 90,\ \ldots.$$

Dividing by 6, we get these numbers:

$$1,\ 3,\ 6,\ 10,\ 15,\ \ldots.$$

These are the triangular numbers.

Therefore, if T_n denotes the n-th triangular number, we have discovered the following relationship:

$$\boxed{6T_n + 1 = (n+1)^3 - n^3.}$$

Editor's note: Sums of squares of natural numbers from the Pascal triangle

We provide an explanation behind the observed relationship. Note that the numbers $1, 4, 10, 20, 35, 56, \ldots$ are the partial sums of the sequence of triangular numbers $1, 3, 6, 10, 15, \ldots$:

$$\begin{aligned} 1 &= 1, \\ 4 &= 1 + 3, \\ 10 &= 1 + 3 + 6, \\ 20 &= 1 + 3 + 6 + 10, \\ 35 &= 1 + 3 + 6 + 10 + 15, \quad \text{and so on.} \end{aligned}$$

So if a_n denotes the n-th number of the sequence $1, 4, 10, 20, 35, \ldots$, then we have

$$a_n = T_1 + T_2 + T_3 + \cdots + T_n.$$

By adding consecutive members of the a-sequence, we obtain

$$a_{n-1} + a_n = (T_1 + T_2 + T_3 + \cdots + T_{n-1}) + (T_1 + T_2 + T_3 + \cdots + T_{n-1} + T_n).$$

This relationship may be written as follows:

$$a_{n-1} + a_n = T_1 + (T_1 + T_2) + (T_2 + T_3) + \cdots + (T_{n-1} + T_n).$$

The following relationships are well-known:

$$T_1 = 1^2, \quad T_1 + T_2 = 2^2, \quad T_2 + T_3 = 3^2, \quad \ldots, \quad T_{n-1} + T_n = n^2.$$

Hence we have the identity:

$$a_{n-1} + a_n = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

This explains the observation made by Adithya.

Box 1

Editor's note: Relationship between cubes and triangular numbers

The triangular numbers are generated by the following formula:

$$T_n = \frac{n(n+1)}{2},$$

so $6T_n + 1 = 3n(n+1) + 1$. This means that we must check whether the following is an identity:

$$3n(n+1) + 1 = (n+1)^3 - n^3.$$

It is an easy exercise to check that this is an identity; both sides simplify to the expression $3n^2 + 3n + 1$. Hence proved.

Box 2

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On Adding Magic Triangles

JAMES METZ

Magic triangles can be added to each other term by term, the same way that magic squares can be added to each other. We show here how two third order magic triangles can yield another third order magic triangle through addition.

Example 1. See Figure 1.

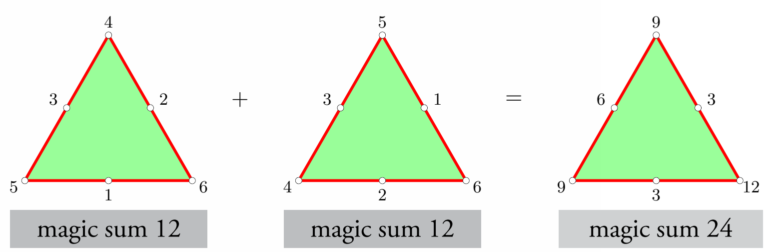

Figure 1. Addition of two third-order magic triangles – I

Note that the resulting magic triangle is non-standard; it does not use all the numbers from 1 till n for some positive integer n. Two of its sides are identical, so we may call it an 'isosceles magic triangle.' Also, the result uses only the numbers 3, 6, 9, and 12; we may also call it a 'hybrid magic triangle.'

The above example may be generalised through the use of an AP (arithmetic progression); see Figure 2. The sequence involved here is

$$2a + d,\ 2a + 4d,\ 2a + 7d,\ 2a + 10d.$$

Keywords: Magic triangle, arithmetic progression, sequence

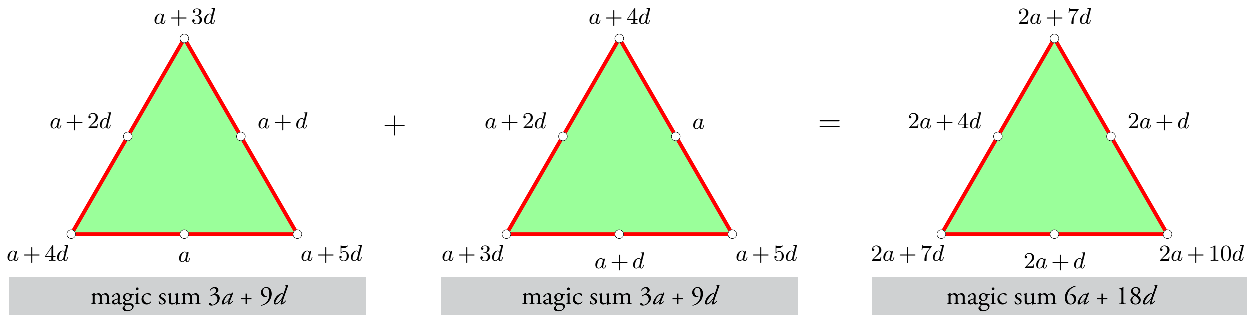

Figure 2. Addition of two third-order magic triangles – I (generalised)

Example 2. Another example is shown in Figures 3 and 4.

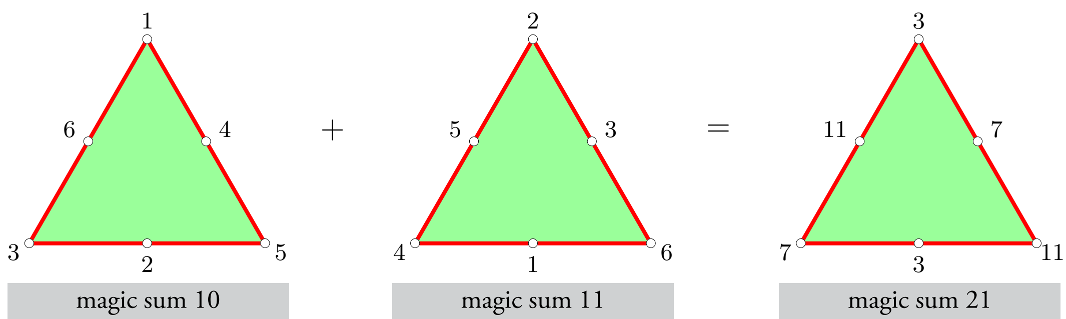

Figure 3. Addition of two third-order magic triangles – II

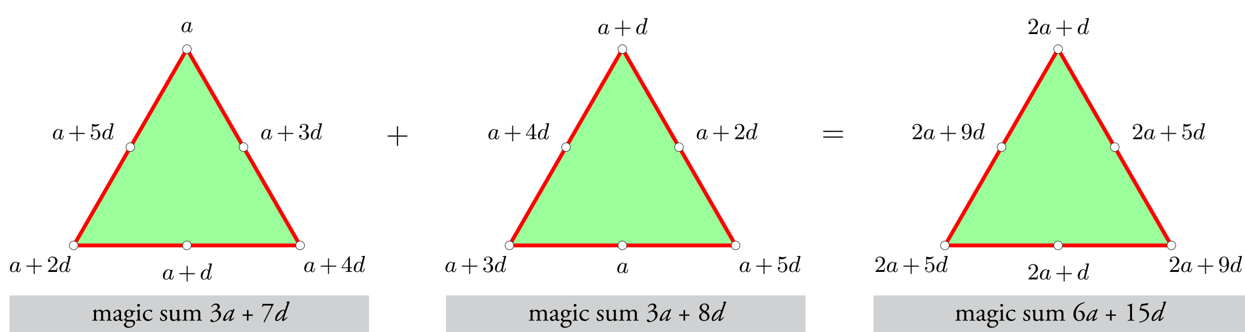

Figure 4. Addition of two third-order magic triangles – II (generalised)

Here the sequence involved is

$$2a + d,\ 2a + 5d,\ 2a + 9d.$$

JAMES METZ is a retired mathematics instructor. He volunteers for Teachers Across Borders Southern Africa for one month each year and enjoys doing mathematics. He may be contacted at metz@hawaii.edu.

PPT EXPLORATIONS

SHAILESH SHIRALI

A classroom observation. A colleague[a] shared an observation that had come up during an exploratory class he was taking at the middle school level. The topic being discussed was ***Pythagorean Triples***, i.e., triples (a, b, c) of positive integers satisfying the relation $a^2 + b^2 = c^2$. If the three integers also happen to be relatively prime to each other, i.e., share no common factors exceeding 1, then the triple is called a 'Primitive Pythagorean Triple' (PPT for short). One of the students pointed out that the most well-known such PPT, namely $(3, 4, 5)$, has the following property:

$$3 + \left(\frac{1}{2} \times 4\right) = 5.$$

Provoked by this observation, we naturally wondered about the existence of Pythagorean triples (a, b, c) which satisfy the relation

$$a + \frac{b}{2} = c.$$

It turns out that the Pythagorean triples that satisfy this relation are all multiples of $(3, 4, 5)$, i.e., $(3, 4, 5)$, $(6, 8, 10)$, $(9, 12, 15)$, ... This means that the *only* PPT which satisfies the stated relation is $(3, 4, 5)$ itself.

It is easy to prove this statement. Suppose that a, b, c are positive integers such that

$$a^2 + b^2 = c^2,$$

$$a + \frac{b}{2} = c.$$

[a]Vinayak Sharma, fellow mathematics teacher in Sahyadri School KFI. Thanks, Vinayak!

Keywords: PPT, coprime, relation, parametrisation

Substituting for c from the second relation in the first one, we get:

$$a^2 + b^2 = a^2 + ab + \frac{b^2}{4},$$

$$\therefore\ 3b^2 = 4ab, \quad \therefore\ 3b = 4a,$$

$$\therefore\ a : b = 3 : 4,$$

implying that $a : b : c = 3 : 4 : 5$.

If a, b, c are to be coprime, then it must be that $(a, b, c) = (3, 4, 5)$. Hence $(3, 4, 5)$ is the only such PPT.

Extending the result. It is easy to extend the result. Let t be any given positive rational number. Suppose that the Pythagorean triple (a, b, c) satisfies the following relation:

$$c = a + tb.$$

We now pose the following question: *What are all the Pythagorean triples for which this relation holds? In particular, what are all the PPTs for which this relation holds?*

The analysis proceeds along the same lines as earlier. Suppose that a, b, c are positive integers such that

$$a^2 + b^2 = c^2,$$

$$a + tb = c.$$

We note in passing that $t > 0$ (because $c > a$) and also that $t < 1$ (because $c < a + b$; this is simply the triangle inequality). For the case considered above, we had $t = 1/2$.

Substituting for c from the second relation in the first one, we get:

$$a^2 + b^2 = a^2 + 2tab + t^2 b^2,$$

$$\therefore\ \left(1 - t^2\right) b^2 = 2tab, \quad \therefore\ \left(1 - t^2\right) b = 2ta,$$

$$\therefore\ a : b = 1 - t^2 : 2t,$$

implying that $a : b : c = 1 - t^2 : 2t : 1 + t^2$.

Now let $t = m/n$ where m and n are coprime integers, $0 < m < n$. Substituting this in the above finding, we get:

$$a : b : c = \frac{n^2 - m^2}{n^2} : \frac{2m}{n} : \frac{n^2 + m^2}{n^2},$$

i.e.,

$$a : b : c = n^2 - m^2 : 2mn : n^2 + m^2.$$

Since we want a, b, c to be relatively prime to each other, we deduce the following:

- If m and n have opposite parity (i.e., one of them is even and the other is odd), the greatest common divisor of the quantities $n^2 - m^2$, $2mn$, $n^2 + m^2$ is 1 (since m and n are coprime), hence

$$a = n^2 - m^2, \quad b = 2mn, \quad c = n^2 + m^2.$$

- If m and n have the same parity (this means that m and n are both odd, as we have already stated that they must be coprime), the greatest common divisor of the quantities $n^2 - m^2$, $2mn$, $n^2 + m^2$ is 2, hence

$$a = \frac{n^2 - m^2}{2}, \quad b = mn, \quad c = \frac{n^2 + m^2}{2}.$$

It is interesting that we have obtained the well-known parametrisation of PPTs in a way which is very different from the usual. All this from the simple observation that the entries of the PPT $(3, 4, 5)$ satisfy the relation $3 + (\frac{1}{2} \times 4) = 5 \ldots$.

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QUILTING

Explorations by a Mathematics Teacher

SNEHA TITUS

What is quilting?

Quilting is the process of sewing two or more layers of fabric together to make a thicker padded material, usually to create a quilt or quilted garment. Typically, quilting is done with three layers: the top fabric or quilt top, batting or insulating material (middle layer) and backing material (inner layer), but many different styles are adopted.[1] Refer to [1] for an explanation of different quilting methods.

I must confess that it was the top layer that attracted me to quilting. This was clearly a case of the whole being greater than the sum of its parts- I was wonderstruck at how scraps of material could be pieced together to make beautiful patterns that were all at once eye-catching and pleasing.

Figure 1

Figure 2

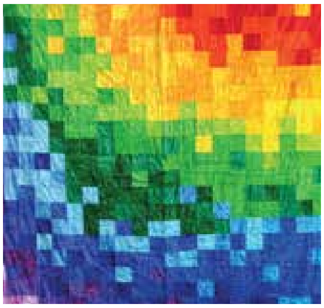

Figure 3

1 https://en.wikipedia.org/wiki/Quilting

Keywords: Craft, quilting, calculation, scale, ratio, transformation, rotation, reflection, quadrilaterals.

In all the quilts shown in Figures 1-3, the patches are square or rectangular. However the choice of patches leads to varied patterns, each attractive in its own way![2]

Quilting has a checkered history and many people, particularly women, have brought their own creativity into designing quilts. In this article, I will describe only my adventures in quilting and how I began to realise that my math teacher hat was beginning to make its presence felt as I explored the craft further. In this article, I will describe three quilts stitched by me, and elucidate how the process of designing the quilts gave me an opportunity to explore specific mathematical concepts and exercise certain basic yet important mathematical skills.

Symmetry and Transformations

Figure 4 shows my first quilt! It was only after I painstakingly hand-sewed the three layers together that I realised why a quilt was also called a comforter! Cosy has taken on a whole new meaning in my house! The top layer continues to intrigue me but I realised the value of the middle layer – the batting- and the lining. And also the cushioning provided by the painstaking hand-stitching which a machine made quilt would never possess.

Figure 4

The process of making this quilt alerted me to the permutations possible when different patches are aligned differently. Though the basic patch was only made of rectangles and squares, numerous possibilities surfaced when these patches were connected in strips and the strips were laid side by side. The impact was really mind-boggling and the designs were indeed a feast for the eyes.

Figure 5 shows the basic patch comprising a white square surrounded by a black L shaped portion and a red strip.

Figure 5: The basic patch

Figure 6 shows a quilt design quite different from the original quilt. This is obtained by rotating the basic patch counterclockwise by 90 degrees, 180 degrees and 270 degrees respectively and creating a larger patch (comprising of 4 copies of the basic patch). The four white squares come together making a larger inner white square and an outer black square with a red surround. This bigger patch is then reflected leading to a new quilt design.

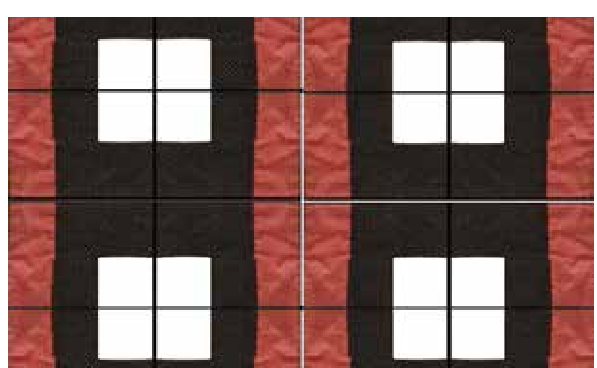

Figure 6: Reflection of the basic patch resulting in a different quilt.

[2] https://www.google.co.in/search?q=images+of+quilts&rlz=1C1RUCY_enIN689IN689&tbm=isch&tbo=u&source=univ&sa=X&ved=2ahUKEwjQ-4Xzx9_eAhUEbo8KHbNOC-8Q7Al6BAgEEB0&biw=1364&bih=617

Measurement, Perimeter, Scaling

My next adventure into quilting was creating a larger version of the one shown in Figure 7. It turned out to be a wonderful exercise in scaling.[3]

Figure 7

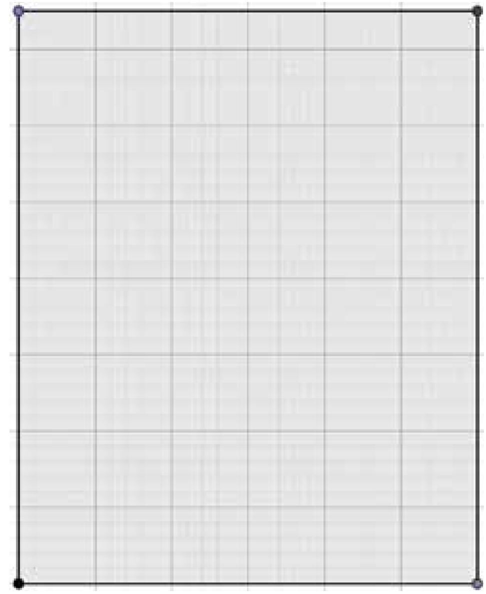

Figure 8

Now that I had some experience, I went about planning this quilt in a very mathematical fashion. My first step was to transfer this sketch on to GeoGebra. This was for my niece's son and I wanted him to use it for a while; so I planned to make it a 72" × 90" quilt. GeoGebra helped me plan the size of the patches nicely! I first outlined the rectangle on the grid (Figure 8).

I needed each small 5 × 5 grid to be 2.4" × 2.4", so that the 30 × 37.5 rectangle shown would represent my 72" × 90" quilt.

I quickly realised that I was getting into very complicated calculations! So I started working with a 48 × 60 grid from the outside-in. The outer yellow border would be 6" all around, the inner blue border would be 3". That meant that the inner square would have to be 54" by 72". I made each 5 × 5 grid 1.5" × 1.5". The joy of using GeoGebra was that my sketch didn't change but my calculations immediately became easier. I was able to move the boundaries of the inner patches to get a pleasing proportion. This was the result. (Figure 9.)

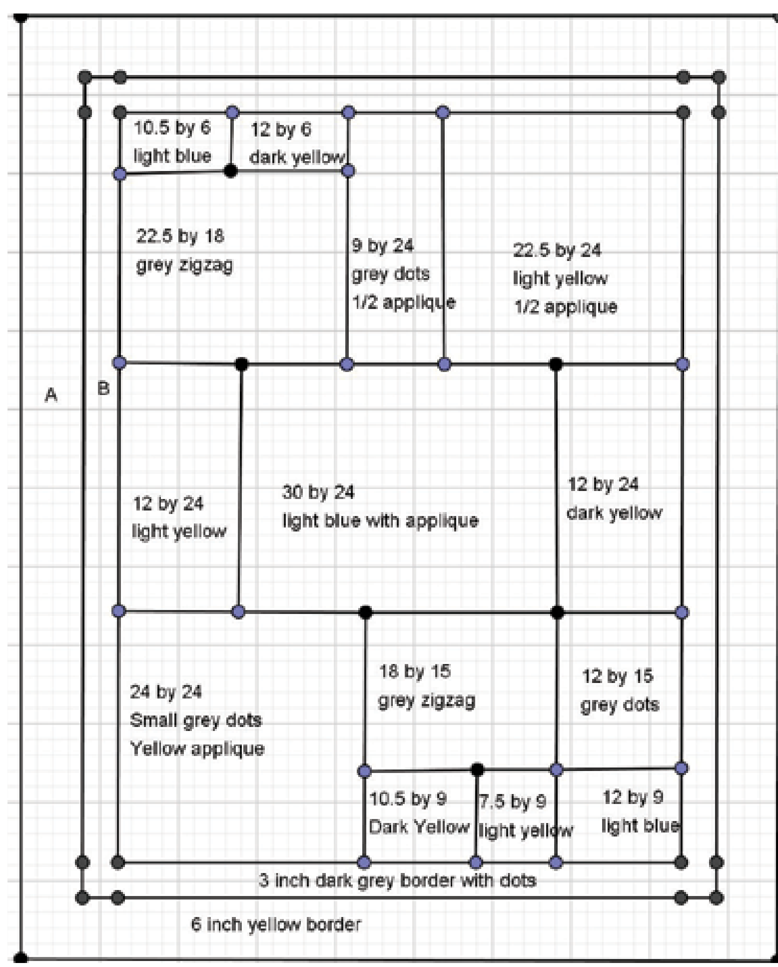

Figure 9: Making the mock-up of the quilt with GeoGebra.

[3]https://in.pinterest.com/pin/373798837804286079/

One problem remained- I usually made each small patch out of 3" squares. So I changed a few of the patches around, for example 10.5”× 9” became 12”× 9” and correspondingly 7.5” × 9” became 6” × 9”. Finally I was ready to start! And I knew that if I ever wanted to resize to a bigger quilt, I could use the same pattern with a different grid size – this was a keeper! I also realised that creating a sketch of a quilt was a good project for a student who needed to practise measurement and scaling and differentiate between area and perimeter.

Quadrilaterals and Triangles

By far, the most interesting quilt that I made was my second one- a wedding present for my daughter. She had selected this pattern from Pinterest[4] (Figure 10).

Figure 10

It looked difficult but really beautiful and since I had a great quilting teacher (a self-confessed math phobic who nevertheless used mathematics unselfconsciously every day), I decided to give it a shot. My very first class was an eye-opener on quadrilaterals, when she taught me how to quickly churn out triangles!

