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Azim Premji University
At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

ISSN 2582-1873

PULLOUT
PERCENTAGES

When will we reach???

This is the question that an adult travelling with a child has often had to handle. How? With puzzles, games, songs, scenery, stories……. Till the journey itself becomes so entertaining and absorbing that the destination seems a distant pleasure with more immediate rewards en route. Mathematical journeys are just the same, a distant destination defines the start, but along the path one sees others on similar journeys, interesting detours, engaging distractions and unlooked for results. And one develops the skill of observation, patience, endurance, discrimination, focus… and most of all simply the joy of the journey. As Robert Louis Stevenson says, *Don't judge each day by the harvest you reap, but by the seeds that you plant.*

From the Editor's Desk . . .

The writing of the editorial for the November 2019 issue had been on my list for several days but today I addressed the task with fresh impetus as the day began with the terrific news that *Azim Premji University At Right Angles* had received its ISSN Number (2582-1873). Another milestone in AtRiA's eight year journey since 2012.

Speaking of journeys, our issue, and cover, this time focus on a unique math journey that is both a destination and a path. Read all about the young Yatris and their journey of discovery in Features. This section also describes more abstract journeys from the *Regular Pentagon to the Icosahedron and the Dodecahedron (Part 2)* and from the familiar 2 and 3 dimensions to n dimensions in *Extension of the Pythagorean Theorem*.

We love it when an article sets readers thinking – and writing spin-offs on the same. And this time *Simple Cryptography* and *Triangles with Integer Sides* do just that. We hope that more of our readers do just that, there is rich fare in our ClassRoom section with *An Unusual Proof of the Centroid Theorem, Modified Pascal Triangle* and *Orthocentre of a Triangle* to name a few. You will find ideas in our regular columns – How to Prove It, Low Floor High Ceiling and TechSpace which features the simulation of a dice game this time. Incidentally, I enjoyed discovering hitherto undiscovered seeds of artistic ability in me with the article on Isometric Sketches in TearOut, do send us your students' creations when you administer this helpful worksheet in class. Readers have been sending in solutions to problems in Problem Corner to our delight. A few of these have been published in this issue.

Our issue is short of 100% without the PullOut, enjoy the learning trajectory on Percentages, defined and illustrated with plenty of examples in this issue. And I for one am queuing to buy *How Craig Barton Wishes he'd Taught Maths* after reading the excellent review by Sir Timothy William Gowers, reprinted from his Weblog.

Bon voyage!

Sneha Titus
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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an ‘academic’ and 'practitioner' oriented magazine.

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Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

LEGENDS - Primary School | Middle School | High School | Pre-University

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Contents contd.

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

A Mathematical Pilgrimage: RAMANUJAN YATRA

VINAY NAIR

All of us have heard anecdotes from the life of Srinivasa Ramanujan through books, friends and teachers. Robert Kanigel presents very beautifully the life of Ramanujan in his book *The Man who Knew Infinity*. A lot about the life of Ramanujan became known after Hollywood brought out the biopic on Ramanujan. Despite this, we felt that it would be worth taking some Math enthusiasts for a tour where they could get immersed in the life of Ramanujan for a few days. The idea took seed sometime in September 2017 while I was talking to Rajith who heads a culture and heritage tour company – *The Traveling Gecko* – and we thought, why not do an experiential tour on the life of Ramanujan? A detailed account of the 5-day *Yatra* (Chennai – Thanjavur – Kumbakonam – Chennai) from 12th to 16th November 2018, which I called *A Mathematical Pilgrimage | Ramanujan Yatra*, is put up on my blog https://vinayrnair.wordpress.com/2018/11/19/ramanujan-yatra-diaries-day-1/ . Hence, I am not writing a travelogue again. What I would like to share in this article are some stories, discussions and thoughts about Ramanujan and the lives of people that revolved around him during and after his time.

Day 1: First halt in Ramanujan Yatra: Ramanujan Museum in Chennai

P. K. Srinivasan (better known as 'PKS') (1924-2005) was a very inspiring Math teacher who dedicated a great part of his life towards popularising the life and works of Ramanujan. It was his dream to start a museum that would depict the life and works of Ramanujan. So with the available resources that he had, which included his own house, he created a museum. The ground floor is an auditorium (where we had our introductory session about the life of Srinivasa Ramanujan) and upstairs was the museum.

Keywords: Mathematicians, experiential learning, common goals, career in mathematics, encouragers and inspirers

PKS was immensely passionate about Ramanujan and he went around talking and writing to people, from top officials to ordinary people. In his own words, "I got 99% appreciation and 1% action." In short, no one offered any help for his idea of starting a Ramanujan museum. At the end he, along with a few of his students, went to Kumbakonam where Ramanujan was born and the students asked every person they met on the road whether they knew anything about Ramanujan. After much searching, somebody told them the address of the house where Ramanujan had lived. (Remember, this was the time before SASTRA University took up the task of preserving Ramanujan's house as a monument.) PKS found the house but by then, it belonged to someone else. The house is now preserved in honour of Ramanujan.

Our Yatris spent a good amount of time going through everything preserved in the museum and getting loads of stories and information from Mrs. Meena Suresh. The museum is a must visit for every Math lover and one can see the hard work done by a one-man army (PKS) in bringing together a huge amount of information about Ramanujan and popularising his work during times when we didn't have the internet.

Day 2: Marabu Foundation, Thanjavur

Marabu is an initiative to promote and preserve old Carnatic musical compositions which are not very well known today; it is run by Dr. Kausalya, a 70-year old musicologist. Our sessions in her 150-year old house started with the first letter Ramanujan sent to Hardy, which starts with the famous words, "I beg to introduce myself as a clerk from the Accounts department." We started with the famous claim made by Ramanujan that the sum of all natural numbers is -1/12 and tried to see why it could be true. The focus was also to read carefully the words that Ramanujan chooses in his letter where he writes, "… under my theory, this is true…" which prompts us to question whether what we have understood is the same as what he refers to as his 'theory.' After some questioning and analysing, we moved on to the topic that might look scary for many – the Nested Radicals[1]. The participants loved the topic and ways to approach solving the nested radicals using simple algebraic identities. The last part of the session was to look at another area in which Ramanujan worked – Continued Fractions[2]. This was very interesting for students as it didn't require knowledge of higher level mathematics. We ended with a story-telling session about *Ramanujan's Lost Notebook*.

In 1918, after Ramanujan left England for India owing to ill health, he started writing down his discoveries on loose sheets of paper which were sent to Hardy after Ramanujan passed away. Hardy worked on this material and later sent it to a mathematician named Watson who worked with another mathematician on Ramanujan's works, named Wilson. Wilson died in 1935 and Watson seemed to have lost interest in the work after some time and left the material somewhere among his papers. British mathematician J. M. Whittaker found the papers in a mess sometime after Watson's death in 1965. He sent them to the Wren library in Trinity College. American mathematician George Andrews learnt about this lost material in the early 1970s but he wasn't able to travel to Europe until 1976. All this material was published as *Ramanujan's Lost Notebook* on 22nd December 1987 to commemorate Ramanujan's birth centenary.

Now, this is a story that is not as popular as the taxicab number 1729 or some of the other popular stories. Many of our *Yatris* were full of questions after the story-telling session. It was hard to believe that a great work of Ramanujan may well have been lost forever had it not been for people like Whittaker and George Andrews. Remember, this work was almost lost *after* Ramanujan's genius had been recognised.

To recognise his genius (which was missed by mathematicians like Baker and Hobson to whom

1 & 2: To learn more, read the article by Utpal Mukhopadhyay in AtRiA March 2018.

Ramanujan had earlier written), there had to be a Hardy to make his work known. We read about great people but seldom do we know about those who were responsible and who played a vital role in making such people great.

Day 3: Town High School, Ramanujan Museum and Ramanujan's house, Kumbakonam

'Great knowledge often comes from the humblest of origins.' Littlewood to Ramanujan, from The Man who knew Infinity.

Ramanujan studied in Town High School, Kumbakonam, a school that has produced many great minds. It is a school where most students come from very humble backgrounds. What I loved was that their alumni support the schooling of many students who cannot afford it.

It was a big day for us as we set foot on the same ground where Ramanujan had walked. We were received with a lot of love by the Headmistress and the mathematics teachers. Following the greetings, we went into the oldest block – the Ramanujan block, which has been there since the time of Ramanujan. While there, we had some interactive sessions with the students of Town High who explained some of the traditional games played in rural areas. Many of these involved strategy and skill. After the session, one of our participants (Hetvi, 9th grade girl from Mumbai) did a session for the young Ramanujans of Town High. The session was on the Theory of Partitions in Numbers, an area in which Ramanujan had made a phenomenal contribution and one of the reasons why he was awarded a Fellowship of the Royal Society. Her presentation was simple and easy to understand and the audience enjoyed working out the questions that she posed.

After Hetvi's presentation, there was a Q & A session between the students of the school and our Yatris. They asked each other questions like, "How does it feel to be studying in the school where Ramanujan studied?" "What aspect of Ramanujan has inspired you?" "How did Ramanujan arrive at the property of 1729?" And so on.

Following lunch, we took a round of the school and it was time to leave. We found it hard to leave the school. The people there were just lovely and they were so happy that we were on this Yatra to study the life and work of Srinivasa Ramanujan. With a heavy heart, we bade good bye to the place and left for the Ramanujan Museum at SASTRA University, Kumbakonam. The museum is very well maintained and has many letters, notes and findings of Ramanujan. After an hour, we left for the house at Sarangapani Street where Ramanujan had lived.

How would someone feel who has waited all their life to visit the religious place closest to their heart and now has arrived at that place? Our feeling was nothing less. All of us went inside the typical Brahmin home. The first room in the left has a window that faces the road. It was written there that Ramanujan used to look out of the window, lost in thought for long hours when he was a child. I tried looking outside just to see if I could feel or see something that Ramanujan saw when he gazed outside. All the Yatris came inside to check out the small house. I could see some of them walking inside quietly, moved by the experience and with an indescribable feeling sinking into their hearts.

Figure 1. Yatris browsing through the Ramanujan Museum

We wondered where Ramanujan sat to have his food, how he worked on mathematics while his mother put into his mouth the rice balls she made for him, and how he continued working on mathematics as he grew up in that house. We stepped out after our hearts overflowed with contentment. As we got out, something suddenly struck me. The lane would have been an *Agraharam* earlier as the plots of land could be seen as elongated rectangles. But there was hardly any traditional house to be seen on the lane, except for Ramanujan's. Had it not been for PKS, would Ramanujan's house have survived today?

We left Kumbakonam after having some *pakodas* and filter coffee and reached Thanjavur to board the bus to Chennai. The faces of all the Yatris were lit throughout the day, for, they had visited their Kashi and Ganga! As pilgrims carry back Ganga water with them, some of us had taken a handful of sand from the ground in the Town High School where Ramanujan's footsteps would have fallen some time.

Day 4: IMSc and CMI, Chennai

'Golf is my life. I get to play golf all day. And I get paid for that. What more do I want?' Tiger Woods, one of the top golf players in one of his interviews.

If you are a math-lover, how would it be to get to 'do mathematics' the whole day and get paid for that? Many students who love mathematics do not know what mathematicians do for a living or whether they can earn a living by pursuing mathematics. As a result, we find many students not pursuing mathematics in college even if they love the subject the most. To get a first-hand experience of what mathematicians do and how *cool* it is to follow one's deepest interest, we visited two premier institutes for mathematics in India – Institute of Mathematical Sciences (IMSc) and Chennai Mathematical Institute (CMI) – and had interactive sessions at both places.

At IMSc, the session was organised by Prof. Ramanujam (popularly known as 'Jam,' who often writes articles for AtRiA). He impressed the Yatris with the library at IMSc which has a collection of more than 75000 books. As the Yatris were mostly school students, they had never seen such a huge library. They were so thrilled when they got to the mathematics section that they refused to come out of the library! Some of them had to be almost threatened and dragged out of the library! The next session was an interaction with Jam, his colleagues and some PhD students. The interactions started with a puzzle posed by Jam and continued with discussions on what researchers do. The professors shared some details about what they were working on and tried to bring down their work to the level of high school students. The research scholars shared their experiences of how they got into research and how wonderful life has been for them after their decision to pursue mathematics. Post the session, we could see some Yatris around some research scholars asking them questions.

After a delicious meal, all of us headed to CMI for the next session. On the way, when I asked the Yatris how they felt about the interactive session, all of them said that it was one of the best sessions that they had attended. I was surprised because I wasn't sure how much they had understood from the session, but later I figured out that the fact that Mathematicians were *cool* and the Math they did was still cooler is what the Yatris really loved.

At CMI, we got an opportunity to interact with three professors: Priyavrat Deshpande, Krishna and Manoj. They were very patient and explained how the courses are generally offered at CMI. The Yatris once again had a great chance to clarify their queries regarding undergraduate studies in places like CMI. One of the first year students of CMI, Sundarraman, shared his experience of being at CMI.

Now, here's something that is commonly seen. When math-lovers are asked in middle school what they want to pursue, many of them say that they want to do something in math. At high school, there's hardly anyone still saying that, even though they still love math the most.

There's a big chunk who aim to do something in engineering. Not that any course is better than the rest, but it could also be that because there is a lot of advertising that happens on coaching for engineering entrance exam, students get so influenced that they fail to see the options available in the pure sciences and mathematics. I feel that visits to places like IMSc and CMI and interactions with the professors and research students there can be very useful in countering this pressure.

Day 5: Closing

Draw parallels between the lives of Hardy and Ramanujan. What are the positives and negatives that you see in the personalities of Hardy, Ramanujan and Komalatammal (Ramanujan's mother)? Come up with three hypotheses on what might have happened if Ramanujan had lived a longer life.

These were some questions that were given to the Yatris to discuss and come out with their thoughts. The discussion and presentations were excellent. The movie screening of *The Man who knew Infinity* the previous night helped the Yatris connect the dots between the different people who played a role in his life. What kind of a person would have Narayana Iyer been, the person who recognised Ramanujan's genius? How much do we owe to Hardy for the credit he brought to Ramanujan? These were questions that sunk into the minds of all the Yatris at the end of the Yatra.

A friend of mine, Sriram, who interacted with the participants on the last day beautifully pointed out something very profound after the participants drew parallels between Hardy and Ramanujan. He said, 'Even though Hardy was an atheist and Ramanujan highly religious, with Hardy coming from a diametrically opposite culture to Ramanujan's, both of them never allowed their personal differences to come in their way. They had great respect for each other and that's a kind of friendship that we need to look up to. There are many stories of great people developing jealousy and enmity amongst them for the sake of pride and glory, and here we have Hardy who took all the risk, put his goodwill at stake, to bring an unknown Indian clerk all the way to Cambridge. He didn't stop there, did he? Hardy took pains to convince an entire band of mathematicians as to why Ramanujan deserved to be a *Fellow of the Royal Society*. Again, from Ramanujan's side, he was willing to sacrifice his personal belief systems and his family for the sake of mathematics and go to England to work with someone whom he had hardly known.' More than mathematics, perhaps it is the stories revolving around mathematicians that can inspire young minds.

During the 5-day tour, all the Yatris lived, thought and talked only about Ramanujan. This happens only in a Pilgrimage. And that is why we felt, despite all the information about Ramanujan being readily available, having a Yatra would be worth it.

Food for thought

Before we concluded the valedictory, there were three questions thrown to the Yatris which might be of interest to the readers of this article.

1. Whom should we thank for Ramanujan's works that we have today? Should it be his wife Janaki, who let him go to England despite all odds and who struggled to live as a widow till her 90's? Should it be Prof GH Hardy? Should it be Narayana Iyer who gave directions and encouraged Ramanujan to go to England? Is it Whittaker? George Andrews? PKS? Bruce Berndt who took the trouble to publish the works of Ramanujan?
2. If Ramanujan's life has been an inspiration to you, how would you want to spread this inspiration?
3. PKS had a dream that every city in India should have a museum dedicated to the life of Ramanujan. Do you think you can contribute in any way to make the dream come true? If so, how?

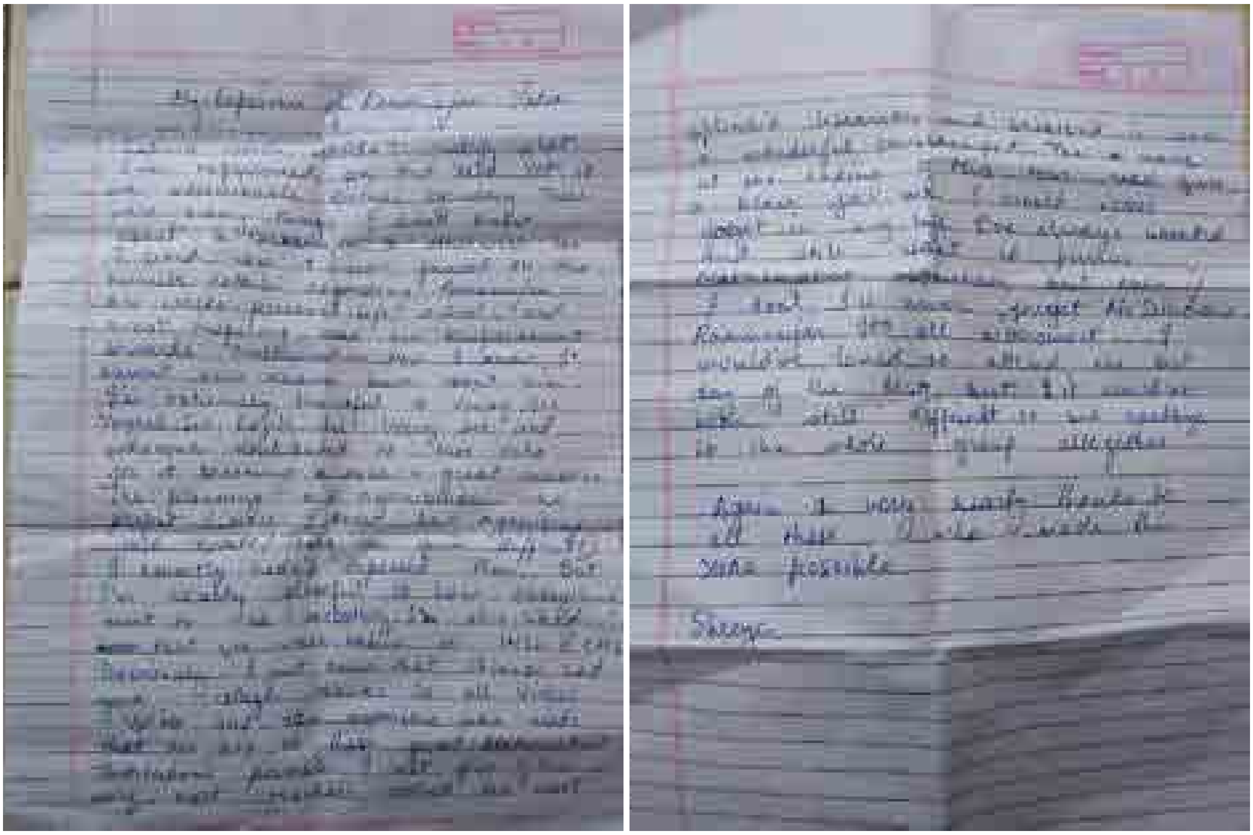

Figure 2. Handwritten letter from one of the Yatris.

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Extensions of the Theorem of Pythagoras

LAXMAN N. KATKAR

Introduction

The study of mathematics as a demonstrative discipline begins in the 6th century BC with the Pythagoreans. Pythagoras was a Greek philosopher-mathematician who considered mathematics to be supreme and all other things secondary. The Pythagorean theorem states that *the sum of the squares of the legs of a right angled triangle is equal to the square of the hypotenuse.* The theorem has a prior history. Propositions like "the square of the diagonal of a rectangle is equal to the sum of the squares of the two adjacent sides of the rectangle" are found in *Baudhāyana* (8th century BC). The theorem plays a crucial role in the development of mathematics. Pythagoras is credited with the first logical proof of the Pythagorean theorem.

Euclid (330-270 BC) was one of the great mathematicians. His name and his geometry go hand-in-hand with the history and development of mathematics. Euclid's geometry is based on axioms, which are "self-evident truths accepted without proof." Some of his contemporaries believed that propositions such as "the sum of two sides of a triangle is greater than the third side" need no proof. However, in axiom-based mathematics, every proposition other than the axioms needs to be proved, even if it seems self-evident. Euclid proved such propositions by rigorous mathematical reasoning in his book "The Elements" which is widely considered to be the most successful and influential textbook of all time. Although much of the content of the book was known during his time, Euclid arranged the results into a coherent logical framework. The irrational nature of numbers such as $\sqrt{2}$, $\sqrt{3}$ was established by using the Pythagorean theorem.

Keywords: Pythagoras, Pythagorean theorem, Euclidean space, Minkowski space-time

Descartes (1596-1650) brought a revolution in mathematics by connecting geometry and numbers/algebra. He introduced coordinates and gave formulas for the distance between two points and the area of a triangle. One outcome of this insight was the description of space in terms of algebraic coordinates on an infinite rectangular grid. A space of 2-dimensions is identified as the set of ordered pairs (x, y), where x and y are real numbers.

In 1830, the Russian mathematician N Lobachevsky and the Hungarian mathematician J Bolyai discovered a new geometry now known as 'hyperbolic geometry.' In 1851, the German mathematician B Riemann obtained another geometry just as consistent and as true as the geometries of Euclid and Lobachevsky and Bolyai. Riemannian geometry is based on the surface of a sphere where the straight lines are the arcs of great circles. In this geometry, the sum of the angles of any triangle exceeds two right angles.

New mathematical developments in interaction with new scientific discoveries were made at an increasing pace; this continues to the present day. Thus, we have the groundbreaking work of I Newton and G Leibniz in the development of infinitesimal calculus towards the end of the 17th century. Newton (1642-1727) discovered the laws of motion and the law of gravitation and used these along with the newly invented calculus to create a revolution in physics. Einstein (1879-1955) brought about a revolution in our understanding of space and time and in the process found a completely different way of looking at gravitation.

In this article, we extend the Pythagorean theorem to n-dimensional Euclidean space. We show how to generate $(n+1)$-tuples of integers satisfying the extended version of the theorem.

Pythagorean Theorem in 2D Euclidean Space

The Pythagorean theorem in 2-dimensional space is well known: "The sum of the squares of the legs of a right-angled triangle is equal to the square of the hypotenuse." It has over 350 proofs. Its mathematical expression is:

$$r^2 = x^2 + y^2, \tag{1}$$

where r is the hypotenuse and x, y are the legs. A triplet $(x, y; r)$ of integers which satisfies (1) is called a *Pythagorean triplet*. Some familiar Pythagorean triplets: $(3, 4; 5)$, $(5, 12; 13)$, $(7, 24; 25)$, $(8, 15; 17)$. There are many methods of finding such triplets. The following formula generates infinitely many such triplets:

$$\left(2k,\ k^2-1;\ k^2+1\right), \quad \text{for every } k > 1.$$

Another such generating formula is $(2k+1,\ 2k(k+1);\ 2k(k+1)+1)$, for $k \geq 1$. The ancient Indian texts *Baudhāyana* and *Apastamba* offer another such family:

$$\left(x, \frac{x^2-1}{2}, \frac{x^2+1}{2}\right), \quad \text{when } x \text{ is odd;}$$

$$\left(x, \frac{x^2}{4}-1, \frac{x^2}{4}+1\right), \quad \text{when } x \text{ is even.}$$

Yet another such formula was given by Diophantus of Alexandria:

$$\left(\frac{2mx}{m^2+1}, \frac{\left(m^2-1\right)x}{m^2+1}, x\right),$$

where m and x are any positive integers. (This gives a triplet of rational numbers satisfying the Pythagorean relation.)

There are infinitely many Pythagorean triplets. Some of these are enumerated in Table 1 for use in the sequel.

$(9,40;41)$,	$(11,60;61)$,	$(12,35;37)$,	$(16,63;65)$,	$(13,84;85)$,
$(17,144;145)$,	$(19,180;181)$,	$(20,21;29)$,	$(20,99;101)$,	$(21,220;221)$,
$(23,264;265)$,	$(24,143;145)$,	$(25,312;313)$,	$(27,364;365)$,	$(28,45;53)$,
$(28,195;197)$,	$(29,420;421)$,	$(31,480;481)$,	$(32,255;257)$,	$(33,56;65)$,
$(33,544;545)$,	$(35,612;613)$,	$(36,77;85)$,	$(36,323;325)$,	$(37,684;685)$,
$(65,72;97)$,	$(85,132;157)$,	$(41,840;841)$,	$(145,408;433)$,	$\ldots$

Table 1. Some Pythagorean triplets

Geometrically the Pythagorean theorem is interpreted thus: "The sum of the areas of squares on the legs of any right-angled triangle is equal to the area of the square on the hypotenuse." Instead of squares, we could also have equilateral triangles or semicircles (or any three shapes that are similar to one another).

Pythagorean Theorem in 3D Euclidean Space

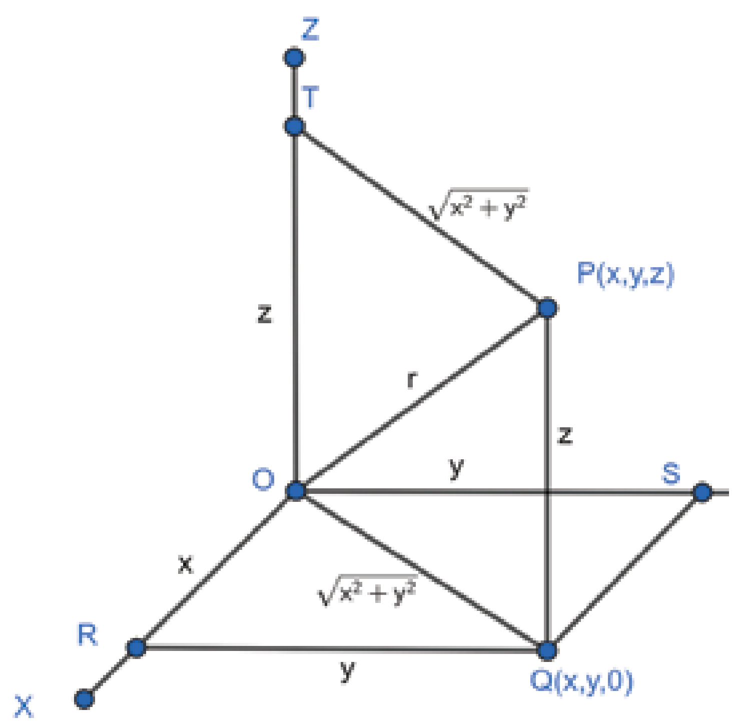

Figure 1.

The Pythagorean theorem in 3-dimensional Euclidean space states that "the sum of the squares of the legs of a tri-rectangular tetrahedron is equal to the square of the hypotenuse." Two applications of the familiar Pythagorean theorem in 2-dimensional space suffice to establish this result. (Figure 1 will perhaps suggest a proof; details are left to the reader.) This is equivalent to stating the following: if $P(x,y,z)$ is any point in 3-dimensional Euclidean space, so that x, y, z are respectively the distances from the origin O to the feet of the perpendiculars from P upon the three coordinate axes, then the distance r of P from O is given by the formula

$$r^2 = x^2 + y^2 + z^2. \qquad (2)$$

A few quadruples of integers satisfying (2) are listed in Table 2. We shall make some remarks later on how such quadruples can be found.

$(3, 4, 12; 13)$,	$(8, 15, 144; 145)$,	$(7, 24, 312; 313)$,	$(12, 35, 684; 685)$,
$(20, 21, 420; 421)$,	$(16, 63, 72; 97)$,	$(33, 56, 72; 97)$,	$(36, 77, 132; 157)$,
$(13, 84, 132; 157)$,	$(12, 16, 99; 101)$,	$(15, 20, 312; 313)$,	$(17, 144, 408; 433)$,
$(9, 40, 840; 841)$,	$(1089, 4840, 6480; 8161)$,	...	...

Table 2. Some Pythagorean quadruples

Generalised Pythagorean Theorem

A pictorial view of the Pythagorean theorem in dimensions higher than 3 may not be possible due to the limitations of human imagination. But there is no such limitation in mathematics. We can think of any n-dimensional space and develop our own mathematical world. The ideas used to extend the theorem to 3-dimensional space can be used in the same way to extend the theorem to n-dimensional space. If $x_i = (x_1, x_2, \dots, x_n)$ are the coordinates of a point P in n-dimensional Euclidean space, then the Pythagorean theorem can be viewed as stating that the distance r from the origin $O(0, 0, \dots, 0)$ to P is given by the formula $r^2 = \sum_{i=1}^{n} (x_i)^2$, i.e.,

$$r^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 + \dots + (x_n)^2. \tag{3}$$

Thus in 3-dimensional Euclidean space, we have

$$r^2 = (x_1)^2 + (x_2)^2 + (x_3)^2.$$

Generating n-tuples of integers satisfying the Pythagorean relation. For each positive integer $n \geq 4$, we can recursively generate infinitely many n-tuples of integers satisfying the Pythagorean relation, starting with any Pythagorean triple. The method is illustrated in Figure 2. For example, for $n = 4$ we may start with the triple $(3, 4; 5)$. We also have the triple $(5, 12; 13)$. Hence we have:

$$3^2 + 4^2 = 5^2, \quad 5^2 + 12^2 = 13^2, \quad \therefore\ 3^2 + 4^2 + 12^2 = 13^2,$$

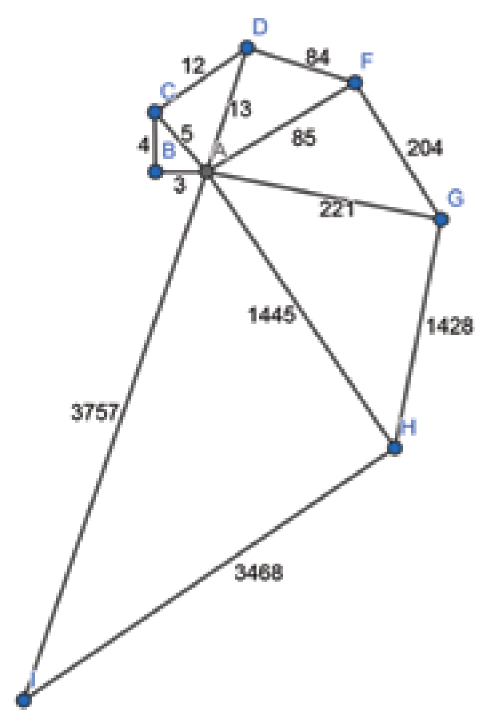

Figure 2.

leading to the quadruple $(3, 4, 12; 13)$. Similarly, we may start with the triple $(8, 15; 17)$. We also have the triple $(17, 144; 145)$. Hence we have:

$$8^2 + 15^2 = 17^2, \quad 17^2 + 144^2 = 145^2, \quad \therefore\ 8^2 + 15^2 + 144^2 = 145^2,$$

leading to the quadruple $(8, 15, 144; 145)$. Both these quadruples had been listed in Table 2. Infinitely more such quadruples can be listed in the same way.

