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Azim Premji University

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

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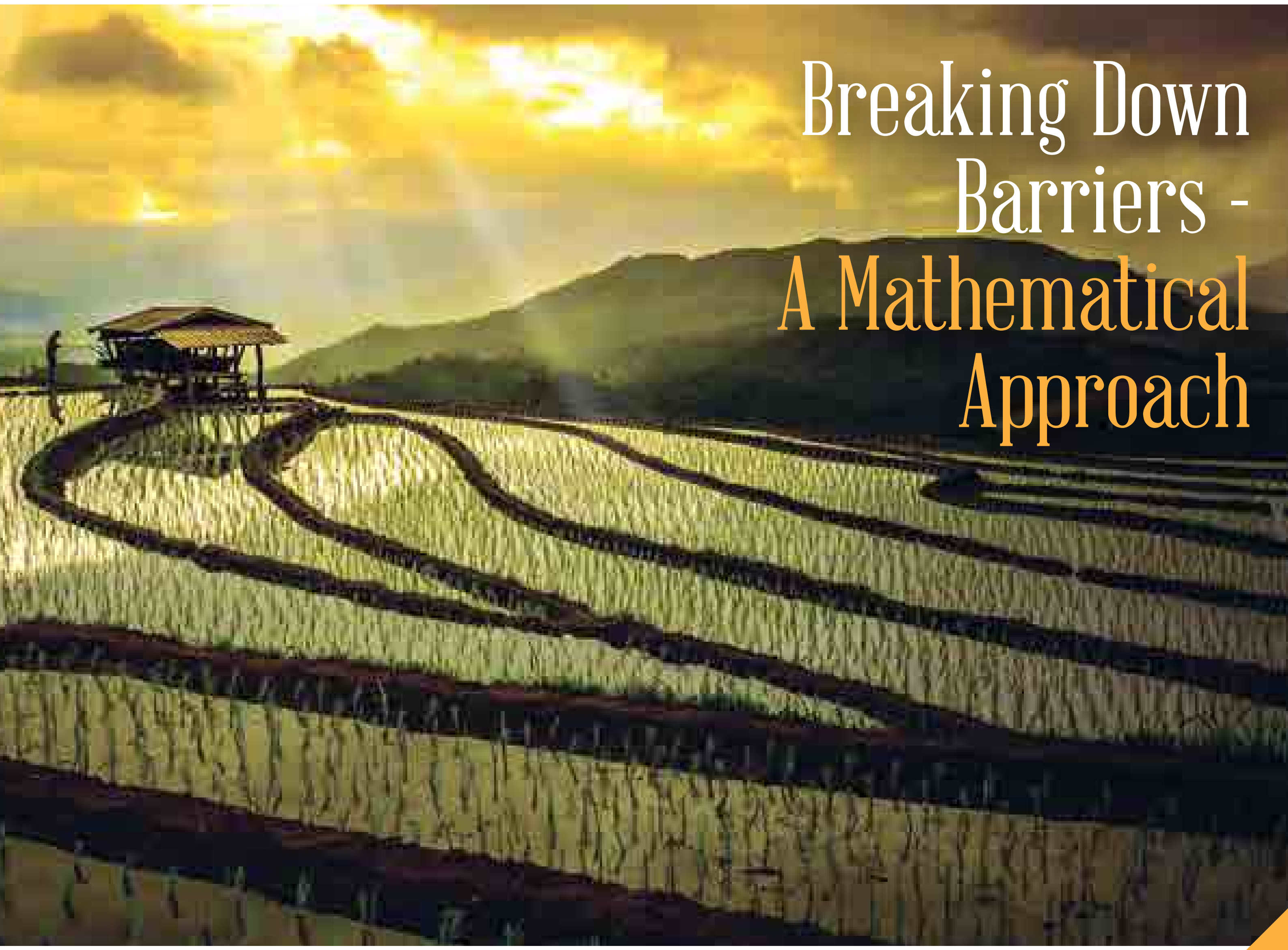

PULLOUT ANGLES

Breaking down walls and networking.

Two much used phrases in the corporate world. And all too relevant in the world of math education, where barriers go up in the classroom, the staff room, between government and private schools, between the exclusive and the plebeian...... the list is endless. With our more inclusive editorial policy, we hope to break down some of these barriers and what better way than to do it mathematically. By minimising the number of gates that need to be made and seeing how efficiently the walls can come down.

From the Editor's Desk . . .

We begin the new year and the new decade by setting a few new directions for *At Right Angles*. It has been in publication now since 2012, and this is a good moment to list a few key questions and identify new goals for ourselves.

The core audience for *At Right Angles* must continue to be the schoolteacher. We must ask ourselves how this target audience receives our publication. We must also ask what we can do to ensure a more dynamic and lively two-way conversation between school teachers and the magazine, and what we can do to draw in more contributors from the vast pool of school teachers across the country.

As a crucial first step in this direction, we have inducted five math resource persons from the Field Institutes of the Azim Premji Foundation into the editorial team. They have been closely involved with mathematics teaching in different states across the country, for very many years. We warmly welcome Shri Arddhendu Dash, Shri Ashok Prasad, Shri Hanuman, Shri Mohammed Umar, and Shri Sandeep Diwakar into the editorial body.

The editors would like to now work consciously towards the following objectives:

a) Increasing the number of contributors and authors

b) Conducting workshops to help those who would like to write for the magazine

c) Expediting translations of AtRiA into some regional languages, chiefly Hindi and Kannada

d) Publishing more articles which are directly usable by teachers in their classrooms

Publishing more articles which are directly usable by mathematics teacher trainers from the Azim Premji Field Institutes in their work with practicing teachers. Suggestions from readers for approaches that will help us move towards these objectives would be very welcome.

AtRiA must also continue addressing the student reader, particularly mathematics students at the high school level. The magazine must continue to be seen as a platform for sharing articles written by students, describing interesting results that they have proved or discovered by themselves, or interesting solutions they have found to challenging, non-routine problems. A particularly important source for such articles would be math clubs or math circles or math problem-solving groups located in schools. As the level of such contributions may differ significantly from the articles referred to above, we are now planning to start an online section of *At Right Angles*. This section will feature articles that will appear only in the online edition and not in the print edition of the magazine.

We hope to publish many more articles now at the middle school level, in the print edition of *At Right Angles*. These could be articles that deal with or feature:

a) The nature of understanding, and the play between conceptual understanding and procedural understanding, i.e., essentially asking what it means to understand mathematics

b) The axiomatic nature of mathematics

c) Pedagogical issues, such as having to teach children of widely differing abilities in one's class

d) Children's experiences of mathematics in the classroom and at home and in the playground

e) Pedagogical approaches that work and pedagogical approaches that don't work

f) Understanding basic procedures better (for example, the square root algorithm, or the GCD algorithm)

g) Interviews with practicing teachers

h) Case studies

I) Connections with other areas of knowledge.

We also hope to publish many more articles from practicing teachers and practicing teacher educators – articles that are reflective, coherent, thoughtful, and authentic, drawing on actual experiences and difficulties.

Lastly, before closing, I would like to place on record our sincere thanks and gratitude to those editors who played an instrumental role in the shaping of *At Right Angles* in its early years and are now leaving us: Shri Athmaraman, who edited the middle school problem section; Shri Tanuj Shah, who wrote many articles for us at the middle school level; Shri D D Karopady, who produced many cross number puzzles for us; and Ms Sneha Kumari, who helped secure an ISSN number for the magazine.

Shailesh Shirali
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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

LEGENDS - Primary School | Middle School | High School | Pre-University

Azim Premji University

A Publication of Azim Premji University
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TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Captured Mathematics

Scattered across this issue, we present to you a collection of several interesting photographs which have captured some mathematical concepts.

VIEWPOINT

Beginning with this issue, we present VIEWPOINT, where we re-examine familiar mathematics concepts and practices through different viewpoints.

Features

Tiresome Paths, Water Gates & Euler's Formula

B. SURY

A hallmark of mathematics is its power to look at seemingly different problems with the same eyes and find a common idea which resolves both. It is not surprising that the two problems we discuss here, about routes to be taken with various constraints and about watering fields, can both be treated using ideas from graph theory.

Where angels fear to tread?

Angel Treading Company has a number of branches all over Malgudi and there are a number of tracks that already exist. Now, the company wants to use all its existing tracks in such a way that it can get to any branch from any other branch, and the cost is minimized. So, it is vital to know the following: *How many routes must the company operate in order to serve all the sections without having more than one route on any section?*

To understand the problem, let us look at a simple situation to begin with. Suppose the branches are at A, B, C, D and P, and the tracks are as in Figure 1.

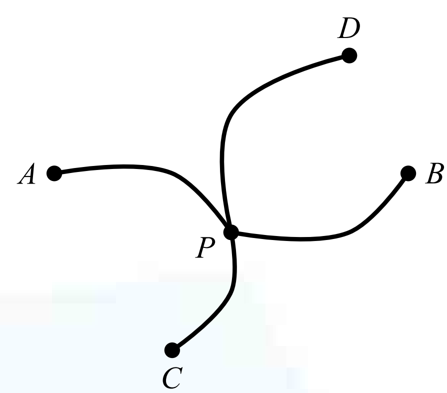

Figure 1

One route could go from A to B via P; another could go from C to D, again via P. It is clear that these two routes suffice and it is also clear that two is the least number solving the problem. A person wanting to go from A to C could then take the first route until P and change over to the route to C from P.

Of course, the above solution is not unique; for instance, one could have a route from A to C via P and another from B to D via P.

Keywords: Networks, routes, vertices, odd, even, edges, faces, Euler's formula

Let us look at another network as in Figure 2 which is slightly more complicated than the previous one.

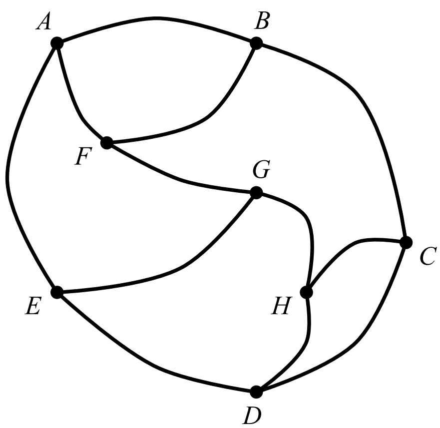

Figure 2

One route could be from *A* and run cyclically around *B, C, D, E* and back to *A*. Another could run from *A* to *F, G, H* and then to *D*. Three other routes *BF, EG, CH* would be needed, making the number of routes five in all. But, as we can see, we could combine the first two routes to make a single route and, therefore, four routes suffice for this network. We will prove in a moment that four is, indeed, the least number of routes needed.

The essential thing in the problem is to consider where the ends of the various routes must lie. Wherever a section of the track has a *free end*, as at *A, B, C, D* in Figure 1, there must be a start or end of a route. Since in Figure 1 there are four free ends, and since each route can have at the most two ends (a closed route has only one), clearly there must be at least two routes between the four free ends. By means of a single consideration, we have obtained the same result which we could obtain earlier only by considering all possible routes!

Let's look again at Figure 2 now. There are no free ends but there are junctions like *A* where three sections come together. At such a place, at least one route must start or end! Why? The reason is that any route passing through *A* has to use one section of the track while coming to *A* and another section while leaving *A*. So, the section of the track is left unpaired with any other section and has to be the start or end of a route. Of course, it might be that all three sections might be where routes start or end. That is why we said that there is at least one route ending or starting at *A*. In Figure 2 there are eight places of this kind, so there must be at least four routes; and as we saw, four routes will suffice.

As a final example, let us look at the network in Figure 3; there are five junctions of order 3 (i.e., where three sections come together) and one junction *F* of order 5. Again, obviously, there must be at least one route starting or ending at *F* since the order 5 of *F* is odd. So, there must be at least 6 route ends and therefore at least 3 routes are needed. Can you find 3 routes which suffice?

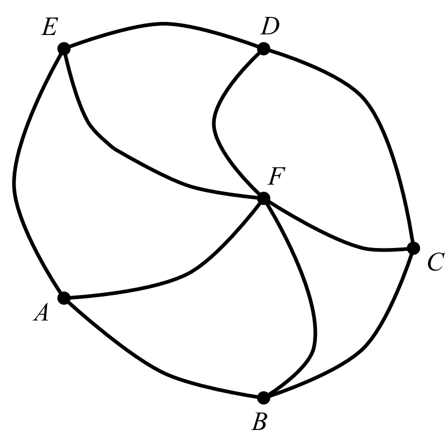

Figure 3

For any network, however complicated, we can count the number of junctions of odd order and divide by 2 to obtain the least possible number of routes. In the three examples, the number of junctions of odd order was always even, and it turned out that half this number was also sufficient.

We can see that for a system of routes to be optimal, the sections at each junction must be paired off, whenever possible. Why? Look at Figure 3 and look at the point *F*. If one route came from *C* to *F* and one from *D* to *F*, then both could be connected at *F* to form a single route and this would reduce the total number of routes. So, the conclusion from this discussion is: *In order for a system to be optimal,* *the sections at each junction must be paired off, whenever possible, and no route must end at an even junction; and then, the total number of ends of routes will be the number of odd order junctions, and the number of routes will be half the number of odd order junctions.*

One point still to be decided is whether a system that is optimal can contain a closed route. In Figure 2, we started with the closed route from *A* via *B, C, D, E* back to *A*, but then we connected it with the route from *A* through *F, G, H* to *D*, to make a single route from *A* to *D* (which is not closed). Such a reduction can be made when a closed route contains a junction of odd order. In fact, a similar reduction can be made when all junctions along the route have even order, as we show now. Let *A* be such a junction as in Figure 4 on a closed route, shown here in the shape of a figure of eight. Some other routes through *A* are also shown here (as dotted curves) and they might continue in any way.

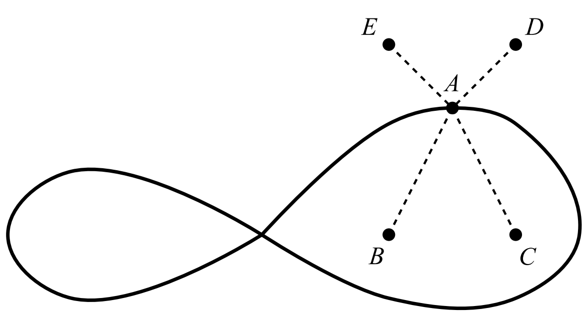

Figure 4

If the system is optimal, no route can end at *A*, so a route from *B* to *A* continues on through, say, to *E*. But then, we can combine these two routes by combining this route from *B* to *A* and along the closed route through *A* and then from *A* to *E*. This reduces the number of routes again. In this way, if we keep reducing closed routes with only even junctions, we will end up at some stage with a closed route with an odd junction and then the next reduction will yield a route that is not closed. Otherwise, *all* junctions in the original network must have been even, and then we can reduce the system to a single closed route.

Summing up our discussion, if a system is optimal, then:

a) Routes start or end only at junctions of odd order.

b) There will be a closed route only if all junctions in the original network are of even order, and then a single closed route will traverse the entire network.

c) The number of junctions of odd order is equal to the number of ends of routes, and is, therefore, an even number.

d) The minimum number of routes is half the number of junctions of odd order, except in the case where all junctions are of even order, when the minimum is one (closed) route.

Euler's Formula

Let us look at a map of fields and dikes (see Figure 5). Any two adjacent fields have a unique dike separating them. Think of the outside as being covered with water. We want to break the dikes one after another until all the fields are under water. (We may also think of this action as "opening the gate".) Suppose there are f fields to start with, e dikes and v vertices (or corners).

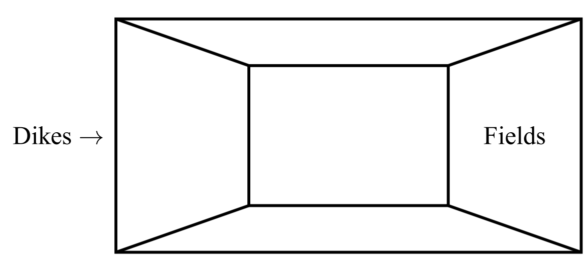

Figure 5. Here $f = 5$, $e = 12$ and $v = 8$

As you can see, it is not necessary to break *all* the dikes in order to water the fields. Any dike that already has water on both sides of it can certainly be left unbroken. If we break dikes that have water only on one side, then at each step we shall destroy one dike and flood one more field. Since this process can be carried out until all the fields have been flooded, and since we shall finally have flooded exactly f fields, we would have destroyed exactly f dikes at the end.

We want to count the number $e - f$ (of dikes left unbroken) in another way.

One can walk dry-footed along the dikes from any vertex to any other vertex. Before any dikes were broken, this could certainly have been done. Suppose in the course of flooding the fields, the destruction of some dike *AB* (as in Figure 6a) would cut the system into two separate islands. If *AB* were destroyed, it would be impossible to walk along dikes from *A* to *B*. This means that water would completely surround each of the two islands. This means that water must have been on *both* sides of *AB* before it is destroyed, and we stated that such a dike should not be destroyed. This shows that we can indeed walk along the dikes from any vertex to any other:

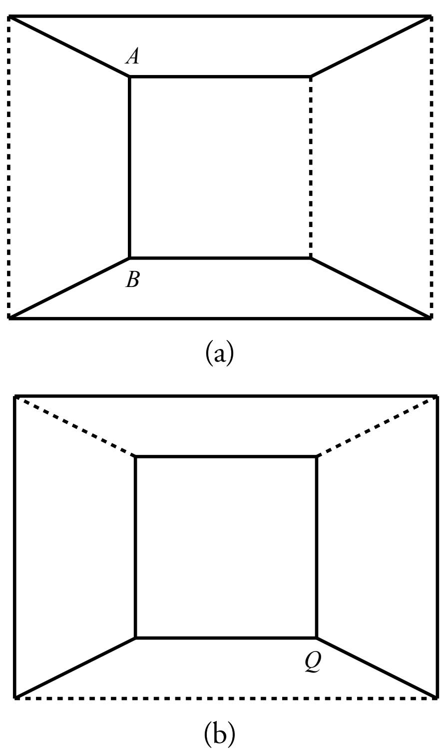

Figure 6. One can walk from any vertex to any other vertex, using the dikes

There is exactly one path going along the dikes from one vertex to another. If there were two paths from *P* to *Q*, they would surround some area (see Figure 6b). The ring of undestroyed dikes surrounding this area will keep the area dry, contrary to the fact that all the fields have been flooded.

From these observations, we see that if we fix any starting point *P*, there is a unique undestroyed dike ending at any vertex (except *P*), and conversely, there is a unique end point for each edge.

To summarize: *There are as many undestroyed dikes as there are end points of paths.*

Since the latter number is $v - 1$ (as *P* is not an end point), we have $e - f = v - 1$.

This is called ***Euler's formula***. To state it in another form, look at a map with *F* faces, *E* edges and *V* vertices. Then $V - E + F = 2$. (In our case, $F = f + 1$ since the water outside the fields is also a face.)

Euler's formula is a result of great power; it can be used to prove that *every map can be coloured with five colours*. What this means is that adjoining faces must have different colours (in a map) and five colours are sufficient to colour any map. Actually, four colours suffice but this is a very deep result proved using methods of topology.

B. SURY got his PhD in Mathematics from TIFR (Mumbai). He has been with the Indian Statistical Institute (Bangalore) since 1999. He has been interested in expository writing at school and college level and in interacting with mathematically talented students. He is the national coordinator for the Math Olympiad in Karnataka and a member of the editorial committee of the newsletter of the Ramanujan Mathematical Society. His research interests are in algebra and number theory. Mathematical limericks are an abiding interest. He may be contacted at sury@isibang.ac.in. His professional webpage is www.isibang.ac.in/~sury.

Sums of Cubes

SHAILESH SHIRALI

We have all heard the taxicab story featuring GH Hardy and S Ramanujan, but we may not know that there are some wonderful problems dealing with sums of cubes which are currently yet to be solved. We describe some recent activity in this area.

Most of us have heard the taxicab story featuring GH Hardy and S Ramanujan and the number 1729. The story [1] concludes with Ramanujan telling Hardy that 1729 is "the smallest number expressible as the sum of two [positive] cubes in two different ways." As a result of this curious episode, numbers with such a property have come to be known as *taxicab numbers*. We have encountered these numbers in an earlier article [2] in AtRiA.

This article deals not with taxicab numbers but with another extremely interesting problem dealing with sums of cubes.

Sums of two cubes

To start with, we ask: *Which positive integers are sums of two cubes?* We must specify at the start whether we are permitted to use cubes of negative integers. We shall opt to do so. So the question we ask is:

> *Which positive integers n can be written in the form $n = a^3 + b^3$ where a, b are integers (which could be positive or negative)?*

Note that we could have opted to use only cubes of non-negative integers. That then becomes another problem, distinct from this one.

For the rest of this article, we shall consistently permit the use of cubes of negative numbers.

There are surely many more numbers which are sums of two cubes than numbers which are cubes. How many more? What can be said about these numbers? Let S represent the set of all positive integers n which can be written in the form $n = a^3 + b^3$ where a, b are integers, i.e.,

$$S = \{n \, : \, n = a^3 + b^3, \;\; a \in \mathbb{Z}, \; b \in \mathbb{Z}\}.$$

Keywords: Cubes, sum of two cubes, sum of three cubes, taxicab number

Note that S contains all the cubes. We display below the first 100 numbers in S (the table has been generated using computer software).

1	2	7	8	9	16	19	26	27	28
35	37	54	56	61	63	64	65	72	91
98	117	124	125	126	127	128	133	152	169
189	208	215	216	217	218	224	243	250	271
279	280	296	316	331	335	341	342	343	344
351	370	386	387	397	407	432	448	468	469
485	488	504	511	512	513	520	539	547	559
576	602	604	631	637	657	665	686	702	721
728	729	730	737	756	784	793	817	819	854
855	866	875	919	936	945	973	988	992	999

Inputs from modular arithmetic. In exploring the structure of a set of integers generated through any arithmetical procedure, it often helps to examine the set through the lens of modular arithmetic. We shall do the same with the set S.

Consider the possible remainders left when the cubes are divided by various natural numbers. On division by 2, remainders of 0 and 1 are possible; no great surprise here! On division by 3, remainders of 0, 1 and 2 are possible; once again, no surprise here. On division by 4, remainders of 0, 1 and 3 are possible, but not a remainder of 2. Continuing, we obtain the result shown in the table below.

Modulus	Remainders	Non-remainders
2	0, 1	
3	0, 1, 2	
4	0, 1, 3	2
5	0, 1, 2, 3, 4	
6	0, 1, 2, 3, 4, 5	
7	0, 1, 6	2, 3, 4, 5
8	0, 1, 3, 5, 7	2, 4, 6
9	0, 1, 8	2, 3, 4, 5, 6, 7

We see that the first really interesting cases are when the moduli are 7 and 9, as there are more non-remainders than remainders in both these cases. This permits the use of these two moduli for making useful characterisations. Let us now see how to make use of these observations.

In the subsequent analysis, we shall use only the modulus 9. The fact that the only remainders possible are $0, 1, 8$ allows us to state the following.

Theorem 1. Every cube is of one of the following forms: $9k$, $9k \pm 1$.

This immediately implies the following two corollaries. Recall that S represents the set of all positive integers n which can be written in the form $n = a^3 + b^3$ where a, b are integers.

Corollary 1. *Every number in S is of one of the following forms:* $9k$, $9k \pm 1$, $9k \pm 2$.

This may be stated in its contrapositive form as follows:

Corollary 2. *If a number is of the form* $9k \pm 3$ *or* $9k \pm 4$, *then it does not belong to S.*

The condition in the above corollary does not eliminate sufficiently many numbers from membership in S. We need to look for better characterisations of S, but these are not readily forthcoming. We may contrast this with the situation for squares, when we have an extremely compact characterisation available, namely: *A number n is expressible as a sum of two squares if and only if, in the prime factorisation of n, all prime factors of the form* $4k + 3$ *occur with even exponent.*

For the sum-of-two cubes problem, though characterisations are available, they are quite involved. For a recent result in this area, see [3].

The prime numbers in S. The prime numbers among the first 100 numbers in S are the following:

$$2,\ 7,\ 19,\ 37,\ 61,\ 127,\ 271,\ 331,\ 397,\ 547,\ 631,\ 919.$$

If we discard the very first number (the prime number 2), then a very curious pattern is noticed about all the remaining numbers. Namely, *they are all of the form* $9k + 1$ *or* $9k - 2$ *for some integer k*. It is worth asking whether this is a genuine pattern, i.e., true for all the odd primes in S, or a misleading pattern that persists only among the first few numbers in S. If it is true, it would imply that numbers in S which exceed 2 and are of the form $9k - 1$ or $9k + 2$ are all composite. So is this strange pattern genuine or not? The answer to this puzzle is not known.

But we have mentioned this observation only in passing, as a by-the-way. The central focus of this article is the sums-of-three-cubes problem, which we now discuss.