We first pinned together a layer of black cloth to a layer of white cloth. Then I drew out a 4 × 4 square on the top layer. I drew all the segments shown, noticing that every square had been bisected. Then I machined along the dotted lines. Next, I cut out the outer square and then carefully cut on either side (1/4” away) of the dotted lines. When I cut out the smaller squares, 16 black and white squares (Figure 11) dropped into my lap. Considering that I needed about 200 of these, this ingenious use of the symmetries of a square was a terrific time save.

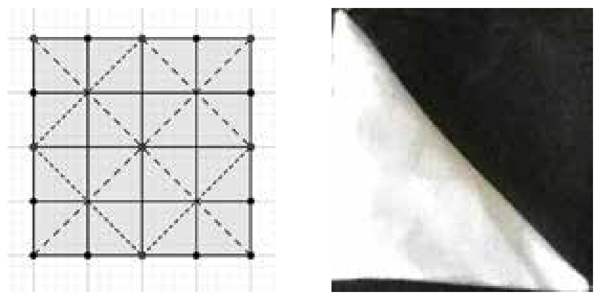

Figure 11

Next we cut out more than 200 plain black and plain white squares. These were slightly smaller than the squares in the 4× 4 grid in Figure 10 so that when cut, their size matched the square in Figure 11. Now came the fun part. Depending on how I aligned the square patches, I got a variety of shapes:

Figure 12: A right angled isosceles black triangle

Figure 13: The same, but with the colours switched

[4] https://in.pinterest.com/pin/365354588504109625/

Figure 14: A black parallelogram

Figure 15: A black isosceles trapezium (adjacent sides equal)

You can imagine the scope for permutations. I painstakingly followed the pattern and, this was the quilt that I finally made. A truly mathematical gift for a couple that met at an undergraduate math programme.

Figure 16

I am looking forward to more mathematical explorations with quilts. Someone once remarked that quilting was nothing but cutting up pieces of cloth into patches and then piecing them together to make a piece of cloth. I hope this article has convinced the reader that quilting is indeed much more!!

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Finding the Square Root of a Four-Digit Perfect Square

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

On the web page [1] belonging to *A2Y Academy For Excellence*, there appears a curious method to find the square root of a four-digit number if it is known that that number is a perfect square. We describe the method here using two examples and then consider how to explain it.

Example 1. To find the square root of 3969 (assuming that this number is a perfect square). We proceed as follows.

- Split the given number into two blocks of two digits each; here we obtain 39 and 69. Call these the 'first' number and the 'second' number, respectively.
- Consider the first number, 39. The largest perfect square less than or equal to 39 is $36 = 6^2$. Inference: the 10's digit of the square root is 6.
- Now consider the second number, 69. Since its 1's digit is 9, the 1's digit of the square root is either 3 or 7. To figure out which one, we proceed as follows.
- We compute the product $6 \times 7 = 42$ (the '6' here is the 10's digit of the square root, and 7 is the integer after 6, i.e., $7 = 6 + 1$) and compare this with the first number, 39. Since $39 < 42$, we select the *smaller* of the two choices $(3, 7)$, i.e., we choose 3. Inference: the 1's digit of the square root is 3.
- Hence the desired square root is 63.
  (Check: $63^2 = 3969$.)

Keywords: Perfect square, square root, algorithm, exploration

Example 2. To find the square root of 5776 (assuming that this number is a perfect square). We proceed as follows.

- Split the given number into two blocks of two digits each; we obtain 57 (the 'first' number) and 76 (the 'second' number).
- Consider the first number, 57. The largest perfect square less than or equal to 57 is $49 = 7^2$. Inference: the 10's digit of the square root is 7.
- Now consider the second number, 76. Since its 1's digit is 6, the 1's digit of the square root is either 4 or 6.
- We compute the product $7 \times 8 = 56$ (the '7' here is the 10's digit of the square root, and 8 is the integer after 7, i.e., $8 = 7 + 1$) and compare this with the first number, 57. Since $57 > 56$, we select the *larger* of the two choices, i.e., we choose 6. Inference: the 1's digit of the square root is 6.
- Hence the desired square root is 76. (Check: $76^2 = 5776$.)

The method described abstractly. The steps can be stated compactly as follows. The task is to find the square root of a given number N, $10^2 \leq N < 10^4$, if it is known that N is a perfect square. Let $N = 100a + b$, where $1 \leq a < 100$ and $0 \leq b < 100$. Let the desired square root be written as $10c + d$ where $1 \leq c < 10$ and $0 \leq d < 10$ (so c and d are single-digit numbers).

1. We find c from $c = \lfloor \sqrt{a} \rfloor$. (Here the symbol $\lfloor\ \rfloor$ denotes the 'floor function' defined as follows: $\lfloor z \rfloor =$ the greatest integer that does not exceed z; e.g., $\lfloor 3.7 \rfloor = 3$, $\lfloor 1.9 \rfloor = 1$.)
2. From the 1's digit of b, we deduce the possible values of the 1's digit of d:

1's digit of b	0	1	4	5	6	9
Possibilities for 1's digit of d	0	1, 9	2, 8	5	4, 6	3, 7

3. Where there is a choice of two values for the 1's digit of d, we choose as follows: if $a < c(c+1)$, choose the smaller number; if $a \geq c(c+1)$, choose the larger number.
4. With c and d found, the desired square root is $10c + d$.

Justification

Steps 1 and 2 are clearly true. Only Step 3 needs to be justified, namely: "if $a < c(c+1)$, choose the smaller number; if $a \geq c(c+1)$, choose the larger number." The justification follows from the steps listed below:

(i) If $d \leq 4$, then $a < c(c+1)$. For example, take the case $d = 4$:

$$\begin{aligned}(10c+4)^2 &= 100c^2 + 80c + 16 = 100c^2 + 100c - (20c - 16)\\ &= 100c(c+1) - (20c-16)\\ &< 100c(c+1), \quad \text{since } c \geq 1.\end{aligned}$$

Hence $N < 100c(c+1)$, implying that $a < c(c+1)$. Similar reasoning takes care of the cases $d = 3, 2, 1$.

(ii) If $d = 5$, then $a = c(c+1)$. This one is easy:

$$(10c+5)^2 = 100c^2 + 100c + 25 = 100c(c+1) + 25,$$

hence $a = c(c+1)$ and $b = 25$.

(iii) If $d \geq 6$, then $a > c(c+1)$. For example, take the case $d = 6$:

$$\begin{aligned}(10c+6)^2 &= 100c^2 + 120c + 36 = 100c^2 + 100c + (20c+36)\\ &= 100c(c+1) + (20c+36)\\ &> 100c(c+1).\end{aligned}$$

Hence $N > 100c(c+1)$, implying that $a > c(c+1)$. Similar reasoning takes care of the cases $d = 7, 8, 9$.

This justifies the stated algorithm.

A side exploration

It is interesting to ask in what cases it happens that $a = c(c+1)$. We have seen that it cannot happen if $d \leq 4$, and it does happen if $d = 5$. What if $d \geq 6$? Let's look at the different cases separately.

- If $d = 6$, then taking forward the working shown above, we see that

$$a = c(c+1) \iff 20c + 36 < 100 \iff 20c < 64 \iff c \leq 3.$$

  So if $d = 6$, then $a = c(c+1)$ provided that $c = 1, 2, 3$. It is easy to verify these claims: $16^2 = 256$, $26^2 = 676$, $36^2 = 1296$. Observe that in each case, we have $a = c(c+1)$.

- If $d = 7$, then we have:

$$(10c+7)^2 = 100c^2 + 140c + 49 = 100c(c+1) + (40c+49).$$

  Hence:

$$a = c(c+1) \iff 40c + 49 < 100 \iff 40c < 51 \iff c \leq 1.$$

  So if $d = 7$, then $a = c(c+1)$ provided that $c = 1$. It is easy to verify this claim: $17^2 = 289$. Observe that we have $a = c(c+1)$.

- If $d = 8$ or 9, then we find by working through the inequalities that there are no cases when $a = c(c+1)$.

In conclusion, we say that the equality $a = c(c+1)$ holds in precisely the following cases: $d = 5$ (i.e., all two-digit numbers whose 1's digit is 5), and the numbers 16, 17, 26 and 36.

References

1. A2Y Academy For Excellence, Finding the Square Root of a number which is a perfect square, http://a2yacademy.com/2018/04/09/finding-the-square-root-of-a-number-which-is-a-perfect-square/

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

The Truncated Icosahedron – An Iconic 3-D Shape

A. RAMACHANDRAN

A polyhedron is a 3-D shape whose boundaries or faces are planar polygons. Two faces meet along an edge while three or more faces meet at a vertex. Among polyhedra we have the five Platonic solids. All the faces of a Platonic solid are congruent regular polygons. These five were identified by ancient Greek geometers who also proved that no other polyhedron has this property (i.e., of having faces that are regular polygons of the same size and the same number of sides). One example of a Platonic solid is the icosahedron, which has 20 faces that are congruent equilateral triangles. It has 30 edges and 12 vertices (Figure 1).

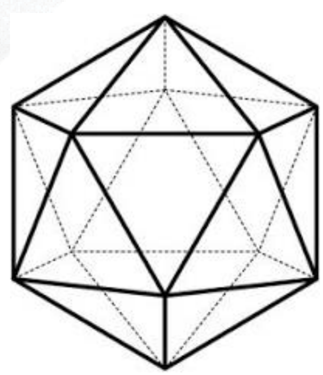


Figure 1

There is another class of semi-regular 3-D shapes - the Archimedean solids. In these the faces are again regular polygons but they are not all congruent, i.e., they differ in

Keywords: polyhedra, Archimedean solids, Euler's formula, angular defect.

the number of sides. One such Archimedean solid is the truncated icosahedron. Now look at the icosahedral shape and imagine that all of its 12 vertices are sliced off by identical and symmetrical cuts. The resulting shape is the truncated icosahedron (Figure 2).

Figure 2

As a result of this action we have the following:

A. The original triangular faces have become hexagons. At every vertex of the original shape there now appears a pentagon. So the new shape has 32 faces – 20 regular hexagons and 12 regular pentagons. No two pentagons touch each other.

B. Each vertex of the original shape is replaced by 5 new vertices, and so the new shape has 60 vertices.

C. Around every vertex of the original shape 5 new additional edges are created. That is, 60 new edges are created in addition to the existing 30, bringing the number of edges of the new shape to 90.

* * * * *

There are two well-known results that apply to all polyhedra, regular or otherwise.

(a) Euler's relation. This states that the sum of the number of faces and the number of vertices of a polyhedron is 2 more than the number of edges. Clearly this holds for the truncated icosahedron ($60 + 32 = 90 + 2$).

(b) The sum of the angles at the vertices of the polygons that meet at any vertex of the polyhedron is necessarily less than 360°, else it would just flatten out. The shortfall, i.e., the difference between 360° and the sum as mentioned above, is termed the angular defect of the said vertex. The sum of the angular defects of all the vertices of a polyhedron always equals 720°. (This statement can be considered as the 3-D analog of the assertion that the exterior angles of any polygon sum to 360°.) In the case of the truncated icosahedron two hexagons and one pentagon meet at each vertex. The sum of the angles at their vertices is $2 \times 120° + 108° = 348°$. The angular defect at one vertex is thus 12°. Since there are 60 similar vertices the total angular defect is $60 \times 12° = 720°$.

* * * * *

The truncated icosahedron is the shape of the soccer ball, one of the best known and best loved objects (Figure 3).

Figure 3

The shape was suggested by Eigil Nielsen in 1962. A standard soccer ball is made of vulcanised rubber units stitched together or thermally sealed together and inflated. Typically the pentagons are black in colour while the hexagons are white. An array of conjoined hexagons would just be planar, while an array of pentagons closes abruptly to form a dodecahedron, not a shape that would roll easily. The above combination of both pentagons and hexagons seems to achieve a balance, giving a well-rounded and pleasing shape.

* * * * *

Among the chemical elements carbon has a unique and special place. This is due to the ability of carbon atoms to form strong bonds with each other enabling the formation of large structures and complex molecules. Carbon is the basis of life itself. The branch of organic chemistry deals with the study of carbon compounds. For a long time pure carbon was known to exist in two forms – graphite and diamond. In the year 1985 a new form of carbon was observed to form in laboratory conditions that mimicked the conditions in red giant stars. Further studies revealed that in this form 60 carbon atoms are bound together in a stable structure - yes, the truncated icosahedron. The carbon atoms occupy the vertices and the C-C bonds form the edges, of which 60 are single bonds and 30 are double bonds. Each carbon atom is linked to two other carbon atoms by single bonds and to one other by a double bond, satisfying its valency of four. The sides of the pentagons are all single bonds. Double bonds radiate from the pentagons, each linking to another pentagon. There are no faces as such, as it is a hollow skeletal arrangement (Figure 4).

Figure 4

This newly discovered form of carbon was named Buckminsterfullerene, or Fullerene (informally, buckyball) for short, in honour of Buckminster Fuller of the 'Geodesic dome' fame. The structure of this form of carbon reminded the chemists of geodesic domes which are shell-like spherical or hemispherical structures made of triangular light-weight units; they are much stronger than one would expect and can be built to an indefinitely large (or small, as above) scale.

Some micro-organisms have external structures that are slight modifications of the icosahedral shape.

The truncated icosahedron is thus a truly iconic shape that links geometry, sport, chemistry and architecture.

Acknowledgments

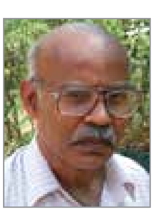
Figures 1-4 are taken from the following websites:

https://shutterstock.com

https://www.korthalsaltes.com/model.php?name_en=truncated%20icosahedron

https://pt.wikipedia.org/wiki/Ficheiro:Soccer_ball_animated.svg

https://solennebv.com/product/c60/



A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

Taking a Chance with a Graphics Calculator

BARRY KISSANE

Graphics calculators have been available to students in secondary school in some countries now for more than thirty years, although of course their capabilities have been developed in various ways to support the school curriculum over that time. The most frequent use of these devices seems to be concerned with the representation of functions, including in particular their graphical representation, which was an important component of a previous paper in this magazine (Kissane, 2016). However, the success of graphics calculators is due in no small part to their use for a much wider range of mathematical capabilities. In this article, the focus is on their potential to help students to learn about chance phenomena, which are generally addressed in schools through the study of probability.

The history of probability in secondary schools is relatively short and generally unfortunate. Unlike many other parts of the secondary school curriculum, such as algebra, geometry, trigonometry and calculus, probability has been studied in schools only recently, and was relatively rare in most countries as little as fifty years ago. One part of the reason for this is likely to be that probability is a relatively recent inclusion in mathematics itself, dating from around the sixteenth century (Hacking, 1975). Until quite recently, much of the probability work in schools has been excessively formal, with a focus on the algebra of probabilities, but with less attention paid to the nature of everyday random phenomena. Yet in recent times, probabilities have become more evident and explicit in our daily world, a good example of which is weather forecasting, now regularly accessed by many people on their smartphones.

Keywords: probability, chance, fair, forecasting, dice, coin, card, Bernoulli event, binomial distribution, technology, graphic calculator

Weather forecast for New Delhi from Weather.com

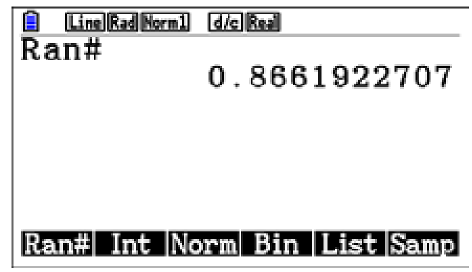
To give an example, the screenshot above is taken from a popular online weather forecasting website (https://weather.com), suggesting that the chance of rain on a certain recent day in New Delhi, India was 40%.

An important part of learning about probability is to understand what such statements mean, and a graphics calculator can be of value for this purpose. A key intention of this article is to explore some of the possibilities now available.

Random numbers on calculators

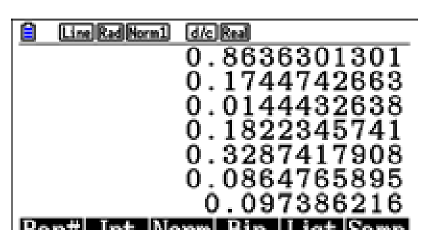
Kissane and Kemp (2014a) claimed that calculators could be of value to education beyond their obvious role to facilitate and undertake computations. Calculators could help develop understanding of mathematical concepts, could allow students to undertake personal explorations in mathematics, and offer opportunities for their hypotheses to be confirmed or to be contradicted, either of which is helpful for learning. In this article, examples of all of these will be offered, using in particular a recent graphics calculator, the CASIO fx-CG50, to illustrate these. Some of the ideas presented are elaborated in more detail in Kissane and Kemp (2014b, Module 7).

The essential ingredient of opportunities to explore chance on calculators is a random number generator, allowing a user to generate a random number between zero and one with a single key press. This capability is present on all graphics calculators and almost all recent scientific calculators. A single press of the relevant command on the CASIO fx-CG50 is shown below.

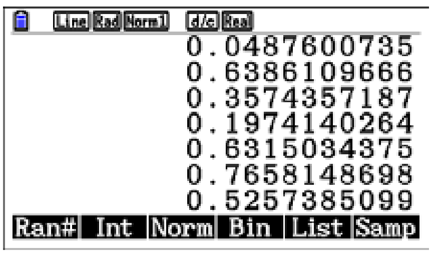


The number generated is not, of course, actually random. It is generated by the calculator, a predictable device, in the form of a *pseudo-random* number, requiring a sophisticated internal algorithm. Importantly, numbers of this kind behave in similar ways to random numbers and thus can be used to simulate and to study random phenomena. On this calculator – and on other calculators – the random numbers are generated with a uniform distribution on the open interval (0,1).

To begin to explore how random phenomena work, you can generate a succession of random numbers, or each of a group of people – such as a class – can each generate a random number and see what happens in the shorter and longer terms. Here is an example of generating seven random numbers in succession (all that will fit on a single screen of this particular calculator):



This screen allows a useful observation to be made about random phenomena: they are much easier to predict in the long run than in the short run. While in theory, for example, half the numbers generated should be larger than 0.5, and half less than 0.5, this is not expected to be evident in such a small collection of observations. In this case, only one of the seven numbers shown is larger than 0.5. Further experimentation will show that other results will occur, such as the following screen, in which four of the seven numbers exceed 0.5.

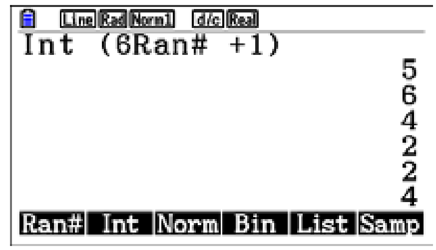


It is rare that random numbers are useful in their basic form, as a number in (0,1). So calculators often include pre-programmed ways of transforming them for various purposes. A common example involves the production of random integers, such as random integers from 1 to 6, to simulate rolling a fair six-sided die. As well as dedicated random commands (which are not used here) the same effect can be achieved with a transformation using the Integer function (Int) in order to obtain the integer part of a number. The table below summarizes this approach:

Command	Random number result
Ran#	between 0 and 1
6Ran#	between 0 and 6
6Ran# + 1	between 1 and 7
Int(6Ran# + 1)	integer from 1 to 6

This kind of transformation is so fundamental to work with simulation that it might reasonably be argued to be an essential part of any modern curriculum in probability, in fact, in an age when technology is often available. Understanding such transformations is a key pre-requisite for designing Monte Carlo simulations, which have become prominent since the age of the computer.

On the calculator, the last of the commands above is used below (six successive times) to produce a set of six simulated dice rolls for a standard die:



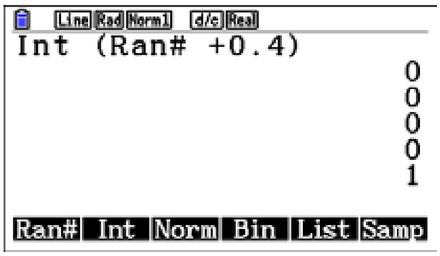
Once again, the essential unpredictability of random phenomena is shown here. It is much quicker and easier (and also quieter) for someone to generate dice rolls in this way on their calculator to study what happens than it is to use actual dice. In this case, each tap of the Execute key on the calculator generates and records another dice roll. When students study probability, and learn to compute probabilities of various results (such as a probability of 0.5 of obtaining an even number on a single roll of a fair die), their understanding is enriched by opportunities to see that this does not mean that an even number will be obtained 50% of the time, with only a few rolls. The result is instead a long-run expectation; such is the intrinsic nature of random events. Expecting long-run patterns to be evident in the short run is perhaps the most common problem people have with random events.

Understanding weather forecasts

Rather than integers, some phenomena are well modelled as Bernoulli events, for which the result is one of two possibilities, usually referred to as 'success' or 'failure'. The weather forecast shown earlier is a good example. The website predicts that it will rain on a particular day with a probability of 40%. That is, they predict that rain will be a 'success' represented by 1, 40% of the time and a 'failure' represented by 0, on the remaining 60% of the time. To understand such predictions, it is helpful to simulate a succession of days of that kind on the calculator. The appropriate command is Int(Ran# + 0.4), which will have the value of 1 (i.e. it rains) on 40%

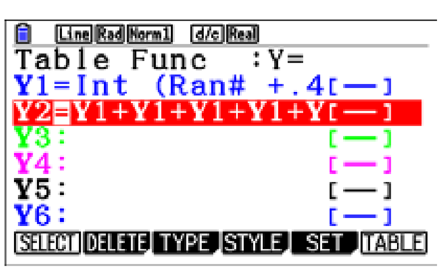
of occasions and 0 (i.e. it does not rain) on the other 60% of occasions.