We can use the same approach to generate 5-tuples of integers satisfying the formula for 4-dimensional space. Thus we have:

$$3^2 + 4^2 = 5^2, \quad 5^2 + 12^2 = 13^2, \quad 13^2 + 84^2 = 85^2, \quad \therefore\ 3^2 + 4^2 + 12^2 + 84^2 = 85^2,$$

leading to the 5-tuple $(3, 4, 12, 84; 85)$. Here are some more 5-tuples of integers satisfying the formula for 4-dimensional space, all generated in the same manner:

$$(8, 15, 144, 408; 433), \quad (7, 24, 60, 156; 169), \quad (8, 15, 144, 348; 377).$$

Here are some 6-tuples of integers satisfying the formula for 5-dimensional space:

$$\begin{array}{ll} (3, 4, 12, 84, 132; 157), & (3, 4, 12, 84, 204; 221), \\ (7, 24, 60, 156, 1092; 1105), & (17, 144, 348, 2436, 5916; 6409). \end{array}$$

They have all been generated in the same manner.

Proceeding in this manner, we can recursively find $(n+1)$-tuples of integers that satisfy the Pythagorean theorem for any $n \geq 3$. Here is an example from 7-dimensional space:

$$(3)^2 + (4)^2 + (12)^2 + (84)^2 + (12 \times 17)^2 + (84 \times 17)^2 + (12 \times 17^2)^2 = (13 \times 17^2)^2.$$

We leave the reader with the task of generating more such examples.

Pythagorean Theorem in 4D Minkowski Space-Time

We conclude this article by considering what happens to the Pythagorean formula in 4-dimensional space-time.

By combining 3-dimensional space and 1-dimensional time into a single entity, called *Minkowski space-time*, Einstein developed his revolutionary special theory of relativity in 1905. His ideas force us to radically change our ideas of space and time. In 4-dimensional Minkowski space-time, the distance r between the origin O and the point P with space-time coordinates (x_1, x_2, x_3, x_4) is given by

$$r^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2. \tag{4}$$

The negative sign in (4) is due to the time coordinate which is taken as imaginary (so its square is negative). This formula results in some oddities; for example, it is possible for the distance between two points to be 0 even when they do not coincide. (This is not possible in ordinary Euclidean space.) The formula (4) can be viewed as the representation of Pythagorean theorem in 4-dimensional Minkowski space-time. The 5-tuples of integers satisfying the formula

$$(x_5)^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2$$

are called *Pythagorean quintics* and denoted by $(x_1, x_2, x_3; x_4, x_5)$.

There exist infinitely many 5-tuples of integers satisfying (4). For example:

$$\begin{array}{lll} (5, 12, 84; 36, 77), & (8, 15, 144; 24, 143), & (15, 20, 60; 33, 56), \\ (12, 16, 549; 101, 540), & (15, 20, 60; 16, 63), & (9, 12, 20; 7, 24), \\ (40, 75, 204; 21, 220), & (84, 112, 225; 23, 264), & (135, 140, 180; 23, 264). \end{array}$$

These may be generated by using the same kind of recursive reasoning as used earlier. For example, we may start with $5^2 + 12^2 = 13^2$ and $13^2 + 84^2 = 85^2$. These two equalities result in the relation $5^2 + 12^2 + 84^2 = 85^2$. We also have $36^2 + 77^2 = 85^2$ (see Table 1). Hence we have $5^2 + 12^2 + 84^2 = 36^2 + 77^2$, i.e.,

$$5^2 + 12^2 + 84^2 - 36^2 = 77^2.$$

This yields the Pythagorean quintic $(5, 12, 84; 36, 77)$. Similarly for the others.

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From Regular Pentagons to the Icosahedron and Dodecahedron via the Golden Ratio – II

SHASHIDHAR JAGADEESHAN

Introduction

In the previous article (At Right Angles, Issue 4, July 2019, pages 5-9) we saw how we could construct a regular pentagon using a ruler and compass, and discovered a nested sequence of pentagons that can be built up by extending the sides of a given regular pentagon (see Figure 1).

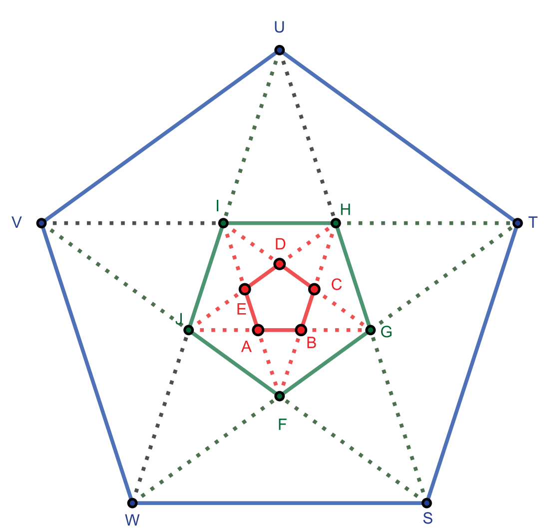

Figure 1.

We also calculated the edges and diagonals of each regular pentagon (see Figure 2)

Keywords: Regular pentagon, Golden Rectangle, Fibonacci sequence, icosahedron, dodecahedron, dual

Features

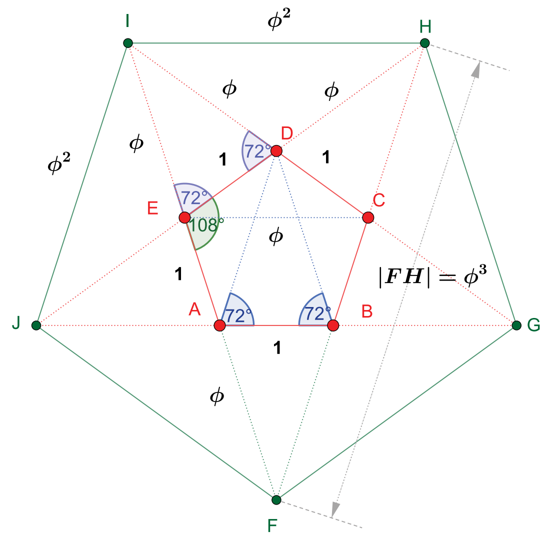

Figure 2.

and got the following infinite sequence of side, diagonal, side, diagonal,...

$$1,\ \phi,\ \phi^2,\ \phi^3, \ldots$$

where ϕ is the Golden Ratio.

In this article [1] we will see that this sequence leads to two famous sequences, the Fibonacci sequence and its cousin, the Lucas sequence. Moreover, we will see how the regular pentagon, the Golden Ratio, the icosahedron and the dodecahedron all come together!

The Fibonacci and Lucas sequences from a regular pentagon

Recall that the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$ and it satisfies the equation $x^2 - x - 1 = 0$, that is, $\phi^2 - \phi - 1 = 0$. We are going to use this equation to compute the powers of ϕ (we saw a hint of this in the last section of the previous article). Let us begin with ϕ^2. Of course we could compute $(\frac{1+\sqrt{5}}{2})^2$, but mathematicians need to be lazy if they can get away with it! We know $\phi^2 = \phi + 1$, so why not use that? This computation is much simpler! Therefore,

$$\phi^2 = 1 + \frac{1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}.$$

What about ϕ^3? We saw in the previous article that $\phi^3 = \phi(\phi^2) = \phi(\phi + 1) = \phi^2 + \phi$. Hence,

$$\phi^3 = \frac{1+\sqrt{5}}{2} + \frac{3+\sqrt{5}}{2} = \frac{4+2\sqrt{5}}{2}.$$

We can similarly see that

$$\phi^4 = \phi^3 + \phi^2 = \frac{7+3\sqrt{5}}{2}.$$

The reader has perhaps begun to notice some patterns. From now on the number n shall refer to a whole number. First of all we can see by induction (keeping in mind that $\phi^0 = 1$) that

$$\phi^{n+1} = \phi^n + \phi^{n-1} \tag{1}$$

[1]I would like to thank Aashotosha Lele, Rishabh Suresh and Gautam Dayal for their suggestions and feed back while writing this article.

Let us denote $\phi^n = \frac{l_n + f_n\sqrt{5}}{2}$ with $n \geq 1$ and see if we can find a pattern in the computation of both l_n and f_n. From ϕ we get $l_1 = 1,\ f_1 = 1$ and from the fact that $\phi^2 = \phi + 1$, we get $l_2 = 3,\ f_2 = 1$. Now for $n > 1$, from Equation (1) we get:

$$\phi^{n+1} = \frac{l_{n+1} + f_{n+1}\sqrt{5}}{2} = \phi^n + \phi^{n-1} = \frac{l_n + f_n\sqrt{5}}{2} + \frac{l_{n-1} + f_{n-1}\sqrt{5}}{2} = \frac{(l_n + l_{n-1}) + (f_n + f_{n-1})\sqrt{5}}{2}, \tag{2}$$

giving us the formulas:

$$l_{n+1} = l_n + l_{n-1},\ n > 1 \tag{3}$$
$$f_{n+1} = f_n + f_{n-1},\ n > 1 \tag{4}$$

yielding $l_3 = 4,\ f_3 = 2,\ l_4 = 7,\ f_4 = 3$ and so on.

It is now time to pay attention to the other root of the equation $x^2 - x - 1 = 0$, namely $\psi = \frac{1-\sqrt{5}}{2}$. Notice ψ also satisfies the equation $\psi^2 = \psi + 1$. Using the same technique as above, we can see that

$$\psi^2 = 1 + \frac{1-\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$$

and

$$\psi^3 = \frac{4-2\sqrt{5}}{2}$$

and

$$\psi^4 = \frac{7-3\sqrt{5}}{2}$$

and so on. Just as in the case of ϕ^n, we get $\psi^n = \frac{l_n - f_n\sqrt{5}}{2},\ n \geq 1$ and

$$\psi^{n+1} = \psi^n + \psi^{n-1}, \text{where } \psi^0 = 1.$$

We can now create the following table:

$\phi = \frac{1+\sqrt{5}}{2}$	$\psi = \frac{1-\sqrt{5}}{2}$
$\phi^2 = \frac{3+\sqrt{5}}{2}$	$\psi^2 = \frac{3-\sqrt{5}}{2}$
$\phi^3 = \frac{4+2\sqrt{5}}{2}$	$\psi^3 = \frac{4-2\sqrt{5}}{2}$
$\phi^4 = \frac{7+3\sqrt{5}}{2}$	$\psi^4 = \frac{7-3\sqrt{5}}{2}$
$\phi^5 = \frac{11+5\sqrt{5}}{2}$	$\psi^5 = \frac{11-5\sqrt{5}}{2}$
$\vdots$	$\vdots$
$\phi^n = \frac{l_n+f_n\sqrt{5}}{2},\ n \geq 1$	$\psi^n = \frac{l_n-f_n\sqrt{5}}{2},\ n \geq 1$

The astute reader might have realised that adding and subtracting each row of the above table gives rise to some familiar sequences:

$\phi + \psi = l_1 = 1$	$\phi - \psi = f_1\sqrt{5} = \sqrt{5}$
$\phi^2 + \psi^2 = l_2 = 3$	$\phi^2 - \psi^2 = f_2\sqrt{5} = \sqrt{5}$
$\phi^3 + \psi^3 = l_3 = 4$	$\phi^3 - \psi^3 = f_3\sqrt{5} = 2\sqrt{5}$
$\phi^4 + \psi^4 = l_4 = 7$	$\phi^4 - \psi^4 = f_4\sqrt{5} = 3\sqrt{5}$
$\phi^5 + \psi^5 = l_5 = 11$	$\phi^5 - \psi^5 = f_5\sqrt{5} = 5\sqrt{5}$
$\vdots$	$\vdots$
$\phi^n + \psi^n = l_n,\ n \geq 1$	$\phi^n - \psi^n = f_n\sqrt{5},\ n \geq 1$

So, the two sequences that are emerging are

$$l_1 = 1,\ l_2 = 3,\ l_3 = 4,\ l_4 = 7,\ l_5 = 11\ldots$$

and

$$f_1\sqrt{5} = \sqrt{5},\ f_2\sqrt{5} = \sqrt{5},\ f_3\sqrt{5} = 2\sqrt{5},\ f_4\sqrt{5} = 3\sqrt{5},\ f_5\sqrt{5} = 5\sqrt{5}\ldots.$$

The first sequence

$$L = \{l_1, l_2, l_3, l_4, l_5, \ldots l_n, \ldots\} = \{1,\ 3,\ 4,\ 7,\ 11\ldots\}$$

is called the Lucas sequence. In the second sequence if we divide by $\sqrt{5}$ throughout, we get

$$F = \{f_1, f_2, f_3, f_4, f_5, \ldots f_n, \ldots\} = \{1,\ 1,\ 2,\ 3,\ 5\ldots\}$$

the famous Fibonacci sequence!

Both these sequences have the same generative principle: you start with two given terms, here $l_1 = 1, l_2 = 3$ and $f_1 = f_2 = 1$ and then generate the sequences using the iterative Equations (2) and (3), namely $l_{n+1} = l_n + l_{n-1}$ and $f_{n+1} = f_n + f_{n-1},\ n > 1$.

Let us now return to the n^{th} term of the Lucas sequence $l_n = \phi^n + \psi^n$ and the n^{th} term of the Fibonacci sequence $f_n = \dfrac{\phi^n - \psi^n}{\sqrt{5}}$. In other words

$$l_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \text{ and } \sqrt{5}f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Now consider $\psi = \frac{1-\sqrt{5}}{2}$; it is easy to see that $-1 < \psi < 0$, and hence $\lim_{n\to\infty} \psi^n = 0$. This then tells us that if n is large $l_n \approx \left(\frac{1+\sqrt{5}}{2}\right)^n$ and that $\sqrt{5}f_n \approx \left(\frac{1+\sqrt{5}}{2}\right)^n$, yielding the following amazing result

$$\frac{l_n}{l_{n-1}} \approx \frac{1+\sqrt{5}}{2} = \phi \text{ and } \frac{f_n}{f_{n-1}} \approx \frac{1+\sqrt{5}}{2} = \phi$$

the Golden Ratio! So for both the Fibonacci sequence and the Lucas sequence the ratio of successive terms approximates the Golden Ratio.

The Icosahedron and the Golden Ratio

It is time to now put together all that we have learned. A rectangle with side lengths 1 and ϕ is called a Golden Rectangle. Let us take three such Golden Rectangles, *ABCD*, *EFGH*, *IJKL* and intersect them mutually perpendicular to each other in three dimensions along the x, y and z axes as shown in Figure 3.

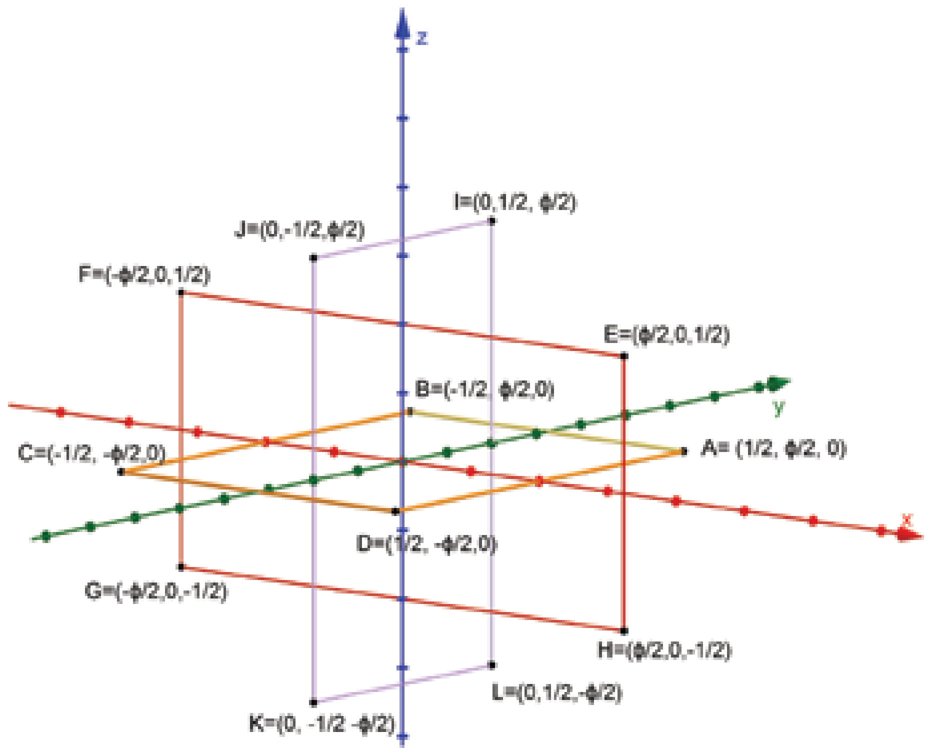

Figure 3.

Notice the special choice of coordinates for our 12 vertices:

$$A = (\tfrac{1}{2}, \tfrac{\phi}{2}, 0), \quad B = (\tfrac{-1}{2}, \tfrac{\phi}{2}, 0), \quad C = (\tfrac{-1}{2}, \tfrac{\phi}{2}, 0), \quad D = (\tfrac{1}{2}, \tfrac{-\phi}{2}, 0),$$
$$E = (\tfrac{\phi}{2}, 0, \tfrac{1}{2}), \quad F = (\tfrac{-\phi}{2}, 0, \tfrac{1}{2}), \quad G = (\tfrac{-\phi}{2}, 0, \tfrac{-1}{2}), \quad H = (\tfrac{\phi}{2}, 0, \tfrac{-1}{2}),$$
$$I = (0, \tfrac{1}{2}, \tfrac{\phi}{2}), \quad J = (0, \tfrac{-1}{2}, \tfrac{\phi}{2}), \quad K = (0, \tfrac{-1}{2}, \tfrac{-\phi}{2}) \text{ and } \quad L = (0, \tfrac{1}{2}, \tfrac{-\phi}{2}).$$

This choice of coordinates ensures that we have Golden rectangles with the correct lengths. You might have also become aware of the amazing symmetry among the vertices.

We now claim that the vertices form the 12 vertices of a regular icosahedron! How do we construct the edges? This is quite intuitively obvious. Take the vertex E for example. There are 11 possible edges that can start at E, but we can reject EB, EC, EF, EG, EK and EL because they are longer than the unit edge EH. This leaves us the edges EA, ED, EH, EI and EJ. Don't worry we will show shortly that these edges do have length 1!

Recall that a regular icosahedron is one of the five Platonic solids. Platonic solids, also called the 'regular solids,' are 3-dimensional geometric solids whose faces are all congruent regular polygons (like equilateral triangles or squares) and in which the same number of polygons meet at each vertex. The amazing fact is that there are only 5 such Platonic solids. For a proof of this please see [3].

The regular icosahedron has 12 vertices, 30 edges and 20 faces (all of which are equilateral triangles). Moreover at each vertex 5 equilateral triangles meet. Notice that the polyhedron $ABCDEFGHIJKL$ has 12 vertices, 30 edges and 20 faces and moreover, at each vertex 5 triangles meet. If we show that each of these triangles is equilateral (and hence congruent), then we would have established that $ABCDEFGHIJKL$ is a regular icosahedron. We will do this by showing that all the 30 edges are of equal length.

Since six of the edge lengths come from our intersecting Golden Rectangles of breadth one, we need to show that all the other twenty four edges are of length one. But we don't need to do 24 calculations! Take

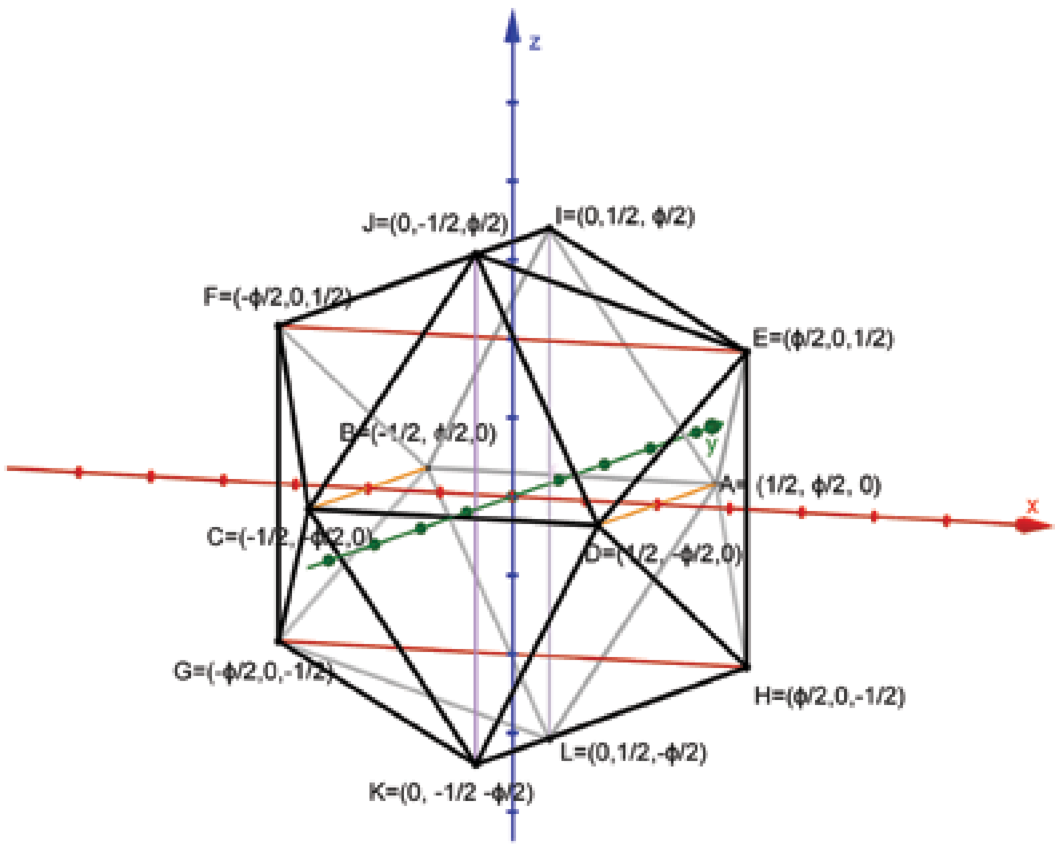

Figure 4.

any edge that forms the icosahedron other than the ones that come from the rectangle; so for example consider the edge *EA* as opposed to *EH*. (See Figure 4.) Using the Pythagorean formula in 3-D (the distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + ((y_1 - y_2)^2 + (z_1 - z_2)^2)}$ we get:

$$|EA|^2 = (\frac{\phi - 1}{2})^2 + (\frac{-\phi}{2})^2 + (\frac{1}{2})^2 = \frac{2\phi^2 - 2\phi + 2}{4} = \frac{\phi^2 - \phi + 1}{2},$$

and from the equation $\phi^2 - \phi = 1$ we get:

$$(|EA|)^2 = 1.$$

You will notice that this very same computation works for every edge that is not a part of a Golden Rectangle, and so we are done!

What about the converse? That is, if we are given a regular icosahedron can we find three intersecting Golden Rectangles, whose vertices coincide with six of those of the regular icosahedron? Since regular icosahedra are essentially determined by their edge lengths, we know that any two regular icosahedra with the same edge length are congruent. Moreover, given an arbitrary regular icosahedron in 3-space, we can always move it (using translations and rotations) in such a way that it coincides with the regular icosahedron *ABCDEFGHIJKL*. The three Golden Rectangles then coincide with our original Golden Rectangles *ABCD*, *EFGH* and *IJKL*.

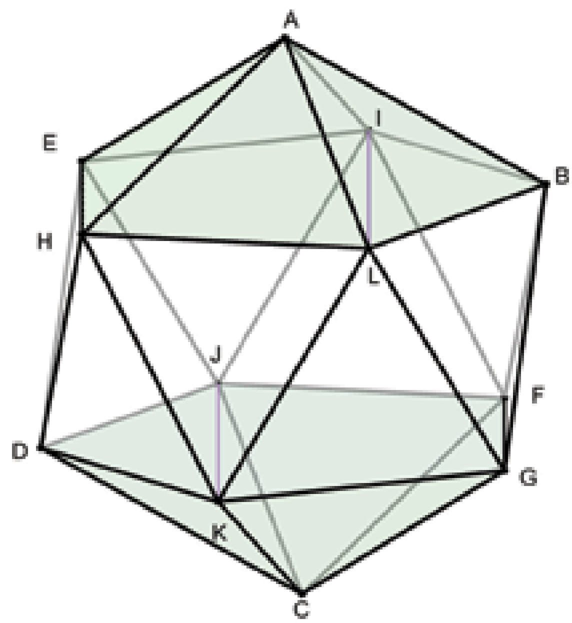

Figure 5.

We claimed that the regular pentagon is also part of the icosahedron. If you have not already noticed it, then you will see the regular pentagon show up in a regular icosahedron in quite a 'natural' manner. Every vertex of an icosahedron forms the apex of a pyramid whose base is a regular pentagon, a sort of 'pentagonal hat' as shown in Figure 5.

We have rotated the regular icosahedron in Figure 4 to obtain Figure 5, so that the vertex A is on top and it is the apex of the pyramid with base the regular unit pentagon $EHLBI$. Now IL is the diagonal of this pentagon and from part I of this article we know it has to be of length ϕ, which of course it is by construction!

The Dodecahedron and the Golden Ratio

Every Platonic solid comes with a dual (see [2] to learn more about duals). The icosahedron is the dual of the dodecahedron and vice-versa (for the rest of this article we will just say icosahedron and dodecahedron without the prefix 'regular', because all of the ones we refer to are going to be regular). One way to construct the dual is to take the centres of all the faces as vertices of the dual solid. So corresponding to the 12 vertices of the icosahedron are the 12 faces of the dodecahedron. The dodecahedron has 20 vertices (one for each face of the icosahedron) and 30 edges.

The following series of figures [2] shows how the regular dodecahedron is built up from the icosahedron. We begin with building four regular pentagons around the vertices A, E, I, and J. We have shaded the regular pentagon surrounding the vertex E.

[2]The reader might be curious as to how the coordinates of the twenty vertices of the regular dodecahedron are to be found, given Figure 4. It turns out that the simplest way is to first recognize that there is a sphere with centre the origin and radius $\frac{\sqrt{\phi+2}}{2} = \frac{\sqrt{5+\sqrt{5}}}{4}$ that envelops the icosahedron formed by the intersecting Golden Rectangles . That is, it passes through all twelve vertices A, B, $\ldots$, L. Let us call this sphere $\mathbb{S}$. Then the faces of the dodecahedron lie on the tangent planes to $\mathbb{S}$ passing through the 12 vertices A, B, $\ldots$ L. To find the coordinates of the twenty vertices of the regular dodecahedron we need to first find the lines of intersection of suitable tangent planes and then find the points of intersection of suitable lines. Obviously this footnote is too small to fit in all the details and providing all the details is an article in its own right!

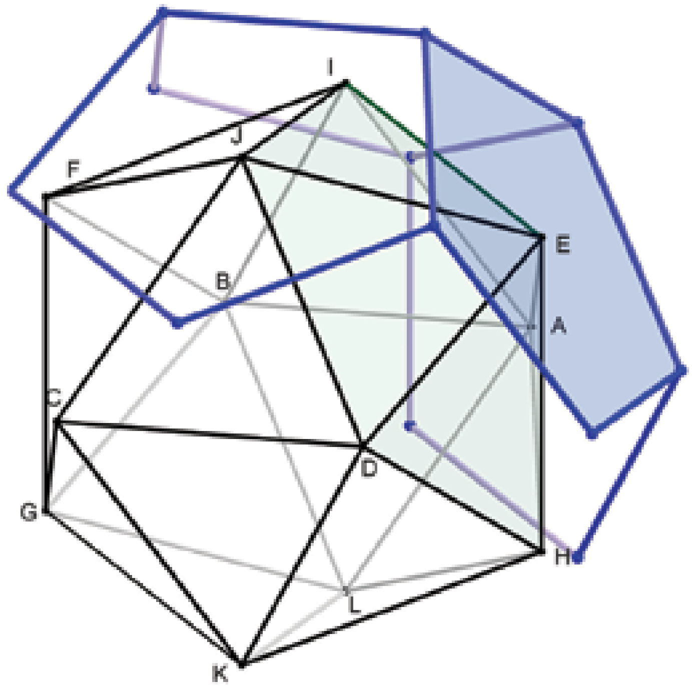

Figure 6.

We then add regular pentagons around the vertices H, D, K, and L.

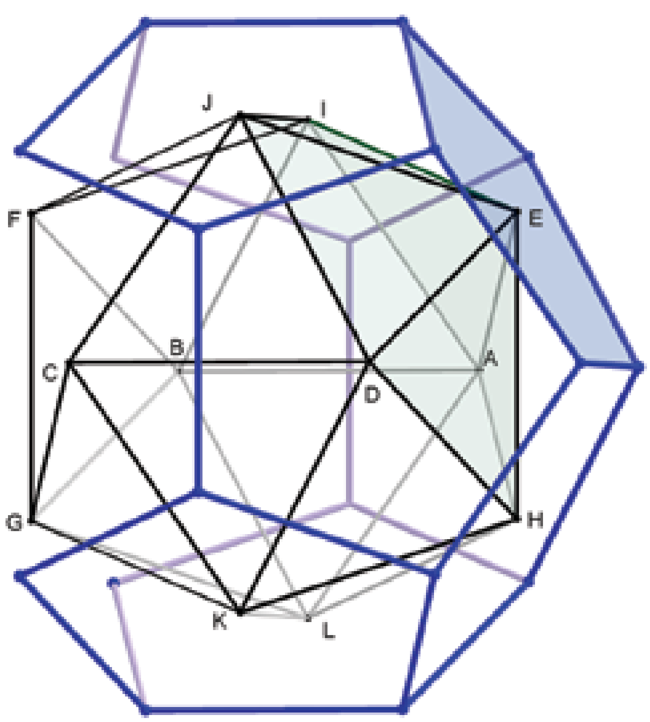

Figure 7.