Sums of three cubes

We move to a consideration of numbers which can be written as the sum of three cubes. Let T represent the set of all natural numbers which are either cubes or sums of two cubes or sums of three cubes, i.e., sums of three or fewer cubes:

$$T = \left\{ n \,:\, n = a^3 + b^3 + c^3,\ a \in \mathbb{Z},\ b \in \mathbb{Z},\ c \in \mathbb{Z} \right\}.$$

Since every cube is of one of the forms $9k$, $9k \pm 1$, it follows that a sum of three or fewer cubes must be of one of the following forms: $9k$, $9k \pm 1$, $9k \pm 2$, $9k \pm 3$. We therefore have the following result:

Theorem 2. If a number is of the form $9k \pm 4$, then it does not belong to T.

It is very easy to generate elements of T, as many as we may want, simply by giving all possible integer values to a, b, c in some specified range (and with $|a| \le |b| \le |c|$ to avoid duplication of elements) and then computing the value of $a^3 + b^3 + c^3$.

But verifying whether a given number belongs to T (or not) is far more difficult. To get a glimpse of the difficulty involved, consider the following. The expressions for 99, 98 and 97 are easily found, as they involve relatively small numbers:

$$99 = 2^3 + 3^3 + 4^3,$$
$$98 = 0^3 + (-3)^3 + 5^3,$$
$$97 = (-1)^3 + (-3)^3 + 5^3.$$

Similarly we have these expressions for 91, 92 and 93:

$$91 = 0^3 + 3^3 + 4^3,$$
$$92 = 1^3 + 3^3 + 4^3,$$
$$93 = (-5)^3 + (-5)^3 + 7^3.$$

There are, of course, no such expressions for 94 and 95, as these are of the forbidden forms $9k \pm 4$. But for 96, we have all of a sudden:

$$96 = 10853^3 + 13139^3 + (-15250)^3.$$

And for 75, we have the following:

$$75 = 435203083^3 + (-435203231)^3 + 4381159^3.$$

To discover such relations, one clearly needs extremely powerful computational facilities. The complexities seem formidable.

At this stage, the following question poses itself quite naturally:

> *If a number n is **not** of the form $9k \pm 4$, then does n belong to T?*

Stated in another (equivalent) form:

> *Are numbers that do not belong to T all of the form $9k \pm 4$?*

In short, is the converse of Theorem 2 true?

Offhand, there does not seem any reason for supposing that it is true (or that it is false). But computational evidence seems to suggest otherwise! *The evidence overwhelmingly suggests that every number not of the form $9k \pm 4$ belongs to T.*

It is interesting to note how this conjecture evolved. Over the decades, mathematicians tried to express various numbers as sums of three cubes (of positive or negative integers), making use of powerful computational resources. By 1960, the only numbers less than 100 which had not yet been expressed in the required form were

$$30,\ 33,\ 39,\ 42,\ 52,\ 74,\ 75,\ 80,\ 84,\ 87,\ 91,\ 96.$$

Then in the 1960s the following relations were discovered:

$$87 = 4271^3 + (-4126)^3 + (-1972)^3,$$
$$96 = 13139^3 + (-15250)^3 + 10853^3,$$
$$91 = 83538^3 + (-67134)^3 + (-65453)^3,$$
$$80 = 103532^3 + (-112969)^3 + 69241^3.$$

In the 1990s, the following relations were discovered:

$$39 = 134476^3 + (-159380)^3 + 117367^3$$
$$75 = 435203083^3 + (-435203231)^3 + 4381159^3$$
$$84 = 41639611^3 + (-41531726)^3 + (-8241191)^3$$

In the first decade of the 2000s, the following were discovered (the numbers keep getting bigger and bigger!):

$$30 = 2220422932^3 + (-2218888517)^3 + (-283059965)^3,$$
$$52 = 23961292454^3 + (-61922712865)^3 + 60702901317^3,$$
$$74 = 66229832190556^3 + (-284650292555885)^3 + 283450105697727^3.$$

By 2019, the only numbers below 100 which were not of the form $9k \pm 4$ and had not yet been expressed in the required form were 33 and 42. Then in mid-2019, the following mind-boggling relations were discovered:

$$33 = 8866128975287528^3 + (-8778405442862239)^3 + (-2736111468807040)^3,$$
$$42 = 80435758145817515^3 + (-80538738812075974)^3 + 12602123297335631^3.$$

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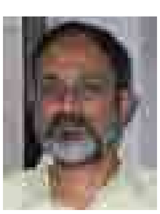

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For those who find it hard to imagine how such relations could ever have been found, here is an account of the story; it is excerpted from [7].

Mathematicians Solve '42' Problem With Planetary Supercomputer

MICHELLE STARR, 9 SEP 2019

Mathematicians have finally figured out the three cubed numbers that add up to 42. This has settled a problem that has been pondered for 65 years …The problem, set in 1954, is …: $x^3 + y^3 + z^3 = k$. Here k is each of the numbers from 1 to 100; the question is, what are x, y and z?

Over the decades, solutions were found for the easier numbers. In 2000, mathematician Noam Elkies of Harvard University published an algorithm to help find the harder ones.

This year, just the two most difficult ones remained: 33 and 42.

After watching a YouTube video [8] about the problem with 33 on the popular maths channel Numberphile, mathematician Andrew Booker from the University of Bristol in the UK was inspired to write a new algorithm. He ran this through a powerful supercomputer at the university's Advanced Computing Research Centre, and got the solution for 33 after just three weeks.

So, we were left with the hardest one of them all: 42. This proved a much more obstinate problem, so Booker enlisted the aid of fellow MIT mathematician Andrew Sutherland, an expert in massively parallel computation.

As you already know from the headline of this article, they figured it out. They also did a fun reveal of their success: according to The Aperiodical, both mathematicians quietly changed their personal websites to the solution, and named the pages "Life, the Universe, and Everything," a fitting nod to Douglas Adams.

Of course, it wasn't simple. The pair had to go large, so they enlisted the aid of the ***Charity Engine****, an initiative that spans the globe, harnessing unused computing power from over 500,000 home PCs to act as a sort of 'planetary supercomputer.' It took over a million hours of computing time, but the two mathematicians found their solution. …*

"I feel relieved," Booker said. "In this game, it's impossible to be sure that you'll find something. It's a bit like trying to predict earthquakes …So, we might find what we're looking for with a few months of searching, or it might be that the solution isn't found for another century."

Is that it, then? Well …no. That's just 1 to 100 covered. Go up an order of magnitude to 1000, and there are still plenty of numbers to solve: 114, 165, 390, 579, 627, 633, 732, 906, 921 and 975 are all awaiting a solution to the sum of three cubes.

Got any ideas?

We'll leave that question for the reader …

Chika's Test for Divisibility by 7

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

At the primary and upper primary levels, students encounter and easily master the tests for divisibility by 2, 5, 3, 9, 4, 6, 8 and 11. They do not always study why the tests work, but they are easy to execute and students do not forget them easily.

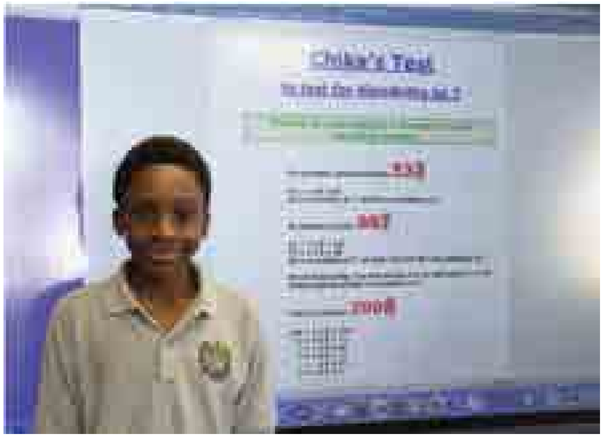
A few observant students wonder at the absence of the number 7 from this list. Occasionally, a keen student may discover such a test for himself or herself. Of course, it is a thrilling experience when this happens, for the students as well as the teachers. Such a test was discovered by Chika Ofili, a Nigerian student in Westminster Under School, London, UK. This article documents the instance.

Chika's test for divisibility by 7. The test is easy enough to perform. We describe it here in the words of his math teacher, Miss Mary Ellis [1]:

> *In a bored moment, Chika had turned his mind to the problem and this is what he came up with. He realised that if you take the last digit of any whole number, multiply it by 5 and then add this to the remaining part of the number, you will get a new number. And it turns out that if this new number is divisible by 7, then the original number is divisible by 7. What an easy test!*

She adds: "The opposite is also true in that if you don't end up with a multiple of 7, then the original number is not divisible by 7." The picture below shows Chika in his classroom demonstrating the test.

Keywords: Divisibility, number, test for divisibility



> **I love Mathematics** Yesterday at 10:15 AM
>
> 12-year-old Nigerian boy based in the UK, Chika Ofili, has been presented with a Special Recognition Award for making a new discovery in Mathematics.
> The little Mathematician just discovered a new formula for divisibility by 7 in Maths.

Illustrative examples. We consider a few examples to illustrate how this works.

- Take the number 532. Its last digit is 2, so the operation prescribed is:

  $$532 \longmapsto 53 + (5 \times 2) = 63.$$

  Note that both 63 and 532 are multiples of 7.

- Take the number 973. Its last digit is 3, so the operation prescribed is:

  $$973 \longmapsto 97 + (5 \times 3) = 112.$$

  We can repeat the same operation with the number 112. Its last digit is 2, so the operation prescribed is:

  $$112 \longmapsto 11 + (5 \times 2) = 21.$$

  Note that both 21 and 973 are multiples of 7.

- Take the number 873. Its last digit is 3, so the operation prescribed is:

  $$873 \longmapsto 87 + (5 \times 3) = 102.$$

  We can repeat the same operation with the number 102. Its last digit is 2, so the operation prescribed is:

  $$102 \longmapsto 10 + (5 \times 2) = 20.$$

  Note that both 20 and 873 are *non-multiples* of 7.

Justification. We now justify why this procedure works. Let N be any number (i.e., a positive integer). Let b be its units digit, and let a denote the remaining part of the number (i.e., the number obtained by deleting the units digit). Then, clearly:

$$N = 10a + b.$$

The procedure replaces N by the number $n = a + 5b$. So we have the following statement to prove:

$10a + b$ is divisible by 7 if and only if $a + 5b$ is divisible by 7.

It may not be immediately obvious how this is to be shown. But the trick is to find a combination of the two expressions, $10a + b$ and $a + 5b$, which is visibly a multiple of 7. Noting that $10 - 3 = 7$, which is a multiple of 7, we may try to subtract 3 times the second expression from the first one, and this immediately works:

$$\begin{aligned}(10a + b) - 3(a + 5b) &= 7a - 14b\\ &= 7(a - 2b).\end{aligned}$$

This means that $(10a + b) - 3(a + 5b)$ is a multiple of 7. That is, $N - 3n$ is a multiple of 7, say $N - 3n = 7r$. Our task is now almost over.

- Suppose that n is a multiple of 7, say $n = 7s$. Then we have:

  $$N = 3n + 7r = 21s + 7r = 7(3s + r),$$

  showing that N is a multiple of 7.

- Suppose that N is a multiple of 7, say $N = 7k$. Then we have:

  $$3n = N - 7r = 7k - 7r = 7(k - r),$$

  showing that $3n$ is a multiple of 7 and therefore that n itself is a multiple of 7 (this works because 3 and 7 are coprime and therefore do not interfere with each other's divisibility).

It only remains to note that the procedure can be iteratively carried forward. At each step, the number of digits decreases by 1, so we soon obtain a number for which divisibility by 7 can be checked mentally. So the procedure is very efficient in its operation.

A final note. There are tests of this kind for many different divisors. Indeed, one can find such a test for any odd divisor which is not a multiple of 5. We leave the proof of this to the reader.

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The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

CAPTURED MATHEMATICS

A Natural Protractor

Mathematical relevance: The plant in the picture is a species of palm tree. Its fan-shaped leaves are symmetrically placed about the central axis. Interestingly, the arrangement of leaves from one end to the other reminds us of a *Protractor* useful for measuring angles.

Photo & ideation:
Kumar Gandharv Mishra

Trials with Triangles

VINAY NAIR

Outline: The article is about some sessions that I had with students of classes 5-8 during the course of some workshops. Most of the students had never 'seen' mathematics outside of the school curriculum. Our intention was to make them think mathematically and at the same time, to see how different students look at a problem and come up with solutions.

Problem

We gave the following problem to the students: use two copies of a scalene right-angled triangle and make shapes by joining them together and without any folding, cutting or overlap of the triangles. The triangles had to be joined such that edges join each other in full.

We split the students into groups and asked them to explore the problem and then to trace whatever shapes they obtained in their notebooks and name some real life objects similar to the shapes they had drawn. They took up the task with enthusiasm. Not even a single student seemed uninterested. We had to repeat the instructions for quite a few students, despite having written the problem on the board and explained it to them through demonstrations.

After 15-20 minutes, we asked them to show us their drawings. Most of them could draw 3-4 shapes. One shape that many of them missed was the *parallelogram* (Figure 6) which was not made by joining the shortest sides. After we had drawn all the possible shapes on the board, we asked the students to make the shapes that they had missed. We noticed that some of them (including teachers) struggled with shapes when they were drawn in an orientation differing from the 'usual' one. For example, we typically draw a parallelogram in such a way that two opposite sides are 'horizontal' (i.e., parallel to the ground). When we showed a slanted parallelogram, many found it difficult to identify the shape. We see from this how important it is for a teacher to develop flexibility in visualisation of shapes.

Keywords: Triangles, exploration, higher order thinking, perimeter, area

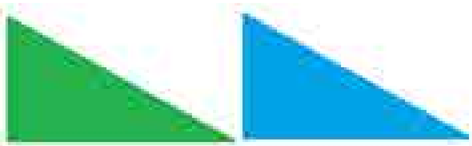

Figure 0

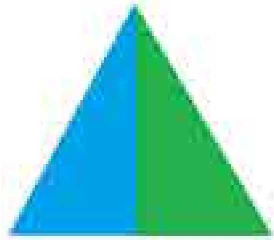

Figure 1: Mountain, birthday cap, nose, pizza slice, paper rocket

Figure 2: Envelope, cell phone, board, table

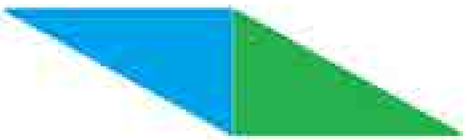

Figure 3: Sailor's compass, thunderbolt, kaju katli, diamond

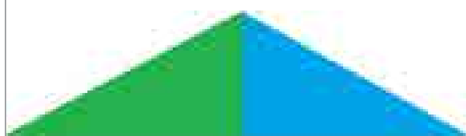

Figure 4: Ray fish, sleeping mountain, glider, roof of a house

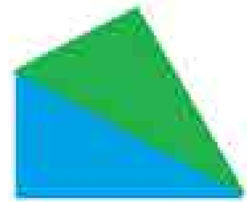

Figure 5: Kite, aerial view of a ship, rocket, ice crystal falling down (hailstones)

Figure 6: Buttons outside an elevator, bogie of a train

When we asked how we could be sure that there are just six possible shapes that can be made using the two scalene right-angled triangles, not many could argue the matter convincingly. But some students were quite clear in their thinking: "There are three sides to each triangle and two congruent sides can be joined in two ways – the second way is by flipping one shape. Thus, three times two gives six ways of constructing shapes." The next question was to check if they would get the same answer if the two congruent right-angled triangles were isosceles. That made them reframe their reasoning in a better way. I leave it to the reader to guess what that reasoning could be.

The next question was this. Imagine these shapes to be racing tracks; you must choose one of them. There are a few ice creams kept for those who finish running around the perimeter of the shape of their choice. Assuming that you are keen to have the ice cream, which track would you choose?

Except for one fifth-grade student who came up with the answer almost instantly, no one else could arrive at the correct reasoning. Some of the common answers given were:

1. As we have made all the shapes using the same two triangles, it should not matter which track you choose.

2. Choose any of the two triangles because the triangles have three sides and all the other four are quadrilaterals.
3. Choose the triangle that is smaller in height.
4. Choose a triangle because then you have to make only three sharp turns (as compared to four sharp turns in the case of a quadrilateral; the sharp turns serve to slow us down).

One boy even said that he would choose a triangle because there is a slope (the hypotenuse) and he can run faster down that slope. I had not expected that someone would think in this way and I just had to remind myself that there are so many ways in which a child thinks and how important it is for a teacher to be alert to all the assumptions that a student can make.

One team just took a scaled ruler, measured the perimeters, and got the answer. That is when I realised that I had never told them that they were not allowed to find the answer by measuring the perimeter. That was really smart of them. At the same time, why didn't others think of it? Did they assume that they weren't supposed to measure? Or did they not think of it at all?

The boy who got the answer almost instantly said (in his own words), "Sir, it is obvious that we need to find the shape with the least perimeter. We will get the shape by joining the longest sides of the triangles." I loved it when he used the word 'obvious.'

For the remaining students, we tried figuring out a way to get the answer. So we took some side lengths a, b, c where $a < b < c$ and tried to compare shapes based on these variables. This convinced the students as to which shapes had the least perimeter.

Outcomes of the session

1. Students observed for themselves that shapes with the same area can have different perimeters. I feel that this learning will stay for a long time as they learnt it on their own, by making mistakes.
2. We pushed the students to give reasons for their claims. They could also listen to others' views and think about them.
3. They started looking at real life objects in a different way. For example, the fifth graders did not know what a right angle was. I told them that it is like an L-shape and showed them a few such shapes. In less than a minute, they identified at least 20 objects in the room that had a right angle. In their subconscious minds, they would have realised that there are numerous real-life situations when we make use of right angles. They also saw geometric shapes in other real-life objects.
4. They were active learners for 75 minutes because they were doing something with their hands, having a dialogue with group members and with the faculty. A classroom where there is a monologue is not interesting for anyone.
5. We did not start by telling the students what they were going to learn in the session. Many a time, if a student has already studied the topic that is going to be taught in class (perhaps with a tuition teacher), they are not too keen to learn in the class. Some of these students also end up getting distracted in the class. As this was new and different from the usual approach followed, they listened. It is important for us teachers to keep this in mind and keep coming up with new and innovative ways of teaching even routine stuff.

Shapes from the eyes of children

Acknowledgements

The motivation to extend the question to figuring out the shortest perimeter was from this video: ***Year 4 Singapore math model lesson: Measuring Area - Maths No Problem***

https://www.youtube.com/watch?v=67Bd_UVsfTU&t=33s

VINAY NAIR is a Math educator and works mainly with students and teachers of middle and high school who are passionate about Math. He is a part of a few educational initiatives viz., Raising a Mathematician Foundation, Vichar Vatika, School of Vedic Maths. He is passionate about the education of gifted students and the history of Indian Mathematics. He can be contacted at nairvinayr@gmail.com

Hara's Triangle and Triple

HARAGOPAL R

Even though the world of numbers and the world of geometry appear to be separate, there are many surprising connections between them. In Euclid's *Elements*, one finds many pretty connections mentioned that connect these two worlds.

A well-known connection is this: if a, b, c are three positive numbers, then a triangle with sides a, b, c can be constructed if and only if the sum of any two of the numbers is greater than the third number, i.e., if and only if $a + b > c$, $b + c > a$ and $c + a > b$.

Another well-known connection: when the sum of the squares of two of the numbers equals the square of the third number, then the triangle is right-angled. Thus, if $a^2 + b^2 = c^2$ for a triangle with sides a, b, c, then the angle opposite side c is a right angle.

While thinking about this, I wondered about the problem of finding solutions in positive integers to the equation $a^n + b^n = c^n$ where n is a negative integer. For example:

- If $n = -1$, then the question becomes: for what positive integers a, b, c is it true that $1/a + 1/b = 1/c$?
- If $n = -2$, then the question becomes: for what positive integers a, b, c is it true that $1/a^2 + 1/b^2 = 1/c^2$?

And so on.

Now let us look more closely at the above two cases.

Keywords: Numbers, geometry, triangle inequality, sum of squares, sum of powers

The case $n = -1$. Here the equation is $1/a + 1/b = 1/c$. Without much explanation, I am just giving solutions to the above equation because I want to focus on the case $n = -2$ as it gives rise to some unexpected surprises in geometry. Here are some solutions for $n = -1$:

$$(a, b, c) = (2, 2, 1),\ (3, 6, 2),\ (4, 12, 3),\ (5, 20, 4),\ (6, 30, 5),\ (7, 42, 6),\ (8, 56, 7),\ \ldots$$

The case $n = -2$. For $n = -2$ the equation can be written as $1/a^2 + 1/b^2 = 1/c^2$.

Before searching for the solutions, let me introduce the connection with geometry.

We know that in any triangle, if the sum of the squares of two sides is equal to the square of the third side, then the triangle is right-angled.

Instead of only considering relationships between the sides, what if we bring in other elements of the triangle, such as its altitudes, its medians and so on?

Imagine a triangle in which *the sum of the squares of two of its altitudes is equal to the square of the third altitude*. Specifically, in a triangle with sides a, b, c, let h_a, h_b, h_c denote the three altitudes (with h_a opposite side a and so on). We now ask: *For what positive integers a, b, c is it true that*

$$h_a^2 + h_b^2 = h_c^2\,?$$

To see the significance of this question, consider the area Δ of triangle ABC:

$$\Delta = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c.$$

From these relationships, we see that

$$h_a = \frac{2\Delta}{a}, \quad h_b = \frac{2\Delta}{b}, \quad h_c = \frac{2\Delta}{c}.$$

It follows from this that if

$$h_a^2 + h_b^2 = h_c^2,$$

then

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2},$$

and conversely. This means that if we are able to construct a triangle whose sides are a, b, c such that $1/a^2 + 1/b^2 = 1/c^2$, then in that triangle the sum of the squares of the altitudes on sides a, b is equal to the square of the altitude on side c. And the converse is also true.

If such a triangle exists, I call it 'Hara's Triangle' and I call such a triple (a, b, c) 'Hara's Triple'.

Now let us look for positive integral solutions of the equation

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}. \tag{1}$$

This reminds us of a Pythagorean triple (x, y, z), i.e., which satisfies the relation

$$x^2 + y^2 = z^2.$$

A general formula which yields such triples is

$$x = 2m, \quad y = m^2 - 1, \quad z = m^2 + 1. \tag{2}$$

So the triple (a, b, c) will satisfy (1) if we can ensure that

$$a : b : c = \frac{1}{2m} : \frac{1}{m^2 - 1} : \frac{1}{m^2 + 1}. \tag{3}$$

A simple way to ensure this is to multiply through by $2m \cdot (m^2-1) \cdot (m^2+1)$ and take

$$a = (m^2-1)(m^2+1), \quad b = 2m(m^2+1), \quad c = 2m(m^2-1). \tag{4}$$

We see that for any integer $m > 1$, the integers a, b, c defined by $a = (m^2-1)(m^2+1)$, $b = 2m(m^2+1)$, $c = 2m(m^2-1)$ satisfy the equation

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}.$$

This means that for any $m > 1$, the triangle whose sides are $2m(m^2+1)$, $(m^2-1)(m^2+1)$ and $2m(m^2-1)$ has the property that the sum of the squares of two altitudes is equal to the square of the third altitude.

Here is a list of some Hara's Triples as generated by the above formula:

m	$a = (m^2-1)(m^2+1)$	$b = 2m(m^2+1)$	$c = 2m(m^2-1)$
2	15	20	12
3	80	60	48
4	255	136	120
5	624	260	240
6	1295	444	420
7	2400	700	672
8	4095	1040	1008
9	6560	1476	1440
10	9999	2020	1980

If we divide out the gcd of a, b, c from a, b, c (thereby obtaining a triangle similar to the original one, so that it has the same geometrical properties), we obtain the following triples a', b', c':

m	a'	b'	c'
2	15	20	12
3	20	15	12
4	255	136	120
5	156	65	60
6	1295	444	420
7	600	175	168
8	4095	1040	1008
9	1640	369	360
10	9999	2020	1980

Closing remark. The above exploration raises many other related questions in our minds. For example, instead of altitudes, we could ask what happens if the same condition is applied to medians or angular bisectors. And so on.

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Low Floor High Ceiling Tasks

Summing V

The Thinking Skills Pullout continues to be our favorite as it packs a lot of food for thought. It has already generated a series of two articles including a LFHC (Dotted Squares) and a TearOut. Here is another LFHC (which we have tried with a group of children from Class 3-4 in Pokhrama, Bihar, and with government school teachers from Telengana).

Make a V with 5 circles as in Figure 1. Pick 5 consecutive numbers (e.g. 13, 14, 15, 16, 17).

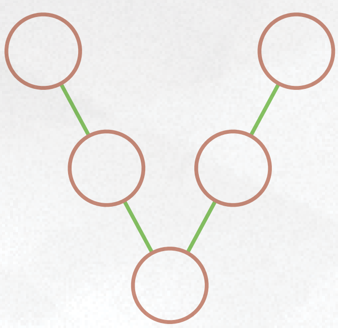


Figure 1

1. Arrange them on this so that the two arms of the V have the same total.
   a. Which number is at the centre? What is the total of each arm?
   b. Keeping the centre fixed, can you rearrange the remaining numbers in another way to get the same total for both arms? Did the total change?
   c. How many arrangements are possible?

