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Azim Premji University

Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

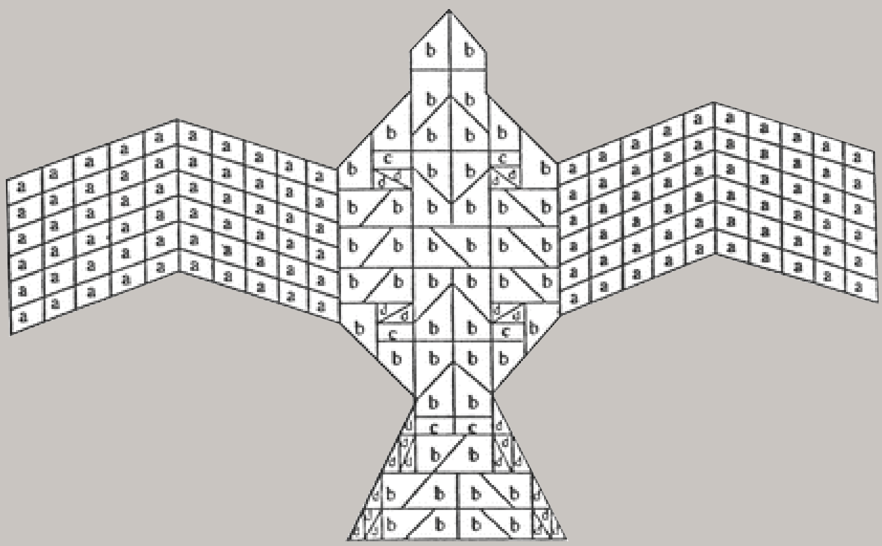
ISSN 2582-1873

Altar-ed Consciousness: From Then to Now

Mathematics in every nook and corner

PULLOUT
TRIANGLES

To see mathematics everywhere you look, is to live in a state of altered consciousness and where better to start than in Vedic times. The cover features an eagle shaped altar in which numbers, geometry and pattern all lend to the beauty of the design. Look at this second image (from George Gheverghese Joseph's book The Crest of the Peacock).



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This shows a Vedic sacrificial altar in the shape of a falcon. The wings are each made from 60 bricks of type *a* and the body from 50 of type *b*, 6 of type *c* and 24 of type *d* (pg 325, The Crest of the Peacock). Here are simple questions that can be framed based on this image:

1. What are the sides of the type *b* bricks?
2. How are the dimensions of type *a* brick related to those of type *b*?
3. Repeat the same for the dimensions of type *c* and type *d* bricks.
4. How much is the acute angle in a type *b* brick?

Isn't mathematics simply divine?

From the Editor's Desk . . .

This first issue of 2021 sees *At Right Angles* go where the pandemic has led everybody – into virtual space. We have always had an online presence with the soft copy of the magazine available (https://azimpremjiuniversity.edu.in/SitePages/resources-at-right-angles.aspx) in its entirety as well as with articles available for individual download. What is different now is that our online version will have more articles than the hard copy. Our print version continues to have the same number of pages, these online articles are a bonus for our loyal readers and allow us to share the increasing number of articles which we are receiving now. Do refer to the Contents page for more details.

Geometry in the Sulvasutras: mathematics which has stood the test of time. This article by professors Dani and Limaye highlights the understanding and applications of mathematics in Vedic times in a most readable and inspiring manner. KG Misra takes a hard look at coaching programs in mathematics and the way the pressures of entrance examinations dumb down elegant reasoning in mathematics and convey a completely false impression about the definition of being successful in mathematics. There is a lot of material in the ClassRoom section from our field authors: Ankit Patodi demystifies the divisibility rules and Amit Chand does the same for the familiar algorithm for finding the square root by long division. Swati Sircar has extended AtRiA's ClassRoom coverage to Statistics with a behind-the-scenes look at the formula for the median. James Metz gives a contextual understanding of exponents and we also have a submission from a reader which details the mathematics behind the folk method used to find the height of a tree which was described in the last issue. Ranjit Desai asks us an interesting 'What-If' question in Journey of a Minute Hand. Speaking of What-If, do check out A Ramachandran's whimsical question in Viewpoint.

Deep Drawing is a fairly new addition to textbooks and many who are teaching this have not studied this when they were in school. TearOut features some unusual solids and unusual exercises on sketching them using dot sheets. We have some lovely problem solving to share with you – Anand Prakash investigates the relationship between the sum and product of a set of distinct primes; Rakshitha looks at a challenging Putnam Geometry Problem, Shailesh Shirali demonstrates a proof of how Kohli's number (described in the November 2020 issue) is a genuine constant and Michael de Villiers and Hans Humenberger describe the journey of discovery triggered by a problem and the many approaches to its solution. Look for the latter three in the online edition.

TechSpace is truly a TPACK special this time – Jonaki Ghosh's article on Conjecture using Dynamic Geometry Software blends Technology, Pedagogy and Content seamlessly and illustrates what teaching with technology is really about.

Two reviews this time – one of an extremely inspiring documentary on Maryam Mirzakhani, the Iranian mathematician, and a mini review on the *Ganitmala* and its 'numerous' uses.

The Triangles PullOut does the impossible, add value to a priceless collection of Padmapriya Shirali's PullOut series.

Enjoy!

Sneha Titus
Associate Editor

Get out of the box! And do send in your feedback AtRiA.editor@apu.edu.in We can also be found on our FaceBook page AtRiuM. The magazine is available for free download on https://azimpremjiuniversity.edu.in/SitePages/resources-at-right-angles.aspx

Hoping to hear from you on all, or any of these platforms!

The Opening Bracket . . .

A year back, in March 2020, when we were getting to know that we were in the middle of a new and strange pandemic, I wrote: "This is not a tale of two cities but a tale of two viruses. One is a tiny being, invisible to the eye, but with the capacity to strike terror. The other is a virus that we carry in the innermost recesses of our hearts – a virus of identity, and a virus of divisiveness. It is highly active all across the Earth, and it seems to be extraordinarily virulent right now in India" (Closing Bracket, March 2020). Who would have thought that these words would hold true in March 2021, with greater virulence than ever?

Looking at this past year, one thing stands out clearly: the stupendous amount of energy that we have put into combating the pandemic. Within the course of 10-11 months, we have been able to make a detailed study of the Covid genome, and we have come up with different vaccines to combat it. It remains to be seen how effective they will be in the years to come.

Another fact stands out just as clearly: that we have done next to nothing regarding the other virus. Unlike Covid 19, which emerged before our eyes, this other virus has been with us from ancient times, in plain sight. Why doesn't humanity devote even a small fraction of the time and energy spent on Covid 19 in understanding *this* virus? It seems beyond belief that we have not done so, all these centuries. Is it that we have no clue how to? That seems clear; we really don't have a clue. But in February 2020 we had no clue about the workings of Covid 19 either; now we do. Why such a difference?

What can we in STEM education do in this regard? Looking around the world, one gets the strong sense that STEM education has not even tried to address this basic human problem. What it has done is to address, in increasingly sophisticated ways, approaches to STEM education; there are whole conferences devoted to such matters. Why can't we contribute some of our time to exploring this far more important issue?

In this regard, it is interesting to note that NEP 2020 (about which we talked briefly in the November 2020 issue of AtRiA) lists as an important objective learning *how to think* (as distinct from learning *what* to think). This is obviously an extremely important matter. But do we know how to think? Not just how to think within the realm of mathematics, but how to think about life, about ourselves. Wouldn't it be wonderful if we teachers got together to thoroughly explore the matter and find out for ourselves how to think? What a difference it could make to the world if we did so. Perhaps we would be able to figure out why this other virus is so extraordinarily virulent and long-lived, and what will make it go away.

Shailesh Shirali
Chief Editor, *At Right Angles*

Editorial Office

The Editor, Azim Premji University
Survey No. 66, Burugunte Village,
Bikkanahalli Main Road, Sarjapura,
Bengaluru – 562 125
Phone: 080-66144900
Fax: 080-66144900
Email: publications@apu.edu.in
Website: www.azimpremjiuniversity.edu.in

Publication Co-ordinator

Shahanaz Begum
Azim Premji University
Email: shahanaz.begum@azimpremjifoundation.org

Print

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Bengaluru 560 062
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Design

Zinc & Broccoli
enquiry@zandb.in

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

VIEWPOINT

The Viewpoint section re-examines familiar mathematical concepts and pedagogical practices through differentviewpoints.

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Continue . . .

Student Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Online Articles

Features

On some Geometric Constructions in the Sulvasutras from a Pedagogical Perspective – I

S.G. DANI
MEDHA LIMAYE

Methodical geometric ideas flourished in ancient India in the context of the engagement with construction of *vedi*s (altars, or platforms) and *agni*s (fireplaces) for performance of *yajna*s (fire worship), which are a hallmark of the Vedic civilization. The *sulvasutras*[1] are compositions giving an exposition of the procedure for erecting the ritual structures involved, which also incorporate along the way descriptions of various geometric principles and constructions. It may be worth recalling here, without going into the details, that the structures involved were of large size (extending to several meters on ground, not amenable to hand drawing) and in intricate shapes representing birds, tortoise, etc. (see [1], [3], [4] or [6] for details), which seems to have generated interest in geometric theory.

The geometry of sulvasutras has many commonalities with Euclidean geometry, and in particular we may recall here that the Pythagoras theorem is stated explicitly in the sulvasutras; there are four sulvasutras that are renowned for their mathematical content and all four of them include the theorem. One of the themes that pervades the sulvasutra

1 In this article we will not use diacritical marks, as commonly used in technical literature in the subject to indicate pronunciation of Sanskrit words, except for certain special words occurring in isolation. A pronunciation guide for the words used is included in a Glossary.

Keywords: History of mathematics, Vedic maths, geometry, constructions, area

geometry is constructing new figures *with the same area* as a given one, or equivalently turning the given shape into other shapes, retaining the same area. Consideration of such an equivalence of geometric figures brings to mind the role played by congruence or similarity in Euclidean geometry, while on the other hand it may also be construed as the issue of constructing desired shapes with a given area.

Our aim in this article is to discuss various instances along this theme, occurring in the sulvasutras. We believe that this would have pedagogical benefits, as the material nicely complements geometry in schools (at 6th to 8th standards) and an exposure to the different perspective involved could enhance the students' interest, and their ability, in getting a better grasp of geometry. The article will be in two parts; in Part I we focus on constructions of basic rectilinear shapes (namely those with straight edges), and in Part II, together with some more general developments concerning these, construction of semicircles and circles with given areas, and certain broader mathematical issues related to the constructions will be discussed.

The instructions in the sulvasutras presume a certain overall familiarity in various respects, on the part of one receiving the instruction. Such a familiarity would be acquired in their time through oral communication, and the compositions were meant to supplement it, as a supporting device. Thus what is put down was mostly intended only to aid recollection, and not aimed at giving a detailed or precise description. A literal translation of the text involved would therefore not be very useful to a modern reader, and a degree of paraphrasing is needed to convey what is meant. It will nevertheless be our endeavour here to stay close to the straightforward meaning of the text[2], keeping the paraphrasing to a minimum, in contrast to the tendency in some of the writing in the area to resort to generous (and sometimes unjustifiable) paraphrasing. This we believe would give the reader a better insight into how the ancients thought of the issues involved.

Typically, the desired transformations were achieved through geometrical procedures, though we do find, as will be seen in Part II of the article, an exception to this at an advanced level. While some of the constructions are based on elementary considerations, or what may be termed 'visual geometry,' others are based on the Pythagoras theorem, and still others (discussed in Part II) involve some arithmetic as well. In developing the theme here we shall begin with the simpler and more basic ideas and proceed gradually, through ascendant sections, to more complex ones, and not adhere to the sequence of their occurrence in the sulvasutras.[3]

As mentioned above there are four sulvasutras known for their mathematical, primarily geometrical, content; Baudhayana sulvasutra, Apastamba sulvasutra, Manava sulvasutra and Katyayana sulvasutra. Of these, Baudhayana is the earliest (ca. 800 BCE) and Katyayana the latest (ca. 200 BCE)[4]; the reader may consult the references cited at the end (of this part) for various details concerning sulvasutras, especially the mathematical aspects. We shall mostly refer to the Baudhayana sulvasutra, which is the most comprehensive and systematic in respect of exposition, even though it is the oldest. There are, however, certain ideas in the other sulvasutras, related to the theme at hand, that are not found in Baudhayana, which also we shall discuss; we do not aim at being comprehensive, however, in covering all instances along the theme, but endeavour to convey the variety and essence involved. References to the original sutras are included for the benefit of the interested reader; for sutras from Baudhayana Sulvasutra the sutra reference is marked with BSS, and similar abbreviations will be used for others, which will be clear from the context.

2 It may be mentioned here that while we have greatly benefited from translations available in literature, the translations presented here are ultimately our own.

3 The considerations that may have gone into the organization of the contents of the sulvasutras, with substantial variations among the individual sulvasutras, are not quite clear, and may provide a worthy topic for exploration by itself.

4 The dates assigned to the sulvasutras are estimates based on rather general considerations, with no specific evidence, and may involve a large error margin.

I. Elementary constructions

In this section we discuss constructions that are based on elementary principles, or what one may call visual geometry.

I.1. Constructing isosceles trapezia. Trapezia are a frequent occurrence in the Sulvasutras, mainly as shapes of various vedis. They are invariably isosceles (symmetrical), symmetric about the line joining the midpoints of the two parallel sides; the line of symmetry is set along the east-west direction, with the face of the trapezium (the smaller of the parallel sides) towards the east. There is no name given to the figure; at many places they turn up as the end product of a construction described (for the shape of the desired vedi), and when a reference to the figure is called for, it is made through a description which goes somewhat like "quadrilateral pointed (*aṇimat*) on one side."

The sutra BSS 2.6 (1.55)[5], from Baudhayana sulvasutra, instructs on how to produce an isosceles trapezium starting with a square (or rectangle), which may be stated as follows:

> When a square is desired to be converted to one that is pointed on one side (in the form of an isosceles trapezium, having the same area as the initial square) keeping a transverse segment of the size desired for the shorter side (at one end), the remaining (rectangular) part is to be divided along its diagonal and the (triangular) excess part is to be inverted and adjoined on the other side.

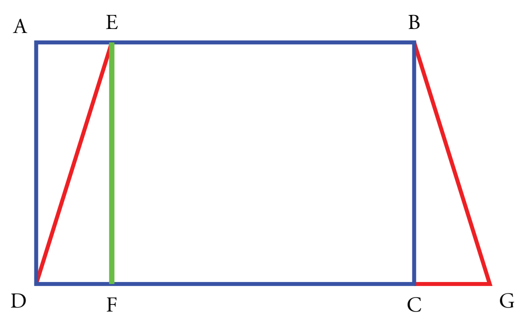


Figure 1. Rectangle ABCD converted to trapezium EBGD

Though the construction is described starting with a square, the same applies just as well to any rectangle; Figure 1 illustrates the procedure involved for a general rectangle; here ABCD is the initial rectangle and EB is the desired reduced size taken as the 'transverse;' the latter term, corresponding to *tiryaṅmānī* in the original sutra, stands for line segments cutting across the verticals (which were set in the east-west direction in their context); EBGD is the isosceles trapezium produced following the procedure, by moving the triangle ADE to the other side.

I.2. Construction of a rhombus. Some vedis are in the shape of a rhombus, namely a quadrilateral with four equal sides, the diagonals being (typically) unequal; in this case the longer diagonal is set along the east-west direction.

BSS 2.8 (1.57) gives a procedure for constructing a rhombus, which goes as follows:

> When a rhombus of the size of a (given) square is desired, produce a rectangle with area twice that of the square and join the midpoints of the sides of the rectangle.

5 The number of the sutra as in [6] is given first, and is followed parenthetically by the numbering adopted in some of the older sources, including [3] and [4].

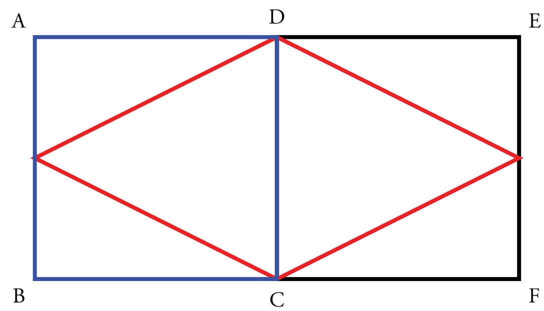


Figure 2. Constructing a rhombus with the same area as a square

The construction is illustrated in Figure 2. To the given square ABCD is adjoined an identical square CDEF, and then the midpoints of the four sides of the rectangle ABFE are joined cyclically. Where we have said "join the midpoints" the original sutra asks for poles to be erected at those midpoints, ropes to be stretched between the poles (tied pairwise, cyclically), and taking away the part outside the ropes – which is a description in a practical context. Though it has not been specified in the sutra, a rectangle with double the desired area would presumably be produced by putting two squares with the given area alongside; as per their convention which for convenience we have chosen not to adhere to in Figure 2, the additional square would be placed towards the south, so that the longer diagonal will be along the east-west direction; this specific detail in the procedure is seen adopted in the construction of a rhombus of a specific size described in Manava sulvasutra (MSS 15.4 (10.3.6.4)).