And finally B, F, G, and C.

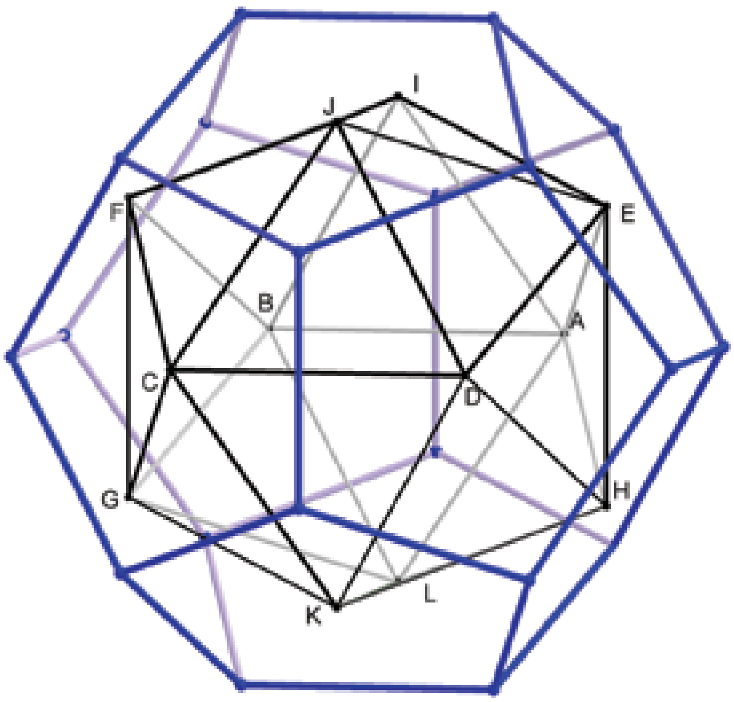

Figure 8.

We can now see how the Golden Rectangles, the icosahedron and the dodecahedron come together.

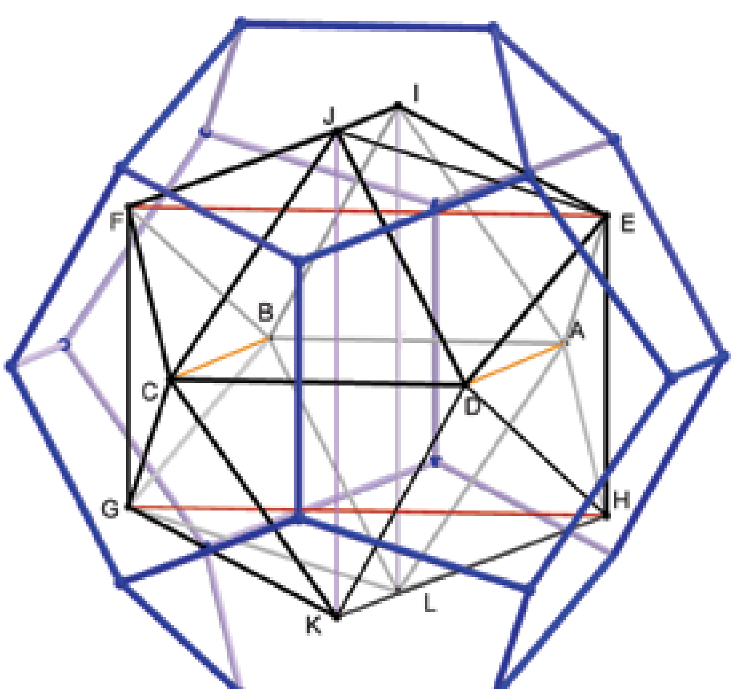

Figure 9.

We hope that in this two part series we have managed to convince the reader that the regular pentagon, the Golden Ratio, the Fibonacci sequence the icosahedron and the dodecahedron all come together so beautifully.

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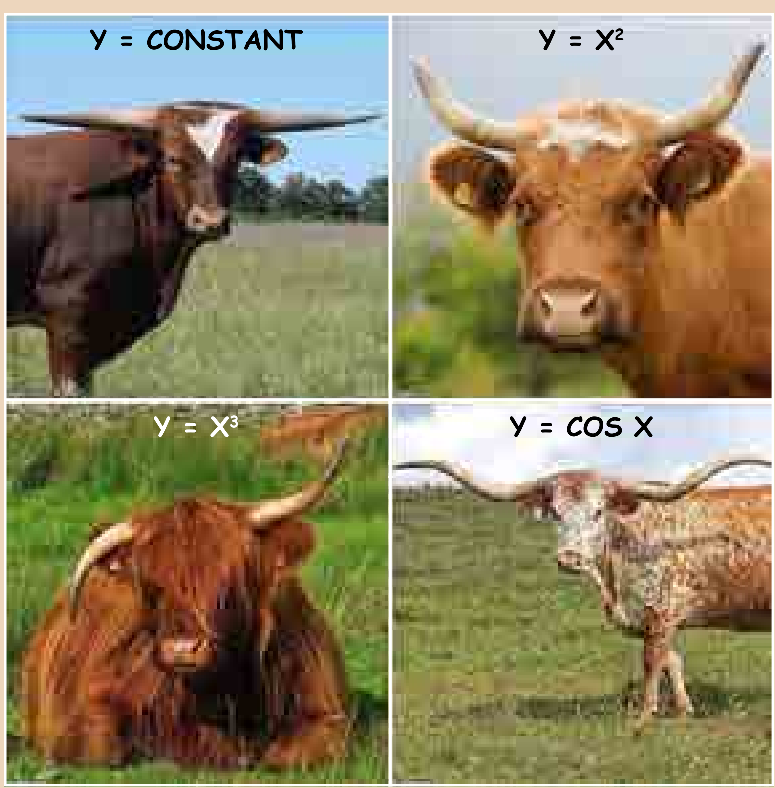
AN INTRODUCTION TO MOO-MATHEMATICAL CURVES

Facebook entry 'Moothematics', posted by Mathnasium of Stony Oak on June 30, 2019



Brahmagupta's Theorem

M NARASIMHA MURTHY

In this short note, we discuss a beautiful theorem first proved by the great seventh century Indian mathematician Brahmagupta—a theorem about a cyclic quadrilateral.

Theorem. *In a cyclic quadrilateral whose diagonals are perpendicular to each other, the line through the point of intersection of the diagonals which is perpendicular to one side bisects the opposite side.*

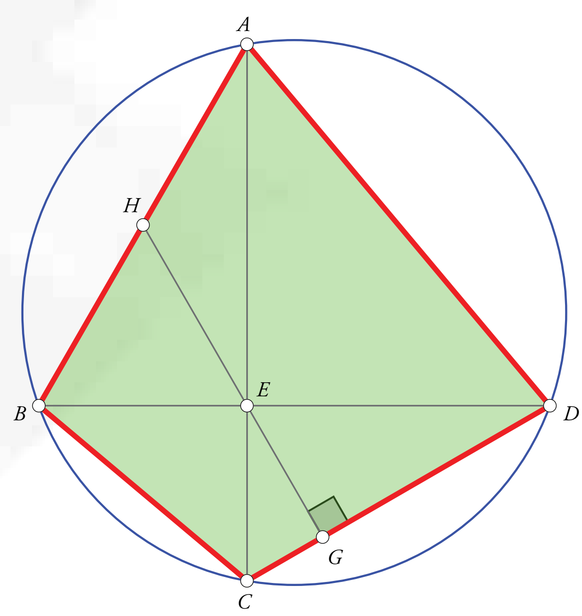


Figure 1.

Figure 1 illustrates what this states. $ABCD$ is a cyclic quadrilateral; E is the point of intersection of diagonals AC and BD. It is given that AC and BD are perpendicular to each other. Through E, a line EG is drawn, perpendicular to CD. It intersects AB at H. We must show that H is the midpoint of AB, i.e., $AH = HB$.

Keywords: Brahmagupta, cyclic quadrilateral, Heron's formula

Proof. In Figure 2, let x denote $\measuredangle CEG$, and let y denote $\measuredangle ECG$. Then x and y must be complements of one another (i.e., they add up to a right angle), since $\measuredangle EGC$ is a right angle. Observe that $\measuredangle AEH = x$ (vertically opposite angles), and $\measuredangle ABD = y$ (using the "angles in the same segment" theorem). Also, $\measuredangle BAE$ is the complement of $\measuredangle ABD$, since $\measuredangle AEB$ is a right angle. Since $\measuredangle ABE = y$, it follows that $\measuredangle BAE = x$. And since $\measuredangle BEH$ is the complement of $\measuredangle AEH$, it must be that $\measuredangle BEH = y$.

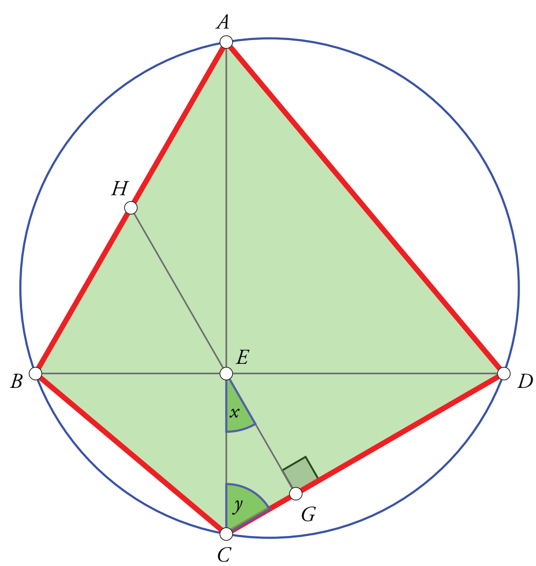


Figure 2.

Therefore we have $\measuredangle HEA = \measuredangle HAE$, and $\measuredangle HBE = \measuredangle HEB$. It follows that $HA = HE$, and $HE = HB$. Hence $HB = HA$, i.e., H is the midpoint of AB, as was to be proved. ■

Further remarks.

- The above proof is a nice illustration of the principle of transitivity: if $a = b$ and $b = c$, then $a = c$.
- Brahmagupta also showed that the area Δ of a cyclic quadrilateral with given sides a, b, c, d is given by the following beautifully symmetric formula:
$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$
where $s = \frac{1}{2}(a+b+c+d)$ is the semi-perimeter of the quadrilateral.
- The above formula reduces to a familiar formula ('Heron's formula') for the area of a triangle in the case when $d = 0$, i.e., one side reduces to zero length (which means in effect that the quadrilateral has collapsed into a triangle).
- Brahmagupta was the first mathematician to state explicitly the arithmetic rules for operating with 0 and the rules for working with negative numbers.

Simple CRYPTOGRAPHY

MEGHRAJ BHATT

This is in reference to the articles [1], [2] and [3] published in At Right Angles. All of them explain clearly the basic mechanism of the science of cryptography for those who have studied Matrices and Modulo arithmetic. What about a student of secondary level?

To answer this question, let us try to develop a simple cryptogram which does not use matrix algebra but uses a simple linear function and a simplified version of modulo arithmetic. We need to prepare a table to convert the letters of the alphabet to numerals and vice-versa. We will use the same conversion table as used in [3]. It is as follows:

Blank Space	.	?	A	B	C	D	E	F	G	H	I	J	K	L
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	
15	16	17	18	19	20	21	22	23	24	25	26	27	28	

Table I

Now let us take a linear algebraic function, say $y = 2x + 3$ … (1).

The person who is sending the message in encrypted form and the person who has to decrypt the message must know the above table, the function given by (1) and the transformation rule (explained in step 3 ahead).

Now, let the message be: "**MATHEMATICS IS EASY.**"

Keywords: Cryptography, coding, encryption, functions

ENCRYPTION

Step 1

Replace all the letters in the message by the corresponding numbers from Table I. The space between the words will be replaced by 0 (zero) and full stop by 1. So, 'M' will be replaced by 15, 'A' by 3, 'T' by 22 and so on. We will insert '–' (dash) between two numbers to separate them. The whole message will be converted to:

15 – 3 – 22 – 10 – 7 – 15 – 3 – 22 – 11 – 5 – 21 – 0 – 11 – 21 – 0 – 7 – 3 – 21 – 27 – 1 … (2)

Step 2

Now, take these numbers as values of x in the function $y = 2x + 3$ and calculate the corresponding values of y, e.g. for $x = 15$, $y = 2(15) + 3 = 33$; for $x=3$, $y = 9$, etc. So the whole message in (2) will be converted into a new string of numbers (i.e. values of y) as follows: 33 – 9 – 47 – 23 – 17 – 33 – 9 – 47 – 25 – 13 – 45 – 3 – 25 – 45 – 3 – 17 – 9 – 45 – 57 – 5 … (3)

Step 3: Transformation Rule

In string (3), we see that all the numbers are odd numbers (because of the nature of the function (1)). We also observe that there are two types of numbers: (a) less than or equal to 28 and (b) greater than 28.

(a) For numbers less than or equal to 28, we can use Table I directly; e. g. for the second number 9, we can replace it with G; we can replace 23 with U, etc.

(b) For numbers greater than 28, we will follow this rule: "If 'n' is the number ($n > 28$) find 'r' ($1 \le r \le 28$) such that $n = 28\,(p) + r$, $p \in N$ and replace 'n' by the letter corresponding to 'r' from Table I. But this letter will have superscript '$+p$' over it; e.g. (i) for the first number 33, we get $33 = 28(1) + 5$; hence p = 1 and r = 5. Hence we will look in the table for 5 and we get 'C '. So in the encryption, we will replace 33 by C^{+1}.
(ii) For the number 57, we get $57 = 28(2) + 1$; hence, p = 2 , r =1 . Table shows '.' for 1 and so we will write '$.^{+2}$' for 57.

So, the **encrypted** script will be:

$C^{+1}\ G\ Q^{+1}\ U\ O\ C^{+1}\ G\ Q^{+1}\ W\ K\ O^{+1}$
$A\ W\ O^{+1}\ A\ O\ G\ O^{+1}\ .^{+2}\ C$ … (*A*)

We will send this text to the receiver.

DECRYPTION

The person receiving the above encrypted message must know

i. Table 1,
ii. The function and
iii. The transformation rule.

First of all, he has to find the inverse function of the function given in (1). It means that, for decryption, he has to use the function $x = \frac{y-3}{2}$ … (4)

Step 4: Transformation Rule (for decryption)

Let us observe the encrypted message carefully. It is made up of two types of letters:

(a) Simple letters like G, U, etc.

(b) Letters having a superscript of the type $+n$ namely, C^{+1}, Q^{+1}, etc.

We decrypt them as follows.

(a) For simple letters, we can use Table I directly to get the value of y; e.g. G = 9, U = 23, etc.

(b) For letters having a superscript $+n$ on it, go to the table and find the number and add ($28*n$) to it to get the value of y; e.g. C^{+1}: from the table $C = 5$ and hence $y = 5 + 28(1) = 33$ or Q^{+1} : from the table $Q = 19$ and hence $y = 19 + 28(1) = 47$ and in particular, $.^{+2}$: from the table . = 1 and hence $y = 1 + 28(2) = 57$. This will give us any number greater than 28. Now replace all letters by these numbers obtained by (a) or (b) and find a string of y values as follows:

33 – 9 – 47 – 23 – 17 – 33 – 9 – 47 – 25 – 13 – 45 – 3 – 25 – 45 – 3 – 17 – 9 – 45 – 57 – 5 … (5)

Step 5

The above string represents y. Now put these values of y in the function $\mathbf{x} = \frac{y-3}{2}$ and get the corresponding values of x. So the string of x values will be:

15 – 3 – 22 – 10 – 7 – 15 – 3 – 22 – 11 – 5 – 21 – 0 – 11 – 21 – 0 – 7 – 3 – 21 – 27 – 1 …(6)

Step 6

At the end, again use Table I and write the proper letter or space or symbol to decrypt and get the original message. It will read: MATHEMATICS IS EASY.

Closing remarks

Here are some points which may enhance the understanding of the procedure.

1. One can choose any linear function which is easy to invert and for which calculations become easy. We want to avoid negative numbers.
2. Table I may be changed to include more punctuation marks. We can even avoid coding all punctuation marks. In that case, they will not be encrypted and will be shown in the encrypted script as they are.
3. One can assign the numbers to letters in any random manner.
4. If the greatest number assigned is different from 28, one has to modify the encryption / decryption rules suitably.
5. If numerals are present in the original message, one has to extend the table and set some other trick for encryption. The simple way is to not encrypt the numerals!
6. The whole work provides exercises in algebra and students will enjoy doing it in a play way method.
7. A teacher can modify this according to his/her requirement.

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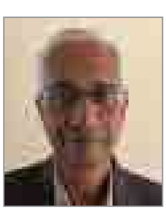

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ClassRoom

Generalising the Divisibility Tests of 9 and 11

A RAMACHANDRAN

Divisibility tests for 9 and 11 are generally taken up at upper primary school level. The test for 9 seems to remain in people's consciousness long after they leave school and move on to other things. The assertion that the sum of the digits of any multiple of 9 is itself a multiple of 9 seems to give a mystical aura to the number 9. The test for 11 is more involved and does not seem to be retained as well, but other patterns in the multiples and powers of 11 are widely appreciated.

The explanations why these tests work calls for a higher level of understanding and does not seem to be part of the curriculum though it may appear under 'enrichment.' Both these tests rely on the fact that we follow the base 10 or decimal system. The place values in a decimal system are 1, 10, 10^2, 10^3,10^4,… Now all these numbers leave a remainder of 1 when divided by 9. So when we add the digits we are actually adding the remainders under division by 9.

For example, when $241 = 2 \times 100 + 4 \times 10 + 1$ is divided by 9, the remainder is $2 + 4 + 1 = 7$. To see why, observe that on division by 9, 200 leaves a remainder of 2, 40 leaves a remainder of 4, and 1 leaves a remainder of 1, so when 241 is divided by 9, the remainder is $2 + 4 + 1$.

In the case of division by 11, the situation is slightly different. 1, 10^2, 10^4, … leave remainders of 1 when divided by 11 ($1 = 0 \times 11 + 1$; $100 = 9 \times 11 + 1$; $10000 = 909 \times 11 + 1$ and so on). On the other hand, 10, 10^3,10^5, … are all one short of a multiple of 11 ($10 = 1 \times 11 – 1$; $1000 = 91 \times 11 – 1$; $100000 = 9091 \times 11 – 1$, etc.). The sum of the digits positioned at the even powers of 10 is thus the sum of the remainders (excesses) under division by 11, while the sum of the digits positioned at the odd powers of 10 is the sum of the shortfalls (deficiencies) from the next multiple of 11. Therefore, the number

Keywords: Divisibility tests, number bases, algebraic factorisation.

under test is divisible by 11 if and only if the difference of these sums is a multiple of 11. (Note that 0 too is a multiple of 11.)

For example, when $143 = 1 \times 10^2 + 4 \times 10^1 + 3 \times 10^0$ is divided by 11, the remainder is $3 - 4 + 1 = 0$. To see why, observe that on division by 11, 1×10^2 and 3×10^0 leave remainders of +1 and +3, while 4×10^1 is 4 short of a multiple of 11. So 143 leaves remainder $1 + 3 - 4 = 0$ when divided by 11. Thus, it is divisible by 11. In the same way, 987 leaves remainder $7 - 8 + 9 = 8$ when divided by 11; it is not divisible by 11.

So these tests of divisibility hinge on the fact that 9 is one less than 10 and 11 is one more than 10, the base of the decimal system. Then a similar situation should arise in other bases too. Let us check it out with base 6 system. Here 5 and 7 play the role of 9 and 11, respectively. The place values in base 6 are 1, 6, 36, 216, 1296, 7776, etc. (That is, in base 10 representation; in base 6 representation they would just be 1, 10, 100, etc.) All these leave a remainder of 1 under division by 5. Even powers of 6, i.e.,1, 36, 1296,….leave a remainder of 1 under division by 7, while odd powers of 6, namely, 6, 216, 7776,….are all one short of a multiple of 7.

We now see if we can show the same for the general case, say base n. The place values here would be 1, n, n^2, n^3, n^4, n^5, … Now $n - 1$ and $n + 1$ take the roles of 9 and 11. Note the following:

- $1 = (n - 1) \times 0 + 1$;
- $n = (n - 1) \times 1 + 1$;
- $n^2 = (n - 1)(n + 1) + 1$; this follows from $n^2 - 1 = (n - 1)(n + 1)$;
- $n^3 = (n - 1)(n^2 + n + 1) + 1$; this follows from $n^3 - 1 = (n - 1)(n^2 + n + 1)$;
- $n^4 = (n - 1)(n^3 + n^2 + n + 1) + 1$; and so on.

In general, $n^k - 1$ is divisible by $n - 1$ for all numbers k.

So all powers of n leave a remainder of 1 under division by $n - 1$, analogous to the situation with 9 in base 10.

Further,

- $n = (n + 1) \times 1 - 1$;
- $n^3 = (n + 1)(n^2 - n + 1) - 1$; this follows from $n^3 + 1 = (n + 1)(n^2 - n + 1)$;
- $n^5 = (n + 1)(n^4 - n^3 + n^2 - n + 1) - 1$; and so on.

In general, $n^k + 1$ is divisible by $n + 1$ for all odd numbers k.

Moreover,

- $1 = (n + 1) \times 0 + 1$;
- $n^2 = (n + 1)(n - 1) + 1$;
- $n^4 = (n + 1)(n^2 + 1)(n - 1) + 1$;
- $n^6 = (n + 1)(n^2 - n + 1)(n^3 - 1) + 1$; and so on.

In general, $n^k - 1$ is divisible by $n + 1$ for all even numbers k.

Thus alternate place values are one more and one less than a multiple of $n + 1$, analogous to the situation with 11 in base 10.

It is hoped that this explanation will not take away from the sense of wonder or magic that a young student may associate with these numbers.

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ClassRoom

Student Corner – Featuring articles written by students.

An Unusual Proof of the Centroid Theorem

RAJATADRI VENKATASUBBAN

Introduction. In this article, I present an unusual proof of the Centroid Theorem. (The theorem states: *For any triangle, the three medians meet in a point. Moreover, the common point of intersection is a point of trisection of each median.*) The standard methods (see [3], pg 65 for a much shorter proof that uses the same base results as this one, or [1], pg 7 for one that uses Ceva's theorem) require nothing but elementary geometry. Another vector-based approach (see [2], pg 19) also exists. This one, however, makes use of an infinite geometric progression to achieve its result.

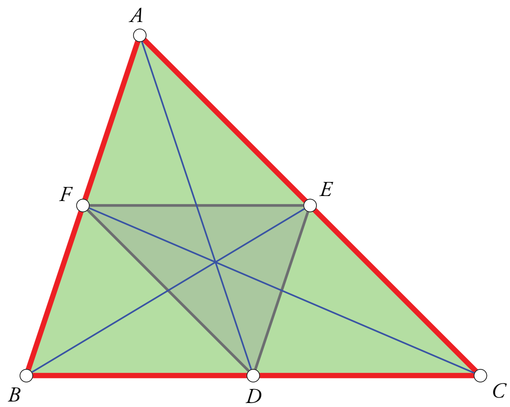
Background. The centroid is a point that has been known since antiquity [4]. It is the meeting point of the three medians of a triangle (lines from each vertex to the midpoint of the opposite side). Interestingly, this point is also the triangle's centre of mass.

It is not obvious that these lines should concur at all. Here I show that they do indeed intersect, and, moreover, that the point G of intersection is one that divides each median in the ratio $1 : 2$.

Concurrence. Observe that the meeting point of each pair of medians of $\triangle ABC$ lies within $\triangle ABC$, and that triangles AFE, FBD, EDC and DEF are congruent to each other. (See Figure 1.)

Observe also that median AD bisects EF, BE bisects DF, and CF bisects DE (all these statements also require proof). So the medians of $\triangle DEF$ lie along the medians of $\triangle ABC$. Therefore, *the meeting point of each pair of medians of* $\triangle ABC$ *lies within* $\triangle DEF$.

Keywords: Median, centroid theorem, Ceva's theorem, vector, infinite geometric series



Given a triangle $\triangle ABC$, we find the midpoints D, E, and F of its sides and construct its medians and the medial triangle, $\triangle DEF$.

Figure 1. $\triangle ABC$ with its medians and the medial triangle, $\triangle DEF$

We can apply exactly the same logic to show that the meeting point of each pair of medians of $\triangle ABC$ lies within $\triangle D'E'F'$ (see Figure 2), the medial triangle of $\triangle DEF$. And then within $\triangle D''E''F''$. …And so forth, until they are constrained to meet within a triangle of infinitesimal proportions (i.e., at a point), thus establishing concurrence. ■

Trisection. Now we show that the centroid divides each median in the ratio 1 : 2. We do this with the median FC, but the same logic would apply with the other medians.

Let F'' be the midpoint of $D'E'$. Then $FF'' = \frac{1}{4}FC$. Repeating the double-medial-triangle procedure that brought $\triangle D'E'F'$ into existence, we get $\triangle D''''E''''F''''$.

Then $F''F'''' = \frac{1}{4}FF'' = \frac{1}{16}FC$. Each successive iteration adds a smaller length one fourth the length of the previous one. Thus each time we add a new smaller medial triangle, we inch that much closer to G (unlabeled).

So we can write down a geometric series (with first term one fourth and an identical common ratio) that gives the distance FG exactly:

$$FG = FC\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\right).$$

Recalling that the sum of the infinite geometric progression $a + ar + ar^2 + ar^3 + \cdots$ is $\frac{a}{1-r}$ if the common ratio r lies between -1 and 1, we deduce that the sum of the infinite progression within the brackets is $\frac{1/4}{1-1/4} = \frac{1}{3}$. This gives us the relation $FG = \frac{1}{3}FC$. ■

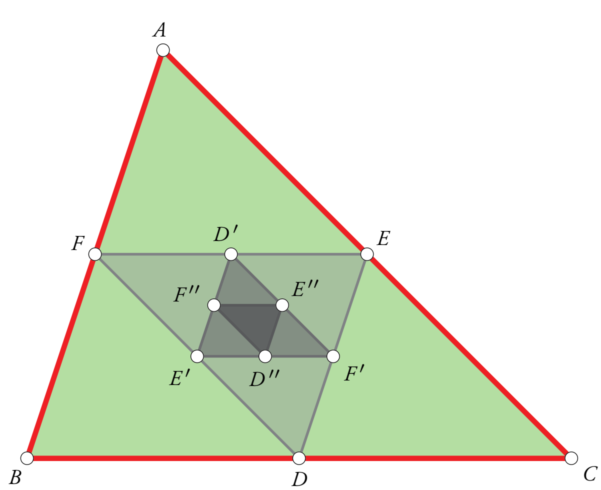


Figure 2. $\triangle D'E'F'$, $\triangle D''E''F''$, …

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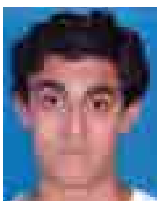

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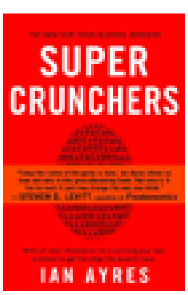
From "Super Crunchers: Why Thinking By Numbers is the New Way To Be Smart" *– By Ian Ayres*

Regression analysis is the original and still most-widely used statistical technique for analyzing data, which today is used prevalently by retailers, banks, internet search engines and even internet dating services to determine customer preferences, tendencies and other correlations:



"A regression is a statistical procedure that takes raw historical data and estimates how various causal factors influence a single variable of interest. In [internet dating services, for example,] the variable of interest is how compatible a couple is likely to be. And the causal factors are twenty-nine emotional, social, and cognitive attributes of each person in the couple.

"The regression technique was developed more than 100 years ago by Francis Galton, a cousin of Charles Darwin. Galton estimated the first regression line way back in 1877. [Take for example] Orley Ashenfelter's simple equation to predict the quality of wine. That equation came from a regression. Galton's very first regression was also agricultural. He estimated a formula to predict the size of sweet pea seeds based on the size of their parent seeds. Galton found that the offspring of large seeds tended to be larger than the offspring of average or small seeds, but they weren't quite as large as their large parents.

"Galton calculated a different regression equation and found a similar tendency for the heights of sons and fathers. The sons of tall fathers were taller than average but not quite as tall as their fathers. In terms of the regression equation, this means that the formula predicting a son's height will multiply the father's height by some factor less than one. In fact, Galton estimated that every additional inch that a father was above average only contributed two-thirds of an inch to the son's predicted height.

"He found the pattern again when he calculated the regression equation estimating the relationship between the IQ of parents and children. The children of smart parents were smarter than the average person but not as smart as their folks. The very term 'regression' doesn't have anything to do with the technique itself. Galton just called the technique a regression because the first things that he happened to estimate displayed this tendency – what Galton called 'regression toward mediocrity' – and what we now call 'regression toward the mean.'

Portrait of Galton by Octavius Oakley, 1840

"The regression literally produces an equation that best fits the data. Even though the regression equation is estimated using historical data, the equation can be used to predict what will happen in the future. Galton's first equation predicted seed and child size as a function of their progenitors' size. Orley Ashenfelter's wine equation predicted how temperature and rain would impact wine quality."

Source: https://mailchi.mp/delanceyplace.com/regression-analysis-122018?e=27
(DelanceyPlace.com is a brief daily email with an excerpt or quote we view as interesting or noteworthy).

A Modified PASCAL TRIANGLE

ANAND PRAKASH

The Pascal triangle is well known. Starting with a single 1 at the top, successive rows are obtained using a simple generating rule. Here is what we get as a result (we have shown only the first few rows):

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

The generating rule used is that *from the second row downwards, each term is the sum of the two numbers immediately above it, to its upper left and to its upper right.*

A Modified Generating Rule

Instead of this familiar rule, suppose we use the following rule: *from the second row downwards, each term is the sum of all the numbers located diagonally from it in the 'north-west' direction and in the 'north-east' direction.* (As earlier, we start with a single 1 at the top.) Here is what we get when we use this generating rule (we have shown only the first few rows):

Keywords: Pascal triangle, modified Pascal triangle, partial sums, generating rule

								1								
							1		1							
						2		2		2						
					4		5		5		4					
				8		12		14		12		8				
			16		28		37		37		28		16			
		32		64		94		106		94		64		32		
	64		144		232		289		289		232		144		64	
128		320		560		760		838		760		560		320		128

Figure 1

For example, we have:

- 5 = 2 + (2 + 1);
- 37 = (4 + 12) + (2 + 5 + 14);
- 94 = (8 + 28) + (2 + 5 + 14 + 37);
- 760 = (16 + 64 + 232) + (4 + 12 + 37 + 106 + 289);

and so on. The generating rule can be used to build the array indefinitely.

