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Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

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Beyond Four Walls...

PULLOUT QUADRILATERALS

The legendary singer Jim Reeves sang

Four walls to hear me
Four walls to see
Four walls too near me
Closing in on me

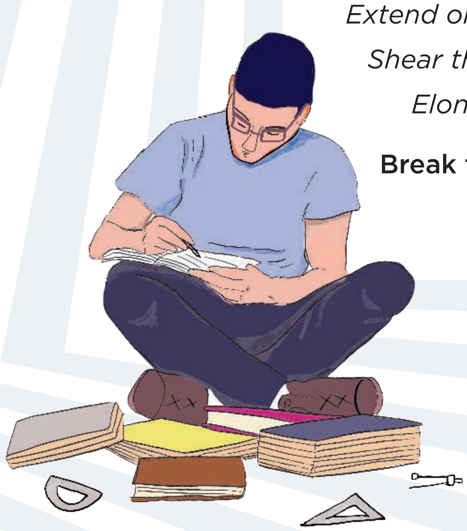
and in our lockdown situation, we can well empathise with these lyrics. What does a mathematician do when confined within the four walls of a room? Well... he could study quadrilaterals couldn't he?

Extend one side of your square to think out of the box!

Shear the square or the rectangle to tilt your world!

Elongate one diagonal to change your angle!

Break that glass ceiling, make diamonds in the sky!



From the Editor's Desk . . .

As the pandemic continues and the four walls of our houses continue to hem us in (and offer a certain measure of protection), we let our minds range freely, exploring the numerous submissions to AtRiA. We would like to thank each and every one of our contributors, some of whose articles have made it to this issue and we hope that over time, we can share with all of you most of the mathematical - and pedagogical- explorations that have been sent to us. We welcome your articles at AtRiA.editor@apu.edu.in, do read the Call for Articles at the back of the magazine to get more details. And now, read on for a quick overview of the articles in this issue.

Why do we study mathematics? Explaining this to our readers may be like preaching to the choir, but surely, lovers of mathematics have been faced with this question far too often! Rahul Tikekar lists possible answers in his article.

Teachers will find plenty of stimulating material in the ClassRoom section. From seven different methods to measure the Height of a Tree and the mysterious iteration which converges to 2997 to the more pedagogical aspects of teaching BODMAS and defining quadrilaterals, you will zoom right out of the textbook and into your students' worlds with the rich content in this section.

We have all seen the role played by technology in breaking lockdown barriers and TechSpace gives you ways in which students can learn more and teach you more. Both our articles in this section are by students and their explorations will certainly give you food for thought. From cellular automata simulating an efficient exit from a crowded space to studying the Golden Ratio using a scientific calculator, it is clear that these students have not let their minds be contained by circumstances.

An unusual series of textbooks which have been produced by the Sikkim SCERT have been reviewed this time. You are sure to be inspired by how local context, the Goals for Sustainable Development and child-centric pedagogy have converged in a government led initiative which makes these textbooks freely available to all and curricular material for the primary classes of the state of Sikkim.

Playing with Quadrilaterals, Definitions and the PullOut inspired our Four Walls theme and as usual, you will find plenty to animate your classroom and get your students thinking about Quadrilaterals and their classification based on different properties.

Get out of the box! And do send in your feedback AtRiA.editor@apu.edu.in We can also be found on our FaceBook page AtRiuM. The magazine is available for free download on https://azimpremjiuniversity.edu.in/SitePages/resources-at-right-angles.aspx

Hoping to hear from you on all, or any of these platforms!

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

The Opening Bracket . . .

The year 2020 will be remembered for many things, most of all Covid 19 and its colossal impact: the lockdowns, the school closures, and the wholesale switch to online teaching. But other notable events took place. One such is the publication of the New Education Policy 2020. A study of the NEP document reveals that it is comprehensive and forward-looking. At the start, it tells us that "… The purpose of the education system is to develop good human beings capable of rational thought and action, possessing compassion and empathy … with sound ethical moorings and values." There is much in it to feel happy about, for example: examinations for classes 10 & 12 are to be conducted twice a year, giving students a second chance to write them; the curriculum is to be reduced to core concepts, with increased focus on critical thinking and conceptual clarity; assessment is to be holistic, taking into account extracurricular and co-curricular activities. Further, from class 6 onwards, there are to be opportunities for vocational training.

However, there are also aspects to be concerned about. One such is the possibility of excessive control from the Centre, with states having very little say in education. Of particular concern is the fact that there are to be Board exams in classes 3, 5 & 8. One can well imagine the 'exam culture' that this will bring about even in the primary classes. There is no mention anywhere in the NEP of the fact that our society is class-ridden and steeped in prejudice, and that education needs to address these matters. These are not the only areas of concern, but for now, we shall not go further into the question.

Let us now take up an important theme which NEP has addressed: *assessment*. It plays a vital role in education, but it has taken a wrong turn, because we have lost sight of the central purpose of education – which, surely, should be to enable the inner growth of the child: learning how to think; learning how to look without an image; learning how to listen without judgement; learning about kindness, learning about generosity, learning about simplicity. But when success and status dominate our mindset, then these aims get thrown out of the window. Assessment, whose aim should be to gain insight into how the student thinks, gets narrowed down; it becomes a mere measurement of skill and mastery of technique. It becomes authoritarian and ruthless, and reinforces hierarchy and power relationships. An unfortunate consequence is the phenomenon of 'teaching to the test' (we have whole industries devoted to this phenomenon). Note that all these are mere corollaries to the factory model of education, which itself is a legacy of the industrial revolution.

What can we do to bring sanity into the educational system, so that assessment becomes a natural part of the learning process? To start with, we as a body of teachers need to ask difficult questions of ourselves (and not accept the answers given by experts, however established they may be). *This is our responsibility*.

It is also our responsibility to work towards ensuring that success and failure do not dominate children's minds. We *must* do this. Why does this matter receive such little attention? Should we not be talking more about it?

Shailesh Shirali
Chief Editor

Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

Student Corner

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online

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resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Why should you study Mathematics?

RAHUL TIKEKAR

Introduction and Motivation

My daughter inspired me to author this essay. She will soon begin her life as a university student. I knew this day would come when she would leave home for college. As a parent, and more specifically, an Indian parent, I tried to steer her toward the engineering and computer science fields. Not only do these fields offer the most opportunities but my wife and I believe that our daughter would do very well and contribute to them. Instead, she is choosing to study mathematics.

I admire my daughter for making this choice. Mathematics is one of the oldest subjects and forms the basis for all problem solving. However, mathematics is not the first choice for many incoming college freshmen. It is, I wager, not their second, or even their third choice. In my opinion, there are three common reasons, or myths, behind this.

First, the general belief is that mathematics is a difficult subject and most of us do not have fond memories of our mathematics classes from school. Second, a student of computer science will pursue a career of a computer scientist; a student of aerospace engineering will seek employment as an aerospace engineer. What is the equivalent profession for a graduate with a degree in mathematics? We do not often see employment adverts for a 'mathematician.' Most of us assume that the logical profession for those with a degree in mathematics would be that of a teacher at a school or college, or at the very high end, that of a professor at a premium research institution. Third, compounding this is the hard reality that none of these professions offer the promise of making you a wealthy person as do other professions like IT and medicine.

Keywords: Mathematics, motivation, history, vignettes, applications of mathematics

How can one then be motivated to study mathematics? This article attempts to break these myths and supply an answer to the question an aspiring college student would ask of us: why should I choose a degree in mathematics and a career as a mathematician?

Let me first get this notion out of the way that mathematics is a difficult subject. I am, by profession, a computer scientist. I study algorithms, applications, systems software, and hardware. When I tell people about my profession, most confide in me that they do not understand computers. Many others confess that they have found programming to be difficult. I find biology and medicine, with all those complex and difficult to spell terms, like hydroxychloroquine, to be harrowing. So, difficult is a relative term: one person's difficult is another person's easy. Once a subject is understood, it no longer remains hard or mysterious. To me, all it takes is motivation and perseverance - this applies to any field, not just to mathematics.

So, why *should* you study mathematics?

To Shake the World

Mathematics offers you the opportunity to achieve greatness. One enviable property of this subject is that discoveries in this field are applicable to so many other areas of life. I will provide three examples, two historic, and one relatively recent, of mathematicians whose discoveries have made incredible and lasting impacts on the world.

Newton [1]

Isaac Newton is generally accepted to be one of the most influential human beings of all time. In 1665 he was a relatively unknown undergraduate student at Trinity College in Cambridge when England was hit by the great plague. Newton had to move out of Cambridge and into his grandmother's home in the country. Most of Newton's great discoveries came during the next two years that he lived at his grandmother's. Influenced by the work of astronomers like Galileo, Newton was studying the motion of planets. As part of this study, Newton was trying to pin down the value of the slope of a curve. This was easy: it is the ratio of rise to the run. However, in the case of a celestial object like a planet, this slope is constantly changing. There was no easy method to finding the exact slope at a given point in time. So, Newton developed a new theory, that he termed the study of fluxions, and what we know today as calculus.

Newton was not sure if his theory – the derivative of a function – would find acceptance from his peers. He kept his ideas to himself and shared them only with a few of his friends. If even Newton could suffer from such insecurities, it is easy to understand our own trepidations. Even though it was not Newton's goal to shake the world, calculus has made an impact on so much of our life that no one graduates from high school without, at least, an introductory course in calculus.

Euler [2]

The year is 1736 and Königsberg is a prosperous town in Russia. The river Pregel runs through the city creating two sides, say A and B. Additionally, the river has created two islands between A and B, say C and D, as it curves through Königsberg. There are seven bridges that connect C and D to each other and to A and B on either side of the river (see Figure 1). A story goes that the mayor of Königsberg enjoyed taking a walk through the city every Sunday. To make his walk more interesting the mayor tried to design it so he would cross each of the seven bridges, but only once. No matter how hard he tried, he found that to be impossible – there was always a bridge that he needed to cross a second time to return home. In his frustration, he wrote to the famous mathematician Leonard Euler who lived in the nearby city of St. Petersburg and sought his help.

Initially, Euler dismissed the problem as trivial because it bore no connection to mathematics but was nonetheless still intrigued by it. He modeled the problem as a graph – a concept that was not invented yet. The landforms, A, B, C,

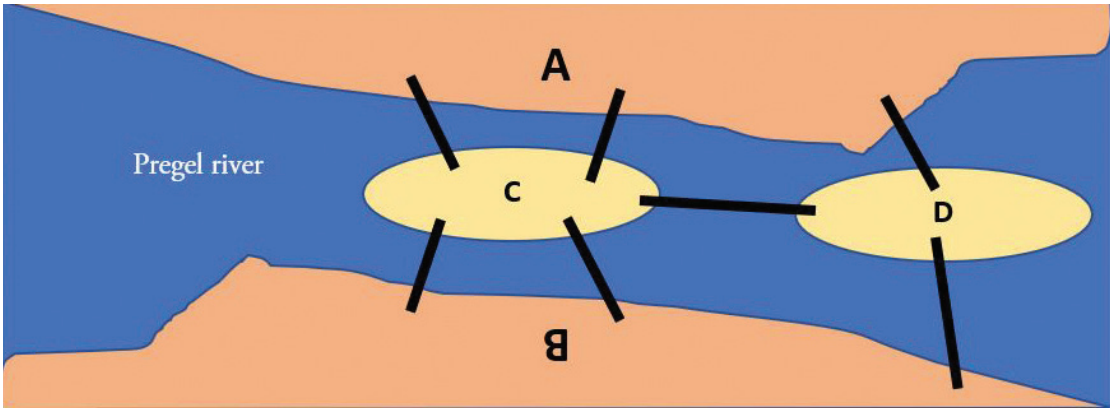

Figure 1

D represented nodes of the graph and the seven bridges connecting them became the edges (see Figure 2). Euler proved that in order to cross each edge just once, the graph must contain either zero or two nodes that have an odd number of edges that lead into them. Because the Königsberg graph had four nodes with an odd number of edges leading into them, it was impossible to come up with a walking path that crossed each bridge just once.

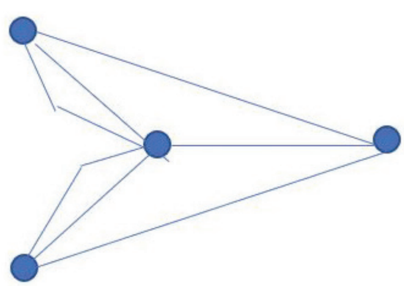

Figure 2

Euler's discovery led to the development of graph theory and topology, and today graph theory has permeated so many branches of knowledge like computer science (graph databases), sociology (social network analysis), linguistics (language analysis), biology (protein and DNA analysis), transportation (shortest path problem), and national security (terrorist and financial networks).

Von Neumann [3]

Poker is a card game that relies on your ability to bluff and deceive your opponents into believing that you hold a stronger hand than them. In 1929, a brilliant young mathematician, Jon Von Neumann, moved to the United States from Germany to take a position at Princeton University. Von Neumann liked playing poker but was not particularly good at it. Poker inspired Von Neumann to formalize the theory of bluffing into what is known today as Game theory.

Game theory seeks to provide a foundation for instances involving multiple parties that make decisions based on predicting the moves of the opposing parties. Situations involving competing players are modeled to produce strategy or help reach an optimal decision. Game theory is a widely used tool in economics. A testament to the power of Game theory is the many Nobel prizes awarded to economists who have applied game theory principles and its refinements to aspects of economics. 'A Beautiful Mind' is a must-see film made on the life of one such Nobel prize winning Game theory economist, John Nash. Game theory finds use in many other fields like computer science (networks and resource allocation), biology (coexistence of wildlife), government (auditing taxpayers, warfare), psychology and, of course, politics.

In all fairness, Von Neumann had an advantage over most other people: like Ramanujan, he was a child prodigy. He possessed a photographic memory, spoke fluent Greek, and by the age of eight was familiar with calculus! So, while Von Neumann may not be the best motivational figure, this story serves to impress upon the reader that mathematicians hold the power to make a big impact on the world.

You may question the fact that the most recent example given is still from a century ago. Aren't there any more recent examples of mathematical discoveries that have 'shaken' the world? The reason behind this is the lag between making a discovery and finding a problem to which it can be applied. It takes time before a result can find its way into other areas. Do not let that discourage you. That is partly the reason why most Nobel prizes awarded are for discoveries the recipients made several years before the award recognized them.

To Solve Common Problems

If shaking the world is not your thing, there still are accomplishments that you as a mathematician can make that will advance a field and help humankind make progress. In this section I shall present three examples of mathematics applied to real world problems that are helping advance our society.

Cryptography [4] (1977)

Cryptography, or the art of obscuring a message so only the intended parties can understand it, is not new. Even Julius Caesar used it for communicating with his generals. However, finding ways to encrypt and decrypt messages that even computers cannot break is challenging. But mathematics, again, came to the rescue of the world.

Whitfield Diffie, Ron Rivest (R), Adi Shamir (S), and Leonard Adelman (A) – all with basic degrees in mathematics – along with Martin Hellman and Ralph Merkel designed the public key cryptographic system to encrypt and decrypt messages. The RSA algorithm uses the concept of prime factors to generate a one-way function (a function that is easy to implement in one direction but almost impossible to perform in the reverse direction) to make this happen. The basic premise is this: you are given a very large number and the challenge is to find two prime factors of that number. If the answer (the two primes) is given to you, then finding their product is trivial, but going the other way is almost impossible for humans and even the fastest computers will take years to find the two primes.

It was not until the invention of the Internet, almost twenty years later, that public key cryptography found widespread use (the lag I mentioned previously). All the applications we use today and take for granted when we browse the Internet, make phone calls, send messages, protect our files, etc., are the result of the efforts of mathematicians applying their skills and knowledge to common problems.

An interesting fact is that a British mathematician, Clifford Cocks, while working for the UK spy agency GCHQ had already developed the idea of public key cryptography. The British government kept the invention classified until finally acknowledging it 24 years later in 1997. Rivest, Shamir, and Adelman independently developed the same result and received the fame and patents that came with it.

Cricket [5] (2000)

When I was a young boy growing up in India, cricket was the only thing that interested me. I could not get enough cricket. If listening to cricket commentary on the radio could nourish a body, I would easily skip lunch. Cricket had one point of universal frustration: the Leg-Before-Wicket (LBW) decision – there was no way to tell if the umpire's decision was correct: that, if not impeded, the ball would have hit, or not hit, the wickets. Good sportsmen usually begrudgingly accepted the umpire's decision if it went against them.

Paul Hawkins, a British student of mathematics and artificial intelligence, faced a similar bad umpiring call during a cricket match. But instead of just getting mad, he decided to do something about it. He invented Hawk-Eye, a system that uses high speed cameras and a complex computer program to predict the path of a ball. It was the cricket commentators who initially used Hawk-Eye to determine if the umpire got it right. TV viewers saw the projected path of the ball superimposed on the image of the game. Cricket governing bodies soon adopted Hawk-Eye as an official tool for use during appeals and times when the umpire is unable to make the call. Sony bought the Hawk-Eye system from Hawkins for millions of dollars. Today, Hawk-Eye has found use in other sports like tennis and soccer.

So how does Hawk-Eye use mathematics? Hawk-Eye uses concepts from Epipolar geometry, linear algebra, and statistics. Many high-speed cameras, separate from the TV cameras, record the ball and then compute its most likely path. Epipolar geometry helps by recognizing corresponding points on two separate images of the same object. Triangulation is the process of identifying the location of a point in 3D space given its location in two or more 2D spaces. In this case, each frame from every camera contains the picture of the ball in 2D space. The system takes that position from one camera and compares it with the position of the ball from other cameras for the same frame. Triangulation then computes the 3D coordinate of the ball. Triangulation is repeated for each frame yielding a set of 3D points that form the path of the ball. Finally, the projected path of the ball is computed to determine if it will hit an object in that path, like the wickets.

Natural Language Processing [6] (2013)

Natural Language Processing (NLP) is the ability of computers to understand human language, be it written or spoken. The last few years have seen incredible advances in this field. Google, Amazon, Apple have products like 'OK Google', Alexa, and Siri that allow us to speak commands to perform actions. You may have noticed Gmail making suggestions for completing your sentences as you type your email. In this case, based on the context of your sentence, Gmail is predicting your next set of words.

Neural Networks are at the center of much of the advances in NLP. While neural networks themselves are based on mathematical concepts of linear algebra and calculus, for this topic, I am going to focus on one aspect of NLP: word embeddings.

Consider the following two sentences:

- I had a wonderful time at the party
- I had a great time at the party

To us, the two sentences express very similar sentiments. However, for a computer the words 'wonderful' and 'great' are very different and making the inference that the two sentences are very close semantically is very difficult. This is because neural networks take as inputs numbers and not characters or words. So how do we represent the words as numbers and then how can we compare their closeness? The answer is word embeddings. Simply put, word embeddings are vector representation of words as in the example below:

Wonderful: [12, 17, 15, 19, 27, 53]

Great: [12, 16, 16, 22, 25, 60]

The words Wonderful and Great are represented as vectors of 6 numbers each (or a 6 dimensional vector). Now that words are represented by vectors, how do we tell if these words are "close" in meaning? Mathematics, once again, provides the solution: cosine similarity. Cosine similarity or cosine distance is a measure of how close two words are and is mathematically computed as the cosine of the angle between the two multi-dimensional vectors. The smaller the distance, the closer the words.

You are probably wondering how I came up with the values in the vectors. In my example, the numbers are made up to illustrate the point. However, from an implementation standpoint, a group of Google researchers led by Czech computer scientist Tomas Mikolov invented the word2vec algorithm in 2013 that generates the vector for a given word.

To Become Rich and Famous

There are many opportunities to achieve fame and fortune through Mathematics. Here are four such avenues.

The Clay Mathematics Institute[7] has identified seven problems, called The Millennium Prize Problems, which have intrigued mathematicians in recent times. The Clay Institute considers these problems sufficiently hard to warrant a prize of $1M (US) per problem for a solution. To date the only Millennium Prize Problem solved is the Poincare Conjecture (from the field of topology, thank you Euler!). The winner of this award, a Russian mathematician named Grigori Perelman, declined to accept the prize money.

If you were wondering why there is no Nobel prize in mathematics, you are not alone. In 1899 when the Swedish philanthropist Alfred Nobel did not create a prize in mathematics, a Norwegian mathematician Sophus Lie proposed to create one. The monarch, who at that time was the king of both Norway and Sweden, agreed to fund the award for mathematics. Unfortunately, Norway and Sweden decided to dissolve their union under a common regent before the prize was instituted. It was not until 2002 that the Government of Norway finally established the Abel prize for contributions to mathematics.

The Abel [8] prize is the Nobel equivalent for mathematics and the cash award is comparable. While the Royal Swedish Academy picks the Nobel winner, the Norwegian Academy of Sciences and Letters picks the winner of the Abel. Remember the Nobel Prize winning Game theory economist John Nash? He won the Abel in 2015. One Indian, S.R. Srinivas Varadhan Iyengar, won the Abel in 2007.

The Fields Medal [9] is another prestigious honor in mathematics. The International Mathematical Union chooses the winners once every four years. Candidates for this award must be under 40 years and are recognized not just for their outstanding contributions to mathematics but also for showing promise of future accomplishment.

The Putnam Competition [10] is an annual mathematics test that is open to undergraduate college students in the USA and Canada. Organized by the Mathematical Association of America this is considered by many to be the toughest mathematics test in America and in the world. There are 12 questions worth 10 points each for a total of 120 points. The test is so hard that the median score is between 0 and 1! The prize money is not high, but the prestige is enormous: the top 5 students are designated Putnam Scholars. These students get scholarships to some of the top universities. Many employers look to steal these scholars by making very generous offers.

With a love for mathematics coupled by hard work and dedication, it is not only possible to become famous and find lucrative jobs, but also win enough money to retire from that lucrative job!

To Become a Problem Solver

A mathematical model is a representation of a system using mathematical symbols, objects, and operators. Solving a real-world problem starts with modeling the problem: stating the problem in a mathematical format. Once you have a model, you apply mathematical principles to solve that problem.

A mathematics education will teach you clarity of thought and good problem-solving skills. More formally, a mathematics education will help you gain expertise in developing mathematical models from problem descriptions, then apply mathematical principles to solve those problems (or as in the case of the Königsberg problem, show that the problem has no solution). It follows that this incredible skill will prove useful in almost any situation, even a domestic one as I will now illustrate. This is a problem that I faced just a few weeks ago.

I want to fertilize the lawn in my front yard. The fertilizer comes in liquid form and I have a spray bottle by which I can apply the fertilizer to my lawn. Here are the basic steps:

1. The spray bottle dispenses 2 litres of water per minute
2. A dial on the spray bottle can be set to mix from 1 tsp to 10 tsp of fertilizer per litre
3. Fill the spray bottle with the required quantity of fertilizer and adjust the dial
4. Attach the water hose to one end of the spray bottle
5. Open the tap and when the water starts flowing, spray the lawn from the other end of the spray bottle
6. Fertilizer from the spray bottle will mix with the water from the hose and feed the lawn.

Question: How much liquid fertilizer do I add to the spray bottle and what dial setting should I use? I do not want to overfeed my lawn and at the same time I do not want to pour back any remaining fertilizer in the spray bottle back into the fertilizer container. Ideally, when I have fertilized the entire lawn, I want the spray bottle to be empty.

Here are the other facts you will need:

1. The area of the lawn in 2000 ft^2
2. The fertilizer instructions say to use 3 tsp mixed with 3 litres of water per 1000 ft^2
3. It takes me about 1.5 minutes to walk from one end of the lawn to the other

My solution was to fill the sprayer with 6 tsp of fertilizer, set the dial to 1 tsp, and cross my lawn twice. You can verify my solution. If it is incorrect, then you know I did not graduate with a mathematics degree. My lawn, however, is doing well.

To Love the Subject for the Sake of the Subject

It is not necessary to love a subject because it offers job opportunities and the promise of big money. Mathematics can keep an interested student happy with a lifetime of problems that have absolutely no bearing on real life. Mathematicians enjoy the subject for its sheer beauty.

One such beautiful flower from the garden of mathematics is the one proposed by French mathematician Pierre de Fermat (Fermat's Last Theorem). He wrote in the margins of a book he was reading: there are no three positive integers X, Y, Z that can solve the following equation:

$$X^n + Y^n = Z^n \text{ for values of } n > 2.$$

An uncorroborated story is that Fermat claimed he knew of a proof but there wasn't enough room in the margin to write it out. Fermat wrote this in 1637 and for over 350 years no one was able to "find" that proof that Fermat thought he had. This conjecture finally became a theorem in 1995 when British mathematician Andrew Wiles, working at Princeton University, supplied the proof. However, the techniques that Wiles used in his proof were not discovered at the time of Fermat.

Another problem that kept mathematicians engaged for a long time was the one stated by South African mathematician Francis Guthrie in 1852. He was coloring a map of the counties in England and noticed that only four colors were sufficient to complete the task. His observation turned into the four-color conjecture: only four colors are sufficient to color a map to ensure that no two adjacent regions are colored with the same color. The proof eluded mathematicians for over 145 years and when the proof was finally published, it did not use maps at all; in fact, it used graphs! Once a proof is proposed, it needs to be verified. This was the first proof that used a computer assisted technique. This technique helps verify the correctness of the proof in a much shorter period than hand, paper, and pencil techniques. Not all mathematicians accept this method of proving theorems.

Two Indian mathematicians deserve special mention under this section for loving mathematics for the sake of the subject. One is the prodigy Srinivasa Ramanujan [11] who lived in the late 19th century - Ramanujan whose contributions are so profound that scholars are still trying to make sense of his work. So deep was his work that it is said that had Ramanujan not lived, nobody would have discovered many of the things that he did. Ramanujan worked very differently from the traditional mathematicians of that time and of today. Traditionalists begin by stating proven results, then extending them, carefully justifying each step. Ramanujan's approach was different: for him Mathematics was pure divinity where concepts just came to him like water from a hose; practicality did not matter, nor did formal proofs. In my opinion, the need for "providing methodical proofs" may even have dampened his capacity for producing more of what he eventually did.

The other is D.R. Kaprekar [12] who was a math schoolteacher in rural India. My father took me to several of Kaprekar's lectures. His enthusiasm for the subject was palpable. He was a man of very modest means: his apparel consisted of a worn white Indian-style *dhoti* held together by a rope, a shirt, and a jacket. He loved numbers and, in his lectures, showed us many interesting properties of numbers that he had discovered. He put in a lot of labor in playing with numbers to find enthralling qualities about them. The number 6174, named after him, is known as Kaprekar's constant. It has a property derived from the following rules:

1. Take any four-digit number, not all digits being equal (e.g., 1234)
2. Rearrange the digits to create two four-digit numbers: one the largest possible and one the smallest possible with those digits (4321 and 1234)
3. Subtract the smaller from the larger (3087)
4. Go back to step 2 and repeat.

This process will, in at most 7 iterations, always reach the constant 6174. Clearly, neither the number nor the process has any real-world implications. Also, where do you begin to find an interesting characteristic like this? Questions such as these did not dissuade Kaprekar; he found beauty in numbers and numbers engaged him for many years. Confessing his addiction for numbers, he would say: "A drunkard wants to go on drinking wine to remain in that pleasurable state. The same is the case with me in so far as numbers are concerned."

Mathematics has the quality to engross you for years and it is yours for the taking. Numbers, anyone?

Conclusion

Galileo has said that "Mathematics is the language in which God has written the universe." Logic tells us that mastering this language will help you understand and decode the universe. Mathematics has the power to discover knowledge that will pave the way for our society's growth and progress. The myth about job opportunities can now be busted: many companies like Google, Facebook, Amazon, Tesla, SpaceX, and organizations like NASA are hungry for graduates with degrees in mathematics. All the technological advances that are happening around us are a result of applying mathematical techniques, results, and models to common problems. Mathematics is the foundation on which these advances rest and it makes sense to hire mathematicians to understand and build this foundation.

My one regret in authoring this essay is not including achievements of women in mathematics. It is unfortunate that only men have been the major players in the history of mathematics. Another amazing film 'Hidden Figures' tells the story of African-American women mathematicians (called computers because they performed computations) who worked at NASA during the years of the space race. While the movie celebrates those women computers, it also points out discrimination they faced, both from men who thought women were not good enough, and from whites who thought black people were not good enough. Our society is evolving, and more women are taking on challenges traditionally tackled by men. Going forward, women will write many of the stories of developments in mathematics. I just cannot wait for that.

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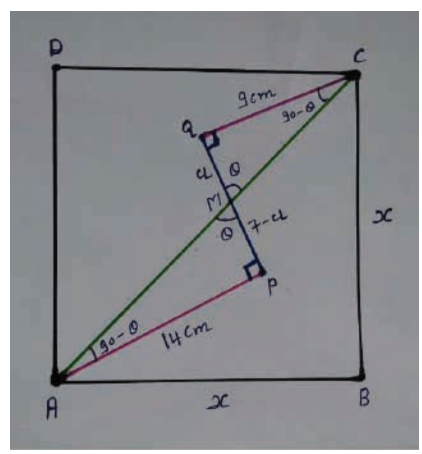
Think out of the box!

Problem on page 68, July 2020 issue:

Find side x of the Square



Solution: As shown in the figure, Let **QM = a** and **PM = 7 – a.** Now we know that ΔAPM and ΔCQM are similar triangles.

$$\therefore \frac{\mathbf{CQ}}{\mathbf{AP}} = \frac{\mathbf{QM}}{\mathbf{PM}} = \frac{\mathbf{CM}}{\mathbf{AM}}$$

$$\therefore \frac{9}{14} = \frac{a}{7-a} = \frac{CM}{AM} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore a = \frac{63}{23}$$

Now $\mathbf{PM} = 7 - a = 7 - \frac{63}{23} = \frac{98}{23}$

$$\therefore \mathbf{PM} = \frac{98}{23}$$

According to the Pythagoras theorem, we can write

$$AM^2 = AP^2 + PM^2 = 14^2 + \left(\frac{98}{23}\right)^2 = \frac{113288}{529}$$

$$AM = \frac{238\sqrt{2}}{23}$$

Now $CM = \frac{9}{14} AM = \frac{9}{14}\left(\frac{238\sqrt{2}}{23}\right) = \frac{153\sqrt{2}}{23}$

Again, according to the Pythagoras theorem

$$x^2 = \frac{AC^2}{2} = \frac{(AM+CM)^2}{2} = \frac{\left(\frac{238\sqrt{2}}{23} + \frac{153\sqrt{2}}{23}\right)^2}{2} = 289$$

$$\therefore x = 17$$

Solution sent in by Tejash Patel, Chanasma Primary School No.2, Patan, Gujarat.