Here is a simulation of five days in succession, using this command, assuming that the chance of rain on each day is 40%:

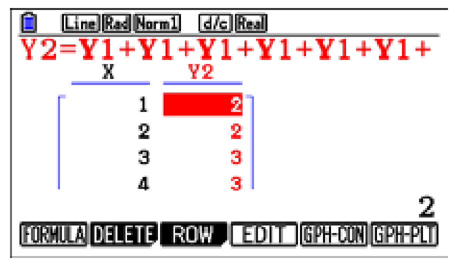


Some students – and some citizens – might regard these forecasts as defective, when there are four days in succession without rain. Instead, they need to recognize that the nature of random phenomena is such that a result of this kind is not especially unlikely: the probability of 40% applies to the long run, but not necessarily to the short run.

These kinds of capabilities, which are available for many scientific calculators, allow students to experience randomness for themselves. Because of its larger screen and other capabilities, a graphics calculator allows more substantial simulations to be undertaken, however, with a better opportunity to see what happens with a relatively large number of events. For example, the CASIO fx-CG50 allows students to generate a sequence of Bernoulli events, and then to add them to produce in effect elements from a simulated binomial distribution, which can then be studied as data. The screen below shows how a single simulation is stored in variable Y1 and then seven successive (different and independent) simulations of that kind accumulated in Y2 to show the number of rainy days in a week, when each of the days independently has a 40% chance of rain.

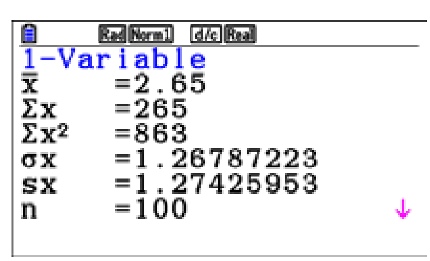


The result of such a simulation in Y2 effectively represents observations from a binomial distribution for which the probability of success is 0.4 and there are seven repetitions. While students might (and should) study the binomial distribution theoretically, there is value in first seeing its origins in this way and examining the consequences of repeated random observations of this kind. On the calculator, when used in this way, results are provided in a table, which can be scrolled easily to see the variation of results. The screen below shows one such simulation of 100 'weeks':



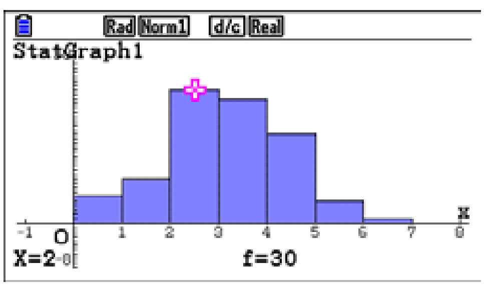
While intuition might expect a 'typical' week to have 7 x 0.4 or somewhere between two or three rainy days per week, of course in reality there are variations, evident from scrolling the table, only the first four elements of which are shown here.

It is difficult to compare tables of 100 elements, however. The capacity of this calculator to readily analyse the (finite) table as a data set and not a function table overcomes this problem. (Some other graphics calculators do not permit this alternative, because tables are not represented as finite lists.) Analyses might take any of several forms. For instance, a numerical analysis shows that, in the longer term (in this case with 100 observations), the mean number of rainy days in a week was 2.65, with a standard deviation of 1.27. At a later point, students might encounter the theoretical mean and standard deviation

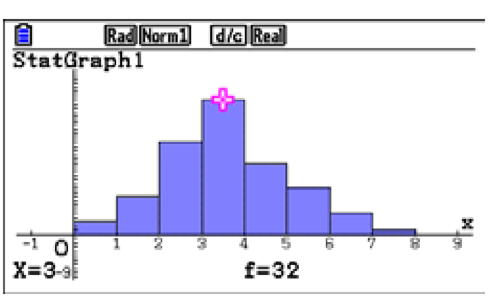


of this binomial distribution (2.8 and 1.30 respectively), but the simulation results give a sense of what might happen in practice, before such theoretical analyses are available.

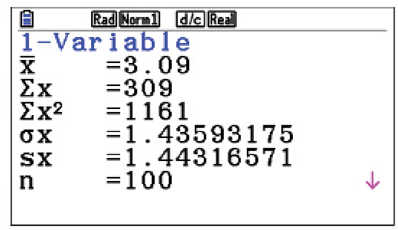
Graphical comparisons can be more evocative than numerical analyses, of course. The calculator routinely provides these as well. In this case, the histogram below gives a sense of what happened in the 100 simulated weeks. Scrolling the histogram shows that there were 30 weeks with two rainy days, slightly more than the number of weeks (28) with three rainy days. However, the graph also shows that there were six weeks with no rainy days at all, one week with six rainy days and no weeks at all for which it rained on all seven days. Such is the nature of randomness.



Importantly, each time the simulation is conducted in this way by someone, a different result is generated; so that one person can undertake experiments like this repeatedly to get a feel for the outcomes and their typical variation. In a classroom, each student will have a different table from every other student, providing a rich opportunity to see what is typical and consistent about a situation that is ultimately random. To illustrate this variation, the screen below shows the results of a second simulation of 100 weeks, conducted in the same way, and using again the same calculator settings.



As each day is simulated at random, the difference between the two simulations is entirely due to the randomness involved. The graph of the second simulation shows both similarities and differences from that of the first. This time, there are more weeks with three wet days than two wet days and there was even a week for which it rained every day. The numerical summary also shows some differences:

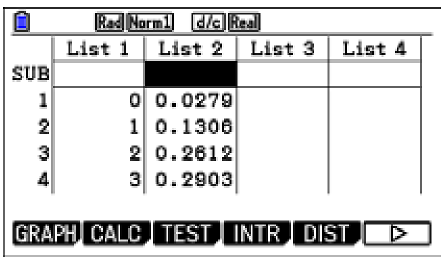


The mean number of wet days (3.09) is larger than previously, and the standard deviation (1.44) is also larger than previously. However, the overall shape of the distributions is similar – peaked in the middle with tails on each end, and with a similar slight skew. While different from each other, the numerical statistics remain close to the long-term theoretical values.

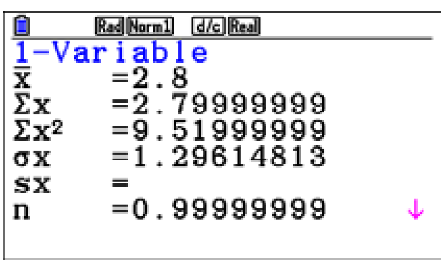
In addition, the situation can be studied with a larger number of 'weeks', in order to appreciate what happens on the longer term. In effect, the calculator is a personal experimental device.

These sorts of experiences – readily repeated on the calculator – provide opportunities to see both short-term and long-term behavior, and to appreciate the difference between the theoretical expectations and their likely practical consequences. They also offer students an opportunity to see for themselves that, even though the events simulated are random, there is a remarkable consistency of results in the longer term – much less visible in the shorter term – which is what makes the formal study of probability valuable, of course.

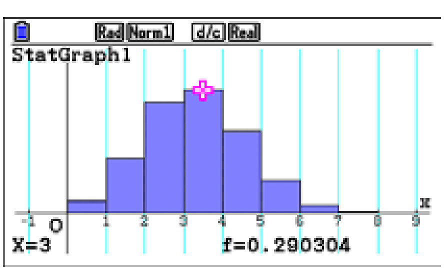
Explorations of these kinds are perhaps most appropriate before theoretical analyses are undertaken, in order to build intuitions about random phenomena, including an expectation for both short-term variation and longer-term stability. However, at a later stage, after theoretical studies have been undertaken, it is also valuable to use the calculator to generate and to show the results. This too is readily done with the CASIO fx-CG50, as shown in the next three screens. In the first screen, the first few terms of the binomial probability distribution for n = 7 and p = 0.4 are shown.



A numerical summary of this distribution is available, showing (within rounding errors) the theoretical mean of 7 x 0.4 = 2.8 and the theoretical standard deviation of 1.30.



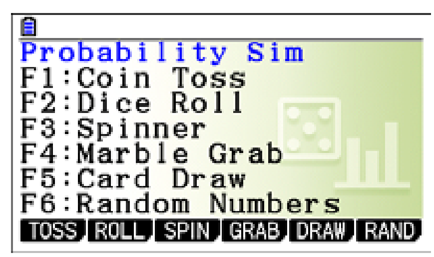
A graphical representation of this distribution shows the characteristic binomial distribution shape, with the vertical axis now showing theoretical probabilities, rather than simulated frequencies:



To understand and analyze situations which involve randomness, and to make predictions about likely outcomes, a theoretical probability model is very important. However, it is also important to build an understanding of the fact that it is a theoretical model, explaining long-term aggregated behavior, and of less practical significance for dealing with the short-term behavior in which we are often interested, such as whether or not it will rain tomorrow, or next week, once we are advised that the probability of rain on any day is 40%. Simulations on a graphics calculator are perhaps even more helpful to build this kind of understanding than are the theoretical models.

A calculator application for simulation

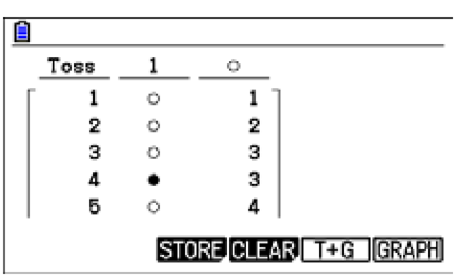
The various calculator explorations described here make use of the standard features of a graphics calculator like the CASIO fx-CG50, including the generation of random numbers, the tabulation of functions and the analysis of statistical data. However, simulations are so helpful for understanding chance phenomena that it is not surprising that the calculator also includes a separate application that is devoted to this area. The *ProbSim* application on the calculator supports various kinds of probability simulations, as suggested by the opening screen shown below:

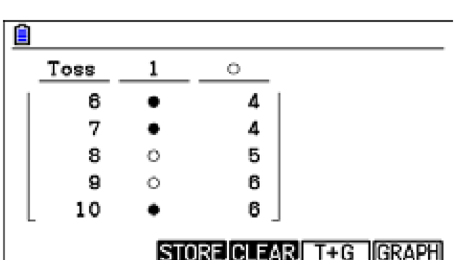


The titles of the various kinds of simulations offered in this application are reminiscent of the typical scenarios discussed in elementary probability studies … tossing coins, rolling dice, playing cards, taking marbles from urns, and so on. However, these can be used to simulate various random phenomena that are consistent with models of those kinds, as well as the actual situations described. The advantages for users of the calculator are that the various simulations are relatively easy to configure in this environment, a large number of results can be obtained fairly quickly and they can be seen in various ways.

As an example of a benefit of this application, consider the analysis of runs of random results. David Moore (1990, p.120) observed that people often intuitively underestimate the probability of runs in random sequences. So, when asked to write down a sequence of heads and tails imitating 10 successive tosses of a fair coin, he suggested that most people write a sequence with no runs of more than two consecutive heads or tails, consistent with this defective intuition. On the calculator, a set of ten successive coin tosses is readily simulated and the results are then available for scrutiny. A summary of one simulation is shown below:

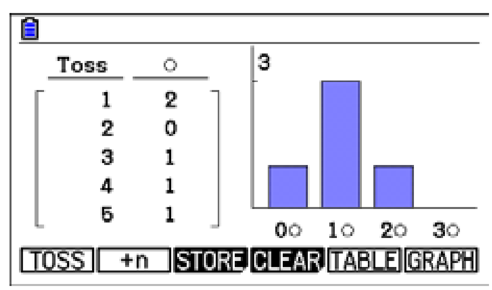
In the screen, 'tails' is represented with a black circle, while 'heads' is represented as a blank circle. Each toss has been recorded, and the screen above shows a graphical summary of the outcome, with four tails and six heads. The table shows the cumulative number of heads after various numbers of tosses. A more thorough investigation of runs is available by choosing to show results in tables, however, as the next two screens illustrate:



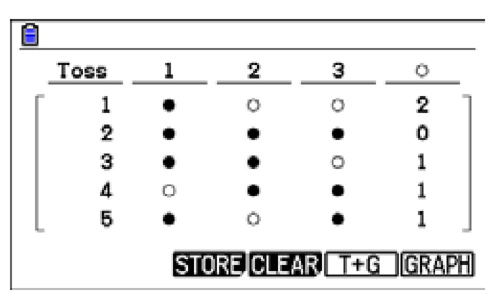


In this case, the tables show clearly that there was a run of three heads (in the first three tosses). It is easy to repeat simulations of these kinds, or for a group of students to independently conduct a simulation, to give a sense of what is 'typical'. At a later stage, such phenomena can be theoretically analysed, but that is too difficult for introductory studies in this area. As Moore notes, the probability of a run of three or more heads is a little over 0.5, so is much less unusual than people think intuitively; the probability of a run of at least three heads or at least three tails in ten coin tosses is even larger – greater than 0.8 – and thus is much more likely to happen than to not happen (1990, p.121).

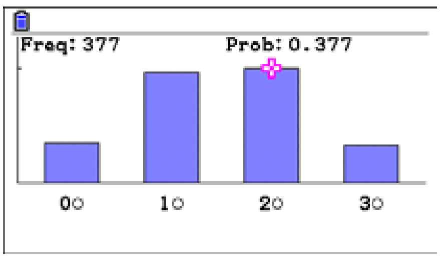
In a similar way, when families are being studied, under an assumption that a newborn baby is equally likely to be a boy or a girl, simulation is a useful tool to examine what possibilities are involved. Thus, the screen below shows a suitable simulation, with boys being represented by dark circles and girls by clear circles. Again, the initial screen shows the number of girls in each of the (five) simulated families:



Alternative screens show the same data differently, however. The screen below shows that even though three simulated families have only one girl, the birth orders in each case are different.

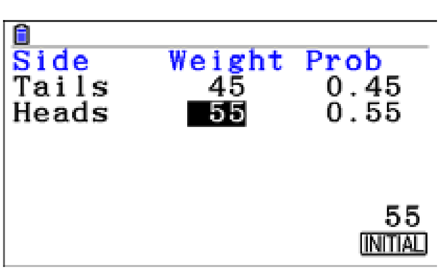


Once again, the graphics calculator is an ideal tool to explore both short-term experiments (like those above) and long-term experiments. For example, the numbers of girls in families is more clearly symmetrical and indeed consistent with theoretical expectations if a large number of families is simulated. The screen below shows one set of results after a thousand three-child families has been simulated.



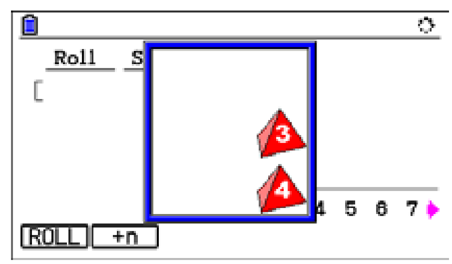
As might be expected, the proportions of families with various numbers of girls after such a large number of trials is close to 12.5%, 37.5%, 37.5% and 12.5% – the theoretical values. But also as expected, the proportions do not exactly equal the theoretical values. Both of these kinds of observations are helpful for learning about randomness.

The *ProbSim* application allows for several other kinds of simulations, as noted above. Space precludes exploring these in fine detail, but some observations about the range of possibilities are appropriate. Although it is possible to conduct simulations with everyday objects, such as actual dice, coins and spinners, it is generally more difficult to do so with processes that are not equally likely. In the case of the calculator, adjustments can be made for this purpose. An example involves tossing an unbalanced coin that lands heads 55% of the time. On the calculator, parameters can be set for this sort of purpose, as shown below:

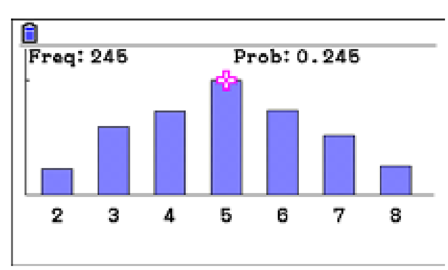


The resulting simulations show that, while a bias of this kind might not at first be clear, it becomes evident after many tosses. In the screen below, for example, the preponderance of heads is clear after the relatively small number of 400 tosses.

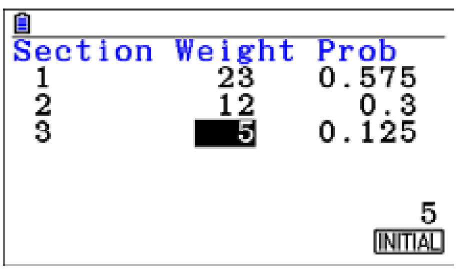
In addition, when students experiment with dice, they are generally restricted to fair six-sided dice, as these are generally the only ones available. However, in the *ProbSim* application, other alternatives are available; the screen below shows the use of a pair of fair tetrahedral (four-sided) dice, one of several choices available.



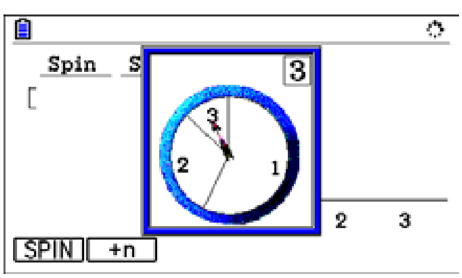
The long-term result of a simulation, after 1000 tosses of these two dice, produces a symmetric distribution, similar to that for a pair of six-sided dice that might normally be accessible. Just as the most likely total for a pair of regular six-sided dice is 7, the most likely total for a pair of four-sided dice is 5, leaving opportunities for students to explain these phenomena.



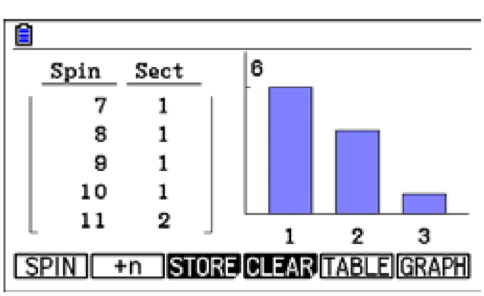
A spinner is another kind of simulation device that is sometimes available in children's games and in classrooms. However, spinners usually comprise a series of equal slices, in order to model equal likelihood. Again, on the graphics calculator application, more flexibility is involved, allowing different probabilities to be modelled easily. Thus, the settings below show a spinner appropriate to modelling the selection of students from a class in which students practice various religions: 23 Hindu, 12 Muslim and 5 Christian.



With an uneven distribution of students into different classes, a spinner that matches the distribution is needed for simulation purposes. As shown below, the calculator automatically uses such a spinner … which under normal circumstances would be hard to accomplish in practice.



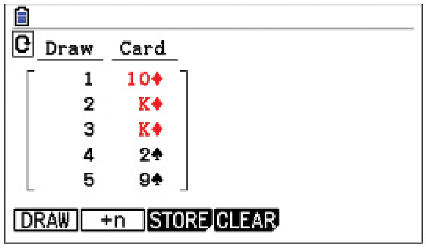
When eleven students are chosen from this class to form a cricket team, the results are generally skewed towards those in the larger groups, as might be expected. A single example is shown below:



With facilities of this kind, students can learn intuitively that random samples might be expected to be similar to their parent populations, although students will also learn from the same source that random samples can also be expected to produce divergent results as well, especially in the short-term, perhaps helping them to understand notions of 'fairness' in such situations.

Finally, the mathematics of probability was not developed in earnest until mathematicians and others became interested in games of chance, including card games. From that interest developed the much more respectable activity of insurance, without which the modern world could not have existed, according to Bernstein (1998). So it is not surprising that the *ProbSim* application also allows users to experiment with regular playing cards in various ways; although card games are often included in probability texts, they are less often included in experimental work, for practical reasons. The next screen shows a five-card poker hand drawn at random from a regular deck of 52 cards.

Again, students can draw successive hands of cards themselves, to explore how often particular events (such as a pair or a flush) happen, and can also compare their observations with those of others. Importantly for this (and other) applications, the calculator permits sampling to be done either with or without replacement, also a feature of probability theory. In addition, students can choose to have a single pack of cards (which will be exhausted after several hands, if sampling is done without replacement) or several packs of cards (as is routine practice in some professional gambling casinos). Should students elect for sampling with replacement, or for several packs of cards, there is of course a risk that the same hand might contain two cards that are identical, as the example below shows:



The availability of simulation of card games on calculators of course is not intended to encourage gambling; instead, it offers an opportunity for the mathematics of gambling to be studied, and thus for past links between games of chance and the mathematics of probability to be addressed. Indeed, it has been suggested by some organisations such as the Tasmanian Government (2019) concerned with reducing both the prevalence and the problematic impact of gambling that a better understanding of mathematics, instead of an uninformed reliance on intuition, is part of a suitable way of dealing with such problems.

Conclusion

In this paper, we have explored some of the opportunities now available to users of graphics calculators, and especially the CASIO fx-CG50 calculator, to undertake experiments related to probability. The main point is that an intuitive understanding of some features of randomness can be developed using simulations on a calculator. Such an understanding is different from – and even complementary to – the formal development of the mathematics of probability. As suggested by Kissane and Kemp (2014a), a modern graphics calculator offers learning opportunities of different kinds, in addition to computation; these have been illustrated throughout the paper. A conceptual understanding of probability as a long-term limit can be supported in various ways. Students can explore ideas for themselves and both design and execute their own experiments. There are many opportunities for students to predict what might happen in this work, some of them readily confirmed and others contradicted – both of which are helpful for learning. Finally, the basic features of graphics calculators might be used productively for this work, but calculator features designed for simulation purposes offer further power and flexibility.

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Student Corner – Featuring articles written by students.