Observations and Conjectures

Looking closely at the resulting triangular array of numbers, we observe a large number of patterns. We list them here under the heading 'observations and conjectures.' While some of these are obviously true, the others have the status of 'conjectures awaiting proof.'

1. The array is symmetric about its central north-south axis, just as in the case of the Pascal triangle.
2. The elements of the left-most diagonal (and the right-most diagonal too, in view of the symmetry noted above) form the sequence 1, 1, 2, 4, 8, 16, 32, . . . and these are all powers of 2. If we leave out the initial 1, they form the successive powers of 2.
3. Labelling the topmost row as the zeroth row, the sum of the entries in the n-th row is equal to $2 \cdot 3^{n-1}$ (for $n \geq 1$).
4. Now let us arrange the array in a left-justified form so that it takes the shape of a right-angled triangle (see Figure 2; to bring out the property more clearly, we have drawn lines separating the rows and columns):

1								
1	1							
2	2	2						
4	5	5	4					
8	12	14	12	8				
16	28	37	37	28	16			
32	64	94	106	94	64	32		
64	144	232	289	289	232	144	64	
128	320	560	760	838	760	560	320	128

Figure 2

Now, using this array, let us compute the sums along the SW-NE (southwest-northeast) diagonals. (One such diagonal is indicated by the numbers shown in red font.) We obtain the following numbers:

$$1, 1, 3, 6, 15, 33, 78, 187, \ldots.$$

For example, we have $3 = 2 + 1$, $6 = 4 + 2$ and $15 = 8 + 5 + 2$. Let the n-th number in this sequence be denoted by a_n (so $a_1 = 1$, $a_2 = 1$, $a_3 = 3$, $a_4 = 6$, . . .). We find the following relation connecting these numbers:

$$a_n = 3 \cdot (1+1 + 3 + \cdots + a_{n-2}) \text{ (for } n \geq 3),$$

i.e.,

$$a_n = 3 \sum_{i=1}^{n-2} a_i \qquad \text{(for } n \geq 3).$$

5. Consider the elements in the **second column** of the array in Figure 2 (note that this corresponds to the second NE-SW diagonal of the array in its original triangular form, Figure 1):

$$1, 2, 5, 12, 28, 64, 144, 320, \ldots.$$

Let the n-th number in the above sequence be denoted by b_n (so $b_1 = 1$, $b_2 = 2$, $b_3 = 5$, $b_4 = 12$, $b_5 = 28$, . . .). We find the following relations connecting these numbers:

$$5 = (2 \times 2) + 1,$$
$$12 = (2 \times 5) + 2,$$
$$28 = (2 \times 12) + 4,$$
$$64 = (2 \times 28) + 8,$$
$$144 = (2 \times 64) + 16, \ldots$$

and, in general,

$$b_n = 2b_{n-1} + 2^{n-3} \qquad \text{for } n \geq 3.$$

6. Now consider the elements in the **third column** of the array in Figure 2 (note that this corresponds to the third NE-SW diagonal of the array in its original triangular form, Figure 1):

$$2, 5, 14, 37, 94, 232, 560, \ldots.$$

Let the n-th number in the above sequence be denoted by c_n (so $c_1 = 2$, $c_2 = 5$, $c_3 = 14$, $c_4 = 37$, $c_5 = 94$, . . .). We find the following relations connecting these numbers:

$$14 = (2 \times 5) + 4,$$
$$37 = (2 \times 14) + 9,$$
$$94 = (2 \times 37) + 20,$$
$$232 = (2 \times 94) + 44,$$
$$560 = (2 \times 232) + 96, \ldots.$$

The pattern here is not readily seen but becomes visible when we focus on the sequence 4, 9, 20, 44, 96, ... We obtain:

$$9 = (2 \times 4) + 1,$$
$$20 = (2 \times 9) + 2,$$
$$44 = (2 \times 20) + 4,$$
$$96 = (2 \times 44) + 8, \ldots.$$

We again see the powers of 2. But to uncover this pattern, we had to go 'one level deeper.'

7. Consider the elements in the **fourth column** of the array in Figure 2 (note that this corresponds to the fourth NE-SW diagonal of the array in its original triangular form, Figure 1). To uncover the pattern here, we have to go still deeper. The elements here are:

$$4, 12, 37, 106, 289, 760, 1944, 4864, \ldots.$$

Let the n-th number in the above sequence be denoted by d_n (so $d_1 = 4$, $d_2 = 12$, $d_3 =37$, $d_4 =106$, $d_5 =289$, . . .). We find the following relations connecting these numbers:

$$37 = (2 \times 12) + 13,$$
$$106 = (2 \times 37) + 32,$$
$$289 = (2 \times 106) + 77,$$
$$760 = (2 \times 289) + 182,$$
$$1944 = (2 \times 760) + 424,$$
$$4864 = (2 \times 1944) + 976, \ldots.$$

Next we have, focusing on the sequence 13, 32, 77, 182, 424, 976, . . .:

$$32 = (2 \times 13) + 6,$$
$$77 = (2 \times 32) + 13,$$

$$182 = (2 \times 77) + 28,$$
$$424 = (2 \times 182) + 60,$$
$$760 = (2 \times 424) + 128, \ldots.$$

Finally we have, focusing on the sequence 6, 13, 28, 60, 128, . . .:

$$13 = (2 \times 6) + 1,$$
$$28 = (2 \times 13) + 2,$$
$$60 = (2 \times 28) + 4,$$
$$128 = (2 \times 60) + 8, \ldots,$$

and we have reached the same pattern again, only it is one layer still further down.

8. Consider again the elements in the second column (Figure 2):

$$1, 2, 5, 12, 28, 64, 144, 320, \ldots.$$

Let us form the *partial sums* of this sequence, i.e., the numbers 1, 1+2, 1+2+5, ... We obtain the following numbers:

$$1, 3, 8, 20, 48, 112, 256, 576, \ldots.$$

Now form the partial sums of *this* sequence. We obtain:

$$1, 4, 12, 32, 80, 192, 448, 1024, \ldots.$$

Let the n-th number in the above sequence be denoted by e_n (so $e_1 = 1$, $e_2 = 4$, $e_3 = 12$, $e_4 = 32$, $e_5 = 80, \ldots$). The formula that generates this sequence appears to be $e_n = n \cdot 2^{n-1}$.

Remark. There appear to be deep connections between the powers of 2 and the modified Pascal triangle. Probably, there are many more such connections to be found.

Editorial comment. The reader will notice the central role of *observed patterns* in this article. In general, the ability to spot patterns—whether numerical or geometric—is vitally important in mathematics. This is so because conjectures are generally made on the basis of observed patterns, and we notice these patterns only through a close study of experimentally generated data. (This is the mathematical equivalent of experiments in science followed by analysis of data.) The reader will also notice that assertions made in the article have not been proved. Hence we refer to them as 'observations' or 'conjectures.' We welcome proofs from our readers and we shall attempt to provide proofs in subsequent issues of the magazine.

ANAND PRAKASH runs a small garment shop at Kesariya village in the state of Bihar. He has a keen interest in number theory and recreational mathematics and has published many papers in international journals in these fields. He also has a deep interest in classical Indian music as well as cooking. In addition, he has written a large number of poems in Hindi. He may be contacted at prakashanand805@gmail.com.

Student Corner – Featuring articles written by students.

Explorations on Triangles with Integer Sides

SHREYAN JHA

In the July 2018 issue of At Right Angles, author A S Rajagopalan had explored the number of triangles with integer sides. He had shown that if two of the sides are the integers *a* and *b*, with *a* > *b*, then the third side *c* can take only integer values from *a* + *b* – 1 to *a* – *b* + 1, leading to the conclusion that there can only be 2*b* – 1 such triangles.

In this article, student Shreyan develops a formula for the number of triangles (*a*, *b*, *c*) with integer sides when *b* is fixed and *a* ≤ *b* ≤ *c*.

Theorem. *Let b be a fixed positive integer. Then the number of integer sided triangles ABC with sides (a, b, c) where $a \le b \le c$ is equal to $b(b+1)/2$.*

Proof. We are going to represent the problem graphically on the coordinate plane, so it is convenient to write x for a and y for c. The task we have before us is therefore the following: to count the number of integer pairs (x, y) satisfying the following inequalities:

$$1 \le x \le b \le y,$$
$$x + b > y.$$

Consider the second inequality, $x + b > y$. As x, b, y are all integers, this inequality is equivalent to $x + b \ge y + 1$, which may be written in the form

$$y - x \le b - 1.$$

Hence the above system of inequalities may be rewritten as follows:

$$1 \le x \le b \le y,$$
$$y - x \le b - 1.$$

We need to count the number of integer pairs (x, y) satisfying these conditions. We shall do so by graphing the inequalities and counting the lattice points (i.e., the points with integer coordinates) within the feasible region. Once we draw the relevant graph, the formula for the total number of lattice points becomes transparently clear.

Figure 1 shows the four constraints plotted on a graph. For illustrative purposes, we have used the value $b = 5$.

Keywords: Integers, triangle inequality, counting, constraints, linear programming

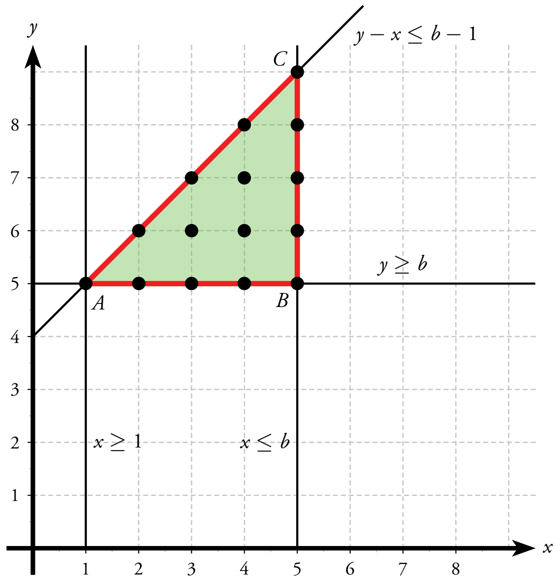


Figure 1.

Observe that the feasible region is an isosceles right-angled triangle with its vertices at $A\,(1, b)$, $B\,(b, b)$ and $C\,(b, 2b-1)$. Observe also that the lattice points within the triangle are neatly laid out in the form of a triangular array, with b lattice points on the base AB, $b-1$ lattice points one unit above the base, $b-2$ lattice points two units above the base, and so on, culminating in a single lattice point at the vertex C. Hence the total number of lattice points within the feasible region is the sum of the following arithmetic progression:

$$b + (b-1) + (b-2) + \cdots + 1,$$

which equals $\frac{1}{2}b(b+1)$, as claimed.

Non-graphical proof. If we wish to avoid arguing with reference to a graph, we could do so as follows. The system of inequalities,

$$1 \le x \le b \le y,$$
$$y - x \le b - 1,$$

may be rewritten as a single extended inequality:

$$y - b + 1 \le x \le b \le y.$$

Since $y < x + b$ and $x \le b$, we obtain $y < 2b$, i.e., $y \le 2b - 1$. Hence there is no harm in further rewriting the above extended inequality in the following form.

$$y - b + 1 \le x \le b \le y \le 2b - 1.$$

We need to count the number of integer pairs (x, y) satisfying these conditions.

- The extended inequality tells us that y can take values from b to $2b-1$ (both endpoints included).
- Suppose $y = b$; then we get $1 \le x \le b$, which means that x can take b possible integer values.
- Next, suppose $y = b + 1$; and we get $2 \le x \le b$, which means that x can take $b - 1$ possible integer values.
- Similarly, if $y = b + 2$, we find that x can take $b - 2$ possible integer values.
- This progression continues till the case $y = 2b - 1$, when x can take just 1 possible integer value.

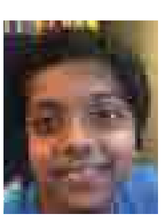
Hence the total number of possibilities is

$$b + (b-1) + (b-2) + \cdots + 1 = \frac{b(b+1)}{2},$$

as earlier.

References

1. A S Rajagopalan, "Triangle inequality — a curious counting result" from *At Right Angles*, http://teachersofindia.org/en/article/triangle-inequality-curious-counting-result



SHREYAN JHA is an 11-year old boy who is being homeschooled. He is an avid reader who finds the world around him intriguing. He loves numbers and science and enjoys conversations about these subjects. He tends to get lost in his thoughts, and his writing is a product of such introspection. Shreyan may be reached at shivajha@gmail.com.

TearOut

Isometric Sketches and More

In this 4th TearOut, we will use the isometric dot sheet to visualize various solid shapes. As before, pages 1 and 2 are a worksheet for students while pages 3 and 4 give guidelines to the facilitator. Since we will be exploring solids, it is a good idea to have some interlocking cubes handy to make some of those solid shapes.

Isometric or triangular dot sheets are different from the square grid in two ways:

I. The square grid looks the same even if the orientation changes from landscape to portrait or vice versa (Figure 1). But the isometric grids look slightly different in each orientation. One of the orientations (Figure 2) has horizontal lines (by joining nearest grid points) while the other (Figure 3) has vertical ones. For isometric sketches of 3D solids, the latter orientation is needed.

Figure 1

Figure 2

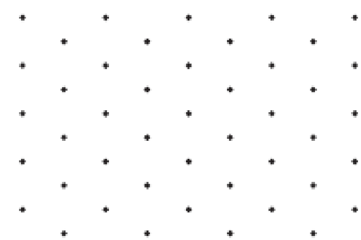


Figure 3

II. If you pick a random point A on the square grid and identify all its immediate neighbours (at most 8 of them) then some of them (the purple ones) would be at a larger distance from A, compared to others (the blue ones) – see Figure 4. But if the same thing is repeated with a random point B on the isometric dot sheet, all the immediate neighbours (at most 6 of them) would be at the same distance from B (Figure 5). This is why it is called isometric.

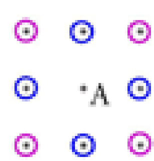


Figure 4

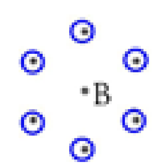


Figure 5

Get some isometric dot sheets, a pencil, an eraser and, ideally, a scale to start. For the later part, it would help to have some interlocking cubes.

1. Cubes and cuboids
   a. Make the isometric sketch of a cube
   b. Make an isometric sketch of a cuboid made by joining 3 cubes. Can you make the sketch in any other way? How?

c. Make an isometric sketch of a cuboid with dimension 3 × 2 × 5. Can you make this in other ways?

d. Make an isometric sketch of a cuboid which has a square face at least 4 square units big. What are the possibilities for the dimension perpendicular to the square face?

e. Make an isometric sketch of any cuboid of your choice.

2. Letters

a. Make isometric sketches of the following letters: E, F, H, I, L, T

Can these be done in any other way? How? [See examples in Figure 6 for T]

b. Make isometric sketches of the following letters: A, K, M, N, V, W, X, Y, Z

How are these different from the above?

c. Make isometric sketches of the remaining letters of the alphabet. How are those different from the 2nd set?

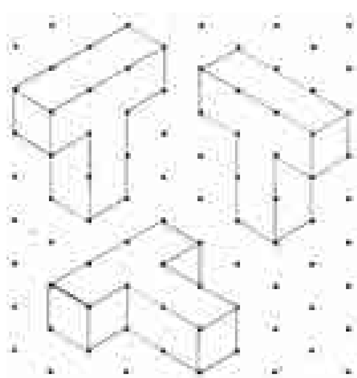


Figure 6

3. 3D ambigrams

a. Pick any 3 letters from 2a and make a 3D ambigram. Make an isometric sketch of it illustrating the 3 letters as projections. Interlocking cubes would be very helpful for this. Multiple dice can be taped together as an alternative. [See example in Figure 7 for H, T and L]

b. Repeat with a new set of 3 letters, at least one from each set 2a and 2b.

In this case, interlocking cubes won't help beyond a point. So, use your imagination and intuition.

c. Repeat with a new set of 3 letters, one from each set 2a, 2b and 2c.

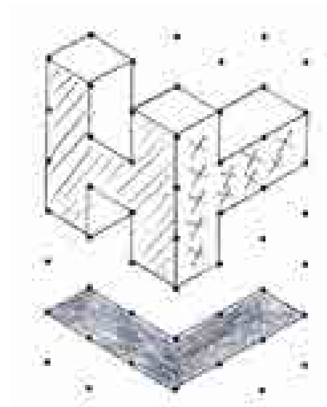


Figure 7

4. Escher

Escher is famous for his drawings that defy common sense, like the triangular bar (Figure 8).

Make your own Escher type drawing.

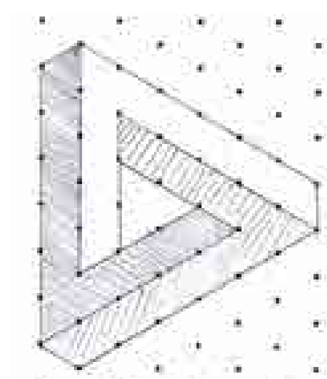


Figure 8

This is a ‘Low Floor High Ceiling’ worksheet that starts simply and within the upper primary syllabus but stretches into imagination that is abstract because it is unreal! The main focus is on developing spatial sense and visualising solid shapes. These ideas are typically ignored in mainstream teaching, because they entered the syllabus and textbooks relatively recently and therefore teachers lack exposure to them and experience in dealing with these concepts. At the same time, visualising 3D space and solids and the ability to map them to 2D are important skills for designing – a must-have for engineering and architecture, to say the least.

The initial questions are suitable for upper primary students. Interlocking cubes will be very useful for one question. It is certainly possible to tackle the question without manipulatives. But that would require a well-developed spatial sense, which may be rare. The last question demands higher order thinking in drawing something that breaks the rule and at the same time allows for artistic freedom.

1. This is a simple question dealing with isometric sketches of cubes and cuboids. The key thing is to realize that three adjacent faces of a cuboid are shown, and each rectangular face becomes a parallelogram (and squares become rhombi). It allows the exploration that a square-faced cuboid can be tall (i.e., the unequal dimension being longer than the equal ones) or short (with a shorter 3rd dimension).

   This skill can be used to draw large cubes (and cuboids) and check the associative property of multiplication with fractions, which was discussed in a poster in the Nov 2018 issue. Figure 9 illustrates the product $4/3 \times 5/6 \times 1/2$ – the unit cube has been stretched to get 4/3, 5/6 of that is shaded by “✓” and 1/2 of the cuboid is shaded by “∘”.

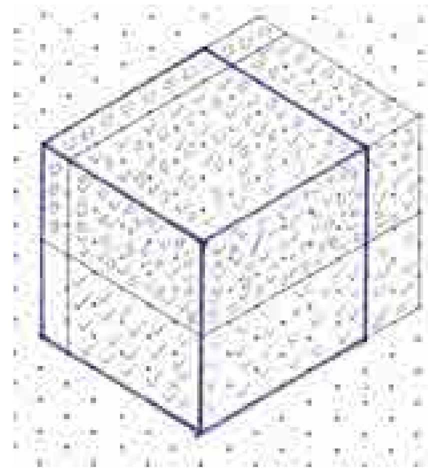


Figure 9

2. This task raises the challenge and does it gradually. The first part involves letters drawn with horizontal and vertical lines only. So, the sketches involve solids made by joining cuboids.

   The second part makes it harder by including letters that have slant lines. Pre-schoolers often play with letters that can be stuck on a magnetic surface like the refrigerator. Such letters can help one see the solids from different perspectives and then draw their isometric sketches. Some of the letters like, say A, may be more complicated than, say V. One can explore the minimum height needed to draw all features. For A, this minimum height is 4 units whereas for E it is 5 and V can be only 2 units tall (see Figure 10). The second part challenges one to imagine how the slant edges of a solid look in its isometric sketch. The distances also get distorted to an extent for the slant edges.

   The third part adds the complexity of curved edges.

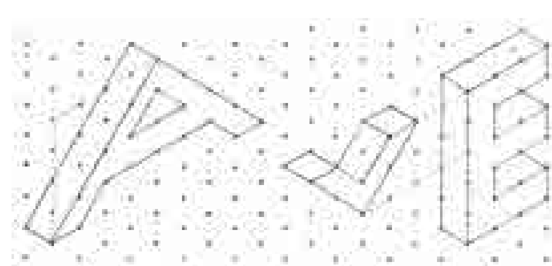


Figure 10

3. This is where it gets quite challenging. Therefore, we strongly recommend trying this question with interlocking cubes. There are two challenges: (i) creating an ambigram that will cast predetermined shadows and (ii) isometric sketch of the solid. Now, (i) requires understanding of top-view, front-view and side-view of a solid because these views are chosen, and one must reverse engineer the solid concerned. It will be an interesting exploration to see if there can be two different ambigrams (excluding rotation) for the same set of three letters. Sometimes there are more than one possibility. For (ii), it is important to select the angle that will generate the best isometric sketch. Figure 7 illustrates an ambigram for L, T and H where the shadow of L is shown, and T and H are marked with "×" and "///" respectively.

   This also starts with only perpendicular edges where interlocking cubes can be used to create the ambigram and then draw its sketches from different angles. This reverse engineering itself can be challenging and therefore engaging.

   Then, the slant edges get added and then interlocking cubes won't be of help. So, at the second level, one has to use the spatial sense developed so far to imagine the solid that can cast the predetermined shadows.

   The last part involves curved edges. Some of the letters in 2c may actually pose less challenge since they can be achieved by just rounding some corners.

   We have suggested three levels – I. all letters from 2a, II. At least one from 2a and 2b each and III. One from each of 2a, 2b and 2c. There are seven more possibilities, e.g. all three from 2b, etc. We encourage exploring them for the adventure-seeking readers. The cover of a famous book *Gödel, Escher, Bach: an Eternal Golden Braid* (or GEB) sports two ambigrams for the initials G, E and B. We hope that this exercise will pave the way for such imaginations.

4. This question can be the most fun! When mapping the 3D space on 2D, we have to abide by some rules. However, in this task, those very rules have to be broken – in a subtle way – to create something that is impossible in the physical world. Classic examples include the Penrose triangle (Figure 8) and the Penrose stairs (Figure 11) based on that.

This TearOut is based on explorations by the following MA Education students at Azim Premji University: Ankita Sharma, Keshvi and Ram Saroj, who got addicted to the isometric dot sheets!

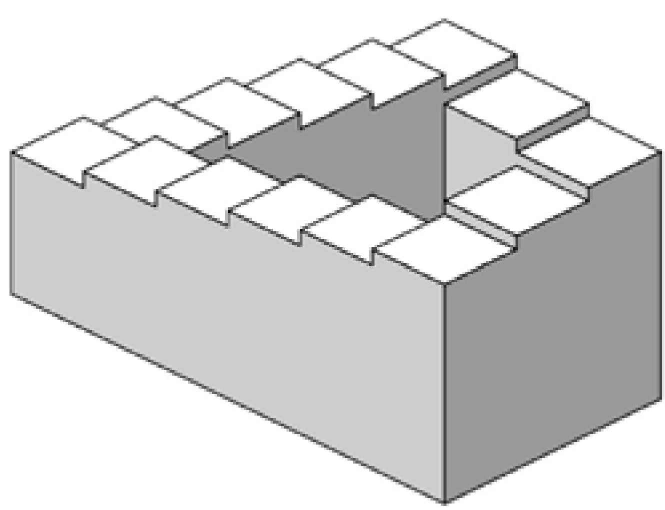


Figure 11

Low Floor High Ceiling Tasks

In-cyclic Quads

SWATI SIRCAR

Every triangle has a circumcircle and an incircle. But is that true for every quadrilateral? Definitely not; there are quads which are not cyclic (recall that a quad with a circumcircle is called 'cyclic'). But what about quads with incircles? In this Low Floor High Ceiling article, we not only explore the *if and only if* condition for a quad to have an incircle, but also reflect a certain duality. It is more fun if students are exposed to Tasks 1 and 2 before they learn the theorems related to cyclic quads.

It is a good idea to start each of these as compass-straight edge construction challenges. Once students get an idea on how to construct, it is a very good idea to replicate them using GeoGebra. That will eliminate the possibilities or doubts whether a point is actually on the circle or just appears to be so. It is not important for students to be exposed to the term cyclic quads for this exploration.

The first task is an easy one of constructing circles around different quads.

Task 1: Circumcircles

This initial task is all about finding circumcircles around various quads.

a. Construct any square ABCD. Find the intersection O of its diagonals. Construct a circle with centre O and radius OA. What do you observe?

b. Repeat the same for any rectangle.

c. Repeat for any rhombus.

d. Repeat for any parallelogram.

Keywords: Quadrilaterals, centre, in-circle, in-radius, generalisation, differentiated teaching, pedagogy.

e. Repeat for any isosceles trapezium ABCD with AB || CD.

f. Construct the perpendicular bisectors of AB and AD of the above isosceles trapezium and let them intersect at X. Construct a circle with centre X and radius XA. What do you observe?

Teacher Note: This is to understand that rectangles and isosceles trapeziums are cyclic whereas general rhombi and general parallelograms may not be. It may be worth exploring when a rhombus can be cyclic. This can be explored with GeoGebra by fixing the sides and varying the angles. Students may be asked to extrapolate their findings and form a conjecture when a parallelogram can be cyclic. Then they can verify their claim with the help of GeoGebra.

In the case of an isosceles trapezium, it may be difficult initially since the centre is not specified. The last part provides the centre and the radius. With their help, students should be able to conclude that an isosceles trapezium also has a circumcircle. The centre need not always be inside the trapezium.

The second task is the reverse one where a circle is the starting point. This demands more construction skills and therefore can be an assessment of the students' understanding of properties of the familiar quads.

Task 2: Quads inside a circle: Draw any circle to begin.

a. Construct a square in the circle such that the vertices are on the circle. Is it unique? What is the ratio of the radius of the circle to the side of the square?

b. Construct a rectangle in the circle in the same manner. Is it unique? If not, construct another with a different aspect ratio. How is the radius related to the sides in each case?

c. Can you construct a trapezium inside the circle? What do you observe?

d. Can you construct a kite inside the circle? How much are the equal angles?

Teacher Note: Given any circle, squares of only one size can be inscribed. The radius: side of the square would be $1 : \sqrt{2}$. So given any circle, the squares that can be inscribed are congruent to each other. However multiple aspect ratios, i.e., length : width are possible for rectangles inscribed within a given circle. If a and b are the sides of an inscribed rectangle and r is the radius of the circle, then $a^2 + b^2 = 4r^2$. For each positive $a < 2r$, there is a unique positive b. But there are infinitely many such pairs (a, b) and each corresponds to a different rectangle. No two such rectangles are congruent. As expected, the inscribed trapezium would be isosceles. It is a good idea to let students reason it out. Note the appearance of kite in this task and the kind of equal angles it has. [This is also the kite that maximizes area among all kites with the same sides. Why?]

We delve into the incircles with the next two tasks which are similar to the previous ones.

Task 3: Incircles

Now we get into the incircles.

a. Construct a square ABCD and the intersection point O of its diagonals. Drop a perpendicular OE from O on AB. Construct a circle with centre O and radius OE. What do you observe? What is the ratio OE : AB?

b. Repeat the same for any rhombus

c. Repeat for any rectangle

d. Repeat for any parallelogram

e. Repeat the same for any kite

f. If ∠A and ∠C are the equal angles of a kite ABCD, construct the bisector of ∠A and let it intersect BD at X. drop a perpendicular XE from X on AB. Construct a circle with centre X and radius XE. What do you observe?

Teacher Note: As in Task 1, we start with the quads and try to construct incircles. The square obviously has one, such that the inradius : side of the square = 1 : 2. Rhombi have incircles whereas rectangles and parallelograms do not. For a kite, the point of intersection of the diagonals is not the incentre. So, the last part indicates the incentre. It is a good idea to discuss why the point of intersection of the diagonals did not work for the kite. Certain parallels can be drawn based on the incentre of a triangle.

The next task, like Task 2, starts with the circle and poses higher levels of construction challenges.

Task 4: Quads outside a circle: Draw a circle to begin.

a. Construct a square around it so that each side of the square touches the circle. Is the square unique? What is the ratio of the radius of the circle to the side of the square?

b. Repeat the same for any rhombus. Is it unique? If not, construct another with different angles.

c. Can you construct a kite?

d. Can you construct an isosceles trapezium ABCD such that AB || CD and AD = BC? Extend AB to A' such that BA' = CD and AD to E such that DE = BC. What kind of triangle is ΔAA'E?

Teacher Note: This is a reverse activity (like Task 2) to understand that certain quads can be wrapped around circles. The constructions will demand some level of critical thinking and problem solving. The circumscribed square is unique for a given circle since radius : side = 1 : 2. However the same circle can have several (actually infinitely many) rhombi around it. Any arbitrary point outside the circle can be the vertex of a circumscribing rhombus. The construction can be fun and challenging. If students are familiar with drawing tangent to a circle, this will provide a good practice of that skill. Otherwise, this may be a good point to introduce that theorem. For a kite, there are more degrees of freedom. Any two points outside the circle and collinear with the centre can generate a circumscribed kite. Note the reappearance of isosceles trapezium. The construction demands visualization. There are infinitely many isosceles trapeziums circumscribing any given circle. Any line segment shorter than the diameter and touching the circle at its midpoint would generate an isosceles trapezium. The triangle should be isosceles. Check the next task to know why!

Task 5: Draw any circle. Construct any quad ABCD around it such that all four sides touch the circle. Prove that AB + CD = AD + BC

Teacher Note: This follows easily from tangent properties and observing that each side is the sum of two tangents.

Task 6: Construct two equal lines XY and PQ. Find any two points Z and R on XY and PQ respectively such that XZ < ZY and PR < RQ.

a. Construct a convex quad ABCD such that AB = XZ, AD = PR, CD = ZY and BC = RQ. Note that you can choose any angle ∠A < 180°. Compare AB + CD and AD + BC.

b. Construct the angle bisectors of any two consecutive angles. Let the bisectors intersect at O. Drop a perpendicular OE from O to AB. Construct a circle with centre O and radius OE. What do you observe?

c. Is this the converse of Task 5? Make a conjecture.

d. Prove your conjecture.