For another Solution please see page 86

Age Digit Reversals

JAMES METZ

Razvi was born in 1974 and his father was born in 1947, so the tens and units digits of their birth years are reverses of each other. In 2021 Razvi will be 47 years old and his father will be 74 years old, again reverses of each other and also reverses of the tens and units digits of their birth years.

We can prove the more general case for a pair of years in the same century, C.

- Let x = the tens digit of the year of birth of the father, which will be the units digit of the year of birth of his son.
- Let y = the units digit of the year of birth of the father, which will be the tens digit of the year of birth of his son.
- Year of birth of the father: $C + 10x + y$.
- Year of birth of his son: $C + 10y + x$.

In year $C + 10x + y + 10y + x$, the age of the father will be $10y + x$ and the age of the son will be $10x + y$, so their ages are reversed. Of course, $10x + y$ must be less than $10y + x$.

In the 20th century, besides 1947 and 1974, what other pairs of years have this reversibility feature? If we make a list, starting with the pair 1901 and 1910, then 1902 and 1920, we soon see that the age difference is a multiple of 9, in fact 9 times the absolute value of the difference of the tens and units digits. We will assume that the age difference must be at least 18 and at most 63. In the first decade, there are 6 practical pairs: 1902 and 1920, with an age difference of 18, 1903 and 1930, with an age difference of 27, continuing until 1907 and 1970, with an age difference of 63. In the second decade there are also 6 pairs, starting with 1913 and 1931, and ending with 1918 and 1981. Likewise, the third decade

Keywords: Age, place value, reversing digits

has 6 pairs. There are only 5 in the next decade, between 1935 and 1939, then 4 between 1946 and 1949, continuing with one less pair each decade until there is only one pair, 1979 and 1997, for a total of 33 practical pairs in a century.

The Magic of 11

Returning to Razvi and his father, because of the magic of 11, their ages are also reverses of each other every 11 years before and after 2021. Since we know the ages are reverses in the year $C + 10x + y + 10y + x = C + 11x + 11y = C + 11(x + y)$, we know that years of the form $C + 11n$ will have the age reversal property, as shown.

When the father is born in 1947 and the son in 1974: difference of ages is 27 years		
Year	Age of Father	Age of Son
1977	30	03
1988	41	14
1999	52	25
2010	63	36
2021	74	47
2032	85	58
2043	96	69

Let us observe another case:

When the father is born in 1926 and the son in 1962: difference of ages is 36 years		
Year	Age of Father	Age of Son
1966	40	04
1977	51	15
1988	62	26
1999	73	37
2010	84	48
2021	95	59

In general, if the difference in the ages is $9k$, with $k = 2, 3, 4..., 7$, there are $10 – k$ age reversals.

This investigation began with a simple observation that opened the door to some beautiful mathematics. Indeed, mathematics resides everywhere.

Reference

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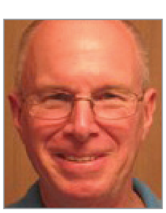

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BODMAS: Much ado about nothing?

ANUPAMA S.M.

BODMAS is taught at upper primary level in most of the states. Teachers have constantly expressed the need for support in teaching BODMAS and their questions have mainly centred around:

- Is this a convention?
- Why the hierarchy in the order of operations?

What is BODMAS?

BODMAS, PEMDAS or PIMDAS are acronyms used as mnemonics for the convention of "the order of operations" in evaluating arithmetic expressions.

According to this convention, when a student encounters an arithmetic expression with more than one operation, the student simplifies the expression in this order:

- First, the Brackets
- Then Exponentiation
- Then, Division or Multiplication whichever comes first when reading from left to right.
- Finally, Addition or Subtraction whichever comes first when reading from left to right.

Note: In 'PEMDAS', E stands for the word Exponent which fits in better than the commonly used 'Of' for the O in BODMAS, which can confuse students. Some teachers refer to the 'O' in BODMAS as Order of exponent and not as 'Of'. If that is used 'BODMAS' can be recommended as the word 'Bracket' is simpler than 'Parentheses' to students. Whatever the choice, it is suggested that the teacher make consistent use of only one acronym.

Keywords: BODMAS, PEMDAS, PIMDAS, arithmetic operations, sequence, convention

Example: In order to simplify $2 \times (3 - 5) + 7^3$

According to BODMAS, we simplify what is in the Bracket first: $2 \times (-2) + 7^3$

Then Order of Exponentiation: $2 \times (-2) + 343$

Then Multiplication: $-4 + 343$

Then Subtraction: 339

Issues and Misconceptions

Many students accept this convention blindly mainly because they are struggling with all the symbols and mistake the convention for a supportive strategy. A few of them question the teacher on the reason for this convention and are often dissatisfied with the responses they get. The fact remains that this convention has become a bogey for both students and teachers and when multiple operations with fractions are introduced, an even greater burden is laid on students to remember the convention and apply it correctly. Obviously, understanding more about the reasoning behind the convention would help a student process the problem better.

Why BODMAS in this order?

Let us begin with a single operator between two or more numbers.

Say, $6 + 2$, 6×2, $6 \div 2$…

Or $6 + 3 + 4 + 5 + 3$……

Or $3 \times 2 \times 6 \times 8 \times 5$….

These are clear to the student, particularly when applied to different contexts related to daily life situations.

However, in real life, the child often comes across situations in which more than one operation is applied. Such as this:

Amma gives 2 biscuits each to Ravi, Reena and Manith. There were 4 more biscuits in the box. How many biscuits were there in all?

a. We could solve this as $2 + 2 + 2 + 4$ and easily get the answer as 10.

b. The same problem could be written as: $2 \times 3 + 4$

Possibilities-

$6 + 4 = 10$
(Here multiplication is followed by addition)

$2 \times 7 = 14$
(Here, addition is followed by multiplication)

We have already seen using only addition that the correct answer is 10. The repeated addition of 2 was condensed as 2×3 and then this was simplified as 6 to which 4 was added. Since the context of the problem was known, it was natural that we could identify the correct answer.

What if the context is not known? Do we simply read from left to right and simplify as we go along?

Let us consider $2 + 3 \times 4$

Options:

i. $5 \times 4 = 20$ (Here, addition is followed by multiplication)

ii. $2 + 12 = 14$ (Here, multiplication is followed by addition)

How would we decide which option is correct?

Clearly, a robust convention is needed. Let us begin with the elementary operations of addition and subtraction.

$3 + 3 + 3 + 3 + 2 + 2 - 7 + 4 - 7 - 7 + 5 + 4$ …. (1)

In this case we could evaluate left – right or right – left or in mixed order. However we do it, we arrive at the same result, provided we use the addition and subtraction of integers correctly.

Knowing that multiplication is repeated addition, the above string could be condensed to:

$4 \times 3 + 2 \times 2 + 3 \times -7 + 2 \times 4 + 5$,
which reads as

3 added 4 times added to

2 added 2 times added to

– 7 added 3 times added to

4 added 2 times

added to 5.

This simplifies to $12 + 4 - 21 + 8 + 5 = 8$

This made our calculation simpler and easier but along with it came an order of operation-precedence of multiplication over addition and subtraction.

Similarly, division being repeated subtraction, it takes precedence over addition and subtraction.

The above example has illustrated that multiplication is repeated addition. Similarly, division is repeated subtraction and exponentiation is repeated multiplication.

This hierarchy has led to the BODMAS convention. Addition and subtraction have the same order in this hierarchy. So do multiplication and division; there is however, a caveat which will be explained a bit later.

This is true for the set of real and complex numbers.

So, this gives a rationale for the origins of the convention ODMAS. What about Brackets? When this hierarchical order needs to be broken, brackets are used to enclose those terms which need to be simplified first.

So, if the given expression in (1) had been: $3 - (5 + 3) - 5 + 7 \times 7 \times 7$, then it would have simplified as:

$3 - 8 - 5 + 7 \times 7 \times 7$

Or $3 - 8 - 5 + 343$

$= 333$

Finally, we consider the internal hierarchy between multiplication and division.

Consider the case of $12 \div 3 \times 2$

Options:

a. $4 \times 2 = 8$ (Here division is followed by multiplication)

b. $12 \div 6 = 2$ (Here multiplication is followed by division)

Though multiplication and division have the same ranking in the hierarchy of operations, clearly we need a convention so as to avoid any ambiguity in simplification. In order to maintain uniformity, the convention is to carry out whichever operation (division or multiplication) comes first as we read from left to right.[1] So, by the BODMAS convention, the correct answer in the above problem is 8 and not 2. The subtlety of this internal hierarchy has caused several controversies, one can read more about it with a simple Google search. Some such are given in the links https://www.popularmechanics.com/science/math/a28569610/viral-math-problem-2019-solved/ and https://twitter.com/kmgelic/status/1155598050959745026

With the correct application of the BODMAS rule and the caveat of reading from left to right when it comes to which of division or multiplication should be applied first, there is no ambiguity in the answer. An incomplete understanding of BODMAS leads to such unnecessary confusion.

Note: In the case of addition and subtraction it does not matter. $5 + 3 - 2$; $8 - 2$ or $5 + 1$ would give the same result, assuming of course, that the rules of operations with integers are applied correctly.

Suggested pedagogical approach:

I. Modelling through word problems

Word problems with 2 operations could be a good start to bring context to symbolic form and thus evaluating to arrive at the answer.

Example 1: Sanu packs 12 apples in a box. She was able to fill 3 boxes, but 2 apples were left out. How many apples did she have?

This in symbolic form is,

$3 \times 12 + 2$

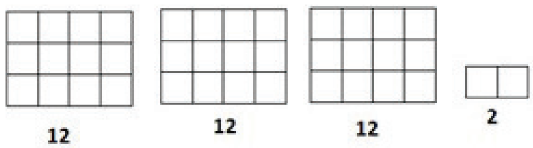
By the given convention, we need to multiply first followed by addition.

$36 + 2 = 38$

1 Though this discussion is confined mainly to integers, the arguments may be extended to rational numbers at a later stage and at that point, it could be pointed out to students that all division ($\div$ n) are multiplication ($\times$ 1/n) and similarly all subtraction ($-$ n) are addition (+ ($-$n))

Note: While modelling the problem,

(i) The child would first draw out the situation



(ii) And then evaluate the answer step wise:

3 x 12 = 36

36 + 2 = 38

Eventually, the teacher could get them to process the solution directly from the arithmetic expression without resorting to a visual representation.

Example 2: 3 out of 24 mangoes in the basket were rotten. Chandar and his two friends share the remaining mangoes. How many good mangoes does each of them get?

This in symbolic form is,

(24 – 3) ÷ 3. 3 taken away from 24 gives us 21. Now, 21 mangoes to be shared among 3 is division.

21 ÷ 3 = 7

The teacher should discuss the role of the brackets in this problem. If the same expression was written without the brackets, then it would be 24 – 3 ÷ 3, which by the rules of BODMAS would be simplified as 24 – 1 which is 23. Such exercises will help students understand the importance of symbolic notations and the need for conventions.

II. Writing a story for a given expression

Example: 5 × 2 + 4

Students could be given this expression and asked to make a story that uses this calculation.

A sample: Meda's teacher asked her to distribute story books to her class. There were 5 children seated at each bench. Meda distributed the books to 2 benches and had 4 books remaining. The story could be in words or sketches. This can also be done in pairs with one student making the expression and the other, the story.

III. Understanding the rationale behind the rule and arriving at it through induction.

Example 1: 53 + 53 + 53 + 53 + 53 + 53 + 27

- The expression could be evaluated in any order.
- Let us condense it to make it simple and easier to calculate. Since 53 is added 6 times, the expression could be written as:

  6 × 53 +27 (getting them to read 6 times 53 added to 27 and hence order of operation being multiplication followed by addition)

  = 318 + 27 = 345

Multiplication first and then addition- this must be brought to the students' notice, along with the answer if the problem was interpreted as addition first, then multiplication.

Example 2: 34 + 102 + 102 + 102

- The expression could be written as 34 + 3 × 102. Which means 34 is added to 3 times 102.
- The rule gives the order of operation, that 102 is multiplied by 3 and then added to 34.

  34 + 306 = 340

Again, we see multiplication preceding addition

Note: Students may represent this problem as 34 + (3 × 102) but they should be led to understand that the bracket is redundant in this problem. Such discussion helps them to see how precise and concise the language of mathematics is.

Example 3: 17 – 25 – 25 + 17 + 17 + 17 + 17 – 25 + 17

- The expression could be written as 17 × 6 – 25 × 3. It is simplified to 3 times 25 to be subtracted from 6 times 17.
- Now, the order of operation is multiplication followed by subtraction; 102 – 75 = 27

Note: The teacher should encourage students to see how the multiplication shortens the process of calculation. In turn, they can create expressions which are shortened by using more than one operation.

Example 4: $12 - 3 - 3 + 2 - 3$

- The expression condenses to $(12 + 2) - (3 \times 3)$. We have 12 added to 2 from which we need to take away 3 times 3.
- So, we simplify the term in the brackets and then subtract.

  $= 14 - 9 = 5$

IV. Understanding the meaning of the expression

Here students will read the expression and state the order of operation.

Example 1: $3 + 4 \times 5$

- 3 added to 4 times 5
- So, we need to multiply 4 and 5 first and then add 3 to it.

  $= 3 + 20 = 23$

Example 2: $6 \times 5 - 8 \div 2$

- Here the convention rule follows reading from left to right we have multiplication followed by division and then subtraction. First we simplify 6×5 to get 30
- Then simplify $8 \div 2$ to get 4
- The answer now becomes $30 - 4$ which is 26.

Students should be able to state: So, as we read left to right we multiply, divide and then subtract.

In summary, BODMAS simply tells that brackets must be first opened to unfold the calculation, after which must be performed any of the two higher operations (Multiplication/Division) before any of the two lower operations (Addition/subtraction). What is important is to ensure that operations of higher order are performed first.

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CRYPTARITHMS

SACHIN VYAVAHARE
& WALLACE JACOB

Here is an interesting type of problem which can exercise your thinking and reasoning skills.

Question. Given that each letter represents a unique digit in the range of 0 to 9, find the values of A, B, D, E, F, G, H, J and K from the computation given below:

```
  ABJ
+ADEF
EGHJK        .......(1)
```

The + operator carries its usual arithmetic meaning.

In such problems, the convention is that ABJ represents a three-digit positive integer, ADEF a four-digit positive integer, and EGHJK a five-digit positive integer.

In questions such as these:

i. There is a one-to-one mapping between each letter (variable) and a particular digit, i.e., each letter corresponds to exactly one digit and each digit corresponds to exactly one letter.

ii. There are no leading zeroes (the zeroes in 003 are leading zeroes and the zeroes in 300 are trailing zeroes).

Equation (1) might be written as:

C1	*C2*	*C3*	*C4*	
		A	B	J
+	A	D	E	F
E	G	H	J	K

Table 1

Here, *C1, C2, C3* and *C4* represent carry digits.

Keywords: Cryptarithm, one-one mapping, place value, addition, reasoning, problem solving

If we add any two digits in the range of 0 to 9, then the maximum carry digit will be 1 and the minimum carry digit will be 0. For example, 2 + 3 = 05 (carry digit is 0), and 8 + 9 = 17 (carry digit is 1). The smallest sum is 0 + 0 = 0 and the largest sum is 9 + 9 = 18.

Since the sum has 5 digits and there are no other digits in the first column (from the left), we can safely conclude that *C1* is 1 and that therefore, the value of E is 1.

For E to be 1, the value of A will necessarily be 9 and the value of *C2* will necessarily be 1. If A = 9, *C2* = 1 and E = 1, then the value of G will be 0.

With these conclusions, we may now rewrite the table as:

1	*1*	*C3*	*C4*	
		9	B	J
+	9	D	1	F
1	0	H	J	K

Table 2

Now we move from conclusions to conjecture:

If J = 2, and F = 3, then the value of K will be 5 (and the value of *C4* will be 0).

Now B + 1 should be equal to J (assumed to be equal to 2; second column from the right in *Table 2*). The value of J will be 2 only if B = E = 1 or B = E = 6. We know that the value of E is 1 and no two letters can have the same value. Hence, let us try with new values for J and F.

Let J = 3, and F = 4. K = J + F, therefore, the value of K will then be 7.

It is easy to infer from the second column (from the right) of *Table 2*, that the value of B will then be 2. On substituting the values in *Table 2*, we obtain:

1	*1*	*C3*	*C4*	
		9	2	3
+	9	D	1	4
1	0	H	3	7

Table 3

We have now arrived at E= 1, A = 9, G = 0, J= 3, F = 4, K = 7, B =2.

The value of D cannot be 7 (because K = 7). The value of D cannot be 8 (because then H would be 7 and we know that the value of K is 7).

If D = 6, then the value of H will be 5. We may now write:

1	*1*	*0*	*0*	
		9	2	3
+	9	6	1	4
1	0	5	3	7

Table 4

Thus, E = 1, A = 9, G = 0, J= 3, F = 4, K = 7, B = 2, D = 6 and H = 5.

It is important to note that certain cryptarithms might have a unique solution and certain cryptarithms may have more than one solution. To arrive at other solutions, we go back to *Table 2*, since we cannot change the values of E, G and A.

The combination J = 5 and F = 6 is not possible, because in that case K would be 1. K cannot be 1, because the value of E is 1.

Let J = 6, and F = 2. K = J + F, therefore, the value of K will be 8 and the value of C4 will be 0. With J = 6, and E = 1, B will have to be 5 and the value of *C3* will be 0.

On substituting the values of J, F, K, *C4*, B and *C3* in *Table 2*, we obtain:

1	*1*	*0*	*0*	
		9	5	6
+	9	D	1	2
1	0	H	6	8

Table 5

If D is 6, then H becomes 5 (and *C2* becomes 1). But this gives us H = B.

If D is 7, then H becomes 6 (and *C2* becomes 1). But this gives us H = J.

If D is 4, then H becomes 3 (and *C2* becomes 1). Then we arrive at a second valid solution set:

1	*1*	*0*	*0*	
		9	5	6
+	9	4	1	2
1	0	3	6	8

Table 6

E = 1, A = 9, G = 0, J = 6, F = 2, K = 8, B = 5, D = 4 and H = 3.

Now, suppose J = 6 and F = 7, then equation (i) might be written as:

1	*1*	*0*	*1*	
		9	B	6
+	9	D	1	7
1	0	H	6	3

Table 7

Thus the value of B would be 4.

The value of D cannot be 6 (because J = 6). The value of D cannot be 7 (because F = 7). So, let the value of D be 8. But if D is 8, then the value of H would be 7 (which is not possible because F = 7).

Problems such as these are invaluable in building reasoning skills for students. They begin to understand the role played by constraints and boundary conditions. They have opportunities to practise mathematical communication by building strong arguments and justifying their conclusions. They learn to listen to others and accept that there is more than one correct solution. Best of all, they learn to frame similar problems for their peers. And they do this all in the spirit of play.

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WALLACE JACOB is the author of *The Unfathomable World of Amazing Numbers, The Merchant of Venice Workbook for ICSE Students and The Tempest Workbook for ISC Students*.

Definitions changing through time

With a closer look at isosceles trapeziums

SWATI SIRCAR

Definitions have changed in textbooks across ages. At one time, squares were not considered rectangles but later they were. Why this change? When we think of classification, we primarily create subsets of a set in a manner that each subset has some specific properties. Essentially there are two ways to do this:

1. **Partition** – where the original set is divided in disjoint i.e. not overlapping subsets.
2. **Hierarchical** – where nested subsets are formed so that a subset with more general properties is the superset of one with a more specific one.

The earlier definition of rectangle, i.e., a parallelogram with a right angle and unequal adjacent sides, followed the Partition classification making it a subset disjoint from squares, which are parallelograms with a right angle and four equal sides (Figure 1). This way of classification makes it easier to depict a shape since it doesn't involve considering various subcases.

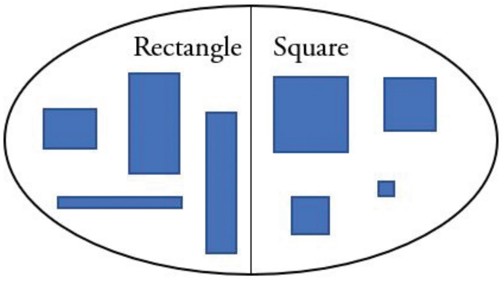

Figure 1: Partition

Keywords: Quadrilaterals, classification, boundaries

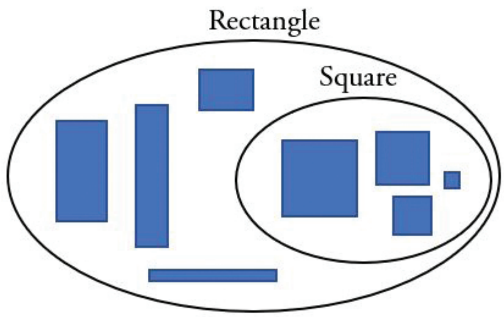

Figure 2: Hierarchy

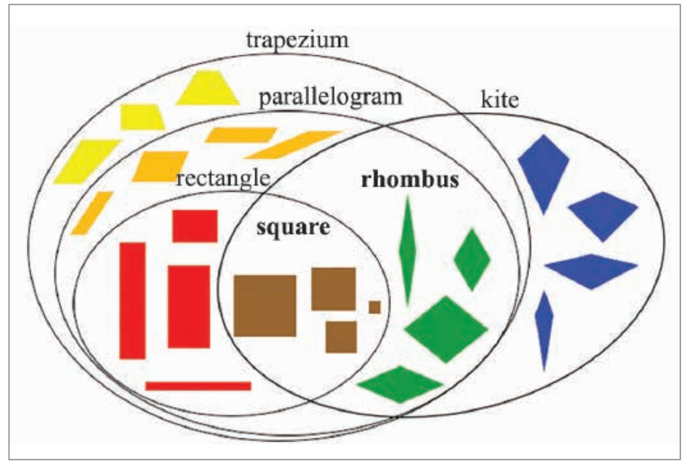

Figure 3

However, the present textbooks have switched to a Hierarchical classification that makes the set of squares a subset of the set of rectangles (Figure 2). A big reason for this change comes from considering the properties of a rectangle. The Partition definition (including unequal adjacent sides) does not provide a rectangle with any property that a square does not have. Or in other words, a square has all the properties of a rectangle. Therefore, it makes sense to consider a square as a (special) rectangle. This does lead to slight complication in depicting such shapes since now subcases may have to be considered e.g. a rectangle can look like a square or like one with unequal adjacent sides.

By the same logic, squares are also rhombi and in fact the intersection of the sets of rectangles and rhombi is precisely the set of squares. Now, the sets of rectangles and rhombi are both subsets of the set of parallelograms, which itself is a subset of the set of trapeziums. Rhombi are also kites. Figure 3 represents these sets. Note that

(i) Rhombus is the intersection of parallelogram and kite i.e. a quad is a rhombus if and only if it is a parallelogram and a kite

(ii) Square is the intersection of rectangle and rhombus i.e. a quad is a square if and only if it is a rectangle and a rhombus

(iii) There is no quad which is a trapezium and a kite but not a parallelogram

We encourage the reader to prove each of the above statements.

But most textbooks are not uniformly adopting this change in definition. By this line of logic, equilateral triangles should be considered special isosceles triangles, but this is not reflected in most textbooks. Isosceles triangles have no property that an equilateral triangle does not have
(See Figure 4). This change however is reflected in some resources available on the web, e.g. https://www.cut-the-knot.org/triangle/Triangles.shtml which includes both cases.

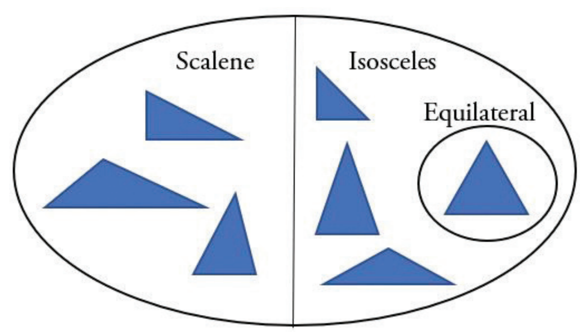

Figure 4

Another interesting case arises w.r.t. rectangles and isosceles trapeziums. Rectangles satisfy all properties of an isosceles trapezium. So, rectangle is the intersection of isosceles trapezium and parallelogram (Figure 5). However, the popular definition of an isosceles trapezium creates a barrier in allowing rectangles to join the set. An isosceles trapezium has a pair of parallel sides and a pair of equal sides. This comes directly from the name 'isosceles' meaning same sides. Now, if the parallel sides are equal, it becomes a parallelogram. So, the standard definition specifies that the non-parallel sides to be equal.

But the non-parallel-ness of the remaining sides excludes rectangles (Figure 6). So, a possible way out is to say, a quad with one pair of parallel sides with the other pair having equal length is an isosceles trapezium. This allows a rectangle to be an isosceles trapezium. But it also leads to one more problem. This alternate definition also allows a parallelogram to be an isosceles trapezium which is not possible. Isosceles trapeziums have line symmetry which a parallelogram need not have. So, there is a problem if we consider only sides:

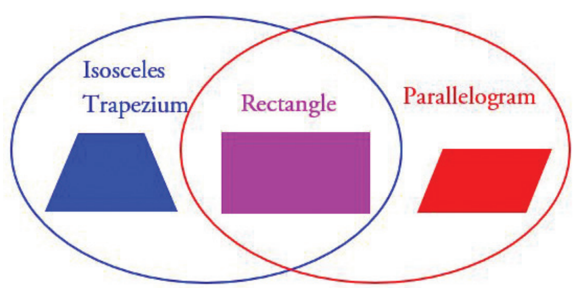

Figure 5

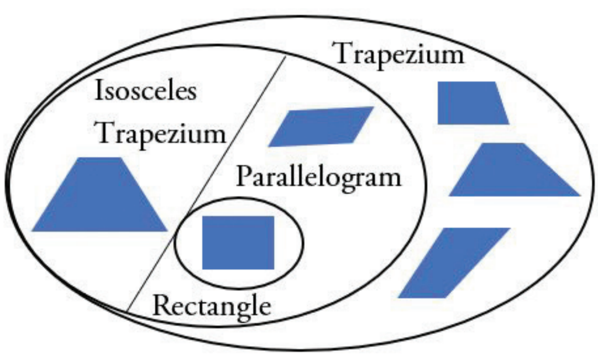

Figure 6

	Desirable	Non-desirable
Other sides are non-parallel	Excludes parallelograms	Excludes rectangles
Other sides can be parallel	Includes rectangles	Includes general parallelograms
Parallel sides are unequal	Excludes parallelograms	Excludes rectangles
Parallel sides can be equal	Includes rectangles	Includes general parallelograms

So, the definitions must go beyond sides and include some statement about the angles. There are several ways to go about this since there are several equivalent conditions:

1. Trapezium with two pairs of equal adjacent angles
2. Trapezium with opposite angles supplementary ⇔ a cyclic trapezium
3. Trapezium with a line symmetry
4. Trapezium with equal diagonals

Each of these includes rectangles but excludes general parallelograms. In the first definition, it is important to mention two pairs since it is possible for a trapezium to have exactly two right angles which must be adjacent. So, just one pair of equal adjacent angles may not ensure an isosceles trapezium. An alternative definition for 1 can be:

5. Trapezium with equal angles adjacent to any of the parallel sides

This excludes the right-angle pair possibility and may be more convenient in terms of construction and pre-requisites compared to 2, 3 and 4. Also it describes the quad in terms of its sides and angles from which properties (like 2, 3 and 4) can be derived, rather than make the derivable property a definition. One may need to understand cyclic quad to grasp the second part of 2 and one must be familiar with the properties of a shape with line symmetry. Whereas, 5 does not require anything beyond sides and angles. If the measure of the equal angles and their common side is given, one can draw the unique isosceles trapezium with these specifications. It involves an ASA type construction and drawing a parallel line. One can thus construct an isosceles trapezium with this definition and then explore its properties. However, it may be more difficult, but not impossible, to construct an isosceles trapezium using 2 or 3. In fact, there is a definition similar to 5 on the web: "An **isosceles trapezoid** (called an **isosceles trapezium** by the British; Bronshtein and Semendyayev 1997, p. 174) is **trapezoid** in which the base angles are equal and therefore the left and right side lengths are also equal." [http://mathworld.wolfram.com/IsoscelesTrapezoid.html]

We end this with the following statement and urge readers to prove it or provide a counterexample: *Any quad with line symmetry is a kite or an isosceles trapezium.*

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PRIME NUMBER GAMES

Prime numbers are an eternal source of fascination for mathematicians, amateurs or otherwise. They offer tantalising glimpses of pattern and symmetry, but these patterns are often elusive and misleading, as many math explorers have found the hard way!

The natural thing to do with prime numbers is to *multiply* them with each other, but mathematicians have also explored what happens when you add prime numbers to one another. There is a famous unsolved problem in this connection, known as Goldbach's conjecture, which states that it is possible to write every even number exceeding 2 as a sum of two prime numbers. For example, 20 = 17 + 3, 30 =17 + 13, 40 = 29 + 11, and so on. This may seem simple to prove – but it has resisted the best efforts of a large number of mathematicians over the past two centuries! (That's why it is called a conjecture.)

We display below some prime number relations found by two budding mathematicians, Harshul and Shresht. Who knows what such number play may lead to? We don't know, and we will not try to guess.

$$
\begin{gathered}
(5-3)=(2\times 1),\\
(7-5)=(3-2)+1,\\
(11-7)\times(3-2)=(5-1),\\
(7-5)+(3-2)=(13-11)+1,\\
(11-7)\times[(5-3)-(2-1)]=(17-13),\\
(7-5)\times[(3-2)+1]=(19-17)+(13-11),\\
(11-7)\times(5-3)\times(2-1)-(17-13)+(23-19),\\
(29-23)+1=(19-17)+(13-11)+(7-5)+(3-2),\\
[(31-29)+(11-7)+(5-3)]\times(2-1)=(23-19)+(17-13),\\
(37-31)+(19-17)+(7-5)=(29-23)+(13-11)+(3-2)+1,\\
(41-37)\times[(23-19)-(31-29)]=\{(17-13)[(11-7)-(5-3)]\}\times(2-1);
\end{gathered}
$$

And so on.