I.3. Converting a square to a rectangle. BSS 2.3 (1.52) describes the following construction to turn a square into a rectangle.

> Wishing to turn a square into a rectangle, cut it diagonally, then divide one part again and adjoin the two halves along the sides (of the original square).

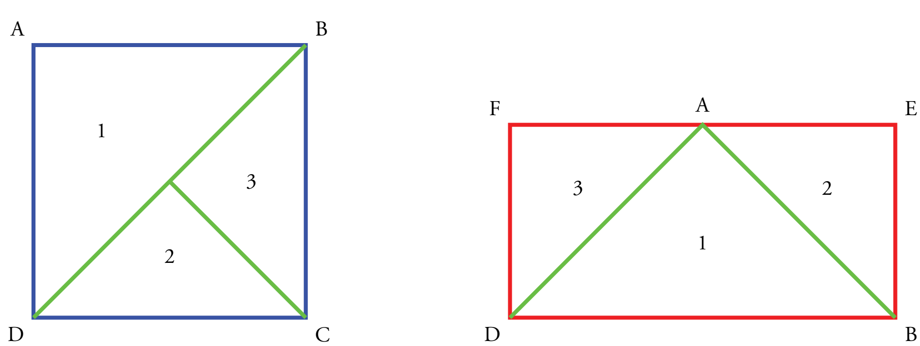


Figure 3. The square ABCD is rearranged into the rectangle FEBD

This construction should be clear from the illustration in Figure 3; the square ABCD is divided into parts and the pieces are reassembled according to the prescription, as indicated by the numbering. The procedure is evidently for producing specifically a rectangle whose sides are in the ratio 2 : 1. BSS 2.4 (1.53) purports to give a construction of a rectangle with more general side length, from a given square. However the description is rather too vague (commented as 'defective' in [6], page 79). The construction described by Apastamba (ASS 3.1 (3.1)) for the same purpose is a bit more specific, mentioning *any desired size* for the side of the rectangle to be produced, but on the whole the purported construction is still not quite clear from the description. Commentators Dwarakanatha of Baudhayana sulvasutra and Sundararaja of Apastamba sulvasutra give a construction in this respect, by way of interpretation of the sutras as above, which is as illustrated in Figure 4.

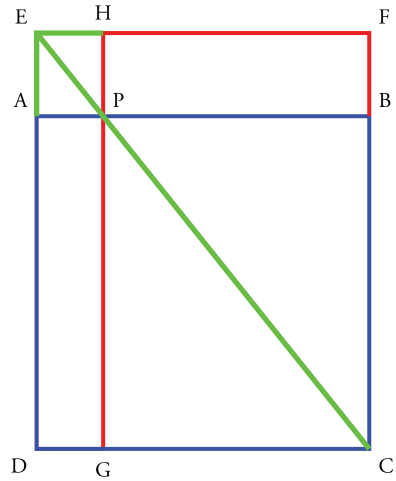


Figure 4. The square ABCD transformed to the rectangle HFCG

Given the square ABCD and a prescribed size for the side, given here by the extended segment DE, one joins E to C and plots the point P of intersection of AB and EC. The points G and H are determined on the vertical (line through P parallel to DE), with G on the segment DC and H at the level of E. A rectangle HFCG is now drawn with GH and GC as the two sides; its area may be readily seen to be equal to that of the given square ABCD. While the construction indeed seems worth noting here in the overall context, it is unclear to what extent the interpretation of the commentators (coming over a thousand years later) may be associated with what the sutrakaras would have had in mind.

Remark I.1. BSS 1.9 (1.45) states:

> The diagonal of a square makes twice the area.

In the sulvasutras, when a line segment is said to 'make' a certain area, it is meant that the latter is the area of the square over the segment. Thus the above statement means that the square over the diagonal has twice the area as the original square. In the context of this property the diagonal of a square acquired a special name, *dvikaraṇī* (side that doubles).

The statement in Remark I.1 is clearly a special case of the Pythagoras theorem when the two sides of the right angled triangle are equal; on the other hand it is more elementary compared to the general case, and visually evident. It has been stated by Baudhayana separately in the form as above, before going to the general statement of the theorem, appearing in BSS 1.12 (1.48), discussed in § 2 below. In particular, it may be observed that the validity of this special case of the theorem is equivalent to the equality of the areas of the rectangle and square in §1.3 as above, as two of those rectangles can be joined along the longer side to get a square with sides equal to the diagonal of the original square.

I.4. Construction of an isosceles triangle from a square. An isosceles triangle of the size of a given square can evidently be constructed in a way similar to that of the construction of a rhombus, described in §1.2, by producing a rectangle with twice the area of the given square and joining the vertices at the base to the midpoint of the opposite side. BSS 2.7 (1.56) describes a specific construction of an isosceles triangle in this respect, in which a *square* is instructed to be chosen in place of the rectangle:

> To construct an isosceles triangle of the size of a (given) square, form a square with twice the area as the given square and join the midpoint of one of the sides to the vertices of the side opposite to it.

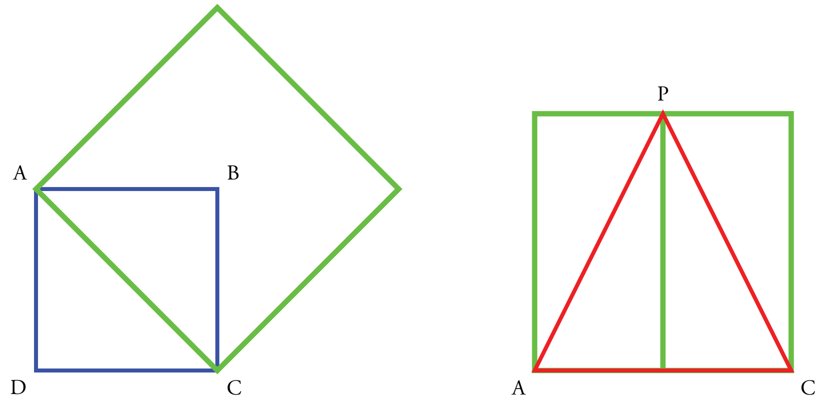


Figure 5. An isosceles triangle with the same area as a given square

Here the sutra specifically alludes to a *square* of twice the area being formed at the intermediate stage, using the term *samacaturasra*; geometrically, this would be mandated, for instance, if the altitude of the isosceles triangle to be produced is required to be of the same size as the base. The procedure is illustrated in Figure 5, where an isosceles triangle PAC is produced with the same area as the given square ABCD. We recall from Remark I.1 that a square with twice the area of the given square can be formed geometrically, taking the diagonal of the given square for its side, thus fulfilling the first step above. The second step, illustrated in the second part of Figure 5, then produces an isosceles triangle with area equal to that of the original square ABCD. As in the comment in §1.2, in the present instance also, joining of the midpoint to the two vertices is described in the sutra in the practical context, mentioning poles and ropes; moreover the midpoint is taken on the side towards the east, the square chosen being arranged along the cardinal directions; the isosceles triangles involved in the ritual constructions had to be pointing to the east.

II. Constructions based on the Pythagoras theorem

To begin with we recall here that the Pythagoras theorem is stated in all the four sulvasutras, mentioned earlier, and that the early ones among them would be considerably prior to Pythagoras (ca. 570 - ca. 495 BCE).[6] The statement in Baudhayana sulvasutra, in BSS 1.12 (1.48), may be translated as

> The diagonal of a rectangle makes both of what the flank and the transverse sides make separately.

The reader would recognise that with the meaning of 'make' discussed in Remark I.1, this is equivalent to the Pythagoras theorem in its general form, stated with respect to rectangles, in place of the right angled triangles. The terms 'flank' and 'transverse' are adopted in the above translation as they closely correspond in their meaning to the original terms *pārśvamānī* and *tiryaṅmānī*, respectively, in the sutra, though in paraphrasing the sutra one may simply refer to the length and breadth of the rectangle; the term used for rectangle is *dīrghacaturasra* and for the diagonal it is *akṣṇayārajju*. The statements in the Apastamba and Katyayana sulvasutras are similar, with minor variations that we need not go into. The Manava sulvasutra (MSS 10.10 (10.3.1.10)) presents the theorem in a different form, which may be worth recalling, for its distinct 'algorithmic' form.

> Product of the width with the width and stretch with the stretch, when added and taken square root of, is the diagonal – this is known.

6 The Pythagoras theorem was also known in the Babylonian civilization, close to 2000 BCE, and perhaps also to the ancient Egyptians.

'Width' and 'stretch' here stand for the two sides of a rectangle ('*āyāma*' and '*vistāra*,' respectively, in the original) whose diagonal is the object to be determined. Incidentally, *karṇa* is the word used in the sutra for the diagonal; it was carried through in later Sanskrit literature, and is now in usage in various Indian languages.

The theorem is used in various transformation problems for figures that we describe through this section.

II.1. Combining two squares into one square. BSS 2.1 (1.50) describes the following procedure for putting together two squares into one:

> To combine two squares of different sizes into one square, plot the rectangle in the bigger square, through the rising point (in the bigger square, when the two are placed side by side); the diagonal of the rectangle becomes the side for the joint square.[7]

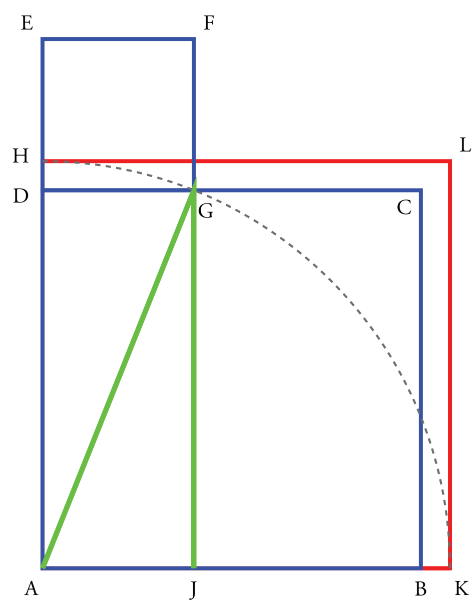


Figure 6. Two squares ABCD and EFGD combined into one square AKLH

The construction is illustrated by Figure 6.[8] Given the squares ABCD and DEFG placed alongside as set in the figure, the rectangle ADGJ through the 'rising point' G is plotted. The side AH of the desired square is taken to be of the size of the diagonal AG of the rectangle ADGJ. As the sides of the latter equal the sides of the two given squares, AH has the desired property, by a straightforward application of the Pythagoras theorem.

II.2. Converting the difference of two squares to a square. BSS 2.2 (1.51) gives the following prescription for converting the difference of two squares, of different sizes, into a square.

> When a square is to be taken away from a square, plot a rectangle in the bigger square along the side of the smaller square; drop the flank side of the rectangle on its other side as the diagonal; the segment which it cuts off is the answer.

7 Regarding this translation we would like to mention that, as there are some apparently technical terms involved in the statement, that have in fact been a subject of discussion going back to Thibaut [7], in this instance we have relied to a larger extent on the received wisdom on what the sutra means, than on our own linguistic resources.

8 In illustrating the construction, often the squares are drawn in an overlapping fashion - in our view it would however be more appropriate, given the spirit of what is involved, to draw them side by side as shown here.

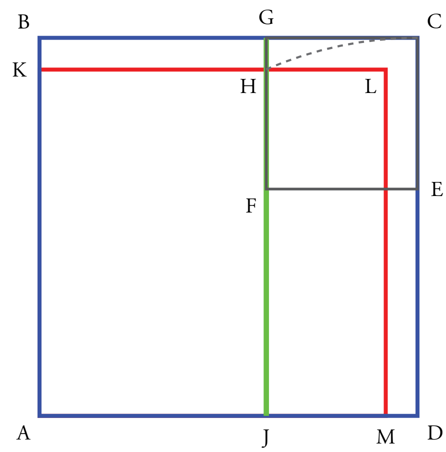


Figure 7. Difference of squares ABCD and CEFG is turned into a square AKLM

The construction is illustrated by Figure 7. Consider a square ABCD, and a square CEFG to be taken away from it, as seen in the figure. Then the rectangle CDJG is plotted with the point J on AD so that JD equals GC. The flank side CD of CDJG is dropped on the opposite side JG, through an arc drawn with centre at D and radius DC, and the point H where it meets FG is marked. A square AKLM is drawn with side equal to JH. This gives a square with the desired property, again as an application of the Pythagoras theorem; here D, J and H form a right angled triangle in which the hypotenuse DH has length equal to the side of the larger square and JD has length equal to that of the smaller square, so the square on the side JH, and equivalently on AK, has area equal to the difference of the two squares.

II.3. Converting a rectangle to a square. BSS 2.5 (1.54) describes the following procedure for converting a rectangle into a square; the same is also described in KSS 3.2.

> When a rectangle is desired to be converted to a square, taking out a square over the transverse side, divide the remaining part of the rectangle into two, and move one part to the flank side; that (the figure formed) can be supplemented by a square to form a square, and how to rectify that (to get a square as desired) has been explained earlier.

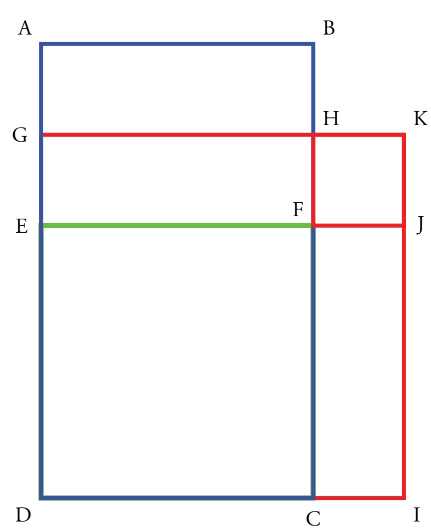


Figure 8. Squaring a rectangle

The construction is illustrated in Figure 8.[9] Given the rectangle ABCD, a square CDEF is marked out at one end and the remaining rectangle is divided in the middle along the shorter side, by joining the midpoints G and H of AE and BF respectively. The rectangle ABHG is moved to the other side, relocating it as IJFC. The rectangle is now transformed to a figure which is readily seen to be a difference of two squares, viz. of GKID and HKJF. The instruction at this point is to follow the procedure described earlier (in §2.2 in the present exposition) to convert the difference into a square.[10]

II.4. Converting an isosceles triangle to a square. Katyayana makes use of the process of conversion of rectangle into square, in describing a procedure to turn an isosceles triangle into a square. KSS 4.5 (4.7) has the following instruction in this respect.

> When an isosceles triangle is sought to be turned to a square, divide it along the bisector, invert one of the parts and place it along the other to get a rectangle, and convert the rectangle to a square.

It should be clear how an isosceles triangle would be turned into a rectangle by dividing along the altitude and reassembling along the slanted sides. After obtaining a rectangle the procedure as in §2.3 is to be followed. Thus altogether we have here a three-tier process of construction!

References

1. S.G. Dani, Geometry in the Sulvasutras, in *Studies in History of Mathematics*, Proceedings of Chennai Seminar, Ed. C.S. Seshadri, Hindustan Book Agency, New Delhi, 2010.
2. Bibhutibhusan Datta, Ancient Hindu Geometry: The Science of the Śulba, Calcutta Univ. Press. 1932; reprint: Cosmo Publications, New Delhi, 1993.
3. Raghunath P. Kulkarni, *Char Shulbasūtre* (in Marathi), Maharashtra Rajya Sahitya Sanskriti Mandal, Mumbai, 1978.
4. Raghunath P. Kulkarni, *Char Shulbasūtra* (in Hindi), Maharshi Sandipani Rashtriya Vedavidya Pratishthana, Ujjain, 2000.
5. T.A. Saraswati Amma, Geometry in Ancient and Medieval India, Motilal Banarsidas, Delhi, 1979.
6. S.N. Sen and A.K. Bag, The Śulbasūtras, Indian National Science Academy, New Delhi 1983.
7. G. Thibaut, The Śulvasūtras, The Journal, Asiatic Society of Bengal, Part I, 1875, Printed by C.B. Lewis, Baptist Mission Press, Calcutta, 1875.

9 The sutra assumes the longer side to be along the east-west direction (vertical in our representation) consistent with a general convention seen in the sulvasutra constructions - of course from a geometric point of view this involves no loss of generality.

10 This is perhaps the oldest instance in which in a mathematical discourse an appeal is explicitly made to a result described earlier, that is now a common practice in mathematics.

S.G. DANI was affiliated with the Tata Institute of Fundamental Research (TIFR), Mumbai until retirement. Subsequently, he was with IIT Bombay, and is currently affiliated with the UM-DAE Centre for Excellence in Basic Sciences, Mumbai. He was the President of the *Indian Society for History of Mathematics* for over a decade, and continues to be the Editor of *Ganita Bharati*, the Bulletin of ISHM. He can be contacted at shrigodani@cbs.ac.in.

MEDHA LIMAYE retired as Mathematics Teacher from the Arvind Gandbhir High School, Jogeshwari, Mumbai. She holds a Doctoral degree in Sanskrit, for her thesis on Cultural Aspects of Ancient Indian Mathematics. She has written a book entitled *Treasure of Interesting Numerical Problems*, and several articles on history of mathematics. She is currently serving as Joint Secretary of *Brihanmumbai Ganit Adhyapak Mandal*, Mumbai. She can be contacted at medhalimaye@gmail.com.