Note on the construction: If one tries to draw a quad ABCD such that AB + CD = AD + BC with four arbitrary side-lengths satisfying the given condition and any given angle, then it is possible that the quad may not close (if ∠A is too large and/or if BC and CD are too small). Also, one must get a convex quad and therefore must choose C to be the point of intersection of the arcs (centred at B and D) further from A. Now any quad with the given side-sum condition, must have the shorter two sides consecutive. So, if one starts with these shorter sides as AB and AD, then the quad would definitely close for any choice of ∠A < 180°. Therefore, the conditions XZ < ZY and PR < RQ have been included.

Teacher Note: This is a way to establish the reverse i.e. if AB + CD = AD + BC in a convex quad ABCD, then it has an incircle. The proof can be done in the following way:

- Let the angle-bisectors of ∠A and ∠B meet at I, drop perpendiculars from I to all sides – IE ⊥ AD, IF ⊥ AB, IG ⊥ BC and IH ⊥ CD, and show that IE = IF = IG
- We need to show IH = IE
- Suppose IH ≥ IE (1) ⇒ DH ≤ DE (2) using Pythagoras
- DE = AD – AE = (AB + DC – BC) – AE = (AF + BF) + DC – (BG + GC) – AE = DC – GC i.e. DE = DC – GC (3)
- DH = DC – HC (4)
- (2) – (4) ⇒ HC ≥ GC ⇒ IH ≤ IG = IE (5)
- (1) and (5) ⇒ IH = IE i.e. I is equidistant from all four sides, allowing an incircle to be drawn.

Supposing IH ≤ IE would lead to the same conclusion with all the inequalities reversed.

Closing reflections: Recall that the condition for a quad to be cyclic is that opposite angles are supplementary. That is equivalent to saying that the sums of opposite angle-pairs are equal, i.e., ∠A + ∠C = ∠B + ∠D (= 180°). This is very similar to the condition for incircle where angles have been replaced by sides! It is also worth noting that the perpendicular bisectors of sides meet at the circumcentre while the angle-bisectors do at the incentre – another exchange of sides and angles – the duality!!

While squares, rectangles, rhombi and parallelograms get their fair share in the syllabus, kites and isosceles trapeziums do not. Here are some of the duality that we observed:

	Kites	Isosceles trapeziums (IT)
1.	Two pairs of equal adjacent sides	Two pairs of equal adjacent angles
2.	Line of symmetry passing through vertices	Line of symmetry passing through sides
3.	Have incircle	Have circumcircle
4.	Some kites have circumcircles	Some ITs have incircles
5.	Rhombi = equilateral quads are special kites	Rectangles = equiangular quads are special ITs
6.	Kites ∩ parallelograms = rhombi	ITs ∩ parallelograms = rectangles
7.	Rhombi with circumcircles = squares	Rectangles with incircles = squares

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HOW TO PROVE IT

SHAILESH SHIRALI

In the episode of "How To Prove It" that appeared in the March 2018 issue of *At Right Angles*, we studied a number of characterisations of a parallelogram; they were also listed in the article on parallelograms elsewhere in that issue. Three assertions had been made in the article without proof. We provide the proofs in this article.

The following question was posed in the article 'Parallelogram' in the March 2018 issue of this magazine: *What characterises a parallelogram?* In other words: *What minimal properties must a quadrilateral have for us to be sure that it is really a parallelogram?* (See [3] and [4].)

The basic definition of a parallelogram is: *A plane four-sided figure whose opposite pairs of sides are parallel to each other.* That is, a plane four-sided figure $ABCD$ is a parallelogram if and only if $AB \parallel CD$ and $AD \parallel BC$ (see Figure 1). An alternative definition, framed in the language of transformations, is: *A parallelogram is a quadrilateral with rotational symmetry of order* 2.

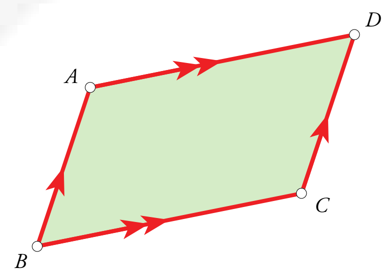

Figure 1.

In the articles mentioned above, we listed five properties possessed by a parallelogram and asked in each case whether the property in question characterises a parallelogram; i.e., *if a planar quadrilateral possesses that property, is it then necessarily a parallelogram*? We reproduce that list below, retaining the numbering from the original articles.

Keywords: Parallelogram, characterisation, congruence, one-way implication

Which of the following properties characterises a parallelogram?

(5) If $ABCD$ is a parallelogram, then each of its diagonals divides it into a pair of triangles with equal area. Does this characterise a parallelogram? That is: *If ABCD is a planar quadrilateral such that each of its diagonals divides it into two triangles that have equal area, then is ABCD necessarily a parallelogram?*

(6) If $ABCD$ is a parallelogram, then $AB = CD$ and $AD \parallel BC$. Does this characterise a parallelogram? That is: *If ABCD is a planar quadrilateral such that* $AB = CD$ *and* $AD \parallel BC$*, then is ABCD necessarily a parallelogram?*

(7) If $ABCD$ is a parallelogram, then $AB = CD$ and $\measuredangle A = \measuredangle C$. Does this characterise a parallelogram? That is: *If ABCD is a planar quadrilateral such that* $AB = CD$ *and* $\measuredangle A = \measuredangle C$*, then is ABCD necessarily a parallelogram?*

(8) If $ABCD$ is a parallelogram, then the sum of the squares of the sides equals the sum of the squares of the diagonals. Does this characterise a parallelogram? That is: *If ABCD is a planar quadrilateral such that*

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2,$$

then is ABCD necessarily a parallelogram?

(9) If $ABCD$ is a parallelogram, then the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point. Does this characterise a parallelogram? That is: *If ABCD is a planar quadrilateral such that the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point, then is ABCD necessarily a parallelogram?*

We then showed—through counterexamples—that statements 6 and 7 *do not* provide characterisations of a parallelogram, and added (without proof) that statements 5, 8 and 9 do provide the asked-for characterisations. We now provide the proofs.

Proofs of statements 5, 8 and 9

Theorem 5. *If ABCD is a planar quadrilateral such that each of its diagonals divides it into two triangles with equal area, then ABCD is a parallelogram.*

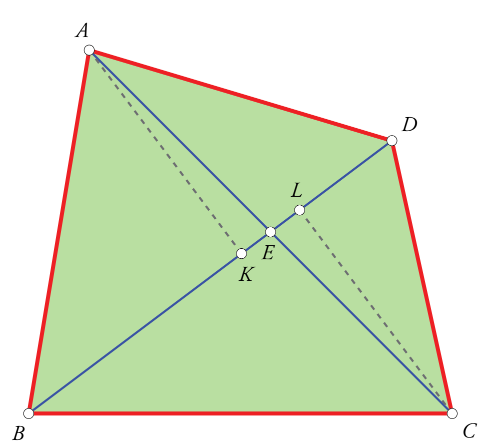

Figure 2.

Proof. Let's start with the statement that diagonal BD bisects the quadrilateral into two triangles with equal area. Figure 2 shows the relevant picture. Drop perpendiculars AK and CL from A and C to diagonal BD.

Since triangles ABD and CBD have equal area, it follows from the formula for area of a triangle ("half base times height") that $AK = CL$.

Now consider $\triangle AKE$ and $\triangle CLE$. It is easy to see that they are congruent to each other ('ASA congruence'). Hence $AE = CE$. In other words, *diagonal BD bisects diagonal AC*. This conclusion follows from the hypothesis that BD bisects the quadrilateral into two triangles with equal area.

In the same way, from the hypothesis that diagonal AC bisects the quadrilateral into two triangles with equal area, we deduce that diagonal AC bisects diagonal BD.

This means that the two diagonals bisect one another. It is well-known (and trivial to prove) that this property implies that $ABCD$ is a parallelogram. ■

Proof using vectors. Taking one vertex of the quadrilateral to be the origin, let the position vectors of the other three vertices be, in cyclic order, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (Figure 3a). The claim that these four points are the vertices of a parallelogram may then be replaced by the equivalent claim that $\mathbf{b} = \mathbf{a} + \mathbf{c}$. So this is what we must prove, under the hypothesis that each of the diagonals bisects the quadrilateral into two parts with equal area.

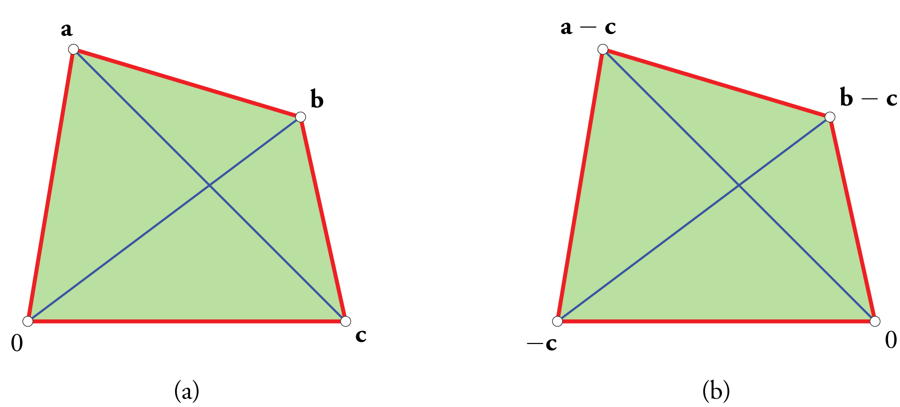

Figure 3.

Let's start with the condition that the diagonal joining the points 0 and $\mathbf{b}$ divides the quadrilateral into two parts with equal area (Figure 3a). Using the vector formula for area, we arrive at the following condition:

$$\frac{1}{2}(\mathbf{b} \times \mathbf{a}) = \frac{1}{2}(\mathbf{c} \times \mathbf{b}).$$

Cancelling common factors in bringing all the terms to one side, we deduce that

$$\mathbf{b} \times (\mathbf{a} + \mathbf{c}) = 0,$$

which implies that $\mathbf{b}$ is parallel to $\mathbf{a} + \mathbf{c}$. Hence $\mathbf{b} = k(\mathbf{a} + \mathbf{c})$ for some real number k.

Now we consider the other condition, that the diagonal joining the points $\mathbf{c}$ and $\mathbf{a}$ too divides the quadrilateral into two parts with equal area. Shifting the origin so that it falls at vertex $\mathbf{c}$, the picture assumes the form shown in Figure 3b. Using the vector formula for area, we deduce that

$$\mathbf{a} - \mathbf{c} = m(\mathbf{b} - \mathbf{c} - \mathbf{c}),$$

for some real number m. (We do not have to go through the steps all over again; we can use the final equality deduced earlier.) Substituting for $\mathbf{b}$ from above, we obtain

$$\mathbf{a} - \mathbf{c} = m(k(\mathbf{a} + \mathbf{c}) - 2\mathbf{c}),$$

$$\therefore\ \mathbf{a} - \mathbf{c} = mk\mathbf{a} + m(k-2)\mathbf{c}.$$

Since **a** and **c** are linearly independent vectors, we may equate their coefficients on the two sides of the above equality and solve for m and k. We obtain $mk = 1$ and $mk - 2m = -1$, and these equations yield $m = 1$ and $k = 1$. Hence $\mathbf{b} = \mathbf{a} + \mathbf{c}$, and the required conclusion has been proved. ■

Remark. It is usually the case that a vector-based proof achieves the desired goal far more efficiently and compactly than an approach based on pure geometry. In the above example, however, the opposite seems to be the case!

Theorem 8. *If ABCD is a planar quadrilateral such that*

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2,$$

then ABCD is a parallelogram.

Proof. We shall make use of a well-known theorem first proved in the second century AD: the theorem of Apollonius. (Indeed, we shall use the theorem as many as six times!)

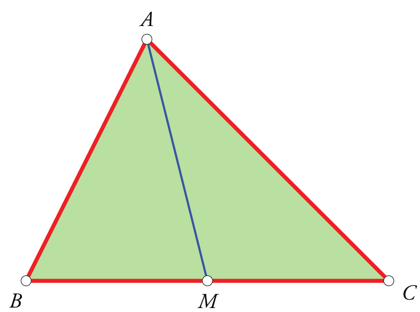

Figure 4.

The theorem provides a relation between the lengths of the medians and the lengths of the sides of a triangle: if ABC is any triangle and AM is a median (i.e., M is the midpoint of side BC; see Figure 4), then

$$AB^2 + AC^2 = 2\left(AM^2 + BM^2\right).$$

Otherwise put: if a, b, c are the lengths of the sides of the triangle, and m_a, m_b, m_c the lengths of the medians (named in the symmetric manner), then $4m_a^2 = 2b^2 + 2c^2 - a^2$ (with similar relations for m_b and m_c).

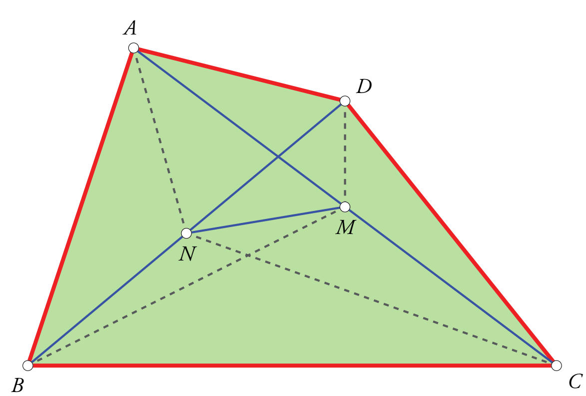

Figure 5.

With these preliminaries in place, we may set out the proof as follows. Let M and N be the midpoints of diagonals AC and BD respectively. Repeated application of the theorem of Apollonius (to triangles ABD,

DAC, CDB and BCA respectively) yields the following equalities:

$$AB^2 + AD^2 = 2AN^2 + 2BN^2,$$
$$DA^2 + DC^2 = 2DM^2 + 2AM^2,$$
$$CD^2 + CB^2 = 2CN^2 + 2DN^2,$$
$$BC^2 + BA^2 = 2BM^2 + 2CM^2.$$

Adding up the corresponding sides of all these equalities, on the left side we obtain twice the sum of the squares of the four sides, i.e.,

$$2\left(AB^2 + BC^2 + CD^2 + DA^2\right).$$

On the right side, note that $BN^2 = DN^2$, and so $2BN^2 + 2DN^2 = 4BN^2 = BD^2$. Similarly, $2AM^2 + 2CM^2 = 4AM^2 = AC^2$. This takes care of four of the terms on the right side.

To process the sum of the remaining terms, $2\left(AN^2 + DM^2 + CN^2 + BM^2\right)$, we invoke the theorem of Apollonius yet again, applying it to triangles NAC and MDB respectively:

$$AN^2 + CN^2 = 2MN^2 + 2CM^2,$$
$$BM^2 + DM^2 = 2MN^2 + 2BN^2.$$

Hence:

$$AN^2 + DM^2 + CN^2 + BM^2 = 4MN^2 + 2CM^2 + 2BN^2$$
$$\therefore\ 2\left(AN^2 + DM^2 + CN^2 + BM^2\right) = 8MN^2 + AC^2 + BD^2.$$

It therefore follows that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2.$$

and *this equality will hold for any quadrilateral ABCD whatsoever*. (Note that this is an interesting result in its own right! It is far from obvious.)

It follows that if $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$ for a given quadrilateral $ABCD$, then it must be that $MN = 0$; in other words, M and N are coincident, i.e., the midpoints of the diagonals are coincident. But this means that the diagonals bisect one another. As is well-known, this condition implies that $ABCD$ is a parallelogram. We have reached the desired conclusion. ■

Proof using vectors. As earlier, we may attempt a vector-based proof; but this time, it does turn out to be more compact and also more insightful than the pure geometry proof!

Let the position vectors of the vertices of quadrilateral $ABCD$ be $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ respectively. For any vector $\mathbf{v}$, we shall (for convenience) use the symbol $\mathbf{v}^2$ to denote the product $\mathbf{v}\cdot\mathbf{v}$.

The sum of the squares of the lengths of the sides is equal to

$$(\mathbf{a}-\mathbf{b})^2 + (\mathbf{b}-\mathbf{c})^2 + (\mathbf{c}-\mathbf{d})^2 + (\mathbf{d}-\mathbf{a})^2$$
$$= 2\left(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2\right) - 2\left(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{a}\right).$$

Next, the sum of the squares of the lengths of the diagonals is equal to

$$(\mathbf{a}-\mathbf{c})^2 + (\mathbf{b}-\mathbf{d})^2$$
$$= \left(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2\right) - 2\left(\mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}\right).$$

Hence, the sum of the squares of the lengths of the sides minus the sum of the squares of the lengths of the diagonals is equal to

$$\left(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2\right) - 2\left(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{a}\right) + 2\left(\mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}\right).$$

A close and searching look at this expression reveals that it is a perfect square! Namely, it is equal to

$$\left[\,(\mathbf{a}+\mathbf{c}) \;-\; (\mathbf{b}+\mathbf{d})\,\right]^2.$$

This may be rewritten more revealingly as

$$4\left[\left(\frac{\mathbf{a}+\mathbf{c}}{2}\right) - \left(\frac{\mathbf{b}+\mathbf{d}}{2}\right)\right]^2.$$

This expression represents four times the square of the length of the segment that connects the midpoints of the two diagonals. This proves the identity proved earlier (using the theorem of Apollonius): $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$. The rest of the proof is the same as earlier. ■

Theorem 9. *If ABCD is a planar quadrilateral such that the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point, then ABCD is a parallelogram.*

Vector-based proof. As it happens, in this particular case, we know only a vector-based proof – but it is a most elegant and pleasing proof!

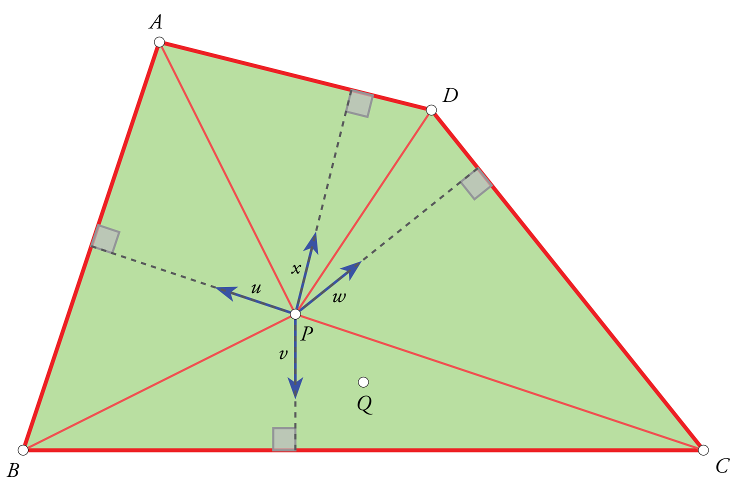

Figure 6.

Let $ABCD$ be a quadrilateral with the given property (see Figure 6), namely: the sum of the perpendicular distances from any point within the quadrilateral to the sides of the quadrilateral is the same. Let P be an arbitrary point within the quadrilateral. Join P to the vertices A, B, C, D and draw unit vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ from P perpendicular respectively to the sides AB, BC, CD, DA of the quadrilateral, as shown.

Now, observe that the perpendicular distances from P to AB, BC, CD, DA are respectively the dot products $\mathbf{PA}\cdot\mathbf{u}$, $\mathbf{PB}\cdot\mathbf{v}$, $\mathbf{PC}\cdot\mathbf{w}$, $\mathbf{PD}\cdot\mathbf{x}$. Therefore:

$$\mathbf{PA}\cdot\mathbf{u} \;+\; \mathbf{PB}\cdot\mathbf{v} \;+\; \mathbf{PC}\cdot\mathbf{w} \;+\; \mathbf{PD}\cdot\mathbf{x} \;=\; k,$$

where k is some constant. If Q is some other point within the quadrilateral and we repeat the construction described above (i.e., draw unit vectors from Q perpendicular to the sides and join Q to the vertices; note that *the unit vectors are the same as earlier*), then we have:

$$\mathbf{QA}\cdot\mathbf{u} \;+\; \mathbf{QB}\cdot\mathbf{v} \;+\; \mathbf{QC}\cdot\mathbf{w} \;+\; \mathbf{QD}\cdot\mathbf{x} \;=\; k.$$

Hence by subtraction we get:

$$\mathbf{PQ}\cdot\mathbf{u} \;+\; \mathbf{PQ}\cdot\mathbf{v} \;+\; \mathbf{PQ}\cdot\mathbf{w} \;+\; \mathbf{PQ}\cdot\mathbf{x} \;=\; 0,$$

i.e.,

$$\mathbf{PQ} \cdot (\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{x}) = 0.$$

Now imagine Q moving along a small circle centred at P (staying entirely within the quadrilateral at all times). The above equality is true for every such point Q. Since the dot product is 0 for vectors pointing in every possible direction, it must be that $\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{x}$ is the zero vector, identically.

If the sum of four vectors is the zero vector, then we can lay them out edge-to-edge to get a closed shape (i.e., we return to the starting point in the end); in this case, we get a quadrilateral. Since the four vectors have equal length, the quadrilateral is in fact a rhombus. But the opposite sides of a rhombus are parallel to each other; hence $\mathbf{u}$ and $\mathbf{w}$ are collinear, as are $\mathbf{v}$ and $\mathbf{x}$. *But this implies that AB and CD are parallel to each other, as are BC and DA.* Hence $ABCD$ is a parallelogram. ■

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ClassRoom

Orthocentre of a Triangle and its Distance from any Point in the Plane

AVIPSHA NANDI

In this note, we derive a formula that gives the distance between the orthocentre and any arbitrary point in the plane of the triangle. We also discuss some inequalities and other consequences that follow from this relation.

Introduction

The identity described here, which gives the distance between the orthocentre of a triangle and any point in the plane of the triangle, came about when we were discussing the various triangle centres in class: the circumcentre, the incentre, the orthocentre, the centroid and the nine-point centre. We were discussing the formulas for the distances between these centres, but found the topic confusing. Our math mentor challenged us to find a generalised identity in this topic, making use of Stewart's theorem. On taking up this challenge, I came up with the identity described below.

Basic notation

Let ABC be any triangle. We denote its side-lengths by $a = BC$, $b = AC$, $c = AB$, its angles by $\measuredangle A$, $\measuredangle B$, $\measuredangle C$, its semi perimeter by $s = \frac{1}{2}(a + b + c)$, and its area by Δ. Its classical centres are the centre N of the nine-point circle, the circumcentre O, the incentre I, the centroid G, and the orthocentre H. We write I_1, I_2, I_3 for the ex-centres opposite A, B, C, respectively. The classical radii are the circumradius R, inradius r, and exradii r_1, r_2, r_3. (See Figure 1.)

Keywords: Stewart's theorem, orthocentre, inequality

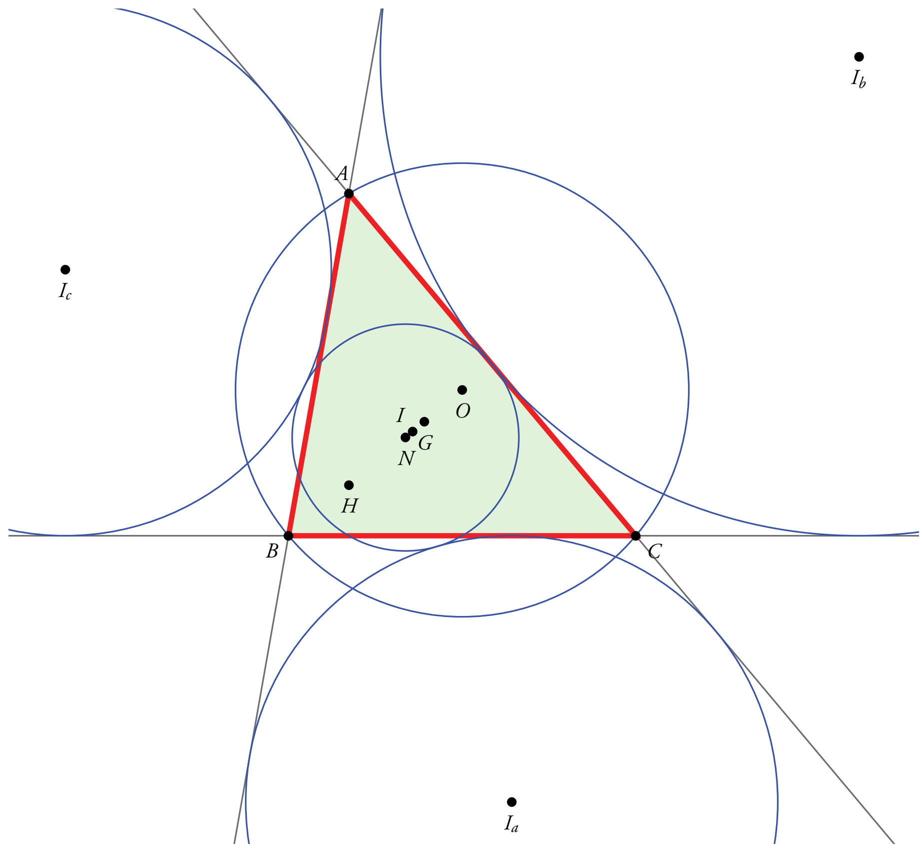

Figure 1.

Let AD, BE, CF be the altitudes of the triangle, concurrent at orthocentre H (Figure 2). The following relations are easily verified using simple trigonometry.

$$BD = c\cos B = 2R\sin C\, \cos B, \qquad CD = b\cos C = 2R\sin B\, \cos C,$$
$$AH = 2R\cos A, \qquad HD = 2R\cos B\, \cos C, \qquad AD = 2R\sin B\, \sin C.$$

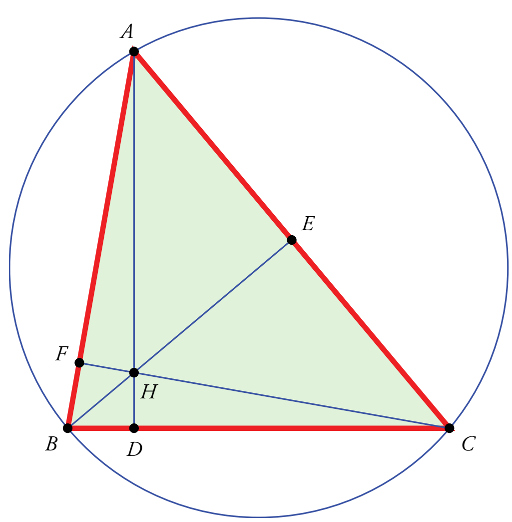

Figure 2.

Connections between different elements of a triangle

We list here a few relations that hold between the elements of a triangle. For the proofs, please refer to the appendix. (Some proofs are given only in outline form.)

Equivalent formulas for the area of a triangle. Four well-known formulas for the area Δ of a triangle (all easily proved):

$$\Delta = rs = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C.$$

Distance from vertex to incentre.

$$AI^2 = \frac{r^2}{\sin^2 A/2}.$$

Similarly for BI^2 and CI^2.

Formulas connecting the lengths of the sides of the triangle.

$$ab + bc + ca = r^2 + s^2 + 4Rr,$$
$$a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right)$$

Formulas connecting the cosines of the angles.

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R},$$
$$\cos A \cos B + \cos B \cos C + \cos C \cos A = \frac{r^2 + s^2 + 4Rr}{4R^2} - \left(1 + \frac{r}{R}\right),$$
$$\cos A \cos B \cos C = \frac{s^2 - r^2 - 4R^2 - 4Rr}{4R^2}.$$

Formulas connecting the sines of the double angles.

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C = \frac{abc}{2R^3} = \frac{2\Delta}{R^2}$$
$$\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C,$$

with similar formulas for $\sin 2A - \sin 2B + \sin 2C$ and $-\sin 2A + \sin 2B + \sin 2C$.

Stewart's theorem

Stewart's theorem gives a relation between the lengths of the sides of a triangle and the length of a cevian of the triangle. (Note. A *cevian* of a triangle is the line segment joining a vertex of the triangle to any point on the side opposite that vertex.) It is named after the Scottish mathematician Matthew Stewart who published the theorem in 1746. See Figure 3 for the notation.

Theorem (Stewart, 1746)**.** *Let a, b, c be the lengths of the sides of a triangle. Let l be the length of a cevian to the side of length a. If this cevian divides a into segments of lengths m and n respectively, with m adjacent to c and n adjacent to b, then*

$$al^2 = mb^2 + nc^2 - amn. \qquad (1)$$

With these preliminary results in place, the main result may be stated as follows.

Theorem 1. *Let M be any point in the plane of an acute triangle ABC with orthocentre H. Then:*

$$HM^2 = \cot B \cot C \cdot AM^2 + \cot C \cot A \cdot BM^2 + \cot A \cot B \cdot CM^2 - 8R^2 \cdot \cos A \cos B \cos C.$$

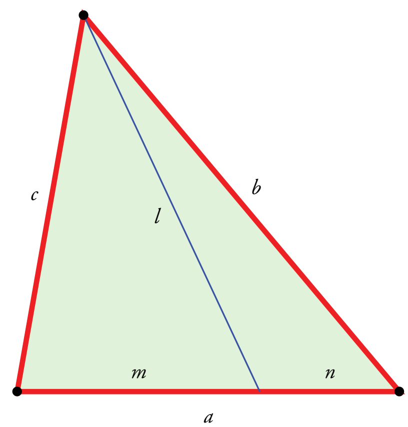

Figure 3.

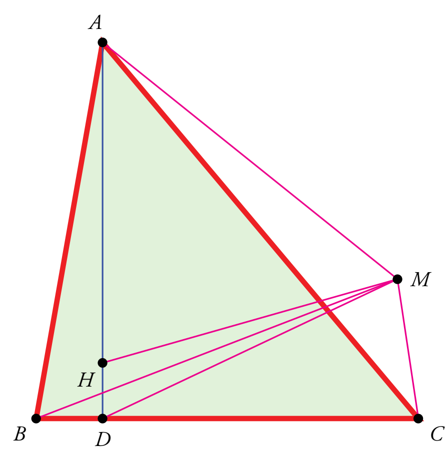

Figure 4.