Pythagorean Triples and Composition

BODHIDEEP JOARDAR

In this discussion I want to demonstrate that:

- If (a_1, b_1, c_1) and (a_2, b_2, c_2) are two Pythagorean triples, then they can be composed to generate the following 6 distinct triples:
    1. $[a_1a_2, (b_1c_2 + c_1b_2), (c_1c_2 + b_1b_2)]$
    2. $[b_1a_2, (a_1c_2 + c_1b_2), (c_1c_2 + a_1b_2)]$
    3. $[(a_1c_2 + c_1a_2), b_1b_2, (c_1c_2 + a_1a_2)]$
    4. $[(b_1c_2 + c_1a_2), a_1b_2, (c_1c_2 + b_1a_2)]$
    5. $[(a_1a_2 - b_1b_2), (a_1b_2 + b_1a_2), c_1c_2]$
    6. $[(a_1a_2 + b_1b_2), (b_1a_2 - a_1b_2), c_1c_2]$
- By such compositions, we can generate infinitely many triples.

We know that Pythagorean triples are infinite in number, and the most common formula for generating triples is to take two relatively prime odd numbers s and t, where $s > t \geq 1$, and produce the triple $(st, \frac{s^2-t^2}{2}, \frac{s^2+t^2}{2})$. *However, can we generate all possible triples from just one triple? Can we generate infinitely many triples from just one triple?* These might be questions worth investigating.

To do this, I take the cue from Brahmagupta's method of composition or *Bhāvanā* as applied to his famous equation *Vargaprakriti* $(x^2 - Ny^2 = 1)$. The method, as we know, can generate all possible solutions from a single solution. So, if I can show that from a known, finite set of roots satisfying the

Keywords: Pythagorean triplets, prime, odd, Vargaprakriti

Pythagorean equation $x^2 + y^2 = z^2$, it is possible to generate other sets of roots, I shall be able to say that the equation has infinitely many roots (see the following section).

Generating triples

For convenience, I begin with two Pythagorean triples (a_1, b_1, c_1) and (a_2, b_2, c_2), instead of one. After developing the formulation, when I know how things are going to develop, I can return to the case of just one triple and proceed from it to generate others.

Composition 1: As the two triples are Pythagorean, they satisfy the equations $a_1^2 + b_1^2 = c_1^2$ and $a_2^2 + b_2^2 = c_2^2$. Rearranging and multiplying the equations, I have,

$$\begin{aligned} a_1^2 a_2^2 &= (c_1^2 - b_1^2)(c_2^2 - b_2^2) = (c_1 + b_1)(c_1 - b_1)(c_2 + b_2)(c_2 - b_2) \\ &= (c_1 + b_1)(c_2 + b_2)(c_1 - b_1)(c_2 - b_2) \\ &= (c_1c_2 + c_1b_2 + b_1c_2 + b_1b_2)(c_1c_2 - c_1b_2 - b_1c_2 + b_1b_2) \\ &= (c_1c_2 + b_1b_2)^2 - (b_1c_2 + c_1b_2)^2 \end{aligned}$$

Therefore, $(a_1a_2)^2 + (b_1c_2 + c_1b_2)^2 = (c_1c_2 + b_1b_2)^2$.

That is, I have a new Pythagorean triple, $(\boldsymbol{a_1a_2, b_1c_2 + c_1b_2, c_1c_2 + b_1b_2})$.

The way I have obtained this new triple is similar to Brahmagupta's *Bhāvanā* (composition) applied to two given triples. If the triples (a_1, b_1, c_1) and (a_2, b_2, c_2) are named t_1 and t_2 respectively, and the resultant triple $(a_1a_2, b_1c_2 + c_1b_2, c_1c_2 + b_1b_2)$ is named T, then t_1 and t_2 composed individually with T will yield two more triples, one from each composition. Using $\odot$ as the symbol for composition, we may represent the situation as below:

$t_1 \odot t_2 \Rightarrow$ T(new triple); $t_1 \odot$ T $\Rightarrow$ another new triple; $t_2 \odot$ T $\Rightarrow$ yet another new triple.

This is enough to indicate that, with this process continued, infinitely many triples will be generated. In other words, Pythagorean triples are infinite in number. Among the generated triples, there may be 'recurrences' under certain conditions which may or may not exist depending on our choice of operations *(see Properties of Composition)*. However, at no point over an infinite range of compositions will the recurrent triples put an end to the never-ending process of generating new triples *(see the section Infinitely Many Triples and Infinite Recurrences)*.

Obviously, if I start with just one triple (a_1, b_1, c_1), composition can well be applied on itself by the above formula so that $(a_1, b_1, c_1) \odot (a_1, b_1, c_1) = (a_1^2, 2b_1c_1, c_1^2 + b_1^2)$, a new triple. And thus again, infinitely many triples can be generated.

Now to continue with the search for triples.

Composition 2: By interchanging the positions of a_1 and b_1 in Composition 1, that is, by composing $(b_1, a_1, c_1) \odot (a_2, b_2, c_2)$ in the same way as above, I get the triple $(\boldsymbol{b_1a_2, a_1c_2 + c_1b_2, c_1c_2 + a_1b_2})$.

Composition 3: Similarly, by multiplying the equations as $b_1^2 = c_1^2 - a_1^2$ and $b_2^2 = c_2^2 - a_2^2$, I get the triple $(\boldsymbol{a_1c_2 + c_1a_2, b_1b_2, c_1c_2 + a_1a_2})$.

Composition 4: Again, by interchanging a_1 and b_1 in Composition 3, that is, by composing $(b_1, a_1, c_1) \odot (a_2, b_2, c_2)$ in the same way, I get the triple $(\boldsymbol{b_1c_2 + c_1a_2, a_1b_2, c_1c_2 + b_1a_2})$.

Composition 5: Now I can proceed to find the triple with the term c_1c_2. Taking the equations $a_1^2 + b_1^2 = c_1^2$ and $a_2^2 + b_2^2 = c_2^2$ as they are and multiplying them, I get,

$$\begin{aligned}(c_1c_2)^2 &= (a_1^2 + b_1^2)(a_2^2 + b_2^2)\\ &= (a_1 + ib_1)(a_1 - ib_1)(a_2 + ib_2)(a_2 - ib_2) \quad \text{[complex factorization; here i} = \sqrt{-1}]\\ &= (a_1 + ib_1)(a_2 + ib_2)(a_1 - ib_1)(a_2 - ib_2)\\ &= [(a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)][(a_1a_2 - b_1b_2) - i(a_1b_2 + b_1a_2)]\\ &= (a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2;\end{aligned}$$

that is, another triple $(\boldsymbol{a_1a_2 - b_1b_2, a_1b_2 + b_1a_2, c_1c_2})$. It does not matter if $a_1a_2 < b_1b_2$; what matters in a triple is the absolute value $|a_1a_2 - b_1b_2|$.

Composition 6: Now, as $(a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2 = (a_1a_2 + b_1b_2)^2 + (b_1a_2 - a_1b_2)^2$, there may be yet another triple, $(\boldsymbol{a_1a_2 + b_1b_2, b_1a_2 - a_1b_2, c_1c_2})$. Again, to avoid negative values, I can take $|b_1a_2 - a_1b_2|$. It may be noticed that this is the same as composing $(b_1, a_1, c_1) \odot (a_2, b_2, c_2)$ according to the formula of Composition 5.

Thus, from the two triples $(\boldsymbol{a_1, b_1, c_1})$ and $(\boldsymbol{a_2, b_2, c_2})$ [or t_1 and t_2], I have generated by compositions $(\mathbf{C_1, C_2, C_3, C_4, C_5, C_6})$ similar to *Bhāvanā* the following six distinct triples:

$\mathbf{C_1}$ $:[a_1a_2, (b_1c_2 + c_1b_2), (c_1c_2 + b_1b_2)]$
$\mathbf{C_2}$ $:[b_1a_2, (a_1c_2 + c_1b_2), (c_1c_2 + a_1b_2)]$
$\mathbf{C_3}$ $:[(a_1c_2 + c_1a_2), b_1b_2, (c_1c_2 + a_1a_2)]$
$\mathbf{C_4}$ $:[(b_1c_2 + c_1a_2), a_1b_2, (c_1c_2 + b_1a_2)]$
$\mathbf{C_5}$ $:[(a_1a_2 - b_1b_2), (a_1b_2 + b_1a_2), c_1c_2]$
$\mathbf{C_6}$ $:[(a_1a_2 + b_1b_2), (b_1a_2 - a_1b_2), c_1c_2]$

It should be noted that, as in Brahmagupta's *Bhāvanā, it is the property that* $(x^2 \pm Ny^2)$ *is 'closed under multiplication' that is made use of in these transformations.*

Notation: C_1, C_2, etc., being the six composition formulas for generating triples, the compositions can be represented as "$C_1(t_1 \odot t_2)$", meaning "t_1 composed with t_2 according to formula C_1", or "$C_3[t_1 \odot C_2(t_1 \odot t_2)]$", meaning "the result of t_1 composed with t_2 according to formula C_2 is composed with t_1 according to formula C_3", and so on.

Primitive or Non-Primitive Triples: Obviously, C_1, C_2, etc., can generate Primitive as well as Non-Primitive Triples. I will be interested in primitive triples (PPT) only, and all numerical examples of triples that I use will be reduced to their corresponding PPTs. This will lead to a search for the conditions under which non-PPTs are generated, and also to the big issue of "recurrence of triples in a continuous process of composition."

How triples multiply

The six formulas, if laid out on an Excel sheet, will go on generating triples if input triples are entered. In Table 1 I have done this, starting with $\mathbf{t} = (3, 4, 5)$, the smallest triple. I have first done the compositions $C_1(t \odot t), C_2(t \odot t), C_3(t \odot t), C_4(t \odot t), C_5(t \odot t), C_6(t \odot t)$, and reduced them to their corresponding PPTs, say, $\mathbf{T_1, T_2, T_3, T_4, T_5, T_6}$ respectively. Then I have proceeded to compose $\mathbf{t}$ with each of these PPTs. *In the whole process, I have reduced every composition to its corresponding PPT (without negative signs).* Note: I could not compose t with T_6 because $T_6 = C_6(t \odot t) = (25, 0, 25) \Rightarrow (1, 0, 1)$, a 'trivial' triple. Also, in composing t with $T_5 = C_5(t \odot t)$ I have not ignored the negative sign in $C_5(t \odot t) = (-7, 24, 25)$ because, as I will show later, keeping or ignoring the negative signs can have quite different consequences.

t	T	C_1	$T_1 = C_1/gcd$	C_2	$T_2 = C_2/gcd$	C_3	$T_3 = C_3/gcd$	C_4	$T_4 = C_4/gcd$	C_5	$T_5 = C_5/gcd$	C_6	$T_6 = C_6/gcd$
3	3	9	9	12	12	30	15	35	35	−7	7	25	1
4	4	40	40	35	35	16	8	12	12	24	24	0	0
5	5	41	41	37	37	34	17	37	37	25	25	25	1
gcd		1	1	1	1	2	1	1	1	1	1	25	1
3	9	27	27	36	36	168	21	209	209	−133	133	187	187
4	40	364	364	323	323	160	20	120	120	156	156	−84	84
5	41	365	365	325	325	232	29	241	241	205	205	205	205
gcd		1	1	1	1	8	1	1	1	1	1	1	1
3	12	36	36	48	24	171	171	208	208	−104	104	176	176
4	35	323	323	286	143	140	140	105	105	153	153	−57	57
5	37	325	325	290	145	221	221	233	233	185	185	185	185
gcd		1	1	2	1	1	1	1	1	1	1	1	1
3	15	45	5	60	60	126	63	143	143	13	13	77	77
4	8	108	12	91	91	32	16	24	24	84	84	36	36
5	17	117	13	109	109	130	65	145	145	85	85	85	85
gcd		9	1	1	1	2	1	1	1	1	1	1	1
3	35	105	105	140	140	286	143	323	323	57	57	153	153
4	12	208	208	171	171	48	24	36	36	176	176	104	104
5	37	233	233	221	221	290	145	325	325	185	185	185	185
gcd		1	1	1	1	2	1	1	1	1	1	1	1
3	−7	−21	21	−28	28	40	5	65	65	−117	117	75	3
4	24	220	220	195	195	96	12	72	72	44	44	−100	4
5	25	221	221	197	197	104	13	97	97	125	125	125	5
gcd		1	1	1	1	8	1	1	1	1	1	25	1

Table 1. C_1, C_2, etc., are the composition formulas applied on t ⊙ T.
The ‘recurrent’ triples are highlighted in different colours.

The table shows that 36 triples have been generated out of which, except a few that recur (after ignoring negative signs), all the others are distinct (including a ‘trivial’ one). The calculations may be carried out on an Excel sheet. The procedure has been explained in the Appendix of this article.

Properties of Composition

Based on the composition formulas and with reference to the triples generated by composition in *Table 1*, I can study the properties of composition.

1. **PPTs and Non-Primitive PTs.** Both PPTs and non-primitive PTs (i.e., Pythagorean triples that are not primitive, which means that the three numbers in the triple have a common factor exceeding 1) are generated by the compositions. Thus I had (9, 40, 41), (35, 12, 37), etc., as well as (30, 16, 34), (40, 96, 104), etc. As PPTs are more fundamental, the non-primitive PTs have all been reduced to their corresponding PPTs in this study (with negative signs ignored).

2. **$b = c - 1$ and $b < c - 1$.** PPTs (a, b, c) being of two kinds, one where the even member $b = c - 1$ and the other where $b < c - 1$, it is found that both kinds are generated by the compositions. So I had (5, 12, 13), (13, 84, 85), etc., as well as (143, 24, 145), (57, 176, 185), etc.

3. **Reversal of order: first two terms.** It is interesting to see what happens when the order of the first two terms is reversed in any one or both of the triples. The composition formulas are so constructed that we have the following consequences (see Table 2 below):

   (1) comparing $(a_1, b_1, c_1) \odot (a_2, b_2, c_2)$ with $(b_1, a_1, c_1) \odot (a_2, b_2, c_2)$: (C_1, C_2) exchange results, (C_3, C_4) exchange results; and (C_5, C_6) not only exchange results, but with the order of the first two terms of the resultant triple reversed;

   (2) comparing $(a_1, b_1, c_1) \odot (a_2, b_2, c_2)$ with $(a_1, b_1, c_1) \odot (b_2, a_2, c_2)$: (C_1, C_4) exchange results with the order of the first two terms of the resultant triple reversed; so do (C_2, C_3) and (C_5, C_6) (ignoring negative signs);

   (3) comparing $(a_1, b_1, c_1) \odot (a_2, b_2, c_2)$ with $(b_1, a_1, c_1) \odot (b_2, a_2, c_2)$: (C_1, C_3) exchange results with the order of the first two terms of the resultant triple reversed; so do (C_2, C_4). But results for (C_5, C_6) remain unchanged if negative signs are ignored.

t_1	t_2	C_1	$T_1 = C_1/\text{gcd}$	C_2	$T_2 = C_2/\text{gcd}$	C_3	$T_3 = C_3/\text{gcd}$	C_4	$T_4 = C_4/\text{gcd}$	C_5	$T_5 = C_5/\text{gcd}$	C_6	$T_6 = C_6/\text{gcd}$
3	35	105	105	140	140	286	143	323	323	57	57	153	153
4	12	208	208	171	171	48	24	36	36	176	176	104	104
5	37	233	233	221	221	290	145	325	325	185	185	185	185
gcd		1	1	1	1	2	1	1	1	1	1	1	1
4	35	140	140	105	105	323	323	286	143	104	104	176	176
3	12	171	171	208	208	36	36	48	24	153	153	57	57
5	37	221	221	233	233	325	325	290	145	185	185	185	185
gcd		1	1	1	1	1	1	2	1	1	1	1	1
3	12	36	36	48	24	171	171	208	208	−104	104	176	176
4	35	323	323	286	143	140	140	105	105	153	153	−57	57
5	37	325	325	290	145	221	221	233	233	185	185	185	185
gcd		1	1	2	1	1	1	1	1	1	1	1	1
4	12	48	24	36	36	208	208	171	171	−57	57	153	153
3	35	286	143	323	323	105	105	140	140	176	176	−104	104
5	37	290	145	325	325	233	233	221	221	185	185	185	185
gcd		2	1	1	1	1	1	1	1	1	1	1	1

Table 2. C_1, C_2, etc. are the composition formulas applied on $t_1 \odot t_2$.

4. **Reversal of sequence of composition: Commutativity check.** In constructing the formulas, I assumed the sequence $t_1 \odot t_2$, where $\mathbf{t}_1 = (a_1, b_1, c_1)$, $t_2 = (a_2, b_2, c_2)$. When the sequence is reversed and $t_2 \odot t_1$ is composed it is found that $C_1(t_1 \odot t_2) = C_1(t_2 \odot t_1), C_3(t_1 \odot t_2) = C_3(t_2 \odot t_1)$, $C_5(t_1 \odot t_2) = C_5(t_2 \odot t_1)$; that is, the compositions C_1, C_3, C_5 are *commutative* under reversal of sequence. In fact, C_6 is also *commutative* if negative signs are ignored in the results. But C_2 and C_4 interchange results with the order of the first two terms reversed. See Table 3 below.

t_1	t_2	C_1	T_1	C_2	T_2	C_3	T_3	C_4	T_4	C_5	T_5	C_6	T_6
171	57	9747	9747	7980	7980	44232	5529	38497	38497	−14893	14893	34387	34387
140	176	64796	64796	70531	70531	24640	3080	30096	30096	38076	38076	−22116	22116
221	185	65525	65525	70981	70981	50632	6329	48865	48865	40885	40885	40885	40885
gcd		1	1	1	1	8	1	1	1	1	1	1	1
57	171	9747	9747	30096	30096	44232	5529	70531	70531	−14893	14893	34387	34387
176	140	64796	64796	38497	38497	24640	3080	7980	7980	38076	38076	22116	22116
185	221	65525	65525	48865	48865	50632	6329	70981	70981	40885	40885	40885	40885
gcd		1	1	1	1	8	1	1	1	1	1	1	1

Table 3. C_1, C_2, etc. are the composition formulas applied on $t_1 \odot t_2$; T_1, T_2, etc. as in Tables 1 & 2.

5. **Chain of compositions: Associativity check.**
Let $t_1 = (3, 4, 5), t_2 = (5, 12, 13), t_3 = (7, 24, 25), t_4 = (15, 8, 17)$. Let the chain of compositions $t_1 \odot t_2 \odot t_3 \odot t_4$ be made under all of $C_1, C_2, \ldots. C_6$. It is found that $C_1[\{C_1(t_1 \odot t_2)\} \odot \{C_1(t_3 \odot t_4)\}] = C_1[(15, 112, 113) \odot (105, 608, 617)] = (175, 15312, 15313)$. Also, $C_1[C_1\{C_1(t_1 \odot t_2)\} \odot t_3\}] \odot t_4] = C_1\{[C_1\{(15, 112, 113) \odot (7, 24, 25)\}] \odot (15, 8, 17)\} = C_1\{(105, 5512, 5513) \odot (15, 8, 17)\} = (175, 15312, 15313)$, which is the same as $C_1[\{C_1(t_1 \odot t_2)\} \odot \{C_1(t_3 \odot t_4)\}]$. This shows that a chain of compositions under C_1 is *associative*. Similarly, compositions under C_3, C_5 are also *associative*. Again, C_6 is also *associative* if negative signs are ignored in composition. Compositions under C_2 and C_4 only are *non-associative*.

6. **Negative values.** As discussed above and as will be seen in Table 4 below, keeping or ignoring negative signs produces quite a different picture in terms of the triples generated. So, strictly speaking, retaining the negative signs ought to be a more authentic process.

t_1	t_2	C_1	T_1	C_2	T_2	C_3	T_3	C_4	T_4	C_5	T_5	C_6	T_6
3	3	9	9	12	12	30	15	35	35	−7	7	25	1
4	4	40	40	35	35	16	8	12	12	24	24	0	0
5	5	41	41	37	37	34	17	37	37	25	25	25	1
gcd		1	1	1	1	2	1	1	1	1	1	25	1
−7	3	−21	21	72	72	40	5	195	195	−117	117	75	3
24	4	220	220	65	65	96	12	−28	28	44	44	100	4
25	5	221	221	97	97	104	13	197	197	125	125	125	5
gcd		1	1	1	1	8	1	1	1	1	1	25	1
7	3	21	21	72	8	110	55	195	195	−75	3	117	117
24	4	220	220	135	15	96	48	28	28	100	4	44	44
25	5	221	221	153	17	146	73	197	197	125	5	125	125
gcd		1	1	9	1	2	1	1	1	25	1	1	1

Table 4. C_1, C_2, etc. are the composition formulas applied on $t_1 \odot t_2$; T_1, T_2, etc. as in Tables 1 & 2.
(We have highlighted differences in values.)

7. **Return to the original triple.** From definition, $C_5[(a_1, b_1, c_1) \odot (a_2, b_2, c_2)] = (a_1a_2 - b_1b_2, b_1a_2 + a_1b_2, c_1c_2) = \mathrm{T}$ (say). Now, $C_6[\mathrm{T} \odot (a_1, b_1, c_1)] = c_1^2. (a_2, b_2, c_2)$, that is, the original triple (a_2, b_2, c_2) returns. Similarly, in $C_6[\mathrm{T} \odot (a_2, b_2, c_2)] = c_2^2. (a_1, b_1, c_1)$, which indicates return of the original triple

(a_1, b_1, c_1). Example: $C_5[(7,4,5) \bullet (15,8,17)] = (-87,416,425)$. Then, $C_6[(-87,416,425) \bullet (15,8,17)] = (2023,6936,7225) \Rightarrow (7,24,25)$, dividing by $gcd = 17^2$; and $C_6[(-87,416,425) \bullet (7,24,25)] = (9375,5000,10625) \Rightarrow (15,8,17)$, dividing by $gcd = 25^2$.