And:

$$
\begin{gathered}
1\times(2+3)=5,\\
(3+7\times 1)\div 5=2,\\
(5-3)\times 1+2=11-7,\\
(13-7)\times(3-2\times 1)=11-5,\\
(17+13)\times 2-(7-3+1)=11\times 5,\\
(2\times 3\times 5)+11+7+1=19+17+13,\\
23+19+17+13=\{(11\times 7)-5\}\times(3-2\times 1),\\
29+23+19+17+13=\{11\times(5+2)\}+\{(7+1)\times 3\},\\
31+29+\cdots+13=\{(7+5)\times 11\}\times(3-2\times 1),\\
37+31+\cdots+13=(11\times 7\times 2)+(5\times 3\times 1),\\
41+37+\cdots+17=11\times(13+2)+(7\times 5)-(3\times 1);
\end{gathered}
$$

Contributed by Harshul Vikam urvi.vikam@gmail.com and Shresht Siddharth Bhat savitharth@gmail.

The Height of a Tree

MOHAMMAD UMAR

This article is based on an interesting activity done with children, in 2008. At that time, I was working with 'Eklavya'- an Institute for Educational Research and Innovations, Hoshangabad. Being new, I picked up some innovative methods to teach science from my seniors here. Also, I started reading some interesting books based on learning theories and activities. I tried some of the experiments and activities with my students – on some occasions, I was successful, on others not. But learning by doing was very enjoyable for all of us.

Our campus was very beautiful and full of greenery and open to the children from the nearby village Malakhedi who came to play games in this garden every day. A banyan tree stood at the center of this garden. There was also a children's library and we organized several activities in this garden for children from classes 3 to 12. In one such activity, we tried to measure the height of the banyan tree by seven different methods. It was an amazing experience. Today, I am sharing some of my reflections from a daily diary, which was used at that time to document my own learnings and experiences.

It is February and I am teaching the measurement of length in Class 6 of the Government School, Malakhedi. Today, I wanted to do some interesting measurement activity with the children. During my school visits, I had observed that most of the children from the primary classes were using their scale only to draw straight lines, they had difficulties in measuring length with it. In the higher classes, the children were familiar with the common units of measuring length: metre, centimetre and kilometre. The banyan tree was just in front of us. A few children asked to measure the height of this tree. We could not measure the height of this banyan tree by using a scale or inch tape, as it was very high. The challenge was to find the height of this tree - without climbing on it.

So, the question was- how can we measure the height? We discussed some ideas together. Here, in this article, I am trying to share the entire process of this activity.

Keywords: Problem solving, estimation, approximation, heights, geometry, trigonometry, folklore

First method – By Estimation. The shadow of the big leaves of the tree was blocking the sunshine. In order to make an estimation, we needed some reference point. So, I stood beside the trunk of banyan tree and a student made a mark on the tree trunk near the top of my head. Now, I walked away from the tree and observed it from top to bottom. I also observed the mark showing my height. Here, I was trying to compare these two heights. It seemed to me that the tree was nearly three times my height. My young companions agreed with my impression when I shared it with them.

A boy ran inside the office and brought a scale and inch tape. We measured the height of the mark on the trunk. It was 172 cm.

So, the height of the tree will be 3 times my own height

$$= 3 \times 172 \text{ cm.}$$
$$= 516 \text{ cm.}$$
$$= 5 \text{ m. } 16 \text{ cm.}$$

That was the very first attempt to say something about the height of the tree, without climbing on it. At this stage we were not able to claim its correctness, as it was based on our estimation only.

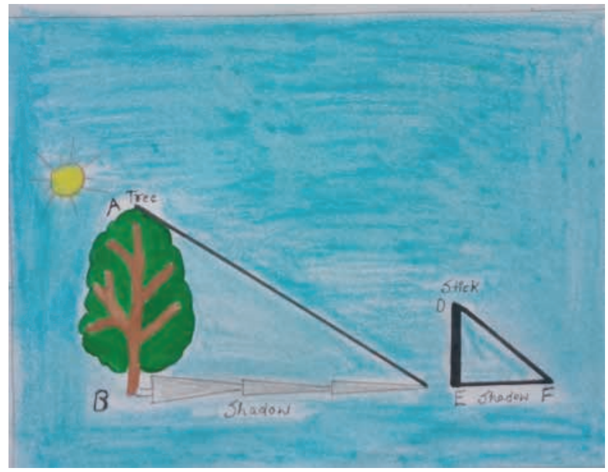
As I said earlier, children from classes 3 to 12 were part of this team. I asked them if they had learned an activity to measure the height of a pole with the help of a stick and its shadow. Two children said that they had, but they were not able to explain any details about the same.

Second Method – Ratio of the lengths of the shadows of a stick and the tree. It was a pleasant winter morning. The sun's rays were casting very beautiful shadows on the green grass. We searched for a straight stick and cut it after measuring off a little more than half a metre.

Now, when we planted this stick vertically in the ground, it stood half a metre above the ground. The stick was casting a straight shadow on the ground.

AB = h metres (height of tree)

DE = height of stick



BC = length of shadow of tree.

EF = Length of shadow of half metre stick.

Here, we could get all three measures (except the height of the tree) easily.

We knew the length of the stick i.e. half metre (= 0.5 m) already. We measured the shadows of the tree and the stick with a measuring tape.

Height of tree = h

Height of Stick = 50 cm

Length of the shadow of tree = 435 cm

Length of the shadow of stick = 33 cm

Δ ABC $\simeq$ Δ DEF (by AA Property)

Hence, AB/DE = BC/EF = AC/DF

(Corresponding sides of similar triangles)

Therefore, AB/DE = BC/EF

Height of tree/height of stick = shadow of tree/shadow of stick

- $h/50$ cm = 435 cm/33 cm
- $h = 435 \times 50/33$
- $h = 659$ cm
- $h = 6$ m. 59 cm.

By this method, we obtained the height of the tree to be 6 m 59 cm. But we were not very happy, as it was very different from our earlier estimate. My companions and I were a little bit confused, as we were not able to choose which option was closer to the actual measurement.

Everybody started thinking again to try and find out some ways to come out from this confusion. Some of the children who were studying in the higher classes were able to contribute well to our discussion.

Third Method – With the help of a Right-angled Isosceles Triangle. Some students who were studying in higher classes told us that they are using trigonometric formulae to get the heights of towers, trees and poles using the formula Tan θ = Perpendicular/Base, where θ was usually 45°. I told them about an activity in which one can measure height by using this principle. Quickly, we cut a right isosceles triangle out of a sheet of cardboard. (One angle 90° and the remaining two 45°.)

Now, I placed this triangle on the ground and tried to find a place from where, the perpendicular side of the triangle can cover the entire tree (from top to bottom). But even after lying prone on the ground, we found it difficult to do this.

So, I decided to observe the tree from a standing position. We again marked the level of my eyes on the trunk of the tree. It was at 158 cm.

Again, I walked away from tree and tried to find an appropriate position. This time, my objective was to cover only the upper part of tree (from the mark on the trunk).

I found one such position. Two children measured this distance, from the tree to my standing position.

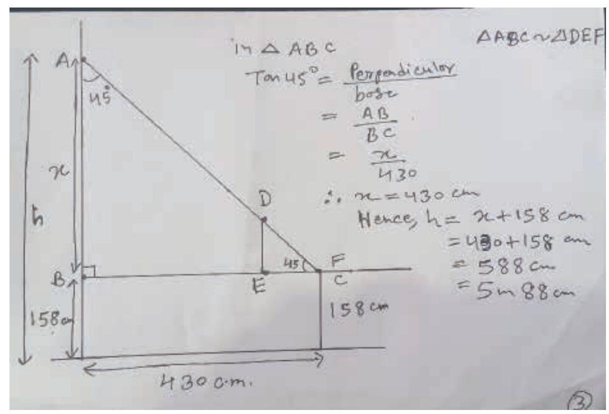
Now, we had to use this information in trigonometric formula.

Height of mark on the trunk (my eye level) = 158 cm

My distance from the tree (where this triangle covered the upper part of tree) = 430 cm

Height of tree = h = (158 + x) cm; here, x is the height of the upper part of the tree.

In Δ ABC, Tan 45° = AB/BC



We know,

Tan 45° = perpendicular/base

= height of upper part of tree/distance from tree to my standing position.

= x cm/430 cm

= $x/430$

Therefore, 1 = x/430

(by trigonometric ratio Tan 45° = 1)

Which gave x = 430 cm

We know,

Total height of the tree = h

= height from ground to the mark on trunk + height of upper part of tree

= 158 cm + x cm

= 158 cm + 430 cm

= 588 cm

= 5 m. 88 cm.

So, in this way, we got one more measurement for the height of the tree. It was different from the first two heights.

Now, we became more interested in finding different ways to get the height of the tree, rather than the accuracy of the height. I had read about one such activity, where one can measure the height of a tree or a pole with the help of a pen.

I shared this activity with the children. They became very curious about it and wanted to try it immediately.

Fourth Method – By using a pen.

I used a simple ball pen to measure the height of this tree.

First, I walked away from the tree and tried to find a position from where this pen can obscure the entire tree, from top to bottom. I remained in the same position and keeping my hands at the same level, I then rotated this pen by 90° (from vertical to horizontal position). [Imagine mentally that the tree topples over to its side and falls to the ground, so that after it has fallen, it continues to be perpendicular to you; it has neither fallen towards you, nor away from you. Let the pen also make exactly the same movement: let it 'topple over' and fall to its side. For this, we have to rotate the pen by 90° till it is horizontal. In its final position, it neither points towards the tree, nor away from the tree. Now let a child position himself so that he, the tip of the pen, and your eye are all in one line. When this position has been found, the child is effectively at the tip of the fallen tree. So, all we need to do is to measure the distance from the tree to the child, and this will give us the height of the tree.]

Finally, we measured this horizontal distance covered by pen on the ground. It was 465 cm.

Hence, the height of the tree was 4 m 65 cm (according to this measurement).

But we were not happy with this finding, as the height of the tree in this case was very different from all the previous results. There was clearly some mistake, but we were unable to trace it.

A few years later, I was sharing these methods with one of my colleagues who had been working in the field of math education for a while. She told me about a minor but crucial mistake, which we had made.

We had to add the height of my eyes (from ground level) with this result, as I was in standing position and the pen was not placed on the ground.

So, we rectified this calculation (better late than never!) by adding my eye level to the previous measure.

Hence,

$$\begin{aligned}\text{The height of tree} &= 465\text{ cm} + 158\text{ cm}\\ &= 623\text{ cm}\\ &= 6\text{ m }23\text{ cm.}\end{aligned}$$

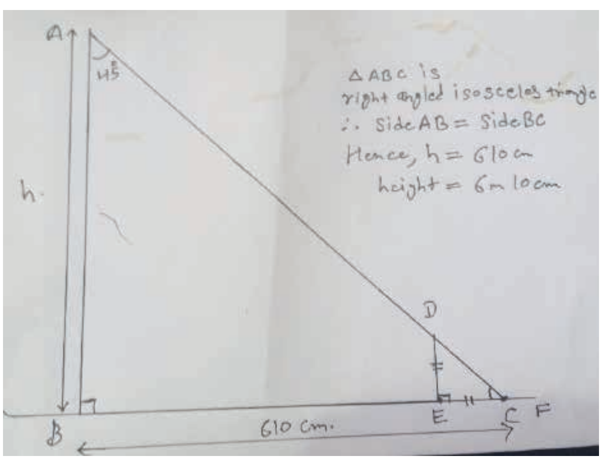
Fifth Method – Using our eyes and hands.

One of my colleagues, who had been working in science education for many years, was watching our efforts silently. He asked about all the methods, which we had already used and then suggested one more method.

'*There is no material required for this method. We have to use our hands and eyes only*'- he said. Children were very happy, as they sensed one more method to measure the height of tree.

This method is mathematically very similar to the third one, where we used the right angled isosceles triangle made from cardboard. But, here, we don't need any triangle. Only our hands and eyes are enough.

In this method, I lay down, a few steps away from the tree with my back on the ground and my feet toward the tree (I could see it easily).



Now, I folded my right arm to make a 90° angle at the elbow. I tried to cover the tree from top to bottom with the vertical part of my right hand (please see picture).

It was a little difficult to get the appropriate location to do so, but after changing positions a few times, I succeeded. Here, my right hand worked as a right-angled isosceles triangle. I checked the distances from shoulder to elbow and from elbow to ring figure of my right hand, they were the same. That was why this whole arrangement provided a natural isosceles right triangle with angles 90°, 45° and 45°.

Therefore, the distance from my head to the tree was equal to the height of the tree. Now, I measured this distance very easily with the help of the tape. It was 610 cm. So, the height of the tree by this method would be = 6 m. 10 cm.

This result gave us a lot of satisfaction, as it was very close to the estimate, which we had made at the beginning.

The children, who were actively involved in all these activities, asked me again for any new method. I remembered that I had read about a tribal community which could estimate heights of trees easily. Their method was very interesting.

Sixth Method – A local device: By folding the body and observing the tree from the folded position. In this small community, some of the senior members could estimate the height of any tree. To do this they turned their backs to the tree, bent their bodies at the waist without bending their knees and observed the tip of the tree through the inverted V formed by their own legs (see picture).

I had read about this method in a book, but had no practical experience of the same.

I invite our readers to justify the mathematics behind this folk method.

When I had shared this method with the children, some of them started laughing, while others immediately started trying it out. I also decided to give this a try.

I moved into position and started walking away from the tree in this folded position. Initially, the trunk of the tree was visible from between my legs. I started walking slowly away from the tree. I was simultaneously searching for an appropriate location from where I could see the entire height of the tree through my legs, i.e., at such a point, the tip of the tree should just touch the vertex of the inverted V, which was formed by my legs.

Suddenly, I was able to see the banyan tree, from top to bottom. One child placed a stone, just to remember this position. Later, we measured the distance from the tree to this stone. The distance between tree and stone was 624 cm

According to the book, the height of tree = 6 m. 24 cm. which was close to the values which we had obtained by the other methods.

At this stage, we had done 6 different methods to measure the height of a tree. Now, it was difficult to think about any new method, other than climbing on it, just like a monkey and measuring it with the tape.

Seventh Method – By a long broom attached to a bamboo. We were planning to wind-up. Suddenly, Sakeena, a girl studying in class 6 suggested that we could use a long broom (used to clean the roofs and walls) and measure it directly. That was a very good idea. Children wanted to confirm all these answers too.

Gopal, a class 5 student, rushed to the office and came back with a very long broom, made longer by attaching it to a bamboo stick.

'*Sir, now we can measure its height*' said Gopal. The other children started laughing, but Sakeena was happy, as we were going to implement the idea given by her. We placed this very long broom, just parallel to the main trunk of the tree. It was a little longer than the height of the tree.

Now, we placed this broom (attached with long bamboo) on the ground and measured its length with the help of the tape. It gave one more estimate of the height of the banyan tree.

Here, I am not going to tell you the height of the tree, which we got by the seventh method as I want to give you an opportunity to learn for yourself. It would be good, if you can do all these activities with your students and let them learn by themselves.

Here, finding the correct answer is not the important thing. As a mathematics teacher my interest is in finding different ways to get the answer. I am sure your students will also enjoy this activity.

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The Mystical Number 2997 (Kohli's Number)

UTTKARSH KOHLI

Introduction

This paper is about discovering and figuring out the working behind the magical number 2997.

Take any three-digit number, say *xyz* and multiply each digit by 111 and take the sum, i.e., $x*111 + y*111 + z*111$.

*[**Note:** Does not have to be a 3-digit number]*

For any number, repeating these steps of multiplying and adding will guarantee that the answer becomes 2997 in a maximum of 4 repetitions.

How does this happen?

It is easy to think that this is pure magic. I resolved to find out the mechanics behind this interesting result.

Consider the number 111, the number we use as a multiplier.

We see that 111 is divisible by 3, this can be confirmed by taking the sum of its digits and checking that it is divisible by 3. $(1 + 1 + 1 = 3$ which is the first multiple of 3.)

Consider any number (say *xyz*). Multiply each digit by 111 and take the sum. The sum is $x*111 + y*111 + z*111$, which can also be written as $111(x + y + z)$. We thus obtain a number that is divisible by 3.

For example, take the case of 999. (Incidentally, this is the only 3-digit number that will end up at 2997 after just one step.)

Step 1: Multiply each digit by 111. The products are 999; 999; 999.

Step 2: Take the sum: 999 + 999 + 999 = **2997**.

For other numbers, the sum may not be 2997 after the first try.

Let the sum $111(x + y + z)$ from the first try be the new number N. As pointed out earlier, N is divisible by 3. Multiplying each digit of N by 111 and taking the sum again, we get a number that is divisible by 9.

Let us consider an example, 4761.

Keywords: numbers, iteration, digital sum, pattern

Step 1: Multiply each digit by 111. The products will be 444; 777; 666; 111.

Step 2: Take the sum: 444 + 777+ 666 + 111 = 1998.

Repeat Step 1 and Step 2: 111 + 999 + 999 + 888 = 2997.

There you have it.

For numbers not divisible by 3, it is necessary to repeat the steps to add another factor of 3. Try 57931 as an example.

The trick is that each time steps 1 and 2 are repeated, the new number gets an additional factor of 3, and once it becomes 111*(27), the number 2997 is obtained.

And when the sum reaches 999 or another number whose digits add up to 27 (the third multiple of 9), the result of multiplying the digits by 111 and taking the sum will be 2997.

Doing this continuously would guarantee an end at 2997.

*The beautiful part is when we multiply each digit of the number 2997 itself by 111 and add the resulting numbers, we get **2997**.*

The number does not have a significance in higher mathematics, but it serves as a recreational and entertaining fact, which shows that even high school mathematics can surprise us with patterns just waiting to be discovered.

Some more examples

1) 29

   222 + 999 = 1221

   111 + 222 + 222 + 111 = 666

   666 + 666 + 666 = 1998

   111 + 999 + 999 + 888 = 2997.

2) 739

   777 + 333 + 999 = 2109

   222 + 111 + 0 + 999 = 1332

   111 + 333 + 333 + 222 = 999

   999 + 999 + 999 = 2997.

3) 6307

   666 + 333 + 0 + 777 = 1776

   111 + 777 + 777 + 666 = 2331

   222 + 333 + 333 + 111 = 999

   999 + 999 + 999 = 2997.

4) 8

   888

   888 + 888 + 888 = 2664

   222 + 666 + 666 + 444 = 1998

   111 + 999 + 999 + 888 = 2997.

General Formulation, to Arbitrary Number of Digits

Let x (an arbitrary positive integer) be the number with say n (another arbitrary variable) digits.

Step 1:

Multiply all digits by 111. We get $111*a_1 + 111*a_2 + \ldots. + 111*a_n$

Step 2:

Take the sum $111(a_1 + a_2 + \ldots + a_n)$.

If $(a_1 + a_2 + \ldots + a_n)$ is divisible by 27

Sum: 2997.

Else:

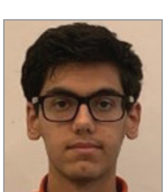
Repeat the two steps. Each time we do this, the quantity acquires an extra factor of 3.

$3*37(a_1 + a_2 + \ldots + a_n)$, i.e., $37*(3a_1 + 3a_2 + \ldots + 3a_n)$

Continue until the sum becomes divisible by 27.

Sum: 2997.

Editor's Note: A formal proof of this result will be published in the next issue, meanwhile we encourage readers to send in their proofs, if they arrive at one.



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Extension of an INMO Geometry Problem

RAHIL MIRAJ

In this work, I extend the result contained in an INMO (Indian National Mathematics Olympiad) geometry problem. This problem refers specifically to a right-angled triangle, but I prove a general result that holds for any triangle. The INMO result is then a special case of this.

Problem (INMO 2016–5). *Let ABC be a right-angled triangle with $\measuredangle B = 90^\circ$. Let D be a point on AC such that the inradii of the triangles ABD and CBD are equal. If this common value is r' and if r is the inradius of the triangle ABC, prove that*

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

The result that we shall prove is the following.

Theorem. *In triangle ABC, let D be a point on side AC such that the inradii of the triangles ABD and CBD are equal. If this common value is r' and if r is the inradius of the triangle ABC, then*

$$\frac{1}{r'} = \frac{1}{r} + \frac{2}{\sqrt{(a+b-c)(b+c-a)}}.$$

Proof of the general result

Let ABC be any triangle, and let D be a point on AC such that the inradii of the triangles ABD and CBD are equal, say r'. Let the inradius of triangle ABC be r. Let the areas of $\triangle ABC$, $\triangle ABD$ and $\triangle CBD$ be Δ, Δ_1 and Δ_2 respectively. (See Figure 1.) Let the semi-perimeters of $\triangle ABC$, $\triangle ABD$ and $\triangle CBD$ be s, s_1 and s_2 respectively. Let $BD = d$, $AD = x$, $CD = b - x$. Also let $\measuredangle ABD = \theta$.

Keywords: Inradius, sine rule, cosine rule

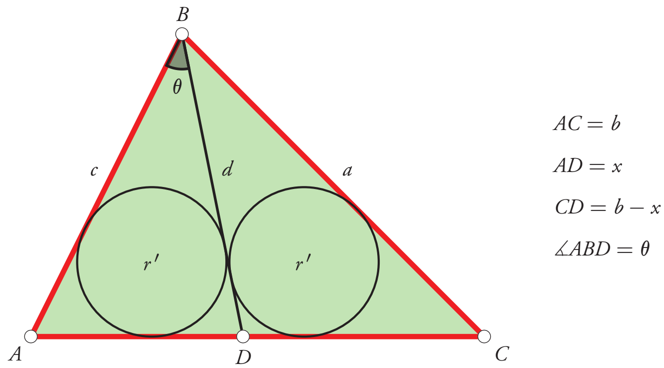

Figure 1. Triangles ABD and CBD have equal inradii (r')

Then we have,

$$\Delta_1 = r's_1, \tag{1}$$
$$\Delta_2 = r's_2. \tag{2}$$

Now, from (1) and (2), we get

$$\frac{\Delta_1}{\Delta_2} = \frac{s_1}{s_2}. \tag{3}$$

Using the fact that for a pair of triangles with the same height, the ratio of their areas is equal to the ratio of the bases, we get from (3),

$$\frac{x}{b-x} = \frac{c+d+x}{a+d+b-x}. \tag{4}$$

From (4) we get:

$$x = \frac{b(c+d)}{a+c+2d}, \tag{5}$$
$$b-x = \frac{b(a+d)}{a+c+2d}. \tag{6}$$

Again, from (1) and (2) we get

$$\Delta_1 + \Delta_2 = r'\left(s_1+s_2\right). \tag{7}$$

Since $\Delta_1 + \Delta_2 = \Delta$ and $\Delta = rs$, this becomes $rs = r'\left(s_1+s_2\right)$, i.e.,

$$r(a+b+c) = r'(a+b+c+2d). \tag{8}$$

From this we get, by division:

$$\frac{r}{r'} = 1 + \frac{2d}{a+b+c} = 1 + \frac{d}{s},$$
$$\therefore\ \frac{1}{r'} = \frac{1}{r} + \frac{d}{rs},$$
$$\therefore\ \frac{1}{r'} = \frac{1}{r} + \frac{d}{\Delta}. \tag{9}$$

Now, applying the sine rule in $\triangle ABD$, we get

$$\frac{d}{\sin A} = \frac{x}{\sin\theta}. \tag{10}$$

Applying the sine rule in $\triangle ABC$ and using (5),

$$\sin\theta = \frac{a(c+d)\sin B}{d(a+c+2d)} \tag{11}$$

Similarly, applying the sine rule in $\triangle CBD$ and using (6), we get

$$\sin(B-\theta) = \frac{c(a+d)\sin B}{d(a+c+2d)}. \tag{12}$$

Expanding $\sin(B-\theta)$ and using (11), we get

$$\cos\theta = \frac{c(a+d)+a(c+d)\cos B}{d(a+c+2d)}. \tag{13}$$

Now, squaring and adding (11) and (13), we get

$$d^2(a+c+2d)^2 = a^2(c+d)^2\sin^2 B + [c(a+d)+a(c+d)\cos B]^2$$

After simplifying, it becomes

$$2d^2 = ac(1+\cos B) \tag{14}$$

Now, applying the cosine rule for $\measuredangle B$ in (14), we get the following very nice result:

$$4d^2 = (a+c)^2 - b^2. \tag{15}$$

Now, putting

$$\Delta = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{b+c-a}{2}\right)},$$

$$d = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)},$$

in (9), we get the desired result

$$\frac{1}{r'} = \frac{1}{r} + \frac{2}{\sqrt{(a+b-c)(b+c-a)}}. \tag{16}$$

Solution to INMO problem using the theorem

The INMO problem is a special case of our general result. Here $\triangle ABC$ is a right-angled triangle with $\measuredangle B = 90^\circ$.

Taking $\cos 90^\circ = 0$ in (14) or $a^2+c^2=b^2$ in (15), we get $d^2 = \frac{1}{2}ac$, but for a right-angled triangle ABC, $\Delta = \frac{1}{2}ac$. Hence,

$$\Delta = d^2.$$

Putting this in (16), we get, as desired:

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{d}.$$

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The N-Queens Problem

DEENS ACADEMY
MATH CLUB

Introduction

The n-queens problem is a generalization of the eight-queens problem of placing eight queens on a standard chessboard so that no queen attacks any other queen. The original eight-queens problem was first posed in 1848 by Bezzel, a German chess player, in the *Berliner Schachzeitung* (or the Berlin *Chess Newspaper*). The generalization is due to Linolet, who asked the same question later in 1869, but now for n queens on an n x n board.

Before delving into this interesting problem, let us recall that a queen placed on a square on a chessboard can attack any other piece that is in its line of sight either horizontally, or vertically, or diagonally. See Figure 1. Thus, to solve the n queens problem, we need to place these n queens on an n x n chessboard such that there is exactly one queen in each row and column, and to ensure simultaneously that no two queens lie on the same diagonal.

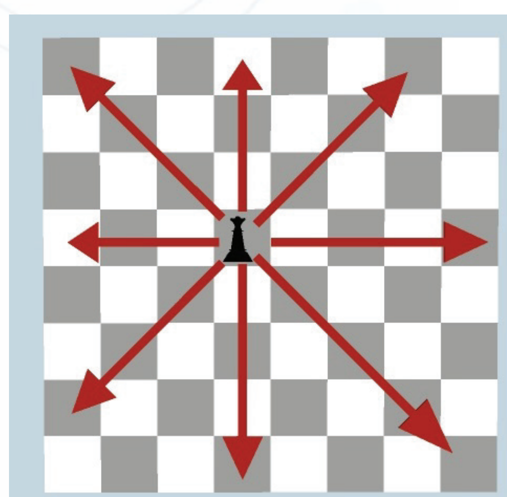

Figure 1. A Queen placed as shown can attack any other piece that is in its line of sight along the arrows shown: horizontally along its own row, vertically along its own column, or along the two diagonals leading out from it.

Keywords: games, strategy, reasoning, logic, mathematics

Nauck in 1850 was the first to give all 92 solutions to the original 8 queens problem. (That there are indeed exactly 92 solutions to the 8 queens problem was claimed by Gauss in a private communication, and proved by Pauls in 1874.)

Pauls was the first to solve the n queens problem for all $n > 3$. (For $n = 2{,}3$ there are no solutions, as is readily observed.) His solutions were followed by several other solutions for all n, including 1) Franel, 2) Scheid, 3) Hoffman, Loessi and Moore, 4) Falkowski and Schmitz, 5) Yaglom and Yaglom, to name a few. Several others gave solutions when n was restricted to lie in various subsets of the positive integers.

The total number of solutions for n from 4 to 11 are 2,10, 4, 40, 92, 352, 724 and 2680 respectively, and this was already known to Sprague in 1899. For larger n, the *backtracking* algorithm can be used to generate all solutions.

The problem still retains much fascination, and continues to be studied. Why study this problem if it has already been solved? It was initially studied for "mathematical recreation." However today, the problem is applied in parallel memory storage schemes, VLSI testing, traffic control and deadlock prevention in concurrent programming. Other applications include neural networks, constraint satisfaction problems, image processing, motion estimation in video coding, and error correcting codes. Additionally, the problem appears naturally in biology, where it was observed that the computation involved in the analysis of the secondary structure of nucleic acids is analogous to that involved in finding solutions to the n-queens problem!

We recommend the paper [1] for a comprehensive survey of the n-queens problem.

Our Solutions

In this article, we will present one solution to the n-queens problem, for all $n > 3$, that we the members of the Deens Academy Math Club obtained independently. We were unaware of previous solutions while we worked on this problem (and indeed were unaware then of the paper [1]), but were pleased to discover later that known solutions contain features of our solution.

We consider squares of size n x n, and write n as $2p$ when n is even, and as $2p + 1$when n is odd. We will divide our solution into two cases: the first where p is congruent to 0 or 2 mod 3 (for example, $p = 6, 9{,}12$, etc. or $p = 5, 8, 11$, etc.), and the second where p is congruent to 1 mod 3 (for example, $p = 4, 7, 10$, etc.).

We will index our squares by the row and the column on which they are situated: for example, a queen with indices (a, b) indicates that it is on the a-th row and b-th column. We will count the topmost row as row 1 and the bottommost as row n; similarly, we will count the leftmost column as column 1, and the rightmost as column n.

Of course, two squares (a, b) and (c, d) are on the same row if and only if $a = c$ and on the same column if and only if $b = d$. Elementary coordinate geometry shows that two squares $(a, b)$and (c, d), not both on the same row (so $c \neq a$), are on the same diagonal if and only if $\frac{d-b}{c-a} = \pm 1$. See Figure 2. We refer to the expression $\frac{d-b}{c-a}$ as the *slope* between the two squares. Recalling the orientation of our coordinate system: $(1, 1)$ is at the upper left and (n, n) is at the bottom right – we see that if the slope between (a, b) and (c, d) is 1 then they are on a diagonal that runs in the Southeast/Northwest direction, and if the slope is -1 then they are on a diagonal that runs in the Southwest/Northeast direction.