Proof. We apply Stewart's theorem to triangle MBC in which MD is a cevian. See Figure 4. We get:

$$a \cdot MD^2 = BD \cdot MC^2 + CD \cdot MB^2 - a \cdot BD \cdot DC,$$
$$\therefore\ a \cdot MD^2 = c \cos B \cdot MC^2 + b \cos C \cdot MB^2 - abc \cos B \cos C.$$

Next, we apply Stewart's theorem to triangle MAD in which MH is a cevian. We get:

$$AD \cdot MH^2 = AH \cdot MD^2 + HD \cdot MA^2 - AD \cdot AH \cdot HD,$$
$$\therefore\ MH^2 = \frac{aR \cos A}{\Delta} MD^2 + \cot B \cot C \cdot MA^2 - 4R^2 \cdot \cos A \cos B \cos C.$$

This further gives:

$$\begin{aligned} MH^2 &= \cot A \cot B \cdot MC^2 + \cot C \cot A \cdot MB^2 + \cot B \cot C \cdot MA^2 \\ &\qquad - \frac{abc\, R \cos A \cos B \cos C}{\Delta} - 4R^2 \cdot \cos A \cos B \cos C \\ &= \cot A \cot B \cdot MC^2 + \cot C \cot A \cdot MB^2 + \cot B \cot C \cdot MA^2 - 8R^2 \cdot \cos A \cos B \cos C, \end{aligned}$$

since $abc = 4R\Delta$. Hence:

$$MH^2 = \cot B \cot C \cdot MA^2 + \cot C \cot A \cdot MB^2 + \cot A \cot B \cdot MC^2 - 8R^2 \cdot \cos A \cos B \cos C.$$

This proves the stated identity.

Corollary 1.

$$HM^2 = \cot B \cot C \cdot AM^2 + \cot C \cot A \cdot BM^2 + \cot A \cot B \cdot CM^2 - 4\Delta \cdot \cot A \cot B \cot C.$$

Proof. The relation follows from Theorem 1, by using the identity

$$\Delta = 2R^2 \cdot \sin A \sin B \sin C$$

and making the necessary substitution.

Corollary 2.

$$\begin{aligned} HM^2 = \frac{1}{16\Delta^2} \Big[& \left(a^2+b^2-c^2\right)\left(a^2+c^2-b^2\right) AM^2 \\ & + \left(b^2+c^2-a^2\right)\left(b^2+a^2-c^2\right) BM^2 \\ & + \left(c^2+a^2-b^2\right)\left(c^2+b^2-a^2\right) CM^2 \\ & - \left(a^2+c^2-b^2\right)\left(b^2+c^2-a^2\right)\left(b^2+a^2-c^2\right) \Big]. \end{aligned}$$

Proof. The relation follows from Corollary 1 by using the following:

$$\cot A = \frac{\cos A}{\sin A} = \frac{\left(b^2+c^2-a^2\right)/(2bc)}{a/(2R)} = \frac{R\left(b^2+c^2-a^2\right)}{abc} = \frac{b^2+c^2-a^2}{4\Delta},$$

with similar relationships for $\cot B$ and $\cot C$. On substituting these into Corollary 1, we obtain the stated result.

Three applications of the identities

Theorem 2.

$$OH^2 = R^2\left(1 - 8\cos A \cos B \cos C\right).$$

Proof. In Theorem 1, replace M by the circumcentre O. We get:

$$OH^2 = \cot B \cot C \cdot AO^2 + \cot C \cot A \cdot BO^2 + \cot A \cot B \cdot CO^2 - 8R^2 \cdot \cos A \cos B \cos C.$$

But $OA = OB = OC = R$, and for any triangle,

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

This implies $OH^2 = R^2\left(1 - 8\cos A \cos B \cos C\right)$, as claimed.

Theorem 3. *For any acute triangle ABC,*

$$\cos A \cos B \cos C \leq \frac{1}{8},$$

with equality when the triangle is equilateral

Proof. This follows from Theorem 2. We know that $OH^2 \geq 0$. Hence it must be that $\cos A \cos B \cos C \leq \frac{1}{8}$.

For equality to hold, we must have $OH = 0$, i.e., the circumcentre coincides with the orthocentre. It may readily be shown that this holds only when the triangle is equilateral. Hence proved.

Theorem 4.

$$IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C.$$

Proof. In Theorem 1, replace M by the incentre I. We get:

$$IH^2 = \cot B \cot C \cdot AI^2 + \cot C \cot A \cdot BI^2 + \cot A \cot B \cdot CI^2 - 8R^2 \cdot \cos A \cos B \cos C,$$

i.e.,

$$IH^2 = \sum_{A,B,C} \cot B \cot C \cdot AI^2 - 8R^2 \cdot \cos A \cos B \cos C.$$

For any triangle, $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$ and $AI^2 = bc - 4Rr$. Hence:

$$\begin{aligned}
IH^2 &= \sum_{A,B,C} \cot B \cot C \cdot (bc - 4Rr) - 8R^2 \cdot \cos A \cos B \cos C \\
&= \sum_{A,B,C} (bc \cot B \cot C) - 4Rr - 8R^2 \cdot \cos A \cos B \cos C \\
&= \sum_{A,B,C} \left(4R^2 \cos B \cos C\right) - 4Rr - 8R^2 \cdot \cos A \cos B \cos C \\
&= 4R^2 \left[\left(\frac{r^2 + s^2 + 4Rr}{4R^2}\right) - \left(1 + \frac{r}{R}\right)\right] - 4Rr - 8R^2 \cdot \cos A \cos B \cos C \\
&= r^2 + s^2 - 4R^2 - 4Rr - 8R^2 \cdot \cos A \cos B \cos C \\
&= r^2 + r^2 + 4R^2 \cdot \cos A \cos B \cos C - 8R^2 \cdot \cos A \cos B \cos C \\
&\qquad \left(\text{since } \cos A \cos B \cos C = \frac{s^2 - r^2 - 4R^2 - 4Rr}{4R^2}\right),
\end{aligned}$$

$$\therefore \; IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C.$$

Hence proved.

We state the following results without proof.

Theorem 5. *Given an acute triangle ABC, the point M in the plane of the triangle which minimises the expression*

$$\cot B \cot C \cdot AM^2 + \cot C \cot A \cdot BM^2 + \cot A \cot B \cdot CM^2$$

is the orthocentre H.

Corollary 3. *For all points M in the plane of the triangle,*

$$\cot B \cot C \cdot AM^2 + \cot C \cot A \cdot BM^2 + \cot A \cot B \cdot CM^2 \geq 8R^2 \cdot \cos A \cos B \cos C.$$

Theorem 6. *Given an acute triangle ABC and any point M in the plane of the triangle, the following inequalities are true.*

$$\frac{AM^2}{\cot A} + \frac{BM^2}{\cot B} + \frac{CM^2}{\cot C} \geq 4\Delta,$$

$$\frac{AM^2}{b^2 + c^2 - a^2} + \frac{BM^2}{c^2 + a^2 - b^2} + \frac{CM^2}{a^2 + b^2 - c^2} \geq 1.$$

Moreover, the equality sign holds precisely when M coincides with the orthocentre H.

Theorem 7. *Given an acute triangle ABC, its incentre I, its ex-centres I_1, I_2, I_3 and any point M in the plane of the triangle, we have:*

$$s \cdot IM^2 = (s - a) \cdot I_1M^2 + (s - b) \cdot I_2M^2 + (s - c) \cdot I_3M^2 - 8R\Delta.$$

Conclusion. We have derived a few relations that hold between the elements of a triangle. Theorem 7 is a beautiful result connecting the incentre and excentres. Proceeding in a similar way, we can discover and prove many new relations and inequalities.

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Appendix: Proofs of some of the background results

Formulas connecting the lengths of the sides of the triangle.

$$ab + bc + ca = r^2 + s^2 + 4Rr,$$
$$a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right).$$

Proof. We start with the relation $rs = \sqrt{s(s-a)(s-b)(s-c)}$. Squaring both sides and dividing by s, we get:

$$\begin{aligned} r^2 s &= (s-a)(s-b)(s-c) \\ &= s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc \\ &= s^3 - 2s\cdot s^2 + (ab+bc+ca)s - 4R\cdot rs, \\ \therefore\ \ ab+bc+ca &= r^2 + s^2 + 4Rr. \\ \text{Next,}\quad a^2+b^2+c^2 &= (a+b+c)^2 - 2(ab+bc+ca) \\ &= 4s^2 - 2\left(r^2+s^2+4Rr\right) \\ &= 2\left(s^2 - r^2 - 4Rr\right). \end{aligned}$$

Distance from vertex to incentre.

$$AI^2 = \frac{r^2}{\sin^2 A/2}.$$

This follows from the definition of sine of an angle (applied to the triangle whose vertices are A, I and the point of contact of the incircle with side AB). The same triangle yields:

$$\begin{aligned} AI^2 &= r^2 + (s-a)^2 = r^2 + s^2 - a(b+c) \\ &= r^2 + s^2 - (ab+bc+ca) + bc = bc - 4Rr, \end{aligned}$$

using the identity proved above for $ab + bc + ca$.

Formulas connecting the cosines of the angles.

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R},$$
$$\cos A\,\cos B + \cos B\,\cos C + \cos C\,\cos A = \frac{r^2+s^2+4Rr}{4R^2} - \left(1 + \frac{r}{R}\right),$$
$$\cos A\,\cos B\,\cos C = \frac{s^2 - r^2 - 4R^2 - 4Rr}{4R^2}.$$

The first formula may be proved using geometrical arguments and the theorem of Ptolemy. We shall prove the second formula and the third formula using the first one. We use the fact that since $A + B + C = 180^\circ$, we have $\cos A = -\cos(B + C)$, and similarly for $\cos B$ and $\cos C$. Hence:

$$-(\cos A + \cos B + \cos C) = \cos(B+C) + \cos(C+A) + \cos(A+B).$$

Now we expand the terms on the right side. We obtain:

$$\begin{aligned}
&\cos(B+C)+\cos(C+A)+\cos(A+B)\\
&=(\cos B\cos C-\sin B\sin C)+\text{two other such terms}\\
&=(\cos B\cos C+\cos C\cos A+\cos A\cos B)-(\sin B\sin C+\sin C\sin A+\sin A\sin B)\,.
\end{aligned}$$

The first bracketed term is the one we want. The second bracketed term is

$$\begin{aligned}
\sin B\sin C+\sin C\sin A+\sin A\sin B &= \frac{bc}{4R^2}+\frac{ca}{4R^2}+\frac{ab}{4R^2}\\
&=\frac{bc+ca+ab}{4R^2}\\
&=\frac{r^2+s^2+4Rr}{4R^2}.
\end{aligned}$$

It follows that

$$\cos A\cos B+\cos B\cos C+\cos C\cos A=\frac{r^2+s^2+4Rr}{4R^2}-\left(1+\frac{r}{R}\right).$$

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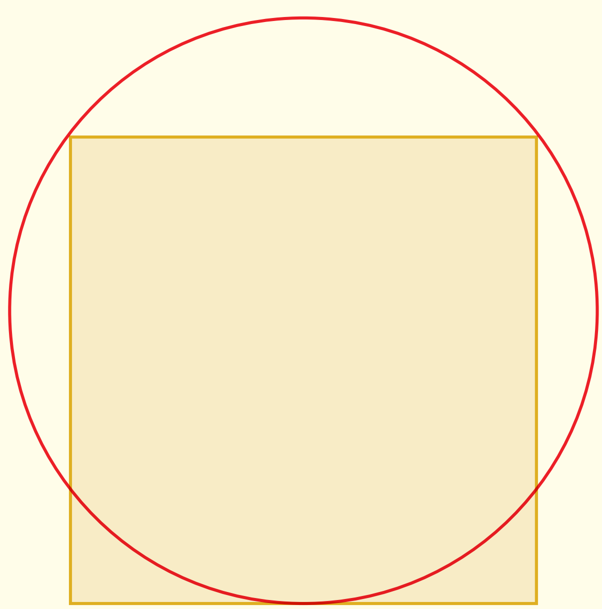
In the figure, you see a square and circle.
The side of the square is 40. What is the radius of the circle?



The Game of Craps

JONAKI B GHOSH

The topic of probability is replete with many interesting problems. Some of these, based on games of chance, can lead to interesting explorations in the classroom. Exploring such games actually enlivens the study of probability and also provides opportunities for students to explore many fundamental concepts in probability. In this article we shall explore a very popular game played in casinos called the *Game of Craps* or just *Craps* which is played simply by rolling a pair of dice. It is said that if you want to double your money quickly on a game of pure chance, one of your best opportunities is to bet all your money on one game of craps! Apparently the probability of winning this game is very close to 50%. In this article we will analyse this game by simulating it on the spreadsheet MS Excel. Of course, the simulation can be done on any spreadsheet such as LibreOffice Calc.

The Power of Simulations

Before we proceed, we need to say a little about simulation although a more extended treatment of simulation is beyond the scope of this article. Simulation is essentially a modeling tool, which is used to imitate real-world problems in order to understand system behavior or the behavior of certain phenomena. In particular, *Monte Carlo Simulation* is a problem solving technique used to approximate the probability of certain outcomes of an experiment by running multiple trial runs (called simulations), using random numbers. Simple simulations of real world problems can be explored through spreadsheets such as MS Excel, which have inbuilt functions for generating random numbers. One of the ways of exploring and analyzing games of chance such as Craps is through simulations since such games, by their very nature, are driven by randomness. In this article, we will illustrate how we can play Craps simply by *generating random numbers* instead of actually rolling a pair of dice.

Keywords: Probability, gaming, craps, simulation

The rules of the game

As mentioned earlier, Craps is played by rolling a pair of dice. When a pair of dice is rolled, there are 36 possible outcomes. However, the sum of the numbers on the two faces can range from 2 (with 1 appearing on both dice), to 12 (with 6 appearing on both dice). Sums of 3, 4, 5, 6, 7, 8, 9, 10 and 11 are also possibilities. We also know that the probability of obtaining each of these sums can be easily computed. For instance, a sum of 9 can be obtained if the dice show up the numbers (3,6), (4,5), (5,4) or (6,3). In the language of probability, we say that these four outcomes are favorable to the occurrence of the sum of 9 and the probability of obtaining this sum is 4/36 = 1/9.

Let us now familiarize ourselves with the rules of the game.

A player rolls a pair of dice. This may lead to three possible outcomes:

1. A total of 7 or 11 is obtained on the first roll. In this case the **player wins** and this outcome is referred to as a '**Natural**'.
2. A total of 2, 3 or 12 is obtained on the first roll. In this case the **player loses** the game. Obtaining a 2 is referred to as **Snake eyes**'
3. A total of 4, 5, 6, 8, 9 or 10 is obtained on the first roll. In this case the player neither wins nor loses but can roll again. The number becomes the player's **point** and the game continues, that is, the player rolls the dice again. Two situations can happen:
   a. In subsequent rolls, the player's point appears before a total of 7. In this case the **player wins** the game.
   b. In subsequent rolls, a total of 7 appears before the point. In this case, the **player loses** the game.

   For example, let us suppose that 6 appears on the first roll which is the player's point. He can continue to roll the dice till either 6 (his point) or 7 appears (other numbers are inconsequential). If 7 comes up before 6, the player loses whereas if 6 comes before 7, the player wins.

The probability of a win

Let us now analyse the game using our knowledge of theoretical probability and compute the probability of winning a single game of craps. Clearly, a win takes place if outcomes 1 or 3a, described above, takes place. Let us begin by computing the probability of **winning on the first roll**, or getting a natural, that is, a total of 7 or 11. The possibilities of obtaining 7 are (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) and those for obtaining 11 are (5,6) and (6,5). Thus there are 8 favorable outcomes out of a total of 36 possible outcomes leading to a probability of 8/36 or 22.22%.

Next we calculate the total probability of winning by getting a total of 4, 5, 6, 8, 9 or 10 (which becomes the player's point). In order to find the probability of winning by getting a total of, say 5, we would need to find the probability of getting 5 in the first roll and then getting 5 before getting 7 in subsequent rolls. This is a case of conditional probability and it can be calculated as follows:

The probability of getting a 5 in the first roll is 4/36 (the possible outcomes being (1,4), (2,3), (3,2) and (4,1)). In subsequent rolls, we need to consider only totals of 5 and 7 which leads to only 10 possibilities (4 for a total of 5 and 6 for a total of 7). Among these, a win is possible only when a total of 5 precedes a total of 7, the probability for which is 4/10. Thus the total probability of winning by getting a 5 in the first roll is 4/36 × 4/10 = 0.044. Similarly we can calculate the probability of winning by getting a total of 4, 6, 8, 9 and 10 in the first roll. Table 1 lists the probabilities of winning for all possible outcomes. The total probablity of winning a single game of craps works out to approximately 0.49!

Table 1: Probabilities of winning a Game of Craps

Initial total	Probability
4	0.027
5	0.044
6	0.063
7	0.167
8	0.063
9	0.044
10	0.027
11	0.056
Total probability of winning	**0.491**

Similarly we can calculate the probability of losing on the first roll, that is, getting a total of 2, 3 or 12 (snake eyes). The possibilities are (1,1), (1,2), (2,1) and (6,6) leading to a probability of 4/36 or 11.11%

Simulating the Game of Craps in Excel

Having worked out the various possibilities of winning the game we can now try to simulate it in Excel. In order to perform the simulation we need to be familiar with the following basic functions of Excel.

IF – a logical function, which specifies a logical test to be performed or a condition to be checked.

OR – a logical function which returns TRUE if *any* of the arguments of a statement is true.

AND – a logical function which returns TRUE if *all* of the arguments of a statement are true.

SUM – a function that adds up all the values of an argument

COUNT – a statistical function which counts how many numbers are in the list of arguments.

RANDBETWEEN – returns a random number between two specified numbers.

We shall now try to simulate the game and find out the average number of throws required to win the game. The steps to be followed are:

Step 1: Enter the numbers 1 to 20 in column A. To do this, we enter 1 in cell A2 and = **A2 + 1** in cell A3. Dragging cell A3 till A21 will create a column of numbers 1 to 20. These indicate the number of rolls of the pair of dice.

Step 2: To simulate the numbers appearing on a pair of dice we enter =**RANDBETWEEN**(1,6) in B2 and C2 as shown in Figure 1.

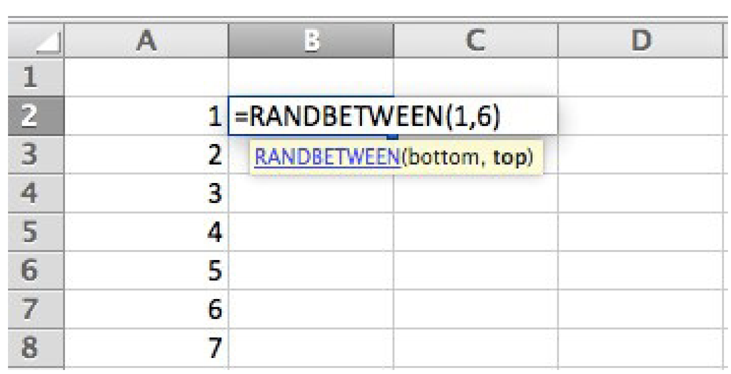


Figure 1: The RANDBETWEEN command in Excel randomly generates an integer between 1 and 6 (inclusive of 1 and 6).

Step 3: After this, we select the cells B2 and C2 and double click in the corner of cell C3. This will fill both columns with numbers from 1 to 6, thus simulating 20 rolls of a pair of dice as shown in Figure 2. To obtain the sum of the numbers appearing on the pair of dice we enter **=SUM(B2,C2)** in cell D2. Column D will show the sums of the dice rolls. The entry in cell D2 (highlighted in red) will determine if the player has won, lost or if the game can be continued.

	A	B	C	D
1		Die1	Die2	Sum
2	1	4	6	10
3	2	6	2	8
4	3	6	2	8
5	4	1	6	7
6	5	6	6	12
7	6	1	5	6
8	7	5	6	11
9	8	3	1	4
10	9	1	5	6
11	10	6	5	11
12	11	2	1	3
13	12	6	1	7
14	13	2	3	5
15	14	1	6	7
16	15	6	3	9
17	16	2	3	5
18	17	5	3	8
19	18	3	4	7
20	19	6	1	7
21	20	4	6	10

Figure 2: Excel simulates 20 rolls of a pair of dice. The entry in cell D2 (shown in red) determines if the player wins, loses or if the game will continue.

Step 4: To check the outcome of the first roll we enter the conditional statement **=IF(OR(D2=7,D2=11), 1,0)** in cell E2 as shown in Figure 3. This will ensure that the value of cell E3 will be 1 if the player obtains a sum of 7 or 11, that is, if the player wins, and will be 0 otherwise.

	A	B	C	D	E	F
1		Die1	Die2	Sum		
2	1	2	4	6	=IF(OR(D2=7,D2=11),1,0)	
3	2	2	6	8		
4	3	3	4	7		

Figure 3: The IF conditional statement is used to check if the player wins in the first roll.

Similarly we will enter the conditional statement **=IF(OR(D2=2,D2=3,D2=12), 1,0)** in cell F2 as shown in Figure 4. This will produce the output 1 in cell F2 if the sum on the first roll is 2,3 or 12, that is, if the player loses and will be 0 otherwise.

Figure 4: The IF conditional statement is used to check if the player loses in the first roll.

Step 5: Finally we enter the conditional statement **=IF(AND(E2=0,F2=0), "YES","NO")** in cell G2. The output of this statement (YES or NO) indicates if the game will continue. The output will be YES if the game continues beyond the first roll and NO otherwise. Note that if the game does not continue beyond the first roll, we will have to ignore the remaining rows of the ouput.

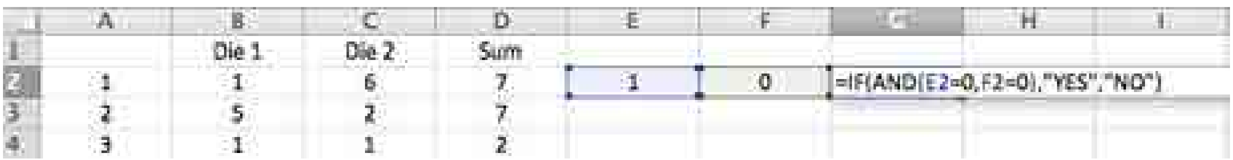


Figure 5: The IF conditional statement is used to check if the game continues beyond the first roll.

Step 6: If the game continues beyond the first roll, we need to insert conditional statements in cells E3, F3 and G3 as follows

In cell E3 enter **=IF(OR(G2="NO",G2=""),"",IF(D3=D2,1,0))**

In cell F3 enter **=IF(OR(G2="NO",G2=""),"",IF(D3=7,1,0))**

In cell G3 enter **=IF(OR(G2="NO",G2=""),"",IF(AND(E3=0,F3=0),"YES","NO"))**

After inserting these statements, we will need to select the cells E3, F3 and G3 and double click in the corner of cell G3 so that the above formulae get evaluated in all the cells till row 21.

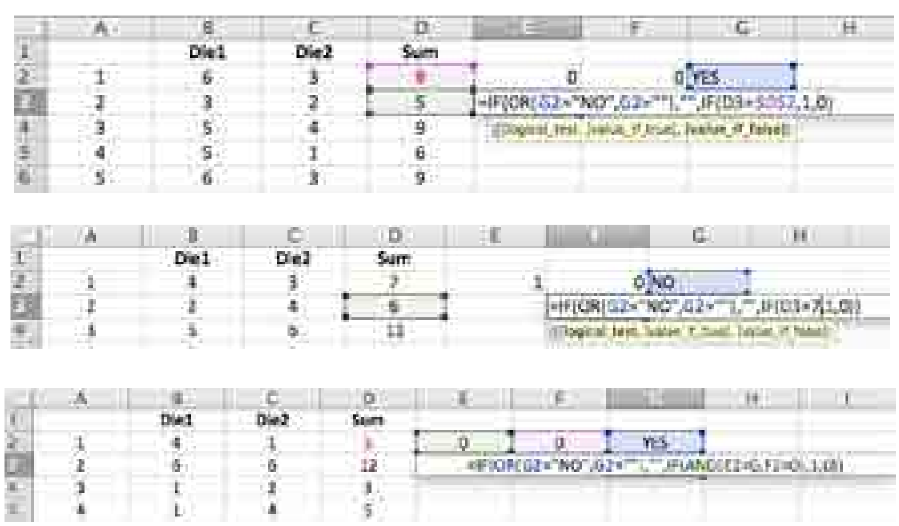


Figure 6: The IF condition is used to compute the cells E3, F3 and G3 to continue the game further.

Step 7: After entering the IF conditions in the cells E3, F3 and G3, the excel sheet is ready for simulation. Click on any cell of column B or C which represent the die rolls. This will generate a new simulation and the outcome of the game can be easily interpreted as we shall see in the next section.

	A	B	C	D	E	F	G
1							
2		Die 1	Die 2	Sum			Game continues?
3	1	2	5	7	1	0	NO
4	2	5	5	10			
5	3	6	6	12			
6	4	3	1	4			
7	5	3	6	9			
8	6	6	4	10			
9	7	3	3	6			
10	8	2	5	7			
11	9	3	4	7			
12	10	2	1	3			
13	11	4	4	8			
14	12	4	6	10			
15	13	3	2	5			
16	14	2	5	7			
17	15	6	4	10			
18	16	1	4	5			
19	17	5	2	7			
20	18	1	1	2			
21	19	2	5	7			
22	20	3	3	6			

Figure 7: The player wins by rolling a 7 in the first roll.

Analysing the game through simulation

Using the above code we will explore four scenarios. Note that each time we click on the Excel sheet a new data set of 20 dice rolls is generated and the sheet gets updated.

Case 1: The sum of the numbers of the two die in the first roll is 7 (Figure 7) and the player wins. The game doesn't continue beyond the first roll. Here we will ignore the output from row 2 onwards as these rows are no longer relevant.

Case 2: The player loses the game in the first roll as the sum is 3 (Figure 8). Once again the game doesn't continue beyond the first roll. Here we will ignore the output from row 2 onwards as these rows are no longer relevant.

Case 3: The player neither wins nor loses in the first roll. In Figure 9, the sum in the first roll is 9, which becomes the player's point. The game continues till either 9 or 7 appears. Since 9 appears in the 5th roll, the player wins the game in this roll. The remaining rows of the output may therefore be ignored.

In Figure 10 the players' point in the first roll is 5. However 7 appears before 5 in the subsequent throws and the player loses the game.

To compute the average number of rolls required to win the game we can conduct several simulations and record the number of rolls each time. Let us suppose that the number of rolls for 10 simulations are recorded. The total number divided by 10 will give us the mean number of rolls. The reader is encouraged to try the experiment for 20 simulations, then 50 and then 100. What do you observe about the mean number of rolls as the number of simulations of the game is increased?

	A	B	C	D	E	F	G
1							
2		Die 1	Die 2	Sum			Game continues?
3	1	1	2	3	0	1	NO
4	2	6	3	9			
5	3	3	5	8			
6	4	6	2	8			
7	5	6	5	11			
8	6	5	4	9			
9	7	1	2	3			
10	8	1	1	2			
11	9	5	3	8			
12	10	1	6	7			
13	11	4	3	7			
14	12	6	4	10			
15	13	1	2	3			
16	14	2	3	5			
17	15	4	1	5			
18	16	6	5	11			
19	17	2	5	7			
20	18	5	3	8			
21	19	3	1	4			
22	20	5	6	11			

Figure 8: The player loses as the sum of the first roll is 3.

	A	B	C	D	E	F	G
1							
2		Die 1	Die 2	Sum			Game continues?
3	1	6	3	9	0	0	YES
4	2	5	3	8	0	0	YES
5	3	4	1	5	0	0	YES
6	4	2	4	6	0	0	YES
7	5	5	4	9	1	0	NO
8	6	6	3	9			
9	7	6	4	10			
10	8	3	6	9			
11	9	5	5	10			
12	10	6	4	10			
13	11	6	2	8			
14	12	6	3	9			
15	13	3	5	8			
16	14	2	5	7			
17	15	3	4	7			
18	16	6	4	10			
19	17	1	3	4			
20	18	3	3	6			
21	19	6	3	9			
22	20	1	5	6			

Figure 9: The player wins in the fifth roll, as 9, the point is obtained before obtaining a 7.

	A	B	C	D	E	F	G
1							
2		Die 1	Die 2	Sum			Game continues?
3	1	1	4	5	0	0	YES
4	2	3	4	7	0	1	NO
5	3	2	5	7			
6	4	2	6	8			
7	5	5	1	6			
8	6	5	3	8			
9	7	3	3	6			
10	8	1	3	4			
11	9	3	6	9			
12	10	2	2	4			
13	11	4	1	5			
14	12	3	4	7			
15	13	3	3	6			
16	14	1	1	2			
17	15	1	6	7			
18	16	6	1	7			
19	17	2	4	6			
20	18	2	3	5			
21	19	3	5	8			
22	20	1	3	4			

Figure 10: The player loses in the second roll, as 7 is obtained before obtaining the point, that is, 5.

In general the probability of a player winning on a point is about 27.07%, that is, about once in every 3.7 rolls. However the probability of winning the game (on the first roll or on a point) is once in every two rolls! That is why it is said that the probability of winning a single game of craps is about 50%.

We hope this article will motivate the reader to further explore this very interesting game and analyse it using the rules of probability.

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Middle School Problems on AVERAGES

A RAMACHANDRAN

Problems

Problem VIII–3–M.1

A trader sells two articles at the same price. On the first he gains 'p' percent ($0 < p < 100$) over his cost price, while he suffers a loss of 'p' percent on the second article. On the whole does he gain or lose? Express his overall gain/loss percentage in terms of 'p.' You could also consider the situation as one of fractional gain/loss with $\frac{p}{100} = x$, say.

Problem VIII–3–M.2

A businessman drives from city A to city B and returns to city A again by the same route every day, driving at a uniform speed both ways. One day he finds that he has to reduce his speed on the onward journey by 'p' percent. However, on the return journey he is able to drive at a speed 'p' percent higher than his usual speed. On this day, does he take more/less time than usual for the round trip? Express the increase/decrease percentage in time taken in terms of 'p.' As earlier, you could think of fractional increase/decrease with $\frac{p}{100} = x$, say ($x < 1$).