8. **Non-Primitive PTs: the GCD.** It can be seen that in each of the composed triples, there is a product term like $a_1a_2, b_1a_2, b_1b_2, a_1b_2, c_1c_2$; the rest are either sums or differences of products, like $(b_1b_2 + c_1c_2) or (a_1a_2 - b_1b_2)$. In generating non-primitive triples, it is always the product term that plays the decisive role. *If $d > 1$, where d is a common divisor between the two factors of the product term, and certain other conditions are satisfied, then the composed triple will be a non-PPT.* In the discussion below, I will look into these *certain other conditions.*

C_1: Take the triple $C_1[(a,b,c) \odot (x,y,z)] = (ax, bz+cy, by+cz)$. I assume that b and y are even. Let a common divisor, not necessarily the *gcd*, of (a,x) be $d > 1$. So ax contains the divisor d^2. Let $a = dma_0, x = dnx_0$. Let $dm > a_0, x_0 > dn$. Then, $b = \frac{1}{2}(d^2m^2 - a_0^2), c = \frac{1}{2}(d^2m^2 + a_0^2)$; and $y = \frac{1}{2}(x_0^2 - n^2d^2), z = \frac{1}{2}(x_0^2 + n^2d^2)$. So, $ax = d^2mna_0x_0$; $bz + cy = \frac{1}{2}d^2(m^2x_0^2 - n^2a_0^2)$; and, $by + cz = \frac{1}{2}d^2(m^2x_0^2 + n^2a_0^2)$. As a and x are both odd, so in the composed triple, the *gcd* $(ax, bz+cy, by+cz) = d^2$. *The necessary condition here is $dm > a_0, x_0 > dn$; if it is not satisfied, the* $gcd = 1$. Thus, $C_1[(15,8,17) \odot (255,32,257)] = (3825,2600,4625)$, a non-PPT with $gcd = 5^2$. Again, $C_1[(3,4,5) \odot (21,220,221)] = (63,1984,1985)$, a PPT; whereas $C_1[(3,4,5) \odot (21,20,29)] = (63,216,225)$, a non-PPT with $gcd = 3^2$.

C_2: Now take $C_2[(a,b,c) \odot (x,y,z)] = (bx, az+cy, ay+cz)$. Let $b = dmb_0, x = dnx_0$. With b even and x odd, d, n, x_0 should be odd. So, $a = b_0^2 - (\frac{md}{2})^2, c = b_0^2 + (\frac{md}{2})^2$; and $y = \frac{1}{2}(d^2n^2 - x_0^2)$, $z = \frac{1}{2}(d^2n^2 + x_0^2)$. Thus, $(bx, az+cy, ay+cz) = [d^2mnb_0x_0, \frac{1}{4}d^2(4b_0^2n^2 - m^2x_0^2), \frac{1}{4}d^2(4b_0^2n^2 + m^2x_0^2]$. To make md even, as d is odd, m should be even; and that also makes the second and third terms integers. Note: Here the necessary condition is $b_0 > \frac{md}{2}, dn > x_0$, and m even. Thus, $C_2[(11,60,61) \odot (15,8,17)] = (900,675,1125)$, a non-PPT with $gcd = 225$.

C_4: Similar to C_2 will be the case for $C_4[(a,b,c) \odot (x,y,z)] = (bz+cx, ay, bx+cz)$.

C_3: In $C_3[(a,b,c) \odot (x,y,z)] = (az+cx, by, ax+cz)$ the terms are all even; so, there is always a common divisor 2. But, let $b = dmb_0, y = dny_0$. Here, if the condition $a = b_0^2 - (\frac{md}{2})^2$, $c = b_0^2 + (\frac{md}{2})^2$, and $x = (nd)^2 - (\frac{y_0}{2})^2, z = (nd)^2 + (\frac{y_0}{2})^2$, is satisfied, then $az + cx = \frac{1}{8}d^2(16n^2b_0^2 - m^2y_0^2)$, $by = d^2mnb_0y_0$, $ax + cz = \frac{1}{8}d^2(16n^2b_0^2 + m^2y_0^2)$. Here the necessary conditions are: md and y_0 are even, and $b_0 > \frac{md}{2}, nd > \frac{y_0}{2}$. So, $gcd(az+cx, by, ax+cz) \geq 2$. Thus, $C_3[(7,24,25) \odot (35,12,37)] = (1134,288,1170)$, with $gcd = 18$; and, $C_3[(5,12,13) \odot (7,24,25)] = (216,288,360)$, which has a $gcd = 72$. But $C_3[(5,12,13) \odot (9,40,41)] = (322,480,578)$ has $gcd = 2$.

C_5 & C_6: If for (a,b,c) and (x,y,z) the $gcd(c,z) = d > 1$, then a non-PPT with $gcd = d^2$ between the terms will be generated by either C_5 or C_6, never both. Obviously, C_5 and C_6 generate different triples with the same term cz. Thus, for $(5,12,13) \odot (33,56,65)$, C_5 generates $(-507,676,845)$, with $gcd = 13^2$ (if negative signs are ignored); and C_6 generates (837, 116, 845), a PPT. But for $(5,12,13) \odot (63,16,65)$, C_5 generates (123, 836, 845), a PPT; while C_6 gives (507, 676, 845), with $gcd = 13^2$.

9. **Recurrence of triples.** The discussion on properties shows that in a process of composition there will be recurrence of triples (as exemplified by Tables 1 - 4). Recurrence of triples will be caused by:

(i) Reversal of the order of the first two terms of any one or both of the composing triples. For instance, (12, 35, 37) and (35, 12, 37) were both generated in the first row in Table 1, and both were used in subsequent compositions.

(ii) Reversal of the sequence of composition of the composing triples: that is, by *commutativity*, wherever it exists.

(iii) *Associativity*, wherever it exists: that is, generating the same triple through different sequences in a given chain of compositions.

[(ii) and (iii) can happen when all possible sequences of compositions are tried during the process.]

(iv) Return to the original triple.

(v) Most importantly, when non-PPTs are reduced to PPTs: that is, when the PPT thus generated may have been generated earlier or will be generated later in the composition process by composing different triples.

Infinitely many triples and infinitely many recurrences

Let us imagine a continuous composition process that starts with one triple (a PPT), which is composed with itself; and all non-primitive PTs that are generated are reduced to their corresponding PPTs. Then every PPT thus produced is composed with itself and with every other triple; and the process goes on indefinitely, and all possible sequences of composition are admitted. What is the outcome of such a process?

It has been seen that:

1. The formulas C_1 to C_6 constitute a system by which infinitely many Pythagorean triples can be generated by compositions, starting from just one triple.
2. Triples generated by the composition formulas are not all unique; occasionally the triples recur.

As the number of compositions increases, the number of recurrences is also likely to increase. But that will depend on the compositions chosen. The big question is: *Will there be a point of 'saturation' when triples are merely repeated and no new ones are generated?*

But we have seen that recurrences occur only when very specific conditions obtain. So, at every composition there will be as much possibility of a new triple generation as of the recurrence of an old one. Therefore, the point of '*saturation*' will never come to be, because *while infinitely many new triples will be generated, there will also be infinitely many recurrences, but this will be a never-ending process*.

What will happen is that:

1. *Over a finite range of compositions, the number of recurrences of triples will vary according to the compositions chosen.*
2. *Over an infinite range of compositions, while infinitely many new triples will be generated, there will also be infinitely many recurrences; but that being a never-ending process, no point of 'saturation' when new triples cease to be generated will ever be reached.*

Appendix

The Compositions on an EXCEL sheet

This is how the Tables have been created on EXCEL. The following table is a representation of one set of compositions ($C_1 - C_6$) in which the cells are marked as A1, A2, A3, B1, B2, B3, etc. The EXCEL formulas in each cell are shown below the table.

A1	B1	C1	D1	E1	F1	G1	H1	I1	J1	K1	L1	M1	N1
A2	B2	C2	D2	E2	F2	G2	H2	I2	J2	K2	L2	M2	N2
A3	B3	C3	D3	E3	F3	G3	H3	I3	J3	K3	L3	M3	N3
gcd		C4	D4	E4	F4	G4	H4	I4	J4	K4	L4	M4	N4

Triple (a_1, b_1, c_1) is laid out in the cells: A1: a_1; A2: b_1; A3: c_1

Triple (a_2, b_2, c_2) is laid out in the cells: B1: a_2; B2: b_2; B3: c_2

C_1: $[a_1a_2, (b_1c_2 + c_1b_2), (c_1c_2 + b_1b_2)]$
C1 = A1*B1; C2 = A2*B3+B2*A3; C3 = A2*B2+A3*B3; C4 = GCD(ABS(C1),ABS(C2),ABS(C3))
D1 = ABS(C1)/C4; D2 = ABS(C2)/C4; D3 = ABS(C3)/C4; D4 = GCD(D1,D2,D3)

C_2: $[b_1a_2, (a_1c_2 + c_1b_2), (c_1c_2 + a_1b_2)]$
E1 = A2*B1; E2 = A1*B3+A3*B2; E3 = A1*B2+A3*B3; E4 = GCD(ABS(E1),ABS(E2),ABS(E3))
F1 = ABS(E1)/E4; F2 = ABS(E2)/E4; F3 = ABS(E3)/E4; F4 = GCD(F1,F2,F3)

C_3: $[(a_1c_2 + c_1a_2), b_1b_2, (c_1c_2 + a_1a_2)]$
G1 = A1*B3+A3*B1; G2 = A2*B2; G3 = A1*B1+A3*B3; G4 = GCD(ABS(G1),ABS(G2),ABS(G3))
H1 = ABS(G1)/G4; H2 = ABS(G2)/G4; H3 = ABS(G3)/G4; H4 = GCD(H1,H2,H3)

C_4: $[(b_1c_2 + c_1a_2), a_1b_2, (c_1c_2 + b_1a_2)]$
I1 = A2*B3+A3*B1; I2 = A1*B2; I3 = A2*B1+A3*B3; I4 = GCD(ABS(I1),ABS(I2),ABS(I3))
J1 = ABS(I1)/I4; J2 = ABS(I2)/I4; J3 = ABS(I3)/I4; J4 = GCD(J1,J2,J3)

C_5: $[(a_1a_2 - b_1b_2), (a_1b_2 + b_1a_2), c_1c_2]$
K1 = A1*B1−A2*B2; K2 = A1*B2+A2*B1; K3 = A3*B3; K4 = GCD(ABS(K1),ABS(K2),ABS(K3))
L1 = ABS(K1)/K4; L2 = ABS(K2)/K4; L3 = ABS(K3)/K4; L4 = GCD(L1,L2,L3)

C_6: $[(a_1a_2 + b_1b_2), (b_1a_2 - a_1b_2), c_1c_2]$
M1 = A1*B1+A2*B2; M2 = A2*B1−A1*B2; M3 = A3*B3; M4 = GCD(ABS(M1),ABS(M2),ABS(M3))
N1 = ABS(M1)/M4; N2 = ABS(M2)/M4; N3 = ABS(M3)/M4; N4 = GCD(N1,N2,N3)

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Problem Corner

Adventures in PROBLEM SOLVING

SHAILESH SHIRALI

In this edition of 'Adventures' we study a few miscellaneous problems, some from the PRMO and some from the AIME (the 'American Invitational Mathematics Examination'). As usual, we pose the problems first and present the solutions later.

Miscellaneous problems

Problem 1. Let a, b be natural numbers such that $2a - b$, $a - 2b$ and $a + b$ are all distinct squares. What is the smallest possible value of b? (Problem 15, PRMO 2018)

Problem 2. Raju's age and his father's age in years are 2-digit integers. When the father's age is written after Raju's age, a 4-digit perfect square is formed. If the father's age 25 years ago is written after Raju's age at that time, another 4-digit perfect square is formed. What are the ages of Raju and his father? (Problem shared with me over email; thank you, Hitha.)

Problem 3. A 5-digit number n is such that when its middle digit is removed, the resulting 4-digit number m is a divisor of n. Find all possible values of n/m. (Purdue, "Problem of the Week")

Problem 4. (a) What is the largest prime factor of the binomial coefficient $\binom{2000}{1000}$?

(b) What is the largest 2-digit prime factor of the binomial coefficient $\binom{200}{100}$? (AIME 1983)

Keywords: Binomial coefficient, prime factor, subset, Pythagorean triple

Problem 5. For each non-empty subset of $\{1, 2, 3, 4, 5, 6, 7\}$, arrange the members in decreasing order with alternating signs and take the sum. For example, for the subset $\{5\}$ we get 5. For $\{6, 3, 1\}$ we get $6 - 3 + 1 = 4$. Find the sum of all the resulting numbers. (AIME 1983)

Solutions to the problems

Solution to problem 1

The problem requires us to look for pairs (a, b) of natural numbers such that $2a - b$, $a - 2b$ and $a + b$ are distinct squares. Let $2a - b = x^2$, $a - 2b = y^2$ and $a + b = z^2$. Since $y^2 \geq 0$, it follows that $a \geq 2b$. If $a = 2b$, then $x^2 = 3b$ and $z^2 = 3b$, so x^2 and z^2 are not distinct squares (contrary to the given information). Hence $a > 2b$, and x^2, y^2 and z^2 are non-zero squares. We now reason as follows.

- Since $(2a - b) - (a - 2b) = a + b$, i.e., $x^2 = y^2 + z^2$, (y, z, x) is a Pythagorean triple.
- We recall the following from number theory: *under division by 3, a square number leaves remainder 0 or 1; the remainder is never 2*. Here are two implications of this: (i) if the sum of two squares is a multiple of 3, then both squares are multiples of 3; (ii) in a Pythagorean triple, at least one number in the triple is a multiple of 3.
- Observe that $x^2 + y^2 = 3a - 3b$, which is a multiple of 3. Hence both x and y are multiples of 3. Therefore z too is a multiple of 3. So 3 divides all three numbers in the triple (y, z, x). This tells us that $(y/3, z/3, x/3)$ too is a Pythagorean triple.
- The problem asks us to find pairs (a, b) of natural numbers satisfying the stated conditions, with b as small as possible. This amounts to finding a Pythagorean triple (y, z, x) in which all three numbers are multiples of 3, with z^2 and y^2 as close to each other as possible.
- It is natural to start by looking at primitive Pythagorean triples in which the two smaller numbers are small and as close as possible. Since no Pythagorean triple exists in which the least number is ≤ 2 (please verify this for yourself), we focus on the triple $(3, 4, 5)$. We clearly cannot do better than this. This means that in any Pythagorean triple, the smallest number is ≥ 3 and the second smallest number is ≥ 4, so the sum of the two smaller numbers is ≥ 7 and the difference between the squares of the two smaller numbers is ≥ 7.
- Since $(y/3, z/3, x/3)$ is a Pythagorean triple, it follows that $(z/3)^2 - (y/3)^2 \geq 7$, and therefore that $z^2 - y^2 \geq 63$. Hence $b \geq 21$.
- Now it is easy to check that the value $b = 21$ can be 'realised.' We start with the triple $(3, 4, 5)$ and scale it up by a factor of 3; we get $(9, 12, 15)$. The squares of these three numbers are 81, 144, 225 respectively. So we write $a - 2b = 81$, $a + b = 144$ and solve for a, b; we get $a = 123$, $b = 21$. For this choice of values for a and b, the numbers $2a - b$, $a - 2b$ and $a + b$ are indeed distinct squares; please check. This justifies the claim that the value $b = 21$ can be realised. Hence the least possible value of b is 21.

Solution to problem 2

Let the son's age be a years, and let the father's age be b years; both a and b are two-digit numbers. As per the data given, the following can be stated:

- the number $100a + b$ is a perfect square;
- the number $100(a - 25) + (b - 25)$ is also a perfect square.

Let $100a + b = u^2$ and $100(a - 25) + (b - 25) = v^2$. Since u^2 and v^2 are 4-digit perfect squares, u and v are 2-digit numbers. This means that $u + v < 200$.

By subtraction, we get $u^2 - v^2 = 2525$. This can be written as $(u + v)(u - v) = 2525$.

Now the number 2525 can be factorised as $2525 = 25 \times 101$. Importantly, 101 is a prime number. In what ways can 2525 be written as a product of two numbers, neither of which exceed 200? Precisely because 101 is a prime number, the only possible way is 25×101. This implies that $u - v = 25$ and $u + v = 101$. By addition we obtain $2u = 126$, hence $u = 63$ and therefore $v = 38$.

Hence $u^2 = 63^2 = 3969$ and $v^2 = 38^2 = 1444$. So the son's age is 39 and the father's age is 69 (current ages).

Solution to problem 3

Let a be the 2-digit number formed by the leftmost two digits of n; let b be the middle digit; and let c be the 2-digit number formed by the rightmost two digits of n. Then we have

$$n = 1000a + 100b + c, \quad m = 100a + c.$$

Since m is a divisor of n, we have

$$\begin{aligned} & 100a + c \mid 1000a + 100b + c, \\ \therefore\ & 100a + c \mid 1000a + 100b + c - 10(100a + c), \\ \therefore\ & 100a + c \mid 100b - 9c. \end{aligned}$$

Now we establish some inequalities. We compute the least possible value of $100a + c$ and the greatest possible (absolute) value of $100b - 9c$. (Note that $100b - 9c$ can be negative.) We have

$$100a + c \geq 100 \times 10 = 1000,$$

i.e., $100a + c \geq 1000$. Also, since $0 \leq b \leq 9$ and $0 \leq c \leq 99$, we have

$$-99 \times 9 \leq 100b - 9c \leq 100 \times 9,$$

i.e.,

$$-891 \leq 100b - 9c \leq 900.$$

Since $100b - 9c$ is required to be a multiple of $100a + c$, the only way this can happen is for $100b - 9c = 0$, which in turn can only happen if $b = 0$ and $c = 0$. But if these conditions hold, then $n = 1000a$ and $m = 100a$, and the requirement that $m \mid n$ is automatically met, for any value of a.

In each case we find that $n/m = 10$; so this is the required answer.

Solution to problem 4 (a)

It should be fairly clear that the number

$$\binom{2000}{1000} = \frac{2000 \times 1999 \times 1998 \times 1997 \times \cdots \times 1003 \times 1002 \times 1001}{1 \times 2 \times 3 \times 4 \times \cdots \times 998 \times 999 \times 1000}$$

is divisible by every prime number between 1001 and 2000 (for this prime number will be a factor of the numerator of the above expression but not a factor of the denominator). Hence the answer is simply the largest prime number between 1001 and 2000. It so happens that 1999 is a prime number. Hence this is the desired answer.

Solution to problem 4 (b)

We require the largest two-digit prime factor of the number

$$\binom{200}{100} = \frac{200 \times 199 \times 198 \times 197 \times \cdots \times 103 \times 102 \times 101}{1 \times 2 \times 3 \times 4 \times \cdots \times 98 \times 99 \times 100}.$$

Here is a list of all the two-digit prime numbers:

$$11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89,\ 97.$$

We start from the 'upper end.' Could 97 be the answer? No, because it is present once in the denominator (as 97 itself), and once in the numerator as well (as a factor of 194). So 97 'cancels' out, which means that the number $\binom{200}{100}$ is not divisible by 97. Could 89 be the answer? Arguing as earlier, we note that it is present once in the denominator (as 89 itself) and once in the numerator (as a factor of 178), so $\binom{200}{100}$ is not divisible by 89. Could 83 be the answer? Yet again the answer is no, for the same reason. We see a way forward now. We clearly require the largest two digit prime number p such that *there are more multiples of* p *between* 101 *and* 200 *than between* 1 *and* 100. This is equivalent to searching for the largest prime number p such that $p < 101 < 2p < 3p < 200$. (This way there are two multiples of p between 101 and 200, and only one multiple of p between 1 and 100.) The number which fits this description is 61 (note that $67 \times 3 = 201 > 200$); hence 61 is the required answer.

Using the same reasoning, we can show that $\binom{200}{100}$ is divisible by each of the following two-digit primes: 59, 53, 37, 17, 13, 11. In the case of 37, we find that it is present twice in the denominator and three times in the numerator. Observe that the primes 41, 43 and 47 are missing from the list. Each of these occurs twice in the denominator and twice in the numerator, thereby canceling out.

Here is the full expression of $\binom{200}{100}$ as a product of primes:

$$\begin{aligned}\binom{200}{100} &= 2^3 \times 3 \times 5 \times 11 \times 13^2 \times 17 \times 37 \times 53 \times 59 \times 61 \times 101 \times 103 \times 107 \\ &\quad \times 109 \times 113 \times 127 \times 131 \times 137 \times 139 \times 149 \times 151 \times 157 \times 163 \\ &\quad \times 167 \times 173 \times 179 \times 181 \times 191 \times 193 \times 197 \times 199.\end{aligned}$$

It is curious that only two primes occur to a power greater than 1; namely, 2 and 13.

Solution to problem 5

This is a truly beautiful problem!

In an effort to get a handle on the problem, we start with smaller sets and build our way upward, all the while looking for a pattern. (We can do this using hand calculation.) Here is what we find.

Set	Sum
$\{1\}$	1
$\{1,2\}$	4
$\{1,2,3\}$	12
$\{1,2,3,4\}$	32

A pattern is already becoming evident: *it appears that if the set is* $\{1, 2, 3, 4, \ldots, n\}$, *then the sum is* $n \cdot 2^{n-1}$. If this pattern is valid, then the required answer is $7 \cdot 2^6 = 448$.

But what could be the explanation for the sum to have this simple form? To find it, we go back to the original problem, in which the largest number is 7.