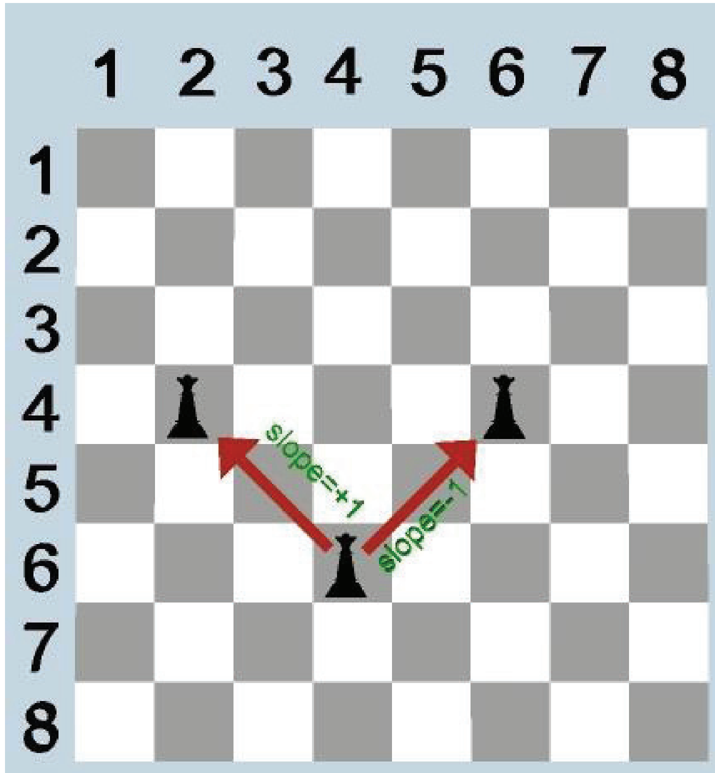

Figure 2. Slope between squares on diagonals: The slope between the squares $(4, 2)$ and $(6, 4)$ is $\frac{4-2}{6-4} = 1$. The slope between the squares $(6, 4)$ and $(4, 6)$ is $\frac{6-4}{4-6} = -1$.

Case A: p congruent to 0 or 2 mod 3
We will first provide the solution for the case $n = 2p$; the solution for the case $n = 2p + 1$ will then be just a simple modification. (The smallest value of p here is 2, corresponding to a 4×4 board.)

We place the queens on squares with the indices $(i, 2i)$ for the upper half of the square, so $1 \le i \le p$. (See Figure 3.) We then rotate this arrangement by 180 degrees about the geometric center of the chessboard to place the queens on the lower half of the square. It is easy to see that the reflected position of the square $(i, 2i)$ is $(n + 1 - i,\ n + 1 - 2i)$, so the queens in the bottom half have coordinates $(n + 1 - j,\ n + 1 - 2j)$, with j once again in the range $1 \le j \le p$. (For example, the rotation of the square at $(8, 16)$ in Figure 3 is the square at $(9, 1) = (17 - 8, 17 - 16)$.) We notice that as j decreases from p to 1, the row index of the rotation, namely $n + 1 - j = 2p + 1 - j$, increases from $p + 1$ to $2p$. Similarly, as j decreases from p to1, the column index of the rotation, namely $n + 1 - 2j = 2p + 1 - 2j$, increases from 1 to $2p - 1$ in steps of 2.

We need to show that no two queens are on the same row, or on the same column, or on the same diagonal. It is clear that two queens cannot be on the same row, as the row indices run from 1 to p in the placements in the upper half and from $p + 1$ to $n = 2p$ in the bottom half. The queens in the upper half are on distinct *even numbered* columns $2i$, $1 \le i \le p$, while those on the bottom half are on distinct odd numbered columns $1, 3, \ldots, 2p - 1$. Thus, all queens are on distinct columns.

To show that no two queens are on the same diagonal, positive or negative, we consider the following cases separately:

Subcase 1. Both queens are on the top half. Quite visually, the queens are not on the same diagonal (see Figure 3 for example), but we can see this formally as follows: since they are both on the top half, one

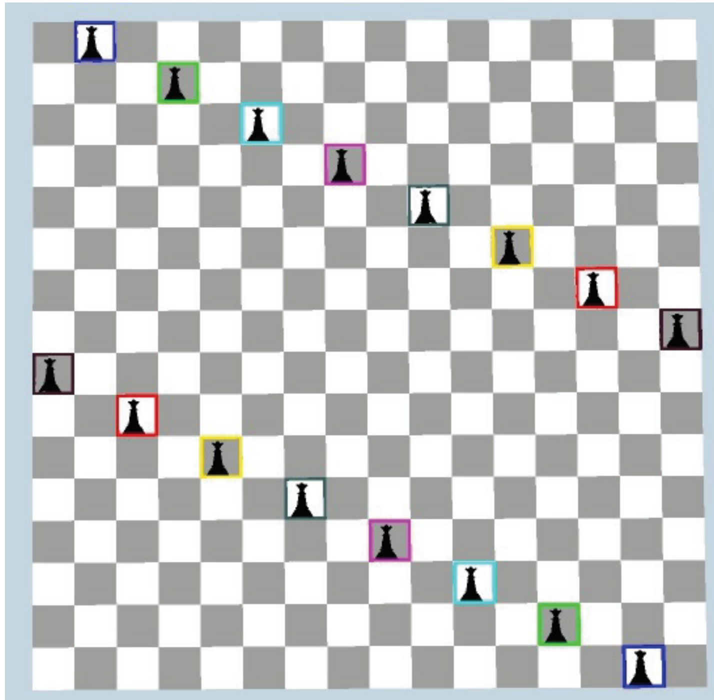

Figure 3. Arrangement when p = 8. The squares that correspond to each other after rotation are shown in the same colour.

queen will have coordinates $(i, 2i)$and the other will have coordinates $(j, 2j)$, where i and j are distinct. If say i <j, then i and j satisfy $1 \le i < j \le p$. The expression for the slope shows that the slope between the pair of points is $(2j - 2i)/(j - i) = 2 \neq \pm 1$, so they are not on the same diagonal.

Subcase 2. Both queens are on the bottom half. This can be proved just as in Case 1, or by observing that Case 2 follows from Case 1 by symmetry: if two queens on the bottom half were on the same diagonal, then their rotations in the top half will be on the diagonal that is the rotation of the first one, and we have already shown above that this cannot happen.

Subcase 3. One queen from the top half and the other from the bottom half. The queen on the top half has coordinates $(i, 2i)$ for some i in the range $1 \le i \le p$, while the queen on the bottom half has coordinates $(n + 1 - j,\ n + 1 - 2j) = (2p + 1 - j, 2p + 1 - 2j)$ for some j in the range $1 \le j \le p$. The slope is thus given by $(2p + 1 - 2j - 2i)/(2p + 1 - j - i)$. If this slope were 1, we would find on cross multiplying that $i + j = 0$, which is impossible as i and j are both positive. If the slope were -1, we would find on cross multiplying that $4p + 2 = 3(i + j)$. Thus $4p + 2\ (= 3p + p + 2)$ is congruent to 0 mod 3, so $p + 2$is congruent to 0 mod 3, or what is the same thing, p is congruent to 1 mod 3. But this violates our assumption that p is congruent to 0 or 2 mod 3, and therefore, the slope cannot be -1 either.

We have thus proved that our arrangement of queens for the case $n = 2p$, p congruent to 0 or 2 mod 3, indeed satisfies the requirement of the puzzle. Finally, when $n = 2p + 1$, with p congruent to 0 or 2 mod 3, we place the queens in rows 1 through $2p$ exactly as described above, and place the $(2p + 1)$-th queen in the square with coordinates $(n,\ n) = (2p + 1, 2p + 1)$. See Figure 4. It is clear that the row and column indices of the squares in the first $2p$ rows are different from $2p + 1$, so this last queen does not share a row or column with the previous $2p$ queens. The only diagonal that this last queen lies on is the "main diagonal" of the square which runs from $(1, 1)$ to (n, n). This diagonal has the property that for any square on it, the row and column indices are equal. It is clear that none of the previous queens are on this diagonal as for each of them, the row and column indices — i and $2i$ in the upper half and

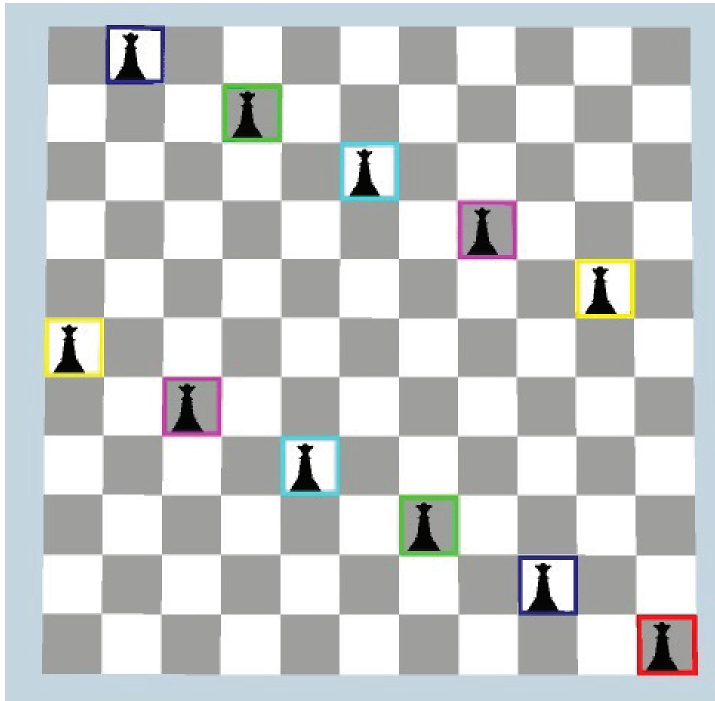

Figure 4. The case of the 11×11 square: here $11 = 2p + 1, p = 5$, and 5 is congruent to 2 mod 3. The solution is built from the 10 X 10 case $(10 = 2p, p = 5)$ by adding a queen at the bottom right square.

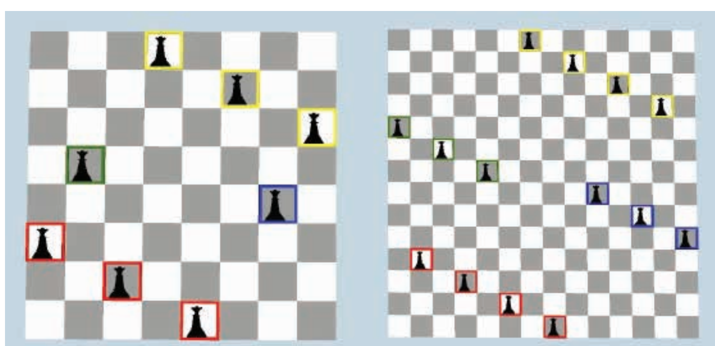

Figure 5. Arrangements when $\text{p} = 4$ (even) and $\text{p} = 7$(odd)

$2p + 1 - j$, $2p + 1 - 2j$ in the bottom half — are unequal. Thus, this arrangement for the case $n = 2p + 1, p$ congruent to 0 or 2 mod 3, indeed satisfies the requirement of the puzzle.

Case B: *p* congruent to 1 mod 3
As in Case A, we will first provide the solution for the case $n = 2p$; the solution for the case $n = 2p + 1$ will then be the same modification as before.

We arrange the queens as shown in Figure 5.

We will suggestively label the four sets of queens shown as Y, G, R, B. In the set Y, we start at the square $(1, p)$, and move in increments of two squares to the right and one square down, till we cannot move any more without falling off the right side of the board. We continue to the set G by "wrapping the pattern in Y around to the left side of the board. Thus, if the last queen in Y is in position (a, b), then the first queen in G has position $(a + 1, (b + 2)\ mod\ n)$, the second has position $(a + 2, (b + 4)\ mod\ n)$, and so on. We stop adding queens to G when the next queen to be added attacks a queen in Y. The sets R and B are then obtained by simply rotating the half of the chessboard containing the sets Y and G by 180 degree about the geometric center of the chessboard as we did in Case A. It is easy to see that the indices for the queens in the various sets are as follows:

Y	$(i,\ p+2(i-1))$	i from 1 to $(p+1)/2$, when p is odd 1 to $(p+2)/2$, when p is even
G	$(k, 2k-p-2)$	k from $(p+3)/2$ to p, when p is odd $(p+4)/2$ to p, when p is even
R	$(2p+1-j, p-2j+3)$	j from 1 to $(p+1)/2$, when p is odd 1 to $(p+2)/2$, when p is even
B	$(2p+1-l, 3p-2l+3)$	l from $(p+3)/2$ to p, when p is odd $(p+4)/2$ to p, when p is even

Table 1: (Row, Column) indices of the queens in Case B.

As in Case A, we compare different queens and show they are not on the same row, or column, or diagonal. It is clear from the row indices that the queens are all on different rows. As for columns, it is clear from the column indices that *within* each of the sets Y, G, R and B, the queens are in different columns. Similarly, the column indices show that every queen in Y is on a different column from every queen in G. By symmetry, the same is true for every queen in R and every queen in B. To compare, for instance, the column of a queen in Y and a queen in R, we notice that when p is even, the column indices of queens in both Y and G are even, while the column indices of queens in R and B are odd. When p is odd the situation is reversed, the queens in Y and G are on odd columns while those on R and B are on even columns. This shows that every queen from either Y or G is on a different column from every queen in R or B.

As for queens being on the same diagonal, it is clear again that *within* each of the sets Y, G, R and B, the queens are on different diagonals. By symmetry it is sufficient to check if a queen from Y and a queen from G, or a queen from Y and one from R, or a queen from Y and one from B, or a queen from G and one from B are on the same diagonal. As in Case A, we need to check if the slope between two squares, one from one colour and the other from the other colour, can equal +/- 1.

Subcase 1. Y and G: Plugging in the respective row and column indices from Table 1, we find the slope between a square on Y and a square on G is given as $(2(k-i)-2p)/(k-i)$. If this is +1, we find $k-i=2p$, which is impossible as the maximum of k is p while the minimum of i is 1. If the slope is -1, we find $3(k-i)=2p$, which forces p to be congruent to 0 mod 3, violating our hypothesis.

Subcase 2. Y and R. The slope works out to $(-2(i+j)+5)/(2p+1-(i+j))$. Setting this to 1 we find $4=2p+i+j$, which is impossible because p is at least 4. Setting the slope to -1 we find $2p+6=3(i+j)$, which forces p to be congruent to 0 mod 3, violating our hypothesis.

Subcase 3. Y and B. Using the row and column indices from Table 1, we find the slope is given by $(2p-2(i+l)+5)/(2p-(i+l)+1)$. If this is +1, we find $4=i+l$. This is impossible, as i is at least 1, and the condition $p > 3$ shows l is at least 4. If the slope is -1, we find $4p+6=3(i+l)$. Writing $4p$ as $3p+p$, we find p congruent to 0 mod 3, violating our hypothesis.

Subcase 4. G and B. The slope here is $(4p-2(k+l)+5)/(2p-(k+l)+1)$. Setting this to 1 we find $2p+4=k+l$, which is impossible as the maximum of k and l are each p. Setting it to -1 we find $6p+6=3(k+l)$,which is impossible for the same reason.

Finally, *when* $n = 2p + 1$, we place the queens exactly as we did in the corresponding situation in Case A: we place the $2p$ queens in rows 1 through $2p$ exactly as described above, and place the $(2p+1)$-th queen in the square with coordinates $(n, n) = (2p + 1, 2p + 1)$. None of the first $2p$ queens are on the main diagonal given by row index = column index, as is visibly clear from Figure 5 and is easily checked from Table 1. It follows from this and the placement of the $(2p+1)$-th queen that it cannot attack any of the first $2p$ queens.

Remark. An analysis of the proof in Case B shows that the solution for p congruent to 1 mod 3 is also a solution for the case p congruent to 2 mod 3. When $p = 2$, the solutions in Cases A and B are the same, but for higher values of p congruent to 2 mod 3, the two solutions are different.

References

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The **DEENS ACADEMY MATH CLUB** began in July 2019 as a small group of tenth graders meeting weekly to discuss mathematics beyond school material and to work on challenging problems. Today it has expanded to double its original membership, and continues to meet weekly using video conferencing. The discussions range from difficult International Mathematics Olympiad material to recreational mathematics. This particular problem was proposed by one of its members Anurag Manoj. He, along with Marina Berchmans, Mohammed Faraaz Ali Khan, Ananya Khare, and Chaitanya Ongole worked on this problem. They solved the problem for small cases of n first (4, 5, 6) and then arrived at the general solution by appealing to analogy. Their mentor Prof. Bharath Sethuraman suggested that they look at slopes of lines joining pairs of queens for their proofs, and they thank him for his guidance.

DIGITAL ROOT OF A PRIME NUMBER

– Biplab Roy

An integer $p > 1$ is called a prime if its only positive divisors are 1 and p.

The **digital root** or **seed number** of a natural number in a given number base is the single digit value obtained by an iterative process of summing the digits of the number, on each iteration using the result from the previous iteration to compute a digit sum. The process continues until a single-digit number is reached.

On examining the digital roots of the first 12 prime numbers, we notice something interesting.

There is no prime number less than 40, except 3 itself, whose digital root is 3 or 6 or 9. Can we prove this for all prime numbers? Try it for yourself and if you can't prove the result, turn to page **52.**

Prime Number	Digital Root	Prime Number	Digital Root	Prime Number	Digital Root
2	2	3	3	5	5
7	7	11	2	13	4
17	8	19	1	23	5
29	2	31	4	37	1

Upper Primary

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Deriving the Golden Ratio using a Scientific Calculator

EYAL N. SHAVIT

In this article, we illustrate the use of a scientific calculator for exploring the Fibonacci sequence and the Golden Ratio. The calculator, equipped with a feature for setting up iterations, enables the process of creating a sequence which rapidly converges to the Golden Ratio.

Introduction

The Fibonacci sequence is one of the best-known mathematical sequences, occurring in nature in a large variety of seemingly unrelated phenomena [1]. The seeds on the spirals of a sunflower and the number of clockwise and anticlockwise spirals in a pinecone exhibit Fibonacci numbers. Each number in the Fibonacci sequence is the sum of the two numbers that precede it. The mathematical equation describing it is

$$F_n = F_{n-1} + F_{n-2} \text{ for } n > 1 \text{ with } F_0 = 0 \text{ and } F_1 = 1.$$

The first 12 terms of the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.

The ratios of successive terms of the Fibonacci sequence, namely, 1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13,. . . approach a value 1.618 (approximately) which is referred to as the Golden Ratio and is represented by the symbol φ. According to Mario Livio [4] the Golden Ratio or φ is "the world's most astonishing number" and can be found not only in natural phenomena but also in a variety of buildings, monuments and works of art.

If we consider three consecutive Fibonacci numbers, $F(i)$, $F(i+1)$ and $F(i+2)$, for sufficiently large values of i, the ratios $F(i+1)/F(i)$ and $F(i+2)/F(i+1)$ are very close to each other and are even identical up to a few places of decimals.

Let

$$\frac{F(i+2)}{F(i+1)} = \frac{F(i+1)}{F(i)} = X \qquad (1)$$

Keywords: iteration, sequence, convergence, exploration, calculator

Replacing $F(i+2) = F(i+1) + F(i)$ we get

$$\frac{F(i+1)+F(i)}{F(i+1)} = \frac{F(i+1)}{F(i)} \tag{2}$$

which further becomes

$$1 + \frac{F(i)}{F(i+1)} = \frac{F(i+1)}{F(i)} \tag{3}$$

In terms of X we may rewrite (3) as

$$1 + \frac{1}{X} = X \text{ which simplifies to } X^2 - X - 1 = 0$$

Solving the quadratic equation in X we get $X = \frac{1 \pm \sqrt{5}}{2}$. We ignore the negative value as it is clearly not applicable here. Thus $X = \frac{1+\sqrt{5}}{2} \approx 1.618033989$ (to 9 decimal places) is the Golden ratio or φ.

Here we assumed that the ratios of two successive pairs of terms of the Fibonacci sequence were equal. This only happens *in the limit*. Thus, as the sequence progresses, the ratios get closer and closer to the limiting value, given by φ. This can easily be demonstrated by using a digital computer. Volume 3, issue 1 of At Right Angles (March 2014) included an article [1] which describes the exploration of the Fibonacci sequence and the Golden Ratio using a spreadsheet.

In this article, we shall derive the Golden Ratio using a scientific calculator. Our approach is to solve the equation, $X = 1 + 1/X$ iteratively. The program is very simple, containing essentially one command, which is repeated several times. The calculation converges very rapidly to the value of the Golden Ratio.

In each step of the iteration process, the corresponding numbers of the Fibonacci sequence are displayed on the calculator screen. In order to understand this equation, we at first show the equivalence between the Golden Ratio and its representation as a continued fraction thus defined by the expression

$$\boxed{\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

Because the continued fraction goes on forever, the denominator of the second term is φ itself. The continued fractions can also be defined recursively. Thus the first term is $1 + \frac{1}{1}$ and the second term is $1 + \frac{1}{1+\frac{1}{1}}$. Thus every continued fraction can be written as $1 + \frac{1}{\text{the previous continued fraction}}$.

The value of $1 + \frac{1}{1}$ is 2, the value of $1 + \frac{1}{1+\frac{1}{1}} = 1 + \frac{1}{2} = \frac{3}{2}$.

Proceeding in this manner, we see that the values of the fractions are 2, 3/2, 5/3, 8/5 . . . and so on, which converge to the Golden Ratio φ.

Thus at any stage, we will get a fraction that is the ratio of two consecutive Fibonacci Numbers. In the iteration process we are computing more and more of this continued fraction.

The Recurrence Relation

Here we make use of a recursion relation for calculating the golden ratio, based on the equation $X = 1 + 1/X$. The recurrence relation is given by

$$R_{i+1} = 1 + \frac{1}{R_i} \tag{4}$$

where i denotes the step number of the recursion sequence, with R_i converging to the value of the golden ratio with increasing i. For the initial choice of $R_1 = 1$ the calculated values of R_i as we go down the sequence gives the ratios of successive Fibonacci numbers. Specifically, where $F(i)$ denotes the i-th Fibonacci number, the calculation yields

$$R_i = \frac{F(i+1)}{F(i)} \tag{5}$$

Using the calculator, these ratios can be presented as ratios of natural numbers, thus displaying the Fibonacci sequence. Interestingly, if we start with $R_1 = 1$ we obtain the continued fraction sequence which we have seen in the earlier section.

$$1 + \frac{1}{1}, 1 + \frac{1}{1 + \frac{1}{1}}, 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}, \ldots$$

However, if we start with $R_1 = 5$ (say) we get the continued fraction sequence as

$$1 + \frac{1}{5}, 1 + \frac{1}{1 + \frac{1}{5}}, 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5}}}, \ldots$$

where the terms lead to the sequence 6/5, 11/6, 17/11,... which converges to the Golden Ratio, more quickly than if $R_1 = 1$.

Programing the calculator. Let us see how to program the iterative procedure on a scientific calculator: The procedure is based on the command **1 + 1/ANS** (ANS is the stored number obtained after pressing the = sign). When we print the above command, each pressing of the = sign repeats the command, with the previous calculated value inserted into ANS. This command provides a natural path for the iterations. Though we have used the Casio fx-82MS calculator, other scientific calculators having the ANS feature may also be used in a similar manner.

The steps to be followed to generate this recursion on the calculator are as follows:

Step 1: Press the number key 1 followed by "=" (this is equivalent to $i = 1$)

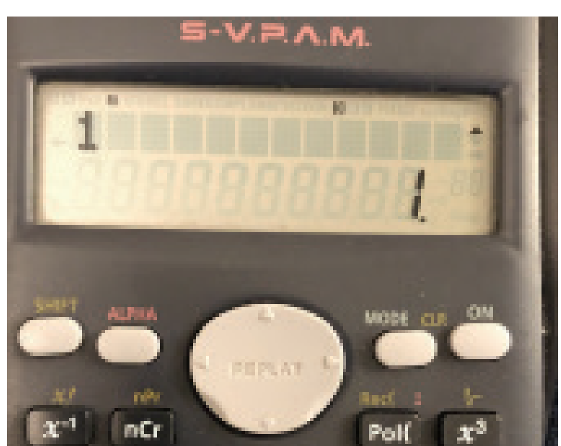

Figure 1. Calculator snapshot for first iteration

Step 2: Type the equation **1 + 1/ANS**, followed by the = sign (this may be repeated as many times as the required number of iterations).

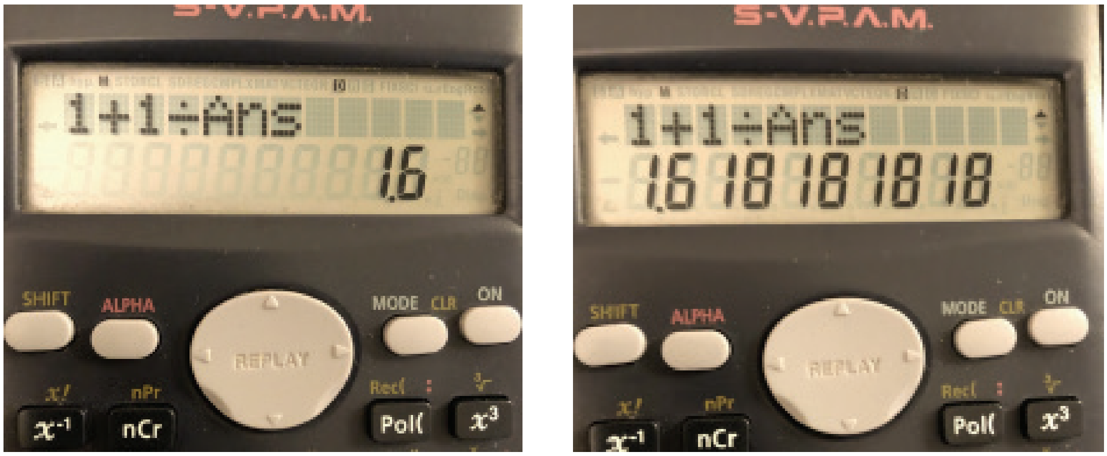

Figure 2. Iteration results for fifth step left and tenth step right.

Each time we press the = key we are asking the calculator to carry out an iteration step of the equation $\boldsymbol{R_i = 1 + 1/R_{i-}}$. Pressing **SHIFT** followed by the **ab/c** key converts the calculated number to a fraction, where the numbers in the numerator and denominator are the consecutive terms of the Fibonacci sequence, the ratios of which give the current value of R.

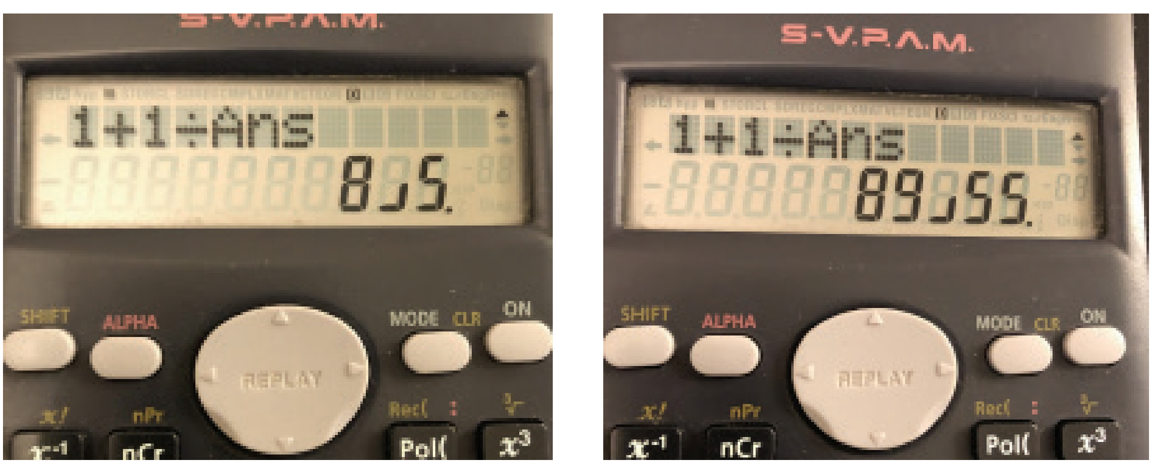

Figure 3. Fifth and tenth iteration values, shown in Figure 2, presented as ratios of Fibonacci numbers.

Results

The result obtained after 23 iterations is the golden ratio, that is, 1.618033989 (up to 9 places of decimals).

Using (4), if $R_1 = 1$ the calculator displays 3/2 in the second iteration, all the way up to 2584/1597 in the 17th iteration.

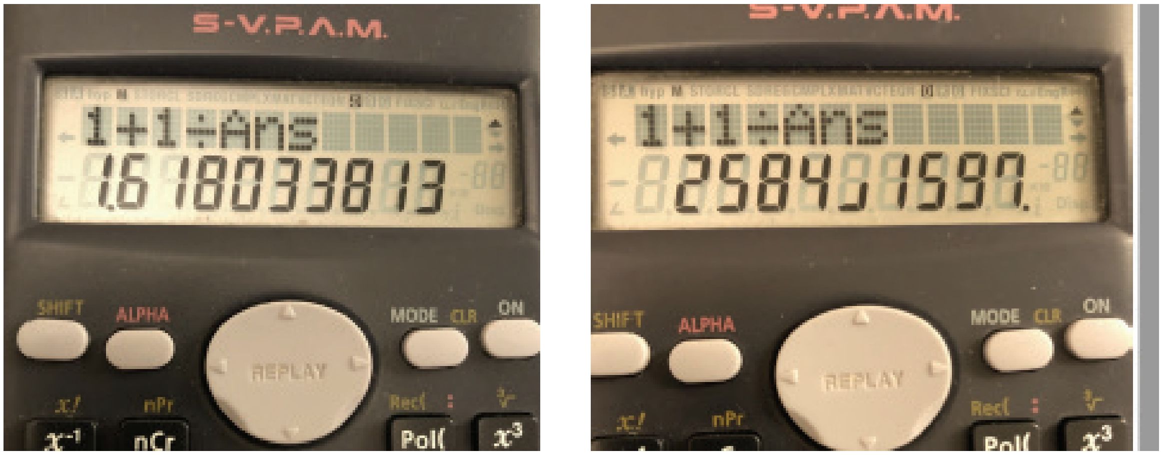

Figure 4. Step 17: On the left iteration value at step 17, on the right this value as presented as ratios of Fibonacci numbers

After the 17th iteration the calculator does not display the Fibonacci numbers.

For values of R_1 not equal to 1 (or to numbers which are given by the ratio of two consecutive Fibonacci terms), the results eventually converge to the above quoted value for the golden ratio, but without the option of displaying the Fibonacci sequence. For $R_1 = 7$ the calculation converges after 24 steps to the value of 1.618033989, the same as for $R_1 = 1$.