Problem VIII–3–M.3

A shop sells chocolate bars of two types priced at ₹ P and ₹ Q per piece. I spend equal *sums of money* buying the two types of bars. What is the average cost of a bar?

Problem VIII–3–M.4

Two cubes of equal *mass* but made of two different metals of densities x and y are fused together. What is the overall density of the object thus formed?

Problem VIII–3–M.5

A person makes a journey from city A to city B in three parts. He travels the first third of the way at a uniform (or average) speed of a km/hr. He travels the second third of the way at a speed of b km/hr and the remainder of the journey at c km/hr. What is the average speed for the entire trip?

Problem VIII–3–M.6

The population of rabbits in a woodland area increased by 20% during a certain year, decreased by 5% the next year and increased by 10% the following year. What is the average growth rate over the three years? In other words, what is the percentage change applied uniformly over the three years that leads to the same end result?

Solutions

Problem VIII-3-M.1

It would be easier to work with fractions rather than percentages. Let S denote the selling price of either article. Then the cost price of the first article is $\frac{S}{1+x}$, while the cost price of the second article is $\frac{S}{1-x}$. The total cost price is then

$$\frac{2S}{1-x^2},$$

which is greater than $2S$, the total selling price, indicating an overall loss. The fractional loss, i.e., loss amount divided by total cost price is then x^2. As a percentage it is $\left(\frac{p}{100}\right)^2$ percent.

Problem VIII-3-M.2

We again work with fractions. Let us take the distance between the cities to be d and the person's usual driving speed to be s. The time taken for a normal round trip is then $\frac{2d}{s}$. The time taken on the day when the speeds for onward and return journeys are different is

$$\frac{d}{s(1-x)} + \frac{d}{s(1+x)} = \frac{2d}{s(1-x^2)},$$

which is greater than $\frac{2d}{s}$, the usual trip time. The fractional increase in time works out to be $\frac{x^2}{1-x^2}$.

Note: In both the above situations, the quantity $(1-x^2)$ plays a key role. This quantity is always less than 1. This fact crops up in other similar situations. For instance, if you increase the dimension of a square in one direction by a certain fraction/percentage and decrease the perpendicular dimension by the same fraction/percentage the area always decreases.

Problem VIII-3-M.3

Let the sum of money spent on each type of chocolate bar be S. Then the number of bars purchased of type 1 is $\frac{S}{P}$ while the number of bars of type 2 is $\frac{S}{Q}$. Therefore the average price of a bar is

$$\frac{\text{total money spent}}{\text{total number of bars}} = \frac{2S}{\frac{S}{P}+\frac{S}{Q}} = \frac{2PQ}{P+Q}.$$

Problem VIII-3-M.4

This problem is similar to the above. Let the mass of each part be M. Then the volumes of the two parts are $\frac{M}{x}$ and $\frac{M}{y}$. The average density is

$$\frac{\text{total mass}}{\text{total volume}} = \frac{2M}{\frac{M}{x}+\frac{M}{y}} = \frac{2xy}{x+y}.$$

Note: In the above two problems, what we find is known as the *harmonic mean*. If the *number of bars* of the two types were the same in Problem 3 or the *volumes* of the two parts were the same in Problem 4, we would just take the arithmetic mean.

Problem VIII-3-M.5

Average speed for the journey $= \frac{\text{total distance}}{\text{total time}}$. Taking the distance from city A to city B to be $3s$, this equals

$$\frac{3s}{\frac{s}{a}+\frac{s}{b}+\frac{s}{c}} = \frac{3abc}{ab+bc+ca}.$$

This is the harmonic mean of a, b, c. Note that if $a = b = c$, then the expression simplifies to a.

Now solve a similar problem where the journey is divided into four parts. Do you see a pattern in the answers?

Problem VIII-3-M.6

If P_O and P_F are the initial and final populations, we have

$$P_F = P_O\left(1+\frac{20}{100}\right)\left(1-\frac{5}{100}\right)\left(1+\frac{10}{100}\right).$$

Then if a is the required average growth percentage, we should have

$$P_F = P_O\left(1+\frac{a}{100}\right)^3.$$

That is, $\left(1+\frac{a}{100}\right)^3 = 1.2 \times 0.95 \times 1.1$, or $1+\frac{a}{100} = \sqrt[3]{1.2 \times 0.95 \times 1.1}$.

Thus, $1 + \frac{a}{100} = 1.078$ approximately (1.078 is the geometric mean of 1.2, 0.95 and 1.1), giving $a = 7.8\%$ approximately.

Problems for the

SENIOR SCHOOL

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem VIII-3-S.1

From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Determine the length of a side of the octagon.

Problem VIII-3-S.2

Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B and C intersect Ω at A_1, B_1 and C_1, respectively, and the internal bisectors of angles A_1, B_1 and C_1 of the triangle $A_1B_1C_1$ intersect Ω at A_2, B_2 and C_2, respectively. If the smallest angle of triangle ABC is 40°, what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees? (See Figure 1.)

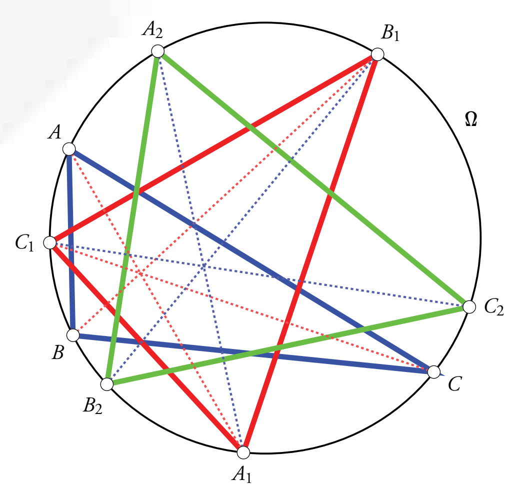

Figure 1.

Problem VIII-3-S.3

The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of ABC. Determine the angles of the triangle ABC.

Keywords: Octagon, circumcircle, internal bisector, least common multiple (LCM), prism

Solutions of Problems in Issue-VIII-2 (July 2019)

Solution to problem VIII-2-S.1

Let ABCD be a parallelogram. Suppose K is a point such that $AK = BD$ and let M be the midpoint of CK. Prove that $\measuredangle BMD = 90°$. [*Tournament of Towns*] (See Figure 2.)

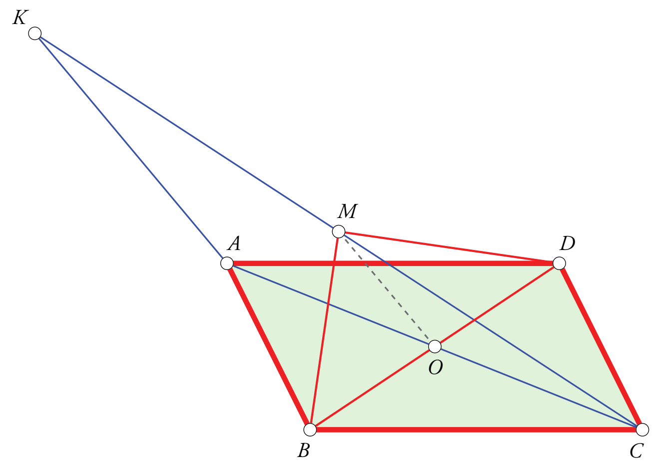

Figure 2.

Let $O = AC \cap BD$. Then in triangle ACK (possibly degenerate), M is the midpoint of CK and O is the midpoint of AC. Thus $OM = \frac{1}{2}AK = \frac{1}{2}BD$ and hence in triangle BMD, $BO = OM = OD$, implying $\measuredangle BMD = 90°$.

Solution to problem VIII-2-S.2

Let A be a finite non-empty set of consecutive positive integers with at least two elements. Is it possible to partition A into two disjoint non-empty sets X and Y such that the sum of the least common multiples of the numbers in X and Y is a power of 2?

The answer is NO.

Any number in A can be written (uniquely) as $2^k m$ for some non-negative integer k and some odd positive integer m. Since there are finitely many elements in A, there are only finitely many non-negative integers k such that 2^k divides some element of A. Let l be the **largest** positive integer such that 2^l divides some element z of A.

As the elements of A are consecutive positive integers, l and z are unique. Without loss of generality we may assume that $z \in X$. Then the LCM of the elements of X is divisible by 2^l but the LCM of the elements of Y is not. Therefore their sum cannot be a power of 2.

Solution to problem VIII-2-S.3

The vertices of a prism are coloured using two colours, so that each lateral edge has its vertices differently coloured. Consider all the segments that join vertices of the prism and are not lateral edges. Prove that the number of such segments with endpoints differently coloured is equal to the number of such segments with endpoints of the same colour. [*Romanian Math Competition*]

Let n be the number of sides of the prism's base. Then the number of segments to be considered is

$$\binom{2n}{2} - n = \frac{2n(2n-1)}{2} - n = 2\left(n^2 - n\right).$$

Let a be the number of the vertices of the upper face which have the first colour and $b = n - a$ the number of the vertices of the upper face which have the second colour. Then the lower face has b points of the first colour and a points of the second colour.

The number of segments with endpoints differently coloured one on the upper face and one on the lower face is $a^2 + b^2 - n$. The number of segments with endpoints differently coloured and on the same face is $2ab$. So, the total number of segments with endpoints differently coloured is $(a+b)^2 - n = n^2 - n$, which is exactly half of the number of all the segments.

(A similar solution was sent to us by Rakshitha of Mangaluru.)

Solution to problem VIII-2-S.4
Let $a, b, c, d \in [0,1]$. Prove that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+d} + \frac{d}{1+a} + abcd \leq 3. \qquad [\textit{Romanian Math Competition}]$$

Since $0 \leq a, b, c, d \leq 1$, it follows that $a, b, c, d \geq abcd$, and hence that

$$\begin{aligned}
&\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+d} + \frac{d}{1+a} + abcd \\
&\leq \frac{a}{1+abcd} + \frac{b}{1+abcd} + \frac{c}{1+abcd} + \frac{d}{1+abcd} + abcd \\
&= \frac{a+b+c+d}{1+abcd} + abcd.
\end{aligned}$$

Using repeatedly the inequality

$$x + y \leq 1 + xy \quad \forall x, y \in [0,1]$$

(this is equivalent to $(1-x)(1-y) \geq 0$ and is therefore true), we get

$$\begin{aligned}
a+b+c+d &\leq 1 + ab + 1 + cd = ab + cd + 2 \\
&\leq 1 + abcd + 2 = abcd + 3.
\end{aligned}$$

Hence

$$\frac{a+b+c+d}{1+abcd} + abcd \leq \frac{3+abcd}{1+abcd} + abcd,$$

i.e.,

$$\frac{a+b+c+d}{1+abcd} + abcd \leq 1 + \frac{2}{1+abcd} + abcd.$$

Let $x = abcd$; then $0 \leq x \leq 1$, so it is enough to prove the following inequality for all $x \in [0,1]$:

$$1 + \frac{2}{1+x} + x \leq 3,$$

that is,

$$\frac{2}{1+x} \leq 2-x, \quad \text{i.e.,} \quad 2 \leq 2+x-x^2,$$

or $x(1-x) \geq 0$, which is clearly true. Hence the stated inequality follows.

Solution to problem VIII-2-S.5

Let a and n be positive integers such that

$$\mathit{Frac}\left(\sqrt{n+\sqrt{n}}\right) = \mathit{Frac}\left(\sqrt{a}\right).$$

Prove that $4a+1$ is a perfect square. (Here Frac (x) = the fractional part of x.) [*Romanian Math Competition*]

The given condition is equivalent to $\sqrt{n+\sqrt{n}} = \sqrt{a}+k$, where k is an integer. This leads to

$$n+\sqrt{n} = a+2k\sqrt{a}+k^2,$$
$$\therefore \quad \sqrt{n} = 2k\sqrt{a}+b, \quad \text{where } b = k^2-n+a;$$

note that b is an integer. We now obtain

$$n = 4k^2a+b^2+4kb\sqrt{a},$$

which implies that $kb\sqrt{a}$ is a rational number.

If $\sqrt{a}$ is rational, then a is a perfect square, as is $n+\sqrt{n}$. This implies that $n=m^2$ for some natural number m. The hypothesis thus implies that m^2+m is a perfect square. But from the obvious inequality $m^2 \leq m^2+m < (m+1)^2$, we obtain $m=0$, that is $n=0$, which is a contradiction (as we had supposed that n is a positive integer).

So, we must have $kb=0$. If $b=0$, we get $n=k^2+a$ and also $n=4k^2a$, hence

$$a = \frac{k^2}{4k^2-1} < 1,$$

which is again a contradiction (as we had supposed that a is a positive integer). The only possibility is $k=0$, and thus $n=b^2$ and $a=b+n$, hence $a=b^2+b$ and $4a+1=(2b+1)^2$, which is a perfect square, as claimed.

Student Corner – Featuring articles written by students.

A Challenging Geometric Construction Problem

JALEEL RADHU

In this article, we study the following construction problem.

Problem. Given any $\triangle ABC$, show that there exist points P on side AB and Q on side AC such that $BP = PQ = QC$ and find a way of locating such a pair of points. (See Figure 1.)

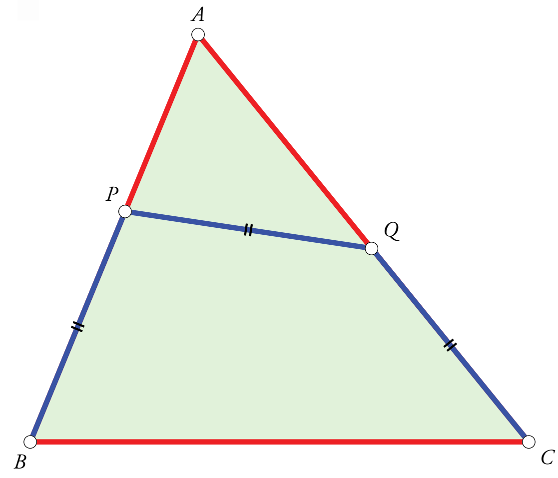

Figure 1.

Remark. The problem claims the existence of a pair of points satisfying the given conditions and also asks for a way of locating the two points. Our solution directly locates the two points and thus also establishes their existence.

Solution. The construction procedure is given below as a series of steps. (See Figure 2.)

(1) Locate any point D on side AB such that $BD < AC$.

(2) Through D, draw a line ℓ parallel to side BC.

Keywords: Construction, triangle, parallelogram, similarity

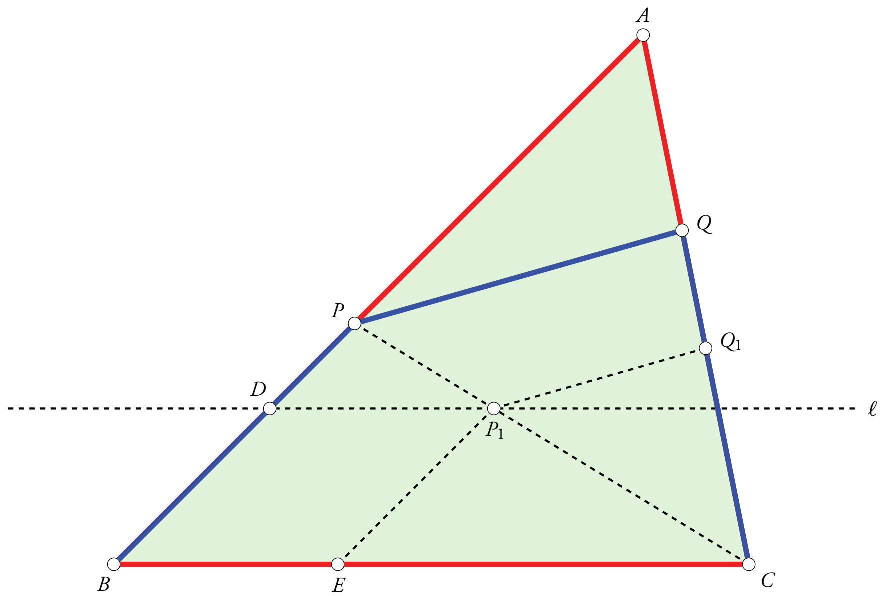

Figure 2.

(3) Locate a point Q_1 on side AC such that $CQ_1 = BD$. (This can be done by drawing a circle with centre C and radius BD.)

(4) Locate a point P_1 on ℓ such that $Q_1P_1 = BD$.

(5) Through P_1, draw a line parallel to AB; let it intersect side BC at E.

(6) Observe that quadrilateral P_1DBE is a parallelogram; so $P_1E = DB$.

(7) Hence $EP_1 = P_1Q_1 = Q_1C$, by construction.

(8) Extend CP_1 beyond P_1; let it intersect AB at P. Through P, draw a line parallel to P_1Q_1; let it intersect AC at Q.

(9) Then P and Q are the required pair of points.

Proof that the construction works. The claim that $BP = PQ = QC$ is proved as follows.

By similarity of triangles (following from $BP \parallel EP_1$ and $PQ \parallel P_1Q_1$), it follows that

$$BP : EP_1 = PC : P_1C, \qquad PC : P_1C = PQ : P_1Q_1.$$

Since $EP_1 = P_1Q_1$, it follows that $BP = PQ$.

In the same way, we have $PQ : P_1Q_1 = QC : Q_1C$. Since $P_1Q_1 = Q_1C$, it follows that $PQ = QC$.

Hence $BP = PQ = QC$. □

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Gowers's Weblog

How Craig Barton Wishes he'd Taught Maths

SIR WILLIAM TIMOTHY GOWERS

A couple of months ago, I can't remember precisely how, I became aware of a book called *How I Wish I'd Taught Maths,* by Craig Barton, that seemed to be highly thought of. The basic idea was that Craig Barton is an experienced, and by the sound of things very good, maths teacher who used to take a number of aspects of teaching for granted, until he looked into the mathematics-education literature and came to realize that many of his cherished beliefs were completely wrong. Since I've always been interested in the question of how best to teach mathematics, both because of my own university teaching and because from time to time I like to pontificate about school-level teaching, I decided to order the book. More surprisingly, given my past history of buying books that I felt I ought to read, I read it from cover to cover, all 450 pages of it.

As it happens, the book is ideally designed for people who *don't* necessarily want to read it from cover to cover, because it is arranged as follows. At the top level it is divided into chapters. Each chapter starts with a small introduction and thereafter is divided into sections. And each section has precisely the same organization: it is divided into subsections entitled, "What I used to believe", "Sources of inspiration", "My takeaway", and "What I do now". These are reasonably self-explanatory, but just to spell it out, the first subsection sets out a plausible belief that Craig Barton used to have about good teaching practice, often ending with a rhetorical question such as "What could possibly be wrong with that?", the second is a list of references (none of which I have yet followed up, but some of them look very interesting), the third is a discussion of what he learned from the references, and the last one is about how he put that into practice. Also, each chapter

Keywords: Mathematics, pedagogy, beliefs, unlearning, learning

ends with a short subsection entitled "If I only remember three things …", where he gives three sentences that sum up what he thinks is most important in the chapter.

One question I had in the back of my mind when reading the book was whether any of it applied to teaching at university level. I'm still not sure what I think about that. There is a reason to think not, because the focus of the book is very much on school-level teaching, and many of the challenges that arise do not have obvious analogues at university level. For example, he mentioned (on page 235) the following fascinating experiment, where people were asked to do the following multiple-choice question and then justify their answers.

Which of these values could not represent a probability?

A. 2/3

B. 0.72315

C. 1.46

D. 0.002

Let me quote the book itself for a discussion of this question.

> Surely the rule *probabilities must be less than or equal to 1* is about as straightforward as it gets in maths? But why, then, did 47% of the 5000+ students who answered this question get it wrong?
>
> A few students' explanations reveal all:
>
> *I think B because it's just a massive decimal and the rest look pretty legit. I also don't see how a number that big could be correct.*
>
> *I think B because you wouldn't see this in probability questions.*
>
> *I think D because you can't have 0.002 as an answer because it is too low.*
>
> If students are only used to meeting `nice-looking' probabilities during examples and practice questions, then it is little surprise they come a cropper when they encounter strange-looking answers.

Could one devise a university-level question that would catch a significant proportion of people out in a similar way? I'm not sure, but here's an attempt.

Which of the following is not a vector space with the obvious notions of addition and scalar multiplication?

A. The set of all complex numbers.

B. The set of all functions from (0, 1) to $\mathbb{R}$ that are twice differentiable.

C. The set of all polynomials in x with real coefficients that have $x^2 + x + 1$ as a factor.

D. The set of all triples (a, b, c) of integers.

E. The set of all sequences $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_1 + \cdots + x_n = 0$ and $x_1 + 2x_2 + \cdots + nx_n = 0$.

I think at Cambridge almost everyone would get this question right (though I'd love to do the experiment). But Cambridge mathematics undergraduates have been selected specifically to study mathematics. Perhaps at a US university, before people have chosen their majors, people might be tempted to choose another option (such as B, because vector spaces are to do with algebra and not calculus), while not noting that the obvious scalars in D do not form a field. Or perhaps they wouldn't like A because the scalar field is the same as the set of vectors (unless, that is, they thought that the obvious scalars were the real numbers).

More generally, I feel that there are certain kinds of mistakes that are commonly made at school level that are much less common at university level simply because those who survive long enough to reach that stage have been trained not to make them. For example, at university level we become used to formal definitions. Once one is in the habit of using these, deciding whether a structure is a vector space is simply a question of seeing whether the definition of a vector space applies, rather than thinking "Hmm, does that look like the vector spaces I've met up to now?" We also become part of a culture where it is common to look at pathological, or at least slightly surprising, examples of concepts, and so on.

Another reason I decided to read the book was that I have certain prejudices about the teaching of mathematics at school level and I was interested to know whether they would be reinforced by the book or challenged by it. This was a win-win situation, since it is always nice to have one's prejudices confirmed, but also rather exhilarating to find out that something that seems obviously correct is in fact wrong.

A prejudice that was strongly confirmed was the value of mathematical fluency. Barton says, and I agree with him (and suggested something like it in my book *Mathematics, A Very Short Introduction*) that it is often a good idea to teach fluency first and understanding later. More precisely, in order to decide whether it is a good idea, one should assess (i) how difficult it is to give an explanation of why some procedure works and (ii) how difficult it is to learn how to apply the procedure without understanding why it works.

For instance, suppose you want to teach multiplication of negative numbers. The rule "If they have the same sign then the answer is positive, and if they have different signs then the answer is negative" is a short and straightforward rule, but explaining why –2 times –3 should equal 6 is not very straightforward. So if one begins with the explanation, there is a big risk of conveying the idea that multiplication of negative numbers is a difficult, complicated topic, whereas if one gives plenty of practice in applying the simple rules, then one gives one's students fluency in an operation that comes up in many other contexts (such as, for instance, multiplying (x - 2) by (x - 3)), and one can try to justify the rule later, when they are comfortable with the rule itself. I remember enjoying the challenge of thinking about why the rule for dividing one fraction by another was correct, but that was long after I was happy with using the rule itself. I don't remember being bothered by the lack of justification up to that point.

As an example in the other direction, Barton gives that of solving linear equations. The danger here is that one can learn a procedure for solving equations such as $2x + 3 = 17$, get good at it, and then be completely stuck when faced with an equation such as $4 - 2x = 3x - 11$. Here a bit of understanding can greatly help. Barton advocates something called the balance method, where one imagines both sides of the equation on a balance, and one is required to make sure that balance is maintained the whole time. I think (but without too much confidence after reading this book) that I would go for something roughly equivalent, but not quite the same, which is to stress the rule *you can do the same thing to both sides of an equation* (worrying about things like squaring both sides or multiplying by zero later). Then the problem of solving linear equations would be reduced to a kind of puzzle: what can we do to both sides of this equation to make the whole thing look simpler?

That last question is related to another fascinating nugget that is mentioned in the book. Barton gives an example of a question concerning a parallelogram ABCD, where the angle at A is 105 degrees. The line BC is extended to a point E, which is then joined by an additional line segment to D, and the angle CED is 30 degrees. The question is to prove that the triangle CED is isosceles.

Apparently, this question is found hard, because one cannot achieve the goal in one step. Instead, one must observe that the angle of the parallelogram at C is also 105 degrees, from which it follows that the angle ECD is 75 degrees. And from that it follows that the angle EDC is 75 degrees as well, and the problem is solved.

But the interesting thing is that if you change the question to the more open-ended question, "Fill in as many angles in this diagram as you can," then many people who found the goal-oriented version too hard have no difficulty in filling out all the angles in the diagram and therefore noticing that the triangle CED is isosceles.

The lesson I would draw from this with the equations question is that instead of asking for a solution to the equation $4 - 2x = 3x - 11$, it might be better to ask "See whether you can make the equation look simpler by doing something to both sides. If you manage, see if you can then make it even simpler. Keep going until you have made it as simple as you can." This would of course come after they had already seen several examples of the kind of thing one can do to both sides of an equation.

Barton isn't content with just telling the reader that certain methods of teaching are better than others: he also tells us the theory behind them. Of particular importance, he claims, is the fact that we cannot hold very much in our short-term memory. This was music to my ears, as it has long been a belief of mine that the limited capacity of our short-term memory is a hugely important part of the answer to the question of why mathematics looks as it does, by which I mean why, out of all the well-formed mathematical statements one could produce, the ones we find interesting are those particular ones. I have even written about this (in an article entitled Mathematics, Memory and Mental Arithmetic, which unfortunately appeared in a book and is not available online, but I might try to do something about that at some point).

This basic point informs a lot of the discussion in the book. Consider, for example, a question that asked you to find the perimeter of a rectangle that had side lengths 2/3 and 3/5. This could be a great question, but it is very important to ask it at the right point in the students' development. If you ask it before they are fluent at adding fractions and at working out perimeters of rectangles, then the amount they have to hold in their heads may well exceed their cognitive capacity: they need to store the fact that you have to add the two lengths, and multiply by 2, and put both fractions over a common denominator. It is to avoid this kind of strain that attaining fluency is so important: it literally makes it easier to think, and in particular to solve the kind of interesting problems we would all like them to be able to solve. Barton absolutely doesn't dispute the value of interesting problems that mix different parts of mathematics — he just argues, very convincingly, that one has to be careful when to introduce them.

An idea he discusses a lot, and that I think might perhaps have a role to play in university-level teaching, is what he calls diagnostic questions, and in particular low-stakes diagnostic tests. These typically take the form of a short multiple-choice quiz, and he tries very hard to create a classroom culture where people understand that the purpose of the quiz is not assessment — the quizzes do not "count" for anything — but a tool to help learning, and in particular to help diagnose problems with understanding.

What makes these questions "diagnostic" is that they are carefully designed in such a way that if you have a certain misconception, then you will be drawn towards a certain wrong answer. That is, the wrong answers people give are informative for the teacher, rather than merely wrong. Here, for example, is a question that fails to be diagnostic followed by a modified version that succeeds.

A triangle has one side of length 6 and two sides of length 5. What is its area?

A. 8
B. 11
C. 12
D. 15
E. 20

A. 6
B. 12
C. 15
D. 16
E. 24
F. 30

With the second set of choices, each answer has a potential route that one can imagine somebody taking. To obtain the answer 6, one chops the triangle into two right-angled triangles, each of height 4 and base 3, calculates the area of one of them, and forgets to double it. The correct answer is 12. To obtain 15, one takes the formula "half the height times the base" but substitutes in 5 for the height. To obtain 16 one calculates the perimeter. To obtain the answer 24 one takes the height times the base.

And to obtain the answer 30 one multiplies the two numbers 6 and 5 together (on the grounds that "to calculate the area you multiply the two numbers together"). Thus, wrong answers yield useful information. With the first set of answers, that just isn't the case — they are much more likely to be the result of pure guesswork.

It's worth mentioning that Terence Tao has created a number of multiple-choice quizzes (http://scherk.pbworks.com/w/page/14864181/FrontPage) on university-level topics. He has also blogged about it here (https://terrytao.wordpress.com/2008/12/14/on-multiple-choice-questions-in-mathematics/). They are not exactly diagnostic in the sense Barton is talking about, but one could imagine trying to make them so.

Barton uses these diagnostic tests to get a much clearer picture of what his class already understands, before he launches into the discussion of some new topic, than he would by simply asking questions to the class and getting answers from a few keen students. If he diagnoses a fairly serious collective misunderstanding, then he will spend time dealing with that, rather than pointlessly trying to build on shaky foundations.

I'm jumping around a bit here, but a semi-counterintuitive idea that he advocates, which is apparently backed up by serious research, is what he calls pretesting. This means testing people on material that they have not yet been taught. As long as this is done carefully, so that it doesn't put students off completely, this turns out to be very valuable, because it prepares the brain to be receptive to the idea that will help to solve that pesky problem. And indeed, after a moment of getting used to the idea, I found it not counterintuitive at all. In fact, it resonates very strongly with my experience as a research mathematician: I find reading other people's papers very difficult as a rule, but if they can help me solve a problem I'm working on, a lot of that difficulty seems to melt away, because I know exactly what I want, and am looking out for the key idea that will give it to me.

There's a great section on the use of artificial "real-world" problems. I think he would agree with me about Use of Maths A-level (https://gowers.wordpress.com/2009/07/11/help-im-stuck-in-my-ivorytower/). As someone he quotes says, "Students are constantly on their guard against being conned into being interested." An example he discusses is

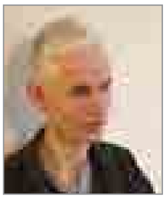
Alan drinks 5/8 of a pint of beer. What fraction of his drink is left?

If the entire point of the exercise is to gain fluency with subtracting fractions, then he advocates just cutting the crap and asking them to calculate 1-5/8, which I agree with 100%.

If, on the other hand, it is intended as an exercise in stripping away the unnecessary real-world stuff and getting at the underlying mathematics, then he has interesting things to say (later in the book than this section) about the relationship between what he calls the surface structure and the deep structure. The former is to do with the elements of the question that present themselves directly to the student — in this case Alan and the beer — while the deep structure is more like the underlying mathematical question. To train people to uncover the deep structure, it is very important to give them pairs of questions with the same surface structure and different deep structures, and vice versa. Otherwise, they may learn a procedure that works for lots of similar examples and lets them down as soon as a new example comes along with a different deep structure.