Teacher Note: The 3rd number of the set can be a natural choice for the number at the centre. The remaining 4 numbers can be split into 2 groups with the same totals since the sum of numbers which are equidistant from the centre is constant (for example, with 13 at the centre, we get 11 + 15 = 12 + 14). This can happen in 2 ways. Since the left and the right arms of the V can be switched, the number of ways doubles to 4. Within each group on each arm, there are 2 choices for the number at the end. So, with the same number at the centre, there are 8 ways of choosing the numbers for the arms. For the set {11, 12, 13, 14, 15}, the eight possibilities are:

11 – 15 – 13 – 12 – 14	15 – 11 – 13 – 12 – 14	11 – 15 – 13 – 14 – 12	15 – 11 – 13 – 14 – 12
14 – 12 – 13 – 15 – 11	14 – 12 – 13 – 11 – 15	12 – 14 – 13 – 15 – 11	12 – 14 – 13 – 11 – 15

2. Now find another arrangement with the same 5 numbers but with a different number at the centre.
   a. Which number is at the centre now? Did the total change? By how much?

b. Can any other number be at the centre? What is the total for that?

c. Are there some numbers which cannot be at the centre? Why?

Teacher Note: Out of the 5 consecutive numbers, the first, the third and the fifth can be at the centre, e.g. for 11, 12, 13, 14, 15, the number at the centre can be 11, 13 or 15. The totals change with the centre and are consecutive numbers as well.

The 2^{nd} and the 4^{th} number cannot be in the centre. For any 5 consecutive numbers, this can be checked by trying out all possible combinations. But something more is needed to generalize for any such set. That can be tackled with parity. If we put, say, the 2^{nd} number at the centre, then out of the 4 remaining numbers, 3 are odd and 1 is even (e.g. 11, 13, 15 odd and 14 even) or 3 are even and 1 is odd (e.g. for 8, 9, 10, 11, 12, if 9 is at the centre, 8, 10, 12 are even and 9 is odd). So, if these 4 numbers are split in 2 groups, then the total for one group is odd and for the other is even. So, the two group totals cannot be the same. Therefore, the 2^{nd} number cannot be at the centre. Similarly, we can show that the 4^{th} number also cannot be at the centre.

3. Try the above with other sets of five consecutive numbers.
   a. Record your findings in this table.
   b. Do you see any patterns? What are they?
   c. If the total is 72, can you find out which five numbers are chosen? Which number is at the centre?
   d. Repeat the same if the total is 17. What if the total is 43?

Teacher Note: We observe the following:

i. The totals are consecutive numbers

ii. The middle total i.e. the total when the 3^{rd} number is at the centre is a multiple of 3

iii. In fact, it is thrice the 3^{rd} number

So, the totals are 3×3^{rd} number – 1, 3×3^{rd} number or 3×3^{rd} number + 1

4. Following the pattern:
   a. Suppose the totals are 12, can you predict the 5 consecutive numbers, and which one is at the centre?
   b. Can you do the same if the total is 23? Or 16?
   c. What can we generalize?

Teacher Note: $12 = 3 \times 4$, so, 4 is the number at the centre and is the 3^{rd} of the 5 numbers. Therefore the 5 numbers are 2, 3, 4, 5 and 6.

$23 = 3 \times 8 - 1$, so, 8 is the 3^{rd} of the 5 numbers. Therefore, the numbers are 6, 7, 8, 9 and 10 with 6 at the centre.

Similarly, $16 = 3 \times 5 + 1$, so, 5 is the 3^{rd} of the 5 numbers which are 3, 4, 5, 6 and 7 with 7 at the centre.

If we represent the 5 numbers as $n - 2$, $n - 1$, n, $n + 1$, $n + 2$ then we get the following totals which explains the above. Note, similar deduction is possible with n, $n + 1$, $n + 2$, $n + 3$, $n + 4$ as well.

Centre	Total
$n - 2$	$(n - 2) + (n - 1) + (n + 2) = 3n - 1$
n	$n + (n - 1) + (n + 1) = 3n$
$n + 2$	$(n + 2) + (n - 2) + (n + 1) = 3n + 1$

Set of numbers	Number at the centre	**Total**	Number at the centre	**Total**	Number at the centre	**Total**
9, 10, 11 12, 13	9	**32**	11	**33**	13	**34**

5. Exploring other kinds of numbers:
   a. How would this change if you take 5 consecutive even numbers?
   b. Or 5 consecutive odd numbers?
   c. Can you generalize this further?

Teacher Note: For 5 consecutive even numbers, we can consider $2n - 4, 2n - 2, 2n, 2n + 2, 2n + 4$ to get the following. For odd, we can consider $2n - 3, 2n - 1, 2n + 1, 2n + 3, 2n + 5$.

Even		Odd	
Centre	Total	Centre	Total
$2n - 4$	$6n - 2$	$2n - 3$	$6n + 1$
$2n$	$6n$	$2n + 1$	$6n + 3$
$2n + 4$	$6n + 2$	$2n + 5$	$6n + 5$

This can be generalized to 5 consecutive numbers in any arithmetic progression i.e. any 5 numbers of the form $n - 2k, n - k, n, n + k, n + 2k$ for any $n, k = 1, 2, 3, \ldots$. We leave the readers to find out what the totals are going to be.

6. Explore similar possibilities for 7 consecutive numbers in a V with 7 circles as in Figure 2 with the same total for each arm.

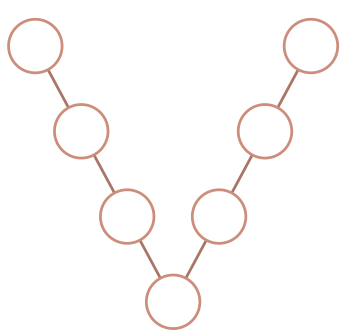


Figure 2

   a. Try with a set starting with an odd number.
   b. Which numbers can be at the centre? Why?
   c. Repeat for a set starting with an even number. Did the pattern change? How?

Teacher Note: The middle most number i.e. the 4^{th} number remains a natural choice. It turns out that the smallest or the biggest numbers can't be at the centre. But the 2^{nd}, 4^{th} and 6^{th} numbers can be. For example, if the numbers are 1, 2, 3, 4, 5, 6, 7, then 2, 4 and 6 can be at the centre. This set has 4 odd numbers and 3 even numbers. Now if an odd number is at the centre, then the remaining 3 odd and 3 even numbers add up to an odd number which can't be halved. So, the centre must be an even number. Similarly, if the numbers are 2, 3, 4, 5, 6, 7, 8, then there are 4 even and 3 odd numbers. So, an odd number can be at the centre and the remaining 4 even and 2 odd numbers can be put in two groups with equal sums. In either case, for 7 circles, the centre can be the 2^{nd}, 4^{th} or the 6^{th} number.

The possible totals are again 3 consecutive numbers with the middle one being 4 times the 4^{th} of the 7 numbers, e.g. totals for 1, 2, 3, 4, 5, 6, 7 are 15, 16 and 17 with 2, 4 and 6 at the centre respectively. Algebraically speaking, if we have $n - 3, n - 2, n - 1, n, n + 1, n + 2, n + 3$ then the possible centres and corresponding totals are:

Centre	Total
$n - 2$	$(n - 2) + (n - 3) + (n + 3) + (n + 1) = 4n - 1$
n	$n + (n - 3) + (n + 1) + (n + 2) = 4n$
$n + 2$	$(n + 2) + (n - 3) + (n + 3) + (n - 1) = 4n + 1$

So, if we take 7 consecutive even numbers, then the totals would be 3 consecutive even numbers with the middle one being 4 times the 4^{th} even number. We encourage the reader to explore the totals for 7 consecutive odd numbers.

7. What happens for 9 consecutive numbers in a V with 9 circles with same total for each arm?
   a. Which numbers can be at the centre?
   b. What is the total for each centre?
   c. For each centre, how many possible arrangements are there?

Teacher Note: With 9 circles, we get similar pattern as for 5 i.e. 1^{st}, 3^{rd}, 5^{th}, 7^{th} and 9^{th} numbers can be at the centre. The totals turn out to be 5 consecutive numbers with the middle one being 5 times the 5^{th} number.

For each centre, we now get more possible combinations for filling the V. If the centre is the 1^{st}, 5^{th} or 9^{th} number then the remaining 8 numbers form 4 pairs with equal sums. So, there are (4/2) × 2 choices for distributing these 8 numbers in the left and the right groups. Within each group, the numbers can be arranged in 4! ways. So, for each of these centres, there are (4/2) × 2 × $(4!)^2$ ways to fill the V! If the centre is the 3rd or 7th number, then the remaining 8 numbers form 4 pairs, 2 with a smaller sum and 2 with a larger one. So, there are 2 × 2 × 2 = 8 choices for the left and right groups. So, combining the arrangements within each group there are 8 × $(4!)^2$ ways to fill the V.

8. Generalize for a V with $2m + 1$ circles and $2m + 1$ consecutive numbers with same total for each arm.
   a. If m is even, what do you get for the answers to questions 6 *a, b* in terms of $m = 2k$?
   b. If m is odd, what do you get in terms of $m = 2k + 1$?

Teacher Note: In general, the middle most number i.e. the $(m + 1)^{th}$ number can always be at the centre. And every alternate number from the $(m + 1)^{th}$ number can be at the centre. So, if m is even, then the 1st, 3rd, … $(m + 1)^{th}$, … $(2m - 1)^{th}$ and $(2m + 1)^{th}$ numbers can be at the centre i.e. $m + 1$ choices. If m is odd, then 2^{nd}, 4^{th}, … $(m + 1)^{th}$, … $(2m - 2)^{th}$ and $(2m)^{th}$ numbers can be at the centre i.e. m choices. Note that it is $2k + 1$ in both cases.

Let the $(2m + 1)$ consecutive numbers be $n - m$, $n - m + 1$, …, $n - 1$, n, $n + 1$, … $n + m$. So, the number in the centre can be

- $n - m$, $n - m + 2$, …, $n - 2$, n, $n + 2$, …, $n + m$ for even m
- $n - m + 1$, $n - m + 2$, …, $n - 2$, n, $n + 2$, …, $n + m - 1$ for odd m

The total with $(m + 1)^{th}$ number i.e. n at the centre is $(m + 1) \times n$. Let us see how this total changes with the number at the centre. The sum of all the $2m + 1$ consecutive numbers $n - m$, $n - m + 1$, …, $n - 1$, n, $n + 1$, … $n + m$ is $(2m + 1) \times n$. Now, if one of the numbers, say x, is at the centre, then the sum of the remaining $2m$ numbers is $(2m + 1)n - x$. This total gets split equally along each arm of the V. So, each arm gets $½[(2m + 1)n - x]$ and x at the centre. So, the total for each arm is $½[(2m + 1)n + x]$. The following table lists the centres with the corresponding totals. So, the $2k + 1$ possible totals are also consecutive numbers viz. $(m + 1)n - k$, … $(m + 1)n - 1$, $(m + 1)n$, … $(m + 1)n + k$ for both even and odd m.

Centre x	Total $½[(2m + 1)n + x]$
⋮	⋮
$n - 4$	$(m + 1)n - 2$
$n - 2$	$(m + 1)n - 1$
n	$(m + 1)n$
$n + 2$	$(m + 1)n + 1$
$n + 4$	$(m + 1)n + 2$
⋮	⋮

We found this to be an interesting exploration to demonstrate the power of algebra in terms of finding the entire set of 5 numbers and which of them is at the centre just from the totals. Later, by the power of algebra further patterns could be explored and proved. It starts with an exploration with adding numbers (with some constraints) but unfolds to touch the sum of consecutive integers and quite a bit of arithmetic progressions. It can be easily generalized for $2m + 1$ numbers in arithmetic progression on a V with $2m + 1$ circles.

It also involves a combinatorial challenge of how many possible Vs can be made with a given set of numbers. We encourage the brave reader to explore that further for the general case!

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] – their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

Conversations on Greatest Common Divisor

ASHOK PRASAD

Today I am going to share with you an interesting conversation between me and my cousin Apoorvi, who is a curious student of class 9. This conversation began after she saw me teaching Greatest Common Divisor to Priyanka. Priyanka lives in my neighborhood and she is a student of class four. Sometimes she visits my home for help in mathematics. One day after helping Priyanka with her homework, I started talking to Apoorvi.

Me: Do you know how to calculate Greatest Common Divisor (GCD)?

Apoorvi: Oh, come on Bhaiya! Of course, I know!! I know five methods of finding it.

Me: Five methods? Even I don't know there were five methods of finding GCD.

Apoorvi: Ok!! Let me explain them to you one by one. We can calculate GCD by first writing the given numbers in the prime factor form. For example, $60 = 2 \times 2 \times 3 \times 5$ and $24 = 2 \times 2 \times 2 \times 3$. After writing them in their prime factor form, the multiplication of common primes i.e. $2 \times 2 \times 3$ is GCD of 60 and 24.

Me: This is the first method of which I am aware. We call this the prime factorization method. Can you tell me why this method works? I mean why we could find GCD by following the steps you explain?

Apoorvi: It's simple and the reason is hidden in the name- GCD. The Greatest Common Divisor of two (or more) numbers is the greatest number that divides both (or all) of them. In the prime factorization method, each number is a unique combination of prime numbers (each one of them divides the number) and all (and only all) the common factors are multiplied to get the GCD by the prime factorization method.

Me: Fair enough!! Tell me another method.

Apoorvi: Suppose we have to find the GCD of two numbers 60 and 24. For this, we first divide 60 by 24 and find the remainder, which is 12. Again 24 is divided by the remainder 12 and we get the remainder 0. The process ends and the last divisor 12 is the GCD of 60 and 24.

Keywords: factor, prime, common factor, GCD/HCF, multiple, algorithm, reasoning why

Me: Fine. This is the long division method of finding GCD, but you have to tell me three more methods, yet.

Apoorvi: Yes! Yes! Have patience! I am going to tell you all the five methods. We can also calculate GCD by square tiles. (Drew Figure 1 and explained).

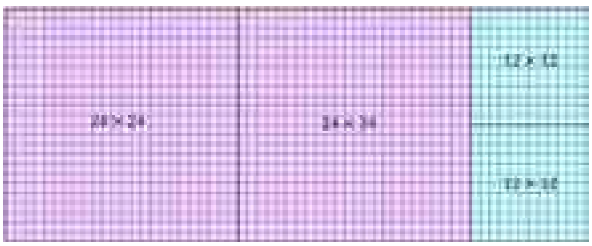


Figure 1: Calculating GCD using square tiles.

To find the GCD of 60 and 24, draw a rectangle of dimension 60 × 24. Now make as many squares of dimension 24 × 24 as possible. What remains is a rectangle of dimension 12 × 24. Again, we make as many 12 × 12 squares as possible. Is there any rectangle left? The answer is no and therefore 12 is GCD of 60 and 24.

(I was surprised that Apoorvi was perceiving this as a completely new method and was unable to see the interrelationships between methods 2 and 3. But instead of discussing the relationship between these two methods, I chose to listen to the next two methods. So, I asked her for the 4th method.)

Me: Interesting!! Tell me the next method of finding GCD.

Apoorvi: We can find GCD by strips also. (Again, she drew Figure 2).

To find the GCD of 60 and 24 we first draw a strip of length 60 and then draw as many times as possible, strips of length 24 starting from one end. We were able to fit two strips of length 24 and on checking the length of the remaining strip, we found it to be 12. Then we draw strips of length 12 on the strip of length 24 (starting again from one end, see Figure 3.). This time, there is nothing left after drawing two strips of length 12. Therefore, the GCD of 60 and 24 is 12.

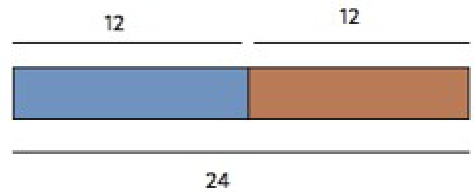


Figure 3: Calculating GCD using strips

Me: I want you to think on some other aspect of the methods shared by you but before that tell me your fifth method.

Apoorvi: I learnt about this method recently. The name of this method is Euclid division algorithm. My teacher told me that this method was first explained by Euclid in his book 'Elements'. Using this method, we can find GCD of 60 and 24 by successive /repeated division till we arrive at zero as remainder, as follows –

$60 = 2 \times 24 + 12$
The remainder 12 is between 0 and 24

$24 = 2 \times 12 + 0$ There is no remainder

Here we get remainder 0, so 12 is GCD of 24 and 60.

Me: Here are two questions for you.

1. Can you explain this method in your own words?
2. Why is 2 not the GCD of 60 and 24?

Apoorvi: (A little puzzled and in a complaining tone) Bhaiya, why do you always ask me to say the methods in words? The method says that for any pair of positive integers a, b, with (say) $0 < b \le a$, we can write $a = q \times b + r$ where $0 \le r$

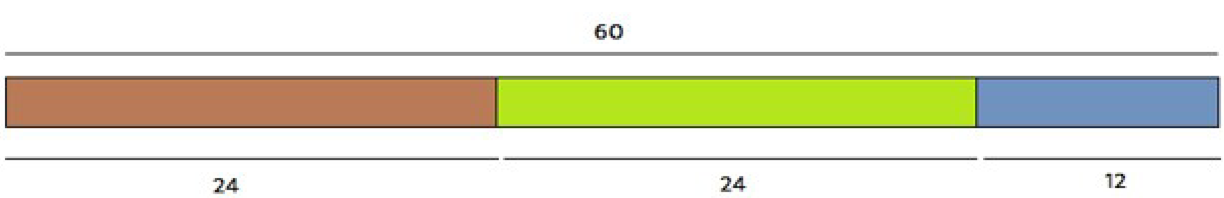


Figure 2: Calculating GCD using strips.

< b; here b is the divisor, q is the quotient, and r is the remainder; b and r are non-negative integers. If the remainder $r = 0$, then it means that a is a multiple of b, so b is itself the GCD of a and b. If $r > 0$, then nothing prevents us from dividing b by r and writing $b = r \times b_1 + r_1$ where $0 \leq r_1 < b_1$. If the remainder $r_1 > 0$, then we continue the division process until the remainder is zero, and then the last divisor is the GCD of a and b.

Me: Ok!! Apoorvi, I am happy that you know so many methods of finding the GCD. But I doubt if they are all distinct methods. So, I am giving you a few questions to think about.

1. *What is the basic argument in all these methods? Why do these methods work? I mean, how do these processes generate the GCD?*
2. *Is there any relationship among these methods?*

Apoorvi: No, No!! This is not fair! You know the answers. Instead of leaving me with your questions, tell me the answers. Otherwise, you know, your questions will keep bothering me until I find the explanations.

Me: Apoorvi, these questions are bothering me as well and right now, I do not have any well-articulated and thoughtful response to them. So, I suggest you also think about them as I will too.

Apoorvi: Ok!! I never thought about these. But I will soon tell you the answer of these questions.

This was my conversation with Apoorvi and I had forgotten about it. But the result of our conversation came after months. I was enjoying the evening, reading the novel 'A Certain Ambiguity', cup of tea at hand and slow Garhwali music in the background. Suddenly, Apoorvi entered the room, note book in hand. She seemed particularly happy. From the brightness in her eyes, it was evident that she wanted to share something exciting with me.

Apoorvi: (With excitement) Bhaiya! Bhaiya! I got it! I got it!

Me: (Little confused) I got it? What did you get, Apoorvi?

Apoorvi: Answers to your questions.

Me: What questions?

Apoorvi: Bhaiya! You remember our conversation on GCD which we had a few months back and at the end of which you left me with some questions?

Me: Conversation on GCD? I can't remember it. Give me some details so that I can recall.

Apoorvi: When I told you that there are five methods of finding GCD of any two numbers and described them to you one by one.

Me: Yes, Yes! Now I remember. I remember even the questions but sorry I did not find time to give thought to them. Well, you tell me.

Apoorvi: Ok, Bhaiya. Now I am going to tell you my findings. First, I found that they are not all distinct methods. The first method is different but the remaining four are based on the same mathematical argument. So, there is a clear relationship among the last four methods.

Me: Oh really! I am curious to know what the mathematical argument is.

Apoorvi: Before telling you the argument, I would like to tell you how I concluded what these methods are based on. I made the following table for finding the GCD which reveals the mathematical argument of methods for finding GCD.

In the above table, I took two arbitrary numbers and filled out the table. In the next line, I took the smaller of the arbitrary numbers and the remainder when the bigger number was divided by the smaller. I continued the same process until I got a remainder of 0. From the table I observed that the GCD by factorization is equal to the last divisor on successive division. This is the argument we applied in the last four methods.

Me: Can you elaborate, Apoorvi?

Apoorvi: Lets us see, one by one. In the second method we first divide 60 by 24 which means that we see how many 24s are there in 60 and find the remainder 12. Then again, we see how many 12s are there in 24 and considering the

Trial No	Numbers a and b ($a > b$)	Prime factorization of bigger number 'a'	Prime factorization of smaller number 'b'	GCD of a & b	Remainder when $a \div b$
1	60 and 24	$2 \times 2 \times 3 \times 5$	$2 \times 2 \times 2 \times 3$	12	$12 = 60 - 2(24)$
	Now taking 24 and 12	$2 \times 2 \times 2 \times 3$	$2 \times 2 \times 3$	12	$0 = 24 - 2(12)$
2	56 and 16	$2 \times 2 \times 2 \times 7$	$2 \times 2 \times 2 \times 2$	8	$8 = 56 - 3(16)$
	Now taking 16 and 8	$2 \times 2 \times 2 \times 2$	$2 \times 2 \times 2$	8	$0 = 16 - 2(8)$
3	165 and 65	$5 \times 3 \times 11$	5×13	5	$35 = 165 - 2(65)$
	Now taking 65 and 35	5×13	5×7	5	$30 = 65 - 1(35)$
	Now taking 35 and 30	5×7	$5 \times 2 \times 3$	5	$5 = 35 - 1(30)$
	Now taking 30 and 5	$5 \times 2 \times 3$	5	5	$0 = 30 - 6(5)$

GCD of 12 and 24 is the same as GCD of 24 and 60. In the rectangle method again we do the same thing that is how many 24s are there in 60. This is equivalent to dividing 60 by 24 and further division of 24 by the remainder 12. Similar process is applied the last two methods also.

Me: Brilliant! You rightly got the core of the argument. But are you sure about your argument?

Apoorvi: No Bhaiya, I am not sure because I have validated it with a few numbers only. I cannot claim that this result is valid for every single case. For this, we need to prove it. Will you help in proving this result?

Me: Ok! Before starting I want to make the declaration that here I am only talking about the set of whole numbers. And since your algebra is quite good I am going to use variables to represent the general case.

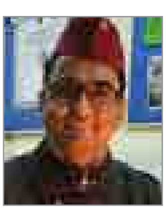
Pedagogical Notes: The table below has two columns. In the left hand column, we will use numbers to illustrate the argument. In the right hand column, we prove the result for the general case. We would advise that the general case is proved for students only when they are comfortable with the algebra. Of course, they must be made to understand that proof by example is not valid.

12 is the GCD of 60 and 24 Then: 1. 12 divides both 60 and 24 $60 = 12 \times 5$ $24 = 12 \times 2$ 2. 12 is the greatest common divisor i.e. 5 and 2 are co-prime, they have no common factors.	If, 'd' is the GCD of given two numbers a and b, then: 1. d divides both a and b $a = d \times \alpha$ $b = d \times \beta$ 2. d is the greatest common divisor i.e. α and β will be co-prime i.e. they will have no common factors.
For whole numbers, say, 12 and 17 $17 = 1 \times 12 + 5$; where $0 \le 5 < 12$ Here 5 is the remainder when 12 divides 17	In general, for whole numbers a and b $a = q \times b + r$, where $0 \le r < b$. Here r is the remainder when b divides a
$60 = 2(24) + 12$ So $12 = 60 - 2(24)$ $= (12 \times 5) - (2 \times 2 \times 12)$ $= 12(5 - (2 \times 2))$	Since $a = q \times b + r$ so $r = a - qb$ $= d\alpha - qd\beta$ $= d(\alpha - q\beta)$

So 12 divides both 60 and 24 and also 12	So d divides a and b and also r.
We have to now show that 12 is also the GCD of 24 and 12. We have seen that 12 divides both 24 and 12. Now we show that 12 is their greatest divisor.	We have to now show that d is the GCD of b and r. We have seen that d' is a divisor of both b and r. Now we show that d is their greatest divisor.
$24 = 12 \times 2$ $12 = 12 \times 1$ And 1 and 2 are co-prime So 12 is the GCD of 12 and 24.	$b = d\beta$ $r = d(\alpha - q\beta)$ Now d' will be the GCD of b and r if β and $(\alpha - q\beta)$ are co-prime
	On the contrary, let us assume β and $(\alpha - q\beta)$ are not coprime. Then we will have a number c which will divide both β and $(\alpha - q\beta)$. Symbolically $c \mid \beta$ and $c \mid (\alpha - q\beta)$.
	we can write $\alpha = \{q\beta + (\alpha - q\beta)\}$ Since c divides $q\beta$ and $(\alpha - q\beta)$ therefore c divides α.
	Thus, we can say that c divides both α and β which is a contradiction to the fact that α and β are co-prime. This implies that our assumption that number c divides both β and $(\alpha - q\beta)$ is wrong. This concludes that β and $(\alpha - q\beta)$ are co-prime. If β and $(\alpha - q\beta)$ are coprime then 'd' is the GCD of b and r too. We can continue like this as we divide successively and prove that the GCD of a and b is also the GCD of the successive remainders.