The convergence is very robust giving the correct result for R even if R_1 is as large as 10,000, also after 24 steps

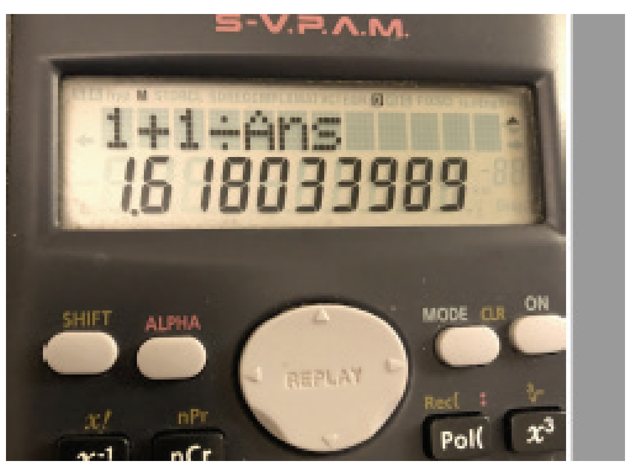

Figure 5. Converged value after 24 steps, with initial value of $R_1 = 10000$

Discussion

While there are software packages available for exploring the Fibonacci sequence and the golden ratio, a simple scientific calculator, if used appropriately, can also lead to deep insights. The recursion feature on the calculator can be effectively utilized to gain insight into the derivation of the golden ratio and other recursive patterns. It is appropriate to mention here, that a similar procedure, although with a larger number of calculator steps, was carried out by Knott [2]. It is also worthwhile to mention the extensive work done by Kissane [3] on the use of scientific calculators in mathematical exploration.

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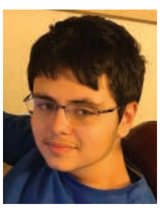

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DIGITAL ROOT OF A PRIME NUMBER

– Biplab Roy

Proof: There is no prime number except 3 itself, whose digital root is 3 or 6 or 9. (See page **46** for the observation and the conjecture)

Let N = $a_m.10^m + a_{m-1}\,10^{m-1} + a_{m-2}.10^{m-2} + \ldots + a_1.10 + a_0$ be an arbitrary prime number other than 3, and let $S = a_m + a_{m-1} + a_{m-2} + \ldots + a_1 + a_0$.

It is obvious that, if S is single digit, then S is the digital root of N, otherwise the digital roots of S and N are same.

Also, we know that $3|S \leftrightarrow 3|\,N$

Since N is a prime number other than 3, 3 is not a divisor of N.

So, 3 is not a divisor of S.

If the digital root of S is 3 or 6 or 9, then 3|S, but this is a contradiction.

So, the digital root of S must not be 3 or 6 or 9.

So, the digital root of N must not be 3 or 6 or 9.

Hence the theorem proved.

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A Cellular Automata Model for Predicting Crowd Movement during Evacuation

ANMOL SINGH DHALIWAL, ANTARA GHOSH, NAMAN MANSUKHANI

> All the wonders of the universe can in effect be captured by simple rules, yet…there can be no way to know all the consequences of these rules, except, in effect, just to watch and see how they unfold.
>
> **– Stephen Wolfram**

Introduction

We usually go for outings to shopping malls, cinema halls, and amusement parks or go to our respective workplaces, such as offices or schools. But have you ever paid attention to that small evacuation map hanging on the side of the corridor of your building? We usually walk past these and take these for granted. But what if an actual earthquake or some other disaster were to happen? How would you manage the situation then?

Many public places attract large crowds of people and are susceptible to stampedes and other disasters. During a panic situation, say an earthquake, if people are inside a building, they would generally move in a random fashion causing crowding and clogging at exit points. Often an evacuation system exists, but it isn't the best option. Hence, having an effective evacuation plan is extremely crucial in such situations. Mathematical modeling can help us develop more efficient evacuation plans.

Researchers have been developing mathematical models for pedestrian movements and evacuation systems. Many of these models are based on differential equations [2]. Such models are highly complex and often difficult to interpret and understand. Numerical methods are used to solve such models and these may be computationally expensive and time consuming.

In this article we will describe a project in which the concept of cellular automata (CA) has been used to develop a simulation model

Keywords: modeling, cellular automata, simulation, math applications

for an evacuation system. Cellular automata is a mathematical tool which can govern complex systems through simple rules. It has applications in traffic flow modeling, structural design, neural networks, forest fires, ant colony activity, crystal growth and in many other fields [1]. Cellular automaton models can be one, two or even three dimensional. They can be generated and explored through simulation and their outputs are very conducive for exploring real life crisis situations.

The aim of the project was to

1. Apply the two dimensional Cellular Automata to simulate the movement of people in a crowded room, hall or corridor.
2. Develop a Python driven CA model to simulate crowd movement in times of a crisis.

Some mathematical preliminaries

In this section we shall explore some basic questions related to cellular automata. To begin with, what is a cellular automaton?

A cellular automaton (CA) is a collection of cells on a grid of a specified shape that evolves through discrete time steps according to a set of rules based on the state (or color) of the neighbouring cells.

- Each cell has a state – dead or alive. Pictorially the dead cells can be represented by white color and the live cells are coloured black. They can also be represented using 0s and 1s, where 0 represents a dead cell and 1 represents a live cell.
- Each cell has a neighbourhood. A neighbourhood of a given cell is a set of cells which are adjacent to it.
- Every CA must have a certain set of rules based on which it evolves through various time steps. These rules are referred to as the defining rules of a CA.

One-dimensional cellular automata

In a one-dimensional cellular automaton (also referred to as an elementary cellular automaton or ECA), the state of a cell can be either 0 or 1 (dead or alive). Such automata are represented on a linear grid of cells, in which each cell has two neighbours (the cell to its left and the one to its right). Each row of the automata represents a different generation or evolution in a different time step. A more detailed account of ECA may be found in the article titled *The Elementary Cellular Automata: A journey into the computational world* [3] published in the March 2018 issue of At Right Angles.

If we consider any three adjacent cells of a linear grid, each of these can be coloured either black or white. Hence, there are a total of 2 x 2 x 2 = 8 possibilities of colouring a set of three adjacent cells. See Figure 1, in which the top rows (of three cells each) represent these eight combinations. Further, each of these combinations can be assigned 0 or 1 (as indicated in Figure 1). Hence, the total number of defining rules in a one-dimensional CA is $2^8 = 256$ possibilities. These possibilities or ECA combinations, are numbered from 0 to 255. Each ECA combination is therefore identified by its *rule number* (from 0 to 255). They can also be represented as binary numbers. For example, Figure 1 represents the defining rule for ECA rule 30 which also has the binary representation 00011110.

The defining rule 00011110 may be treated as a binary number whose decimal representation (namely, 30) is obtained as follows

$0 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 30.$

The evolution of a one-dimensional CA starts with an initial state (generation 0), and evolves according to one of the 256 ECA rules. Different

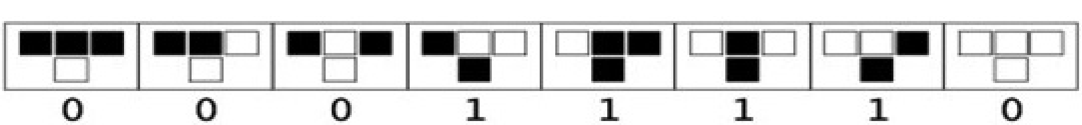


Figure 1. Defining rule for ECA rule 30.

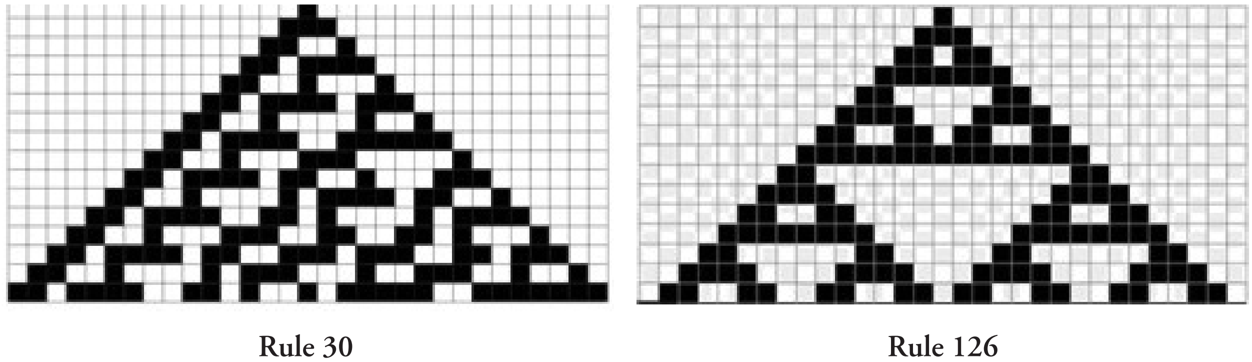


Figure 2. Evolution of ECA rule 30 and rule 126 over 15 time steps starting with a single live cell in generation 0 (indicated by the topmost row).

rules generate different patterns. For example, rule 30 leads to a chaotic pattern, and rule 126 leads to a Sierpinski triangle like pattern. Figure 2 shows the first 20 generations (where each row of the grid represents a different generation) of the ECA rules 30 and 126 respectively.

Two-dimensional Cellular Automata

Like one-dimensional CA, the state of a cell in two-dimensional CA is also either dead or alive. A more detailed analysis of two – dimensional cellular automata may be found in [4]. However, a two-dimensional automaton evolves on a grid of square cells. One way to visualise a two-dimensional cellular automaton is to imagine an infinite sheet of squared paper and a set of rules according to which the cells evolve from one generation to the next. Each cell in a grid has neighbourhood cells which are the cells adjacent to it.

The two types of neighbourhoods widely used in two-dimensional CA are (a) the Von Neumann neighbourhood, and (b) the Moore neighbourhood. As shown in Figure 3, the

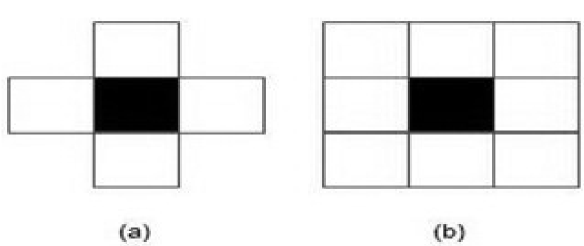


Figure 3. Two dimensional cellular automata neighbourhoods

Von Neumann neighbourhood of a given cell comprises the cells adjacent to it and positioned just above, below, to the left and to the right. In the case of the Moore neighbourhood, all the eight cells surrounding a given cell comprise the neighbourhood.

The state of the center cell in the next generation depends on all its neighbourhood cells in the current one. If we take the Moore neighbourhood, each of the nine cells can be either black or white. Hence, the total combinations would be $2^9 = 512$ combinations. The central cell of each of these 512 possibilities can change in any way in the next time step according to defined rules. This leads to a total of 2^{512} possibilities which is a very huge number!

Game of Life

One of the very well-known and popular two dimensional cellular automata is the *Game of Life*. It is a game based on cellular automata that was invented by the British mathematician John Horton Conway [4] in the1960s. It is a one-player game, that is, only the initial configuration of cells is required and the game continues according to certain rules.

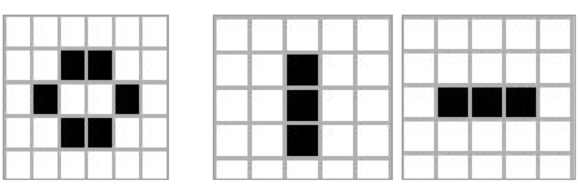
The rules of the Game of Life are as follows:

1. Any live cell with less than two live neighbours dies (due to lack of resources).
2. Any live cell with two or three live neighbours lives on to the next generation.

3. Any live cell with more than three live neighbours dies (due to excess resources and overcrowding).
4. Any dead cell with exactly three live neighbours becomes a live cell (by reproduction).

This implies that any initial configuration of cells (referred to as generation 0) will be subjected to the above four rules and will lead to a new configuration, that is, generation 1. Generation 1 will further undergo the rules (1 to 4) and so on. Interesting patterns are seen to emerge after a few generations. Some common pattern types include: *still lifes*, which do not change from one generation to the next; *oscillators*, which return to their initial state after a finite number of generations; and *spaceships*, which translate themselves across the grid. A more detailed account is available in [5].

Figure 4 illustrates one example of each type. Pattern (a) represents a still life called *beehive*, (b) represents an oscillator called *blinker* which oscillates between the two states and (c) represents a *glider*. The reader is urged to play the game online at https://playgameoflife.com/



(a) Beehive (still life) (b) Blinker (oscillator)

(c) Glider (spaceship)

Figure 4. Examples of patterns emerging from John Conway's Game of Life

A Two-dimensional Cellular Automata model to simulate the behaviour of people during evacuation

Let us imagine that the floor of a hall is a rectangular grid consisting of small square cells (Figure 5(a)). Each person in the hall occupies exactly one cell in a given time step. Thus a cell is *live* if it is occupied by a person and is *dead* if it is unoccupied (Figure 5 (b)). The state or configuration of the system at a given time step is represented by the black and white cells.

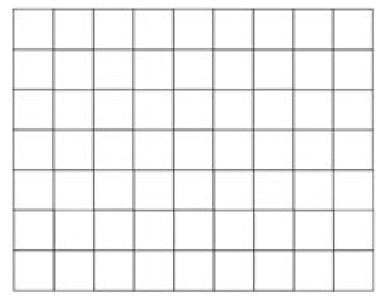


Figure 5(a). A grid of square cells representing the floor of a hall

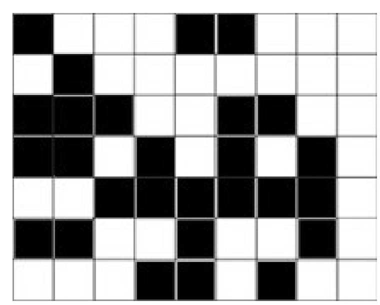


Figure 5(b). Black cells represent people standing inside the hall

The eight cells surrounding a given cell will be referred to as its neighbours (similar to Moore neighbourhood). In the next time step a person can move to any one of its 8 neighbouring cells.

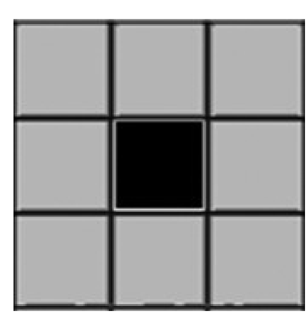


Figure 6. The black cell represents the central cell and the grey cells are its neighbours

Movement Rules

The rules determining the movement of the people occupying the room are as follows:

1. A person can move **only one cell at a time** and it can move only to its defined neighbourhood cells.
2. Logically, a person would want to take the **shortest route to the exit**. So, (s)he will move to one of its neighbourhood cells which is closest to the exit. If an obstacle is present, (s)he will move to the neighbourhood cell second closest to the exit and so on.
3. A person **cannot move in the direction opposite to the exit**. In case his/her movement is completely restricted by other neighbouring cells, (s)he does not move.
4. If there is a conflict between two people for one cell, the one **closer to the exit will prevail**, that is, will get to move.

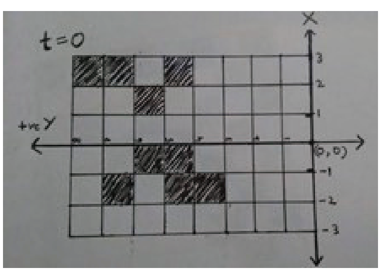


Figure 7. The state or configuration of the room at time t=0

In Figure 7 we have an 8 x 6 sized grid (representing a room) with the exit positioned at the origin. The black cells represent cells occupied by people at time t = 0.

Each cell has specific coordinates assigned to it. Figure 8 represents the configuration at time t = 1, after applying the movement rules 1 to 4.

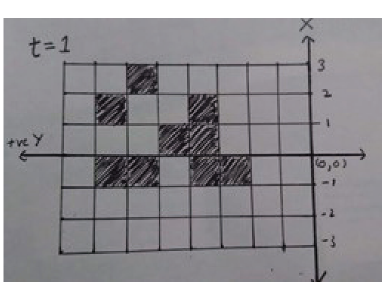


Figure 8. The configuration of the hall at time t=1

Movement rules (such as rule 1 to 4 mentioned above) can be used to simulate the movement behaviour of people during an evacuation.

The Python-driven CA Model for a single room

The goal of this model was to simulate the movement of people exiting a single room. The model was simulated using the Python mode for Processing, an open-source graphics library. The model is based on an m x n grid, with the exit at the cell corresponding to the mth row and the nth column (the bottom right corner). The model takes in another argument x, the number of people in the room.

Occupied and empty cells are represented by black and white colours respectively. For each iteration of the model, x random positions are coloured black at t = 0. The configuration of the grid at t = n is a function of the configuration at t = n – 1. In order to determine the configuration at the next step, the cells adopt a 'greedy' strategy. Each cell tries to move along the diagonal if possible; if this is not possible, then the cell moves randomly either to the bottom or to the right, depending on whether the desired cell is empty. For each iteration, the cell at the front gets the preference in movement, as is seen in real-world situations as well. Figure 9 illustrates the simulation where people are exiting a room (represented by an 8 by 8 grid enclosed within the red square) where the exit is located in the bottom right corner of the square.

The algorithmic complexity in order to determine the configuration at the next time-step is O(mn), where m = number of rows and n = number of columns.

The Python driven CA Model for a multi-room floor evacuation

The goal of the second simulation was to extend the first one and study floor-based evacuations rather than room-based ones. It was developed purely in Python using industry-standard addon libraries like 'numpy' and 'matplotlib', and the inbuilt libraries 'time' and 'random'. Typically

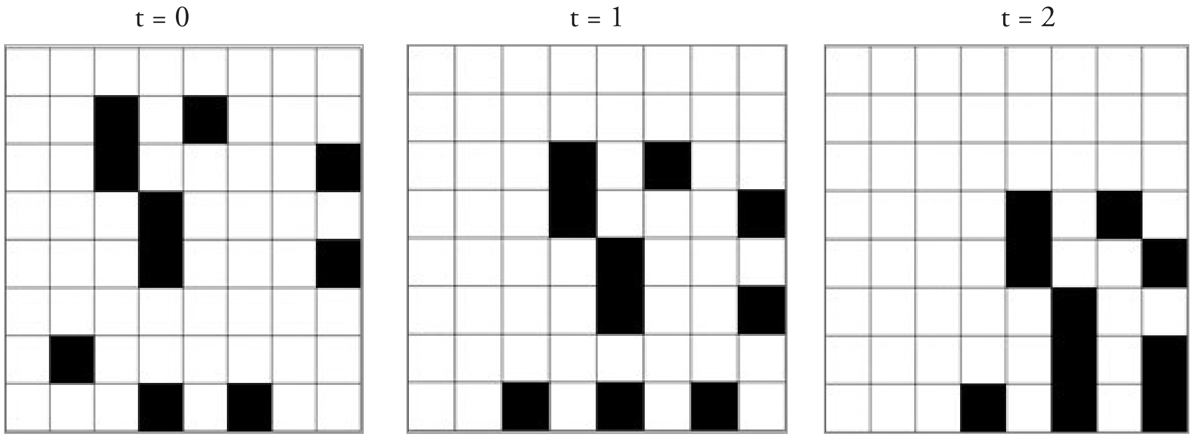


Figure 9: Model with m = n = 8 and k = 10 at time steps t = 0, t = 1 and t = 2

such problems would require the use of graph theory to find the shortest path for each evacuee. However, our model simplifies this by dividing the coordinate frame into unit cells, and using 2D Euclidean distance measurements between cells, binary matrices, and sorting algorithms to generate the movements at each time step.

Simple shifting of origin concepts are used to convert evacuee positions in each room to global coordinates for the floor, and a 'global' binary matrix is maintained to keep track of occupied cells for the entire floor.

In this model [x, y] denotes a matrix index, (x, y) denotes a 2D coordinate, and {cx, cy} denotes a movement pair of cx units horizontally, and cy units vertically.

Any closed space (a room or a corridor) is defined using (a) a bottom left coordinate, and (b) its size as a width and height pair. An example definition would be a bottom left coordinate of (1, 1) and a size of (20, 20), indicating a room/corridor that starts at (1, 1) in the global reference frame, and has a size of 20 x 20. Thus, it would be a square having a diagonal between the points (1, 1) and (21, 21).

Figures 10 and 11 are outputs of the Python simulation. Figure 10 shows three rooms of size 5 by 5 which open into a corridor while Figure 11 shows three rooms of varying sizes along with a corridor. The boundaries of the rooms and the corridor are indicated in red. Instead of narrowing down the frame to the smallest bounding box that would enclose all rooms and the corridor, five rows and five columns, respectively, have been padded around to improve aesthetic appeal.

The exits, represented by blue coloured lines, are also defined as coordinate pairs, in the frame of reference of each room, and then for the corridor as well. Exits of different lengths can be defined just by declaring their component edges of unit length. Rooms and corridors must have exactly one common edge, and the exit must lie on that edge. This is ensured while defining the model.

Steps of simulation

1. The number of evacuees in each room can be defined, and their locations are generated randomly without any overlaps. The locations are returned as coordinate pairs, local to each room. These occupied cell locations are assigned a value of 1 in the binary matrix, with empty cells having value 0.
2. A separate 'labels' array is also maintained for assigning numbers to each cell in each room based on order of random generation while in the room. The label for a cell is subsequently prefixed with a specific letter of the alphabet as soon as it exits the room. For example, in the frame of reference of room 'a', cell 1 refers to the first randomly generated coordinate cell. While it is in room 'a', it will remain as '1', and as soon as it exits the room, it will become 'a1' in the corridor.

- For example, visualise a unit square based grid, with a movement of [2, 2] → [1, 3] in a 3 x 3 room having bottom left corner as (5, 5). This will result in the transfer of colour black from the cell with bottom left corner as (6, 6) to the cell with bottom left corner as (7, 7) in the global frame. The underlying concept is just that matrix row indices increase in a top - bottom manner, whereas Cartesian y coordinates increase in a bottom - top manner, with row indices in matrices being indicative of the y coordinate, and column indices being indicative of the x coordinate.

The algorithm

The algorithm starts by calculating the euclidean distance between the center of each exit and the corresponding cell location for each cell. It has the capability of simulating two kinds of scenarios. Firstly, where each cell independently moves to its optimal cell in every timestep i.e. directly moving to the one that will take it closest to the exit (this may result in many collisions), and secondly, where each cell moves to a distinct cell depending on its closeness to the exit, factoring in neighbour awareness, which means this will result in no collisions.

In both scenarios, in every timestep, the cell may only move to any cell in its Moore neighbourhood (scenario 1), or additionally also have the capability of staying at its position (scenario 2). To exit, the cell must reach the unit cell having a common edge with the corridor and lying inside the particular room.

In the first scenario (Figure 10), a cell's final position is simply assigned as the one that will take it closest to the exit. The best step of the cell is therefore just a movement of +1, -1 or 0 in each of the two x and y directions, depending on the location of the exit. However, in this scenario, since it is effectively moving blindly i.e. simulating panic, it must move at least one unit in every timestep, so a {0, 0} i.e. static (no movement) step is not allowed.

In the following parts, when we say that an exit is located at a matrix index, we imply that that cell will have the exit(s) as its edge(s).

Consider an exit at [2, 5] in a room matrix. For an evacuee located at [5, 4], the best movement path will be [5, 4] → [4, 5] {-1, +1} → [3, 5] {-1, 0} → [2, 5] {-1, 0} → exit. However, if there is another cell located at [5, 5] in the first time step, it will also strive to move to [4, 5], and in this scenario, a *collision* will occur. The number of collisions are counted by determining the number of cells that have the same final coordinates at each time step, and then dividing by 2. Each collision has been given a one second penalty to simulate the real-world delay it causes, and in the plots, the labels for those colliding cells are overlaid on top of each other to also give a visual effect. The simulation continues until the last cell exits the corridor.

In the second scenario (Figure 11), the final positions are calculated for each cell in a hierarchical format depending on their closeness to the exit. This means that a cell closer to the exit will have the chance to move first and choose its preferred coordinate before the others. Once it chooses the final location, that location is invalidated for the other cells, implying an absolute *collision-free ideal evacuation*. If two cells are equidistant from the exit, e.g. lying in the left and right upper diagonals of the Moore neighbourhood of the target exit cell, the one having lower label value will get to move first. This also means that there is a possibility for a cell to remain at its current position in a timestep if all of its preferred locations have already been occupied by the cells that moved before it.

Screenshots of simulations

Scenario 1: Note the overlaps (in Figure 10 (a); an overlap can be seen in the lower room 'b', two units directly below the exit and in Figure 10 (b) one overlap can be seen in the cell lying in the middle row of the corridor just two units away from the long blue line center). The overlaps depict collisions and hence delays.

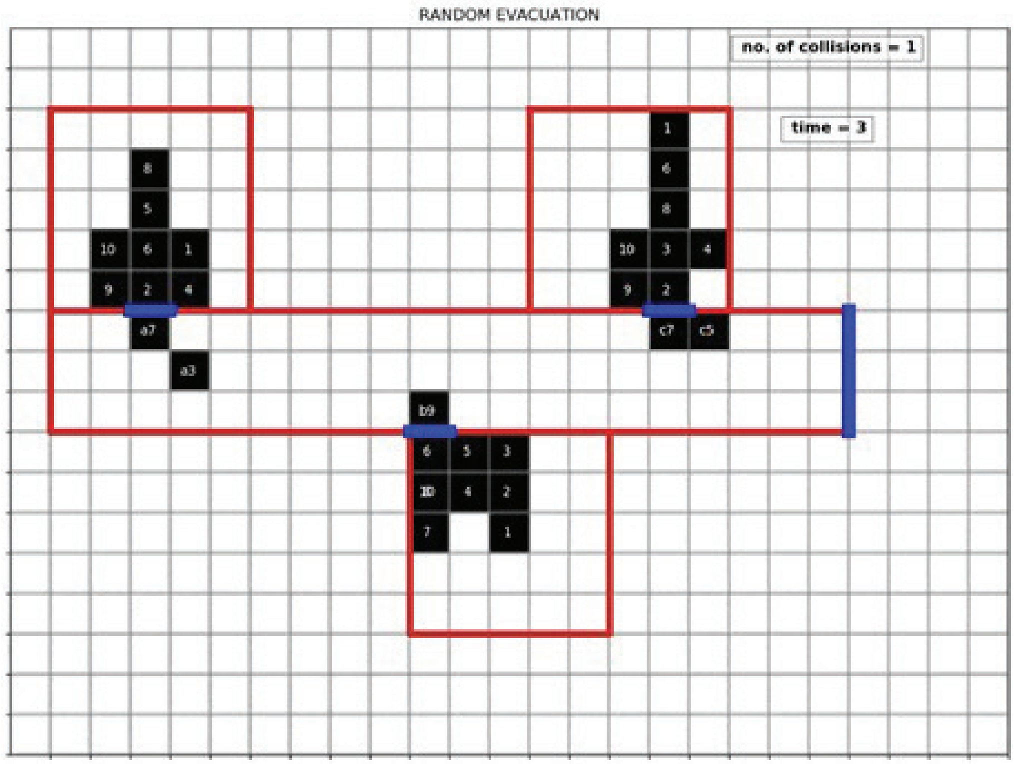


Figure 10(a). Scenario 1 simulation at t = 3 seconds

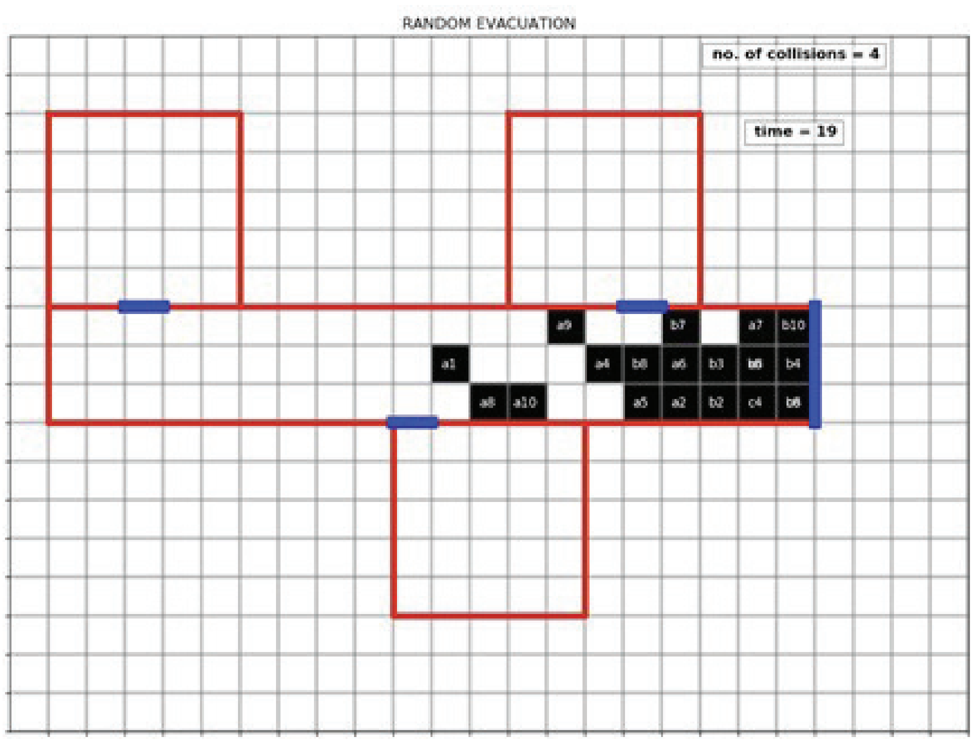


Figure 10(b). Scenario 1 simulation at t = 19 seconds

Scenario 2: A perfect orderly movement of the cells is visible. Also note the different room sizes and exit locations, showing the versatility of the algorithm.