Consider any subset A of $\{1, 2, 3, 4, 5, 6, 7\}$ such that $7 \in A$. Let B be the subset obtained by removing 7 from A, i.e., $B = A \setminus \{7\}$. Let the alternating sums associated with sets A and B, computed the way described in the problem, be a and b respectively. What will $a + b$ be equal to? A moments reflection reveals that the answer must be 7. To see why, we look at a simple instance. Suppose that $A = \{2, 5, 7\}$ and $B = \{2, 5\}$. Then the alternating sums associated with the two sets are $a = 7 - 5 + 2$ and $b = 5 - 2$ respectively. Adding them, we see that a beautiful cancellation takes place, and the sum is 7.

We infer from this phenomenon that the sum of all such alternating sums is equal to 7 times the number of subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ not containing the element 7, i.e., the number of subsets of $\{1, 2, 3, 4, 5, 6\}$. This number is $2^6 = 64$. Hence the required sum is $7 \times 64 = 448$.

Generalising, if the initial set is $\{1, 2, 3, \ldots, n\}$, the required sum is $n \cdot 2^{n-1}$.

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Letter to the Editor

This is in connection with the article "Mathematics Olympiads in India" by Phoolan Prasad (PP) in the November 2018 issue of AtRiA.

In the last line of page 87, PP mentions that the IMO training camp in India started at IISc from 1986 and lasted until 1993 and was later taken over by NBHM and HBCSE. Modesty has forbidden PP not to mention the role played by himself, his Department and his Institute for initiating and nurturing the IMO training camps. PP was enthusiastically supported by his colleagues like V G Tikekar, Renuka Ravindran and R Vittal Rao in planning and executing the entire program.

I remember PP expressing concerns over lack of encouragement to high school students talented in mathematics. He was very keen to do something about this deficiency and even had an exchange of letters with the then Prime Minister Rajiv Gandhi in 1984. This may have had some influence in setting up the Jawahar Navodaya Vidyalayas in 1985 by an initiative taken personally by the PM.

A great impetus to this activity came when the MO program became an official activity of IISc in 1979 through the Centre of Continuing Education headed by L S Srinath. As documented in the article, this team of four in IISc intensified the activity further and continued planning the training camps for participation in IMO with great success. Looking back, this training program initiated by PP and his three colleagues is extremely laudable and should not go unreported.

A. S. Vasudeva Murthy
Bangalore

Problems for the SENIOR SCHOOL

Problem Corner

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem VIII-1-S.1

An altitude AH of triangle ABC bisects a median BM. Prove that the medians of the triangle ABM are side-lengths of a right-angled triangle.

Problem VIII-1-S.2

There exists a block of 1000 consecutive positive integers containing no prime numbers, namely, $1001!+2, 1001!+3, \ldots, 1001!+1001$. Does there exist a block of 1000 consecutive positive integers containing exactly 5 prime numbers?

Problem VIII-1-S.3

Initially, the number 1 and two positive numbers x and y are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose a non-zero number on the blackboard and write its reciprocal on the blackboard. Is it possible to write on the blackboard, in a finite number of moves, the numbers x^2 and xy?

Problem VIII-1-S.4

For which positive integers n can one find distinct positive integers $a_1, a_2, \ldots, a_n$ such that the number

$$N = \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is also an integer?

Problem VIII-1-S.5

In triangle ABC, $\measuredangle A = 2\measuredangle B = 4\measuredangle C$. Their bisectors meet the opposite sides at D, E and F respectively. Prove that $DE = DF$.

Keywords: Altitude, median, prime number

Solutions of Problems in Issue-VII-3 (November 2018)

Solution to problem VII-3-S.1

Let $f(x) = x^2 + bx + c$ where b is a negative integer and c is a real number. Suppose the sum of the roots of $f(f(x))$ is a prime number. Prove that $f(f(x))$ has no real root in the interval $(0, 1)$.

Observe that

$$f(f(x)) = x^4 + 2bx^3 + (b^2 + b + 2c)x^2 + (2bc + b^2)x + (c^2 + bc + c).$$

Since the sum of roots $-2b$ is a prime number, $b = -1$. Hence

$$f(f(x)) = x^4 - 2x^3 + 2cx^2 + (1 - 2c)x + c^2 = (x(x-1) + c)^2 + x(1-x)$$

and this is positive for $x \in (0, 1)$.

Solution to problem VII-3-S.2

Let k be a given positive integer. Determine all real x, y, z such that $xyz \neq 0$ and

$$x^k + y^{k+1} = z^{k+2}, \quad x^{k+1} + y^{k+2} = z^{k+3}, \quad x^{k+2} + y^{k+3} = z^{k+4}.$$

Observe that

$$\left(x^k + y^{k+1}\right)\left(x^{k+2} + y^{k+3}\right) = z^{k+2} \cdot z^{k+4} = \left(z^{k+3}\right)^2 = (x^{k+1} + y^{k+2})^2,$$

which implies

$$x^k y^{k+1}\left(x^2 - 2xy + y^2\right) = 0,$$

i.e., $x = y$ (since $xyz \neq 0$). Let $x = y = t$. Then $t \neq 0$ and

$$z = \frac{z^{k+3}}{z^{k+2}} = \frac{t^{k+1}(1+t)}{t^k(1+t)} = t.$$

Thus $1 + t = t^2$ and $t = \dfrac{1-\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}$. Hence

$$(x, y, z) = \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right), \quad \left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right).$$

There are two solutions.

Solution to problem VII-3-S.3

A quadratic polynomial $f(x) = ax^2 + bx + c$ has no real roots. It is given that b is a rational number, and exactly one of c and $f(c)$ is a rational number. Is it possible for the discriminant of $f(x)$ to be a rational number? [Russian Mathematical Olympiad]

Suppose that c is a rational number. Then, by hypothesis, $f(c) = c(ac + b + 1)$ is irrational. Since b and c are rational, a must be irrational. Therefore the discriminant $D = b^2 - 4ac$ is irrational.

Suppose that $f(c)$ is rational but c is irrational. Note $f(c) \neq 0$, since f does not have any real root. Then $(ac + b + 1) \neq 0$ and is irrational. But b is rational. Therefore ac is irrational and hence $D = b^2 - 4ac$ is irrational.

It follows that D is irrational.

Solution to problem VII-3-S.4

The sequence $\{a_n\}_{n\geq 0}$ is defined as follows:

$$a_0 = 1, \quad a_1 = 3, \quad a_{n+1} = a_n + a_{n-1} \text{ for all } n \geq 1.$$

Find all integers $n \geq 1$ for which $na_{n+1} + a_n$ and $na_n + a_{n-1}$ share a common factor greater than 1.

Let $d > 1$ be a common factor of $na_{n+1} + a_n$ and $na_n + a_{n-1}$ for some $n \geq 1$. Then we see that d divides

$$(na_{n+1} + a_n) - (na_n + a_{n-1}) = na_{n-1} + a_{n-2}.$$

Continuing this way we see that d divides $na_1 + a_0$ and $na_2 + a_1$. Thus d divides both

$$n(a_2 - a_1) + (a_1 - a_0) = n + 2$$

and

$$3(na_2 + a_1) - 4(na_1 + a_0) = 3(4n + 3) - 4(3n + 1) = 5.$$

Since $d > 1$ and 5 is prime, $d = 5$ and $n + 2$ is a multiple of 5. Thus for any n of the form $5k + 3$, where k is a non-negative integer, $na_{n+1} + a_n$ and $na_n + a_{n-1}$ share a common factor greater than 1.

Conversely, if n is of the form $5k + 3$ for some non-negative integer k, then $n + 2$ is divisible by 5 and retracing the steps we see that $na_{n+1} + a_n$ and $na_n + a_{n-1}$ are both divisible by 5.

Solution to problem VII-3-S.5

Consider the sequence $\{10^n\}_{n\geq 1}$. Prove that the sum of no two terms of the sequence is a perfect square.

Let $a_n = 10^n$ for $n \geq 1$. Then $a_n \equiv 1 \pmod 3$ for all n, hence

$$a_j + a_k \equiv 2 \pmod 3.$$

Since the square of any integer is either $0 \pmod 3$ or $1 \pmod 3$, a number of the form $2 \pmod 3$ is never a perfect square. Therefore $a_j + a_k$ is never a perfect square for any $1 \leq j \leq k$.

Problem Corner

Problems for the MIDDLE SCHOOL

Problems in Pure Geometry

A RAMACHANDRAN

Pure geometry or Euclidean geometry is a body of theorems and corollaries logically derived from certain axioms and postulates as presented in Euclid's *Elements*. Later geometers, both Greek and others, have added to this. Occasionally some algebra is brought in but not trigonometry. Abraham Lincoln is said to have read the *Elements* just for the reasoning.

Problems

Problem VIII–1–M.1

A fallacy is an argument that sounds logical but is actually erroneous. So the conclusion drawn is false. Often the error is not easy to spot – these could be erroneous assumptions, unwarranted generalisations or applying logic out of context. Fallacies are fun to go through and can help us gain greater alertness in reasoning. Here is a fallacy in geometry – a 'proof' that "all triangles are equilateral." The question is to spot the error.

In an arbitrary triangle ABC (Figure 1) the bisector of $\measuredangle A$ meets the perpendicular bisector of side BC at point O. Join OB and OC. OB = OC as any point on the perpendicular bisector of a line segment is equidistant from the two ends of the line segment.

Now drop perpendiculars OY and OZ from O to AC and AB respectively. OY = OZ as any point on an angle bisector is equidistant from the arms forming the angle.

Now consider ΔBOZ and ΔCOY. We have OB = OC, OZ = OY, and $\measuredangle BZO = \measuredangle CYO$ (right angles).

So ΔBOZ ≅ ΔCOY (RHS congruency), and therefore BZ = CY. But we have AZ = AY and so AB = AC.

Similarly we could show that AC = BC and thereby claim that ΔABC is equilateral.

Keywords: Pure geometry, fallacy, proof

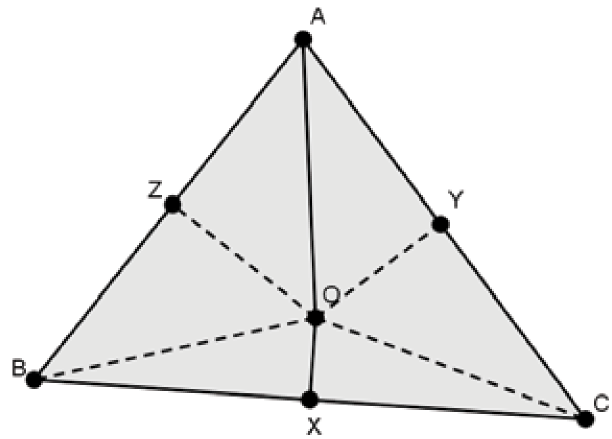


Figure 1

Problem VIII–1–M.2

Figure 2 shows two triangles ΔABC and ΔPQR. AB = PQ, AC = PR, and $\measuredangle BAC$ and $\measuredangle QPR$ are supplementary.

Give a geometric proof that the triangles are equal in area.

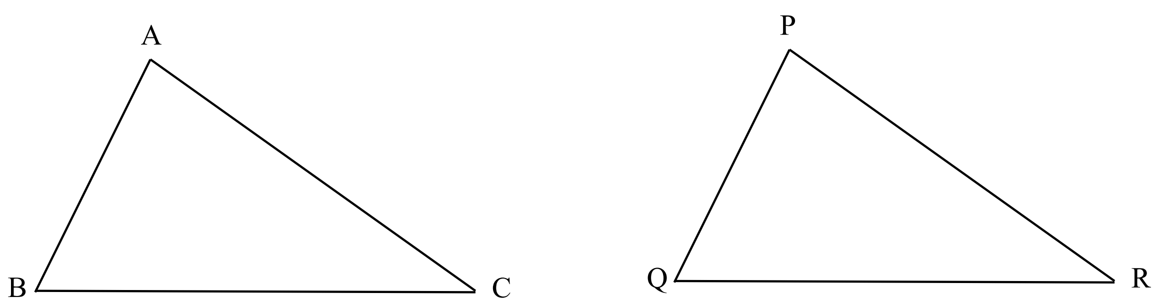


Figure 2

Problem VIII–1–M.3

Figure 3 shows a rectangle composed of three squares with some additional lines drawn. Give a geometric proof that $\measuredangle EAD + \measuredangle EBD = \measuredangle ECD (= 45^\circ)$.

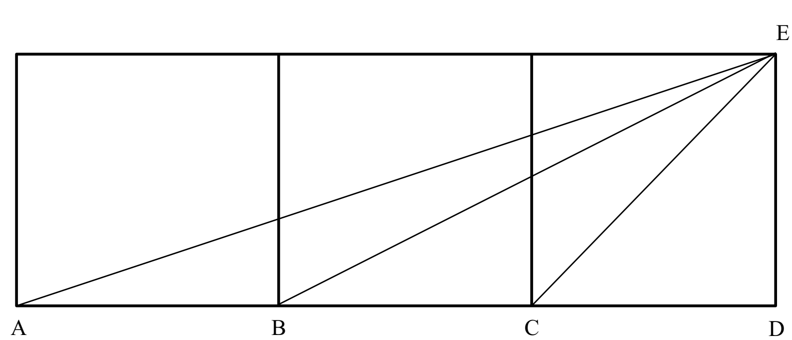


Figure 3

Problem VIII–1–M.4

Figure 4 shows three semicircles on a shared base and on the same side of it. The sum of the diameters (or radii) of the two smaller semicircles equals the diameter (or radius) of the largest semicircle ($BA + AC = BC$). We can consider the radius of the largest semicircle to be unity and that of one of the smaller to be r, while the radius of the other is $1 - r$. AH is drawn perpendicular to BC, with H on the largest semicircle. HB intersects one of the smaller semicircles at D, while HC intersects the other at E. DE intersects AH at O.

Prove the following:

(a) The area of the circle with diameter AH equals the area of the region enclosed by the three semicircles (shaded blue).

(b) AH and DE are equal in length and bisect each other.

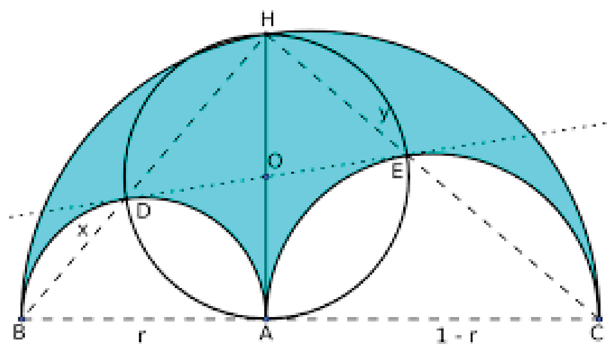


Figure 4

Problem VIII–1–M.5

The angles at the 5 corners of a pentagram (5-pointed star) total to 180°. This is easy to prove for a symmetrical figure. The central part is a regular pentagon. Knowing its interior angle to be 108° one can see that each base angle of the isosceles triangles is 72° and hence calculate the angle at each apex to be 36°. So the 5 apex angles add to 180°. The question here is to prove the same for any (asymmetrical) pentagram.

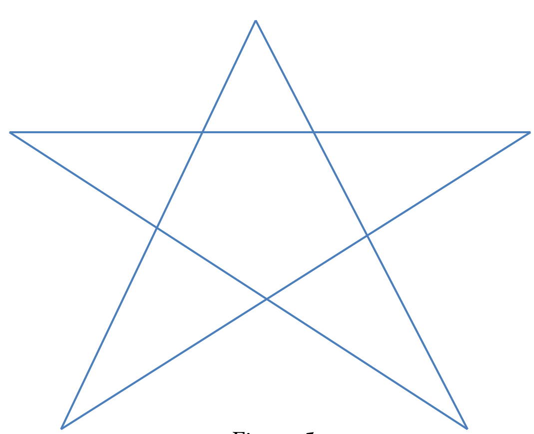


Figure 5

Solutions

Problem VIII-1-M.1

The fallacy lies in the assumption that the bisector of ∡A and the perpendicular bisector of BC meet inside the triangle. If AB ≠ AC, they meet outside the triangle. This can be seen if we note that the bisector of ∡A divides BC in the ratio AB : AC. Hence if AB < AC say, then bisector of ∡A will meet BC at a point closer to B than to C. When extended it will meet the perpendicular bisector of BC at point O outside the triangle. Do make an actual construction and observe the following (Figure 1.1):

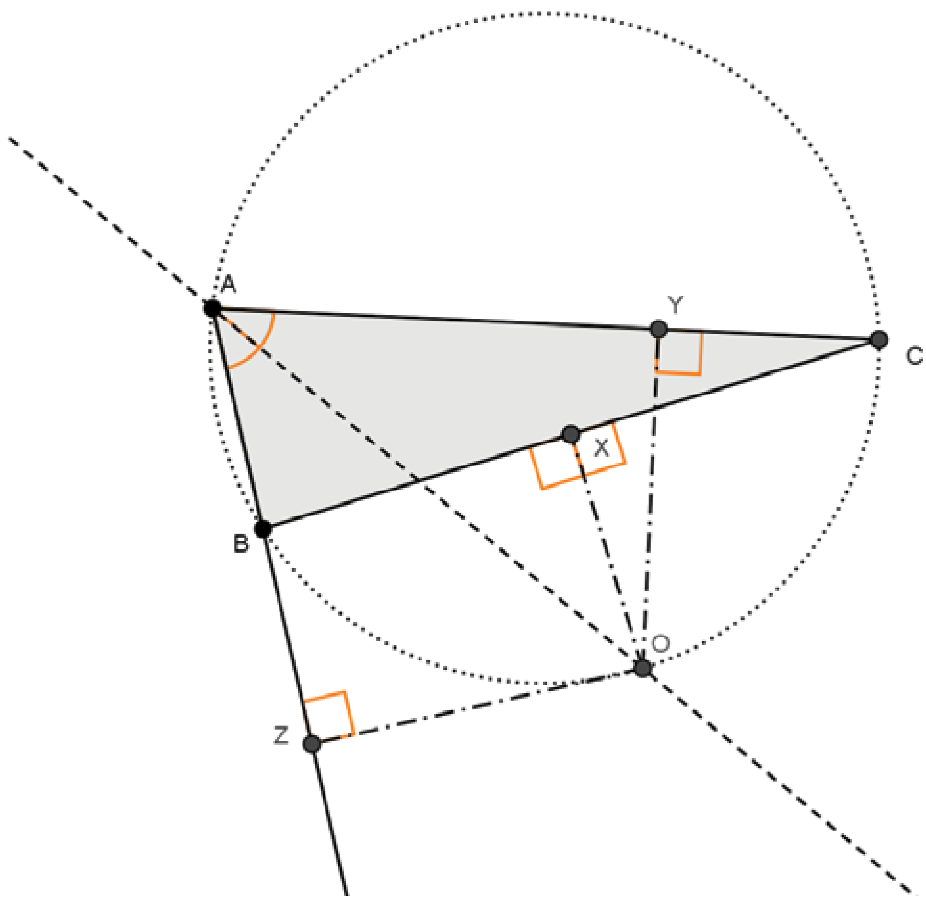


Figure 1.1

Point O lies on the circumcircle of ΔABC.

Point Y lies on AC while point Z lies on AB extended.

Congruent triangles COY and BOZ do get formed but we have AC − CY = AB + BZ.

If AB = AC, the two lines merge into a single line of symmetry.

Figure 1.1 does not represent a valid geometrical situation and therein lies the fallacy.

Additional comment from the editor. The reader may wish to explore with this configuration further, as there is a subtle issue involved. What we need to show is that (as in Figure 1.1), it can never happen that both Y and Z lie in the interiors of AC and AB respectively, or that both lie on the extensions of these sides respectively. It will always happen that one of the points lies in the interior of its respective side and the other point lies on the extension of the side. We need to show that this will *always* be the case. (If a situation occurred when both Y and Z lie on the extensions of their respective sides, the original reasoning would continue to work and the same fallacy would result.) We invite the reader to explore further.

Problem VIII-1-M.2
Figure 2 shows two triangles ΔABC and ΔPQR. AB = PQ, AC = PR, and $\measuredangle BAC$ and $\measuredangle QPR$ are supplementary.

Give a geometric proof that the triangles are equal in area.

Solution. To prove that ΔABC and ΔPQR are of equal area we carry out the following construction. Extend CA to point D such that AC = AD (Figure 2.1). Join BD.

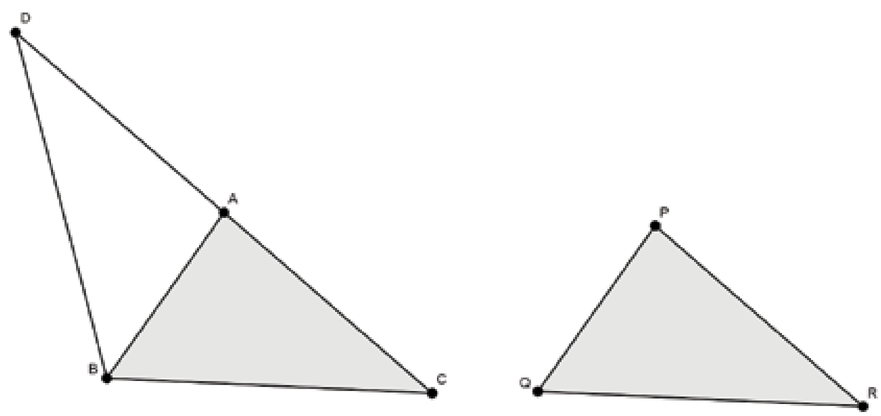


Figure 2.1

Considering ΔABD and ΔPQR, we have

$$AB = PQ,\ AD = AC = PR \text{ and } \measuredangle DAB = 180^\circ - \measuredangle BAC = \measuredangle QPR.$$

Hence the triangles are congruent. Since BA is a median of ΔBCD, ΔABC, ΔABD and therefore ΔPQR are of equal area.