Glossary of names and terms

As in text	As in technical literature	In Devanagari script
Apastamba	Āpastamba	आपस्तम्ब
Baudhayana	Baudhāyana	बौधायन
Circle	Maṇḍala	मण्डल
Diagonal (1)	Akṣaṇayā/ Akṣaṇayārajju	अक्षणया/अक्षणयारज्जु
Diagonal (2)	Karṇa	कर्ण
Flank (longitudinal) side	Pārśvamānī	पार्श्वमानी
Isosceles triangle	Prauga	प्रउग
Katyayana	Kātyāyana	कात्यायन
Manava	Mānava	मानव
Pointed	Aṇimat	अणिमत्
Puruṣa (height of man with uplifted arms)	Puruṣa	पुरुष
Quadrilateral	Caturasra	चतुरस्र
Rectangle	Dīrghacaturasra	दीर्घचतुरस्र
Rhombus	Ubhayataḥ prauga	उभयतः प्रउग
Rope or cord	Rajju, Śulva/Śulba	रज्जु, शुल्व/शुल्ब
Semicircle	Ardhamaṇḍala	अर्धमण्डल
Square (1)	Caturasra	चतुरस्र
Square (2)	Samacaturasra	समचतुरस्र
Stretch	Vistāra	विस्तार
Sulvasutra	Śulva-sūtra/ Śulba-sūtra	शुल्बसूत्र
Sutra (statement in aphoristic style)	Sūtra	सूत्र
Sutrakara (composer of sutras)	Sūtrakāra	सूत्रकार
Transverse (lateral) side	Tiryaṅmānī	तिर्यङ्मानी
Width	Āyāma	आयाम
Yajamana (master of ceremony)	Yajamāna	यजमान
Yajna (fire worship/ritual)	Yajña	यज्ञ

A Utilitarian Math World: The Tail that Wags the Dog

KUMAR GANDHARV MISHRA

One of the priorities of mathematics educators and teachers has been to improve mathematics learning in the classroom: to experiment with different teaching methods. The objective is to engage children with the mathematics they study by allowing them to explore and enabling them to think about the problem in multiple ways rather than memorising procedures and formulas. But, what about the world outside this classroom? What is happening in this world and how is it going to influence and affect classroom efforts in the future? This article takes the readers through a short journey of such a world and tries to highlight what the world often offers for mathematics in general, and mathematics education, in particular. Roaming around streets and markets in India, you would often come across posters, pamphlets and hoardings of coaching centres which claim to have ***Remedies*** *for all your* **Math Problems**. It is exciting to see these 'Mathematics' hoardings which stand tall along with the hoardings of cars and apartments for sale, near highways, streets, roads or in markets. Thousands pass by and see them daily. They would have absorbed some impression about mathematics. What is the (perhaps half-baked and partial) impression that people who have passed by these hoardings must have absorbed about mathematics?

A look at all the advertisements during a walk through Mukherjee Nagar, Delhi (a famous coaching hub in Delhi) made me wonder: **'Is Mathematics really frightening?'**, **'Does Mathematics need to be taught from the beginning?'**, **'Do students who pass out from college and school need to learn mathematics from the very beginning even at this stage?'**, **'Is mathematics that subject which needs to be taught with a guarantee stamp?'** or **'Are 10 ways sufficient to learn Mathematics?'** I am sure you have come across similar advertisements. Perhaps in some way, these statements

Keywords: Mathematics education, aims, competitive examinations, shortcuts.

reflect the status of mathematics in society and what mathematics means to society. If you happen to visit bookstores at a railway station or a bus station, you would often find books on mathematics tricks and formulas. Most of the coaching centres or coaching platforms mainly focus on 'competitive mathematics,' also known as quantitative aptitude which is an integral part of almost all competitive examinations leading to employment opportunities in India. During the last 4-5 years, the use of smartphones with affordable data services (internet) has significantly increased among students and graduates in India. For example: you would find YouTube flooded with videos on tricks. Many of these channels have grown with millions of subscribers (users or learners) as well as a million views. With an aim to gain maximum views and subscribers (monetary benefit), most of these videos appear with catchy thumbnails/ captions to solve mathematics problems within *'seconds'* - not *'minutes.'* Many of these coaches 'guarantee' the selection of every subscriber (user) for various competitive examinations. (Though the number of seats is pre-known and applicants are in lakhs, they guarantee selection of every subscriber with their tricks.) A significant leap in revenue, number of subscribers and number of registered users in these coaching platforms and other Edtech companies during the times of the Covid-19 pandemic has also made headlines. The phrases used across various platforms (online and offline) confirm certain societal notions about mathematics:

i. Mathematics is important and an integral part of any competitive examination. (Mathematics as a gatekeeper in various employment opportunities.)
ii. Success in mathematics is measured by the speed in which the correct response is given.

These also popularise some notions (fallacies) about mathematics:

i. Mathematics is frightening (Math Phobia).
ii. People who are good in mathematics give the correct answer quickly.
iii. You become good at mathematics if you learn 'shortcuts,' 'magic formulas' and 'tricks.'

These examples also reflect how the widespread 'fear of mathematics' is shrewdly (mis)used by these platforms and it also shows how mathematics is being made synonymous with formulas, shortcuts and tricks in the public domain.

A short glimpse of some of the popular tricks

The syllabus for most of the competitive examinations is almost the same. Generally, the syllabus comprises of these topics: number system and arithmetic operations, divisibility, surds, exponents, GCD and LCM, elementary algebra, ratio and proportion, rates, commercial arithmetic, alligation/mixture problem, etc. Some examinations also include probability, data handling, mensuration, trigonometry, etc.

The terms 'tricks' and 'shortcuts' are often used without any distinction in day to day conversations. In colloquial language, a *shortcut* suggests a route which takes less time to reach the destination. In the context of a mathematics problem, any approach which helps find the answer quickly compared to another approach (conventional) is a shortcut. One approach may be a shortcut for one problem but not for another. For example: factorisation, formula or trial and error are all methods to solve quadratic equations, but the shortest method would depend on the quadratic itself.

Example 1: A popular trick[1] on multiplication of 2 two-digit numbers close to 100 is discussed here. Illustration and presentation of these tricks varies from coach to coach. Some present it as ***'multiplication of any 2 two-digit numbers'*** while some present it as ***'multiplication of 2 two-digit numbers close to 100'***.

Problem 1: Multiply 88 and 92.

Multiplication through conventional method

```
  88
× 92
----
 176
7920
----
8096
```

Figure 1

Multiplication through a trick

Step 1: Subtract each number from 100 and note down the corresponding difference. Place these below the two numbers.

88	**92**
12	**08**

Figure 2

Step 2: Multiply the differences. Denote this result by *A*.

$$12 \times 08$$

Figure 3

A = 96. Note: If the result is a one-digit number (say 2), write it as 02.

Step 3: Subtract one of the differences from the other number (diagonally opposite position). Denote this result as *B*.

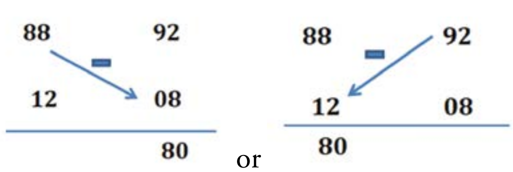


Figure 4

Here, *B* = 80.

Step 4: Place **B** before **A** i.e. *BA*, and this (8096) is the answer.

How does this trick work?

Let us understand the reasoning behind this trick through a general case. Let the 2 two-digit numbers 88 and 92 be P and Q respectively.

88	**92**
12	**08**

As per Step 1, the representation is:

P	**Q**
100-P	**100-Q**

Figure 5

Let the difference when P and Q are subtracted from 100 be x and y respectively i.e. $x = 100 - P$ and $y = 100 - Q$. In other words, $P = 100 - x$ and $Q = 100 - y$.

So, another equivalent representation (diagram) of Figure 5 is:

100-*x*	**100-*y***
x	***y***

Figure 6

Now, as per step 2 of the trick, Step 2: *A* (*A* used in trick) $= xy$ Step 3: $B = (100 - x) - y$ or $(100 - y) - x = 100 - (x + y)$. [From Figure 6, you can see why the difference of the two pairs of diagonally opposite numbers in the Figure 4 yields the same result.]

As per the trick, placing ***B*** before ***A*** i.e. ***BA*** yields the result. Let us see how the rule of 'placement' works in this case.

The product of ***P*** and ***Q*** can be written as

$PQ = (100 - x)(100 - y) = 10000 - 100x - 100y + xy = 10000 - 100(x + y) + xy = 100\{(100 - (x + y)\} + xy = 100B + A$

The product (100B) after multiplication of **B** by 100 will always have zero in its tens and units place and so when you add a **two-digit number** ***A*** to **100B**, the result will be the same as placing **B** before *A* (left of *A*). That's how a visual representation ***BA*** becomes an answer without any explicit operation like addition. (This is why *A* is always written as a two-digit number.)

Conditions for A to be a two-digit number (to be represented by two digits): $0 \le xy \le 99$

If the product $xy > 99$, the 'placement' rule will not work. For example: if 88 changes to 87, the product of the differences of 87 and 92 from

100 (13 and 8 respectively) becomes 104 which is a three-digit number. A three-digit number can't be placed or fit into '00', so the trick doesn't work in this case. In fact, the two 2-digit numbers used for illustration are intentionally selected to suit this condition (that $xy \leq 99$). Cases where $xy > 99$ are conveniently omitted in spite of the trick being advertised (and the audience fooled) as a general ***multiplication trick for 2 two-digit numbers which finds the product within seconds.***

Example 2: Figure 7 depicts a famous trick used in the coaching world for *Alligation* (mixture) problems. Here, a mixture problem where the Cost Price of Product A < Mean Price of mixture of A&B < Cost Price of Product B is discussed:

Problem 2: How many kg of rice at ₹ 52 per kg should a shopkeeper mix with 25 kg of rice at ₹ 24 per kg so that on selling the mixture at ₹40 per kg, the shopkeeper can gain 25% on the outlay?

Solution using conventional method

Selling Price (SP) = 40; A gain of 25% has been made by selling the mixture at ₹ 40/kg.

Here, CP + 0.25CP = SP, (Cost Price = CP)

So, 1.25 CP = 40 or CP = 32

The cost price of the resulting mixture is ₹ 32/kg. Let x kg of rice at ₹ 52/kg be mixed with 25 kg of rice at ₹ 24/kg.

As per the conditions:

$$52x + 25 \times 24 = 32(x + 25)$$

Solving this equation, we get $x = 10$. So, the required amount of rice should be 10 kg.

Solution using Trick

Step 1: $CP = \frac{100\,SP}{100+P}$ (A readymade formula for calculating CP, here P is the profit percent). Using this formula, $CP = \frac{100 \times 40}{100+25} = 32$ (which is the mean cost price of the mixture of two varieties of rice).

Step 2: ***Alligation Diagram***

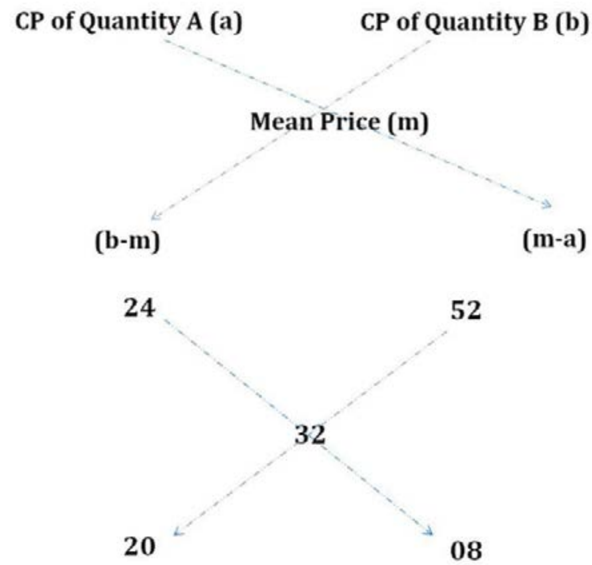


Figure 7
Quantity of A: Quantity of B = (b – m): (m – a)

Quantity of rice A (₹ 24/kg) : Quantity of rice B (₹ 52/kg) = 20:8 which is 5:2.

Step 3: For every 5 kg of rice **A**, 2 kg of rice **B** is used. So, for 25 kg of rice **A**, 10 kg of rice **B** will be used.

This trick (Figure 7) is a modified version of the conventional method. The *Alligation Diagram* is nothing but a visual aid for solving linear equations (at least for those who haven't understood or applied the conventional approach). The *trick* is to be mugged up by those students who are not well aware of the reasoning process involved in this problem.

By using tricks repeatedly, children and even adults begin to experience the satisfaction of getting the right answer, but in the process, they begin to think that *'this is the way mathematics works and coaches/teachers who 'invent' or come up with these tricks are definitely champions or kings of mathematics'*. Coaches are described with adjectives like Magician, King and Wizard and called 'Mathematicians' and 'Educators' in the public domain. You may come across such discourse among social media users, common public and students in coaching classrooms about glorification of similar tricks as well as of their math coaches.

Some Consequences

In general, the audience which attends coaching classes or virtual platforms for competitive examinations are not school children. Wide circulation of billboards, posters, hoardings in public spaces and circulation of videos (both online and offline mode) with thumbnail-captions emphasising shortcuts helps create and strengthen a utilitarian and narrow impression of mathematics among parents, general public and college and school students (future parents, teachers and mathematicians). Some of the far reaching and long-lasting consequences on children, parents and society in general can be:

A. Parents: When parents absorb such impressions about mathematics, their expectation from their children of being good in mathematics would be reduced to being good with tricks and of giving the right answer quickly. A notion of competition is prevalent in society where parents try to compare their children's ability and success with that of other children. Sadly, a child who is dubbed a 'young mathematician' is often nothing more than a trained performer.

B. Children: Children who don't have the privilege to be taught through various pedagogical interventions, classroom engagement, teaching resources, games and activities are more vulnerable to rely on the power of tricks and formulas and absorb similar beliefs about mathematics which carry on into their adulthood. This culture has also started making its way to school students by targeting the syllabus of school mathematics. For example: Trick based or readymade solutions are being made available in the name of NCERT or CBSE mathematics problems. Though tricks are not emphasised in an ideal school classroom, these have the potential to divert students in getting readymade answers once they get exposed to these. It can help them create their own impression about mathematics: a narrow impression that mathematics means only numbers, tricks, mechanical procedures and correct answers. Based on these, a contrast between the teacher's and parent's expectations can create an uncomfortable situation for children.

Children have also joined YouTube and other platforms where you would find them publishing their own channel and videos on multiplication tricks, addition tricks, etc. with the use of phrases like that used by math coaches. They are also appreciated and glorified for the presentation of tricks. Peer pressure leads to such practices spreading. The prevalent culture and its consequences for children are also contrary to the vision envisaged in the position paper of the teaching of mathematics (NCF 2005) which explicitly states that equating mathematics with formulas and mechanical procedures does great harm. Unfortunately, such activities are taking place outside the classroom. However, it would be a good initiative if children are asked to explore the reasoning behind working of their tricks and formulas. They should also be asked to explore the conditions under which their tricks work or fail. Developing understanding behind the working of many of these techniques can also help them become problem solvers in the true sense of the word.

Concluding comments

No doubt, tricks and readymade formulas come to rescue in various competitive examinations and it is unfair to expect anyone to refrain from tricks and formulas in tests where speed matters. In such an environment, everyone has the freedom to teach and solve the way they want until they get correct answers, but it is unfortunate that tricks and formulas have become a ubiquitous part of mathematics. Many a time, tricks are glorified in the name of mathematics education which is contrary to the goals of mathematics education and voices of mathematics educators across the globe. One may say that it depends on discretion of viewers and learners to follow and use these tricks. But, irrespective of the usage of tricks, marketing and popularising tricks with misleading thumbnails and posters spreads a wrong and partial impression about mathematics. The unfair depiction of mathematics in public spaces through these advertisements can construct such a perception in society where **getting answers within the shortest possible time = being successful** may emerge as the new equation.

Acknowledgement

I would like to thank Dr Shailesh Shirali for the encouragement provided by him to highlight this issue in *At Right Angles*. In 'The Closing Bracket' of the March 2018 issue of *At Right Angles*, he has also raised a similar concern on the prevalent utility factor in society regarding the goals of studying mathematics. I also thank Sneha Titus for her valuable suggestions.

References

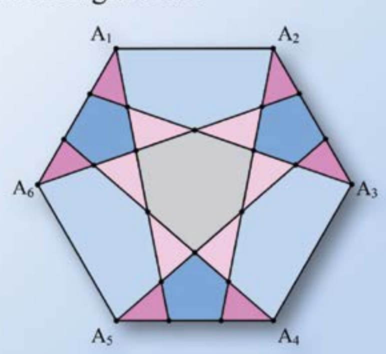
[1] Mishra, K. G. (2019). The Art of Selling Mathematics. *Proceedings of the Tenth International Mathematics Education and Society Conference*. Hyderabad.

[2] *Position paper on teaching of mathematics*. (2006). National Council of Educational Research and Training.