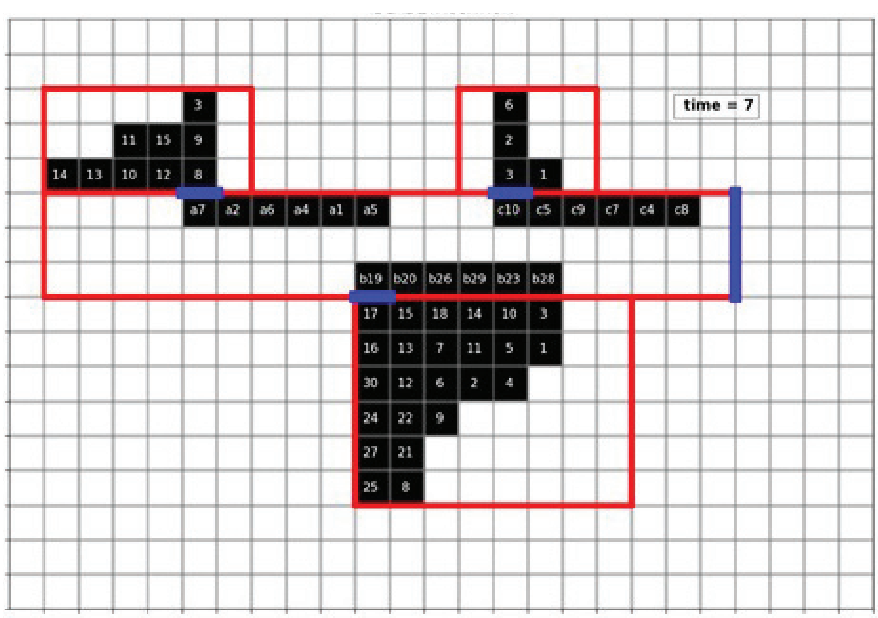


Fig 11(a): Scenario 2 simulation at t = 7 seconds

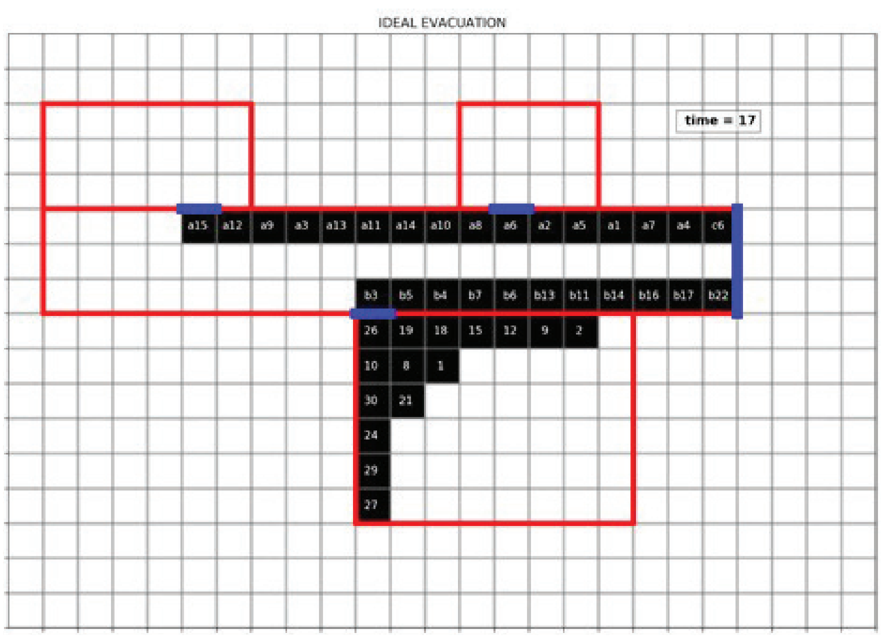


Figure 11(b). Scenario 2 simulation at t = 17 seconds

Supported by intuition, it was observed that the evacuation time for scenario 2 simulations was always less than scenario 1 simulations, for the same initial configuration of cells. Smaller rooms would have greater collisions, and thus the difference between the evacuation times between the two scenarios was more in those cases.

Conclusion

In this article we have tried to show that Mathematics and Computer Science can prove to be the ideal tools for studying phenomena such as evacuation during a crisis situation. Initially, we thought of using one dimensional ECA to model crowd behaviour in corridors but it proved to be unrealistic and less efficient. The use of two dimensional CA was a necessity in this case.

The Python implementations verified that panic caused by random movements of people inevitably lead to longer evacuation times and can be harmful in the long run. The proposed model of rule-based evacuations with neighbour awareness is an example of a robust model that will lead to ideal, collision-free evacuations with least possibility for injuries and loss of life.

This project can be extended to simulation of people exiting a hall with two or more exits and identifying an ideal path for evacuation. We also plan to increase the complexity of the model by studying bidirectional crowd movements (such as at a subway or a metro station) and finding an efficient means of crowd management.

Finally, we would like to thank Mr. Mukesh Kumar, Head of Computer Science Department, Delhi Public School, R K Puram, for his support and guidance in developing the algorithms and Python implementations. We would also like to acknowledge Mr. Anil Kathuria, Head, Department of Mathematics, and our teachers Ms. Tandeep Kaur, Ms. Shalini Monga and Ms. Nagalakshmi for their encouragement and support.

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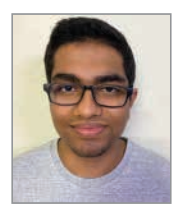

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The authors are students of class 12. Anmol Singh Dhaliwal has three published articles related to using Wolfram Mathematica in robotics and English language teaching to underprivileged children. His interests are coding, robotics and mathematical modelling. Antara Ghosh has interests varying from dancing and playing basketball to physics and mathematics and solving real life problems. Naman Mansukhani enjoys spending time on websites like Codeforces and Brilliant and is interested in machine learning and neural networks. He enjoys learning about other cultures and their languages.

Miscellaneous problems for the MIDDLE SCHOOL

A. RAMACHANDRAN

Here are some problems collected from various sources. They can be termed 'word problems' – some of them can be termed 'story problems' as they present an episode leading to a question.

Problem–IX–3–M–1. Three travelers meet at an oasis one evening and quickly become friends. As the day draws to a close, one of them (say traveler A) says, "Friends, I haven't brought any food with me. If you have brought some, could you share with me? I can pay for it." Traveler B says he has 3 loaves of bread while traveler C says he has 5 loaves with him. They then pool the loaves and share them equally. Next morning traveler A gets up early, while the other two are still asleep. He leaves 8 silver coins and departs. When the others wake up and see the coins, traveler B remarks, "Since I had 3 loaves and you had 5, I shall take 3 coins and you can have 5."

Do you think that is a fair deal? If not, what is a fair division of the coins?

Problem–IX–3–M–2. A farmer goes to market with a basket of eggs for sale. His first customer buys half the eggs in the basket and half an egg. His next customer buys half the eggs left and half an egg. Then a third customer buys half of what is left now and half an egg. If 3 eggs are now left in the basket, how many were there at the start?

Problem–IX–3–M–3. A bakery sells cookies of three types priced at Rs.3, Rs.2 and Rs.0.50 per piece. A person buys 20 cookies (including all types) and is charged Rs.20. How many does he buy of each type?

Keywords: Word problems, logic, visualisation, reasoning

Problem–IX–3–M–4. A census-taker knocks at a door, which is opened by a lady. On being questioned, she says, "I have three children. The product of their ages is 36. The sum of their ages is this door number." After a pause, the other says, "Sorry. I am unable to work out their ages." The lady then says, "My eldest child plays the piano." The other then says, "Oh, good. Now I know their ages."

What are the ages of the three children? (The term 'age' refers to the number of years completed by the person, i.e., it is an integer.)

Problem–IX–3–M–5. Boss calls his peon and says, "Take this cheque to the bank and encash it. On the way back, please take 20 paise for a cup of tea and bring me the rest of the amount." The cashier at the bank, in a fit of absentmindedness, interchanges the figures given in the cheque under rupees and paise, and hands over the amount. After spending 20 p. the peon hands over the rest to Boss. On checking the amount, Boss exclaims, "Very strange. This is twice the amount I had indicated."

What was the amount mentioned in the cheque?

Pedagogical Note

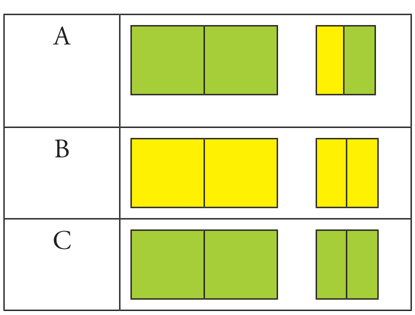
These word problems can become very interesting for students if they are encouraged to model the situation. Teachers may also provide the representations as a stimulus to jump start their thinking. This process of problem solving can help take away their aversion to word problems.

Solutions to Problem–IX–3–M–1.

8 loaves

B's contribution C's contribution

Each person's share

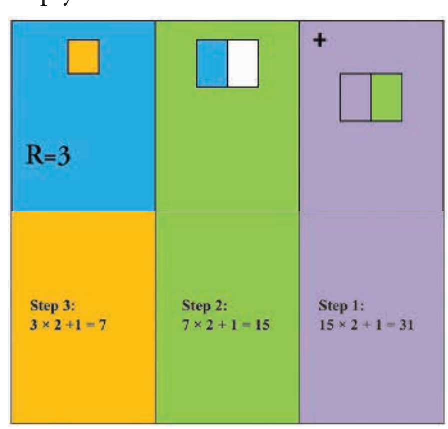


A gets $\frac{1}{3}$ of the bread loaf from B and $\frac{7}{3}$ from C. So, of the 8 coins B should get 1 part which is 1 coin and C 7 parts, that is, 7 coins.

Alternative Solution: 8 loaves were shared by three persons equally. So each consumed $\frac{8}{3}$ loaves. Traveler B brought 3 loaves, out of which he consumed $\frac{8}{3}$ and so his contribution towards traveler A's share is $3-\frac{8}{3}=\frac{1}{3}$. Traveler C brought 5 loaves, out of which he consumed $\frac{8}{3}$ and so his contribution towards traveler A's share is $5-\frac{8}{3}=\frac{7}{3}$. So it would be fair if B takes 1 coin and C takes 7 coins.

Solutions to Problem–IX–3–M–2.

Working backwards from the final answer, we reverse all the operations. So, instead of halving, we double and instead of dividing or subtracting, we multiply or add.



Alternative Solution: If the original number of eggs in the basket is taken as x, we can form the following equation:

$\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}x-\frac{1}{2}\right)-\frac{1}{2}\right)-\frac{1}{2}=3$, yielding $x = 31$.

Solutions to Problem–IX–3–M–3.

Here, the 3 types of cookies should add up to 20 cookies and the cost of the 20 cookies should also sum to ₹20

Number of cookies	p	q	r	$p+q+r=20$
Cost	$a=3p$	$b=2q$	$c=½\,r$	$a+b+c=20$

Guess and check now:

Case 1: If c is ₹ 10, then $r = 20$ (not possible with the given condition)

Case 2: If $c =$ ₹ 9, we have $r = 18$; then $p + q = 2$

p	q	$a+b$
1	1	$3 + 2 = 5$; and $a + b + c < 20$

Case 3: If $c =$ ₹ 8, we have $r = 16$; then $p + q = 4$

p	q	$a+b$
1	3	$3 + 6 = 9$; and $a + b + c < 20$
3	1	$9 + 2 = 11$ and $a + b + c < 20$
2	2	$6 + 4 = 10$ and $a + b + c < 20$

Case 4: If c is ₹ 7, we have $r = 14$; then $p + q = 6$

p	q	$a+b$
1	5	$3 + 10 = 13$; and $a + b + c = 20$ (meets the condition)
5	1	$15 + 2 = 17$ and $a + b + c > 20$
4	2	$12 + 4 = 16$ and $a + b + c > 20$
2	4	$6 + 8 = 14$ and $a + b + c > 20$
3	3	$9 + 6 = 15$ and $a + b + c > 20$

Alternative Solution: If we take the number of cookies of the first type to be x and the number of cookies of the second type to be y, then we can form the following equation:

$$3x + 2y + \tfrac{1}{2}(20 - x - y) = 20.$$

Upon simplification this yields $5x + 3y = 20$, with the following possible integer solutions:

$x = 1, y = 5$, and $x = 4, y = 0$.

Rejecting the second solution as we are told that all three types were bought, we have (1,5,14) as the solution set.

You can try similar strategies for the remaining problems:

Solution to Problem–IX–3–M–4.

If the product of three numbers is 36, we have the following possible sets of factors:

(1,1,36), (1,2,18), (1,3,12), (1,4,9), (1,6,6), (2,2,9), (2,3,6) and (3,3,4). The sums of the numbers in the above sets are 38, 21, 16, 14, 13, 13, 11 and 10. Of these, all except 13 are unique. So the door number must be 13, else the census-taker would have obtained the ages of the children at that stage. The last statement of the lady tells us that there is *an* eldest child, not two. This leads to the solution (2,2,9).

Solution to Problem–IX–3–M–5.

Assuming that the cheque was made out for ₹ X and Y paise, the expected amount was 100X + Y paise. The amount issued by the cashier was ₹ Y and X paise, i.e., 100Y + X paise. After 20 paise was spent from this it became 100Y + X – 20 paise. This is twice the amount indicated in the cheque. So we can form the equation 2(100X + Y) = 100Y + X – 20.

On simplification this yields 199X = 98Y – 20. We need to obtain integer values of X and Y that satisfy this equation. As this appears to be a formidable task, we try a different approach.

When we assume that the cheque was made out for ₹ X and Y paise, it gives rise to two situations: (i) Y < 50 (ii) Y ≥ 50.

In case (i) we get the following relations: Y = 2X and X – 20 = 2Y. (That is, there is no carryover from paise to rupees in doubling the amount of the cheque.) Solving these we obtain $\left(X = -20/3,\ Y = -40/3\right)$ which we reject though it satisfies the equation obtained earlier: 199X = 98Y – 20.

In case (ii) there is a carryover and so we get the relations Y = 2X + 1 and X – 20 = 2Y – 100. Solving these we get (X = 26, Y = 53). As expected, this solution set too satisfies the equation obtained initially. The cheque was made out for ₹ 26.53.

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Acknowledgement: The help extended by Ms.S.M. Anupama in suggesting alternative approaches to some of the problems is hereby gratefully acknowledged.

Secondary

Problem Corner

Problems for the SENIOR SCHOOL

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem IX-3-S.1

Consider the quadratic function $f(x) = x^2 + bx + c$ defined on the set of real numbers. Given that the zeros of f are two distinct prime numbers p and q, and $f(p - q) = 6pq$, determine the primes p and q, and the function f.

Problem IX-3-S.2

Find all positive integers a, b, c satisfying the equation

$$(a+1)^4 \cdot (b+1)^4 \cdot (c+1)^4 = (40a+1) \cdot (40b+1) \cdot (40c+1).$$

Problem IX-3-S.3

In a right-angled triangle ABC, point D lies in the interior of side AC, and point E lies on the extension of hypotenuse AB beyond B. The second intersection of circles ADE and BCE (different from E) is F. Show that $\measuredangle CFD = 90^\circ$.

Problem IX-3-S.4

The areas of two faces of a cuboid are 40 sq.cm and 56 sq.cm. The length of its main diagonal is $\sqrt{138}$ cm. Given that, numerically, the total surface area of the cuboid is a positive integer, determine its volume.

Problem IX-3-S.5

Solve for real x:

$$4^x + 9^x + 36^x + \sqrt{\frac{1}{2} - 2x^2} = 1.$$

Keywords: Quadratic function, cuboid, divisor, multiple

Solutions of Problems in Issue-IX-2 (July 2020)

Problem IX-2-S.1

The numbers 4 and 52 share the following features: both are sums of two squares; both exceed another square by 3. Thus:

$$4 = 0^2 + 2^2, \qquad 4 - 3 = 1^2;$$
$$52 = 4^2 + 6^2, \qquad 52 - 3 = 7^2.$$

Show that there are infinitely many numbers that have these two characteristics. [CRUX]

Solution. We need to show that there are infinitely many integer values of k such that $k = a^2 + b^2$ and $k = c^2 + 3$ for non-negative integers a, b, c. Here is a way of generating such values. Let $a = 2n$, $b = 2n^2 - 2$, and $c = 2n^2 - 1$ for a positive integer n. Then

$$a^2 + b^2 = 4n^2 + 4(n^2 - 1)^2 = 4(n^4 - n^2 + 1),$$
$$c^2 + 3 = (2n^2 - 1)^2 + 3 = 4(n^4 - n^2 + 1).$$

Therefore $a^2 + b^2 = c^2 + 3$ and there is an infinite sequence of numbers

$$k = 4(n^4 - n^2 + 1), \quad n = 1, 2, 3, \ldots$$

with these two characteristics. (The numbers 4 and 52 correspond to $n = 0$ and $n = 2$ respectively.)

Problem IX-2-S.2

Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer M such that $f(n)$ is divisible by M for every positive integer n. [CRUX]

Solution. Observe that

$$f(n+1) - f(n) = 24 \cdot 25^n - 72 = 24 \cdot (25^n - 3) = 24 \times \text{an even number}.$$

This shows that $f(n+1) - f(n)$ is a multiple of 48 for every positive integer n. Since $f(1) = -48$, which is a multiple of 48, it follows by the principle of induction that $f(n)$ is a multiple of 48 for every positive integer n. This shows that $M \geq 48$. On the other hand, since $f(1) = -48$, M cannot exceed 48. Hence $M = 48$.

Additional comment. Solutions to this problem came from students in Mangalore. We describe them briefly here. It is striking to note the variety of solutions for this problem.

Recall that we need to find the largest integer M such that $f(n)$ is divisible by M for every positive integer n.

Praneetha Kalbavi (Class XI): Let n be any positive integer, and let $f(n) = Mk$ and $f(n+1) = Ml$ for some positive integers k, l; i.e.,

$$25^n - 72n - 1 = Mk, \qquad 25^{n+1} - 72(n+1) - 1 = Ml.$$

We now have:

$$\begin{aligned} Ml = 25^{n+1} - 72(n+1) - 1 &= 25 \cdot 25^n - 72n - 72 - 1 \\ &= 25 \cdot (25^n - 72n - 1) - 72n - 73 + 25 \cdot 72n + 25 \\ &= 25 \cdot Mk - 1728n - 48. \end{aligned}$$

This implies that M is a divisor of $1728n + 48$ for every positive integer n. Since $1728n + 48 = 48(24n + 1)$, it follows that $M = 48$.

Akshar Kumar N (Class XI): First we note $f(1) = -48$ and $f(2) = 480$. Hence $M \leq 48$. Noting that $48 = 16 \times 3$ and $\gcd(16, 3) = 1$, we shall now prove that $3 \mid f(n)$ and $16 \mid f(n)$ for every positive integer n. If we do this, then it will follow that 48 is a divisor of $f(n)$ for every positive integer n, and hence that $M = 48$.

To see why $3 \mid f(n)$, observe that

$$f(n) = 25^n - 72n - 1 \equiv 1 - 0 - 1 \equiv 0 \pmod 3.$$

We prove that $16 \mid f(n)$ by mathematical induction. Note that the hypothesis is true for $n = 1$. Now suppose that $16 \mid f(m)$ for some $m \geq 1$. Let

$$f(m) = 25^m - 72m - 1 = 16k,$$

where k is an integer. We now have:

$$\begin{aligned} f(m+1) &= 25^{m+1} - 72(m+1) - 1 = 25 \cdot 25^m - 72(m+1) - 1 \\ &= 25 \cdot (16k + 72m + 1) - 72(m+1) - 1 \\ &= 25 \cdot 16k + 72m \cdot 25 + 25 - 72m - 73 \\ &= 25 \cdot 16k + 1800m - 72m - 73 + 25 = 25 \cdot 16k + 1728m - 48 \\ &= 16(25k + 36 - 1), \end{aligned}$$

proving that $16 \mid f(m+1)$. This justifies the claim that the largest integer dividing $f(n)$ for all n is 48.

Rakshitha (Class XII): Since $f(1) = -48$, it follows that $M \leq 48$. To show that $M = 48$, we first write

$$\begin{aligned} f(n) &= 25^n - 1 - 72n \\ &= 24 \cdot (25^{n-1} + 25^{n-2} + \cdots + 1) - 24 \cdot 3n \\ &= 24 \cdot (25^{n-1} + 25^{n-2} + \cdots + 1 - 3n) = 24 \cdot g(n), \end{aligned}$$

where $g(n) = 25^{n-1} + 25^{n-2} + \cdots + 1 - 3n$. Now

$$\begin{aligned} g(n) &= 25^{n-1} + 25^{n-2} + \cdots + 1 - 3n \\ &\equiv n - 3n \pmod 2 \equiv -2n \pmod 2 \equiv 0 \pmod 2. \end{aligned}$$

It now follows that $48 \mid f(n)$ for all positive integers n. It follows that $M = 48$.

Problem IX-2-S.3

Nine (not necessarily distinct) 9-digit numbers are formed using each digit 1 through 9 exactly once. What is the maximum possible number of zeros that the sum of these nine numbers can end with? [Kvant]

Solution. The answer is eight. Since

$$8 \times 987654321 + 198765432 = 8100000000,$$

the answer is at least 8. But the largest possible value that the sum can take is

$$9 \times 987654321 = 8888888889,$$

so the only other possibility is to have nine zeros.

Now each number whose digits are a permutation of $1, \ldots, 9$ is a multiple of 9, since the sum of their digits is 45 (which is a multiple of 9). Therefore any sum of these numbers must also be a multiple of 9.

But the only 10-digit number ending in nine zeros that is a multiple of 9 is 9000000000, and this is larger than our upper bound.

Problem IX-2-S.4

Note that $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$. *Determine conditions for which* $\sqrt{a\frac{b}{c}} = a\sqrt{\frac{b}{c}}$, *where* a, b, c *are positive integers.* [CRUX]

Solution. Assume that a, b, and c satisfy

$$\sqrt{a\frac{b}{c}} = a\sqrt{\frac{b}{c}}, \quad \text{i.e.,} \quad \sqrt{\frac{ac+b}{c}} = a\sqrt{\frac{b}{c}}.$$

Squaring both sides, we get

$$\frac{ac+b}{c} = \frac{a^2b}{c}, \quad \therefore \; ac = b(a^2-1).$$

Since $\gcd(a, a^2-1) = 1$, a cannot divide a^2-1. Therefore a divides b.

Let $b = ka$ for some integer $k \geq 1$. Then $c = k(a^2-1)$.

It is easily verified that for any choice of integers $a \geq 2$ and $k \geq 1$, the triple

$$(a, b, c) = \big(a,\; ka,\; k(a^2-1)\big)$$

will satisfy the condition.

Problem IX-2-S.5

Find all positive integers n *satisfying the following condition: numbers* $1, 2, 3, \ldots, 2n$ *can be split into pairs such that if the numbers in each pair are added, and the sums are then multiplied together, the result is a perfect square.* [Tournament of Towns]

Solution. We claim that n satisfies the condition if $n > 1$. We first observe that $n = 1$ fails the condition. For $n = 1$ the only pairing is $\{1, 2\}$, the sum of which is 3, i.e., not a perfect square. We now consider separately the cases when n is even and when n is odd.

Case 1, n even: Then $n = 2k$ where $k \geq 1$. By pairing i with $2n+1-i$ for $i = 1, 2, \ldots, n$, we get a product of $((2n+1)^k)^2$.

Case 2, n odd: Then $n = 2k+1$ where $k \geq 1$. When $k \geq 1$, we pair 1 and 5, 2 and 4, 3 and 6, and $6+i$ with $2n+1-i$ for $i = 1, 2, \ldots, n-3 = 2k-2$. The product is then

$$(1+5)(2+4)(3+6)(2n+7)^{2k-2} = \left(18 \cdot (2n+7)^{k-1}\right)^2.$$

Bisecting the Perimeter of a Triangle using Ruler and Compass

A S RAJAGOPALAN

In this article, we discuss three geometric construction methods for dividing the perimeter of a given triangle into two equal parts. The methods require only a ruler and a compass.

We may express the problem as follows:

Given an arbitrary triangle, how can we locate pairs of points on the sides that divide the perimeter into two equal parts? (We permit one of the points to be a vertex of the triangle.)

> *Given any polygon and a pair of points on its perimeter, if we travel along the perimeter from one point to the other, we refer to the path as a 'polygonal path.'*

For an equilateral triangle, the midpoint of any one side together with the vertex of the opposite side is one such pair of points, as the length of the path from the midpoint to the opposite vertex (along the perimeter) is the same in both directions.

If the given triangle ABC is isosceles with AB = AC, and D is the midpoint of side BC, then paths DBA and DCA (once again, along the perimeter) have equal length, so the points {A, D} have the required property.

The problem becomes more interesting when the triangle is scalene. We offer three different methods to deal with this case. In the first two methods, one of the points is a vertex of the triangle. In the third method, neither of the points is a vertex of the triangle.

Keywords: Construction, triangle, perimeter, dividing, path length, reasoning

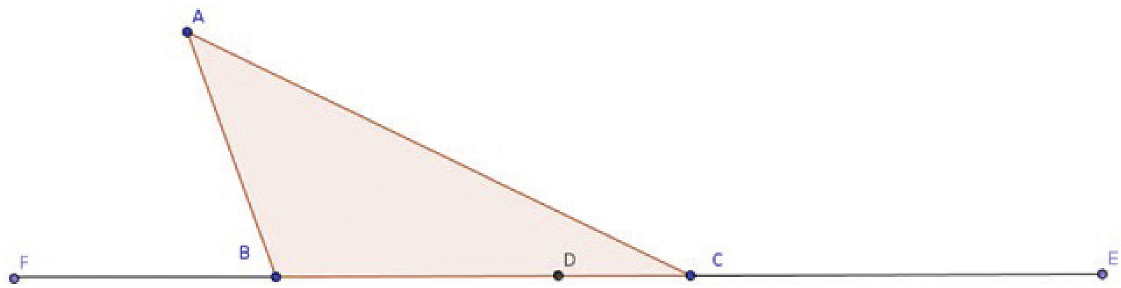


Figure 1

Method 1

We are given a scalene triangle ABC.

Extend BC to point E by length CA and extend CB to point F by length BA (so CE = CA, BF = BA; see Figure 1). Locate the midpoint D of EF, by constructing its perpendicular bisector. Then points A and D divide the perimeter of triangle ABC into two halves. The polygonal paths DCA and DBC have equal length since DC + CA = DE; DB + BA = DF; and DE = DF.

How can we be sure that D lies between B and C? To see why, we use the triangle inequality.

Proof

We have,

$$ED - EC = \frac{(AB + AC + BC)}{2} - AC = \frac{(AB + BC - AC)}{2}.$$

Since AB + BC > AC by the triangle inequality, it follows that ED > EC. This means that D cannot lie between C and E.

Also:

$$EB - ED = \frac{(AC + BC) - (AB + AC + BC)}{2} = \frac{(AC + BC - AB)}{2}.$$

Since AC + BC > AB by the triangle inequality, it follows that ED < EB. This means that D cannot lie between B and F.

Therefore, D lies between B and C.

Method 2

Here we have named the scalene triangle PQR. Construct an escribed circle ('ex-circle') on any side, say side QR. (An ex-circle of a triangle touches one side of the triangle and the extensions of the other two sides, as shown in Figure 2.)

Construction: Extend sides PQ and PR. Construct angle bisectors of the exterior angles at vertices Q and R of ΔPQR. Let the two angle bisectors meet at H. Drop a perpendicular HG from H to QR. Since angles HQR and HRQ are acute, G lies on side QR (and not on its extension on either side).

Since H lies on the angle bisectors of the exterior angles at Q and R of ΔPQR, H is equidistant from PQ extended and QR. Similarly, H is equidistant from QR and PR extended. If we draw a circle with centre H and radius HG, the extensions of PQ and PR will be tangents to the circle. Let the points of tangencies be S and T, respectively.

Since tangents to a circle from a point outside it are equal, PS = PT.

In the same way, {QS, QG} and {RG, RT} are pairs of tangents to the circle from points Q and R, respectively. Therefore QS = QG, and RG = RT. Now we have:

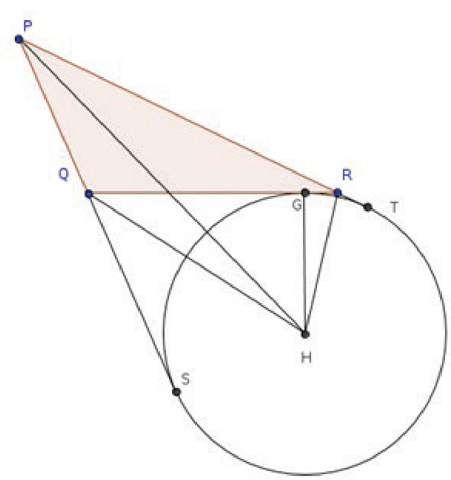


Figure 2

PS = PQ + QS = PQ + QG, and
PT = PR + RT = PR + RG.

Hence PQ + QG = PR + RG.

Therefore, the polygonal paths PQG, PRG (traversed along the perimeter) have equal length.

It follows that points P and G bisect the perimeter of ΔPQR.

Method 3

This time, we have named the scalene triangle XYZ. Assume that XZ > XY (Figure 3).

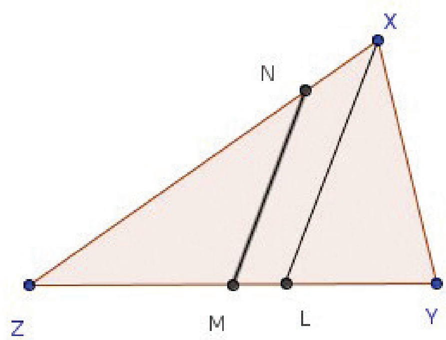


Figure 3

Locate midpoint M of side ZY of ΔXYZ. Let L be the point on ZY where the angle bisector of the opposite angle to side ZY meets side ZY. Draw line through M, parallel to XL. Let this line meet side XZ at N. We shall show below that polygonal path MZN = polygonal path MYXN. Therefore, the points M and N bisect the perimeter of ΔXYZ.

We now justify our claim that ZM + ZN = MY + YX + XN (see Figure 4).

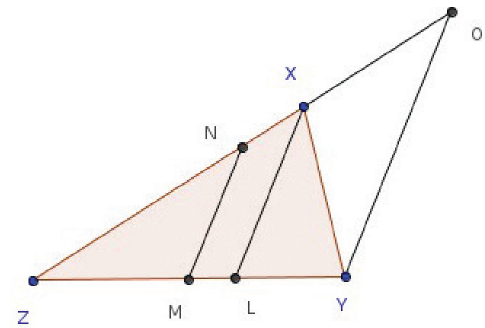


Figure 4

Through Y, draw a line parallel to MN and let it meet ZX extended at O. Since M is the midpoint of ZY and MN is parallel to YO, by the midpoint theorem, N is the midpoint of ZO. Therefore NO = ZN.

MN, XL and YO are parallel lines and ∠XOY = ∠NXL = ∠LXY = ∠XYO. From these relations, it follows that ∠XOY = ∠XYO, so XO = XY. Next: ZN = NO = NX + XO = NX + XY Also, ZM = MY, since M is the midpoint of ZY.