Apoorvi: Oh! Thank you Bhaiya! I got it. I never thought small concepts like GCD might have such relationships and insights.

Note from Author: The author would like to express his gratitude to Swati Sircar for encouraging him to write this article and helping him hone his writing skills.



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ClassRoom

Justification of 90° Angle Construction Procedure

ARDDHENDU SHEKHAR DASH

Geometrical construction is one of the interesting topics in geometry and most of us like constructing geometrical figures using compass & ruler. We follow certain processes to complete the construction of the geometrical figures and this process follows certain logical arguments based on properties of shapes and the relation between different geometrical shapes. However, most of our experience reflects two major challenges in the construction of geometrical figures. The first one is following certain processes without understanding the justification of the processes and the second one is not exploring different ways of constructing the same geometrical figure. Therefore, it is necessary to take care of these two major challenges during our classroom teaching.

If we will recall our curriculum, generally we start compass and ruler geometrical construction from class 6. The deductive or axiomatic reasoning behind some geometrical constructions might be difficult to grasp for children at that level so we should use verification to make the children understand the justification at that level.

In this article, we will explore two processes of constructing a right angle and their justification.

Keywords: Geometric construction, angles, justification, alternatives

Process 1 (Construction of 90° angle)

Step 1: Draw a line l and mark a point B on it.

Figure 1

Step 2: Mark another point O outside the line l.

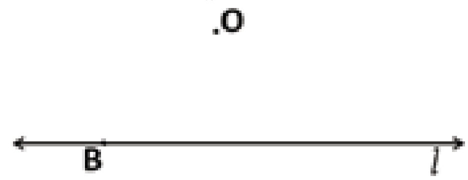

Figure 2

Step 3: Place the point of the compass at O and draw an arc that passes through B. Let it intersect the line l again at point C.

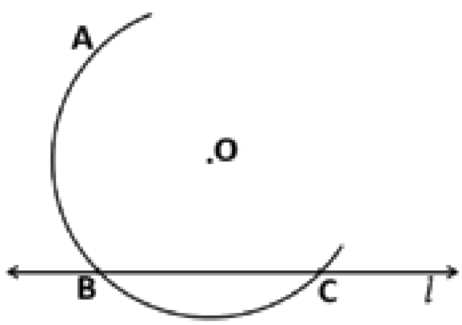

Figure 3

Step 4: Draw the line segment from point C through O and let it meet the arc at point A. Join AB.

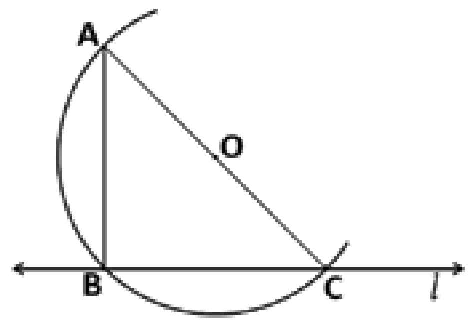

Figure 4

Note: Here we have used the word arc as part of the circumference of a circle.

So, what is the measure of the angle ABC?

The question is whether it is 90°. If so, how do we justify it?

One way of justification could be by measuring the angle using a protractor, based on the level of understanding of the children. Note that using the protractor to measure the angle may give only an approximate value. However, how will we justify it logically?

Proof 1: Given l is a line. A, B and C are three distinct points on the circle γ_1 with centre at O. AC is a diameter of the circle. We have to prove that $\measuredangle ABC$ is a right angle.

Join BO. Here OA, OB and OC are radii of the circle γ_1.

In the triangle AOB, AO = BO,

Hence, triangle AOB is an isosceles triangle.

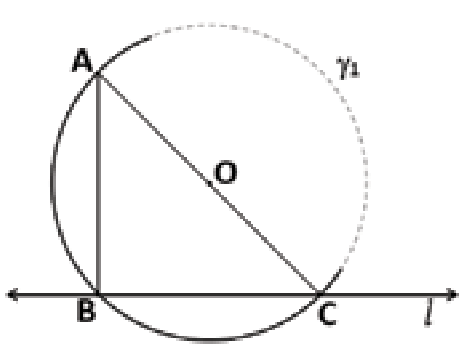

Figure 5

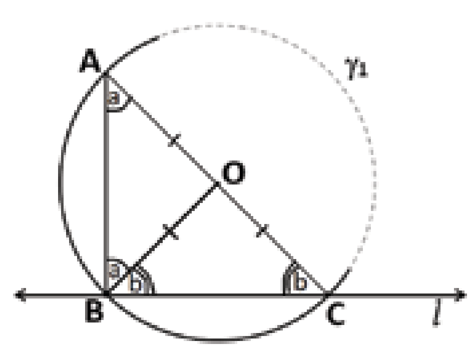

Figure 6

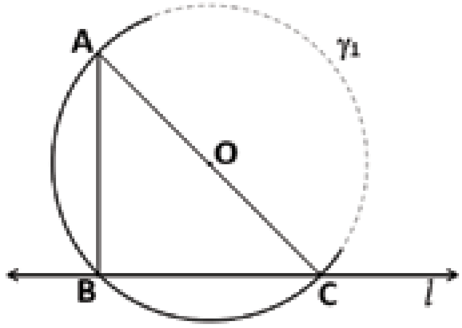

Figure 7

So, $\measuredangle OAB = \measuredangle ABO$ (angles opposite to equal sides of an isosceles triangle are equal).

Let $\measuredangle OAB = \measuredangle ABO = a$

Similarly, BOC is an isosceles triangle as BO = CO,

So, $\measuredangle OBC = \measuredangle BCO$ (angles opposite to equal sides of an isosceles triangle are equal).

Let $\measuredangle OBC = \measuredangle BCO = b$

Since ABC is a triangle, the sum of the interior angles will be equal to two right angles.

$\measuredangle CAB + \measuredangle ABC + \measuredangle BCA = 180°$

$\measuredangle CAB + (\measuredangle ABO + \measuredangle OBC) + \measuredangle BCA = 180°$ (as $\measuredangle ABC = \measuredangle ABO + \measuredangle OBC$)

$$a + a + b + b = 180°$$
$$2a + 2b = 180°$$
$$2(a + b) = 180°$$
$$a + b = 90°$$
$$\text{i.e., } \measuredangle ABC = 90°$$

So, we have proved that $\measuredangle ABC$ is a right angle.

Proof 2: Given that l is a line. A, B and C are three distinct points on the circle γ_1 with centre at O. AC is a diameter of the circle. We have to prove that $\measuredangle ABC$ is a right angle.

Here, figure 7 is rotated to Figure 8 to visualize the coordinates easily.

Let the co-ordinates of O be (0,0), r be the radius of the circle and CA be on the X-axis (as in figure 8); so the co-ordinates of A are (r, 0) and the coordinates of C are (-r, 0). B is a point on the circle of radius r and OB makes an angle θ with

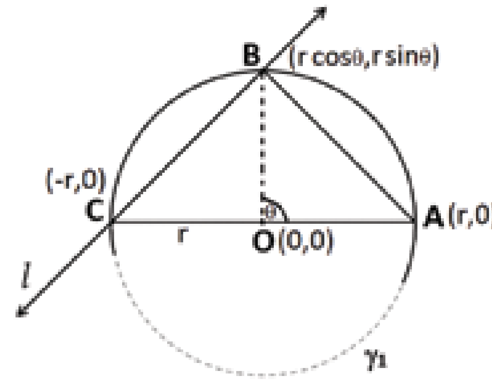

Figure 8

the positive X-axis. Using the polar coordinate system, the coordinates of point B will be ($r\cos\theta$, $r\sin\theta$). We can also find the coordinates of points A and C using the polar coordinate system by applying θ as 0° for point A and 180° for point B in the coordinates ($r\cos\theta$, $r\sin\theta$).

Now the slope of AB $= \dfrac{r\sin\theta}{r\cos\theta - r}$

The slope of BC $= \dfrac{r\sin\theta}{r\cos\theta + r}$

To prove $\measuredangle ABC$ is a right angle, one way could be to show that AB is perpendicular to BC. This is possible, if one could show that the product of slopes of both the lines is −1.

$$\begin{aligned}
&(\text{Slope of AB}) \,.\, (\text{Slope of BC}) \\
&= \frac{r\sin\theta}{r\cos\theta - r} \cdot \frac{r\sin\theta}{r\cos\theta + r} \\
&= \frac{r^2\ \sin^2\theta}{r^2\ \cos^2\theta - r^2} \\
&= \frac{r^2\ \sin^2\theta}{r^2\ (cos^2\theta - 1)} \\
&= \frac{\sin^2\theta}{\cos^2\theta - 1} \\
&= \frac{\sin^2\theta}{-\sin^2\theta} = -1
\end{aligned}$$

So, AB is perpendicular to BC. Hence $\measuredangle ABC$ is a right angle.

This construction is based on Thales's theorem which states: If A, B, and C are distinct points on a circle in which the line segment AC is a diameter, then $\measuredangle ABC$ is a right angle. Or generally we say that the angle in a semi-circle is a right angle.

Process 2 (Construction of 90° angle)
This process of constructing a right angle (90°) is common in most of our state and NCERT textbooks. This construction includes the two concepts (as in Figure 9) – construction of an angle of 60° (or multiple of 60°) and bisector of an angle. Here it is the bisector of 60°. As in Figure 9, $\measuredangle CAB = 60°$ and $\measuredangle EAC = 30°$ EA being the bisector of angle $\measuredangle DAC$.

$$\text{So, } \measuredangle EAB = \measuredangle EAC + \measuredangle CAB = 30° + 60° = 90°.$$

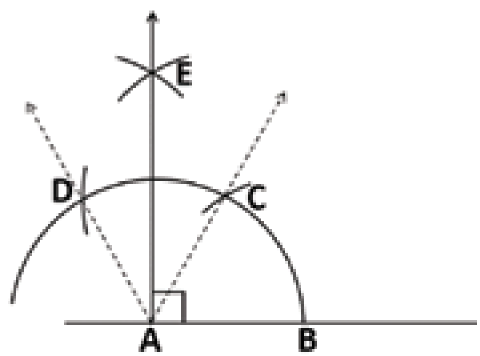

Figure 9

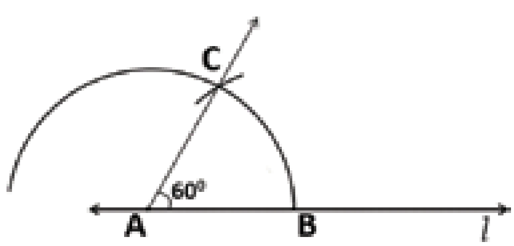

Figure 10

To justify that $\measuredangle EAB$ is 90°, we have to justify (i) the construction of 60° angle and (ii) the bisection of an angle.

Construction of angle 60°
Step 1: Draw a line *l* and mark a point A on it.

Step 2: Place the point of the compass at A and draw an arc of convenient radius that cuts the line *l* at a point B.

Step 3: Keeping the width unchanged, place the point at B and draw an arc that cuts the previous arc at C.

Step 4: Draw the ray AC.

Here angle CAB is 60°.

Justification: We have to prove that $\measuredangle CAB$ is 60°. Join CB as in figure 11. In the steps of the construction of an angle of 60°, we do not change the radius of the arc. So, the circles γ_1 and γ_2 have the same radii and the centres of both circles lie on the endpoints of AB, and AB is the radius of both the circles.

In circle γ_1, AB = AC as they are radii of the circle,

In circle γ_2, BA = BC as they are radii of the circle,

so, AB = BC = AC.

Hence, the triangle ABC is an equilateral triangle. Since the sum of the interior angles is 180° and all the angles are equal, each angle will be equal to 60°.

Construction of Angle Bisector. Given an angle CAB, suppose we wish to bisect the angle. With B as centre, draw an arc whose radius is more than half the length BC. With the same radius and with C as centre draw another arc. Let the two arcs intersect at D (as in figure 14). Draw the ray AD, from A through D. Now the ray AD bisects the angle CAB.

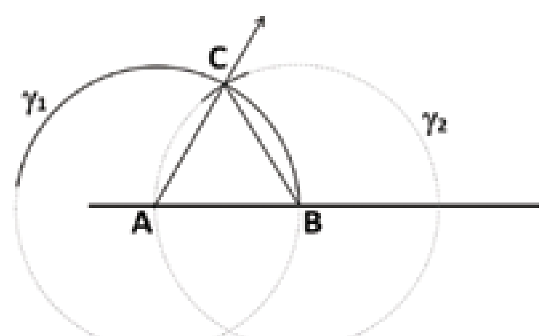

Figure 11

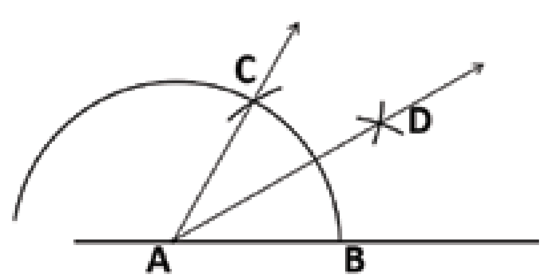

Figure 12

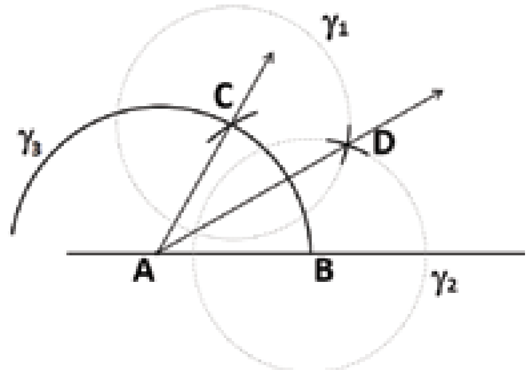

Figure 13

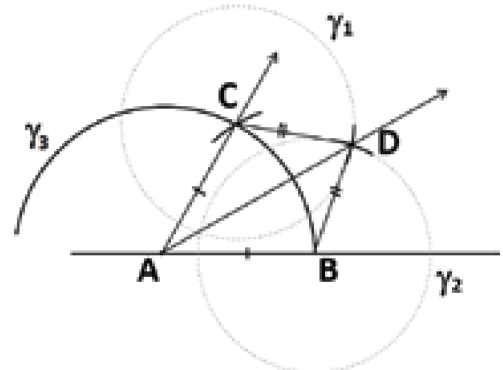

Figure 14

Justification: In figure 14, AB = AC being the radii of circle γ_3. CD = BD as we have chosen the same radius for the arcs $\gamma_2 \& \gamma_1$.

We have to prove that the ray AD bisects the angle CAB,

Now consider the triangle ACD and triangle ABD

AC = AB

CD = BD

AD is common

So, by the side-side-side property, triangle ACD is congruent to triangle ABD.

Hence $\measuredangle$CAD = $\measuredangle$DAB

So, AD is the bisector of the angle $\measuredangle$CAB.

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TWO PROBLEMS

PROBLEM 1

Someone computed the factorial of some large number (recall that if *n* is a natural number, then '*n* factorial' (written *n*!) is the product of all the numbers from 1 till *n*) and obtained the following gargantuan number:

81591528324789773■34■5611269596115894272000000000.

Unfortunately, some blobs of red paint fell on two digits right in the middle (shown as red rectangles). Can you figure out what those two digits could be?

PROBLEM 2

Look at the following beautiful relation which expresses 2019 in terms of the digits 1,2,3,…,9:

$$9 + (8 + 7) \times \left(\frac{6!}{5} - 4 - 3 - 2 - 1\right) = 2019.$$

Can you find a similar expression for 2020?

Product Numbers

VISHAK VIKRANTH

The 'balancing problem' asks us to find a pair of natural numbers m, n (with $m < n$) such that the sum of all the natural numbers from 1 till $m-1$ equals the sum of all the natural numbers from $m + 1$ till n. (We can think of this as a kind of balancing problem. The number m acts as a point of balance.) This problem was first published in the English magazine *Strand* in December 1914. It was solved by P. C. Mahalanobis by trial and error; Ramanujan came up with a general solution. The simplest solution is

$$1 + 2 + 3 + 4 + 5 = 7 + 8.$$

Here the middle term (the point of balance) is $m = 6$, and $n = 8$. The problem has infinitely many solutions.

Product balance problem

In just the same way, we formulate a *product balance problem*. That is, we seek a pair of natural numbers m, n (with $m < n$) such that the product of all the natural numbers from 1 till $m - 1$ equals the product of all the natural numbers from $m + 1$ till n.

For example, we may take $m = 7$ and $n = 10$. It may be checked that

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 = 8 \times 9 \times 10 \quad \text{(both sides are equal to 720).}$$

But trying to find more such pairs of numbers gets difficult - and this is because there are no other solutions! Indeed, (7, 10) is the only such pair.

Keywords: Product number, Strand Magazine, Mahalanobis, Ramanujan, balancing problem

Proof of claim

We may check by looking at all the cases that there exists no solution with $m < 7$. And we already know that $m = 7$, $n = 10$ is a solution.

Now suppose there exists a pair (m, n) satisfying the condition of the problem, with $m > 7$. Let p be the largest prime number less than m, so $p \geq 7$. Then the product of the natural numbers from 1 till $m - 1$ is necessarily a multiple of p.

Hence the product of the natural numbers from $m + 1$ till n must also be a multiple of p. For this to happen, there must be a multiple of p among the numbers from $m + 1$ to n. Therefore $n \geq 2p$, as the least natural number after p which is a multiple of p is $2p$.

Now it is known that for any natural number $k \geq 2$, there exists a prime number q such that $k < q < 2k$. For more on this remarkable result, which is known in number theory as *Bertrand's postulate,* please see this reference: https://en.wikipedia.org/wiki/Bertrand's_postulate.

Stronger results are known. Ramanujan showed that for any natural number $k \geq 6$, there exist *at least two prime numbers* q_1, q_2 such that $k < q_1 < q_2 < 2k$.

Applying Ramanujan's result to the present situation, we see that there exist two prime numbers q_1, q_2 such that $p < q_1 < q_2 < 2p$. Of these two primes, it might happen that $q_1 = m$. (This will be the case if m itself is a prime number. If not, we will have $q_1 > m$.) But in any case we will have $q_2 > m$. This means that the product of the natural numbers from $m + 1$ till n must be divisible by q_2.

But the product of the natural numbers from 1 to $m - 1$ cannot be divisible by q_2, as q_2 is a prime number and $q_2 \geq m$. Hence it cannot be that the product of all the natural numbers from 1 till $m - 1$ equals the product of all the natural numbers from $m + 1$ till n.

Therefore there exists no solution other than $m = 7$, $n = 10$.

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Bihar Teacher's Innovative Technique of Teaching Math Tables

https://www.hindustantimes.com/education/bihar-teacher-s-innovative-math-teaching-goes-viral-on-social-media-srk-anand-mahindra-appreciate/story-PmdNIDGBYEp6pcK6wePOxM.html

Click on the link to read how this teacher has made fans of the likes of Anand Mahindra and Shahrukh Khan. More impressively, she is a government school teacher in a tiny village Banka in Bihar. This is heart-warming evidence of the impact of the government's program to provide access to quality education through technology.

Other readers have contributed many similar tricks in the comment section. As math-o-philes, our readers would be part of the group which immediately analyses why this trick works. We invite our readers to share their thoughts on how math tricks should be taught in class, given their unmistakable magnetism. Write in to AtRiA.editor@apu.edu.in

Beginning with this issue, we present VIEWPOINT, where we re-examine familiar mathematics concepts and practices through different viewpoints. We encourage you to write in with your thoughts on the viewpoints expressed here. Send in your mail to AtRiA.editor@apu.edu.in.

A Word and an Idea

A RAMACHANDRAN

It may be a truism to say that a word can make or mar matters. New words are constantly coming into vogue while some go into disuse. I want to highlight a word that could make life a bit simpler for math teachers and learners. When I first joined an Anglo-Indian school about six decades ago there were no sharp divisions between subjects. In one craft/math class we students cut out 2D shapes from coloured paper and stuck them in scrap books, besides naming objects around us that resembled those shapes. The first shape we learnt was the square, a figure with four equal sides and similar-looking corners. The next was the 'oblong' – a four-sided figure with opposite pairs of sides equal but one set longer than the other, and with all corners appearing similar. (We did not know about right angles then.) We could find plenty of objects around of that shape – table top, door frame, window frame, book, etc. - many more than we could for the square. It was after a few years that I encountered the word 'rectangle' – introduced as a figure with four right angles, which could be a square or an oblong shape. I was seldom troubled by the existential question 'Is the square a rectangle?'

So, as a math teacher of three decades, and reflecting on my own student days, I urge all math educators at primary level to help resurrect this word 'oblong' and use it up to class 3 or 4 (to stand for a non-square rectangle). The word rectangle could be introduced at class 5 level. This should help clear the air considerably.

Keywords: Viewpoint, opinion, sharing, pedagogy, vocabulary, quadrilateral, classification

Geometrical Proof of an Application of Ptolemy's Theorem

RADHAKRISHNAMURTY PADYALA

Introduction. A recent article [1] discussed Ptolemy's theorem and applications of the theorem. The author noted that for the first application, there was also a trigonometric solution based on the identity $\sin(60° - \theta) + \sin\theta = \sin(60° + \theta)$. In this note, we present a simple and elegant geometrical proof for this theorem.

Theorem. *Let ABC be an equilateral triangle, and let P be any point on the minor arc BC of its circumcircle. Then* $PA = PB + PC$.

A geometrical proof. Figure 1 depicts the situation. On BP as base, draw an equilateral triangle EBP, with E on the same side of BP as A. Join AE.

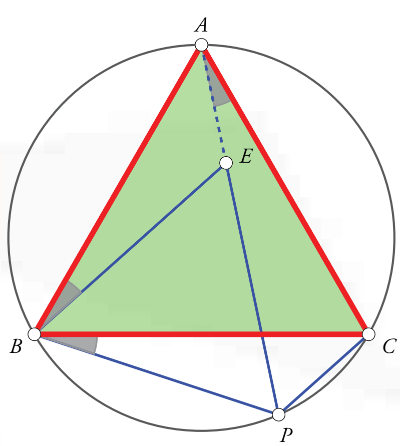

Figure 1. Construction: Triangle EBP is equilateral. Join AE.

Observe that we have shown AE using a dashed line and EP using a solid line. This is to ensure that we do not unconsciously assume that points A, E, P lie in a straight line.

Keywords: Ptolemy's theorem, equilateral triangle, circum-circle, point, relationship

Since $\measuredangle ABC = \measuredangle EBP$, both being $60°$, it follows that $\measuredangle ABE = \measuredangle CBP$ (both angles are marked).

Consider $\triangle ABE$ and $\triangle CBP$. They are congruent to each other (side-angle-side or SAS congruence), therefore $\measuredangle BAE = \measuredangle BCP$. As we also have $\measuredangle BAP = \measuredangle BCP$ ("angles in the same segment"), it follows that $\measuredangle BAE = \measuredangle BAP$, and hence that points A, E, P are collinear. Hence $PA = PE + EA$.

But $PE = PB$, since $\triangle EBP$ is equilateral, and $EA = PC$, by the triangle congruence just proved.

Hence $PA = PB + PC$. ■

Acknowledgement. I thank Mr. Arun Rajaram of AECOM India, for his support and encouragement in every possible way in my research pursuits. I also thank Prof. Shailesh Shirali, whose article was the inspiration for this article, for his help in shaping the article to the present form from a lengthy original.