KUMAR GANDHARV MISHRA is an independent researcher in Mathematics Education. He completed his Masters in Mathematics Education – M.Sc. (Mathematics Education) from Cluster Innovation Centre, University of Delhi. He often writes on social and cultural aspects of mathematics, and mathematics education. He can be reached at mishrakumargandharv@gmail.com

The whole is equal to the sum of the parts!

A regular hexagon A has its vertices at points A_1, A_2, A_3, A_4, A_5, and A_6. Two points on three of the sides of hexagon A divide that side into three equal parts. These six points on three of the sides of hexagon A are connected to the six vertices of hexagon A as shown. These six lines divide hexagon A into 19 disjoint polygons. If the area of hexagon A is 1260, find the area of each of the 19 disjoint polygons inside hexagon A.



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Interpretation of Divisibility Rules

ANKIT PATODI

Divisibility rules which help to quickly identify if a given number is divisible by 2, 3, 4, 5, 6, 8 and 10 are taught in the upper primary classes. However, the conceptual background or the proofs behind the rules are rarely dwelt upon. One of the reasons could be that most of us, i.e., the teachers, are not aware of the logic behind the rules. Even if we are, we may believe that the proofs for these rules are beyond the scope of understanding of children as they involve complex algebraic expressions and interpretation. These two reasons together distance children from learning the logic behind the rules.

This challenged me to put the explanation behind these rules in a simple manner. I shared this with a group of teachers and saw that it helped them further in their classrooms. I used a few basic rules to justify my reasoning:

RULE 1: If any number is divisible by another, all its multiples will also be divisible by that number. For example, if 10 is divisible by 2 then all multiples of 10 viz. 20, 30, 100, 1000 and so on will also be divisible by 2.

Let us take an example to understand this rule. If we say that 10 is divisible by 2, it means that when we divide 10 objects into groups of 2 each, no object is left out. The representation below (Figure 1) depicts the same:

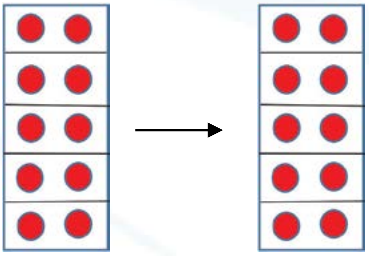


Figure 1

Keywords: Patterns, divisibility, place value, multiples, visualisation, generalisation

Now, suppose we have objects in any multiple of 10, we can divide the objects into smaller groups of 2 in the same fashion (Figure 2). Similar relationships can be visualized for any number and its multiples.

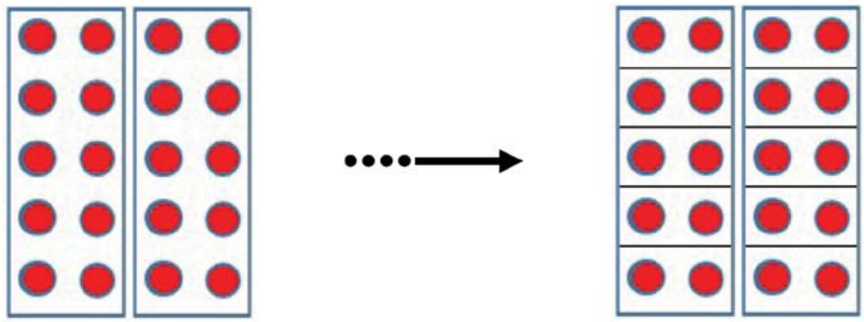
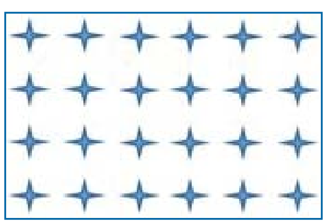


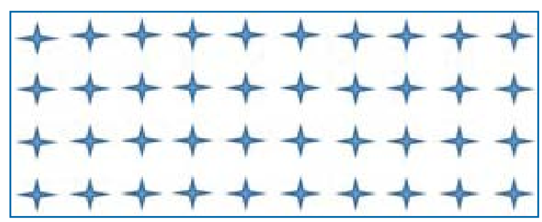
Figure 2

RULE 2: If two or more numbers are all divisible by the same number, then their sum will also be divisible by that number. For example, if 24 and 40 both are individually divisible by 4, their sum i.e., 24 + 40 = 64 will also be divisible by 4. This can be easily shown using counters (Figure 3):

$24 = 4 \times 6$



$40 = 4 \times 10$



$64 = 4 \times (6 + 10)$

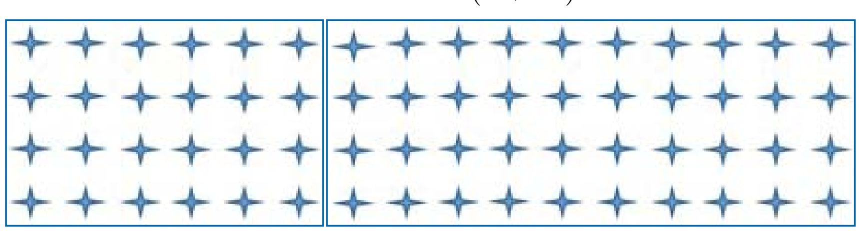


Figure 3

This simple representation could be generalized for the combination of three or more numbers. We will be using a few basic expansions of numbers in order to understand how the concept of place value plays an important role in understanding divisibility rules. Let us take an example:

We have a number 13455, which we call thirteen thousand, four hundred and fifty-five. We can expand this number in many ways using the place value concept. So, 13455 can be written in any of the following ways:

- $10000 + 3000 + 400 + 50 + 5$
- $13000 + 400 + 50 + 5 = 13000 + 455$
- $13400 + 50 + 5 = 13400 + 55$
- $13450 + 5$; and so on.

Divisibility by 10

Any number is divisible by 10 if the last digit of that number is 0.

Here we would try to understand the divisibility rule for 10 in the first place as this rule will act as the base to interpret the rules for the other numbers.

Observe the way we write the numbers below:

1	11	21	31	41	51	61	71	81	91	101	111	121
2	12	22	32	42	52	62	72	82	92	102	112	122
3	13	23	33	43	53	63	73	83	93	103	113	123
4	14	24	34	44	54	64	74	84	94	104	114	124
5	15	25	35	45	55	65	75	85	95	105	115	125
6	16	26	36	46	56	66	76	86	96	106	116	126
7	17	27	37	47	57	67	77	87	97	107	117	127
8	18	28	38	48	58	68	78	88	98	108	118	128
9	19	29	39	49	59	69	79	89	99	109	119	129
10	20	30	40	50	60	70	80	90	100	110	120	130

Figure 4

All numbers are made of ones and tens where the number of ones must be less than 10 but the number of tens can be as large as we want. For example, 473 is made with 3 ones and 47 tens, while 6850 is made with 685 tens and zero ones. If children have experience in making numbers with bundles (representing tens) and sticks (representing ones), this would be easy for them to understand. Now the tens are all divisible by ten of course but the ones are not. So, the only way a number would be divisible by 10 is if there are no loose ones (i.e., outside the tens) which is when it has 0 as its ones digit (i.e., the last digit).

Divisibility by 5 and 2

A number is divisible by 5 if the last digit of the number is 0 or 5.

Similarly, *a number is divisible by 2 if the last digit of the number is 0, 2, 4, 6 or 8.*

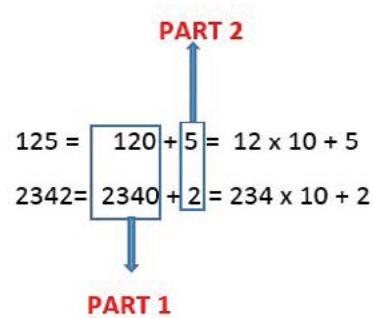
One way of looking at these rules can be through the 10×10 grid.

Here, multiples of 2 are highlighted in blue, those of 5 in red and those of both 2 and 5 in purple.

1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	15	25	35	45	55	65	75	85	95
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99
10	20	30	40	50	60	70	80	90	100

Figure 5

A way of justifying these two rules could be through RULE 1 and RULE 2. Any number greater than 10 can be expressed in the form of a sum of a multiple of 10 and the remaining last digit. For example:



In the case of divisibility by 5, for any such number, Part 1 which is a multiple of 10 is always divisible by 5 (RULE 1), and for the whole number to be divisible by 5, the left out last digit should be divisible by 5 (RULE 2). This is possible only if the last digit is either 5 or 0. Hence, the condition for divisibility by 5.

Can you make a similar argument for divisibility by 2?

Divisibility by 4

A number is divisible by 4 if the number formed by its last two digits is divisible by 4.

This is similar to the divisibility rules of 5 and 2. But here, the last two digits of the number play an important role. The reason for this is that while 4 is not a factor of 10, it is a factor of 100. So, in this case, Part 1 should be a multiple of 100 (instead of 10), and Part 2 is a 2-digit number whose divisibility it remains to be checked.

For example:

$$464 = 400 + 64 = \overset{\text{Part 1}}{\boxed{4 \times 100}} + \boxed{64} \rightarrow \text{Part 2}$$

$$4596 = 4500 + 96 = 45 \times 100 + 96 = \overset{\text{Part 1}}{\boxed{45 \times 25 \times 4}} + \boxed{96} \rightarrow \text{Part 2}$$

Now can you use similar reasoning to understand the rule for divisibility by 8?

Divisibility by 9 and 3

A number is divisible by 9 if the sum of all the digits of the number is divisible by 9.

Similarly, *a number is divisible by 3 if the sum of all the digits of the number is divisible by 3.*

Till now, we have been using 10 or 100 as one of the multiples in Part 1 to justify the divisibility rules of 5, 2 and 4. In the case of 9 and 3, Part 1 is written as a multiple of 9, 99, 999, and so on. Let us look through an example:

Take a number, say 873, to check its divisibility by 9.

$$873 = 800 + 70 + 3 = 8 \times 100 + 7 \times 10 + 3 = 8 \times (99 + 1) + 7 \times (9 + 1) + 3$$

$$= (8 \times 99 + 8) + (7 \times 9 + 7) + 3 = (8 \times 99 + 7 \times 9) + (8 + 7 + 3)$$

Here Part 1 $= 8 \times 99 + 7 \times 9$ is clearly a multiple of 9. The remaining part or Part 2 i.e. $8 + 7 + 3 = 18$ is the sum of the digits of number 873. (Do you see why you get the digits of the given number in Part 2?) So, to check divisibility of 873 by 3 or 9, we need to check only if the digit-sum 18, is divisible by 3 or 9. Since 18 is divisible by 9 (and therefore by 3 also), 873 is divisible by both 9 and 3.

Similarly, $4{,}83{,}720 = 4 \times 100000 + 8 \times 10000 + 3 \times 1000 + 7 \times 100 + 2 \times 10 + 0$

$$= 4 \times (99999 + 1) + 8 \times (9999 + 1) + 3 \times (999 + 1) + 7 \times (99 + 1) + 2 \times (9 + 1) + 0$$

$$= (4 \times 99999 + 8 \times 9999 + 3 \times 999 + 7 \times 99 + 2 \times 9) + (4 + 8 + 3 + 7 + 2 + 0)$$

Again Part 1 is clearly divisible by 9 and Part 2 is the digit-sum $= 4 + 8 + 3 + 7 + 2 = 24$, which is not divisible by 9 but is divisible by 3. So, 483720 is not divisible by 9, but it is divisible by 3.

On the other hand, $5273 = (5 \times 999 + 2 \times 99 + 7 \times 9) + (5 + 2 + 7 + 3)$ and Part 2 or its digit-sum $= 5 + 2 + 7 + 3 = 17$ is not divisible by 3 while Part 1 clearly is. So, 5273 is not divisible by 3 or by 9.

Now what do you think should be the rule for divisibility by 6? For 12? For 15? For 20?

Some important points to summarise

For each divisibility rule, the idea is to break the number into Part 1 and Part 2 such that

(i) Part 1 is divisible by the concerned number usually by RULE 1

(ii) Part 2 is very small compared to the original number

(iii) We need to check only Part 2 for divisibility and apply RULE 2 for the whole number as depicted below:

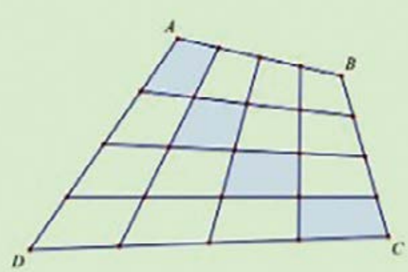
Number	Part 1	Part 2
2, 5, 10	10m (Multiple of 10)	Remaining last digit in units place
4	100m (Multiple of 100)	Remaining last 2-digit part
8	1000m (Multiple of 1000)	Remaining last 3-digit part
3 and 9	9m (Multiple of 9)	Digit-sum

So, we have discussed divisibility by 2, 3, 4, 5, 6 (by extension), 8, 9 and 10. Though we have not proved the rules algebraically for any number, these visualizations will provide enough of a spark to the minds of young learners on why such rules/tests work.

ANKIT PATODI is a Resource Person at Azim Premji Foundation. He has a BE in Computer Science and MA in Development from Azim Premji University, Bangalore. Ankit works with government teachers across various districts of Chhattisgarh on issues related to mathematics and he conducts workshops focusing on conceptual understanding as well as pedagogical strategies used in teaching mathematics. He has been doing mathematics with children for more than 5 years and is interested in exploring and designing teaching resources to address misconceptions in mathematics. Ankit may be contacted at ankit.patodi@azimpremjifoundation.org

Playing in the Quad

Polygon *ABCD* is an *arbitrary* quadrilateral with an area of 4. Three points on each side of *ABCD* divide the side into four equal parts. These twelve points are connected as shown by six line segments. These six line segments divide quadrilateral *ABCD* into sixteen non-overlapping quadrilaterals as shown. Four of the smaller quadrilaterals have been shaded. Find the sum of the areas of the four shaded quadrilaterals.



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How the Square Root Algorithm works

AMIT CHAND

One of the algorithms taught at the elementary level is deriving the square root of a number using the long division method. Like many algorithms taught in high school, the focus is on the 'how' of performing the steps rather than the 'why' these steps produce the square root. This article focuses on developing conceptual understanding of the Square Root Algorithm.

The difference between the factorization method and the division algorithm is that the former gives only the exact value of the square root of a whole number which is a perfect square whereas the latter may be used to find the square root of any positive number. Secondly, while factorization works for numbers with small factors, it becomes tedious when numbers have large prime factors.

We will find $\sqrt{576}$ by the division algorithm and in the process, we will raise some questions.

	2 4
2	5 76
	- 4
44	176
	-176
	0

Figure 1

Step 1: Make pairs of digits from the right, in this case we get 5 and 76.

Step 2: Select the left most pair (5 here) and find the greatest number (2 here) whose square is equal to or less than it. Subtract this square from the selected pair. Write 2 on the left as well as above (as part of the result). This is the first digit which will appear in the quotient.

Keywords: algorithm, square root, reasoning, visualization, understanding

Step 3: Write the second pair from the left (76 here) on the right side of subtracted result (1 here), to get a new dividend (176 here).

Step 4: Double the first number (the quotient 2) and write it down on the left of the new dividend.

Step 5: Now, by trial and error find the greatest digit __ such that $(4\_\_ \times \_\_) \leq 176$. Write this second new digit (4 here) on the right side of 2 in the quotient, now the quotient is 24.

Step 6: Subtract this product $(4\_\_ \times \_\_)$ from the dividend. Repeat steps 3 to 6 until there are no more pairs to be brought down.

Step 7: If the last difference is zero, the square root is the quotient (here the square root of 576 is 24). If not, append pairs of zeroes and find the square root to the required degree of accuracy. At the same time place a decimal point to the right of the quotient at this stage. Future entries will go to the right of the decimal point.

Questions raised here (and often not addressed)

- Why do we make such pairs, and why do we start making pairs from the right?
- How do we make pairs if the number has a decimal part?
- Why do we double the quotient each time and write it on the left as part of the divisor?
- Why do we append a new digit to the divisor and why do we append the same digit to the quotient and multiply the new divisor with the appended digit?

Theoretical part of Algorithm

The algorithm is based on estimating the largest square within the given number (with the help of place-value) and then refining that estimate by adding layers to the estimated square (again with the help of place-value) to get as close to the given number as possible.

There are three main important parts to understand the algorithm.

i. Understanding why we make pairs
ii. Understanding the idea of the formula $(a + b)^2 = a^2 + 2ab + b^2$
iii. Understanding the iteration

i. Understanding why we make pairs

- Square of the smallest 2-digit number i.e. 10 is the smallest 3-digit number i.e. 100. So, any square number of 1-digit or 2-digit must be the square of a 1-digit number
- Similarly, square of the smallest 3-digit number i.e. 100 is the smallest 5-digit number i.e. 10000. So, any square number of 3-digit to 4-digit is the square of a 2-digit number
- Similarly, any square number of 5-digit to 6-digit is the square of a 3-digit number
- And any square number of 7-digit to 8-digit is the square of a 4-digit number and so on…

So, generally speaking, the square of the smallest n digit number (i.e. 10^{n-1}) is the smallest $(2n - 1)$-digit number (i.e. 10^{2n-2}). Therefore, a square number of $(2n - 1)$-digits or $2n$ digits must be the square of an n-digit number. So, if the digits of a number of $2n - 1$-digits or $2n$ digits are paired, then the number of pairs provides the number of digits of the square root.