Problem VIII-1-M.3
Figure 3 shows a rectangle composed of three squares with some additional lines drawn. Give a geometric proof that $\measuredangle EAD + \measuredangle EBD = \measuredangle ECD (= 45^\circ)$.

Solution. We take the squares in Figure 3 to have sides of unit length. In ΔECA the side lengths, in increasing order, are $\sqrt{2}$, 2, $\sqrt{10}$, while the side lengths of ΔBCE are 1, $\sqrt{2}$, $\sqrt{5}$. That is, CE/BC = AC/EC = AE/BE. As the corresponding lengths are in proportion, the triangles are similar and the angles opposite corresponding sides are equal. Hence ∡EAC = ∡BEC. Therefore ∡EAD + ∡EBD = ∡ BEC + ∡EBD = ∡ECD = 45°.

Problem VIII-1-M.4
Figure 4 shows three semicircles on a shared base and on the same side of it. The sum of the diameters (or radii) of the two smaller semicircles equals the diameter (or radius) of the largest semicircle (BA + AC = BC). We can consider the radius of the largest semicircle to be unity and that of one of the smaller to be r, while the radius of the other is $1 - r$. AH is drawn perpendicular to BC, with H on the largest semicircle. HB intersects one of the smaller semicircles at D, while HC intersects the other at E. DE intersects AH at O.

Prove the following:

(a) The area of the circle with diameter AH equals the area of the region enclosed by the three semicircles (shaded blue).

(b) AH and DE are equal in length and bisect each other.

Solution.

(a) It is a well-known result that the length AH is the geometric mean of the lengths BA and AC. You may be aware that the geometric mean of two positive quantities a and b is $\sqrt{ab}$. As mentioned earlier, we take the radius of the largest semicircle to be unity and the radius of the semicircle BDA to be r. Then length BA $= 2r$, while length AC $= 2(1-r)$. Then AH $= \sqrt{[2r \cdot 2(1-r)]} = 2\sqrt{r(1-r)}$. The area of the circle with AH as diameter would then be $\pi r(1-r)$.

Area of region enclosed by the three semicircles is area of large semicircle diminished by sum of areas of the other two, which is $\frac{\pi}{2} - \frac{\pi}{2}(r^2 + (1-r)^2)$ which simplifies to $\pi r(1-r)$.

(b) $\measuredangle$BHC, $\measuredangle$BDA and $\measuredangle$AEC are all right angles being angles in semicircles, making quadrilateral HDAE a rectangle. Hence HA and DE are equal and bisect each other.

Figure 4 was studied by the ancient Greeks. Archimedes named it "Arbelos," which is the Greek word for the 'shoemaker's knife,' which the figure resembles.

Problem VIII-1-M.5
The angles at the 5 corners of a pentagram (5-pointed star) total to 180°. This is easy to prove for a symmetrical figure. The central part is a regular pentagon. Knowing its interior angle one can calculate the angle at the apex of any of the triangles in the figure. The question here is to prove the same for any (asymmetrical) pentagram.

Solution. One way to prove the above statement is to note that the 10 angles at the bases of the 5 triangles form two sets of 5 exterior angles of the central pentagon. So they add up to $2 \times 360° = 720°$. The sum of all the angles of the 5 triangles would total $5 \times 180° = 900°$. So the apex angles should add up to 180°. An alternative proof is given below as a self-explanatory figure.

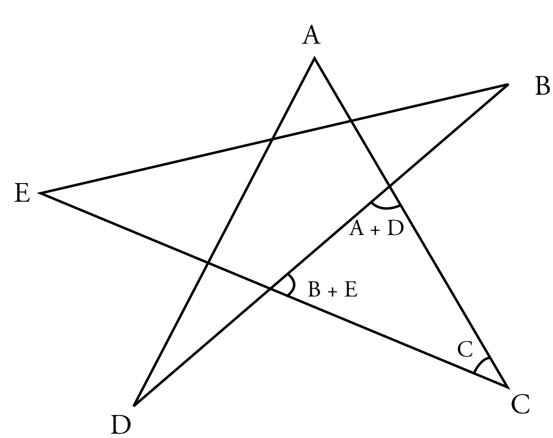


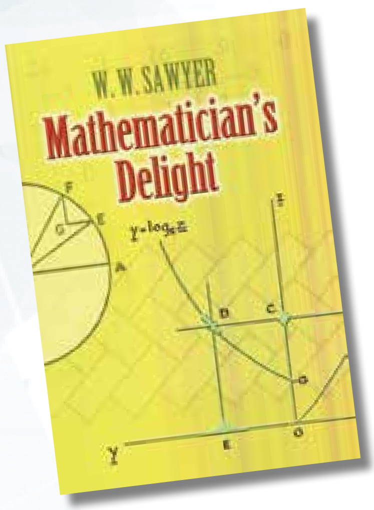
Figure 6

Acknowledgements. Figure 4 is taken from Wikipedia.

Review

Book Review of W.W. Sawyer's Mathematician's Delight

Reviewed by Mohan. R



It was during lunch on a pleasant day that I was told about this popular book on Mathematics called "Mathematician's Delight." I was chatting with a professor who said that his choice to become a mathematician was influenced by this book. The story went like this: When the professor was a teenager, just after high school, during the summer vacation, he found this book and wanted to give it a try. He could follow most of it without much difficulty, and solved most of the exercises which led him to 'experiment' with mathematical ideas on his own. When he started his higher secondary school, there was a problem in the mathematics textbook asking him to prove a formula for the sum of the squares of the first n natural numbers using mathematical induction. He neither knew what mathematical induction was, nor how to use it to prove the formula. Yet he knew how to find the formula on his own; not just for the sum of the squares of the first n numbers, but also for sums of cubes, or of fourth powers and so on. He was using the calculus of finite differences that he had learnt from a chapter in the book. This particular incident, he said, made him decide to be a mathematician. I wondered what was in the book that made a teenager not just understand mathematics, but realise that doing mathematics is a rewarding experience.

The book was first published in 1943 by Walter Warwick Sawyer, and is still in print. When I had the book in my hands I had many questions. How is a book written during World War II still relevant today in the age of information? Is the language easy to follow? Who was the intended audience? Were the topics covered relevant (say to a school teacher)?

Most popular mathematics authors try to educate the public on how much fun mathematics can be, on amusing mathematical facts/tricks or on adventures of mathematicians. It is assumed that the reader is already motivated enough to learn these. But

Keywords: mathematization, abstraction, applications, curiosity, commonsense, beauty

that is not often the case. Most of us are afraid of mathematics and we tend to mostly *read about mathematics* than *learn actual mathematics*.

W.W. Sawyer, however, distinguishes himself from the rest. He is known for the mathematics books he authored and his views on how the subject should be taught. His writings seem to arise from a deep understanding of how we learn. He believes,

> *"Education consists in co-operating with what is already inside a child's mind. The best way to learn geometry is to follow the road which the human race originally followed: Do things, make things, notice things, arrange things, and only then reason about things." (page 17)*

In Mathematician's Delight, in addition to 'what and how', he spends quite a lot of time on 'why'. The book is divided into two parts. Part 1 is on 'the approach to mathematics' and Part 2 is on 'certain parts of mathematics'. Part 1 consists of four chapters on how mathematics should be approached and his reasons for doing so. In Part 2, he reconstructs much of school level mathematics along with some advanced topics. Throughout the book, he adopts a questioning approach and engages the reader in mathematical thinking. The topics are built upon the readers' experience of real life situations to which mathematics may be applicable. By starting at the level of simple arithmetic and algebra and then proceeding step by step through graphs, logarithms and trigonometry to calculus and the dizzying world of imaginary numbers, the book makes an effort to constructively encounter the mystery in math.

In Part 1, the author argues that the fear of mathematics is felt not due to the nature of the subject, but due to how it is taught. He gives a brilliant example of how learning is reduced to imitation in a scenario of a hearing and speech impaired child learning to play the piano: the child would have learned the imitation of music without being able to appreciate the music. It makes a point of why there is a need for revolt against the tradition of dull education. In a chapter on geometry he illustrates how most mathematical ideas occur by putting common sense into action rather than by a stroke of genius.

To illustrate with an example, Sawyer argues how the first mathematicians would have obtained the first theorems and definitions on triangles: The Egyptians knew that a triangle of side lengths 3, 4, and 5 is always a right triangle. But they never bothered to know 'why?' For them it was a god given fact and that fact was useful for building pyramids. But when Greeks learnt the fact from Egyptians, they wondered 'why 3, 4 and 5? why not 7, 8 and 9? What does happen if one tries any three numbers?' This led them to experiment. It would be natural to take three numbers and see what happens to the triangle with those numbers as sides. One would start with fairly small numbers such as (1,1,1), (1,1,2), (1,1,3), (1,2,2), (2,2,2) and so on. As soon as one starts experimenting, they would start to discover some 'facts'. For instance it is impossible to make triangles when one side is bigger than the sum of the other two sides - a simple observation! Hence it can be concluded immediately that $(1, 1, n)$ cannot form a triangle if $n > 1$ - this is just common sense (here n is a natural number). Comparing the triangle of sides (2,2,2) or (3,3,3) with that of (1,1,1) one gets the idea of similar triangles. This way, instead of giving a list of definitions and then another list of examples and theorems, he lets the reader rediscover them more naturally through experiments.

He also discusses why reasoning and imagination play an important role in mathematical thinking. He illustrates the meaning and stages of mathematization (he uses the word abstraction). This vividly brings out how mathematicians first have to work with their 'hands' before working with their 'minds'. To illustrate this he brings up the definition of a straight line. The words straight and line come from Old English for 'stretched' and 'linen' respectively. The first mathematicians were practical men (like carpenter and builders) and they made practical use of straight lines. Yet Euclid's definition of straight line says they have no thickness. It is mathematization of the already existing idea of how to make straight lines. In laying out a table or in building a house, one is not interested in the size of the rope. Hence, to define straight line, neglect the thickness in order to keep the subject reasonably simple. In the last chapter of Part 1, he acts as a guide and gives a lot of practical advice on how to learn mathematics with available resources. Particularly, he spends a good amount of time explaining how reading around mathematics is better than reading from textbooks. In fact he lists quite a lot of books on different subjects which could be used before learning mathematics directly.

By the end of Part 1, the reader is well prepared to face the subject confidently. Although very engaging, the prose is dense and the author's way of wandering into philosophy might not suit everyone. If the reader wants to dive right into the subject it is recommended that they start the book right away from Part 2, returning to Part 1 whenever it is convenient (except maybe for chapter 2 which deals with geometry).

In Part 2, various topics at school level (and some at an advanced level) are dealt with in detail. Sawyer rightly justifies the claims he makes in Part 1. The reader begins to wonder why her school teacher did not teach like this. What is unusual is the order in which the topics are organized. There is arithmetic, followed by logarithms and algebra, then comes the calculus of finite differences (which is not dealt with at school level), followed by graphs, calculus and then trigonometry, and finally series and complex numbers. There is a clear emphasis on building the new on what is existing already. Thus, for example, the burden of learning all of trigonometry before calculus is removed.

Three distinctive features of the book are (i) developing any mathematical idea in the most 'natural' form, using 'common sense,' (ii) providing the reader with enough hands-on experience that the abstraction follows naturally, and (iii) a great selection of doable exercises interspersed throughout the book that creates interest to do mathematical experiments.

Here are a few experiments from the book:

1. *If 7 teams enter for a knock-out competition how many matches will have to be played?* The author helps the reader to arrive at a problem solving strategy and then poses the following question: *replace 7 by 2,176,893 (or by any n).*
2. In chapter 6, the author explains in detail how slide-rules were invented and then asks the reader to make one.
3. In the chapter on Trigonometry, a model is described to demonstrate the meaning of sine and cosine. Then the reader is asked to *make an actual model from the design and then make a table giving the sines and cosines of 5º, 10º, ..., upto 90º, to two decimal places and check the results from printed tables.*

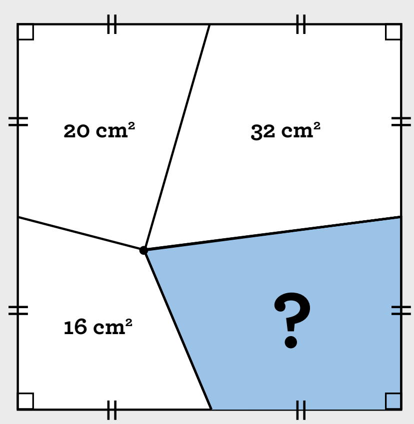
Rather than trying to explain mathematics in a conventional manner, Sawyer engages with those of us who did not get to appreciate the beauty of mathematics in a school classroom by using what was then (and still is, to an extent) a revolutionary approach to explaining maths: tell the student what the problems will be used for, and offer concrete examples, before explaining the mechanics of the concepts. Here is an example from the book: In making a motor headlamp or a searchlight it would be inconvenient to have an electric lamp which spreads out light equally in all directions. One would prefer to have all the light coming out in one direction. It can be achieved by placing a reflector behind the lamp. What shape should the reflector be, if the reflected light is to come out in a perfect beam? Then he explains how to arrive at the answer (parabola) to the above question, in step by step fashion.

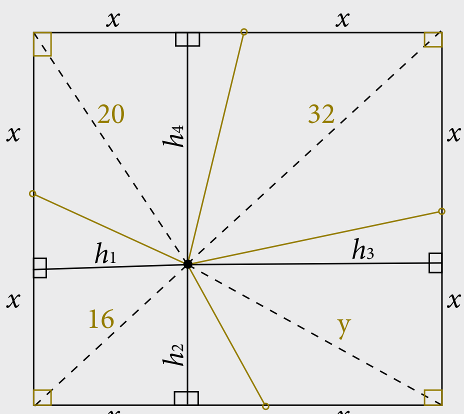
However, the book feels a bit outdated as the world has changed enough to feel like a very different place today. The use of imperial units and references to slide rules makes it a bit hard to digest for a modern reader. Hence some of the practical examples and analogies were less helpful then they presumably once were, and the words used to explain them are stylistically unusual. Most of the examples he uses are in the context of war and there is a subtle glorification of war. To illustrate a few: to introduce negative numbers the author uses examples of a bomb falling into a sea and of an army retreating; to introduce logarithms he supposes a situation in which you are fire-watching on a roof, and have to lower an injured comrade by means of a rope. One starts to wonder if the book was written in and for a battlefield!

Overall the book realizes the objective it originally set to achieve: to dispel the fear of mathematics, and to convince the reader that mathematics is a tool for thinking and a language for common sense. Though this book could be advertised as a popular mathematics book, it is actually a book on mathematics education. It is specifically designed to address the issue of putting mathematics in real-life contexts. More than a student, a teacher would benefit immensely from reading this book. The book is truly a delight to read!

MOHAN. R is a doctoral student at Indian Statistical Institute, Bangalore. He is also a media editor for *Bhāvanā* mathematics magazine and runs the YouTube channel *Math Nomad*. His other interests are mathematics education, photography, painting, and bird watching. He may be contacted at rmohan689@gmail.com.

Solution to the puzzle on page 110 of the November 2018 issue





$$20 = \frac{1}{2}(h_4 \times x) + \frac{1}{2}(x \times h_1) \quad \text{—— (1)}$$

$$16 = \frac{1}{2}(h_1 \times x) + \frac{1}{2}(h_2 \times x) \quad \text{—— (2)}$$

$$32 = \frac{1}{2}(x \times h_4) + \frac{1}{2}(x \times h_3) \quad \text{—— (3)}$$

$$y = \frac{1}{2}(x \times h_2) + \frac{1}{2}(x \times h_3)$$

$$48 = \frac{1}{2}(h_1 \times x) + \frac{1}{2}(x \times h_4) + y$$

$$48 = 20 + y$$

$$\boxed{y = 28}$$

Submitted by Hiren Makwana

In the TearOut for the November 2018 issue (page 43 Fun with Dot Sheets) Swati Sircar had ended with a tantalizing observation and the promise of a proof. The observation was: Note that one type of triangle (with all three vertices on lattice points) is not possible. This is the equilateral triangle - An equilateral triangle can't be drawn with lattice points as vertices. Elsewhere in this issue, Shailesh Shirali has provided four proofs of the same result. Here is one more from Swati!

We will do this in two parts:

1. Triangle with one side horizontal or vertical
2. Triangle with no side horizontal or vertical

   a. This is quite easy using Pythagoras and the fact that for an equilateral triangle with base b and height h, $h = \left(\frac{\sqrt{3}}{2}\right)$ b. Let's assume without loss of generality that one side is horizontal. Again without loss of generality we can take the vertices on this side to be $(-n, 0)$ and $(n, 0)$ where n is a natural number. So the 3rd vertex will be of the form $(0, m)$ where m is a natural number and is the height of the triangle. So $m = \frac{\sqrt{3}}{2} \times 2n = \sqrt{3}n$ which is irrational. Similarly, if one side is vertical we can take the vertices as $(0, n)$ and $(0, -n)$ and $(m, 0)$ and again get $m = \sqrt{3}n$.

   So it is not possible to locate the third vertex on a lattice point if one side is either horizontal or vertical.

   b. If an equilateral triangle does not have any side horizontal or vertical, then each side is slant i.e. each has a non-zero slope. So the triangle can be translated along the axes if needed to get two of its vertices lie of the two axes. We can take these two vertices to be M = $(m, 0)$, N = $(0, n)$ and let the 3rd vertex be L (x, y). Since we are talking about lattice points, m, n, x and y are non-zero integers. Let P be the midpoint of MN i.e. P = $\left(\frac{m}{2}, \frac{n}{2}\right)$. Now, LP ⊥ MN and LP = $\frac{\sqrt{3}}{2}$ × MN. This gives us two equations:

   $$\frac{y - \frac{n}{2}}{x - \frac{m}{2}} = \frac{m}{n} \Rightarrow y - \frac{n}{2} = \frac{m}{n}\left(x - \frac{m}{2}\right) \;\ldots\; (1) \text{ and}$$

   $$\left(x - \frac{m}{2}\right)^2 + \left(y - \frac{n}{2}\right)^2 = \frac{3}{4}\left(m^2 + n^2\right) \;\ldots\; (2)$$

   Using (1) and substituting for $y - \frac{n}{2}$ in (2) and simplifying, we get $\left(x - \frac{m}{2}\right)^2 = \frac{3}{4}(n^2)$

   $$\Rightarrow x = \frac{m}{2} \pm \left(\frac{\sqrt{3}}{2}\right)n \text{ and therefore } y = \frac{n}{2} \pm \left(\frac{\sqrt{3}}{2}\right)m$$

   Since m and n are non-zero, x and y are irrational. Therefore L can't be a lattice point regardless of our choice of m and n.

   It is interesting to note the symmetry in the coordinates of an equilateral triangle of this kind $(m, 0)$, $(n, 0)$ and $\left(\frac{m}{2} \pm \left(\frac{\sqrt{3}}{2}\right) n, \frac{n}{2} \pm \left(\frac{\sqrt{3}}{2}\right) m\right)$.

The Closing Bracket . . .

Looking around at world events, reading the newspapers, watching the TV news, browsing through news portals, we see a continuing dissonance in human society. Divisions are on the increase all across the world. The threat of devastating climate change is ever on the increase. A general sense of anger among people is becoming ever more visible, especially in the Western countries. All across, our treatment of nature and animals continues to be abysmal. The phenomenon of fake news – of believing what one wants to believe regardless of the truth – is spreading at frightening speed, all over the world. In our own country, caste and religious divisions are deepening and hardening. In every direction, we get a sense of things falling apart.

On the other hand, we have the miracles of mathematics and the sciences and technology and yoga and medicine and surgery and poetry and classical music. These are products of our creativity, just as much as the deep divisions and the despairing conflicts across the world. How can such contrary phenomena exist side-by-side, both born within us? What relationship is there between these two phenomena? How are we able to live with so great a dissonance in our lives? Does it not indicate a serious schism in us? What is our responsibility in this regard, as mathematics teachers? What are we to do if we are to address this crisis in human consciousness?

Perhaps the phenomena are much too large for us to do very much. But in our individual classes, in our relationship with children, in our relationship with colleagues, we can surely have talk about these matters, we can create an awareness of the schism, we can nurture a feeling for truth and beauty that extends beyond the confines of our subject, we can nurture a love for goodness, a love for wholeness and integrity. As mathematics teachers, should we not be talking about such matters? Or is our focus exclusively on the content of mathematics? If so, I fear that we will be failing in our work.

But in order to do this, mere talk is not enough; one needs to be very honest, one needs to be keenly aware of the schisms and contradictions that lie within, one needs to want to discover the truth of these matters for oneself.

Of late, we see a few signs of hope. There is much greater interest in teaching now than a little while back, a much greater interest in citizens' initiatives and a feeling that “it is our responsibility, our work, we need to do something about it”. One sees this also in the field of environmental protection and care. Let us pray that these initiatives live on and become self-sustaining. But in themselves, they are not enough, for the crisis is much too deep. It demands action from each one of us.

We are living in extraordinary times, facing an unprecedented crisis.

Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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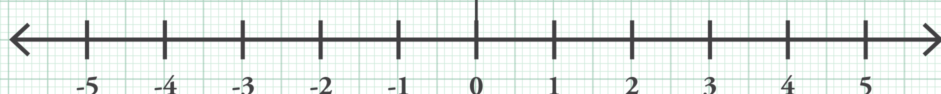
At Right Angles

A Resource for School Mathematics

INTRODUCTION TO ALGEBRA - IV

PADMAPRIYA SHIRALI



Azim Premji University

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INDICES AND IDENTITIES

This is the fourth article in the series, *Introduction to Algebra*. The first one looked at algebra as a 'pattern language' and focused on noticing patterns and expressing patterns using terms and expressions. The second one looked at algebra as a 'design language' and focused on expressing designs using algebraic language. The third one explored solving simple equations using the 'balance approach' and the 'machine approach.' However, the topic of equations is vast and gets steadily more complex as the student progresses through higher algebra. Similarly, the topic of solving and simplifying expressions of higher degree steadily grows in complexity.