References

1. Shailesh Shirali, "How to Prove it: Ptolemy's Theorem" from *At Right Angles*, Vol.5, No. 3, Nov. 2016. Pp 53-57. http://www.teachersofindia.org/en/ebook/how-prove-it-ptolemys-theorem

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PRIME NUMBER RELATIONS

1. (5 – 3) = (2 × 1)
2. (7 – 5) = (3 – 2) + 1
3. (11 – 7) × (3 – 2) = (5 – 1)
4. (7 – 5) + (3 – 2) = (13 – 11) + 1
5. (11 – 7) × [(5 – 3) – (2 – 1)] = (17 – 13)
6. (7 – 5) × [(3 – 2) + 1] = (19 – 17) + (13 – 11)
7. (11 – 7) × (5 – 3) × (2 – 1) = (17 – 13) + (23 – 19)
8. (29 – 23) + 1 = (19 – 17) + (13 – 11) + (7 – 5) + (3 – 2)
9. [(31 – 29) + (11 – 7) + (5 – 3)] × (2 – 1) = (23 – 19) + (17 – 13)
10. (37 – 31) + (19 – 17) + (7 – 5) = (29 – 23) + (13 – 11) + (3 – 2) + 1
11. (41 – 37) × [(23 – 19) – (31 – 29)] = {(17 – 13)[(11 – 7) – (5 – 3)]} × (2 – 1)
12. (37 – 31) + (43 – 41) + (13 – 11) + 1 = (29 – 23) + (19 – 17) + (7 – 5) + (3 – 2)
13. (47 – 43) + (41 – 37) + (31 – 29) + (5 – 3) = [(23 – 19) + (17 – 13) + (11 – 7)] × (2 – 1)
14. (53 – 47) + (29 – 23) + (7 – 5) = (37 – 31) + (43 – 41) + (19 – 17) + (13 – 11) + (3 – 2) + 1
15. (47 – 43) + (31 – 29) + (23 – 19) + (11 – 7) = [(59 – 53) + (41 – 37) + (17 – 13)] × [(5 – 3) – (2 – 1)]
16. [(61 – 59) + (53 – 47) + (29 – 23)] × 1 = [(43 – 41) + (37 – 31) + (19 – 17) + (13 – 11) + (7 – 5)] × (3 – 2)
17. (67 – 61) + (47 – 43) + (23 – 19) + (11 – 7) = [(59 – 53) + (41 – 37) + (17 – 13) + (31 – 29) + (5 – 3)] × (2 – 1)
18. (53 – 47) + (29 – 23) + (19 – 17) + (7 – 5) + 1 = (71 – 67) + (61 – 59) + (37 – 31) + (43 – 41) + (13 – 11) + (3 – 2)

Pretty Power Play

SHAILESH SHIRALI

A few months back, while browsing through the pages of LinkedIn (I am, unfortunately, unable to recall the URL), I came across the following set of striking equalities:

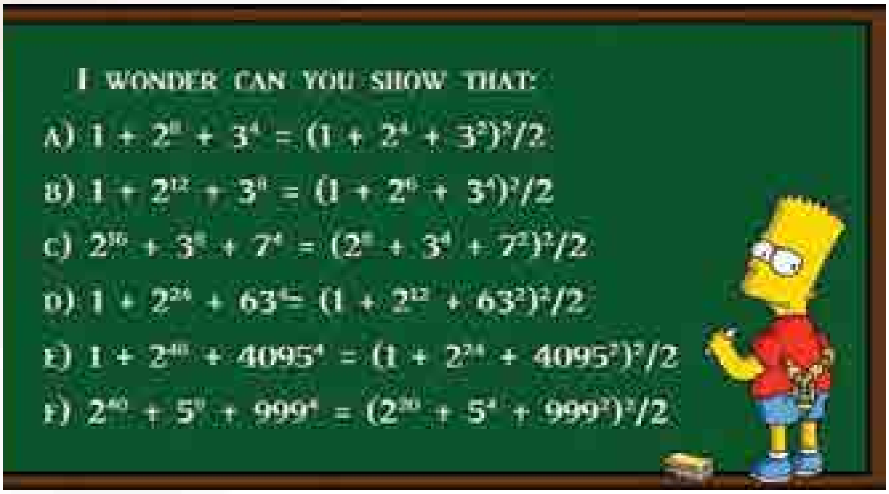

Figure 1. Unusual power equalities

A most remarkable set of equalities! What could be the general law behind them? (There clearly does seem to be some kind of general result hiding behind these separate instances.) Let us see if we can uncover the secret.

We observe that each identity is of the following form, for some positive integers a, b, c and some positive integers k, m, n:

$$a^{2k} + b^{2m} + c^{2n} = \frac{1}{2}\left(a^k + b^m + c^n\right)^2 .$$

Moreover, a, b, c are clearly connected through some relation.

Keywords: Power identity, difference of squares, quadratic

Observe further that in all the cases shown, the exponents on the left side (i.e., $2k, 2m, 2n$) are not just even numbers but are multiples of 4.

This being the case, perhaps it will be easier to spot a pattern if we rewrite the identities in a more uniform manner, so that each term on the left side is a fourth power (i.e., of the form k^4), and each term on the right side is a squared quantity (i.e., of the form k^2). Accordingly, we rewrite the six identities as shown below.

$$\begin{aligned}
1^4+4^4+3^4 &= \frac{1}{2}\left(1^2+4^2+3^2\right)^2,\\
1^4+8^4+9^4 &= \frac{1}{2}\left(1^2+8^2+9^2\right)^2,\\
16^4+9^4+7^4 &= \frac{1}{2}\left(16^2+9^2+7^2\right)^2,\\
1^4+64^4+63^4 &= \frac{1}{2}\left(1^2+64^2+63^2\right)^2,\\
1^4+4096^4+4095^4 &= \frac{1}{2}\left(1^2+4096^2+4095^2\right)^2,\\
1024^4+25^4+999^4 &= \frac{1}{2}\left(1024^2+25^2+999^2\right)^2.
\end{aligned}$$

Each identity now has the following form:

$$a^4+b^4+c^4=\frac{1}{2}\left(a^2+b^2+c^2\right)^2. \tag{1}$$

Looking carefully at the numbers involved in the different instances, we quickly notice a pattern connecting a, b, c; namely: in each case, one of the three numbers is equal to the sum of the other two numbers; please check! We have clearly made progress.

To clinch the issue, let us now approach the problem from the opposite end; let us find out what relation must exist between the quantities a, b, c in order that relation (1) reduces to an identity. Here is what we obtain:

$$\begin{aligned}
a^4+b^4+c^4=\frac{1}{2}\left(a^2+b^2+c^2\right)^2 &\iff 2\left(a^4+b^4+c^4\right)-\left(a^2+b^2+c^2\right)^2=0\\
&\iff a^4-2a^2(b^2+c^2)+b^4-2b^2c^2+c^4=0.
\end{aligned}$$

It is not obvious how to factorise the expression $a^4-2a^2(b^2+c^2)+b^4-2b^2c^2+c^4$. To this end, let us write $x=a^2$ and treat the last equality as a quadratic equation in x. The resulting equation is easy to solve, as the discriminant turns out to be a perfect square ($\Delta=4\left((b^2+c^2)^2-(b^2-c^2)^2\right)=16b^2c^2$). Here is what we get (we have skipped the in-between steps):

$$\begin{aligned}
&x^2-2x(b^2+c^2)+b^4-2b^2c^2+c^4=0,\\
&\implies x=(b-c)^2 \text{ or } x=(b+c)^2.
\end{aligned}$$

And now, using the ever-versatile difference-of-two-squares factorisation formula,

$$\begin{aligned}
&a^4-2a^2(b^2+c^2)+b^4-2b^2c^2+c^4\\
&=\left(a^2-(b-c)^2\right)\cdot\left(a^2-(b+c)^2\right)\\
&=(a-b+c)\cdot(a+b-c)\cdot(a-b-c)\cdot(a+b+c).
\end{aligned}$$

It follows that

$$\begin{aligned}
&a^4+b^4+c^4=\frac{1}{2}\left(a^2+b^2+c^2\right)^2\\
&\iff (a-b+c)\cdot(a+b-c)\cdot(a-b-c)\cdot(a+b+c)=0.
\end{aligned}$$

Well: the secret is now fully revealed! Namely, identity (1) holds if and only if any of the following conditions holds:

$$\begin{aligned} a-b+c&=0,\\ a+b-c&=0,\\ a-b-c&=0,\\ a+b+c&=0. \end{aligned} \tag{2}$$

The first three possibilities may be described compactly by stating that one of the numbers a, b, c equals the sum of the other two numbers.

The remaining possibility (in which the sum of the three numbers is 0) may create the impression of a new relation under which the identity holds, unnoticed earlier; but a closer look reveals that this is not so. For, all the exponents occurring in relation (1) are even numbers, implying that we can freely change the signs of any of the numbers a, b, c, without equality being affected. And if we change the sign of any one of the numbers, we are back in one of the first three possibilities.

Using (1) as a base, arithmetical relations of interest can be deduced; for example:

$$1^2+16^2+25^2=\frac{1}{2}\left(1+16+25\right)^2.$$

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Impossible Triangles on Dot Sheets

SWATI SIRCAR

In the March 2019 issue of AtRiA, there were several proofs given explaining why it is not possible to draw an equilateral triangle on a rectangular dot sheet. Similarly, it is not possible to draw a right isosceles triangle (or a square) on an isometric dot sheet. We provide a proof of the latter.

Right Isosceles (or Square) on Isometric Dot Sheet

The rectangular dot sheet looks the same regardless of its orientation. But that is not the case for the isometric one. Usually, we take the minimum distance between two grid points as the unit distance. So, in one orientation the grid points along any horizontal line are unit distance apart but along any vertical line they are further apart (Figure 1). If we change the orientation, then the points along any vertical line become unit distance apart (Figure 2). Without loss of generality, we can orient the isometric dot sheet (or grid) so that on the y-axis, grid points are unit distance apart (see Figure 2).

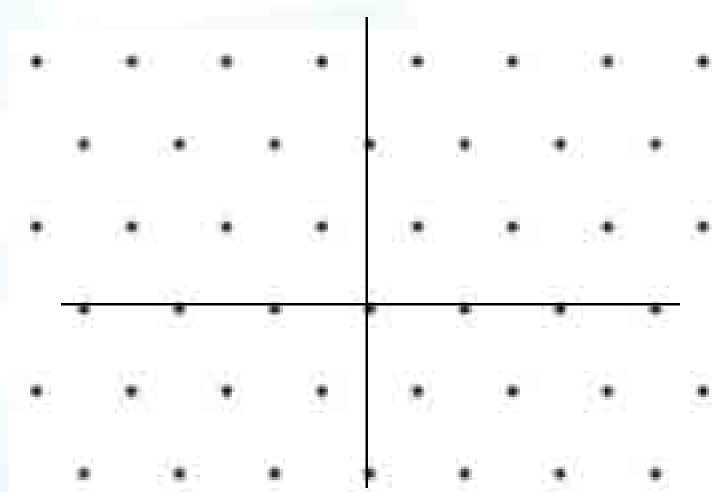

Figure 1

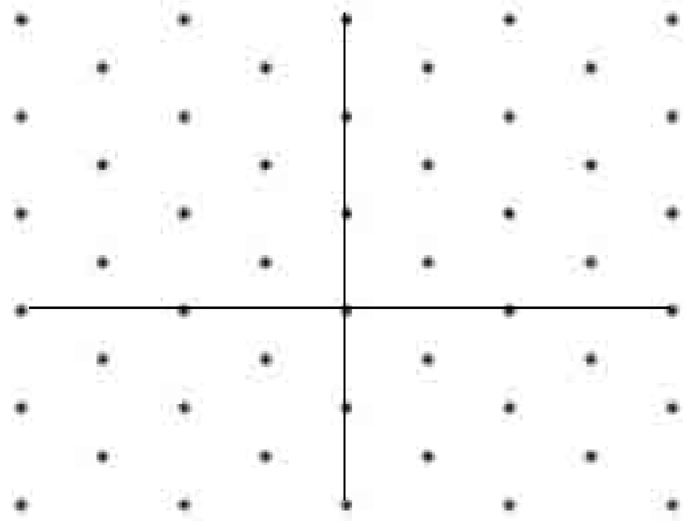

Figure 2

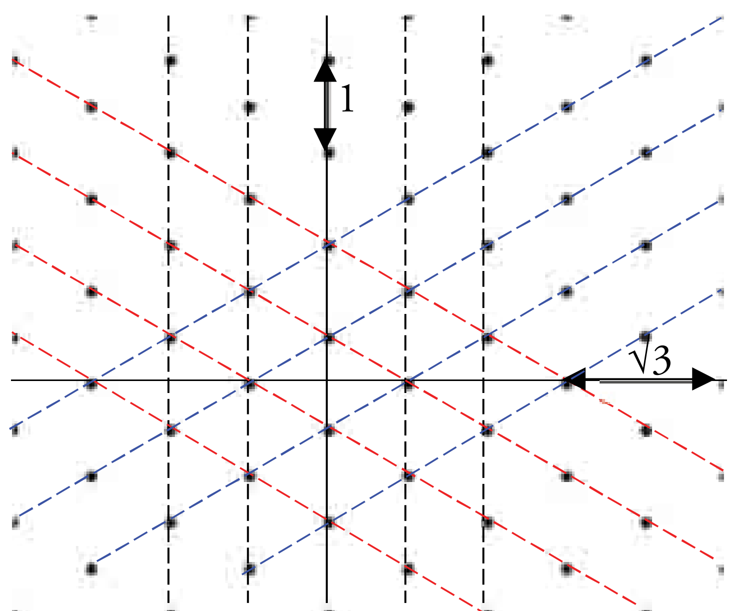

Figure 3

Note that there are 3 sets of parallel lines that we can get by joining the nearest points – the black lines parallel to the y-axis, the blue ones and the red ones (Figure 3). These three sets form equilateral triangles. So, the blue lines form 60° (and 120°) angles with the y-axis, and therefore 30° (and 150°) angles with the x-axis. Similarly, the red lines form 120° (and 60°) and 150° (and 30°) angles with the y- and the x-axis respectively. Therefore, the distance between two consecutive dots along the x-axis is twice the altitude of an equilateral triangle with unit sides i.e. $\sqrt{3}$ units.

So, as shown in Figure 4, the points along the x-axis then have x-coordinates $0, \pm\sqrt{3}, \pm 2\sqrt{3}, \pm 3\sqrt{3}, \ldots,$ i.e., their coordinates have the form $\left(m\sqrt{3}, 0\right)$ for some integer m. Similarly, points on any row with integer y-co-ordinates are of the form $\left(m\sqrt{3}, n\right)$ for some integers m and n.

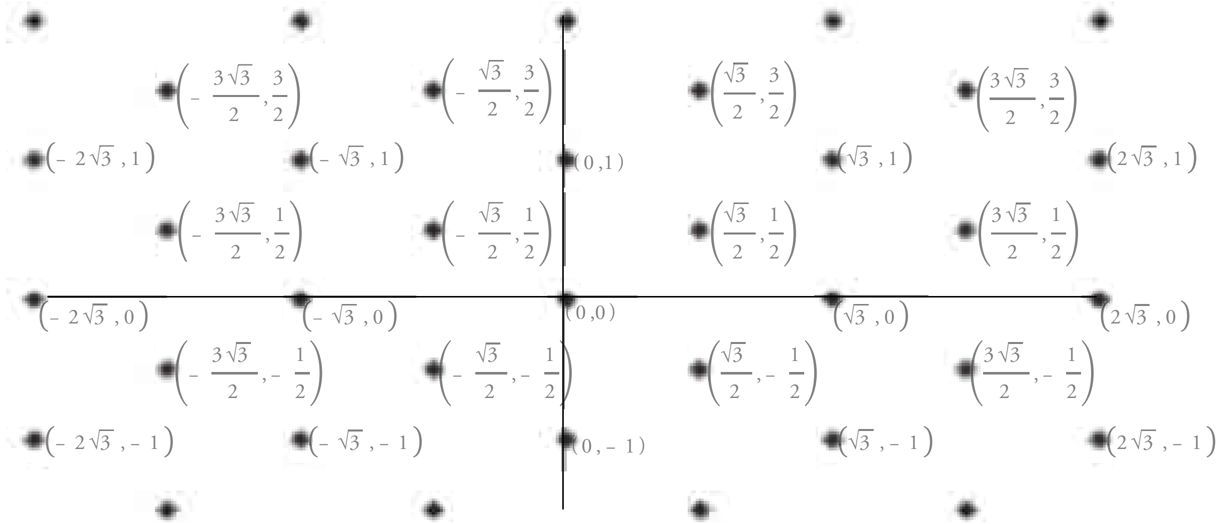

Figure 4

The points on the rows immediately above and below the x-axis have x-coordinates $\pm\frac{\sqrt{3}}{2}, \pm\frac{3\sqrt{3}}{2}, \pm\frac{5\sqrt{3}}{2} \ldots$, i.e., they are all odd multiples of $\frac{\sqrt{3}}{2}$. And the y-coordinates of these points are $\pm\frac{1}{2}$. This pattern continues for all rows with non-integer y-coordinate. For such rows, the points have coordinates $\left(\frac{(2m+1)\sqrt{3}}{2}, n+\frac{1}{2}\right)$ for some integers m and n. In short, the coordinates of points on an isometric grid are of the form $\left(\frac{m\sqrt{3}}{2}, \frac{n}{2}\right)$ where m, n are either both even or both odd.

Now, if we can draw a right isosceles triangle on such a grid, we may assume without loss of generality that the vertex of the triangle corresponding to the right angle coincides with the origin of the grid. (To accomplish this, we translate the triangle parallel to itself so that the vertex corresponding to the right angle coincides with the origin.) Let P and Q be the remaining two vertices. Then $OP = OQ$ and $OP \perp OQ$.

Now there are three possibilities considering the parity of the coordinates of P and Q; namely, their m, n values may be (i) both even, (ii) both odd, (iii) one even and the other odd. We consider each of these in turn, starting with (i).

Let $P = \left(m\sqrt{3}, \text{n}\right)$ and $Q = \left(r\sqrt{3}, s\right)$ where m, n, r, s are integers.

The product of the slopes of OP and OQ is -1, so

$$\frac{n}{m\sqrt{3}} \times \frac{s}{r\sqrt{3}} = -1, \quad \therefore s = -\frac{3mr}{n}. \tag{3}$$

And $OP^2 = OQ^2$, so

$$n^2 + 3m^2 = s^2 + 3r^2, \quad \therefore s^2 = n^2 + 3\left(m^2 - r^2\right). \tag{4}$$

Combining (3) and (4), we get:

$$n^2 + 3\left(m^2 - r^2\right) = \frac{9m^2r^2}{n^2}, \quad \therefore n^4 + 3\left(m^2 - r^2\right)n^2 - 9m^2r^2 = 0,$$
$$\therefore \left(n^2 + 3m^2\right)\left(n^2 - 3r^2\right) = 0, \ \therefore n = \pm r\sqrt{3}.$$

This is not possible since $\sqrt{3}$ is irrational.

For (ii), let $P = \left(\frac{(2m+1)\sqrt{3}}{2}, n + \frac{1}{2}\right)$, $Q = \left(\frac{(2r+1)\sqrt{3}}{2}, s + \frac{1}{2}\right)$. So, the slope equation becomes:

$$\frac{\left(n + \frac{1}{2}\right)\left(s + \frac{1}{2}\right)}{\frac{3}{4}\left(2m+1\right)\left(2r+1\right)} = -1, \ \therefore \left(2n+1\right)\left(2s+1\right) = -3\left(2m+1\right)\left(2r+1\right).$$

Note that this is similar to what we got in (i), but with the following changes:

$$m \to 2m+1, \quad r \to 2r+1, \quad n \to 2n+1, \quad s \to 2s+1.$$

Therefore, this reduces to $2n + 1 = \pm\left(2r+1\right)\sqrt{3}$, i.e., an impossibility as earlier.

For (iii), let $P = \left(\frac{(2m+1)\sqrt{3}}{2}, n + \frac{1}{2}\right)$ and $Q = \left(r\sqrt{3}, s\right)$, without loss of generality. The slope equation now becomes

$$\frac{\left(n + \frac{1}{2}\right)s}{\frac{3}{2}\left(2m+1\right)r} = -1, \ \therefore s = -\frac{3\left(2m+1\right)r}{2n+1}.$$

This is again similar to (i), but with the changes $m \to 2m+1$ and $n \to 2n+1$. Consequently, we get $2n + 1 = \pm r\sqrt{3}$, an impossibility as earlier.

We conclude that constructing a right isosceles triangle on an isometric grid is not possible.

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Triangle Centres – Barycentric Coordinates

A. RAMACHANDRAN

In an earlier article we had presented a way to characterize well known triangle centres – by their 'trilinear coordinates.' (Ref. 1) In this article we take up another system of characterising triangle centres, where each triangle centre is considered as the location of the centre of mass of a system of three point masses placed at the vertices of the triangle. The ratio of the masses then forms the 'barycentric coordinates' of the point in question. This approach was suggested by the German mathematician August Ferdinand Mobius in 1827.

Centroid

We first look at the centroid. This point is the centre of mass of a system of three equal masses placed at the vertices of the triangle, as discussed below. The centre of mass of the masses at B and C lies at the midpoint of BC, say D. So the centre of mass of all three must lie on (median) AD. Similarly we could say that the overall centre of mass should lie on the medians BE (E, midpoint of CA) and CF (F, midpoint of AB) as well. So, the overall centre of mass lies on the point of concurrence of the medians, a fact guaranteed to us by the converse of Ceva's theorem, since $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$. Hence the barycentric coordinates of the centroid are $1 : 1 : 1$.

We also note that the point of concurrence of the medians, the centroid G, divides AD in the ratio $2 : 1$, which is the inverse of the ratio of the mass at A to the combined masses at B and C.

Keywords: Coordinates, barycentric, centre of mass, centroid, ratio

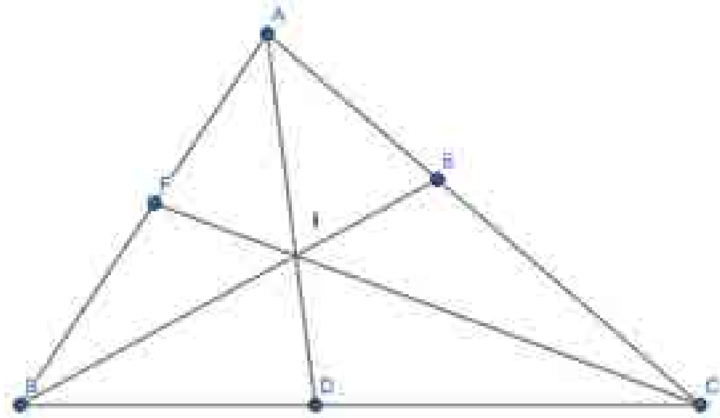

Figure 1

Incentre

We now take up the incentre. Let AD, BE, CF be the angle bisectors in Δ ABC (Figure 1). It is a well-known result that D divides BC in the ratio of the sides AB : AC or $c : b$. So if masses (proportional to) b and c were placed at points B and C, respectively, their centre of mass would lie at D. If now mass a were placed at A, then the centre of mass of all three would lie on the angle bisector AD. Similar arguments lead to the conclusion that the centre of mass of the system would lie on the other angle bisectors as well, or at their point of concurrence, the incentre I, the fact of concurrence guaranteed by the converse of Ceva's theorem, as $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = \frac{c}{b} \times \frac{a}{c} \times \frac{b}{a} = 1$. So the barycentric coordinates of the incentre are $a : b : c$, or sin A : sin B : sin C.

Note that in Δ ABD, BI bisects ∡B, and so AI : ID = AB : BD = $c : \frac{ac}{b+c} = (b+c) : a$. Thus, I divides AD in a ratio that is the inverse of the ratio of the mass at A to the combined masses at B and C.

Orthocentre

We now consider the orthocentre. We first look at an acute angled triangle, say Δ ABC, with altitudes AD, BE, CF, meeting at H (Figure 2).

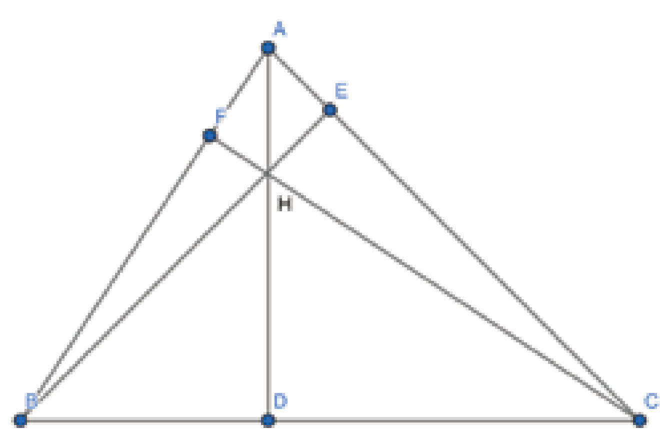

Figure 2

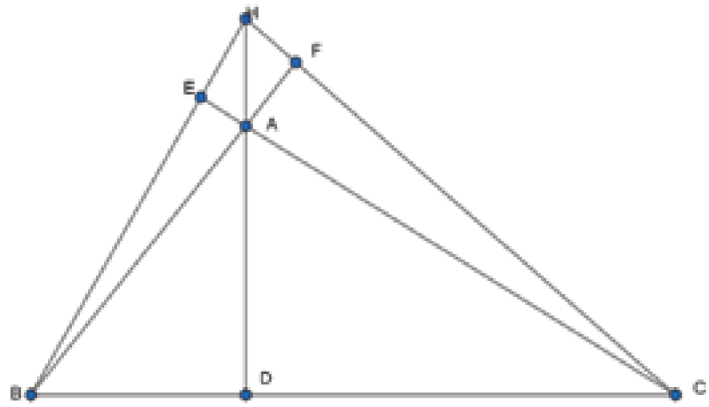

Figure 3

We have $\dfrac{BD}{DC} = \dfrac{AD/DC}{AD/BD} = \dfrac{\tan C}{\tan B}$. We could say that if masses (proportional to) tan B and tan C were placed at B and C, respectively, their centre of mass would lie at D. If now mass tan A were placed at A then the centre of mass of all three would lie on the altitude AD. By similar arguments we could say that the overall centre of mass lies on altitudes BE and CF too, or at their point of concurrence. Again, concurrence is guaranteed by the converse of Ceva's theorem. So the barycentric coordinates of the orthocentre are tan A : tan B : tan C.

Also note that $\dfrac{AD}{HD} = \dfrac{AD}{DC} \times \dfrac{DC}{HD} = \tan B \ \tan C$ and

$$\frac{AH}{HD} = \frac{AD}{HD} - 1 = \tan B \ \tan C - 1 = \frac{\tan B + \tan C}{\tan A}.$$

(The last step follows from the relation tan A tan B tan C = tan A + tan B + tan C, for A + B + C = 180° .)