Now, if the pairing is done from the left then sometimes the units digit would be left alone (for odd number of digits e.g. for 729, 72 and 9) and otherwise it would be along with the tens digit (for even number of digits e.g. for 1296, 12 and 96). This is not consistent. But if the pairing is done from the right, then we get a consistent system since the units digit is always clubbed with the tens. Also, when we pair the digits, we are essentially splitting the number in n parts as follows: 729 = 700 + 29, 1296 = 1200 + 96, 21316 = 20000 + 1300 + 16, 182329 = 180000 + 2300 + 29, etc.

ii. Application of the formula. $(a+b)^2 = a^2 + 2\,ab + b^2$

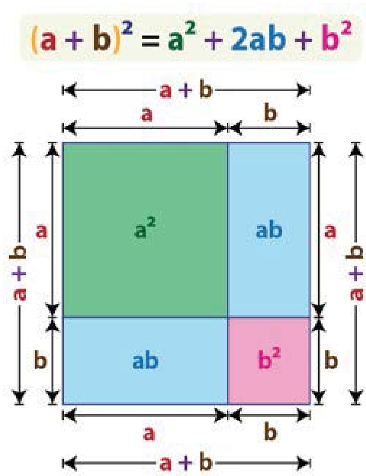


Figure 2

Any square number is of the form $(a+b)^2 = a^2 + 2ab + b^2$ which also can be written as $a^2 + (2a+b)\,b\ldots$(A)

(We are interested in square numbers ≥ 100)

In order to find the square root $(a+b)$ from the square number $(a+b)^2$, we need to find an a and will subtract a^2 from $a^2 + (2a+b)\,b$ which will leave $(2a+b)\,b$. Then our work is to find such b for which $(2a+b)\,b$ holds.

Let us take an example to understand this part.

We have to find the square root of the number 4225.

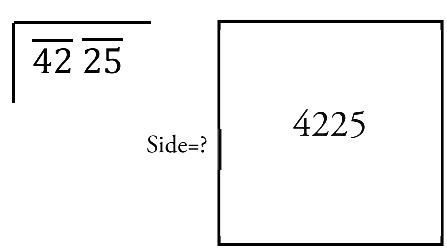


Figure 3

From 4225, we have to find a square number of the form $100x^2$ (since $100 < 4225 < 10000$)

The largest square of the form a^2 which is a multiple of 100 and less than 4225 is 3600

Therefore, remaining area $= 4225 - 3600 = 625$

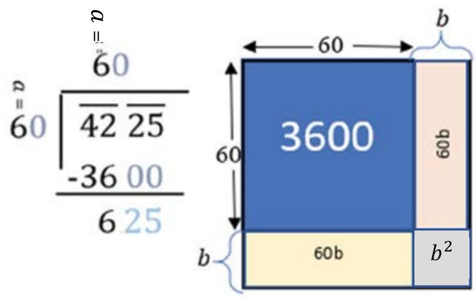


Figure 4

So. we need to find b such that

$$\left(2 \times 60b + b^2\right) = (120 + b)b = 625$$

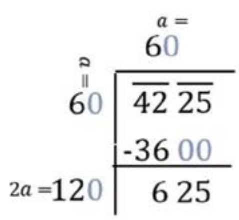


Figure 5

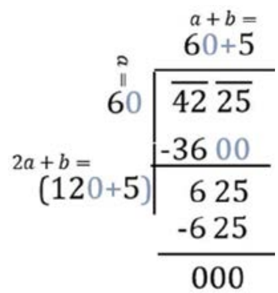


Figure 6

The process to find b involves some trial and error, and with some educated guess, especially if the given number is a perfect square.

We can check if b = 5 then $(120 + 5) \times 5 = 625$ (the last digits give us a clue to find b).

To find the square root of numbers with 5 or more digits, iteration is required where a' is the side of the next layer of the square i.e. $a + b$ and b' is the 2^{nd} iteration.

iii. Understanding iteration. We will understand the iteration in two cases. The first case will be with the square of a 3-digit number and more. The later case will be with a number which is not a perfect square.

Case 1: Square of a 3-digit number. (The square will be between 10000 and 1000000.)

The 3 digit number may be written as $100x + 10\,y + z$ (where x, y and z are single digit numbers). The first objective is to find the first square of the form $(100x)^2$. So, we will find a number $a = 100x$, the square of which may be 10000, 40000, 90000, 160000, 250000, 360000, 490000, 640000 or 810000.

The next part (orange in Figure 8) depends on the selection of $b = 10y$.

b should be chosen so that the orange part does not exceed the remaining part (as in Figure 9).

This orange and gold part together represent $(2a + b)\ b = (2 \times 100x + 10y)\ 10y$

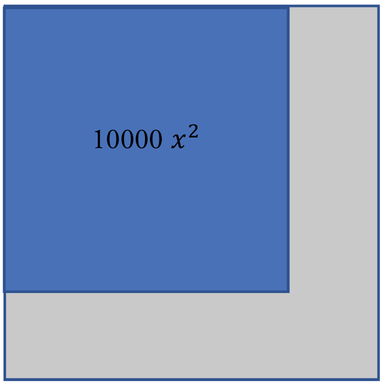


Figure 7

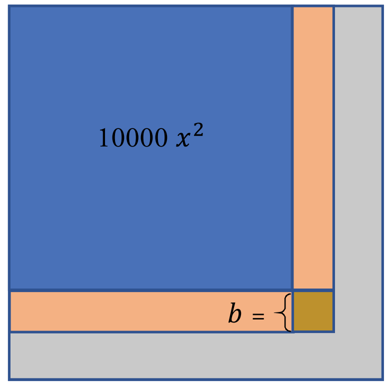


Figure 8

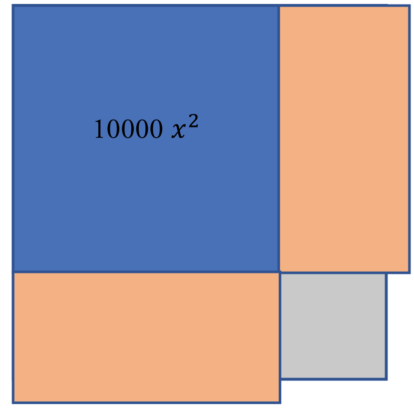


Figure 9

Finally, take $a' = a + b = 100x + 10y$ and then select $b' = z$ to cover the remaining gray part (as in Figure 8) which represents $(2a' + b')\, b' = \{2 \times (100x + 10y) + z\}z$

Now, $(100x)^2 + (2 \times 100x + 10y)\, 10y + \{2 \times (100x + 10y) + z\}\, z = (100x + 10y + z)^2$

Let us take an example (Figure 10) of finding the square root of 21316.

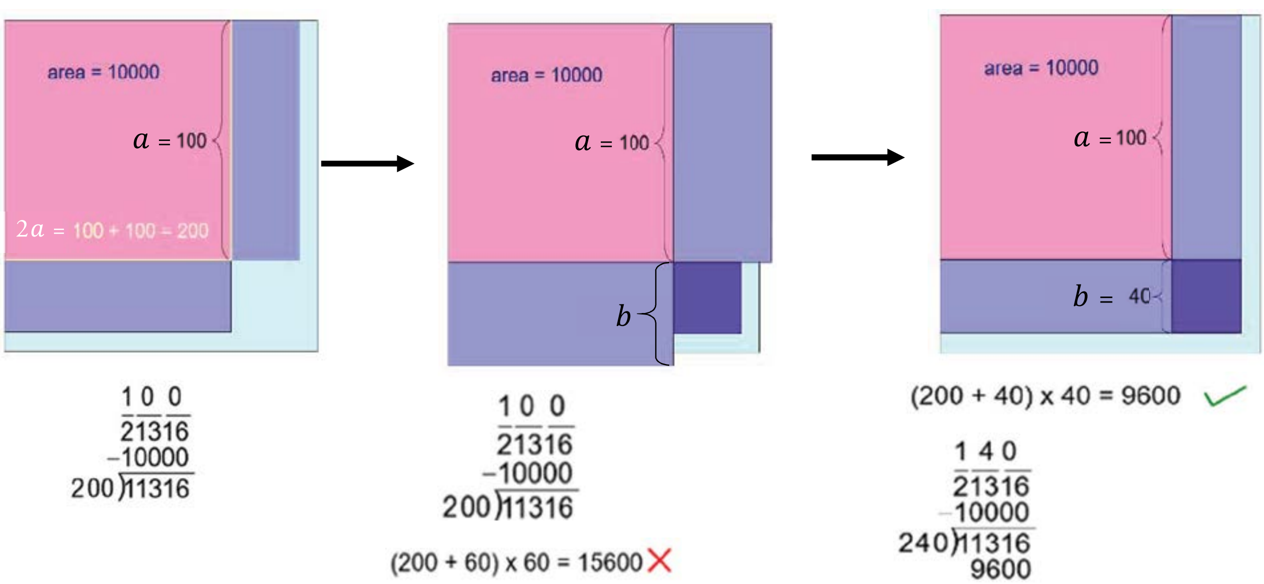


Figure 10

So, after choosing a suitable $b = 10y = 40$ in this example, we get $a' = a + b = 140$

Now we are to find b'.

Therefore, the square root is $a' + b' = (a + b) + b' = 146$ after two iterations (in Figures 10-11).

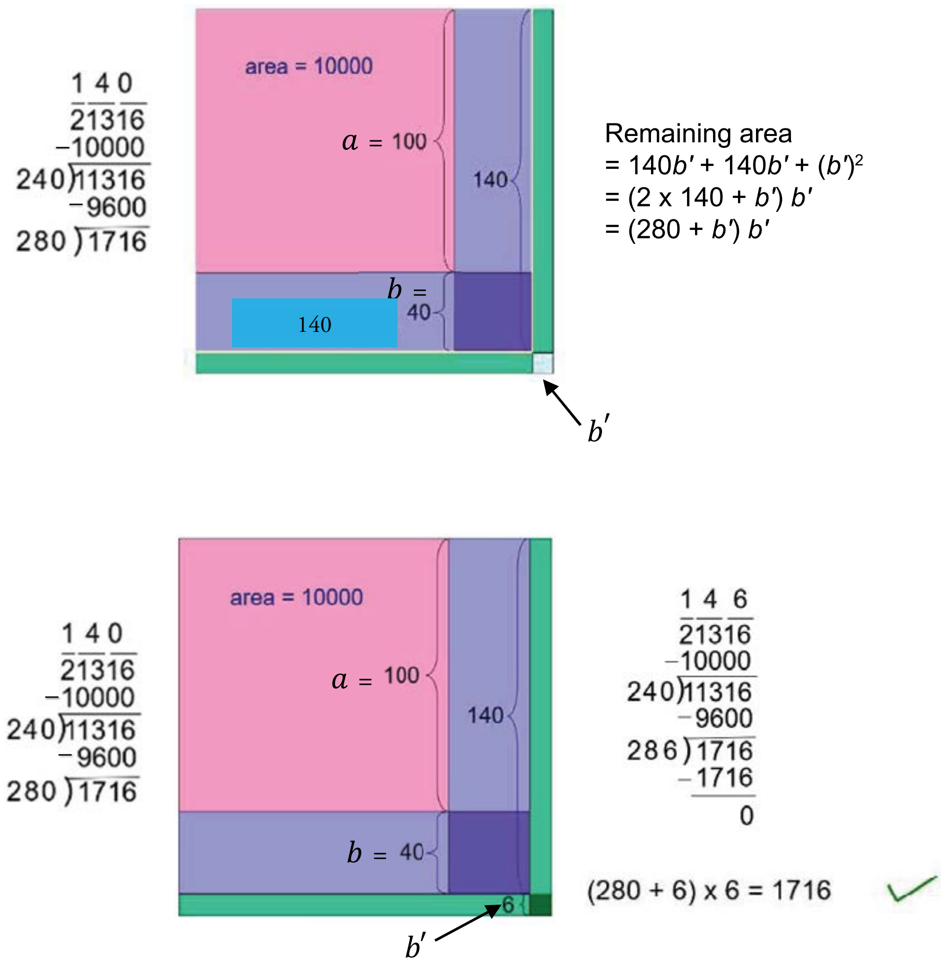
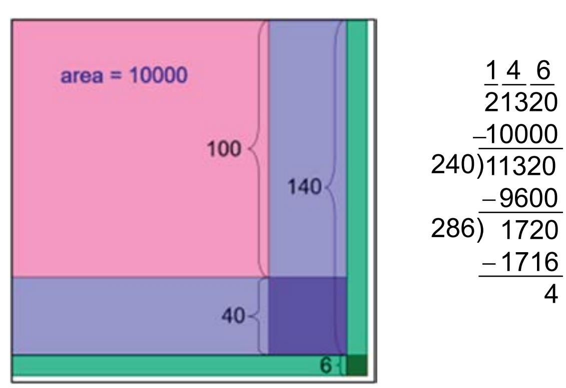


Figure 11

Case 2: Now we will see its use to find the square root of a number which is not a perfect square. The iteration process is the same. Let us try to find the square root of 21320.

From the above case of finding square root of 21316, we can reach up to the step when 4 square area remains.



So, we will continue the iteration process by representing the number with decimal places like 21320.00 00 000...

```
          1 4 6 0 1 3
          21320.000000
         -10000
 24     ) 11320
          -9600
 286    )  1720
          -1716
 2920   )     400
               -0
 29201  )     40000
             -29201
 292023)      1079900
              -876069
```

Here is the square root of 2

	1.41421
1	2.000000000000 1
24	100 96
281	400 281
2824	11900 11296
28282	60400 56564
282841	383600 282841
2828423	10075900

$$\Rightarrow \qquad \sqrt{2} = 1.41421......$$

It is interesting to observe that the key theory behind this algorithm, i.e., $(a + b)^2$ is integral to upper primary algebra. However, even when its application, i.e., the division algorithm of finding the square root is included in the syllabus or textbooks, the connection is rarely made. This is an attempt to fill that gap.

References

1. http://teachersofindia.org/en/presentation/getting-root
2. https://ask.learncbse.in/t/prove-that-2-is-not-a-rational-number-by-long-division-method/44944, picture only
3. https://www.homeschoolmath.net/teaching/sqr-algorithm-why-works.php#:~:text=Each%20time%20the%20algorithm%20is,value%20of%20the%20square%20root.
4. https://www.youtube.com/watch?v=aaOO8DUsnl4
5. https://xlinux.nist.gov/dads/HTML/squareRoot.html

AMIT CHAND is a Resource Person at Azim Premji Foundation. He has an M.Sc in mathematics from Burdwan University, Burdwan, West Bengal. He has been closely working with teachers on issues related to Mathematics and conducts workshops focusing on conceptual as well as pedagogical strategies in teaching mathematics. He has been doing mathematics with children for more than 6 years and is deeply interested in exploring the base of mathematical ideas. He may be contacted at amit.chand@azimpremjifoundation.org.

Complete Journey of a Minute Hand

RANJIT DESAI

We are familiar with wall clocks. We discuss here the movement of the minute hand of a wall clock that behaves differently from the usual kind of clock.

Let the minute hand of this different clock start to move from some position. We say that it has "completed the journey" when it returns to its starting position for the first time.

In a usual clock, the minute hand moves 1 minute/60 seconds. In our clock, we shall suppose that the minute hand jumps through m minutes/60 seconds, where m is a whole number between 1 and 60. We shall say that "the minute hand moves stepwise with step value m."

We discuss here the number of steps taken by the minute hand to complete the journey. We claim the following.

- If m is a divisor of 60, then the minute hand takes $60/m$ steps to complete its journey. This is so because $m \times \frac{60}{m} = 60$.
- If m is not a divisor of 60, then we compute the number $k = \frac{\text{LCM}(m,60)}{m}$. The number of steps taken by the minute hand to complete its journey will then be k.

Examples

- For $m = 7$, we get $k = \frac{\text{LCM}(7,60)}{7} = \frac{420}{7} = 60$; so 60 steps are required to complete the journey.
- For $m = 8$, we get $k = \frac{\text{LCM}(8,60)}{8} = \frac{120}{8} = 15$; so 15 steps are required to complete the journey.

Keywords: Clock, stepwise movement.

RANJIT DESAI started his career as a mathematics teacher in Bai Avabai High School, Valsad in 1960, and retired in 1998 as Incharge of P. G. Centre (Mathematics) from B. K. M. Science College, Valsad. In 1968, he started the Ganit Milan to connect students and teachers of schools and colleges with one another. He is actively associated with the Gujarat Ganit Mandal. He may be contacted at ranjitraimdesai@gmail.com

Deciphering the Median Formula

Part 1: From the Histogram

MATHEMATICS CO-DEVELOPMENT GROUP, Azim Premji Foundation

At the secondary level, data handling becomes statistics. There is a new graph – the ogive and two quite complicated formulas (i) $M = l + \frac{\frac{N}{2}-m}{f} \times c$ and (ii) $m = l + \frac{f_1-f_0}{2f_1-f_0-f_2} \times c$ for computing the median and the mode respectively from grouped data, i.e., data that is organized in class intervals. Textbooks usually do not explain how those formulas work even though the logic can be completely worked out within secondary mathematics content. In this series, we are going to decipher these formulas. We shall also investigate a question related to the mean. So, there are four parts as follows:

1. Median formula from the histogram
2. Median formula from the ogive, and why the ogives intersect at the median
3. Mode formula from the histogram
4. A 'Mean' question

For ungrouped quantitative data, median splits the entire data set in two parts – each with the same number of data points.