Therefore,

NZ + ZM = NX + XY + YM

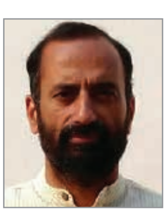
Therefore, points M and N bisect the perimeter of triangle XYZ, as claimed.

Remark

The idea behind the third method comes from the broken chord theorem of Archimedes.

References

Episodes in Nineteenth and Twentieth Century Euclidean Geometry, by Ross Honsberger (Mathematical Association of America)



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Secondary

Problem Corner

A Very Special Congruency Problem

KOUSIK SETT

Introduction

The following question appears in Chapter 7 of the Class VII NCERT Mathematics textbook (page 150):

> *Draw a rough sketch of two triangles such that they have five pairs of congruent parts but still the triangles are not congruent.*

Here, 'parts' refers to the three sides and three angles of the triangle; 'congruent parts' means that the corresponding parts of the two triangles are identically equal to each other. The questions that quickly come to mind are: "Why a rough sketch? Why not the exact figure?" "Is it possible to draw such pairs of triangles exactly?" We will find the answers to these questions in this article and prove some related results.

The four congruence theorems

Students are familiar with four congruence criteria, i.e., SSS, SAS, ASA (equivalently, AAS), and RHS. So if three parts of one triangle are equal to three parts of another triangle, as per these criteria, then the triangles are congruent. But if someone asks, *Is it possible that two triangles have five pairs of congruent parts and yet are not congruent?* the students' impulsive answer is, *NO, it is not possible*. At that moment if you tell them *YES, it is possible*, they just cannot believe it and they ask the immediate question, *How?* Well, let's investigate the reason behind it.

Keywords: NCERT, congruent, congruence criteria, SAS, SSS, ASA, AAS, RHS

Analysis

We will describe how one can construct such pairs of triangles. If we make three pairs of sides equal, then the triangles will be congruent. So in the list of five congruent pairs, three pairs of congruent sides cannot be included. Rather, it must be two pairs of congruent sides and three pairs of congruent angles. In this case, the triangles will be similar but not necessarily congruent. Now, if the equal pairs of sides correspond to equal pairs of angles, then the triangles will be congruent to each other, by SAS or ASA. As we do not want congruence, we must ensure that corresponding sides of the triangles are not equal to each other. So we have to make two pairs of non-corresponding sides of the triangles equal in length. Then it is possible to create such constructions. The following two figures show one such instance.

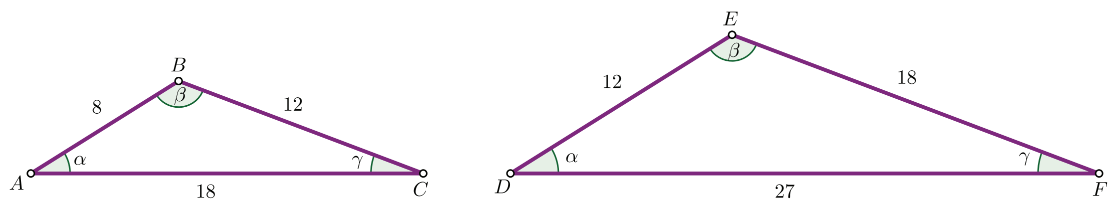

Figure 1.

From the above figures, we see that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{2}{3}.$$

So the triangles are similar but non-congruent.

Again in both triangles, two pairs of sides are equal, i.e., $BC = DE$ and $AC = EF$, but they are non-corresponding sides.

Now we will characterize all such pairs of triangles.

Let the sides of the first triangle be a, b, c, and let the sides of the second triangle be b, c, d. (Observe that there are two pairs of equal sides.) Since the triangles must be similar to each other, with a, b, c corresponding to b, c, d respectively, we must have:

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}. \tag{1}$$

Now we must choose a, b, c and b, c, d in such a way that they must form genuine triangles. Let

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k, \tag{2}$$

where $k > 0$ but $k \neq 1$ (else the triangles are equilateral).

So clearly, $c = dk$, $b = ck = dk^2$ and $a = bk = ck^2 = dk^3$. The sides of the two triangles are

$$(a, b, c) = (dk^3, dk^2, dk) = (ck^2, ck, c), \tag{3}$$

$$(b, c, d) = (dk^2, dk, d). \tag{4}$$

We know that three line segments form a triangle if (and only if) the sum of the lengths of any two line segments is greater than the third. So $a + b > c$, $b + c > a$ and $c + a > b$ must hold simultaneously, with similar conditions for b, c, d. Both lead to these inequalities:

$$k^2 + k > 1, \tag{5}$$

$$k + 1 > k^2, \tag{6}$$

$$k^2 + 1 > k. \tag{7}$$

From (5) we have $k^2 + k > 1$, hence

$$k^2 + k - 1 > 0, \quad \therefore \left(k + \frac{\sqrt{5}+1}{2}\right)\left(k - \frac{\sqrt{5}-1}{2}\right) > 0.$$

Since $k > 0$ we must have

$$k > \frac{\sqrt{5}-1}{2}. \tag{8}$$

From (6) we have, $k + 1 > k^2$, hence

$$k^2 - k - 1 < 0, \quad \therefore \left(k - \frac{\sqrt{5}+1}{2}\right)\left(k - \frac{1-\sqrt{5}}{2}\right) < 0.$$

Since $k > 0$ we must have

$$k < \frac{\sqrt{5}+1}{2}. \tag{9}$$

From (7) we have, $k^2 + 1 > k$, hence $k^2 - k + 1 > 0$, but this is always true for all real k, as the discriminant of the corresponding quadratic $k^2 - k + 1$ is $(-1)^2 - 4 = -3 < 0$.

From (8) and (9) we get:

$$\frac{\sqrt{5}-1}{2} < k < \frac{\sqrt{5}+1}{2}. \tag{10}$$

But we know $k \neq 1$. So we have:

$$\frac{1}{\phi} < k < 1 \quad \text{and} \quad 1 < k < \phi, \tag{11}$$

where ϕ is the *golden ratio*:

$$\phi = \frac{\sqrt{5}+1}{2}.$$

By choosing different values of k between these intervals, we can generate infinitely many such pairs of triangles.

We know that the golden ratio $\phi = \frac{\sqrt{5}+1}{2} = 1.61803398874$, so the second interval $1 < k < \phi$ yields $1 < k < 1.618\ldots$.

To construct pairs of triangles having the required property and the added property that the lengths of the sides of the two triangles are integers, we make use of the above analysis and the following sequence,

$$\left\{\frac{n+1}{n}\right\},$$

which steadily decreases and converges to 1:

$$1 < \cdots\cdots < \frac{6}{5} < \frac{5}{4} < \frac{4}{3} < \frac{3}{2} < \frac{2}{1}. \tag{12}$$

Except for the first term, $\frac{2}{1}$, all other terms of the sequence lie in the second interval, and the first value of the sequence which satisfies (11) is $\frac{3}{2}$.

Choosing $k = \dfrac{3}{2}$, we get the side lengths as

$$(a, b, c) = (dk^3, dk^2, dk) = \left(\frac{27d}{8}, \frac{9d}{4}, \frac{3d}{2}\right),$$
$$(b, c, d) = (dk^2, dk, d) = \left(\frac{9d}{4}, \frac{3d}{2}, d\right).$$

To ensure that all the lengths are integers, the least value of $d = 8$. This gives a pair of triangles whose side lengths are $(27, 18, 12)$ units and $(18, 12, 8)$ units respectively. These two triangles are shown in Figure 1.

If we choose $k = \frac{4}{3}$ and $d = 27$, we get two triangles whose side lengths are $(64, 48, 36)$ units and $(48, 36, 27)$ units respectively. These two triangles also satisfy the required condition.

Now in general, if we choose $k = \frac{n+1}{n}$ where $n \geq 2$ is an integer, we get,

$$(a, b, c) = \left(\frac{(n+1)^3}{n^3}d,\ \frac{(n+1)^2}{n^2}d,\ \frac{(n+1)}{n}d\right),$$
$$(b, c, d) = \left(\frac{(n+1)^2}{n^2}d,\ \frac{(n+1)}{n}d,\ d\right).$$

If we take $d = n^3$, then we get two integer-sided triangles whose side lengths are given by

$$(a, b, c) = \left((n+1)^3,\ n(n+1)^2,\ n^2(n+1)\right),$$
$$(b, c, d) = \left(n(n+1)^2,\ n^2(n+1),\ n^3\right).$$

Notice that among these six sides, the smallest side is n^3, and the biggest side is $(n+1)^3$.

So from the above results, we can state the following theorem,

Theorem. *Let $n \geq 2$ be a positive integer. Then there exists a pair of integer-sided triangles with five pairs of congruent parts, such that the triangles themselves are not congruent to each other, the shortest side of the smaller triangle being n^3, and the longest side of the larger triangle being $(n+1)^3$.*

That means that for every two consecutive integer cubes, there exists a pair of integer-sided triangles with five pairs of congruent parts, such that the triangles themselves are not congruent to each other, the shortest side of the smaller triangle being the smaller cube, and the longest side of the larger triangle being the larger cube.

Note that if we choose the first interval $\frac{1}{\phi} < k < 1$, we arrive at the same conclusion. In this case, we could choose the sequence $\left\{\frac{n}{n+1}\right\}$ and the starting value $k = \frac{2}{3}$, instead of $k = \frac{3}{2}$.

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Playing with Quadrilaterals

PRITHWIJIT DE

In school we are introduced to quadrilaterals as four-sided figures enclosing a region and usually, while discussing their properties, we work only with quadrilaterals each of whose interior angles is less than 180°. In keeping with this tradition, all quadrilaterals discussed in this article are assumed to have this property.

Let $ABCD$ be a quadrilateral. The diagonals AC and BD intersect at X. Characterise all quadrilaterals $ABCD$ in which

(a) the areas of the triangles ABC, BCD, CDA, and DAB are equal.

(b) the areas of the triangles ABX, BCX, CDX, and DAX are equal.

(c) the perimeters of the triangles ABC, BCD, CDA, and DAB are equal.

(d) the perimeters of the triangles ABX, BCX, CDX, and DAX are equal.

(e) (a) and (c) hold simultaneously.

(f) ((b) and (d)) or ((a) and (d)) hold simultaneously.

(g) (c) and (d) hold simultaneously.

Let us investigate. See Figure 1.

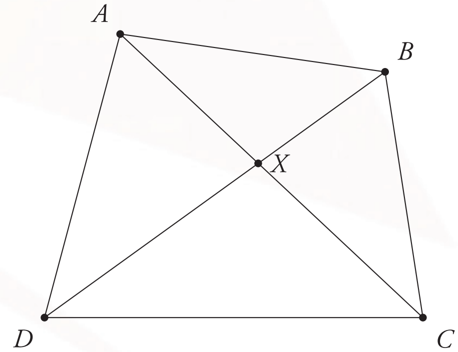

Figure 1.

Keywords: Quadrilateral, bicentric, cyclic, inscriptible, GeoGebra

(a) The triangles ABC and BCD are on the same base and as their areas are equal, $AD \parallel BC$. Similarly, equality of the areas of triangles ABC and DAB together with the fact that they are on the same base AB imply $AB \parallel DC$. Therefore, $ABCD$ is a parallelogram.

(b) The observation that X is the midpoint of AC as well as BD is immediate which shows that the diagonals of $ABCD$ bisect each other. Hence $ABCD$ is a parallelogram.

(c) Equating the perimeters of the pairs of triangles (BCD, DAB) and (ABC, CDA) leads to

$$DA + AB + BD = BC + CD + BD \tag{1}$$

and

$$AB + BC + CA = AD + DC + CA. \tag{2}$$

Adding (1) and (2) and canceling the common terms yields $AB = CD$. Using this in (1) shows $AD = BC$. Thus the opposite sides of $ABCD$ are equal. Now, by equating the perimeters of the triangles ABD and ABC we get

$$DA + AB + BD = AB + BC + AC \tag{3}$$

which immediately yields $AC = BD$, that is, the diagonals of $ABCD$ are equal. SSS congruence now shows that the triangles CDA and ABC are congruent, and so are the triangles BCD and DAB, leading us to conclude that $ABCD$ is a parallelogram with equal diagonals. Therefore, $ABCD$ is a rectangle (the reader may prove this at leisure!).

(d) Let $AD = x$, $AB = y$, $BC = z$, $CD = w$, $AX = p$, $BX = q$, $CX = r$ and $DX = s$. Equating the perimeters of the four triangles yields

$$x + p + s = y + q + p = z + q + r = w + r + s. \tag{4}$$

But

$$x + p + s = z + q + r = \frac{1}{2}(x + z + p + q + r + s),$$

$$\text{and} \quad y + p + q = w + r + s = \frac{1}{2}(y + w + p + q + r + s),$$

whence

$$x + z + p + q + r + s = y + w + p + q + r + s,$$

which shows

$$x + z = y + w. \tag{5}$$

Thus in quadrilateral $ABCD$ the sums of opposite sides are equal and therefore it has an incircle. In literature, a quadrilateral with an incircle is called a *tangential* quadrilateral or an *inscriptible* quadrilateral. We will return to tangential quadrilaterals later.

(e) $ABCD$ is a rectangle.

(f) $ABCD$ is a rhombus.

(g) $ABCD$ is a square.

Problem 1. *Characterise (if it is possible to characterise!) all quadrilaterals ABCD in which*

(a) the areas of ***some two*** *of the triangles ABC, BCD, CDA, and DAB are equal;*

(b) the areas of ***some two*** *of the triangles ABX, BCX, CDX, and DAX are equal.*

Interestingly, if the areas and perimeters of DAX and CDX are equal, then it follows that $AX = CX$ and $DA + AX = DC + CX$ which gives $DA = DC$, and as DX is the common side of triangles DAX and CDX, by SSS congruence it follows that ADX is congruent to CDX, whence $BD \perp AC$. The triangles ABX and BCX turn out to be congruent by SAS congruence and we get $BA = BC$. Therefore, $ABCD$ is a quadrilateral with BD perpendicular bisector of AC, $DA = DC$ and $BA = BC$. It must be a kite. Note that a kite is also a tangential quadrilateral.

Problem 2. *If AC bisects $\measuredangle DAB$ and $\measuredangle BCD$, what can we say about $ABCD$?*

Evidently, in this case, the triangles ACD and ACB are congruent by AAS congruence, and we have $AD = AB$ and $CD = CB$. If the diagonals AC and BD intersect at X, then by SAS congruence the triangles AXD and AXB are congruent, and so are the triangles CXD and CXB. Moreover, observe that the diagonals intersect at a right angle. Therefore, $ABCD$ is a kite.

Here is an exercise for the reader.

Problem 3. *If AC bisects $\measuredangle DAB$ and $\measuredangle BCD$, and BD bisects $\measuredangle ABC$ and $\measuredangle ADC$, what type of a quadrilateral is $ABCD$?*

Tangential quadrilaterals

As promised earlier let us talk about tangential quadrilaterals. A quadrilateral $ABCD$ is tangential if, and only if, $AB + CD = AD + BC$. This is known as Pitot's Theorem and we shall not prove it here.

Suppose $ABCD$ is a tangential quadrilateral and let I be the centre of the incircle. Then, as I is equidistant from the four sides, it lies on the internal bisectors of the four angles of $ABCD$. Therefore, the internal angle bisectors of a tangential quadrilateral are concurrent at the centre of the incircle.

The converse also holds. That is, if the internal bisectors of the angles of a quadrilateral are concurrent, then the quadrilateral is tangential and the point of concurrence is the centre of the incircle.

There is a very simple way to obtain a tangential quadrilateral through a construction. Take a circle and a point outside it. Draw the tangents and join the points of contact to the centre of the circle. The quadrilateral thus obtained is a kite which is tangential. In fact this quadrilateral is cyclic too. See Figure 2.

A quadrilateral which is both cyclic and tangential is called a *bicentric* quadrilateral. Note that in a bicentric kite, the centres of the incircle and the circumcircle lie on one of the diagonals.

In general, is it true that the point of intersection of the diagonals of a bicentric quadrilateral lie on the line joining the centres of its incircle and circumcircle? The reader may indulge in some GeoGebra explorations to find out.

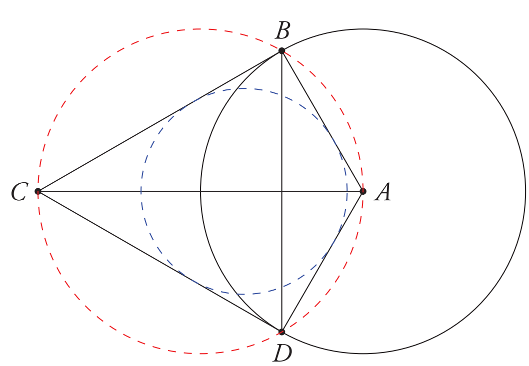

Figure 2. A bicentric kite $ABCD$

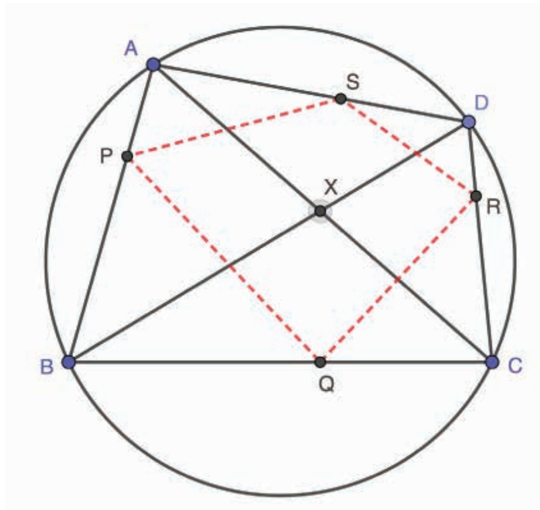

Figure 3. *PQRS* is a tangential quadrilateral

There is a very nice way to obtain a tangential quadrilateral from a cyclic quadrilateral and vice-versa. Let *ABCD* be a cyclic quadrilateral and suppose the diagonals intersect at *X*. If *P*, *Q*, *R*, and *S* are respectively the feet of perpendiculars drawn from *X* onto *AB*, *BC*, *CD*, and *DA* then *PQRS* is a tangential quadrilateral. See Figure 3.

To prove this just observe that the quadrilaterals *XSDR* and *XSAP* are cyclic. This implies $\measuredangle XDR = \measuredangle XSR$ and $\measuredangle XAP = \measuredangle XSP$. But since *ABCD* is cyclic,

$$\measuredangle XDR = \measuredangle BDC = \measuredangle BAC = \measuredangle XAP.$$

Therefore

$$\measuredangle XSR = \measuredangle XSP,$$

that is, *XS* bisects $\measuredangle PSR$. Similarly, it can be shown that *XP* and *XQ* are internal bisectors of $\measuredangle SPQ$ and $\measuredangle PQR$, respectively, showing that *X* is the centre of the incircle of *PQRS*.

Interestingly, if the diagonals *AC* and *BD* intersect at right angles then *PQRS* turns out to be cyclic, and hence bicentric. Maybe the reader can explore to find a proof of this observation.

On the other hand, to obtain a cyclic quadrilateral from a tangential quadrilateral, start with a tangential quadrilateral with mutually perpendicular diagonals and drop perpendiculars from their point of intersection on to the sides . The quadrilateral obtained by joining the feet of the perpendiculars is cyclic. Can you find a proof of this?

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Developing Textbooks for Primary Mathematics

Background to the Sikkim Classes I to III textbooks

Swati Sircar and Anupama S.M.

Review

In 2018, Sikkim wanted to revamp their primary textbooks for language, mathematics and environmental studies (EVS). The state was not satisfied by the existing textbooks for Classes 1-5. (They used NCERT textbooks from Class 6 onwards.) Sikkim SCERT took help from Mahatma Gandhi Institute of Education for Peace and Sustainable Development (MGIEP), UNESCO and Azim Premji University for this venture. MGIEP's objective was to embed the Education for Sustainable Development (ESD) goals in textbooks. This was the first time such an embedding was attempted for primary level textbooks anywhere in the world. The state had also partnered with Jodo Gyan[1] in the past. So, most of the resource persons were exposed to both how interactive and fun mathematics can be, as well as to various teaching-learning aids. The team of authors included teacher-educators from the SCERT and DIET(s) and school teachers.

The mathematics group had 12 authors divided into three smaller equal groups for Classes 1-3. Dr. Wagner from UNESCO and two members of the Azim Premji University Resource Centre worked with these groups of authors. The first few meetings were to orient the authors to various aspects of textbook writing, giving them an understanding of ESD and also to develop a syllabus for these three classes. We met for almost a week each month once the chapter writing started. Gangtok DIET had already received a set of number kits (for whole numbers, fractions, decimals and integers) from Math Space, the mathematics laboratory of Azim Premji University.

While the authors had good command over mathematics, it was not always easy to come up with stories and examples rooted in the local contexts and with a flavor of ESD. Each example had to be thought through. Each topic had to be introduced with the

1 See https://jodogyan.org/

need for the concept, while connecting it to the life of the local people. The second challenge was to make the pedagogy constructivist – bringing in a flavor where children are encouraged to discover rather than the textbook (or an adult facilitator) telling them what to do. So, conversations among children as well as those with teachers and other adults were used often. Many a time, the initiating question would be posed by a child and the adult attempted to help the child (or children) figure things out rather than giving a direct answer. Games and activities were also included wherever possible. Many of these were done as stories, so that the teacher (and the children) could connect both to their world and to mathematics through play.

While the textbooks were based on the NCERT Math Magic series, they were modified so that the local food, fruits, trees, games, festivals and various other aspects of their culture got represented in the chapters. In addition, materials were included to explain concepts. All the authors felt the need for materials and understood the benefits they offer. We borrowed from the works of Maria Montessori, Rohit Dhankar in Digantar, Jodo Gyan and the Pullouts from the Azim Premji University and CoMaC publication *At Right Angles*.

The entire project got further enriched as we were joined by a group of designers from the state. Thanks to their efforts, the textbooks turned out to be full of colorful pictures representing the local people and their culture.

We added thematic chapters in both Class 2 and Class 3 to showcase how different mathematical concepts could be used in day-to-day lives. ESD was new for all of us. Almost all exemplars in the MGIEP guidebook involved higher level mathematics – using algebra and functions to model and optimize. So, we had to figure out what could be done with the basic concepts in the beginning years of primary mathematics. Apart from going local, we focused on fruits and vegetables rather than candies, rural situations in addition to urban ones, local measures used for weight and capacity and issues (like landslide) that people commonly face in Sikkim. Moreover, we included choices people make based on mathematics – e.g. why are wheels round, or what would happen if the football was a cube.

The best example of this effort was the following: We decided to include a child with one leg in one chapter of Class 2 to fulfill a mathematical need as well as to be inclusive. Another author liked the idea and added a child with one arm as a character in another chapter in Class 1. So, we added Braille in the patterns chapter in Class 3.

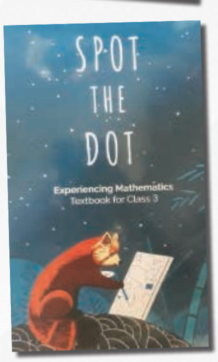
We were aware that these textbooks would be very different from the existing ones and therefore the teachers would need a lot of hand holding. We decided to address that in two ways – (i) include a "Teacher Note" wherever needed within the chapters, and (ii) add "Teacher Pages" for selected chapters in each textbook. In addition, we added a "Note for Parents and Teachers" at the beginning of each textbook to help them understand the thoughts behind the approach. The learning outcomes for the respective class were also added to each textbook. Sikkim also got the textbooks reviewed by experts from the state. Their feedback was also incorporated.

After one year of hard work, we had an orientation workshop with the teachers from 40 pilot schools, who would use these new textbooks for a year. They went through various selected portions of the textbook, many activities and uses of various materials embedded therein. They agreed that the new textbooks were very different from what they had been using so far, and that it would require them to work hard. However, they were all willing to put in the effort. And they did!

In the end, we got enriched by the experience of working deeply in primary mathematics, had the privilege to build on the work of many others, stretched our horizons w.r.t. Education for Sustainable Development and most importantly could contribute meaningfully to the lives of school children of an entire state.

Finding Mathematics, Playing Mathematics Experiencing Mathematics

Reviewer: Padmapriya Shirali



Concealed behind children's daily activities, there is a lot of mathematics comprising counting, comparing, estimating, recognising patterns, sequencing, logical thinking, and reasoning. By including all these aspects, the Sikkim SCERT Math books make the subject more child centred, less abstract, and more appealing.

Textbooks, particularly in mathematics education, play an important role in guiding teachers in the use of effective teaching methodology. A lot of the teachers' instruction in the areas of introduction of concepts, usage of mathematical contexts, development of a topic, teaching of procedures and exposure to a variety of problems and skill building are directly drawn from the text. Young students who are not yet literate are guided through the text by the teacher apart from drawing the students' attention to the illustrations and pictures. Sadly, mathematics is often taught in a dry and uninspiring way at school, usually with an emphasis on computations. Through illustrations, dialogs, activities, games, puzzles and stories, this set of books will engage children in enjoying learning mathematics.

By drawing from the local culture and environment in Sikkim, and by using stories and contexts relating to sustainable development, the authors have brought an integrated approach which is definitely the need of the day.

Keywords: textbooks, pedagogy, sustainable development, child-centric, local context

Organisation

The books are divided into several small chapters, each falling into one of three main categories: number, shape, and measurement.

The Standard I book teaches numbers, shapes, measurement, time telling, money, and graphs.

The Standard II book teaches addition and subtraction with exchange, place value, measurement, money, time telling, graphs, and geometric shapes.

The Standard III book has more advanced work on the three arithmetic operations, measurement, graphs, time, and geometry.

The authors have been able to meet several guidelines that had been laid out in the initial note.

- **From the foreword:** "The content of the books is embedded in the socio-cultural context and experiences of the children." Anyone going through this book will also learn about the culture and the distinctive characteristics of this beautiful state.
- **From the note to teachers and parents:** "It aims to help children engage with mathematics, construct their own knowledge and think mathematically." The text makes good usage of number lines for operations, bamboo strips for building multiplication tables and other materials in geometry and measurement topics.

Teacher support features

There is also a note to parents and teachers that outlines the vision of mathematics education and the underlying thinking behind the organisation of this book, and a list describing the expected learning outcomes. The teacher pages for selected chapters are well written and provide ample suggestions in the usage of manipulatives.

At several places in the text, there are pointers for the teachers that provide suggestions. For example: "Teacher's Note: Get the children to find out – How many members are there in your family? Who is the shortest among them? Who is the tallest in your family?" By acting on these, teachers can help children link the concepts learnt in the classroom to home situations.

Student centred features

- The material makes connections to students' prior knowledge, skills, experiences, and interests.
- It consists of various forms of representations, diagrams, graphs, models, tables, pictures, manipulatives and symbolic expressions that define mathematical relationships in order to help students organize their thinking and interpret mathematical situations.
- The language used is simple and clear, and within the grasp of students. The fonts used are large and readable, and the books have been designed with an appealing cover. Basic mathematical vocabulary has been introduced with nice illustrations in the Grade 1 book.
- Understanding of concepts precedes the ability to write the signs and symbols and it is good to see that words like more and less have been used before the usage of symbols like <,>.
- Familiar games like *Pittu* have been used to talk about small numbers and operations with small numbers.
- Themes like Neeraj's kitchen garden provide scope for building various mathematical problems. A lot of mathematical discussion can happen around these pictures. A good teacher can use the illustrations in these books in creative ways to help children to think further.
- Open-ended questions have been incorporated in various places.
- Word problems are incorporated at all levels and require the students to develop facilities with mathematical language in real life contexts. By repeatedly encountering mathematical language, children will be able to express their understanding and communicate in precise terms.
- There is also sufficient variety in the word problems to apply different concepts so that

the students reflect on the context and do not solve them mechanically.

- Activities have been incorporated at appropriate places to demonstrate concepts and help in building skills.
- It is difficult to teach someone the joy of something until they experience it themselves. There are various fun problems throughout the book for the students to solve. Students of a wide range of mathematical abilities will be able to find activities and tasks that interest them, and which they can engage with productively. There are many worked examples to build confidence in the students. There is enough scope for mathematics concepts to be communicated through representation, discussion and drawings.
- Students also see multiple ways of solving problems. There are also questions that get students to do some problems in different ways. 'Fingers: Can you show 8 in another way?'
- Thematic problem solving has been used in all the books (for example, *MeghaMela*), and several concepts have been covered under each theme. This is an attractive replacement for normal review exercise and reinforces the concepts and skills covered.
- The intention of any mathematics course is to get children to think mathematically rather than just having them memorize the mechanics of problem solving. A good attempt has been made by requiring students to experiment, observe and reason things in measurement, geometry, shapes, etc. Reasoning and understanding are critical to the learning of mathematics.

Suggestions

- While place value has been explained well through the usage of various materials, more work in preceding numbers and more emphasis on transition of numbers to the nearest tens and hundreds would have helped in the Class 3 book. Transition points such as the passage from 299 to 300 or 419 to 420 (understanding the manner in which digits change) are always tricky areas for students. Introduction of hundred and other hundreds could have emphasised the relationship of tens to hundreds.
- A discussion on how to find the greatest number could have helped. Problems which require ordering of number sets such as 101, 9, and 45 would have tested the students' understanding of place value.
- For young children, a program needs to lay a solid foundation for conceptual understanding using a three-step process. It should take the children from concrete, to pictorial, and finally to the abstract form. This order could have been maintained in the book by having exercises linked to concrete material precede problems involving abstraction. For example, in Book 2, *churpis* (nuggets of local cheese) could have been used before discussing number charts and the number line.
- In Book 2, the topic of number line addition could have been built up gradually: adding single digit numbers, followed by double digit and single digit numbers, followed by double digit and multiples of ten, and finally by double digit and double digit. The same holds for *Ganitamala* exercises.
- Subtraction could have started with usage of concrete materials before being demonstrated on a number line. Again on the number line, it would have been good to build up the topic gradually, using both forward counting and backward counting.
- A fundamental principle involved in teaching mathematics is to derive new facts from known ones. Understanding the relationship of addition and subtraction processes strengthens the ability of the child to do problem solving in addition and subtraction. A little more emphasis could have been given to the fact that each addition fact gives rise to two subtraction facts.
- Missing digit problems in vertical addition and subtraction problems of 3 digits would have been a good challenge.