Thus, H divides AD in a ratio that is the inverse of the ratio of the mass at A to the combined masses at B and C.

In the case of an obtuse angled triangle (see Figure 3), we have $\dfrac{CE}{EA} = \dfrac{CE/HE}{EA/HE} = \dfrac{\tan\,(180^\circ - A)}{\tan C} = -\dfrac{\tan A}{\tan C}$, the negative sign indicating an external division of a line. Also,

$$\frac{AH}{HD} = 1 - \frac{AD}{HD} = 1 - \tan B \ \tan C = -\frac{\tan B + \tan C}{\tan A}.$$

When one angle, say ∡A, approaches 90° , we have $\dfrac{AH}{HD} = (\tan B + \tan C)\cot A = 0$. That is, A and H coincide, with AH = 0. As tan 90^0 is not defined, we work with its reciprocal, cot A.

Circumcentre

We now turn our attention to the circumcentre. In Figure 4, O is the circumcentre of acute angled Δ ABC. AO produced meets BC at K. BO produced meets CA at L, while CO produced meets AB at M. Now, BK : KC = area Δ BOK : area Δ COK = $\frac{1}{2}$ r(OK) sin ∡BOK : $\frac{1}{2}$ r(OK) sin ∡COK = sin ∡BOK : sin ∡COK = sin ∡AOB : sin ∡AOC = sin 2C : sin 2B.

(Angle subtended at the centre by a chord is twice that subtended at a point on the circumference.)

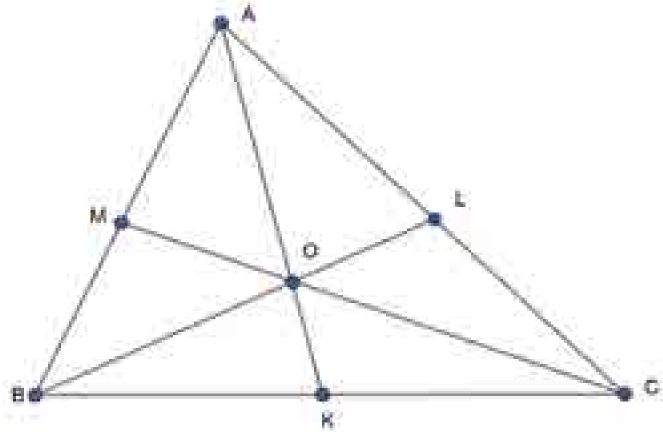

Figure 4

So if masses (proportional to) sin 2B and sin 2C were placed at B and C respectively, their centre of mass would lie at K. If now a mass proportional to sin 2A were placed at A the centre of mass of all three would lie on line AK. By similar arguments we could say that the overall centre of mass would lie on lines BL and CM as well, i.e., at their point of concurrence, again guaranteed by the converse of Ceva's theorem. Thus the barycentric coordinates of the circumcentre are sin 2A : sin 2B : sin 2C or sin A cos A : sin B cos B : sin C cos C.

Also note that AO : OK = area Δ ABO : area Δ OBK = area Δ ACO : area Δ OCK.

We could then say, adding quantities in the same ratio, AO : OK = (area Δ ABO + area Δ ACO) : area Δ OBC = $(\frac{1}{2}r^2 \sin 2C + \frac{1}{2}r^2 \sin 2B) : \frac{1}{2}r^2 \sin 2A$ = (sin 2C + sin 2B) : sin 2A. So O divides AK in a ratio that is the inverse of the ratio of the mass at A to the combined masses at B and C.

In the case of one angle, say ∡A, being obtuse, a negative sign appears in some of the relations indicating an external division in the given ratio.

In Figure 5, CL : LA = area Δ COL : area Δ LOA = $\frac{1}{2}$ r (OL) sin ∡COL : $\frac{1}{2}$ r (OL) sin ∡LOA = sin [180° − (2B + 2C)] : sin (180° − 2C) = sin (2A − 180°) : sin 2C = −sin 2A : sin 2C.

Further, AO : OK = area Δ ABO + area Δ ACO : area Δ KBO + Δ KCO = $\frac{1}{2}$ r^2 (sin 2B + sin 2C) : $\frac{1}{2}$ r^2 sin 2(180° − A) = −(sin 2B + sin 2C) : sin 2A.

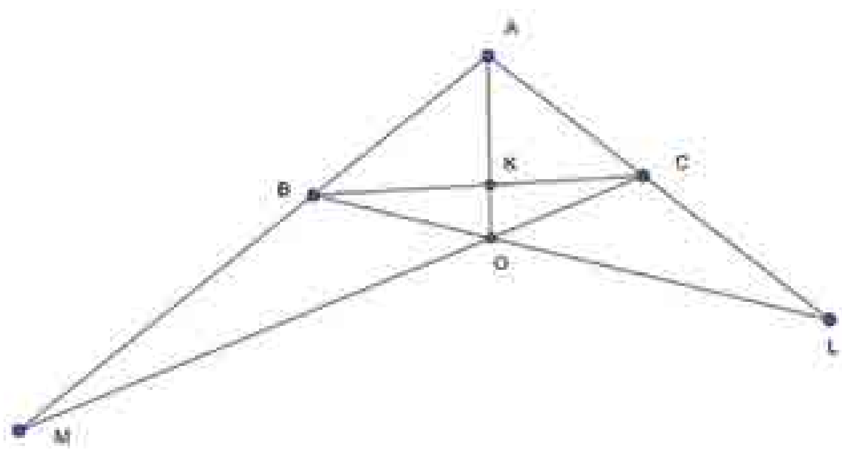

Figure 5

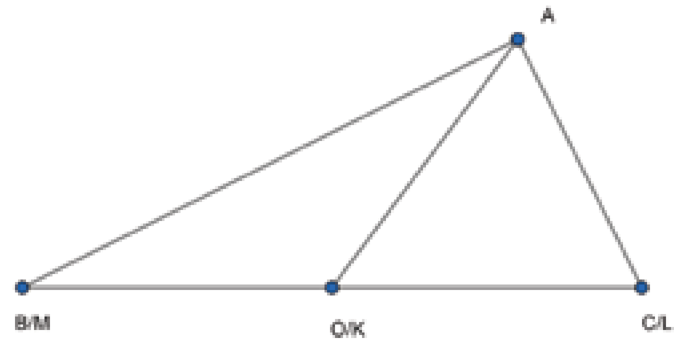

Figure 6

If one angle of the triangle, say A, is a right angle, then O, K coincide, as do L, C and M, B (Figure 6).

We then have KO/OA = sin 2A / (sin 2B + sin 2C) = 0.

However, BO/OL = (sin 2A + sin 2C) / sin 2B = 1. Note that sin 2A = 0, while 2B and 2C are supplementary, as B and C are complementary angles in this context.

The table below summarises the observations made above as well as in the earlier article (Ref.1).

Triangle centre	***Trilinear coordinates***	***Barycentric coordinates***
Incentre	1 : 1 : 1	sin A : sin B : sin C
Centroid	cosec A : cosec B : cosec C	1 : 1 : 1
Orthocentre	sec A : sec B : sec C	tan A : tan B : tan C
Circumcentre	cos A : cos B : cos C	sin A cos A : sin B cos B : sin C cos C

Note that the barycentric coordinates of any of the above triangle centres can be obtained by multiplying each component of the corresponding trilinear coordinates by the appropriate sine function (for instance, in the case of the orthocentre, multiplying sec A, sec B, sec C by sin A, sin B, sin C respectively, yields tan A, tan B, tan C).

We hope that the reader now has a feel for the idea of a triangle centre. Such a point would retain its position relative to the vertices of the triangle under transformations such as reflection, rotation, enlargement, or contraction, or when we permute the vertices among themselves. The reader can access the online "Clark Kimberling's Encyclopedia of triangle centres" (Ref. 2) for more information on the above and many, many more triangle centres.

References

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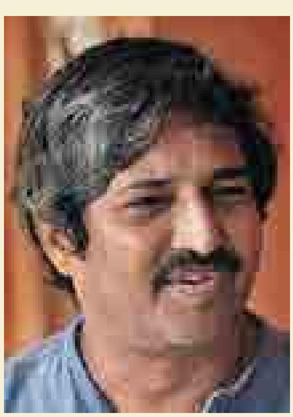
A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

The Editors of 'At Right Angles' congratulate Prof R Ramanujam on his being awarded the INSA Science Popularisation Prize 2020

We are delighted to share with our readers the news that renowned mathematician Prof R Ramanujam (better known by his other name, Jam) of the Institute of Mathematical Sciences, Chennai has been awarded the Indira Gandhi prize for Popularisation of Science for the year 2020, for his sustained outreach work on science and mathematics, stretching back for over three decades.

The prize was instituted by Indian National Science Academy in 1986 and is given once in three years to a career media professional or career scientist who has made outstanding contributions to science popularization in English or in other languages. Previous winners include physicist Jayant V Narlikar, scientist and writer G. Venkataraman, and scientist D. Balasubramanian.



For more details, please see the following: https://www.thehindu.com/sci-tech/science/chennai-scientist-chosen-for-insa-science-popularisation-prize/article30449804.ece

Long-time readers of At Right Angles will know that Prof Ramanujam has written for us on several occasions – in the July 2013, November 2014 and July 2015 issues. We look forward to more articles from him!

Euler's Inequality for the Circumradius and Inradius of a Triangle

SHAILESH SHIRALI

For any arbitrary triangle ABC, let R denote its circumradius and r its inradius (Figure 1). It was the Swiss-German mathematician Leonhard Euler who first observed that regardless of the shape of the triangle, the following inequality is invariably true:

$$R \geq 2r,$$

equality precisely when the triangle is equilateral.

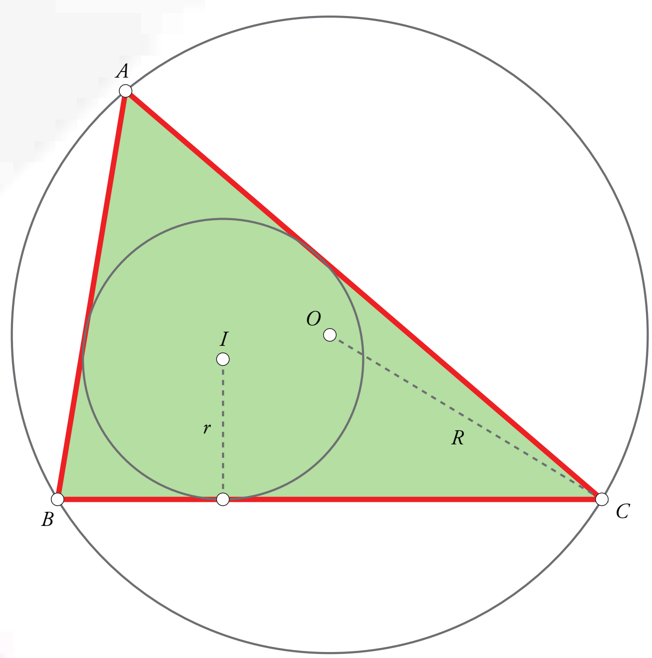


Figure 1.

Keywords: Euler, inequality, circumcircle, circumradius, incircle, inradius

The statement being so simple, one naturally longs for an equally simple proof of the result. Unfortunately, this is not readily forthcoming. (There are elegant and short proofs, but not simple proofs!) In this article, we present two very different proofs.

Notation. We use standard symbols for the various elements of the triangle: A, B, C for the three angles; a, b, c for the three sides (named appropriately); s for the semi-perimeter; R for the circumradius; r for the inradius; and Δ for the area of the triangle.

First proof. We first present a proof which has a strong component of algebra. The following formulas are all very well-known:

$$\begin{aligned} 2R &= \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \\ \Delta &= \frac{1}{2}\, bc \sin A = \frac{1}{2}\, ca \sin B = \frac{1}{2}\, ab \sin C, \\ \Delta &= rs, \\ \Delta &= \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned}$$

Combining the results in the first two lines, we obtain the additional result

$$\Delta = \frac{abc}{4R}.$$

In addition, we shall need the most basic result in the theory of inequalities, namely, the arithmetic mean-geometric mean inequality. This is the statement that if x, y are any two non-negative real numbers, then

$$\frac{x+y}{2} \geq \sqrt{xy},$$

with equality precisely when $x = y$. This gives rise to the following nice result. Let x, y, z be any three non-negative real numbers. Then we have:

$$\begin{aligned} x + y &\geq 2\sqrt{xy}, \\ y + z &\geq 2\sqrt{yz}, \\ z + x &\geq 2\sqrt{zx}. \end{aligned}$$

Hence by multiplication of the respective sides we obtain:

$$(x+y)(y+z)(z+x) \geq 8xyz. \qquad (1)$$

Moreover, equality will hold in (1) precisely when $x = y = z$. Note that this is an interesting result in its own right.

Next we obtain the lengths of some segments associated with the incircle of a triangle (see Figure 2). Let D, E, F denote the points of contact of the incircle with the sides of the triangle, and let x, y, z denote the lengths of the segments as indicated.

It is easy to obtain x, y, z in terms of a, b, c. We have:

$$\begin{aligned} y + z &= a, \\ z + x &= b, \\ x + y &= c. \end{aligned}$$

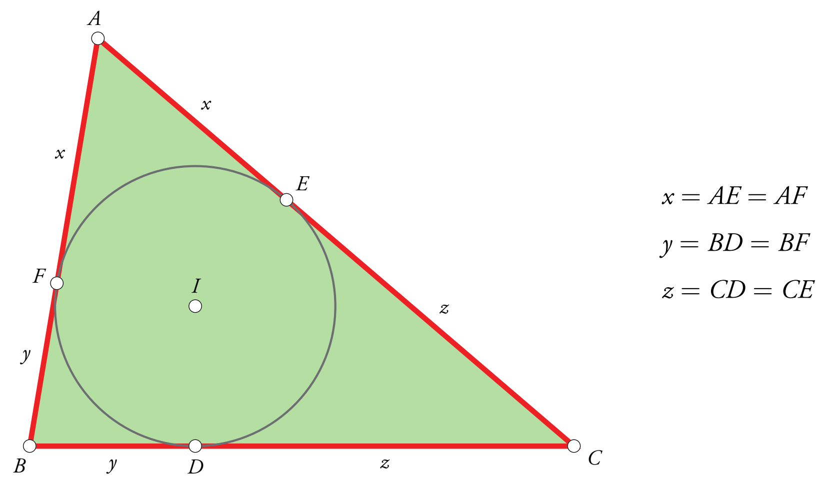


Figure 2.

By addition we get $2(x+y+z) = a+b+c = 2s$, hence $x+y+z = s$. Therefore:

$$x = s-a, \quad y = s-b, \quad z = s-c. \tag{2}$$

We now use these expressions for x, y, z in the inequality (1) (note that x, y, z are all positive; this follows from the triangle inequality, $b+c > a$, which yields $s-a > 0$; similarly for $s-b$ and $s-c$). We obtain:

$$abc \geq 8(s-a)(s-b)(s-c). \tag{3}$$

This too is an interesting result in its own right; it holds for any triangle. Moreover, equality holds precisely when $a = b = c$, i.e., when the triangle is equilateral.

We are now in a position to obtain Euler's inequality. From the formulas stated earlier for the area of a triangle, we have:

$$\Delta = \frac{abc}{4R},$$
$$\therefore \quad abc = 4R\Delta,$$

and:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$
$$\therefore \quad (s-a)(s-b)(s-c) = \frac{\Delta^2}{s}.$$

Therefore we have:

$$4R\Delta \geq \frac{8\Delta^2}{s},$$
$$\therefore \quad R \geq \frac{2\Delta}{s}.$$

Since $\Delta = rs$, the last line yields the desired result:

$$R \geq 2r, \tag{4}$$

with equality precisely when the triangle is equilateral.

Second proof. In contrast to the above, we now present a proof which is highly geometric. This is Euler's original proof (1765). He obtains the inequality as an easy consequence of an important geometric result.

Theorem (Euler)**.** *The distance d between the circumcentre and the incentre of a triangle is related to its circumradius R and its inradius r by the following relation:*

$$d^2 = R(R-2r).$$

Remark. Before proceeding with the proof of the theorem, we note that the result instantly provides a proof of Euler's inequality; for, we must have $d^2 \geq 0$, and this yields $R \geq 2r$.

Proof of Euler's theorem. In Figure 3, let AI extended meet the circumcircle at L; let LO extended meet the circumcircle at M; let segment IO extended in both directions meet the circumcircle at P and Q; and finally, let F be the foot of the perpendicular from I to AB.

Consider $\triangle AFI$ and $\triangle MBL$. They are similar to each other, for $\measuredangle AFI$ and $\measuredangle MBL$ are right angles, and $\measuredangle FAI = \measuredangle BML$ ("angles in the same segment"). Hence:

$$\frac{FI}{BL} = \frac{AI}{ML}, \quad \text{i.e.,} \quad \frac{r}{BL} = \frac{AI}{2R}, \tag{5}$$

which yields $2Rr = AI \cdot BL$. Next, we claim that $BL = IL$, i.e., that $\triangle LBI$ is isosceles. This follows from a simple computation of angles. For we have,

$$\measuredangle LBI = \measuredangle LBC + \measuredangle IBC = \measuredangle LAC + \measuredangle IBC = \frac{\measuredangle A}{2} + \frac{\measuredangle B}{2},$$

and

$$\measuredangle LIB = \measuredangle LAB + \measuredangle ABI = \frac{\measuredangle A}{2} + \frac{\measuredangle B}{2}.$$

It follows that $BL = IL$ and so:

$$\boxed{2Rr = AI \cdot IL.} \tag{6}$$

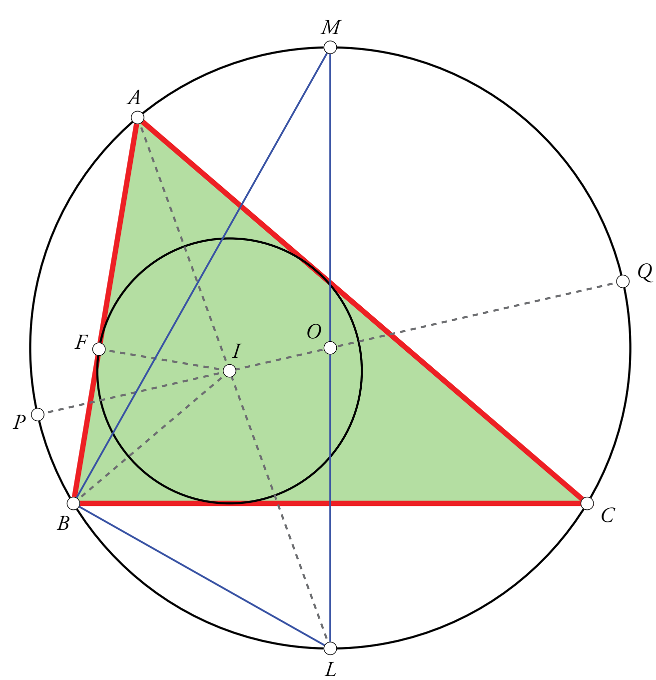


Figure 3.

Finally, from the intersecting chords theorem it follows that

$$AI \cdot IL = PI \cdot IQ.$$

Since $PI = OP - OI = R - d$ and $IQ = QO + OI = R + d$, we obtain:

$$2Rr = (R - d)(R + d) = R^2 - d^2,$$

i.e.,

$$d^2 = R^2 - 2Rr = R(R - 2r). \tag{7}$$

Euler's inequality now follows.

For more proofs, the three references listed below may be consulted.

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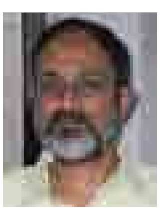
3. Samer Seraj, "A Short Proof of Euler's Inequality", *Resonance – Journal of Science Education*, Volume 20, Issue 1, January 2015 pp 75-75, https://www.ias.ac.in/article/fulltext/reso/020/01/0075-0075



SHAILESH SHIRALI is the Director of Sahyadri School (KFI), Pune, and heads the Community Mathematics Centre based in Rishi Valley School (AP) and Sahyadri School KFI. He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Math Jokes and Puns

1. Why was the fraction apprehensive about marrying the decimal?
   Because he would have to convert…
2. Why do plants hate math?
   Because it gives them square roots…
3. Why did the student get upset when his teacher called him average?
   Well, it was a pretty mean thing to say!
4. Why was the math book depressed?
   Poor thing, it had too many problems.
5. Why is the obtuse triangle always so frustrated?
   Because it is never right.
6. Why can you never trust a math teacher holding graphing paper?
   He must be plotting something.

Contributed by Harin Hattangady, Azim Premji University, M.A. Education, batch of 2011-13

Introducing Robocompass

A nifty tool for Geometrical Construction

M SRINIVASAN

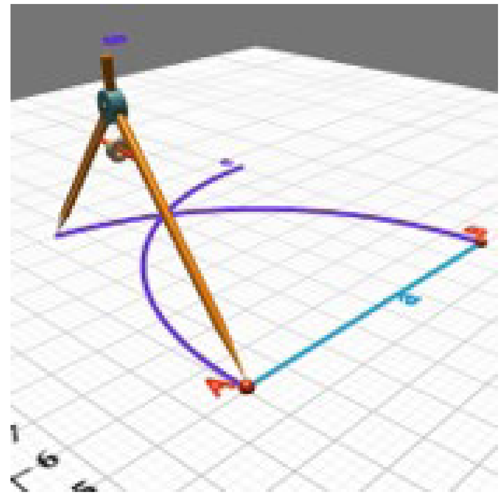
How Geometrical Constructions are taught in Schools

Euclid's *Elements* – one of the most influential mathematical textbooks ever to have been written – is primarily a compendium of geometrical constructions created using straightedge and compass. But are we sure that these two geometrical tools which lie at the heart of such foundational ideas are being used effectively in the classrooms? The current practice is actually to use large wooden geometrical instruments in the classrooms, as the size of the real physical compass (in the 'geometry box') is not large enough to use conveniently as a demonstration tool.

From a student's perspective, pencil smudges, torn papers, hurt fingers due to sharp edges, changing measurements as they move the instruments are some of the inconveniences they need to deal with when working with a physical geometry box. Spatial reasoning is one of the skills all students should inculcate, but are the challenges of using these physical instruments holding students back?

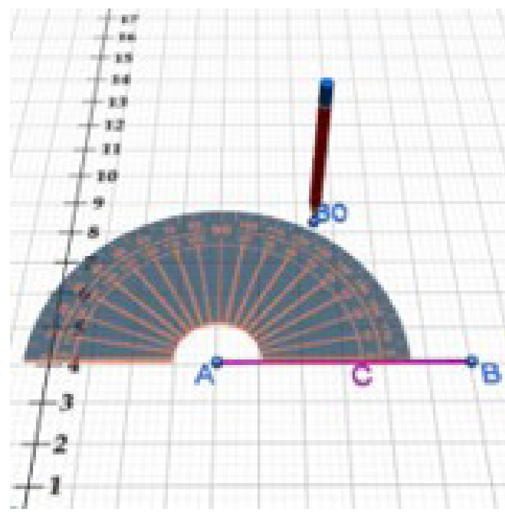
Keywords: geometry, construction, space, tools, software

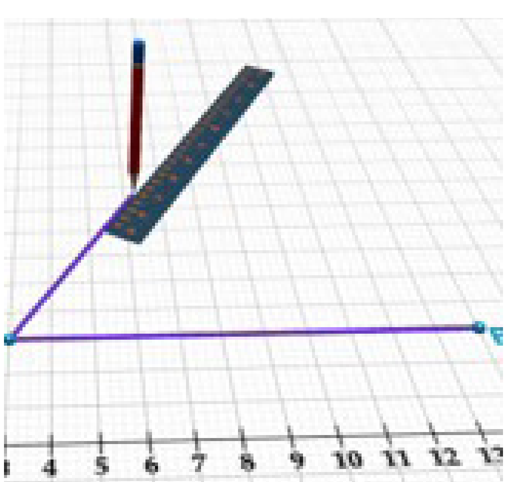
The question we are trying to address in this article is, with increasing computerisation of our classrooms, would a digital equivalent of a physical compass make it easier for teachers to demonstrate geometrical steps of a construction? Would it help students understand geometrical concepts better and appreciate the beauty behind geometrical constructions more fully? In this article, we explore these questions using a geometrical software called Robocompass, https://www.robocompass.com/. It is freely available on the web.



Technology based Approach

If not used appropriately, technology in the classroom can end up serving more to distract than to enable. This is because of two reasons: (i) technology introduces a radically new way of learning that students find difficult to adapt to; (ii) the technology comes up with a steep learning curve which teachers find difficult to cope with. Robocompass addresses these two primary concerns by simulating the behaviour and movements of geometrical instruments in 3D exactly as they would work in the real world. From a learning perspective, Robocompass uses 10 simple text-based commands which can be used to create the code for building constructions. The geometrical construction unfolds as the user types the commands one by one. The tool is very user friendly and correcting or modifying the commands can be done easily without having to remember too many menus/buttons or other distracting user interface operations.