If the daily salaries of 7 employees in a company (in decreasing order) are ₹5000, ₹500, ₹450, ₹430, ₹400, ₹150 and ₹100, then the median is the 4^{th} salary ₹430 which splits the salaries in two groups {₹5000, ₹500, ₹450} and {₹400, ₹150, ₹100} – each with 3 salaries (or data points).

Similarly, if the salaries are ₹5000, ₹500, ₹450, ₹430, ₹410, ₹400, ₹150 and ₹100, then the median is the mean of the 4^{th} and the 5^{th} salaries i.e. ₹420 which splits the salaries in two groups {₹5000, ₹500, ₹450, ₹430} and {₹410, ₹400, ₹150, ₹100} – each with 4 data points.

Keywords: Data, Median, Histogram, Developing the formula

So, if there are an odd number $2n + 1$ data points, then we order them and pick the $(n + 1)^{th}$ data point as the median. If there are an even number $2n$ of data points, then we take the mean of the nth and the $(n + 1)^{th}$ data points as the median. For both cases i.e. odd and even number of data points, the first n data points have values lower than the median and the remaining n points have values higher than it. Note that, for the even case, any value in between the n^{th} and the $(n + 1)^{th}$ data points would have worked as median in terms of splitting the data set in two subsets with equal cardinality. The above examples illustrate these with $n = 3$ for odd and with $n = 4$ for even number of data points.

Marks	No. of students
20	6
25	20
28	24
30	28
33	15
38	4
42	2
43	1

Table 1

Marks	Cumulative frequency
≤ 20	6
≤ 25	$6 + 20 = 26$
≤ 28	$26 + 24 = 50$
≤ 30	$50 + 28 = 78$
≤ 33	$78 + 15 = 93$
≤ 38	$93 + 4 = 97$
≤ 42	$97 + 2 = 99$
≤ 43	$99 + 1 = 100$

Table 2

Let us consider the frequency table (Table 1) for marks (out of 50) obtained by 100 students. The marks of these 100 students can be put in decreasing order: 43, 42, 42, 38, 38, 38, 38, 33,… 25, 20, 20, 20, 20, 20, 20. So, it is possible to find the 50^{th} and the 51^{st} marks. We can find it more easily with a cumulative frequency table (Table 2). The 50^{th} mark = 28 and the 51^{st} mark = 30. So, the median is 1/2(28 + 30) = 29.

Marks		No. of students	Marks	Cumulative frequency
150-154	149.5-154.5	5	≤ 154.5	5
155-159	154.5-159.5	2	≤ 159.5	$5 + 2 = 7$
160-164	159.5-164.5	6	≤ 164.5	$7 + 6 = 13$
165-169	164.5-169.5	8	≤ 169.5	$13 + 8 = 21$
170-174	169.5-174.5	9	≤ 174.5	$21 + 9 = 30$
175-179	174.5-179.5	11	≤ 179.5	$30 + 11 = 41$
180-184	179.5-184.5	6	≤ 184.5	$41 + 6 = 47$
185-189	184.5-189.5	3	≤ 189.5	$47 + 3 = 50$

Table 3

Now consider another frequency table (Table 3) for marks (out of 100) obtained by 50 students. In this case, we have no way of finding the exact marks, the x_1, x_2, … x_{50}. So, we can't order them to find the median using the above method. So, we turn to the graphs. As mentioned above, in this article we are going to use the histogram. So, we make the histogram (Figure 1) for the grouped data in Table 3 (and make the class intervals continuous i.e. 149.5-154.5, 154.5-159.5, etc., to draw the graph).

Note that median halves the dataset. We will use something similar with histogram. The area of each rectangle = the frequency of the corresponding class × the class width. Since the width is uniform across classes, the heights of the rectangles are proportionate to the respective class frequencies. So, the total area of the histogram is proportional to the total frequency.

Therefore, median should be that value of x that cuts the histogram in two parts (blue and pink) with equal areas. In other words, M is the median if the line $x = M$ cuts the histogram in two parts each with half the area of the total histogram. This line cuts through the rectangle corresponding to the class where the cumulative frequency crosses half of total frequency.

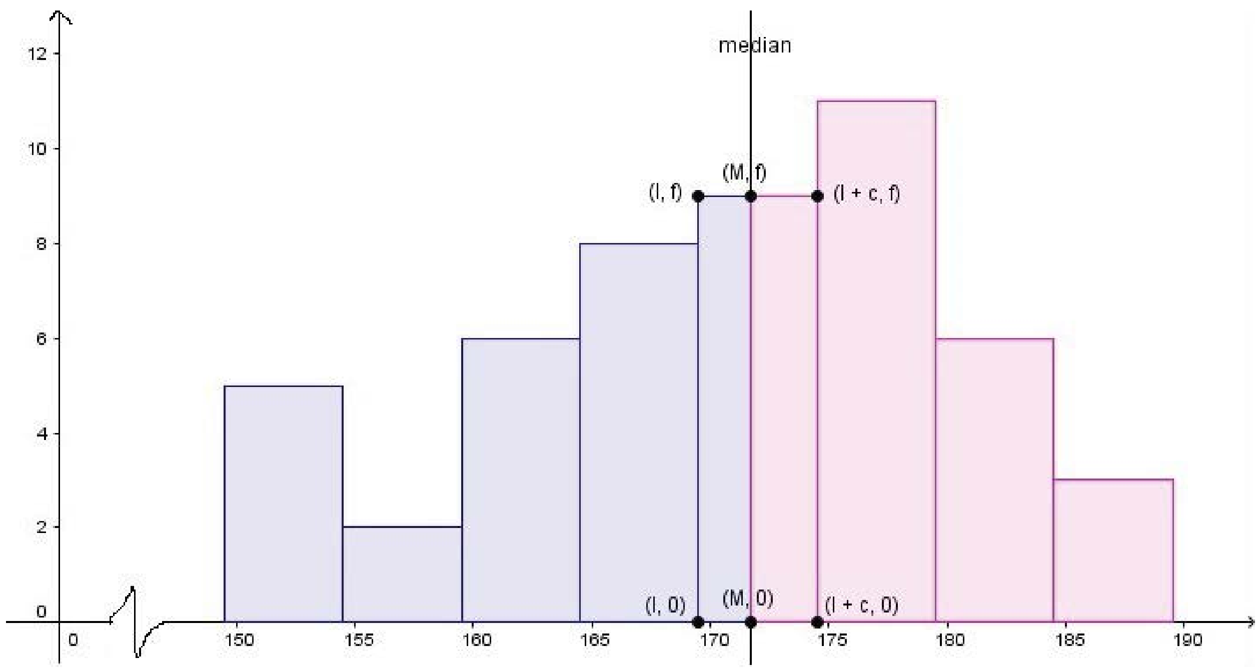


Figure 1

In this case, half of the total frequency is $50/2 = 25$. The cumulative frequency up to the class 169.5-174.5 is $21 < 25$ and after this class it becomes $30 > 25$. So, the median i.e. the 25th data value should be somewhere in the class 169.5-174.5. We call this class, 169.5-174.5, the **median class**. So, the cumulative frequency crosses 25 for 174.5 which is the upper limit of the median class.

The total area of the histogram is the sum of areas of all rectangles.

The area of each rectangle is the class-width × the corresponding frequency.

This histogram has uniform class-width 5.

So, the total area is $5 \times (5 + 2 + 6 + 8 + 9 + 11 + 6 + 3) = 5 \times 50$, i.e., class-width × total frequency.

So, the area of each part (blue or pink) is $1/2 \times 5 \times 50$.

Now the blue part consists of

(i) the full rectangles corresponding to the classes before the median class with area $5 \times (5 + 2 + 6 + 8) = 5 \times 21$

(ii) a part of the median class with area $(M - 169.5) \times 9$

Note that:

- 169.5 is the lower limit of the median class
- 21 is the cumulative frequency (less than) for 169.5
- 9 is the frequency of the median class.

So, the area of the blue part is

$$5 \times 21 + (M - 169.5) \times 9 = \frac{1}{2} \times 5 \times 50 \Rightarrow (M - 169.5) \times 9 = 5 \times (50/2 - 21)$$

$$\Rightarrow M - 169.5 = 5/9 \times (50/2 - 21) \Rightarrow M = 169.5 + (50/2 - 21) \times 5/9$$

Now, doesn't this look similar to the median formula mentioned at the beginning?

Let us generalize by algebraizing as follows:

Symbol	Meaning	In the example
N	Total frequency	50
c	(uniform) class-width	5
l	Lower limit of median class	169.5
f	Frequency of median class	9
m	(Less than) cumulative frequency for l	21

Table 4

- The total area of the histogram is Nc
- The area of the full blue rectangles is mc and that of the blue part of the median class is $(M - l)f$

Therefore, the total area of the blue part is

$$mc + (M - l)f = \frac{1}{2}Nc \qquad \Rightarrow (M - l)f = \frac{1}{2}Nc - mc = \left(\frac{N}{2} - m\right)c$$

$$\Rightarrow M - l = \frac{\left(\frac{N}{2} - m\right)c}{f} = \frac{\frac{N}{2} - m}{f} \times c \qquad \Rightarrow M = l + \frac{\frac{N}{2} - m}{f} \times c$$

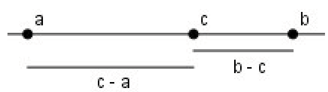
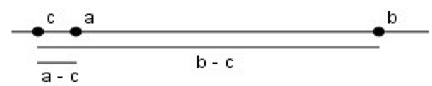
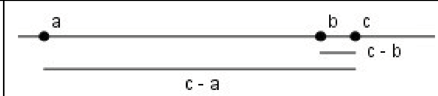
It may make sense to let different groups of students work with different histograms, collate their work in a table like Table 4 and then crystalize the algebraic form of the formula from them.

It is worth mentioning that median also minimizes mean deviation i.e. M = median minimizes $\frac{1}{n}\sum_{k=1}^{n} |x_k - M|$ where $x_1, x_2, x_3 \ldots x_n$ are the data points.

We will prove this in three steps [c represents a constant in the remaining part, not class-width]:

1. Mean deviation from c i.e. $\dfrac{|x_1 - c| + |x_2 - c| + \cdots + |x_n - c|}{n}$ is minimized when the sum of the deviations i.e. $S = |x_1 - c| + |x_2 - c| + \ldots + |x_n - c|$ is minimized since n is independent of c

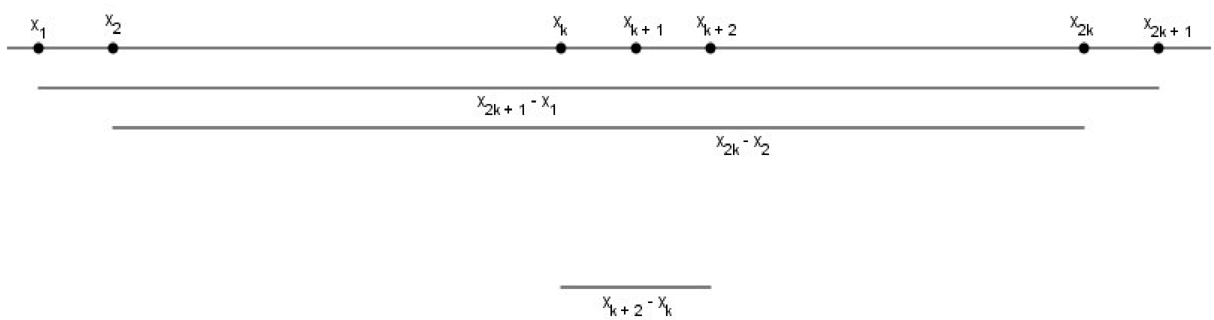
2. Show that if $a < b$ then $|a - c| + |b - c|$ is minimized when a $< c < b$
   There are three possibilities:

$a < c < b$	$c < a < b$	$a < b < c$
$\|a - c\| + \|b - c\|$ $= c - a + b - c$ $= b - a$	$\|a - c\| + \|b - c\|$ $= a - c + b - c = a + b - 2c$ $= b - a + 2a - 2c$ $= b - a + 2(a - c)$ $> b - a$ since $a - c > 0$	$\|a - c\| + \|b - c\|$ $= c - a + c - b = 2c - (a + b)$ $= 2c - 2b + b - a$ $= b - a + 2(c - b)$ $> b - a$ since $b < c$
		

Therefore $|a - c| + |b - c|$ is minimized when $a < c < b$

3. We now prove that the median minimizes the mean deviation

Let us first look at the situation for odd $n = 2k + 1$ with $x_1 < x_2 < \ldots < x_{k+1} < \ldots < x_n$ without loss of generality. Since $(n + 1)/2 = k + 1$, so x_{k+1} is the median.



We need to show that the sum of the deviations i.e. $S = |x_1 - c| + |x_2 - c| + \ldots + |x_n - c|$ is minimized when $c = x_{k+1}$

Now this sum can be regrouped in pairs with

- the 1^{st} pair corresponding to the extreme values x_1 and x_{2k+1}
- the 2^{nd} pair corresponding to the next two i.e. x_2 and x_{2k} etc. till
- the k^{th} pair corresponding to x_k and x_{k+2} and
- the single term corresponding to the median x_{k+1} i.e.

$$S = (|x_1 - c| + |x_{2k+1} - c|) + (|x_2 - c| + |x_{2k} - c|) + \ldots + (|x_k - c| + |x_{k+2} - c|) + |x_{k+1} - c|$$

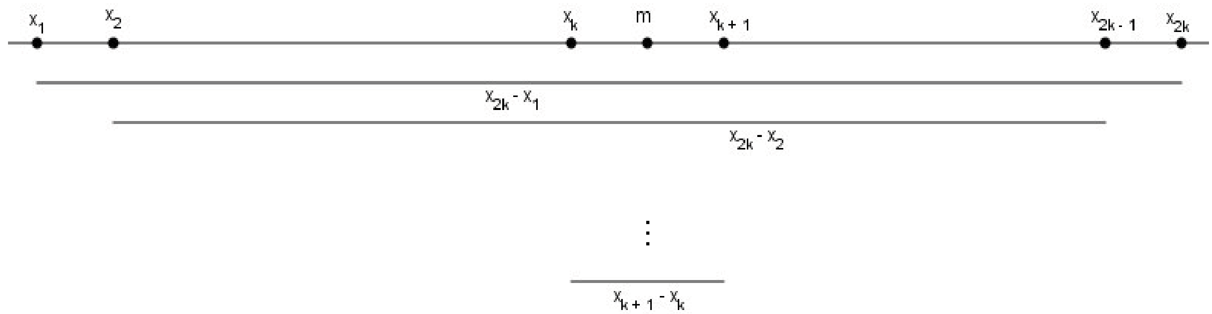
$$= S' + |x_{k+1} - c|$$

Now S' will be minimized if $x_k < c < x_{k+2}$ from 2. above

And $|x_{k+1} - c|$ will be minimized if $c = x_{k+1}$

Since $x_k < x_{k+1} < x_{k+2}$, $c = x_{k+1}$ will minimize both S' and $|x_{k+1} - c|$ and therefore S, and that minimal value will be $(x_{2k+1} - x_1) + (x_{2k} - x_2) + \ldots + (x_{k+2} - x_k) = (x_{2k+1} + \ldots + x_{k+2}) - (x_1 + \ldots + x_k)$ i.e. the difference between the sums of the upper and lower halves of the data-values

For even $n = 2k$, $S = (|x_1 - c| + |x_{2k} - c|) + (|x_2 - c| + |x_{2k-1} - c|) + \ldots + (|x_k - c| + |x_{k+1} - c|)$ which will be minimized if $x_k < c < x_{k+1}$



In this case, median $m = 1/2(x_k + x_{k+1})$ and obviously $x_k < m < x_{k+1}$ and therefore median minimizes S and that minimal value will be $(x_{2k} - x_1) + (x_{2k-1} - x_2) + \ldots + (x_{k+1} - x_k) = (x_{2k} + \ldots + x_{k+1}) - (x_1 + \ldots + x_k)$ or the difference between the sums of the upper and lower halves of the data-values

Note that for even n, median is not the ONLY number that minimizes the mean deviation, but for odd n, it is.

In the next part, we will look into the relations between the ogives and the median…

MATHEMATICS CO-DEVELOPMENT GROUP

Math Co-dev Group or more elaborately **Mathematics Co-development Group** is an internal initiative of Azim Premji Foundation where math resource persons across states put their heads together to prepare simple materials for teachers to develop their understanding on different content areas and how to transact the same in their classrooms. It is a collaborative learning space where resources are collected from multiple sources, critiqued and explored in detail. Math Co-dev Group can be reached through yashvendra@azimpremjifoundation.org.

MATH DATES

This WhatsApp forward appeared on February 12, 2021: a date that is sure to warm a mathematician's heart much more than one on Valentine's Day!

Today's date is both a palindrome and an ambigram! Which means you can read the date from left to right, from right to left and also upside down!

12022021

Dreaming On……..