The text stresses conceptual understanding over math-fact drill at this level. Teachers may need to supplement the text with additional exercises in some of these areas to provide the necessary level of practice.

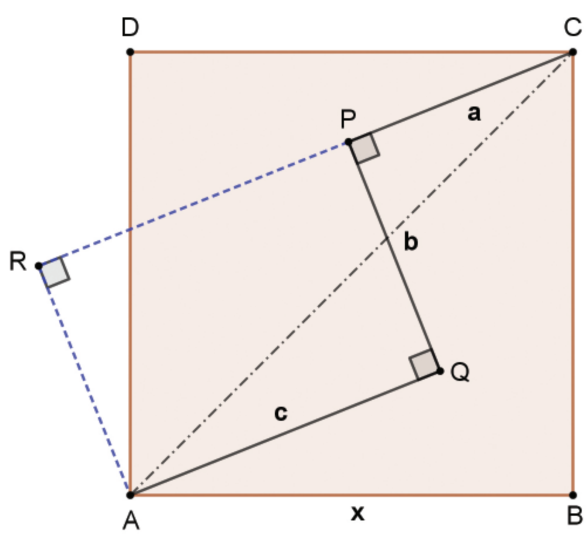
Closing remarks

On the whole, this set of three books has been successful in making an attractive presentation of the various mathematical concepts from which all students will benefit. The concrete introduction of each concept makes new units approachable even without complete memorization of math facts. The attractive pictorial layout will draw students into the lessons, and make learning enjoyable.

It is commendable that the books have been designed both pictorially and in content in an appealing manner for children. They are the kind of books that children would be happy to leaf through on their own.

Here is our solution to the Think Out of the Box problem from the July 2020 issue.

Special thanks to reader Tejash Patel who sent in a solution based on Coordinate Geometry.



CP = a
PQ = b
AQ = c
AB = x

Extend CP.
Draw a perp to CP from A
Let it meet CP extended at R
PQAR is a rectangle
PR = c, CR = c+a, AR = b
AC is hypotenuse of rt triangle ARC

$$AC^2 = b^2 + (a+c)^2, \quad \therefore \quad x^2 = \frac{b^2 + (a+c)^2}{2}$$

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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Editor: Shailesh Shirali

A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

Foundational topics include, but are not limited to, the following:

- Number systems, patterns and operations
- Fractions, ratios and decimals
- Proportional reasoning
- Integers
- Bridging Arithmetic-Algebra
- Geometry
- Measurement and Mensuration
- Data Handling
- Probability

We invite articles from teachers, teacher educators and others that are helpful in designing and implementing effective instruction. We strongly encourage submissions that draw directly on experiences of teaching. This is an opportunity to share your successful teaching episodes with AtRiA readers, and to reflect on what might have made them successful. We are also looking for articles that strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

Articles in this section may address key questions such as -

- What challenges did your students face while learning these fundamental mathematical topics?
- What approaches that you used were successful?
- What preparations, in terms of knowing mathematics, enacting the tasks and analysing students work were needed for effective instruction?
- What contexts, representations, models did you use that facilitated meaning making by your students?

Send in your articles to
AtRiA.editor@apu.edu.in

Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to 'At Right Angles'.

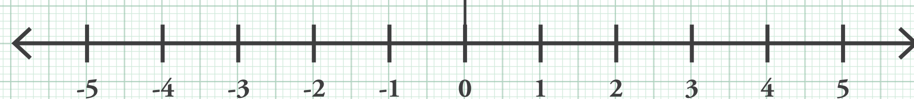
While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

At Right Angles

A Resource for School Mathematics

QUADRILATERALS

PADMAPRIYA SHIRALI



Azim Premji University

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Rishi Valley

QUADRILATERALS

Quadrilaterals is a topic which need not be 'taught' by a teacher. As a topic it presents ample opportunities for holding discussion based classes. Geometry topics in general need to be approached as explorations. Quadrilaterals is a topic where students can freely construct, measure, and compare to discover the properties of different shapes and establish the relationships between them.

Discussion amongst students is absolutely necessary for formulating ideas and justifying their reasoning. The teacher's role changes to providing meaningful tasks to the students in order to create an exploratory environment, and thereby to spark a discussion leading to understanding and learning.

Constructions can be done in a hands-on manner or in a virtual manner, using dynamic geometry software. Properties of the shapes can be studied and arrived at through observation and experimentation. Through various activities which involve comparison, students can notice the finer points and deepen their understanding.

Most of the activities given here are well known. The focus here is to find ways of generating good discussion and creating opportunities for sharing one's thinking, argumentation, and categorisation.

The crucial work of the teacher in such a class is to observe students as they discuss the terms and to watch out for misconceptions and incorrect usage of vocabulary. By asking clarifying questions like 'why' and 'can you justify', the teacher can encourage students to articulate their reasoning and thereby clear up any possible confusion.

Class organisation plays an important role to have a discussion centered approach. Class and work tables can be organised in such a way that students can work in pairs, or in groups of three or four, to share materials and notes, and to compare and discuss their findings. The group will need to work cooperatively, and a teacher may have to lend a hand if group members are working in isolation or if the group is being dominated by one student.

Each class can end with group presentations and consolidation of learnings from the activity.

There are many ways of initiating this topic; each has its own merits. Hence, the sequence of activities suggested here is not linear, and teachers can opt for what works best for them.

Before starting the activities the teacher:

- Informs the students that quadrilaterals are shapes that have four straight sides.
- Points out the difference between concave and convex quadrilaterals through drawings.
- Pins the names of various shapes with corresponding drawings on a bulletin board for students' reference.

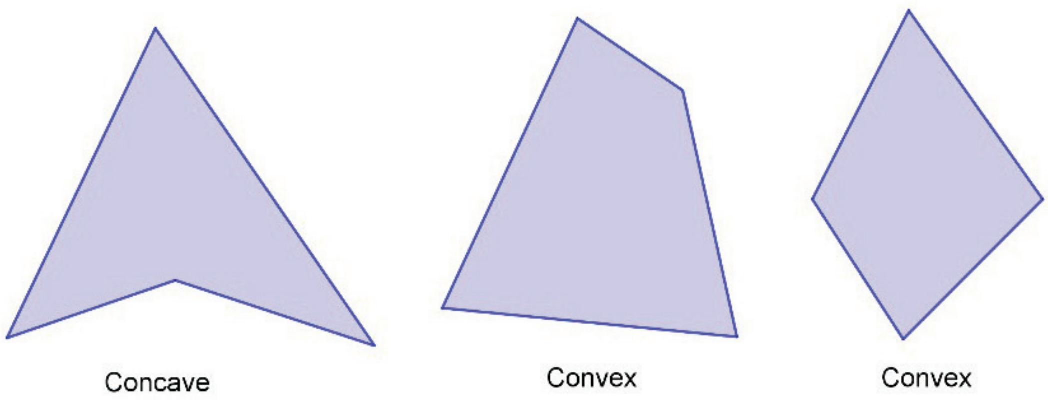


Figure 1

PRIOR KNOWLEDGE

- How to measure angles; how to use a compass
- Angle categories (right, acute, obtuse, straight, reflex angle, linear pair, supplementary, complementary)
- Triangle categories (equilateral, isosceles, right-angled)
- Sum of the angles of a triangle; sum of the angles of a four-sided figure
- Parallel lines and their properties
- Lines of symmetry of a figure

GROUP ACTIVITY 1

Objective: Tracing quadrilateral shapes on paper and discovering properties

Materials: Quadrilateral shapes or stencils (plastic, wooden or cardboard), Ruler, Protractor; one set per student group

Students can use the shapes as stencils to outline them on a paper and cut the shapes.

They experiment with these shapes and discover as many properties as possible and record them on the shapes themselves.

(Square, Rectangle, Rhombus, Parallelogram, Kite, Trapezium, Isosceles trapezium, Irregular quadrilateral)

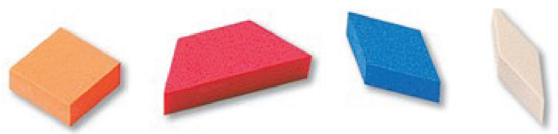


Figure 2

Some students may choose to measure the lengths and the angles. Some may fold the shapes to discover lines of symmetry. Some may measure the diagonals. Some attributes may not get used, for example, parallel lines.

The activity can be followed by a discussion on what each group has noticed about a particular shape.

Is there any property which is common to all quadrilaterals?

Since it is a preliminary activity, the teacher need not attempt to make an exhaustive list of all properties. At a later point the students can build a table with a complete list of properties based on various attributes.

GROUP ACTIVITY 2

Objective: Building Quadrilaterals and noticing transformations

Materials: Straws and thread

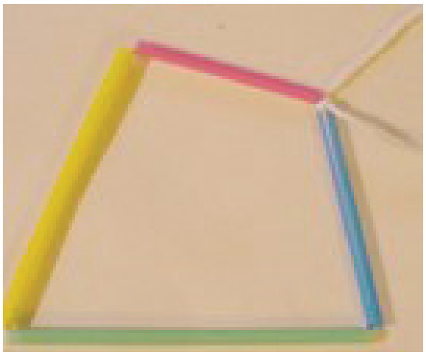


Figure 3

Students can build a quadrilateral shape with straws and thread. They will notice that the shape is not rigid – unlike a triangle.

Let them build a square shape. They can bend the shape and notice the change in shape.

What is the new shape? How have the properties changed in the new shape? What happens to the sides? How have the angles changed? How do the diagonals of the new shape compare with the diagonals of the previous shape? Are they equal?

Let students build a rectangle, bend the shape and observe in a similar manner how the properties change. What is the new shape? How do the angles change? Are the diagonals of the same size in the new shape?

Take the rectangle apart. Use the same four straws to make a quadrilateral which is not a parallelogram. What shape is it now?

Let students build other shapes, play around with them and note down what they notice.

Can a convex quadrilateral be changed to a concave quadrilateral? Why? What happens to the angles?

See also http://teachersofindia.org/en/article/exploring-geometric-shapes-straw-models.

GROUP ACTIVITY 3

Objective: Symmetry of Quadrilaterals

Materials: Paper shapes made in group activity 1

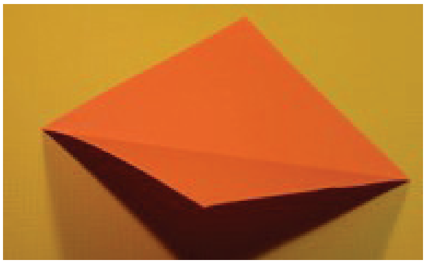


Figure 4

Can you fold a square in half so that one half fits exactly over the other?

Does a square have reflection symmetry? Show the line of symmetry with a dotted line. How many such lines are there?

Can you fold a rectangle in half so that one half fits exactly over the other?

Does a rectangle have reflection symmetry? Show it with a dotted line.

Can you fold a parallelogram in half so that one half fits exactly over the other?

Does a parallelogram have reflection symmetry? If yes, show it with a dotted line.

In a similar way, work out lines of symmetry for all the other shapes.

Each group can share its findings and compare it with other groups' findings.

GROUP ACTIVITY 4

Objective: Drawing different types of quadrilaterals on square and isometric dot paper, naming them

Materials: Square dot and isometric dot paper

Let each student in a group draw different types of quadrilaterals on dot paper.

They can name their shapes by comparing them with the shapes pinned on the board.

As a challenge, teacher can draw a line of a specific quadrilateral on dot paper and ask the students to complete it as a square, rectangle, kite, or any other shape.

At the end of the activity all students within a group compare their drawings.

Can there be multiple answers? Does orientation affect the shape?

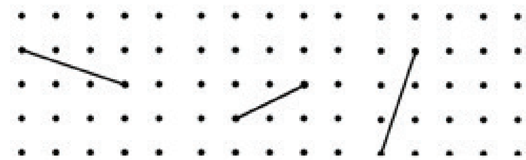


Figure 5

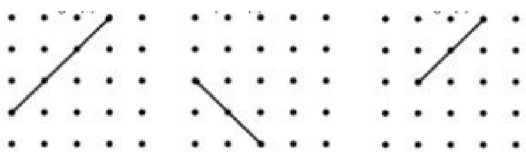


Figure 6

GROUP ACTIVITY 5

Objective: Classifying quadrilaterals on the basis of one attribute

Materials: Cutouts of different quadrilateral shapes of multiple sizes

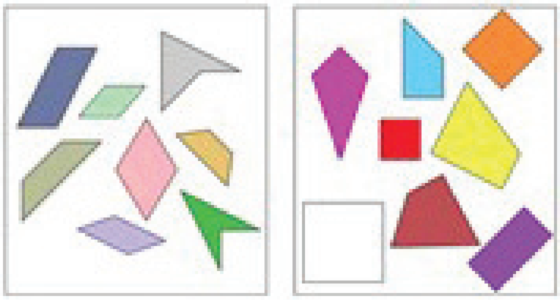


Figure 7

What is the basis for the way the shapes in the drawing have been sorted?

Each group can select an assortment of various shapes and sort them using some attribute.

It could be based on number of equal angles (all four equal, opposite angles equal, etc.), number of equal sides (all four equal, opposite sides equal, etc.), presence of parallel lines, lines of symmetry, etc.

Other groups need to figure out the attribute based on which the shapes have been sorted.

GROUP ACTIVITY 6

Objective: Building a table of attributes for different quadrilaterals

Materials: Cutouts of different quadrilateral shapes of multiple sizes

Let students fill the properties they have noticed based on each attribute given, in the table.
Here is an example of a table that is constructed based on diagonal properties.

	Square	Rectangle	Rhombus	Kite	Trapezium	Isosceles trapezium	Parallelogram
Equality of diagonals							
One diagonal bisects the other							
Both diagonals bisect each other							
Diagonals meet at right angles							
One diagonal bisects the angles at the vertices							
Both diagonals bisect the angles at the vertices							

Tables can be made comprehensive by including attributes like sides, angles, symmetry, etc.

GROUP ACTIVITY 7

Objective: Representing quadrilaterals as sets in a Venn diagram

Prerequisite: Familiarity with Venn diagrams

Materials: Name cards of different quadrilaterals

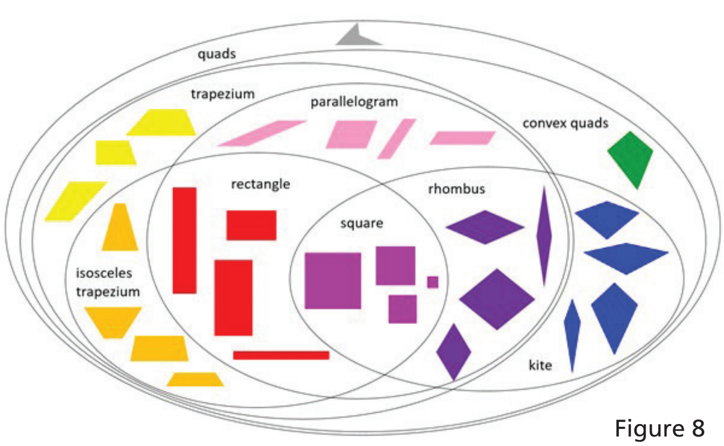


Figure 8

Let each group work out a Venn diagram to represent the relationships of different kinds of quadrilaterals and place the name cards in the appropriate circles. It is not an easy task and students will have to do a fair amount of rearranging the cards before they come up with a satisfactory solution.

If the students are finding it difficult to get started, teachers can ask a few leading questions. What shape set is a part of the parallelogram set?

How will you show that? Is any other set part of parallelogram?

Each group can present their drawing to the other groups for raising questions and clarifying doubts.

Are the representations of all the groups similar or different? If some diagrams are different, can the students explain the logic behind it?

As an extension, the teacher can give copies of a Venn diagram based on any one attribute. Students try to find an attribute which will correspond to the given drawing.

What attribute has been used to create these sets?

What is the property common to all of the figures in the left circle? What is the property common to all of the figures in the right circle?

The teacher can give a hint if necessary.

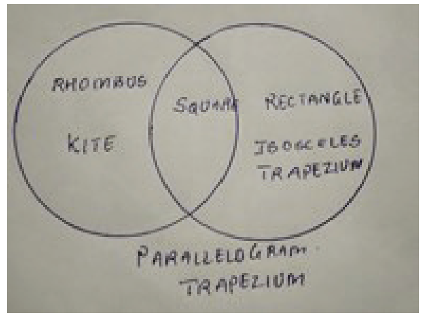


Figure 9

GROUP ACTIVITY 8

Objective: Classifying quadrilaterals with a tree diagram

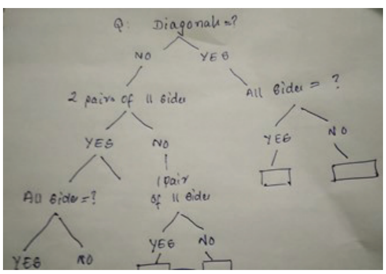


Figure 10

Students can be taught ways of drawing tree charts with Yes /No questions. They must understand that all possible shapes need to be covered.

Teachers can give hints to the kind of questions they can begin with to draw the tree. 'Does the shape have equal angles?' 'Does the shape have a pair of parallel sides?'

GROUP ACTIVITY 9

Objective: Guessing the shape

Materials: Different quadrilateral shapes

One group selects a shape. The other group must deduce the shape by asking questions.

The other groups are allowed to ask questions that will require only a Yes/No answer.

For example: 'Are both pairs of opposite sides parallel?' 'Are all the angles equal?' 'Is any angle more than 180 degrees?'

Group which asks the minimum number of questions to identify the shape wins.

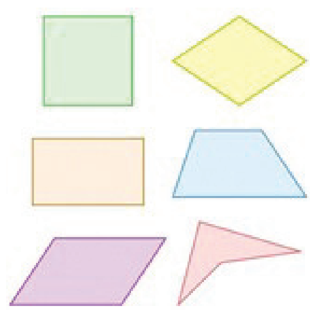


Figure 11

GROUP ACTIVITY 10

Objective: Match properties of quadrilaterals with their names

Materials: Cards describing the properties and corresponding name cards

Example: The properties can be written as follows:

The shape has two pairs of parallel sides

The shape has four right angles

The length of each side is the same

The shape is irregular

The shape has two lines of symmetry

The shape is a parallelogram with a right angle

GROUP ACTIVITY 11

Objective: Building quadrilaterals with triangles

Materials: Equilateral triangle, isosceles triangles (acute, obtuse, right-angled) and right angled triangles (isosceles, scalene)

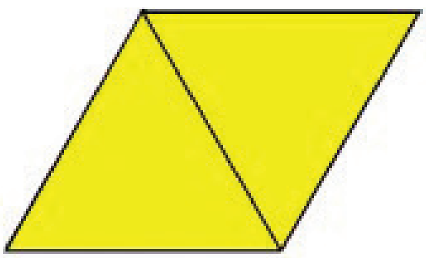


Figure 12

Let students try building the various quadrilaterals using different types of triangles

Can you create a rectangle using equilateral triangles? Using four identical isosceles triangles? Using two pairs of identical isosceles triangles? Using right-angled triangles?

Can you build a rhombus with two isosceles triangles? With two equilateral triangles?

Can you build a square with two equilateral triangles? Why not?

Can you create any other quadrilaterals in a similar manner?

GROUP ACTIVITY 12

Objective: Making different quadrilaterals on a circle

Materials: Compass

Let students construct a large circle with 8 equally spaced dots.

How many differently shaped quadrilaterals can be made by joining the dots on the circle?

Each group can share their work with the other groups, and the class can work out the total number of differently shaped quadrilaterals that are possible.

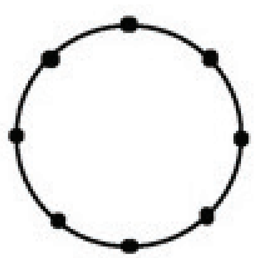


Figure 13

GROUP ACTIVITY 13

Objective: Learning to justify

Materials: Cards with statements

Here are a few samples of statements:

Quadrilaterals will always be rectangles.

A rhombus can sometimes be a trapezoid.

A trapezium is a parallelogram.

An equilateral parallelogram is equiangular.

No quadrilateral has 3 lines of symmetry.

Teacher distributes these cards to different groups, and each group has to say whether the statement is true, sometimes true or false.

Most importantly, they should justify their answer.

INDIVIDUAL ACTIVITY 14

Objective: Practice writing and understanding the symbols of a drawing

Materials: Quadrilaterals of all types with appropriate markings

Let the students record the information as sentences. Here are a few samples:

A trapezium has ____ pair of parallel sides, ____ pairs of equal sides, and _____ right angles.

A parallelogram has _____ pairs of parallel sides, _____ equal sides, and ____ pairs of congruent angles.

As the sides are equal and the angles are not _____ the shape is a _______.

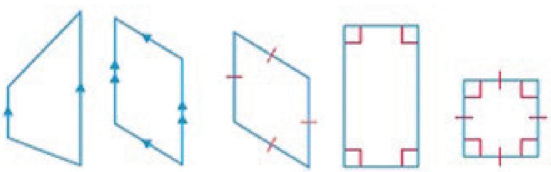


Figure 14

In a convex quadrilateral like a square or a rectangle, the two diagonals lie __________.

GROUP ACTIVITY 15

Objective: Quiz

Materials: Cards with questions

Here are a few sample questions:

When isn't a rectangle a rhombus?

Can a figure be a square, a rectangle and a parallelogram at the same time? Why?

The diagonals of a four-sided shape bisect each other. What shape can it be?

When all the sides are different in a quadrilateral, are all the angles always different as well?

"A quadrilateral which has two right angles is always a trapezium." Is this so?

Can a quadrilateral have three obtuse angles? Can it have four acute angles?

If a quadrilateral has two right angles, what can the other two angles be?

What are the similarities between a rectangle and a rhombus? What are the differences?

Each group is required to pick up a card, discuss the question amongst themselves, and present their answer to the class.

GROUP ACTIVITY 16

Objective: To strengthen visualisation

Materials: Pose questions which require visualisation

Ex. Draw two more lines to make this shape a kite.

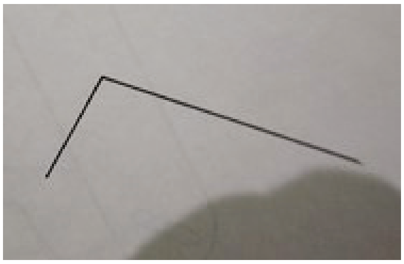


Figure 15

How many parallelograms do you see in this figure?

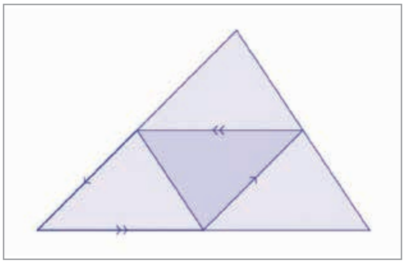


Figure 16

How many isosceles trapeziums do you see in this figure?

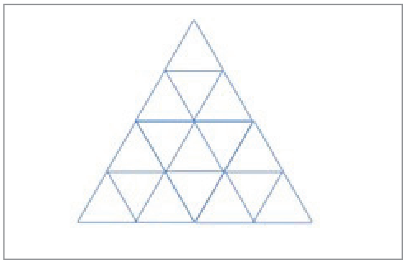


Figure 17

GROUP ACTIVITY 17

Objective: Playing with quadrilaterals

Materials: Tangrams

Students can try to create different quadrilaterals using Tangram shapes. Teacher can raise some questions.

Can you use two Tangram pieces to make a square?

Can you use two Tangram pieces to make a trapezium?

Can you use three Tangram pieces to make a rectangle?

Can you use three Tangram pieces to make a square?

Can you use three Tangram pieces to make a parallelogram?

Can you use four Tangram pieces to make a square? Is there more than one way?

Various other combinations can be tried.

GROUP ACTIVITY 18

Objective: Quadrilateral playing cards

Materials: Set of cards with shapes, names and two properties for each shape.

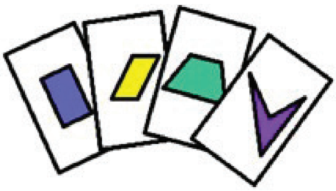


Figure 18

As a reinforcement exercise, students can create a set of playing cards with names of quadrilaterals, quadrilateral shapes and cards with properties of the shapes.

The winner will be the first who puts together the shape card, name card and two property cards.

Source: https://nrich.maths.org/2924

GROUP ACTIVITY 19

Objective: Tiling patterns with Quadrilaterals

Materials: Multiple (12 to 15) sets of each shape

Students can explore how quadrilaterals are used to make tiling patterns.

The teacher can raise questions while they are doing the activity.

In a tiling pattern, the tiles must fit together exactly at each vertex of the pattern.

Do the students see that the angles at each vertex must add up to 360 degrees?

Can four identical squares fit together in more than one way? Why do four squares fit together?

Can four squares of different sizes fit together?

In how many ways can four identical rectangles fit together? What can you say about the angles at the vertex?

Can four identical rhombi fit together? What do you notice about the angles at the vertex?

Students can explore kites, parallelograms,

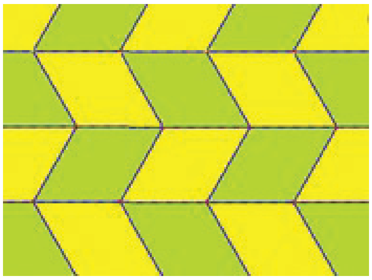


Figure 19

isosceles trapeziums, trapeziums, concave quadrilaterals and irregular quadrilaterals in a similar way.

Let them combine two types of shapes and explore the possibilities.

Can two squares and two rhombi fit together around a point? When is that possible?

Are there combinations that are not possible?

GROUP ACTIVITY 20

Objective: Constructing Quadrilaterals

Materials: Compass, protractor and ruler

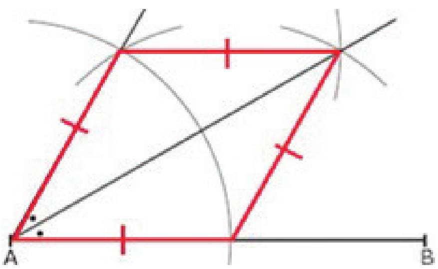


Figure 20

Ask 'How many pieces of data can be found in a quadrilateral?' Four sides, four angles and two diagonals.

Points for discussion

What piece of data is needed to make a square? Would it be possible to construct a square if you knew the diagonal length?

What pieces of data are needed to make a rectangle? Would any other data help to make a rectangle?

Would it be enough to know the length of the side of a rhombus to construct it? What else is needed? Is there any other way?

Let the students explore data needed for other shapes.

To construct a general quadrilateral, data about five parts is needed. For example, the data could be: four sides and one diagonal, three sides and two diagonals, three sides and two included angles, two adjacent sides and three angles, and so on.

Let students construct various quadrilaterals that satisfy certain conditions. Methods of construction can be brainstormed with students and demonstrated on the board.

Padmapriya Shirali

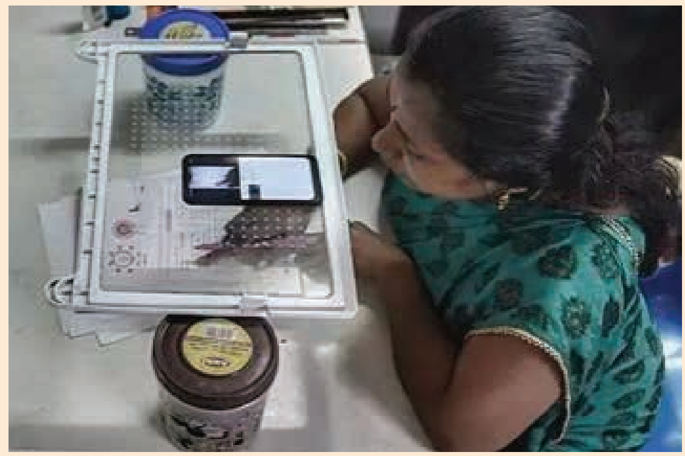
Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box' Padmapriya may be contacted at padmapriya.shirali@gmail.com

The Closing Bracket . . .

Where we Celebrate Mathematics Teachers

The difficulties of teaching mathematics online can be imagined by all and experienced by a few. At Right Angles would like to celebrate those who used their creativity to solve these problems in unique and innovative ways. We present here a collection of news items to salute these determined teachers.

Jeyaishwari Nadar, a mathematics teacher at Gandhi Memorial English School, Mumbai perched a refrigerator tray atop two kitchen containers so that she could speak into her phone and at the same time work out problems on a sheet of paper placed below the phone. What a cool screen share!



https://www.momspresso.com/parenting/article/after-teacher-s-jugaad-of-refrigerator-tray-goes-viral-she-says-we-are-doing-it-for-students-xfgaiqlnr7e2

This visual from Jharkand has had innumerable forwards but this is exactly the kind of problem solving that mathematicians strive to emulate.

For students who don't have access to smartphones, this government school in Dumarthar village, Dumka district, Jharkand arranged classes with social distancing norms being followed.

https://www.hindustantimes.com/education/teachers-in-jharkhand-s-dumka-turn-village-into-classroom/story-jNz8eaZQWGoaFbgzpfZiHI.html

There are more inspiring stories: the teacher who calls students in pairs to her home to take class; the headmaster who arranged loudspeakers at different spots and broadcast classes for two hours every day- these are the stories that are emerging on social media.

Leo Buscaglia famously said that *change is the end result of all true learning*.

Here we see that true learning has been the end result of change.

Azim Premji University
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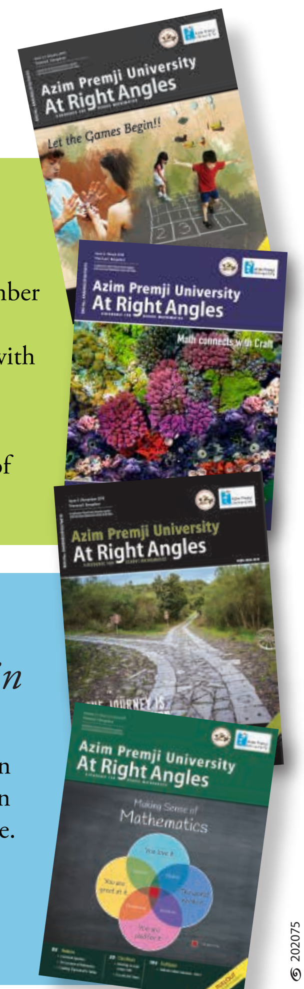

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On e-mail: AtRiA.editor@apu.edu.in

We welcome submissions and opinions at AtRiA.editor@apu.edu.in The policy for articles is published on the inside back cover of the magazine.

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