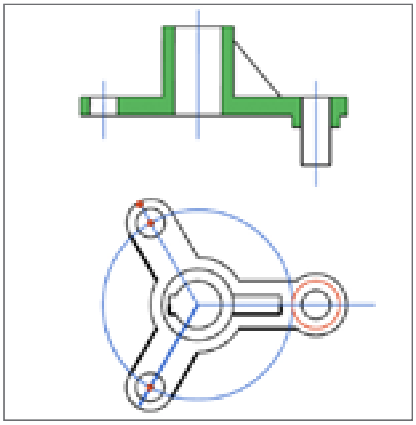


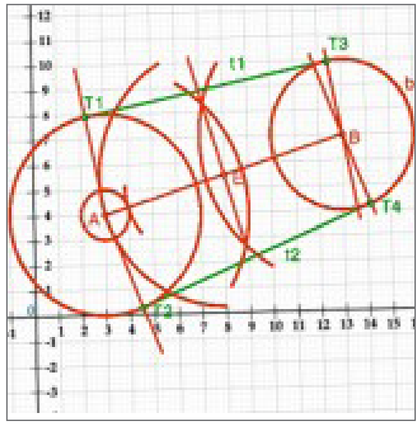
Having coded the steps, we can 'play' the construction, just like a Video. A teacher can import someone else's construction and add/remove or modify the commands, and include hints while demonstrating the steps of geometrical constructions; similarly, students can replay the whole construction or play a single step at a time.

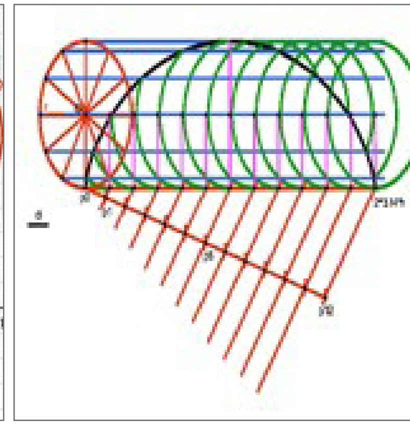
Apart from basic constructions, we may also demonstrate key geometrical ideas such as dilation of a geometric figure using Robocompass.

Building Complex Geometrical Constructions

The simplicity of the commands and the look of the 3D environment might give Robocompass a game-like aura, but many complex geometrical constructions are already being developed and published by expert mathematicians.







That the geometrical primitives can be colour-coded differently and commented upon is another aspect that greatly enhances the comprehensibility of complex constructions.

Code the Construction

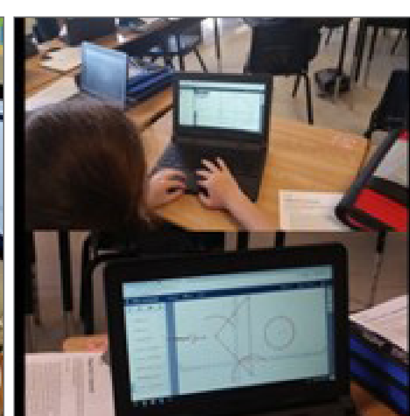
One of the positive (but unintended!) consequences of the Robocompass command-based system is how gently it introduces *coding practices* to students. Naturally, to create the correct code for a construction process, the student needs to understand the steps that will lead to the final desired output.

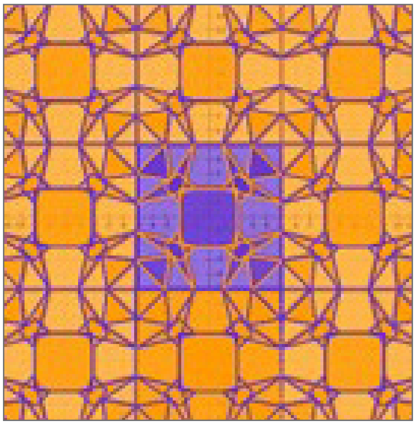
Against the backdrop of many developed countries already incorporating coding into the curriculum, many teachers have appreciated this aspect of Robocompass. Today coding is the new expression of literacy, and Robocompass can facilitate the exposure of such a vital skill to students early on through curriculum oriented geometrical concepts.

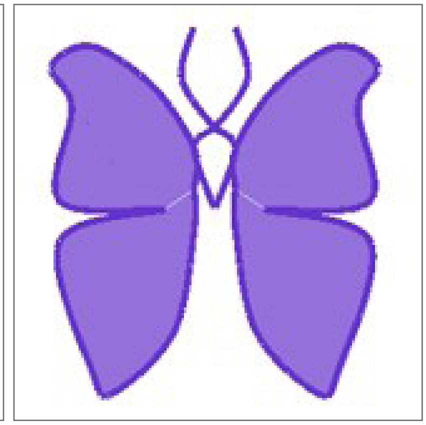
Robocompass as a Creative Tool

Beyond the basic commands used for building geometrical constructions, Robocompass comes with a rich set of commands to do creative mathematical art and tessellations using basic geometrical transformations such as reflection, translation, and rotation. This has encouraged many schools in the US and Canada to give creative project assignments to students using Robocompass.

The flexibility of Robocompass to deconstruct each step of the actual mathematical art in animated fashion reinforces the essential ideas such as symmetry, perpendicularity, and parallelism.







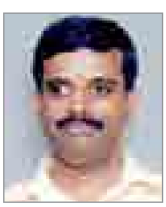
Classroom Integration

Teachers who are new to Robocompass do not have to create any new worksheets from scratch. They can download many ready to use worksheets (based on the NCERT syllabus) from the Mumbai website of the Kendriya Vidyalaya Sangathan, Zonal Institute of Education and Training:

http://zietmumbai.gov.in/homedir/public_html/OldResources.html

(Solutions to constructions: Class VI to X using Robocompass)

Teachers can modify these worksheets as required and project them to a smartboard. Teachers will find Robocompass to be a great supplementary tool in teaching Geometrical Constructions and this will lead to greater student engagement in the classroom.



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Middle School Problems Requiring Exhaustive Search

A. RAMACHANDRAN

There are problems where one is required to find examples or instances that satisfy some given conditions, from many possibilities. One would have to search among several possibilities taking care not to leave out any possibility. Here are a few such problems.

Problem 1. We generally record dates in the format DD-MM-YY, i.e., the day followed by the month and the last two digits of the year. In some cases you may find that (DD)(MM) = (YY), i.e., the product of the date and month equals the (last two digits of the) year. An example is 15-4-60. Find all such dates in a century and present them in chronological order.

Problem 2. In an article on magic squares in the July 2014 issue of AtRiA, the following was stated: 'There are 86 ways of picking out four numbers from the set 1-16 to give a sum of 34.' Find all these combinations.

Problem 3. What is the least natural number that has exactly 100 factors? (Here we include all the factors of the number, not just the prime factors.)

Problem 4. Observe the factorisations below:

$$x^2 + 10x + 24 = (x + 4)(x + 6)$$

$$x^2 + 10x - 24 = (x + 12)(x - 2)$$

The expressions on the left differ only in the sign of the constant term. They are both factorisable. Find all such instances with the constant term under 100, i.e., values of p and q, $q < 100$ that make both $x^2 + px + q$ and $x^2 + px - q$ factorisable.

Keywords: constraint, compliance, search, magic square, quadratic, factor

Pedagogical Notes: The value of the skill of searching is so obvious (even to students) that one may not need to emphasise this. However, the process during which a random search becomes systematic is worth observing and developing in students. Not only do students learn to reflect on their thinking, but they also begin to analyse and select more efficient ways of conducting their searches. It is important that they document this process and talk about it, rather than just focus on the answers that they get.

Some Hints/Pointers

As these problems are somewhat time consuming, we provide some tips below to ensure that you are on the right track. You may decide not to read this part and proceed on your own. It may be good to work on these as a group.

Hints for problem 1

This is essentially an exercise in expressing the numbers 1 to 99 as products of two numbers in as many ways as possible. Among all such factorisations, you need to pick the ones permissible in view of the restrictions on the dates and month orders.

Hints for problem 2

To start with, we make a ground rule: we present the numbers in any combination in ascending order; this helps avoid repetition.

Start with the lowest combination, (1, 2, 3, 4), which, of course, is not admissible as the sum is less than 34. Keep moving up, first, by increasing the last number, then the third number, followed by the second number and finally, the first number, till you hit the ceiling, i.e., with any further increase the total would exceed 34.

Hints for problem 3

To address this question, you should be aware of the following rule from number theory: "If $N = a^p \times b^q \times c^r \ldots$ is the prime factorisation of a natural number N, where $a, b, c, \ldots$ are prime numbers and $p, q, r, \ldots$ are natural numbers, then the number of factors of N is given by $(p + 1)(q + 1)(r + 1)\ldots$" This includes the factors 1 and the number N itself.

Now, factorise 100 in as many ways as possible. In each factorisation, reduce each factor by 1 and assign these as exponents of suitable prime numbers with a view to keep the products as low as possible. For example, starting with the factorisation $100 = 4 \times 5 \times 5$, we reduce each factor by 1, obtaining the numbers 3, 4, 4. Using the rule stated above, we see that each of the following numbers has exactly 100 factors:

$$2^3 \times 3^4 \times 5^4,\ 2^4 \times 3^3 \times 5^4,\ 2^4 \times 3^4 \times 5^3.$$

These are not the only possibilities; there are several others. Now, from among all these possibilities, we need to choose the smallest.

Hints for problem 4

The factorisation exercise you carried out for Problem 1 should aid you now. Examine the factors of each number from 1 to 99 and look for instances where the sum of two complementary factors equals the difference of two complementary factors. In other words, values that satisfy the equations:

$$A \times B = C \times D = N < 100,$$

$$A + B = C - D.$$

We hope that these exhaustive searches do not turn out to be exhausting searches!

Solutions on page 75

Problem Corner

Problems for the Senior School

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem IX-1-S.1

The midpoints of two sides of a triangle are marked. How can the midpoint of the third side be found using only a pencil and a straightedge?

Problem IX-1-S.2

Is it possible to cut several circles out of a square of side 10 cm, so that the sum of the diameters of the circles is 5 metres or more?

Problem IX-1-S.3

Suppose in a given collection of 2020 integers, the sum of every 100 of them is positive. Is it true that the sum of all the 2020 integers is necessarily positive?

Problem IX-1-S.4

Suppose integers a, b and c are such that $ax^2 + bx + c$ is divisible by 5 for any integer x. Prove that each of a, b and c is divisible by 5.

Problem IX-1-S.5

The altitude dropped from A onto BC in triangle ABC is not shorter than BC, and the altitude dropped onto AC from B is not shorter than AC. Find the angles of triangle ABC.

Keywords: Triangle, midpoint, circle, integer, altitude, divisible, octagon, circumcircle

Solutions of Problems in Issue-VIII-3 (November 2019)

Solution to problem VIII-3-S.1

From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Determine the length of a side of the octagon.

Let t be the length of a side of the regular octagon. The four triangular pieces removed are four congruent right-angled isosceles triangles with hypotenuse t. It follows that

$$t + 2\left(\frac{t}{\sqrt{2}}\right) = 5$$

whence $t = 5(\sqrt{2} - 1)$.

Solution to problem VIII-3-S.2

Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B and C intersect Ω at A_1, B_1 and C_1, respectively, and the internal bisectors of angles A_1, B_1 and C_1 of the triangle $A_1B_1C_1$ intersect Ω at A_2, B_2 and C_2, respectively. If the smallest angle of triangle ABC is $40°$, what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees? (Figure 1.)

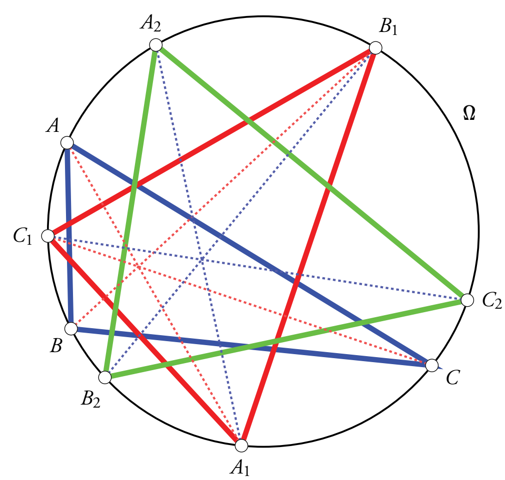


Figure 1.

Simple angle chasing yields

$$\measuredangle A_1 = 90° - \frac{1}{2}\measuredangle A, \quad \measuredangle B_1 = 90° - \frac{1}{2}\measuredangle B, \quad \measuredangle C_1 = 90° - \frac{1}{2}\measuredangle C,$$

$$\therefore \measuredangle A_2 = 90° - \frac{1}{2}\measuredangle A_1 = 45° + \frac{1}{4}\measuredangle A,$$

and in the same way,

$$\measuredangle B_2 = 45° + \frac{1}{4}\measuredangle B, \quad \measuredangle C_2 = 45° + \frac{1}{4}\measuredangle C.$$

If $\measuredangle A$ is the smallest angle of $\triangle ABC$ then $\measuredangle A_2$ is the smallest angle of $\triangle A_2B_2C_2$. Therefore, $\measuredangle A_2 = (45 + 40/4)° = 55°$.

Solution to problem VIII-3-S.3

The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of ABC. Determine the angles of the triangle ABC. (Figure 2.)

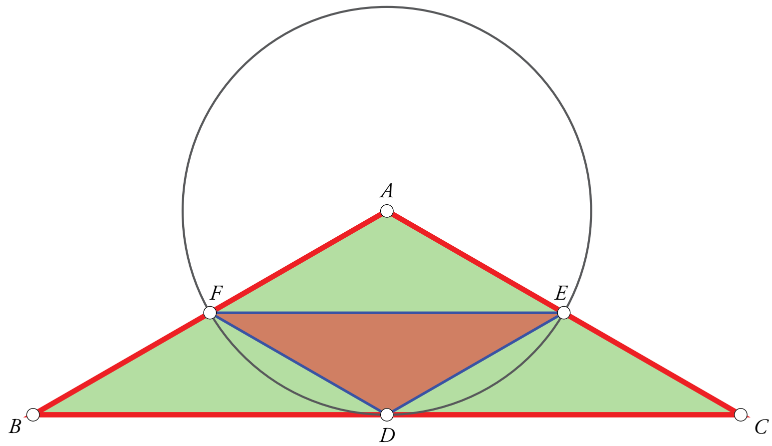


Figure 2.

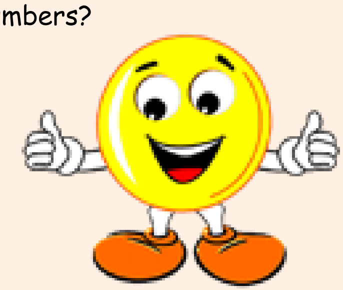
In $\triangle ABC$, let $AB = AC$. Suppose D is the midpoint of BC, E is the midpoint of AC and F is the midpoint of AB. Then DEF is similar to ABC with $\measuredangle D = \measuredangle A$. Also, EF is parallel to BC. The perpendicular bisector l of BC passes through A and it is also the perpendicular bisector of EF. Therefore the circumcentre of $\triangle DEF$ lies on l. As it lies on the circumcircle of ABC, it is outside $\triangle DEF$. Thus $\measuredangle D$ is obtuse which implies that the circumcentre lies on the opposite side of EF as D. It is that point of intersection of l and the circumcircle of $\triangle ABC$ which is on the opposite side of EF as D. Therefore it must be A. Hence:

$$360^\circ - \measuredangle A = 2\measuredangle A,$$

so $\measuredangle A = 120^\circ$. Thus the angles of ABC are $120^\circ, 30^\circ, 30^\circ$.

Math Jokes and Puns

1. Why was the equal sign so humble?
   Because she knew that she wasn't greater than or less than anyone else.
2. What do you call the number 7 and the number 3 when they go on a date?
   The odd couple (but 7 is in her prime).
3. I'll do algebra, I'll do trig, I'll even do statistics…
   … But graphing is where I draw the line!
4. Why should you never talk to Pi?
   Because he goes on and on and on forever…
5. Why are parallel lines so tragic?
   Because they never get to meet…
6. What is the best way to flirt with a math teacher?
   Use acute angle.
7. Did you hear about the mathematician who is afraid of negative numbers?
   He stops at nothing to avoid them.
8. How do you stay warm in any room?
   Just huddle in the corner, where it's always 90 degrees.
9. Why is six afraid of seven?
   Because seven eight ("ate") nine!
10. Why DID seven eat nine?
   Because you're supposed to eat 3 squared meals a day!



Contributed by Harin Hattangady, Azim Premji University, M.A. Education, batch of 2011-13

Adventures in Problem Solving

SHAILESH SHIRALI

In this edition of 'Adventures' we study three problems. As always, we pose the problems first and present the solutions later. The different approaches used to solve the problems should be studied with care by the student.

Miscellaneous problems

Problem 1. Determine the dimensions of all integer-sided cuboids whose surface area is 100 square units.

Problem 2. Rectangle $ABCD$ has sides $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\measuredangle BPC = 90^\circ$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC and CPD, what is the value of $r_1 + r_2 + r_3$?

Problem 3. Let O be the midpoint of the base BC of an isosceles triangle ABC. A circle is drawn with centre O and tangent to the equal sides AB and AC. Let P be a point on AB and Q a point on AC such that PQ is tangent to this circle. Prove that $BP \cdot CQ = \frac{1}{4}BC^2$. Discuss the converse of this result. (Australia, 1981)

Solutions to the problems

Solution to problem 1

Let the dimensions of the cuboid be a, b, c where a, b, c are positive integers, labelled so that $a \le b \le c$. Then we have $2ab + 2bc + 2ca = 100$, and therefore

$$ab + bc + ca = 50.$$

Keywords: Cuboids, surface area, rectangles, incircles, isosceles triangles

Since $a = \min(a, b, c)$, it must be that $ab + bc + ca \geq 3a^2$. Therefore we get $3a^2 \leq 50$. This implies that $a \leq 4$. So a must take one of the values $1, 2, 3, 4$. We consider each possibility in turn. In each case, the clinching argument will be through a suitable factorisation.

a = 1: The equation in this case reduces to $bc + b + c = 50$. We employ a clever but familiar trick now, by adding 1 to both sides. The left side then becomes $bc + b + c + 1$, and this expression factorises as $(b + 1)(c + 1)$. Hence it must be that

$$(b + 1)(c + 1) = 51.$$

Therefore $b + 1, c + 1$ are a pair of positive integers whose product is 51, and $1 \leq b \leq c$. The factorisation 1×51 does not work, as we must have $b \geq 1$, i.e., $b + 1 \geq 2$. The only other factorisation available is 3×17, and this yields $b + 1 = 3$, $c + 1 = 17$, so the triple in this case is $(a, b, c) = (1, 2, 16)$.

a = 2: The equation in this case reduces to $bc + 2b + 2c = 50$, or

$$(b + 2)(c + 2) = 54.$$

Now we must look for factorisations of 54, the smaller factor being at least 4 (since we must have $b \geq 2$). The only possibility available is $6 \times 9 = 54$, which gives $b + 2 = 6$ and $c + 2 = 9$. Hence the triple in this case is $(a, b, c) = (2, 4, 7)$.

a = 3 or 4: The remaining two cases may be analysed in the same way but they do not yield any fresh solutions. (Details left to the reader.)

So there are just two such cuboids, their dimensions being $(1, 2, 16)$ and $(2, 4, 7)$. It should be clear that this approach will work for any specified surface area.

Solution to problem 2

Please see Figure 1. Observe that

$$r_1 = \frac{AP + AB - BP}{2}, \quad r_2 = \frac{BP + PC - BC}{2}, \quad r_3 = \frac{DP + CD - PC}{2}.$$

Note that these relations hold because $\triangle BAP$, $\triangle BPC$ and $\triangle CPD$ are all right-angled. (The general statement is the following: if $\triangle ABC$ is right-angled, with the right angle at A, then the inradius r is given in terms of the sides a, b, c by the relation $2r = b + c - a$.) Adding these we obtain

$$r_1 + r_2 + r_3 = \frac{AD + AB + CD - BC}{2} = AB = 8.$$

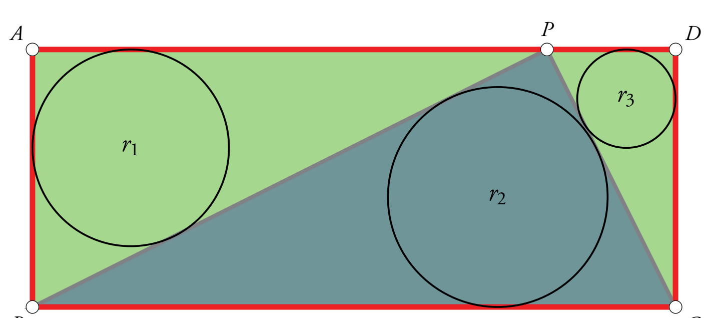


Figure 1.

Solution to problem 3

Please see Figure 2. Let L, M be the points of tangency of AB, AC with the semicircle, and let PQ touch the semicircle at D. Join OP and OQ, as shown.

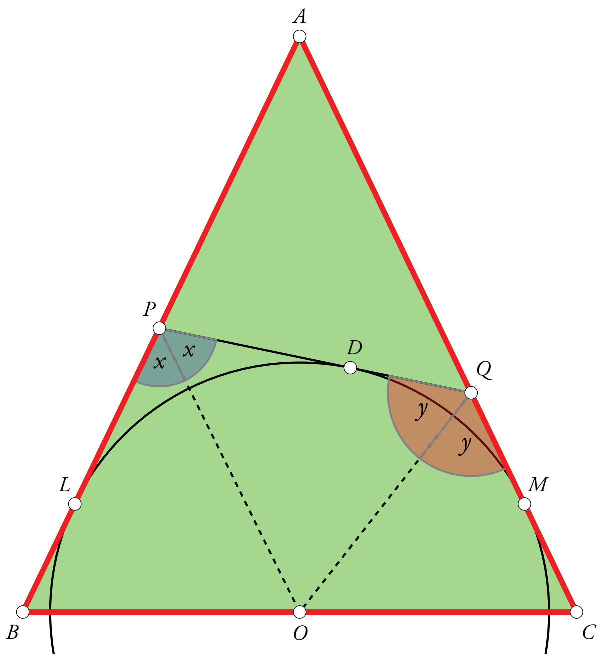


Figure 2.

Let $\measuredangle ABC = \theta = \measuredangle ACB$, $\measuredangle BPQ = 2x$, $\measuredangle CQP = 2y$. Consider quadrilateral $BPQC$. Its angles add up to 360°, therefore

$$2x + 2y + \theta + \theta = 360^\circ, \quad \therefore \; x + y + \theta = 180^\circ.$$

We now have:

$$\measuredangle BPO = x = \measuredangle OPQ, \quad \measuredangle OQP = y = \measuredangle OQC,$$
$$\therefore \; \measuredangle POQ = 180^\circ - x - y = \theta.$$

It follows that

$$\measuredangle BOP = y, \quad \measuredangle COQ = x,$$
$$\therefore \; \triangle PBO \sim \triangle POQ \sim \triangle OCQ,$$
$$\therefore \; \frac{PB}{BO} = \frac{OC}{CQ},$$
$$\therefore \; BP \cdot CQ = BO \cdot OC = \frac{1}{4} BC^2.$$

Note that we have not answered the question about the converse. We leave this part for the reader to explore.