1. *Which months can have ambigram dates?*
2. *When is the next such date?*
3. *When is the next ambigram date? [It may not have another type of symmetry]*
4. *What other ambigram and palindrome dates can you think of? How many were/can be in your lifetime?*

How To Prove It

SHAILESH SHIRALI

In Part I of this article, we remarked that there are essentially three components of a proof by induction:

Step 0: Framing the hypothesis or conjecture.

Step 1: Anchoring the induction, i.e., verifying the initial step.

Step 2: The bridge step, i.e., establishing the link between successive propositions of the induction hypothesis.

We dwelt at length on Step 0 (framing the hypothesis or conjecture), remarking that this step is generally completely ignored in the teaching of mathematics, thereby giving the impression that the formula to be proved literally comes out of nowhere. To illustrate this, let me quote an actual experience that I have often had as a mathematics teacher at the class 11 level: I ask the class to prove, using the principle of induction, that the sum of the squares of the first n natural numbers is equal to $n(n+1)(2n+1)/6$. Most of them are able to do so successfully. And then, a student comes after the class and asks, "Excuse me, we have proved this formula using the principle of induction; but where did we get this formula in the first place?" This innocent question captures the precise point that we are trying to make.

Having dwelt on this point in detail earlier, we now dwell on the remaining two steps by focusing on a few case studies. We shall study some examples that may not be so familiar to readers.

Abstract structure of a proof by induction

To start with, we describe in abstract the essential features of Steps 1 and 2 of a proof by induction.

Our intention is to establish that a certain given proposition $P(n)$ is true for all positive integers n. Note that $P(n)$ is a *proposition* for each positive integer n; it is either true or false.

Keywords: Proof by induction, hypothesis, conjecture, anchor, bridge step

We start (Step 1) by showing that $P(1)$ is true. This typically simply consists of verifying a numerical equality.

Next (Step 2) we show that for an arbitrary positive integer k, the truth of $P(k)$ implies the truth of $P(k+1)$. Expressed compactly: we show that the implication

$$P(k) \implies P(k+1) \tag{1}$$

is true for any arbitrary positive integer k.

Here, the emphasis on the word 'arbitrary' needs to be noted. Precisely because k is arbitrary, what the above establishes is the following infinite chain of implications:

$$P(1) \implies P(2) \implies P(3) \implies P(4) \implies \cdots. \tag{2}$$

The conclusion is now: $P(1)$ is true, therefore $P(2)$ is true, therefore $P(3)$ is true, therefore $P(4)$ is true, and so on, indefinitely. In short, $P(n)$ is true for every positive integer n. So we have proved what we set out to prove. In practice, we do not bother to write this final sentence. We simply write: "As (1) and (2) have been proved, it follows that $P(n)$ is true for every positive integer n."

A few familiar examples

Example 1 (Sums of squares of the natural numbers)**.** The sum of the squares of the first n natural numbers is equal to $n(n+1)(2n+1)/6$.

This is the result about which the student had registered a protest! But we have already described in the first part of this article (the July 2020 issue of AtRiA) how it is possible to hit upon this formula by playing with the sequence.

Note carefully the sequence of steps set out below. Also note the notation that we use.

We start by defining $P(n)$ to be the proposition (or assertion),

$$\textit{The sum of the squares of the first } n \textit{ natural numbers is } \frac{n(n+1)(2n+1)}{6}. \tag{3}$$

Step 1 (anchor): We check that the conjectured relationship or proposition is true for some initial value of n, typically $n = 1$. In this instance, it amounts to checking that the first squared number (i.e., 1^2) is equal to

$$\frac{1 \times 2 \times 3}{6}.$$

This is clearly true (both sides are equal to 1). So $P(1)$ is true.

Step 2 (inductive step): We proceed to show that for any arbitrary positive integer k, the truth of $P(k)$ implies the truth of $P(k+1)$. So we must verify that

$$P(k) \implies P(k+1),$$

for an arbitrary positive integer k. Now, $P(k)$ is the assertion that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6},$$

while $P(k+1)$ is the assertion that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Therefore, proving that $P(k) \implies P(k+1)$ is the same thing as proving that

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

This in turn is equivalent to proving that

$$\frac{(k+1)(k+2)(2k+3)}{6} - \frac{k(k+1)(2k+1)}{6} = (k+1)^2,$$

$$\text{i.e.,} \quad (k+2)(2k+3) - k(2k+1) = 6(k+1).$$

The verification of the last line takes an instant. It follows that the stated formula giving the sum of the squares of the first n positive integers is true.

Example 2 (A divisibility problem)**.** The quantity $9^n - 2^n$ is a multiple of 7 for every positive integer n.

As earlier, we start by defining $P(n)$ to be the proposition,

$$\textit{The quantity } 9^n - 2^n \textit{ is a multiple of } 7. \tag{4}$$

Step 1 (anchor): We check that the proposition is true for $n = 1$. In this instance, it amounts to checking that $9 - 2$ is a multiple of 7. This is true. So $P(1)$ is true.

Step 2 (inductive step): We must show that for any arbitrary positive integer k, the truth of $P(k)$ implies the truth of $P(k+1)$. Now, $P(k)$ is the assertion that

$$9^k - 2^k \textit{ is a multiple of } 7,$$

while $P(k+1)$ is the assertion that

$$9^{k+1} - 2^{k+1} \textit{ is a multiple of } 7.$$

This implication can be shown in several ways. Here is one such. Since $9^k - 2^k$ is a multiple of 7, we write

$$9^k - 2^k = 7m$$

for some integer m. We now have:

$$\begin{aligned} 9^{k+1} - 2^{k+1} &= 9 \cdot 9^k - 2 \cdot 2^k \\ &= 9 \cdot \left(2^k + 7m\right) - 2 \cdot 2^k \\ &= 7 \cdot 2^k + 63m = 7\left(2^k + 9m\right). \end{aligned}$$

The last line shows clearly that $9^{k+1} - 2^{k+1}$ is a multiple of 7. So we have shown that $P(k) \implies P(k+1)$. Since $P(1)$ is true, it follows that $9^n - 2^n$ is a multiple of 7 for all positive integers n.

A few not-so-familiar examples

Before continuing, we note that there are two obvious variations which may occur in the standard proof by induction.

- The conjecture may have to be proved not from $n = 1$ but from some subsequent point. For example, we may have a statement like this: "Such and such a statement is true for all integers $n \geq 3$." (Example: "For all integers $n \geq 3$, the sum of the angles of a n-sided convex polygon is equal to $(n-2)180°$.") In such cases, we must anchor the induction suitably, i.e., start by verifying the conjecture for the base value of the argument.

- The conjecture may have to be proved not for all positive integers n but for some suitable subset of the positive integers; for example, for all odd positive integers n; or for all positive integers n of the form 1 (mod 3); and so on. In such cases, the inductive step has to be modified suitably. The way this has to be done will depend on the specifics of the situation.

Some of the examples shown below exhibit these features.

Example 3 (Another divisibility problem)**.** The quantity $2^n + 3^n$ is a multiple of 5 for all odd positive integers n.

As earlier, we start by defining $P(n)$ to be the proposition,

$$\textit{The quantity } 2^n + 3^n \textit{ is a multiple of } 5. \tag{5}$$

So we have to prove that the propositions $P(1)$, $P(3)$, $P(5)$, $P(7)$, ... are all true.

Step 1 (anchor): We check that the proposition is true for $n = 1$. In this instance, it amounts to checking that $2 + 3$ is a multiple of 5. This is true. So $P(1)$ is true.

Step 2 (inductive step): We are required to prove the proposition for the *odd* integers. To move from each odd integer to the next one requires an addition of 2. So our task is the following. We must show that for any arbitrary positive integer k, the truth of $P(k)$ implies the truth of $P(k+2)$. Now, $P(k)$ is the assertion that

$$2^k + 3^k \textit{ is a multiple of } 5,$$

while $P(k+2)$ is the assertion that

$$2^{k+2} + 3^{k+2} \textit{ is a multiple of } 5.$$

This implication can be shown in several ways. Here is one such. Since $2^k + 3^k$ is a multiple of 5, we write

$$2^k + 3^k = 5m$$

for some integer m. We now have:

$$\begin{aligned} 2^{k+2} + 3^{k+2} &= 4 \cdot 2^k + 9 \cdot 3^k \\ &= 4 \cdot 2^k + 9 \cdot \left(5m - 2^k\right) \\ &= 45m - 5 \cdot 2^k = 5\left(9m - 2^k\right). \end{aligned} \tag{6}$$

The last line shows clearly that $2^{k+2} + 3^{k+2}$ is a multiple of 5. So we have shown that $P(k) \implies P(k+2)$. Since $P(1)$ is true, it follows that $2^n + 3^n$ is a multiple of 5 for all odd positive integers n.

Comment. A little tweak to the above analysis shows that we can get more from this line of thinking than had been asked for at the start. From the relation (6) obtained above, we see that:

$$2^{k+2} + 3^{k+2} = 9 \cdot \left(2^k + 3^k\right) - 5 \cdot 2^k. \tag{7}$$

Since $5 \cdot 2^k$ is a multiple of 5, and 9 is coprime to 5, relation (7) implies the following:

$$2^{k+2} + 3^{k+2} \text{ is a multiple of } 5 \iff 2^k + 3^k \text{ is a multiple of } 5. \tag{8}$$

Since $2^2 + 3^2 = 13$ is not a multiple of 5, the above relation (8) shows that the quantities

$$2^4 + 3^4, \quad 2^6 + 3^6, \quad 2^8 + 3^8, \quad 2^{10} + 3^{10}, \quad \ldots$$

are all non-multiples of 5.

Example 4 (Yet another divisibility problem)**.** The quantity $1^n + 2^n + 3^n + 4^n$ is a multiple of 5 for all positive integers n except the multiples of 4.

We start by defining $P(n)$ to be the proposition,

$$\textit{The quantity } 1^n + 2^n + 3^n + 4^n \textit{ is a multiple of } 5. \tag{9}$$

So we must prove that the propositions $P(1)$, $P(2)$, $P(3)$, $P(5)$, $P(6)$, ... are all true.

As the proposition to be proved is of a more complex nature, we can expect to have to apply the inductive approach in a more flexible manner.

Step 1 (anchor): In general, this step consists of a single verification. However, the situation is of an unusual nature here. So we shall examine whether the propositions $P(1)$, $P(2)$, $P(3)$, $P(4)$ are true. Here are the results:

n	$1^n + 2^n + 3^n + 4^n$	Divisible by 5?	Conclusion
1	10	Yes	$P(1)$ is true
2	30	Yes	$P(2)$ is true
3	100	Yes	$P(3)$ is true
4	354	No	$P(4)$ is **not** true

Step 2 (inductive step): We have just seen that $P(n)$ is true for $n = 1, 2, 3$, and false for $n = 4$. The wording of the proposition suggests that this pattern will repeat: $P(n)$ will be found to be true for $n = 5, 6, 7$, and false for $n = 8$. Observe that the numbers in the second group are 4 more than the numbers in the first group. This suggests the strategy we need to use. Instead of advancing from n to $n + 1$ (as we generally do), or from n to $n + 2$ (as we did in the previous example), why don't we advance from n to $n + 4$? That is, why don't we try to prove the following? —

$$P(n) \implies P(n + 4). \tag{10}$$

This is just the task that we now take up. We have:

$$\begin{aligned}
&\left(1^{n+4} + 2^{n+4} + 3^{n+4} + 4^{n+4}\right) - \left(1^n + 2^n + 3^n + 4^n\right) \\
&= \left(1 + 16 \cdot 2^n + 81 \cdot 3^n + 256 \cdot 4^n\right) - \left(1^n + 2^n + 3^n + 4^n\right) \\
&= 15 \cdot 2^n + 80 \cdot 3^n + 255 \cdot 4^n.
\end{aligned}$$

The above result shows that

$$\left(1^{n+4} + 2^{n+4} + 3^{n+4} + 4^{n+4}\right) - \left(1^n + 2^n + 3^n + 4^n\right) \tag{11}$$

is a multiple of 5. Hence:

$$1^{n+4} + 2^{n+4} + 3^{n+4} + 4^{n+4} \text{ is a multiple of } 5 \tag{12}$$

$$\iff 1^n + 2^n + 3^n + 4^n \text{ is a multiple of } 5. \tag{13}$$

The above relation shows that we have proved more than what was required! We had set out to prove that $P(n) \implies P(n + 4)$; instead we have proved:

$$P(n) \iff P(n + 4). \tag{14}$$

We already know that $P(1)$ is true. From the above relation, we deduce that all of the following are true as well:

$$P(5), \quad P(9), \quad P(13), \quad P(17), \quad P(21), \quad \ldots.$$

Similarly, as we already know that $P(2)$ and $P(3)$ are true, we deduce that all of the following are true as well:

$$P(6), \quad P(10), \quad P(14), \quad P(18), \quad P(22), \quad \ldots,$$
$$P(7), \quad P(11), \quad P(15), \quad P(19), \quad P(23), \quad \ldots.$$

And finally (and most importantly), as we already know that $P(4)$ is not true, we deduce that all of the following are not true as well:

$$P(8), \quad P(12), \quad P(16), \quad P(20), \quad P(24), \quad \ldots.$$

We have now proved proposition (9) in full.

In Part III of this article, we shall consider a few more non-standard examples.

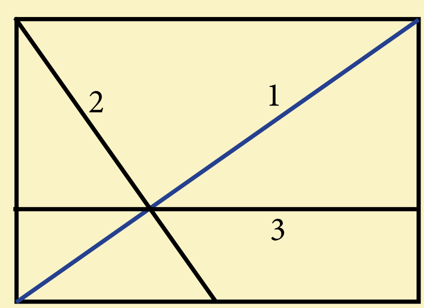
References

1. Wikipedia, "Mathematical induction" from https://en.wikipedia.org/wiki/Mathematical_induction

SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Take any rectangular sheet of paper. By making at most 3 creases, can you find the point of trisection of one of the shorter edges?

Here is a solution for an A4 sheet



Try to justify this claim!
For a general rectangle, there is a solution with 4 creases. Let's see if you can find it!

Winning Ways

JAMES METZ

The following practical application of exponents easily leads to a consideration of special cases when the exponent, the base, or both are zero.

Yunus, Laurie and Jim play a simple game of tossing a ball into a basket. The first person to put the ball in the basket wins the game. They decide to play a tournament with the champion the first one to win 3 games. What is the total number of different ways in which Laurie can win?

Since Laurie wins 3 games, Yunus and Jim each win either 0, 1 or 2 games (3 options for each of them), so the total is 3 times 3 or 9 ways in which Laurie can win. See Figure 1.

Yunus	Jim
0	0
0	1
0	2
1	0
1	1
1	2
2	0
2	1
2	2

Figure 1. The 9 ways for Laurie to be champion by winning 3 games

If the champion must win 4 games, then there are 4 times 4 or 16 ways for Laurie to win. In general, for a championship in which Laurie must win n games, there are n^2 ways for Laurie to win.

If Nadia joins the tournament, there will be n^3 ways, since each of the 3 losers can win 0, 1, 2, ... $n - 1$ games. In general, if there are p players and Laurie must win n games to be champion, there are

Keywords: Experiment, observation, documentation, exponents, indeterminate forms

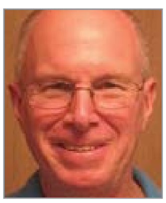
n^{p-1} ways. For example, if 3 people play and 8 victories are needed for the champion, there are $64 = 8^2$ ways for Laurie to win. (There are also 64 ways if 4 people play and 4 victories are needed for the champion.)

What if Laurie plays alone and decides she must win 3 games to be champion? Then there is only one way that she can win. Also, by our formula, there are 3^0 ways. So we have that $3^0 = 1$, and we have found a nice practical way to show why it is a good idea to define $n^0 = 1$ for $n \neq 0$.

It makes no sense for Laurie to play alone and be a champion if the game is not defined by a fixed number of wins, so it makes sense that 0^0 is indeterminate.

There is no way for 2 or more people to have a tournament with 0 victories required for a champion, so $0^{p-1} = 0$, that is $0^k = 0$ for k = 1, 2, 3, ...

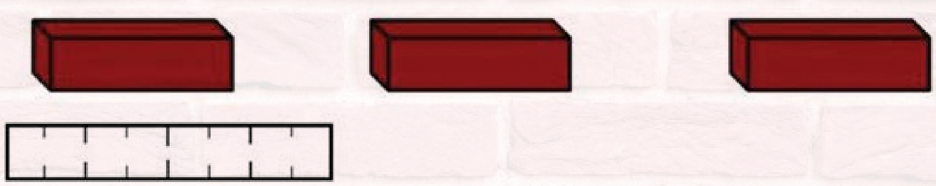
Putting exponents in a practical context may help make sense of special cases.



JAMES METZ is a retired mathematics instructor. He volunteers for Teachers Across Borders, Southern Africa for one month each year and enjoys doing mathematics. He may be contacted at metz@hawaii.edu

Corner to Corner

You have several bricks and a ruler and a large enough sheet of paper.



You want to measure the distance between opposite corners of a single brick without using any formulas. How can you do it?