The precursor to simplification, factorisation and expansion of expressions of higher degree is a study of the laws of indices and basic identities. The index laws come into play while using large numbers as well, which are often written in scientific notation and involve powers. When these numbers are multiplied, divided or raised to a power, the laws of indices are applied.

In this article, we study indices (only positive whole number indices) and basic identities. The article does not go into the application of these concepts in problem situations.

Note: While getting students to learn concepts, it is crucial to cover only one aspect at a time, build it up gradually and not bring in several concepts or variations all at once.

ACTIVITY 1

Objective: Expressing higher powers of 10 in exponential form
Purpose of writing large numbers in a compact form.

Prerequisites:

- Familiarity with large numbers extending to millions and crores.
- Prime factorisation.

Let students bring some data collected from newspapers or books (atlas, geography book) which makes use of large numbers (rounded to a suitable multiple of a power of 10).

Check if they are comfortable reading and writing large numbers.

Select some large figures to write on the board and discuss the difficulty of writing and reading such numbers.

Example: The Sun is at a distance of **150,000,000** km from Earth.

Proxima Centauri, the closest star to our Sun, is **40,208,000,000,000** km away.

Show them how to represent these numbers using powers.

1,00,000 can be written as
$10 \times 10 \times 10 \times 10 \times 10$. This can be written as 10^5.

It is highly important to read it out aloud as '10 raised to the power of 5'.

Students have all along been used to seeing a sign between two numbers. This is their first encounter with two numbers where there is no explicit sign. It is going to take some time for the students to internalise this representation and interpret it correctly.

20,00,00,000 can be written as
$2 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$.

This can be written as 2×10^8. Read it out as '2 times 10 raised to the power of 8'. Point out that it is not the same as $(2 \times 10)^8$.

What would this expand into?
$20 \times 20 \times 20 \times 20 \times 20 \times 20 \times 20 \times 20$.

4,000,000,000 can be written as
$4 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$.

This can be written as 4×10^9. Read it out as '4 times 10 raised to the power of 9'.

Note: Formal words like *base* and *index* can be brought in a little later.

ACTIVITY 2

Objective: To contrast $a \times 3$ and a^3

Let students study the following pattern and describe it using pattern language.

- $5 + 5 + 5 = 15 = 3 \times 5$
- $2 + 2 + 2 = 6 = 3 \times 2$
- $7 + 7 + 7 = 21 = 3 \times 7$
- $a + a + a = 3a = 3 \times a$

Now let them study this pattern.

- $5 \times 5 \times 5 = 125$
- $2 \times 2 \times 2 = 8$
- $7 \times 7 \times 7 = 343$
- $a \times a \times a = ???$ (Are the students able to say what will come here?)

Let students state the difference that they notice between these two patterns. The first one is an additive relationship where a quantity is repeatedly added to itself, whereas the second one is a multiplicative relationship where a quantity is repeatedly multiplied by itself.

The teacher can then show the standard form of writing and reading it.

- $5 \times 5 \times 5 = 125 = 5^3$ is read as '5 raised to the power of 3'.
- $2 \times 2 \times 2 = 8 = 2^3$ is read as '2 raised to the power of 3'.

- 7 × 7 × 7 = 343 = 7^3 is read as '7 raised to the power of 3'.
- a × a × a = a^3 is read as 'a raised to the power of 3'.

Caution: Evaluating 4^3 as 4 × 3 is a very common mistake that occurs in the initial stage. This is because the students have not fully internalised the meaning of a^3.

Let there be plenty of oral exercises which reinforce the meaning of 'to the power of'.

Thirty second exercises which require mental calculations can be used to calculate powers of small numbers.

Teacher says '3 to the power of 4'. Students have to give the value within thirty seconds.

Teacher says '64'. Students have to respond using the exponential form, '2 to the power of 6' or '4 to the power of 3' or '8 to the power of 2.'

Visualising indices

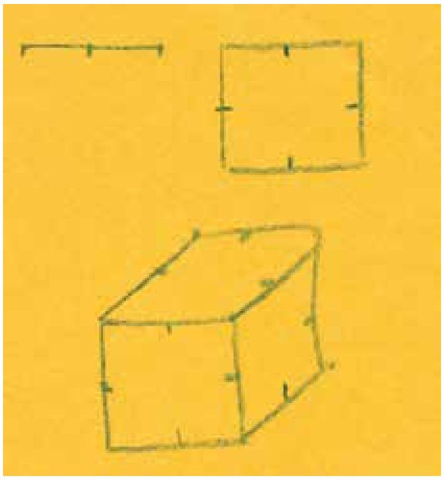

Figure 1

2^1 can be thought of as a line of length 2.

2^2 can be thought of as a square of side 2.

2^3 can be thought of as a cube of edge 2.

How does one visualise 2^4?

Here is one way!

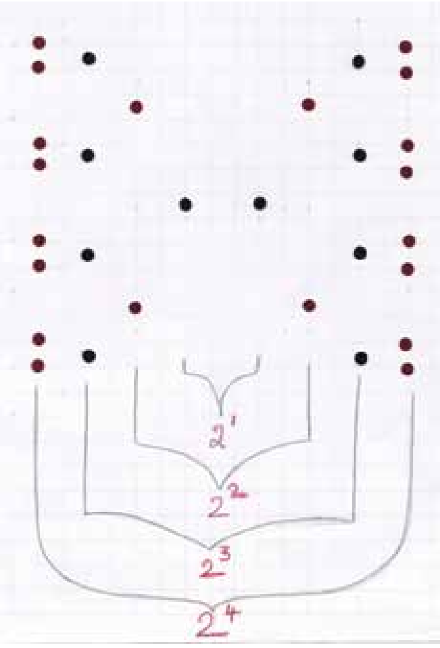

Figure 2

ACTIVITY 3

Objective: Exploration of numbers using indices

Prior knowledge: Prime factorisation

Ask students to find a number which can be expressed using indices in two ways.

Example: $16 = 2^4 = 4^2$.

How many such numbers can be found between 1 and 100?

Ask students to find a number which can be expressed using indices in three ways.

How many such numbers can be found between 1 and 100?

Which number between 1 and 200 can be so expressed in the largest number of ways?

Can the students pose more such challenges for themselves?

Is there any pattern to be found in such an exploration?

Game 1

Objective: Practice of index notation

Materials: Set of 16 cards

Prepare a set of 16 cards (8 matching pairs) where each pair consists of a number in index form and another its value.

(**Example:** 2^4, 16, $4 \times 4 \times 4$, 2^6, $3 + 3 + 3$, 3^2, 4^2)

The set of cards can now be placed face down and played as a memory game.

Note: Take care not to use power 0 until it has been properly introduced.

ACTIVITY 4

Objective: To study index laws when the base is common, $a^m \times a^n = a^{(m+n)}$

Let students study each of the following examples and discover the law.

Ex. 1. What does $5^3 \times 5^2$ become?

5 repeated as a factor 3 times × 5 repeated as a factor 2 times = 5 repeated as a factor (3 + 2) times.

$(5 \times 5 \times 5) \times (5 \times 5) = 5 \times 5 \times 5 \times 5 \times 5$

$5^3 \times 5^2 = 5^{(3+2)} = 5^5$

Ex. 2. What does $2^4 \times 2^5$ become?

Ex. 3. What does $3^4 \times 3^3$ become?

Let the students note down the result as given below, notice and express the pattern in a generalized manner.

- $5^3 \times 5^2 = 5^{(3+2)}$
- $2^4 \times 2^5 = 2^{(4+5)}$
- $3^4 \times 3^3 = 3^{(4+3)}$
- $a^m \times a^n = a^{(m+n)}$

Caution: Students often make mistakes in applying this law.

They need to be very clear that $a^m \times a^n$ is **not** equal to $a^m + a^n$.

It is important to state what $5^3 \times 5^2$ represents, as has been done earlier (5 repeated as a factor 3 times multiplied by 5 repeated as a factor 2 times = 5 repeated as a factor (3 + 2) or 5 times).

This can be done in the context of several examples, to avoid the above kind of mistake.

The teacher needs to help students to focus on the factor being repeated and the number of times it is being repeated.

Another point which needs to be made clear is that $a^m + a^n$ is not equal to $a^{(m+n)}$.

- What approaches can help to minimise these errors?

- Would visuals help? Would drill exercises help? Would true or false questions help?

I have found it useful to make a collection of such common errors and ask groups of students to discuss these errors among themselves, one type of error per group, and then share what was discussed with the rest of the class, giving reasons for its incorrectness.

Students can also verify such results by computations with small values.

Note: Students can now be introduced to the words *base* and *index*.

The **base** is the quantity that is repeated.

The **index** or **power** shows the number of times that quantity is repeatedly multiplied by itself. If it becomes confusing for a student to use two different words for the same thing, the teacher can stick to one of them.

At whichever point one introduces *coefficient*, care should be taken to see that the students do not confuse *coefficient* with *index*.

ACTIVITY 5

Objective: To study index laws when the base is common: $a^m \div a^n = a^{(m-n)}$

Let students study each of the following examples and discover the law.

Ex. 1. What does $4^5 \div 4^2$ become?

4 repeated as a factor 5 times ÷ 4 repeated as a factor 2 times is the same as 4 repeated as a factor (5 – 2) = 3 times.

$$\frac{(4 \times 4 \times 4 \times 4 \times 4)}{(4 \times 4)} = 4 \times 4 \times 4$$

$4^5 \div 4^2 = 4^{(5-2)} = 4^3$

Ex. 2. What does $7^8 \div 7^3$ become?

Ex. 3. What does $3^4 \div 3^3$ become?

Let the students note down the result as given below, notice and express the pattern in a generalized manner.

- $4^5 \div 4^2 = 4^{(5-2)} = 4^3$
- $7^8 \div 7^3 = 7^{(8-3)} = 7^5$
- $3^4 \div 3^3 = 3^{(4-3)} = 3^1$
- $a^m \div a^n = a^{(m-n)}$

Some students may wonder what happens if there is a higher or equal power in the denominator. The teacher can tell them that this would be taken up soon.

Caution: Again, watch out for common mistakes and make sure that students understand this.

$a^m \div a^n$ is **not** equal to $a^m - a^n$ and $a^m - a^n$ is **not** equal to $a^{(m-n)}$

Occasionally, students make errors like $3^4 \times 2^3 = 6^7$. It is important to go back to basics in such situations and explain the misconception.

Give examples as well as non-examples to demonstrate laws of indices.

For instance, it is not possible to apply the laws of indices to $3^4 \times 2^3$.

ACTIVITY 6

Objective: To study index laws when the index is common, $a^m \times b^m = (a \times b)^m$

Let students study each of the following examples and discover the law by themselves.

Ex. 1. What is $3^3 \times 5^3$?

- What is $2^4 \times 3^4$?
- What is $6^2 \times 4^2$?

Let the students write them in an expanded form.

- $3^3 \times 5^3 = 3 \times 3 \times 3 \times \ 5 \times \ 5 \times 5$
- $2^4 \times 3^4 = 2 \times 2 \ \times 2 \times 2 \times \ 3 \times 3 \ \times \ 3 \times 3$
- $6^2 \times 4^2 = 6 \times 6 \ \times 4 \times 4$

Using the associative and commutative laws, the first one can be rearranged as $3 \times 5 \times 3 \times 5 \times 3 \ \times 5$.

This can now be written as $15 \times 15 \times 15$ which is 15^3.

Let the students work out the other two in a similar manner.

- $3^3 \times 5^3 = 15^3$
- $2^4 \times 3^4 = 6^4$
- $6^2 \times 4^2 = 24^2$

Do the students see the pattern?

$$a^m \times b^m = (ab)^m.$$

ACTIVITY 7

Objective: To show $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$

Let students study each of the following examples and discover the law.

Ex. 1. What is $5^8 \div 2^8$?

- What is $3^7 \div 5^7$?
- What is $6^9 \div 2^9$?

Let the students write the expressions in an expanded form.

$$\frac{5^8}{2^8} = \frac{(5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5)}{(2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2)}$$

This can be written as

$$\frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} = \left(\frac{5}{2}\right)^8.$$

Similarly $\frac{3^7}{5^7}$ becomes $\left(\frac{3}{5}\right)^7$.

$\frac{6^9}{2^9}$ becomes $\left(\frac{6}{2}\right)^9$

We see that $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$.

ACTIVITY 8

Objective: To show $a^0 = 1$

Explanation 1:

This makes use of students' knowledge of fractions and the law $\frac{a^m}{a^n} = a^{(m-n)}$.

Ask students to work out the following problems using fractions.

- What is $\frac{3^4}{3^4}$?
- $\frac{(3 \times 3 \times 3 \times 3)}{(3 \times 3 \times 3 \times 3)}$
- What is $\frac{2^5}{2^5}$?
- What is $\frac{7^2}{7^2}$?

In each case, the students will notice that the answer is 1.

Now let them use the law $\frac{a^m}{a^n} = a^{(m-n)}$

Using that each of the above examples results in

- $\frac{3^4}{3^4} = 3^{(4-4)} = 3^0$
- $\frac{2^5}{2^5} = 2^{(5-5)} = 2^0$
- $\frac{7^2}{7^2} = 7^{(2-2)} = 7^0$

Using fractions, it has already been established that they are equal to 1.

Hence $3^0 = 1$

$2^0 = 1$

$7^0 = 1$

Generalising, we get $a^0 = 1$.

Explanation 2:

Let students write the values of each of these and notice the pattern.

- $2^5 = 32$
- $2^4 = 16$
- $2^3 = 8$
- $2^2 = 4$
- $2^1 = 2$
- $2^0 = ?$

Each successive answer is $\frac{1}{2}$ of the earlier answer.

- $\frac{1}{2}$ of 32 is 16.
- $\frac{1}{2}$ of 16 is 8.
- $\frac{1}{2}$ of 8 is 4.
- $\frac{1}{2}$ of 4 is 2.

Following the pattern, the next number needs to be $\frac{1}{2}$ of 2, which is 1.

Hence 2^0 is 1.

At a later point, the same approach can be used to handle negative indices, e.g., $2^{(-1)}$ or $3^{(-2)}$.

ACTIVITY 9

Objective: To extend the index laws to varied problems

The students' understanding and practice can be further enhanced by giving them problems of the following type to simplify.

- $a^l \times a^m \times a^n$
- $\frac{(a^c \times a^d)}{a^e}$
- $\frac{a^x}{(a^y \times a^z)}$

Varied problems involving comparison, sequencing and simplification can be given.

ACTIVITY 10

Objective: To extend the index laws for factors with coefficients.

Students need to be shown the difference between coefficients in this example.

Is $4x^2$ same as $(4x)^2$?

Is $-16x^2$ the same as $(-4x)^2$?

Proper usage of brackets and correct interpretation of the quantity being squared needs to be focused on.

ACTIVITY 11

Objective: To show the identity $(a + b)^2$

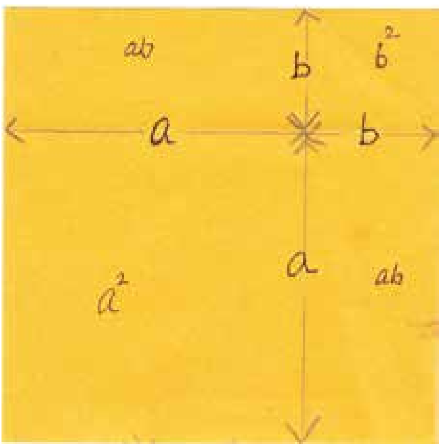

Figure 3

Ask students to take a square paper and fold them along the two indicated lines.

The two different lengths can be labelled as *a* and *b* as shown in Figure 3.

- What is the side of the original square? $a + b$
- What is the area of the original square? $(a + b)^2$
- What is the area of the big square? a^2
- What is the area of the small square? b^2
- What is the area of each rectangle? *ab, ab*
- What do they all sum up to? $a^2 + 2ab + b^2$

Hence $(a + b)^2 = a^2 + 2ab + b^2$

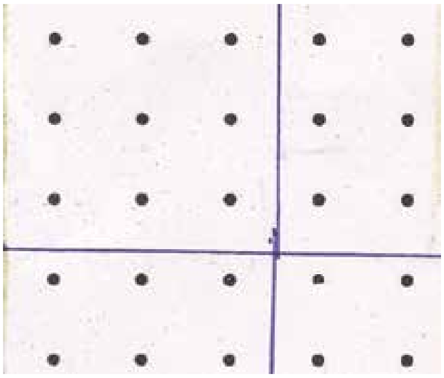

Figure 4

The teacher can also show this on a square dot paper as shown in Figure 4.

$$(3 + 2)(3 + 2) = 3 \times 3 + 3 \times 2 + 3 \times 2 + 2 \times 2$$

which is $(3 + 2)^2 = 3^2 + 2 \times 3 \times 2 + 2^2$

Again if $a = 3$ and $b = 2$ then $(a + b)^2 = a^2 + 2ab + b^2$

ACTIVITY 12

Objective: To show the identity $(a - b)^2$

This method works for positive numbers *a* and *b*, with $b < a$.

Ask students to take a square paper and fold them along the two indicated lines.

The two different lengths can be labelled as *a* and *b* as shown in Figure 5.

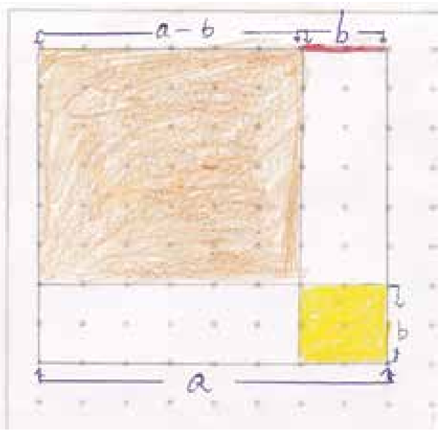

Figure 5

- What is the side of the original square? *a*
- What is the area of the original square? a^2
- What is the length of the portion which is being cut? *b*
- What is the length of the remaining part of the line? $a - b$
- What is the area of the square brown portion?

Each of its sides is $a - b$. Area of the brown square is $(a - b)^2$

- What is the area of the small yellow square? b^2
- What is the area of each outlined rectangle? *ab*
- Is it possible to remove both these two rectangles (of size *ab*) ?

Removal of two such rectangles will mean that b^2 will end up being removed twice.

In order to compensate for that we need to put back one b^2.

Hence $(a - b)^2 = a^2 - 2ab + b^2$

This can also be visualised as shown here.

×	a	$-b$
a	a^2	$-ab$
$-b$	$-ab$	b^2

ACTIVITY 13

Objective: To show the identity $a^2 - b^2 = (a + b)(a - b)$

This method works for positive numbers a and b, with $b < a$.

Ask students to take a square paper and fold them along the two indicated lines.

The two different lengths can be labelled as a and b as shown in Figure 6.

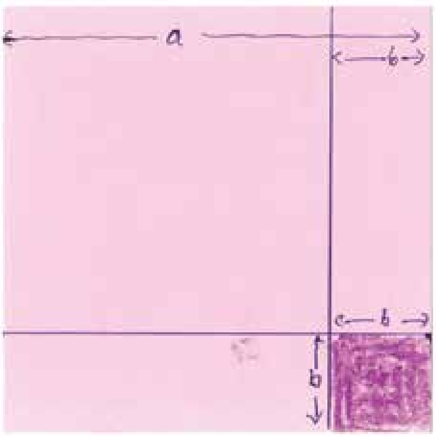

Figure 6

- What is the side of the original square? a
- What is the area of the original square? a^2
- What is the length of the portion which is being cut? b
- What is the area of the square purple portion? b^2

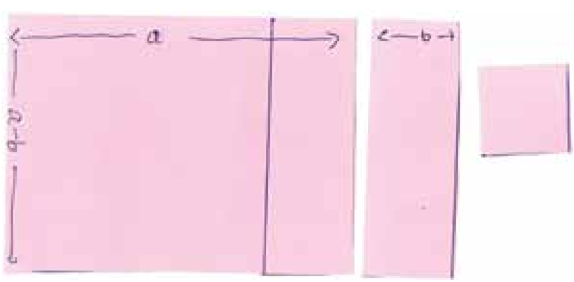

Figure 7

The remaining portions can be rearranged as shown in Figure 7.

- What is the length of the rearranged rectangle? $a + b$
- What is the breadth of the rearranged rectangle? $a - b$
- What is the area of this rearranged rectangle? $(a + b)(a - b)$

Hence $a^2 - b^2 = (a + b)(a - b)$

For further reading

Indices:

- http://www.teachersofindia.org/en/video/cloth-clips-powers-two
- http://www.teachersofindia.org/en/video/straw-powers-3
- http://www.teachersofindia.org/en/video/straw-powers-5
- http://www.teachersofindia.org/en/video/powers-2-3-and-5
- http://www.teachersofindia.org/en/video/exponential-identities
- https://www.youtube.com/watch?v=0fKBhvDjuy0

Identities:

- http://teachersofindia.org/en/presentation/algebraic-identities-visualized-one-more-time

With algebra tiles:

- http://teachersofindia.org/en/presentation/visual-proof-ab2-2ab-0
- http://teachersofindia.org/en/presentation/visual-proof-a-minus-b-whole-square
- http://www.teachersofindia.org/en/presentation/make-algebraic-proofs-visual-treat

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