SHAILESH SHIRALI is the Director of Sahyadri School (KFI), Pune, and heads the Community Mathematics Centre based in Rishi Valley School (AP) and Sahyadri School KFI. He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Solutions to Search Problems

Solution 1

01/01/01	02/05/10	03/06/18	13/02/26	12/03/36	12/04/48	10/06/60
02/01/02	01/10/10	02/09/18	27/01/27	09/04/36	08/06/48	06/10/60
01/02/02	11/01/11	19/01/19	09/03/27	06/06/36	06/08/48	05/12/60
03/01/03	01/11/11	20/01/20	03/09/27	04/09/36	04/12/48	21/03/63
01/03/03	12/01/12	10/02/20	28/01/28	03/12/36	07/07/49	09/07/63
04/01/04	06/02/12	05/04/20	14/02/28	19/02/38	25/02/50	07/09/63
02/02/04	04/03/12	04/05/20	07/04/28	13/03/39	10/05/50	16/04/64
01/04/04	03/04/12	02/10/20	04/07/28	20/02/40	05/10/50	08/08/64
05/01/05	02/06/12	21/01/21	29/01/29	10/04/40	17/03/51	13/05/65
01/05/05	01/12/12	07/03/21	30/01/30	08/05/40	26/02/52	22/03/66
06/01/06	13/01/13	03/07/21	15/02/30	05/08/40	13/04/52	11/06/66
03/02/06	14/01/14	22/01/22	10/03/30	04/10/40	27/02/54	06/11/66
02/03/06	07/02/14	11/02/22	06/05/30	21/02/42	18/03/54	17/04/68
01/06/06	02/07/14	02/11/22	05/06/30	14/03/42	09/06/54	23/03/69
07/01/07	15/01/15	23/01/23	03/10/30	07/06/42	06/09/54	14/05/70
01/07/07	05/03/15	24/01/24	31/01/31	06/07/42	11/05/55	10/07/70
08/01/08	03/05/15	12/02/24	16/02/32	22/02/44	05/11/55	07/10/70
04/02/08	16/01/16	08/03/24	08/04/32	11/04/44	28/02/56	24/03/72
02/04/08	08/02/16	06/04/24	04/08/32	04/11/44	14/04/56	18/04/72
01/08/08	04/04/16	04/06/24	11/03/33	15/03/45	08/07/56	12/06/72
09/01/09	02/08/16	03/08/24	03/11/33	09/05/45	07/08/56	09/08/72
03/03/09	17/01/17	02/12/24	17/02/34	05/09/45	19/03/57	08/09/72
01/09/09	18/01/18	25/01/25	07/05/35	23/02/46	20/03/60	06/12/72
10/01/10	09/02/18	05/05/25	05/07/35	24/02/48	15/04/60	25/03/75
05/02/10	06/03/18	26/01/26	18/02/36	16/03/48	12/05/60	15/05/75

19/04/76	16/05/80	21/04/84	22/04/88	10/09/90	24/04/96	09/11/99
11/07/77	10/08/80	14/06/84	11/08/88	09/10/90	16/06/96	
07/11/77	08/10/80	12/07/84	08/11/88	13/07/91	12/08/96	
26/03/78	27/03/81	07/12/84	30/03/90	23/04/92	08/12/96	
13/06/78	09/09/81	17/05/85	18/05/90	31/03/93	14/07/98	
20/04/80	28/03/84	29/03/87	15/06/90	19/05/95	11/09/99	

Solution 2

(1,2,15,16)	(1,3,14,16)	(1,4,13,16)	(1,4,14,15)	(1,5,12,16)	(1,5,13,15)	(1,6,11,16)
(1,6,12,15)	(1,6,13,14)	(1,7,10,16)	(1,7,11,15)	(1,7,12,14)	(1,8,9,16)	(1,8,10,15)
(1,8,11,14)	(1,8,12,13)	(1,9,10,14)	(1,9,11,13)	(1,10,11,12)	(2,3,13,16)	(2,3,14,15)
(2,4,12,16)	(2,4,13,15)	(2,5,11,16)	(2,5,12,15)	(2,5,13,14)	(2,6,10,16)	(2,6,11,15)
(2,6,12,14)	(2,7,9,16)	(2,7,10,15)	(2,7,11,14)	(2,7,12,13)	(2,8,9,15)	(2,8,10,14)
(2,8,11,13)	(2,9,10,13)	(2,9,11,12)	(3,4,11,16)	(3,4,12,15)	(3,4,13,14)	(3,5,10,16)
(3,5,11,15)	(3,5,12,14)	(3,6,9,16)	(3,6,10,15)	(3,6,11,14)	(3,6,12,13)	(3,7,8,16)
(3,7,9,15)	(3,7,10,14)	(3,7,11,13)	(3,8,9,14)	(3,8,10,13)	(3,8,11,12)	(3,9,10,12)
(4,5,9,16)	(4,5,10,15)	(4,5,11,14)	(4,5,12,13)	(4,6,8,16)	(4,6,9,15)	(4,6,10,14)
(4,6,11,13)	(4,7,8,15)	(4,7,9,14)	(4,7,10,13)	(4,7,11,12)	(4,8,9,13)	(4,8,10,12)
(4,9,10,11)	(5,6,7,16)	(5,6,8,15)	(5,6,9,14)	(5,6,10,13)	(5,6,11,12)	(5,7,8,14)
(5,7,9,13)	(5,7,10,12)	(5,8,9,12)	(5,8,10,11)	(6,7,8,13)	(6,7,9,12)	(6,7,10,11)
(6,8,9,11)	(7,8,9,10).					

Solution 3

Factorisations of 100: (i) 100 as a standalone factor, (ii) 50×2, (iii) 25×4, (iv) 20×5, (v) 10×10, (vi) $25 \times 2 \times 2$, (vii) $10 \times 5 \times 2$, (viii) $5 \times 5 \times 4$, (ix) $5 \times 5 \times 2 \times 2$. These should lead to the following numbers with exactly 100 factors:

(a) 2^{99} (b) $2^{49} \times 3$ (c) $2^{24} \times 3^3$ (d) $2^{19} \times 3^4$ (e) $2^9 \times 3^9$ (f) $2^{24} \times 3 \times 5$
(g) $2^9 \times 3^4 \times 5$ (h) $2^4 \times 3^4 \times 5^3$ (i) $2^4 \times 3^4 \times 5 \times 7$. Of these, the last one, 45360, is the least.

Solution 4

The required values of (p,q) are (5,6), (10,24), (13,30), (15,54), (17,60), (25,84) and (20,96).

The Mathematics of India by P.P. Divakaran

Reviewed by M. S. Narasimhan

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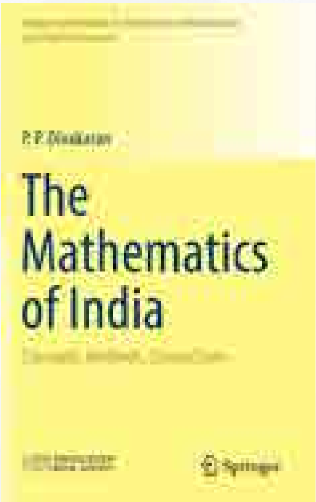

The Mathematics of India: Concepts, Methods, Connections. P. P. Divakaran. Hindustan Book Agency (India), P 19 Green Park Extension, New Delhi 110 016, India and Springer Nature Singapore Pte Ltd, 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore. 2018. xi + 441 pages. Price: Rs 980/74,96€.

While it is well known that India has a long and rich tradition in Mathematics, it is hard to come by books which explain the specific contributions in detail, trace the evolution and continuity of mathematical ideas, and survey the historical and social background in which research in mathematics was carried out. Divakaran's excellent book, which is readable, scholarly and well-researched, fills this need.

André Weil, in his lecture 'History of mathematics: why and how?' given at the International Congress of Mathematicians (1978) discusses the aim and content of a work on the history of mathematics. The first aim is to keep before us instances of first-rate mathematics, highlighting the mathematical ideas involved and their interconnections. A biographical sketch bringing alive mathematicians, their environment, in addition to their writings would be desirable. Weil says that 'An indispensable requirement is an adequate knowledge of the language of the sources; it is a basic and sound principle of all historical research that a translation can never replace the original when the latter is available.' It is important to give an exposition of the results and methods using modern mathematical notation, concepts and language. Weil also says that it is necessary not to yield to the temptation of concentrating only on the work of past great mathematicians, neglecting the work of lesser mathematicians. The book under review fulfills these requirements.

Incidentally, one of the reasons for a professional mathematician's interest in the history of mathematics is the hope that it may reveal hidden ideas which may be useful for further research. Thus the purpose of the history is 'to serve as an inspiration and promote the act of discovery' and not just for 'aesthetic enjoyment'.

Keywords: Indian mathematics, mathematicians, history

The book deals mainly with three periods of mathematical tradition in India: (1) the ancient period beginning approximately in BCE 1200, (2) the 'Golden Age' spanning the fifth to twelfth centuries and (3) the period of the Kerala School of Mathematics, i.e., the fourteenth to sixteenth centuries. (The book refers to this third phase as the Nila School.)

The ancient period

The principles of a decimal enumeration system of numbers date from this period. The system was perfected over a millennium and evolved as the decimal place value system for denoting numbers, namely 'the ingenious method of expressing every possible natural number as a set of ten symbols (0, 1, 2,…, 9), including zero, each symbol having a place value'. The impact of this innocent looking invention cannot be overestimated in view of its utility in commerce as well as in the development of mathematics in India and in Europe. Newton defined power series, and in particular, polynomials in one variable, in analogy with this 'new doctrine of numbers' and observed that these algebraic expressions in one variable can be manipulated (added, multiplied…) in the same way as 'common numbers'.

The main source of mathematical knowledge of this period is the *Sulbasutras* (rules of the cord), a manual for building ritual altars. The book under review contains a detailed analysis of the contents of *Sulbasutras*. The *Sulbasutras* deal with elements of plane geometry including the theorem of the diagonal (i.e. 'Pythagoras' theorem') and rectilinear figures and their transformations into one another with a given relationship between the areas of the figures. They also describe a geometric method for finding a good approximation to the radius of the circle whose area is that of a given square (and its converse construction).

Numbers like the square root of 2 and π were considered but the distinction between rational and irrational numbers does not seem to have been perceived. Indian mathematics never adopted the decimal representation of fractions throughout the course of history, as the author remarks. The conception of zero as a number in its own right as any other number (and not just a placeholder in the decimal system of enumeration) and its introduction into calculations count among the most original contributions to Mathematics from India. This topic is briefly discussed in section 5.3, 'Infinity and zero' of the book.

The Golden Age

The second period, sometimes called the Golden Age of Indian Mathematics, lasted from the fifth to twelfth centuries CE and was dominated by the names of Aryabhata, Brahmagupta and Bhaskara II. Much of the mathematics during this period was closely related to astronomy (the study of the motion of celestial objects). A great contribution of this period was the development of plane trigonometry by Aryabhata. He defined the sine function and set up the difference equation for the sine function. This work of Aryabhata was very influential and was a major input in the work of the Kerala school of mathematics which will be described later.

A subject to which all these three mathematicians made fundamental contributions is the study of indeterminate equations (also called Diophantine equations). Here one seeks integer solutions (not just rational solutions) of a polynomial equation whose coefficients are integers.

(One can also consider systems of polynomial equations of this type. This is a hard problem of much current interest. Think of Fermat's 'last theorem' concerning the integer solutions of the equation $x^n + y^n = z^n$!) Indeterminate equations of first and second degrees were studied during this period and this work received much acclaim as a high point of mathematical contributions from India.

The linear Diophantine equation of the type $ax + by = c$, where a, b, c are integers and x and y are integers to be determined, was essentially solved by an extension of the Euclidean algorithm. Such problems arose from astronomy.

Much more difficult to treat was the quadratic Diophantine equation $Nx^2 + 1 = y^2$, N being a positive integer, considered by Brahmagupta. (Centuries later this equation was named Pell's equation by Euler as a result of a misunderstanding. Pell was not the first person to notice this equation, nor did he find a solution.)

To deal with this problem, Brahmagupta considered the more general problem $Nx^2 + C = y^2$, where the integer C will be treated as an auxiliary parameter. He showed that if (x, y, C) is a solution of $Nx^2 + C = y^2$ and (x', y', C') is a solution of $Nx'^2 + C' = y'^2$ one can write down an explicit solution of $Nx^2 + CC' = y^2$.

(Thus he considered the set S of all solutions (x, y, C) for all values of C and defined a binary operation on S. Thus he defined a structure on the set of all solutions, in this case an algebraic structure, a very modern way of thinking. This operation has been called *Bhavana*.) Using this operation he found solutions for Pell's equation in some cases. While he could not solve the equation in general (which was done by Lagrange centuries later, when the integer N is not a perfect square), he found that he could solve the equation provided one can solve one of the equations

$$Nx^2 + C = y^2,$$

where $C = -1, 2, -2, 4, -4$.

Bhaskara's famous book *Lilavati* served as a basic textbook for generations of mathematicians to study arithmetic and geometry. It appears that Bhaskara wrote this book to teach mathematics to his daughter, a progressive act in an age when knowledge was passed on primarily from father to son.

His book *Bijaganita* also contains an algorithm, due to Jayadeva, to find a solution of Pell's equation, called *cakravala* or cyclic method. Starting from a known solution of $Nx^2 + C = y^2$, it sets up an algorithm to find a solution of $Nx^2 + C = y^2$, where $C = -1, 2, -2, 4, -4$. From here we can find a solution of Pell's equation, by the result of Brahmagupta alluded to above. However it seems that it was proved only in the nineteenth century that this algorithm yields the desired result.

This work is still of interest, as Pell's equation is related to quadratic number fields and binary quadratic forms. *Bhavana* is the manifestation of the multiplicative property of the norm in a quadratic number field and solutions of Pell's equation yield 'units' in a real quadratic field.

Brahmagupta had facility in dealing with negative numbers and stated the rule for multiplication of signs. This is remarkable since it took many more centuries for negative numbers to be accepted. Brahmagupta had some beautiful results on cyclic quadrilaterals, that is, quadrilaterals inscribed in a circle.

While much of mathematics during this period was driven by astronomy, the examples of quadratic Diophantine equations and cyclic quadrilaterals show that mathematics was also cultivated for its own sake.

The author gives a detailed account of the contents of Aryabhata's book *Aryabhatiya*, particularly the *Ganitapada* portion. Since *Aryabhatiya* is difficult to read, the author draws upon commentaries on the text, especially by Nilakantha. The 'kuttaka' method of solving linear Diophantine equation is explained. The ideas which go into the 'invention of trigonometry' are explained in sections 7.3 and 7.4.

For a clear exposition of the quadratic Diophantine problem, *Bhavana*, and the *Cakravala* one may refer to sections 8.2 and 8.3.

The Kerala (or Nila) school of mathematics

It was believed for some time that mathematical activity and creativity in historical India ceased in the twelfth century. As a matter of fact, during the fourteenth to sixteenth centuries there was a burst of mathematical activity in Kerala giving rise to what is arguably the finest of Indian mathematics. A small number of mathematicians living on the banks of the river Nila, in Kerala, constituted what is now known as the Kerala School of mathematics. (The author prefers to call this the Nila School to distinguish it from an earlier School in Kerala, see Chapter 9.) The founder of the School was Madhava. He and his School discovered the power series expansions for the functions sine, cosine and arctangent, and developed infinitesimal calculus for trigonometric functions, polynomials and rational functions. In particular the famous formula

$$\pi/4 = 1 - 1/3 + 1/5 - \ldots\ldots\ldots,$$

which was found by Gregory and Leibniz centuries later, was known to the Kerala School. This series is now known as the Madhava–Gregory series. Madhava has been compared with Newton and Leibniz, the discoverers of calculus in Europe.

The primary source for the path-breaking work of the School is *Yuktibasha*, by Jyeshthadeva. This text is written in Malayalam, and not in Sanskrit in which scholarly books used to be written. Divakaran, who can read Malayalam, says of the book: 'Motivations, conceptual inventiveness, technical advances (including proofs) are all given in a meticulous and sophisticated treatment in unambiguous Malayalam prose, a far cry from the enigmatic sutras of earlier masters.'

Divakaran has spent several years studying and researching material about the School. He writes knowledgeably, passionately and authoritatively about the members of the school, their work and their social background. The whole of part III of the book is devoted to a detailed and comprehensive description of the work of the School, wherever appropriate in modern mathematical language.

Divakaran has studied the continuing influences of the idea of recursion on Indian mathematics and believes it is one of the major features of *Indian Mathematics*. David Mumford says elsewhere that '*Yuktibasha* gives a unique insight into Indian methods: these are recursion, induction and careful passage to the limit'. All these three come together in the mathematics of the Kerala School.

For instance, the calculus for the sine function is built up by starting with Aryabhata's result on the difference equation for the sine function and passing carefully to the limit. This anticipates d'Alembert's dictum: 'The true meta-physics of infinitesimal calculus is nothing else than the notion of limit.'

It took more than two centuries for the work of this School to be recognized. As the author says, 'It has taken a long time for modern scholars to go from relative ignorance to puzzled admiration to an informed appreciation of the brilliance and originality of this achievement.'

Soon after the discovery of calculus by Newton and Leibniz, there was in Europe an explosion of calculus and its applications to natural sciences, powered by great mathematicians like Euler, Lagrange, Laplace, Cauchy and others. On the other hand, there was hardly any echo in India (and elsewhere) of the work of Kerala School. In fact, mathematics as a creative activity ceased to exist in India till modern times.

The rediscovery owes a great deal, on the one hand, to Indian mathematicians, K. Balagangadharan and C. T. Rajagopal among them, who studied this work and wrote (starting from the 1940s) expositions of the work in modern language, which brought this work to the attention of the international mathematical community. On the other hand (at around the same time) a critical edition of *Yuktibasha* was published by scholars. The name 'Kerala School' is now familiar to the general public in India. Could it be that the long delay in the recognition of the remarkable contributions of the school was due to the hegemony of Sanskrit, because *Yuktibasha* was written in the local language, Malayalam?

Conclusion

The last part, titled 'Connections' treats many topics which are relevant to a proper appreciation of the course and sociology of the development of mathematics and is not easy to summarize. It does not avoid dealing with vexed questions like priorities, originality, transmission of ideas, and the role of proof in Indian mathematics. These are questions which evoke much passion among historians of mathematics (and mathematicians); for instance some eurocentric mathematicians would denigrate Indian mathematics and some Indian mathematicians would exaggerate Indian contributions. The author analyses the different viewpoints and presents his own conclusions which are sensible and non-dogmatic.

He also discusses the role of faith in individual mathematicians. Some successors of Madhava and also his biological descendants are thought to have adhered to Lokayata philosophy, but about Madhava we do not have enough information to know if he also did.

Concerning the transmission of knowledge, it is striking that while Indian mathematicians were receptive to Greek astronomy, there was no influence of Euclid's 'Elements', and no trace of any of the following in Indian mathematics: prime numbers, prime factorization ('the fundamental theorem of arithmetic'), the treatment of incommensurables as in Euclid's 'Elements' and a familiarity with axiomatic and deductive methods. Other intellectual activities in India which might have been relevant for mathematics also had no influence. The author says: 'It is futile but fascinating nevertheless to ponder how a whole-hearted adoption of Paninian structural methods might have transformed India's mathematical landscape.'

As for the transmission from India, Indian mathematics, especially Algebra, was studied and developed by the Arabs (a generic term which included inhabitants of present day Iran, Central Asia and some Arabic speaking countries) and transmitted by them to Europe. The development of Algebra in the sixteenth century in Italy, influenced by the mathematics originating in India and Islamic countries, started modern mathematics and the renaissance of mathematics in Europe.

The knowledge of the decimal place value system was also transmitted to Europe by the Arabs.

Exposition

The exposition in the book is tuned to the matter under discussion. Those with little mathematical background can get a gentle introduction to what natural numbers are (Peano axioms), and what recursion and induction mean. They can also learn about the decimal system of enumeration (section 4.2). Those with some mathematical background would enjoy reading in modern notation and mathematical language, how the power series expansion for the sine function was derived by the Kerala School (section 12.2). Even someone with no interest in mathematics or history of mathematics can read with pleasure (in sections 9.2 and 9.3) a fascinating social history of Kerala at a certain period of its history.

I have passed over other topics treated in the book, like mathematics in the Indus Valley civilization, the influence of Greek Astronomy, Jaina and Buddhist Mathematics, and the Bakhshali manuscript.

The book is highly recommended for anyone interested in understanding in depth the history of mathematics of India. While the material in certain sections is somewhat densely packed, reading these sections with close attention would be a rewarding experience,

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The Closing Bracket . . .

A Tale of Two Viruses

Shailesh Shirali

At a juncture of time when the coronavirus epidemic threatens to bring social and economic activity everywhere to a sudden, grinding halt, we find that human conflicts are continuing to proliferate all across the earth, their intensity, malevolence, and stupidity undimmed.

This is not a tale of two cities but a tale of two viruses. One is a tiny being, invisible to the eye, but with the capacity to strike terror. The other is a virus that we carry in the innermost recesses of our hearts – a virus of identity, and a virus of divisiveness. It is highly active all across the Earth, and it seems to be extraordinarily virulent right now in India.

There is no need to write about the first virus here (the coronavirus); there is enough written about it everywhere. What do we do about the second virus? After thousands of years of civilisation, with stunning advances in science and technology that boggle the imagination, why is it that this virus has stayed intact in us? Why is it that in our hearts, we remain as primitive as we have ever been?

This is a magazine for mathematics educators, and we – we who spend so many years teaching children so many different and wonderful things – must put this question to ourselves: what is our responsibility in such a matter? Is our responsibility merely that of turning out technicians, people who are technically proficient at solving equations and drawing graphs and solving all kinds of baffling problems, but in whom that virus remains as active as ever, because we teachers do not ever bother to address it?

The problem of identity has become one of the most serious in modern times. Unless we take responsibility for our lives and for our behaviour, and question the very desire for identity, we are going to perpetuate what is happening now. Unless we question the very roots of our desire for identity – whether religious identity, or nationalistic identity, or ethnic identity, or any other kind of identity – we will carry this urge with us to eternity. Unless we see for ourselves how poisonous and destructive the very notion of identity is, unless we see its divisiveness with the same intensity as the pain we feel when we step on a thorn with bare feet, we will never be free of this urge.

It is our responsibility as teachers to talk about these matters with children, with the same love and dedication that we show in teaching them how to write, how to pronounce words correctly, how to compute correctly, or how to use a pair of compasses correctly.

Shailesh Shirali
Chief Editor, At Right Angles

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Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.

2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.

3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.

4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.

5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.

6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.

7. Provide a compact list of references, with short recommendations.

8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.

9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.

10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.

11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).

12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.

13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.

14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.

15. Submit articles in MS Word format or in LaTeX.

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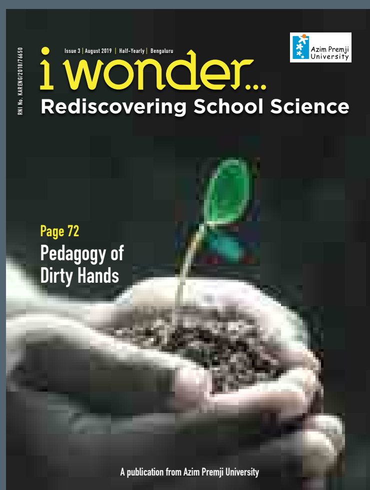

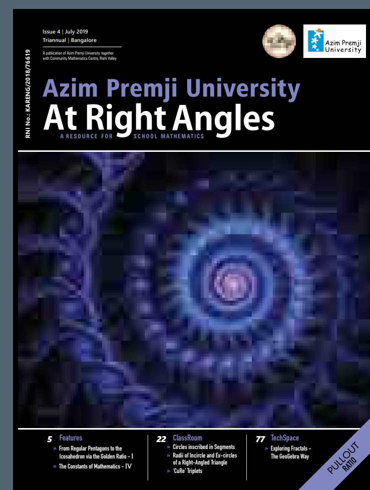

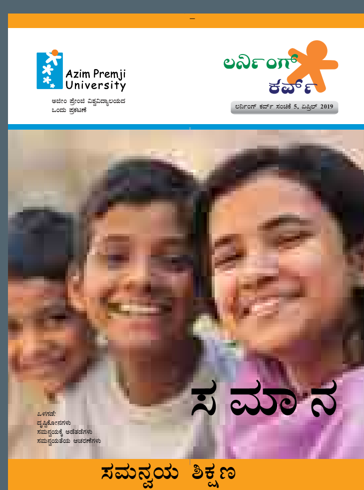

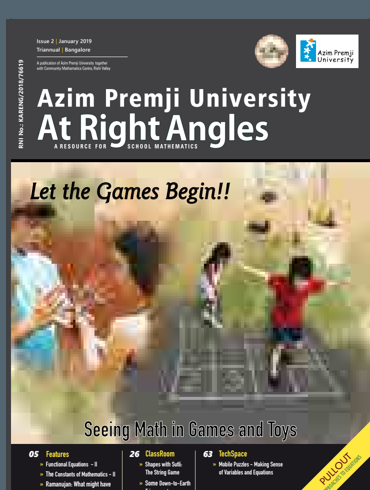

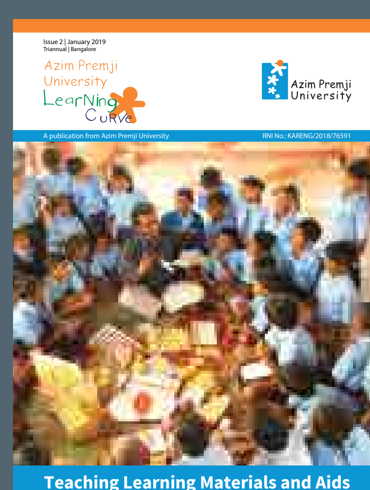

Call for Articles

At Right Angles welcomes articles from math teachers, educators, practitioners, parents and students. If you have always been on the lookout for a platform to express your mathematical thoughts, then don't hesitate to get in touch with us.

Suggested Topics and Themes

Articles involving all aspects of mathematics are welcome. An article could feature: a new look at some topic; an interesting problem; an interesting piece of mathematics; a connection between topics or across subjects; a historical perspective, giving the background of a topic or some individuals; problem solving in general; teaching strategies; an interesting classroom experience; a project done by a student; an aspect of classroom pedagogy; a discussion on why students find certain topics difficult; a discussion on misconceptions in mathematics; a discussion on why mathematics among all subjects provokes so much fear; an applet written to illustrate a theme in mathematics; an application of mathematics in science, medicine or engineering; an algorithm based on a mathematical idea; etc.

Also welcome are short pieces featuring: reviews of books or math software or a YouTube clip about some theme in mathematics; proofs without words; mathematical paradoxes; 'false proofs'; poetry, cartoons or photographs with a mathematical theme; anecdotes about a mathematician; 'math from the movies'.

Articles may be sent to :
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Please refer to specific editorial policies and guidelines below.

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'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

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While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

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On e-mail: AtRiA.editor@apu.edu.in

We welcome submissions and opinions at AtRiA.editor@apu.edu.in The policy for articles is published on the inside back cover of the magazine.

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