There is lots more in the book — obviously, given its length — but I hope this conveys some of its flavour. The only negative thing I can think of to say is that the word "flipping" is overused — the sentence "Teaching is flipping hard" occurs several times, when once would be enough for one book. But if you're ready for a bit of jocularity of that kind, then I recommend it, as I found it highly thought provoking. I don't yet know what the result of that provocation will be, but I'm pretty sure there will be one.

This review has been reprinted with kind permission from Prof Timothy Gowers. The original review was published in Prof Gowers's Weblog, https://gowers.wordpress.com/2018/12/22/how-craig-barton-wishes-hed-taught-maths/.



SIR WILLIAM TIMOTHY GOWERS, FRS is a Royal Society Research Professor at the Department of Pure Mathematics and Mathematical Statistics at the University of Cambridge, where he also holds the Rouse Ball chair, and is a Fellow of Trinity College, Cambridge. In 1998, he received the Fields Medal for research connecting the fields of functional analysis and combinatorics. Prof Gowers is deeply interested in mathematics education and his thoughts and insights might be found at his weblog (https://gowers.wordpress.com/).

ASPECTS OF AN ALGEBRAIC IDENTITY

– Laxman Katkar

In the article [1] (*At Right Angles*, July 2019), a visual justification is provided for the relation $(a+b)^2 \neq a^2+b^2$, the reference being to a commonly made error in algebra, in schools everywhere. In this context, it is of interest to note that there are some contexts in mathematics where the relation $(a+b)^2 = a^2+2ab+b^2$ is not true, and some contexts where the relation $(a+b)^2 = a^2+b^2$ is true.

Vectors. The first such context occurs in vector algebra. For vectors **a** and **b**, let $|\mathbf{a}|$ and $|\mathbf{b}|$ denote their respective lengths. Then we have:

$$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b}.$$

From this, we obtain three special cases of interest:

- If **a** and **b** are perpendicular to each other, then
$$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2.$$
- If **a** and **b** point in the same direction (so the angle between them is 0°), then
$$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}|\,|\mathbf{b}|\,.$$
- If **a** and **b** point in opposite directions (so the angle between them is 180°), then
$$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}|\,|\mathbf{b}|\,.$$

Matrices. If A and B are square matrices of the same order, then the products AB and BA are both well-defined. In this case we have:

$$(A+B)^2 = A^2 + B^2 + AB + BA.$$

Here we note the following.

- In some special cases, it may happen that $AB = BA$ (i.e., the matrices commute with each other). In this case the usual formula will hold, $(A+B)^2 = A^2+B^2+2AB$.
- In other special cases, it may happen that $AB = -BA$. In this case it will happen that $(A+B)^2 = A^2+B^2$.

References

[1] James Metz, "Seeing Why," *At Right Angles*, July 2019, http://www.cut-the-knot.org/triangle/PedalTriangle.shtml

The Closing Bracket . . .

Market and Mathematics

S. Giridhar*

The 2019 winner of the Shanti Swarup Bhatnagar Award for Mathematical Sciences is Prof. Neena Gupta. Her field of study and research is cutting-edge commutative algebra and affine algebraic geometry. And yet interestingly, when journalists met her to congratulate her and get some sound bites for their reports, Gupta talked to them about the importance of good foundational maths for children at a young age. She felt the phobia for maths develops because the subject is not introduced properly to kids. But in a constructive follow up sentence, she also mentioned how, 'the market is the best tutorial for kids. One is amazed with the ease with which vegetable sellers tackle calculations.'

As I read these views over my cup of morning coffee, it brought back to me, my own experiences of observing some of the most committed teachers recently over the past two years as a part of a deep-dive qualitative study of good schools and teachers. During this study, I have visited over 120 schools across six districts, where the Azim Premji Foundation is deeply and intensively engaged with government schools and contributing to improving the quality of education in these institutions. My colleagues in these districts pointed me to the teachers, who in their judgment, were doing a great job – committed to their own development, constantly refining and reviewing their methods and materials and most importantly, held a burning conviction that every child can learn. And I thought I must share what I saw and observed.

Out in the government schools of rural India, more than fifty percent of the students come from socio-economically disadvantaged communities. Many are first generation learners; for many the free mid-day meal at school is the only hot meal of the day. Language barrier, migration of parents for livelihoods are other challenges that our government school teachers grapple with, not to speak of their own hardship with reaching their schools which are often in remote places far from their homes.

Many of the schools I visited were primary schools, i.e., classes 1 to 5, while some were middle schools (also known as upper primary school) till class 8. Therefore, what I observed were only the early classes, where maths concepts are concrete, with activities and material to understand numbers, basic operations, place value, fractions, decimals and basic geometry. Some amount of data collation and graphical representations too. I did not observe teachers taking children from the concrete to the abstract, grappling with algebra or with more demanding geometry. So bear this in mind as I select a few stories from the hundreds of inspirational teachers I met in the course of my 'pilgrimage'.

First, the primary school in the village of Nadya-ki-Dhani in the district of Tonkin Rajasthan. It was almost 3 p.m. by the time we reached the school. The approach to the school – as with many rural schools – is through a messy, slushy path, with buffaloes on either side up to the school gate and a tabela (cattle shed) right outside the school. Inside, is a fine, clean premise with a large, well-maintained playground, and a perfect avenue of neem trees leading up to the main school building. We climbed the steps into the veranda and heard excited voices from the classroom on the right where Ashok Sohail was teaching maths to the children of Classes III, IV and V.

There were twenty-three children in Ashok's classroom. Usually, when a visitor comes in, the children get up and sing out the rehearsed, 'Good morning/afternoon, sir, how are you?' in a chorus. Not here, for they were completely immersed in the maths puzzles that their teacher was tossing at them. Ashok grinned a welcome at us but continued with the lesson. The ensuing thirty-five minutes were a treat. Ashok was challenging the children with questions that required them to mentally do a variety of operations with two-digit numbers. The context Ashok created was that of the market. He 'sent' the children to buy various stationery items. On the board, the price of each item was listed. Ashok asked, 'If you bought three pencils and four erasers, how much would you spend?' 'If you spent Rs 90, what combination of items would you have bought?'. 'If you took Rs 100 and bought these three items in these quantities, what is the balance you will receive?' Children from across the room answered; some questions had more than one correct answer; some children made errors but they were doing the calculations in their heads with an agility that could have only come with a complete comprehension of the question, as well as the operation.

I could read Ashok's pride and happiness as his students responded enthusiastically. He moved to bigger numbers and 'sent' the children to the hardware market to buy cement, steel and bricks for the construction. I noted that there were four children quick as lightning with their answers and seventeen of the twenty-three children answered at least one question. Every child had such an anticipatory smile that Ashok might well have been distributing Deepavali sweets and not asking maths questions. It was past school closure time but no one seemed to care. Ashok was now helping children construct an understanding of decimals. In primary classes, two concepts that both teachers and students struggle with are the concepts of 'fractions' and 'decimals.' But when children have a strong comprehension of numbers; their construction and basic operations, then these concepts can be easily built. Ashok spent a few minutes asking the children to construct a variety of currency denominations to make up Rs 100. The children started in chorus with the simple, 'two Rs 50 notes' and moved on to make combinations of fifty, twenty, ten, five, two- and one-rupee notes. At a particular point in this raucous discussion, Ashok stopped them by saying that the shopkeeper had a problem. He had notes only for Rs 99 and only coins in small denomination for the remaining one rupee. Children now made up the remaining rupee with fifty, twenty-five and ten paisa coins. It was the perfect time for Ashok to introduce the concept of decimal and fraction with the explanation of the fifty paise as half a rupee and as 0.5 rupee. The children constructed the new number as 99 + 0.5 + 0.25 + 0.25. Even as Ashok decided it was enough for the day, the children were racing ahead creating fractions and decimals for two and three rupees.

Let me now take you to the Government Primary School in Barethi village of Uttarkashi district and into the classroom where Mukesh Nautiyal is teaching children the concept of 'place value'. He begins by reinforcing the difference between a digit and a number by asking the students to give examples of single, double- and triple-digit numbers, to bring in the concept of units, tens and hundreds. As he did this, Nautiyal introduced numbers and values in the context of money. From his wallet, he pulled out a couple of notes of different denominations. Very naturally, the discussion veered to the topic of Mahatma Gandhi's picture on our currency notes. A short discussion about Gandhi, India and independence, about India and our society and what Gandhi stood for, followed. He asked the children to connect this to what they had learnt in their weekly general knowledge class and pointed to how two different subjects are related. When he felt that the children could construct four-digit numbers, he asked them to announce their roll numbers—single- and double-digit, odd and even—by turns. Using their roll numbers, he helped children construct a variety of four-digit combinations. While some of the students were still trying to absorb this, he engaged those who had grasped the concept with a more challenging task- to place the roll numbers in a manner that the highest number was formed. The energy in the class was sustained, there was also a kind of shared excitement between the teacher and the students about the discovery and understanding of a new concept.

Many miles south of Uttarkashi, I was at an early morning school assembly. The school had a culture of including very interesting language, environment and math questions / quizzes as part of the assembly. That morning, as the teacher asked the children a variety of questions, most children attempted an answer. The wrong answers too were accepted with equanimity. The teacher wrote '5555' on the board and asked if each of the '5s' has the same value since it is the same digit. The children knew the concept of place value well and one of them explained, 'Sir, *every 5 has a different seat because each seat has* a different value. The children called out the place values and their chorus of units, tens, hundreds, thousands resounded across the room.

While Mukesh Nautiyal is not the usual Math teacher – he is highly qualified in the subject – many of the primary school teachers in our rural government schools are not graduates in Maths. They usually complete the D.Ed. diploma after their XII class. So, it is remarkable to see these ordinarily endowed people have not only become good teachers but are resolutely striving to become better at their profession. The complexities and scale of our education system are such that in-service professional development for teachers is inadequate. Teachers in rural postings fend for themselves and those who are determined seek out opportunities to learn. Many of these teachers make up for their inadequacies in subject knowledge by their commitment to continuous self-development. There can be no greater evidence of this than the fact that some maths teachers, acknowledged for their excellence in teaching the subject, have no formal degree in mathematics. Their lesson plans and activities, the worksheets and experiments that they had constructed were indicative of their efforts to learn and teach the subject. Over time, this keen desire to improve has become a lifelong habit. What I saw was moving and humbling.

S. Giridhar, the Chief Operating Officer of Azim Premji University, is the author of the forthcoming book, 'Ordinary People, Extraordinary Teachers' (Westland Books)

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2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.

3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.

4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.

5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.

6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.

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10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.

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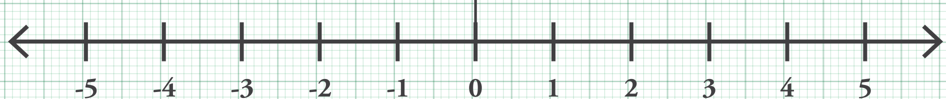
At Right Angles

A Resource for School Mathematics

PERCENTAGES

PADMAPRIYA SHIRALI



Azim Premji University

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INTRODUCTION

The word ***percentage*** has become part of our everyday language. 'I will come for the party, 100 percent.' 'There is no way I will get selected for the team, 0 percent chance.' 'I don't go to the gym regularly, maybe 50 percent of the time.' We use the word ***percent*** in a loose manner and often only as an approximation. However, through its usage we manage to communicate our intentions.

Many of our daily activities involve calculations involving percentages or comparisons based on percentages. 'How much will a shirt cost after the discount?' 'What is the highest percentage of marks in a subject?' 'What percentage of the school attends football coaching?' And so on.

Percentage is not a new concept. Percent is a form of fraction – a special fraction where the denominator is 100. Technically, percent means 'out of 100' and refers to a hundredth in decimal numbers. It is a way of expressing a number 'out of 100' using the symbol %.

What is the purpose of learning percentages? What does its usage achieve?

It builds the relational thinking of the students; a percentage is used to express how large or small one quantity is relative to another quantity.

Percentage is a topic closely connected with fractions and decimals. Students often encounter this topic after exposure to fractions and decimals. The teacher makes use of the prior knowledge of the students in the areas of fractions and decimal place value while introducing percentages. While solving percentage problems these earlier concepts are revisited and reinforced. Hence it is important to establish the connections between fractions, decimal place value and percentages in multiple ways at the start, so that students understand the linkages thoroughly.

The teaching of percentages incorporates estimation skills and building mental agility and dexterity in converting simple percentages to fractions and vice versa. Students should see the usage of percentage in varied contexts, as visuals and pie charts and should be able to model percentage problems.

I have long used the approach presented here and have found it to work well in developing students' mental arithmetic skills. As the approach proceeds in a gradual step-by-step manner, it focuses on one skill at a time and proceeds to build the connections.

Procedures and usage of notation are taken up at the second stage. Converting complex fractions and decimals to percentage and vice versa is taken up at the third stage.

Before starting on the topic, the teacher can focus on the word and its meaning.

The word 'percent' can be broken up into two parts: per and cent. Cent reminds us of words like century, centimetre, etc. It is common knowledge that a Century stands for 100 years and that 100 centimetres make a metre. 'Cent' stands for 100. The word 'Per' means 'out of.' So percent means 'out of 100,' a quantity expressed out of 100.

Students may be familiar with the word 'centipede.' (However, even though it is called a centipede it does not really have 100 legs!)

ACTIVITY 1

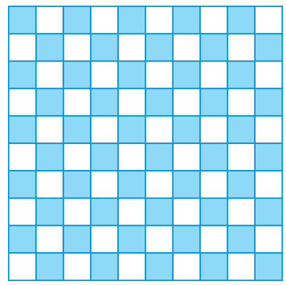
Objective: Build the students' capacity to use 50% in different contexts

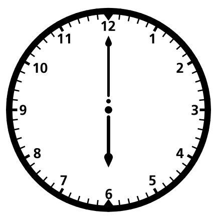
Materials: For demonstration: 100 square grid paper with 50% shaded, clock dial, transparent measuring jars, interlocking cubes, half drawn figures on square grid.

For students: Square papers cut from newspapers, square grid paper sheets.

Show 50% in various contexts.

In a square grid of 100 where both 50 and 100, part and total are clearly visible.



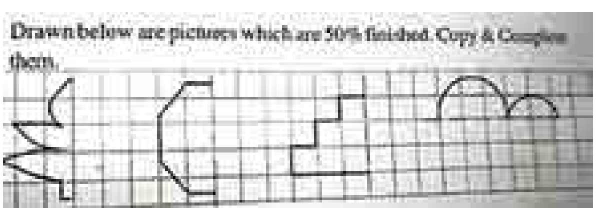


On a clock dial, show 50% of a circle.

Show 50% of a cake.

Show a measuring jar which is half filled with sand or water.

- Ask the students to show 50% of a square paper through paper folding. How many ways can this be done?
- Give them some polygon shapes, ask them to trace the outline and colour 50% of the shape.
- Give them the half-drawn figure cards, ask them to copy and complete.



Do the students grasp the relationship between 50% and ½?

Pose some mental arithmetic questions.

What is 50% of 60?

50% of 30? 50% of 10?

50% of 48? 50% of 124?

You can gradually raise the challenge!
Ask what is 50% of 3? 50% of 17? 50% of 101?

Pose questions where the 50% is given and the students have to find the 100%.

What is 100% if 48 is 50%?

100% if 28 is 50%? 100% if 9 is 50%?

100% if 112 is 50%? 100% if 4008 is 50%?

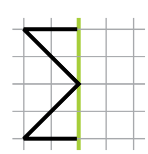
Raise the challenge gradually.

What is 100%, If 50% is 10 ½?

100% if 50% is 2.5? 100% if 50% is 99.5?

Pose some questions based on visuals.

If this shape is 50%, what is 100%?



Use some word problems as oral questions.

Example: There are 12 children left in a classroom. The other 50% are in the library. What is the strength of the class?

ACTIVITY 2

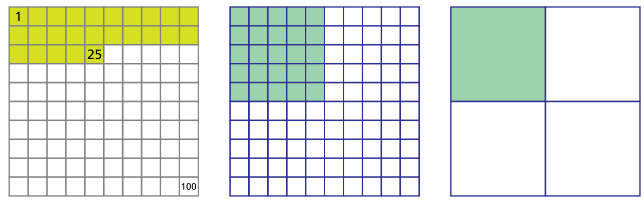
Objective: Build the students' capacity to use 25% in different contexts.

Materials: For teacher demonstration: 100 square grid paper with 25% shaded, clock, transparent measuring jars, interlocking cubes, quarter figures on square grid.

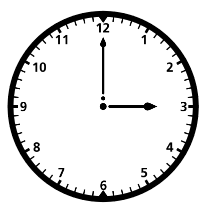
For students:
Square papers cut from newspapers, square grid paper sheets.

Show 25% in various contexts.

In a square grid of 100 where both 25 and 100, part and total are clearly visible.



On a clock face, show 25% of a circle.



Show a measuring jar which is quarter filled with sand or water.

- Ask the students to show 25% of a square paper through paper folding. How many ways can this be done?
- Give them some polygon shapes, ask them to trace the outline and colour 25% of the shape.
- Give them the quarter figure cards, ask them to copy and complete.

Do the students grasp the relationship between 25% and ¼?

Pose some mental arithmetic questions.

What is 25% of 20?
25% of 120? 25% of 96?
25% of 200? 25% of 1000?

What method did the students use to find the answer?

Let students share with the class how they obtained the answers.

Some would have directly calculated ¼ of the number in a single step.

Some would have calculated ½ first and then calculated ½ of the ½.

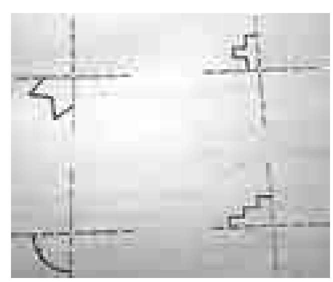
Example: ½ of 120 is 60 and 1/2 of 60 is 30.

Gradually raise the challenge.

Ask what is 25% of 2? 25% of 14? 25% of 25?

Pose some visual questions.

If this shape is 25%, what is 100%?



Pose numerical questions where the 25% is given and the students have to find the 100%.

What is 100% If 16 is 25%? If 30 is 25%?
If 19 is 25%? If 105 is 25%?

Raise the challenge gradually.

What is 100%, If 25% is 12 ½? If 25% is 1¼?
If 25% is 3.2?

If 25% of a number is 60, what is 50% of that number?

ACTIVITY 3

Objective: Build the students' capacity to use 75% in different contexts

Materials: Show 75% on a square grid and on a clock face.

Pose the question What is 75% of 40?'

How do the students respond to it?

If they are able to give the right answer, ask them to explain the method they have used.

If not, pose some additional questions as a hint.

What is 50% of 40?   What is 25% of 40?

Some students may have realised that 75% is the same as ¾ and calculated ¾ of 40 in a single step.

Some may have computed 50% first and then 25% and added the two results.

Let the students discuss this and realise that both methods give the same answer.

Pose some mental arithmetic questions.

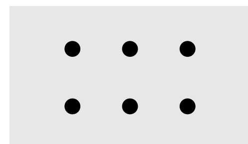
What is 75% of 48?   75% of 120?
75% of 400?   75% of 12?

You can gradually raise the challenge.

Ask what is 75% of 2?   75% of 18?

Pose some visual questions.

If this shape is 75%, what is 100%?



Pose questions where the 75% is given and the students have to find the 25% and 50%.

- If 75% of a number is 12, then what is 25% of that number? What is 50% of that number?
- What is the actual number?
- If 75% of a number is 1.5, then what is 25% of that number? What is 50% of that number?
- What is the actual number?

ACTIVITY 4

Objective: Build students' capacity to use 10% in different contexts

Pose the question 'How will you find 10% of a number?'

By now the students see the relationship between percentage and fractions clearly to be able to state that 10% is the same as 1/10 of a whole.

Finding 10% is the same as dividing the number by 10.

Ask 'what happens to the digits of a number when you divide by 10?' Do the students see that when you divide a number by 10, each digit shifts by one place to the right? From a tens place, it shifts to the units place. From a units place, it shifts to the tenths place. And so on.

What is 10% of 60?   10% of 500?
10% of 45?   10% of 2?

If 5 is 10% of a number, what is the number?

What is 10% of 100?
10% of 200?
10% of 300?

Do the students see the pattern?

ACTIVITY 5

Objective: Build students' capacity to use 20% in different contexts

Pose the question 'What is 20% of 80?' How do the students respond to it?

Some students may have realised that 20% is the same as ⅕ and calculated ⅕ of 80 in a single step.

Some may have computed 10% first and then doubled the result.

Has anyone tried any other method?

Let the students discuss this and realise that there are different ways of solving the problem.

Pose some mental arithmetic questions.

What is 20% of 40? 20% of 120?
20% of 300?

If 10% of a number is 5,
what is 100%? What is 50%?

If 20% of a number is 4,
what is 100%? What is 60%?

ACTIVITY 6

Objective: Students to develop methods to calculate 5%

Ask the students if they can develop a method for computing 5% of a number. Discuss the methods they develop.

How will they compute 1% of a number?

What happens to the digits of a number when the number is divided by 100?

How will they compute ½ % of a number?

Introduction of Symbol: Teacher can now introduce the symbol % for percentage and show that a whole stands for 100%.

ACTIVITY 7

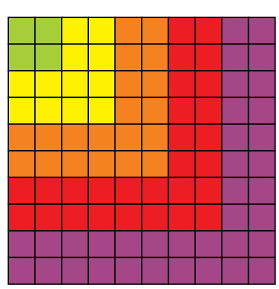
Objective: To reinforce that a whole represents 100%

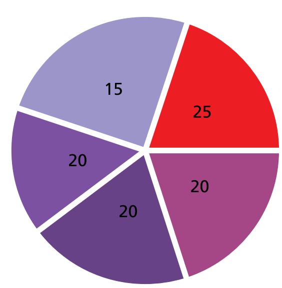
Materials: 100 square grid, Circle divisions.

To reinforce that parts of a whole add up to 100%

Students to make a coloured design in a 100 square grid and write the percentage of each colour.

Let them total the percentages of all the colours and notice that the percentage figures of the different colours add up to 100%.





ACTIVITY 8

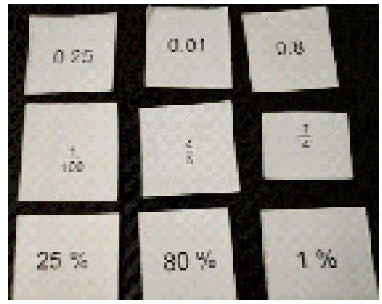
Objective: To create sets of matching cards

Materials: Percentage kit consisting of equivalent percentage, fraction and decimal cards

The kit can be used for multiple activities. Show the percentage card and have students pick up the matching fraction and decimal card.

Show the fraction card and have the students pick up the matching percentage card.

Give two percentage cards and have the students find a fraction or decimal card which lies between the two percentages.

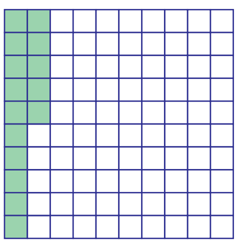


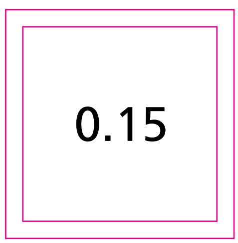
GAME 1: Four Set!

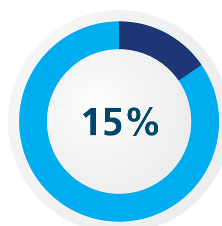
Objective: To make a full set of equivalent cards.

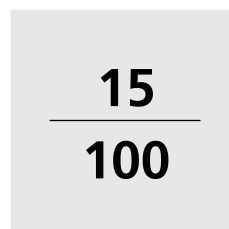
Materials: Percentage Kit: Several sets of equivalent percentage, fraction, decimal and visual cards

No. of players: 4









Four set is a card game in which you try to improve the hand that you have been originally dealt. You can do this whenever it is your turn to play, either by drawing cards from a pile (or stock) or by picking up the card thrown down by the opponent and then discarding a card from your hand.

It is quite similar to the game of Rummy (for those who are familiar with this game).

Distribute the cards to students so that each student gets 4 cards. Leave one card open and place the remaining cards in the centre, face down. Each student is allowed to pick up the topmost card from the pile or the card just thrown down by the opponent. Each time he picks up a card, he has to discard one card. The second player may pick up the discarded card or pick up a new card from the pile. This continues until one of the players manages to get a full equivalent set.

If all the cards in the closed pile finish, they can shuffle the cards left on the ground, pile them face down and repeat the process. And so on.

ACTIVITY 9

Objective: Converting fractions with denominators which are factors of 100 to percentages.

While using the percentage kit, students would have already come across how percentages are represented by fractions which are reduced to the lowest form. Example: 50% is 50/100, i.e., ½.

The teacher can reinforce this using several examples. She need not take up fractional or decimal percentages at this point. That can come later, once the percentage concept has been thoroughly understood.

The teacher begins with fractions where the denominator is a factor of 100.

Example: ⅖, ¾, 7/10, 11/20, etc.

Students are familiar with equivalent fractions and they will be able to write:

⅖ = 40/100 = 40%

¾ = 75/100 = 75%

7/10 = 70/100 = 70%

11/20 = 55/100 = 55%

ACTIVITY 10

Objective: Converting fractions to percentages. Converting decimals to percentages.

Discuss the method the students used for fractions such as 2/5 or 3/10 to generate the fractions 40/100, 30/100, etc.

Help them notice that they multiplied the numerator and denominator by the same number to get a fraction with denominator 100.

Point out that multiplying numerator and denominator by the same number does not change the value of the fraction.

Now ask the students to multiply 2/5 (the numerator and denominator) by 100.

- It will be ⅖ × 100/100. By 100 (/100) can be replaced by % symbol as the symbol stands for /100.
- It can now be written as ⅖ × 100%.
- Now they can do cancellations to simplify the answer.
- Teacher can now explain to the students that converting fractions to percentage can be done by multiplying by 100% (which is the same as 100/100).
- ¾ × 100/100 is the same as ¾ × 100% = 75%

Note: Most textbooks and teachers ask students to multiply the fraction by 100. This is incorrect as it alters the value of the fraction, whereas multiplying by 100% is the same as multiplying by 1 which does not alter the value of a fraction.

Students can now use the method of multiplying by 100% to convert fractions to percentages.

Note: The teacher can show that the same method of multiplying by 100% works for converting decimals to percentages.

ACTIVITY 11

Objective: Calculating given percentages of a quantity.

Students can prepare a set of matching cards for practice.

Teacher can create a grid to be filled in with answer cards. Students can note down the time they take to complete a grid.

	120	1080	1800	640
10%				
50%				
25%				
75%				
90%				

GAME 2: Percentage dominoes

Materials: Domino cards

Objective: Matching fractions and percentages

Prepare a set of domino cards as shown.

Let students arrange the matching cards to complete the domino train.

ACTIVITY 12

Objective: Given the percentage of a number to find the original quantity.

Questions can be posed in the following manner and the students figure out the original quantity.

40% of a number is 64. What is the number?

30% of a number is 27. What is the number?

ACTIVITY 13

Objective: Understanding of percentages that are more than 100.

Materials: Cubes or buttons.

Understand that a percentage can be zero

Modelling percentages

Call 6 students. Give 8 cubes to each student.

Tell the first student to increase the cubes by 50%. The student will take 4 more.

Tell the second student to increase the cubes by 25%. The student will take 2 more.

- Ask the third student to increase the cubes by 75%. The student will take 6 more.
- Ask the fourth student to increase the cubes by 100%. The student will take 8 more.
- Ask the fifth student to increase the cubes by 200%. What should the student do?
- Ask the sixth student to increase the cubes by 0%. What will the student do?

Discuss various percentages greater than 100 like 150%, 250%, 300% and so on.

Discuss some real life examples.

I planned to finish 10 problems today. I did 15 problems. This was 150% of my plan!

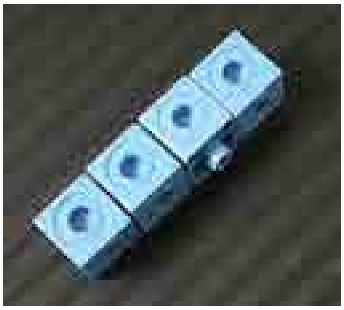
I expected the meal to cost ₹ 60. But it cost ₹ 120. This was 200% of the expected price.

ACTIVITY 14

Objective: Percentage increase/decrease through modelling

Materials: Shapes, cubes, problem cards

Pose problems that require application of percentage increase and decrease using modelling.



Example: Yagya ate 4 strawberries yesterday. Today he ate 75% more than yesterday. How many strawberries did he eat today?

Let students pair up, share their understanding of the problems and explain their thinking.

They can model the problem using materials.

They could also represent large numbers on a graph which will aid their visual appreciation.

- 450 people watched a film on the first day. On the second day the number of people who watched the film is 10% less than the number who watched on the first day. How many watched on the second day?
- If 150 decreased to 120, by what percentage has it decreased?
- The number of students in the school today is 744. That is 7% less than the number present in the school yesterday. How many were present yesterday?

ACTIVITY 15

Objective: Estimations

Materials: Problem cards

Estimations in word problems

Students should be encouraged to use estimation while solving problems. Again, the teacher can give some problem cards to students and ask them to work in pairs.

Have students explain in pairs how they would estimate the following.

9% of 140 (10% of 140 is 14, so 9% would be less than 14)

14% of 180 (10% is 18, 5% is 9, so 15% is 18 + 9 = 27, so 14 percent is around 25)

26% of 320 (25% is 80 so 26% will be around 84)

In a classroom, 7 of the 49 students like movies based on true stories. What percentage is this?

First ask students to estimate a percentage.

(7/49 = 1/7

1/5 is the same as 20%, and 1/8 is the same as 12.5%

So 1/7 will be in-between 12.5% and 20% but closer to12.5%)

ACTIVITY 16

Objective: Practice in fraction, decimal and percentage conversions (including mixed fractions)

Materials: Matching kit with common percentage and fraction cards.

Let students get plenty of practice through matching activities.

Both the fractions and percentage cards given above can be modified to include mixed fractions and corresponding percentages. Ex. 1 ½, 150%, 2 1/4, 225%.

Provide some tables as shown for students to practice conversions.

Fraction	Decimal	Percentage
$\frac{1}{2}$		
	0.75	
		30%
$\frac{3}{100}$		
		$33\frac{1}{3}$%
	0.625	

The teacher can collect clippings from newspapers of usage of percentages in real life and discuss them.

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at padmapriya.shirali@gmail.com