Folk Method Analysis
(Sixth Method – a local device)

Mahit Warhadpande
http://www.jigyasujuggler.com

(Submission from a reader with reference to the article: Mohammad Umar, "The Height of a Tree", AtRiA November 2020, pp. 33-34)

The amount by which different people can bend their bodies forward at the waist is different. Those who are highly flexible may be able to put their heads through their legs and look almost straight up behind themselves as in Figure 1. Those who are less flexible though would probably end up with a position as in Figure 2.


Figure 1


Figure 2

Further differences in posture could be caused by spreading the legs apart by different amounts or if one inadvertently leans a little forward or back (i.e., the legs are not in a perfectly vertical plane w.r.t. the ground). Thus, it would seem that different people attempting this measurement (or even the same person repeating the experiment) could get different results.

Keywords: folk mathematics, analysis, reasoning, similarity

Assumptions

We shall assume that the 'average' person will assume a body posture similar to that of Figure 2 and the same person will be able to replicate the same posture every time. Then, the following parameters (see Figure 3) will remain constant for a particular person:

- $\angle VKE = \angle VFX = 90°$
- E = eye level above the ground in this position
- V = inverted V-tip level above the ground
- EK = horizontal distance between legs and eyes
- By implication, $\angle VEK$ is constant.

It is then claimed that the height OY of an object will be equal to the distance OF.

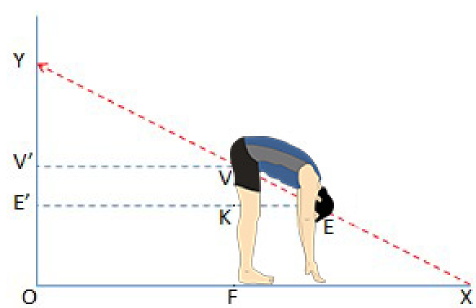


Figure 3

Analysis

Let us now analyse the mathematical implications of this method under the assumptions above. Figure 4 shows objects of 2 different heights ($OY_1$and OY_2) and two different viewer positions (F_1 and F_2) superimposed on each other. Now, if $OY_1 = OF_1$ and $OY_2 = OF_2$, then $\angle Y_1F_1O$ and $\angle Y_2F_2O$ are each 45°. Then $\angle V_1F_1Y_1$ and $V_2F_2Y_2$ must also be 45°.

With a constant posture, $V_1F_1 = V_2F_2$ (as per our assumption). Also, $\angle V_2Y_2F_2 < \angle V_1Y_1\ F_1$ (this can be proved using the sine rule and the fact that $Y_2V_2 > Y_1V_1$). This means that $\angle Y_2V_2F_2 > \angle Y_1V_1F_1$. Hence, the supplementary angles $E_1V_1F_1$ and $E_2V_2F_2$ are different (the former is larger). *However, these two angles in actuality must be the same, as they are completely determined by the posture, which we have assumed to be constant.* Thus, in general, we will **not** get the correct result with our assumption.

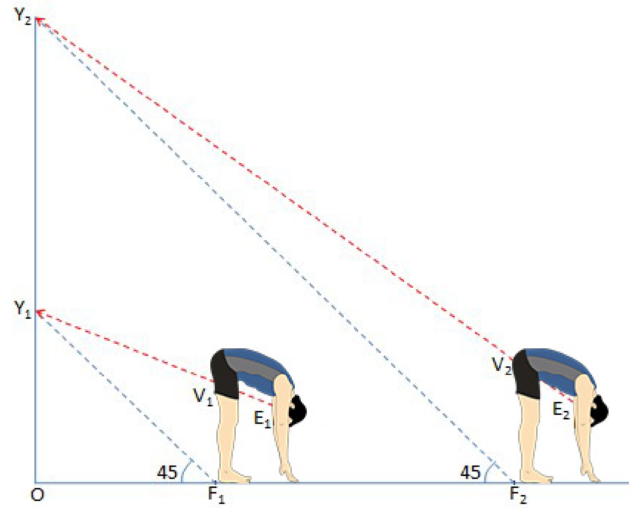


Figure 4

Refining our assumption

If the height to be measured (and therefore OF as well) is much larger than the dimensions of our body, then $\angle YFO$ will almost equal $\angle YEE'(=\angle VEK)$, or equivalently, $\angle VYF$ will be very small (see Figure 5). If we also have $\angle VEK = 45$, then OF (which will be nearly equal to OX if the dimensions of the body are negligible compared to OF) will give a close estimate of the object height OY.

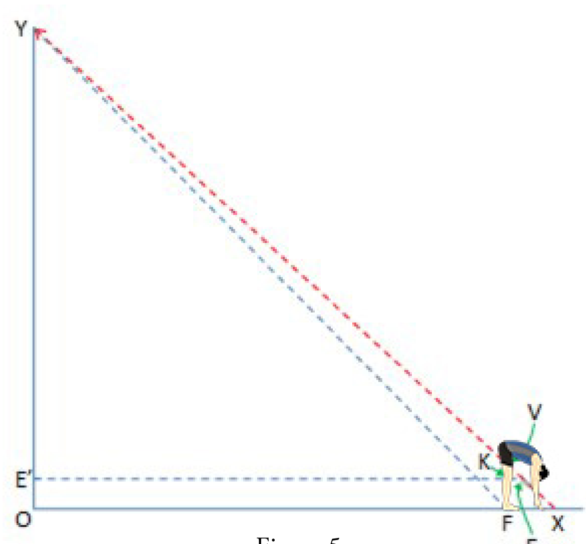


Figure 5

So, one way to get the 'folk method' (i.e., Method 6) to work is to refine our assumptions as follows:

1. One can replicate a body posture such that $\angle VEK \approx 45°$; i.e., $VK \approx EK$.
2. The height to be measured is much larger than the dimensions of the human body.

These two assumptions make Method 6 mathematically equivalent to Method 3 described in the article. In particular, under assumption 2, Method 3 will also work well even without accounting for the eye-level above ground since it will be negligible.

Alternatively, instead of the second assumption, we could claim $OX = OY$ (the object height), where point X is obtained by extending line VE to ground level. Assumption 1 and the claim that $OX = OY$ then make Method 6 mathematically equivalent to Method 5.

Finally, we can even relax the requirement for replicating the same posture with $\angle VEK \approx 45°$ for every measurement by noting that triangles VEK and YXO are similar, and VK, EK and OX are measurable. Then we can determine the object height OY by using the similarity relation $OY : OX = VK : EK$. The same applies to Methods 3 and 5 as well.

Example calculations

Let's assume that the height measured through Method 6 was 6 m (it actually gave 6.24 m, but we round it off for convenience) and that this was indeed the correct height of the tree. The relevant body dimensions (EK, KF in Figure 6) could realistically be assumed to be roughly 0.5 m. Then, in Figure 6, if $OY = OF = 6\text{m}$, we'll get $EE' = 6.5\text{m}$ and $YE' = 5.5\text{m}$. This means $\angle YEE' = \angle VEK = \tan^{-1}\frac{5.5}{6.5} \approx 40°$ which is a different posture from our assumption of $\angle VEK = 45°$. One possibility of ending up with such a posture when $EK = 0.5\text{m}$ is that $VK = EK \cdot \tan 40° = 0.5 \cdot \tan 40° \approx 0.42\text{m}$. Then for such a person to achieve $\angle VEK = 45°$, the posture would have to be adjusted (e.g., by bending the back and neck differently) such that $EK = VK = 0.42\text{m}$. The table below compares the results we get for various tree height estimates with these postures for $\angle VEK = 40°$ and $45°$. For clarity, a sample calculation has been worked out in the "Calculation Details" section at the end.

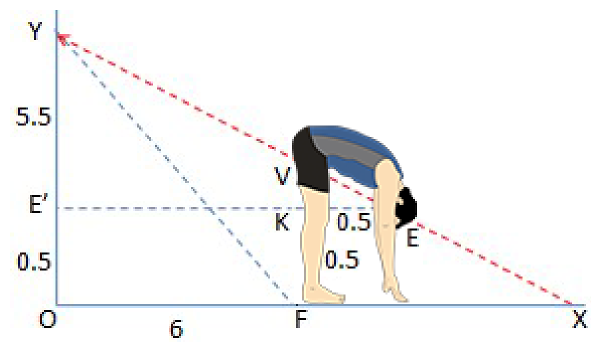


Figure 6

As expected, in both cases, $\angle YFO$ approaches $\angle VEK$ as the object height increases. Though the estimate for a tree of 6 m height turns out to be better when $\angle VEK = 40°$, using this posture means that the error magnitude keeps increasing with object height while relative error approaches the value $\frac{\tan 45}{\tan 40} - 1 \approx 19\%$. On the other hand, when $\angle VEK = 45°$, the absolute error remains constant (and in fact can be eliminated if we use $OY = OX$) while the relative error keeps reducing as we move further for higher objects. Thus, having $\angle VEK = 45°$ would be the more desirable posture for making height measurements with Method 6. As indicated while stating assumption 1, $\angle VEK = 45°$ can be achieved in practice by bending in such a way that we get $VK = EK$.

Calculation details (true object height $OY = 60$ m, $\angle VEK = 40°$)

In Figure 6, if we use the posture with $\angle VEK = 40°$, then as calculated above, we'll have $VK = 0.42\text{m}$. Note that $\angle VXF = \angle VEK$. Since $VF = 0.5 + 0.42 = 0.92$, we get in triangle VXF, $FX = \frac{VF}{\tan\angle VXF} = \frac{0.92}{\tan 40°} \approx 1.1\text{m}$. Also, in triangle YXO with $OY = 60\text{m}$, we get $OX = \frac{OY}{\tan\angle YXO} = \frac{60}{\tan 40°} \approx 71.5\text{m}$. Then, $OF = OX - FX = 71.5 - 1.1 = 70.4\text{m}$ will be the estimated object height. Finally, in triangle YFO, $\angle YFO = \tan^{-1}\frac{YO}{FO} = \tan^{-1}\frac{60}{70.4} \approx 40.44°$.

Other results listed in the table are calculated similarly. In particular, for the case when $\angle VEK = 45°$, we have assumed that the posture is such that $EK = 0.42 = VK$.

True Object Height (m)	$\angle VEK = 40°$			$\angle VEK = 45°$		
	Height Estimate (m)	Error (%)	$\angle YFO$	Height Estimate (m)	Error (%)	$\angle YFO$
6	6.05	+0.8	44.76	5.08	-15.3	49.75
60	70.4	+17.3	40.44	59.08	-1.5	45.44
100	118	+18	40.28	99.08	-0.9	45.26

WhatsApp forwards are a great opportunity to test your critical thinking. Here's one for the sceptics:

Is this what the theorem of Pythagoras really says? What is the mathematics behind this shortcut?

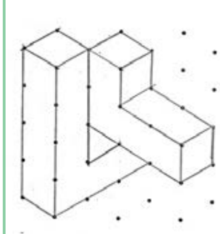
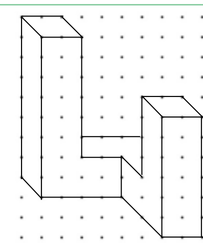
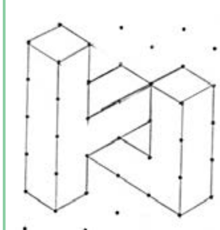
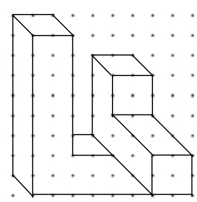
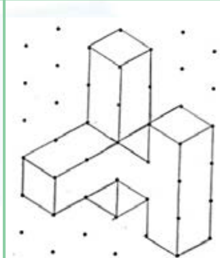
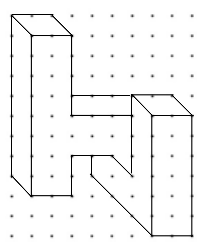
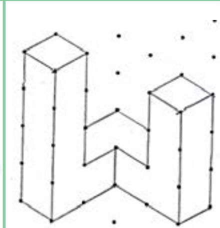
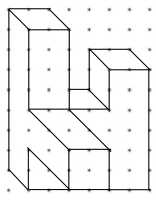
TearOut

Sketches and Views:

Mapping 3D to 2D and vice versa

In this 5th TearOut, we will use the isometric and the rectangular dot sheets to visualize various solid shapes. As before, pages 1 and 2 are a worksheet for students while pages 3 and 4 give guidelines to the facilitator. Since we will be exploring solids, it is a good idea to have some interlocking cubes[1] handy to make some of those solids.

The first part involves matching the isometric sketches with the oblique ones and then drawing the front, top and side views of each solid. The second part does the reverse. You start with the views and make the solids and then draw the sketches for each one. You can refer to NCERT mathematics textbooks for Class 7, chapter 15[2], for isometric and oblique sketches. Isometric sketches are made on isometric dot sheets while the oblique sketches are made on rectangular dot sheets.

	Isometric sketches	Oblique sketches
I		
II		
III		
IV		

Part A

1. Isometric and Oblique Sketches
    a. Match the isometric sketches with the oblique ones
    b. Mention the side lengths in the oblique sketches
    c. Draw the top, front and side view of each solid – the front is according to the oblique sketch
2. (Optional) Make a solid with interlocking cubes
    a. Make the isometric sketch
    b. Make the oblique sketch using the isometric sketch without looking at the solid
3. (Optional) Make another solid with interlocking cubes

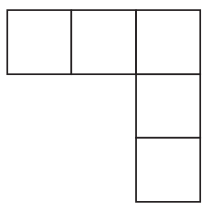
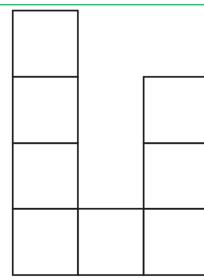
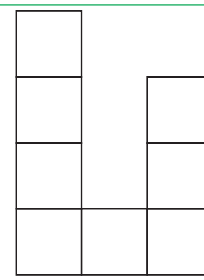
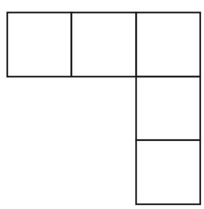
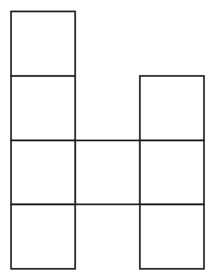
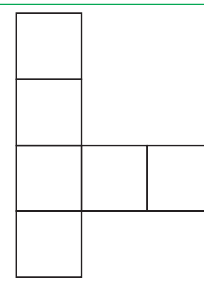
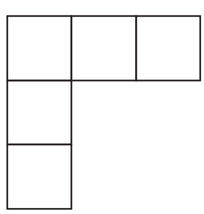
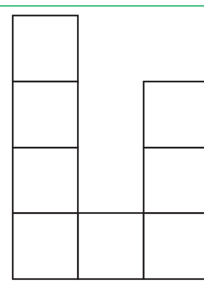
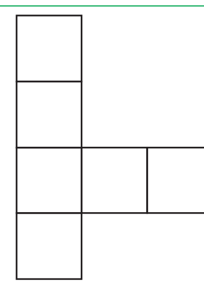
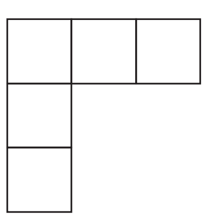
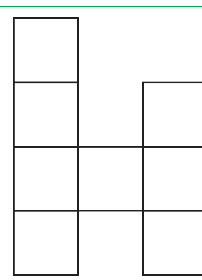
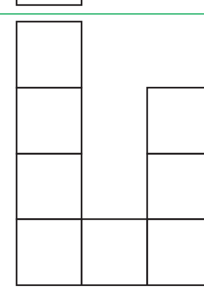
2 NCERT textbooks: https://ncert.nic.in/textbook.php

a. Make the oblique sketch
b. Make the isometric sketch using the oblique sketch without looking at the solid

Part B

4. Views and Sketches

a. Make each of the following solids
b. Draw the isometric sketch of each
c. Draw the oblique sketch of each mentioning the side lengths

	Top view	Front view	Side view
I			
II			
III			
IV			

5. (Optional) Views to solids

a. Take three 3 × 3 spaces on a rectangular dot sheet (i.e. each space is a square of side 3 units) and shade at least 5 unit squares in each. Consider these shaded parts as the top, front and side views respectively.
b. Using these views, make the solid with interlocking cubes.

This TearOut can be considered a continuation of the last one since the focus is on developing spatial sense and visualising solid shapes. This worksheet combines oblique sketches of solids. These sketches maintain some of the side proportions and some of the angles but distorts the rest. In addition, front, top and side views of the solids are also explored. While top view is clear from both isometric and obliques sketches, the latter is better at defining the front view – the part without any distortion, where rectangular faces are preserved. Side and front views are interchangeable for isometric sketches.

(published in the Nov 2019 https://azimpremjiuniversity.edu.in/SitePages/resources-ara-vol-8-no-5-november-2019-isometric-sketches-and-more.aspx)

Part A

This part focuses on the sketches. The first question uses the sketches as the starting point while the remaining two use the solids to start off.

1. Isometric and Oblique Sketches:
   a. This is a simple matching task to assess understanding of the sketches.
   b. This task is to deepen the understanding of oblique sketches by focussing on the side-lengths. The horizontal and vertical lines are stretched by a factor of 2. So, a vertical or horizontal line that is 6 units long actually represents a length of 3 units for the solid. All oblique lines are at 45° angle with horizontal and vertical lines. These lines are stretched by a factor of √2. So, the smallest possible oblique line segment represents unit length for the solid.

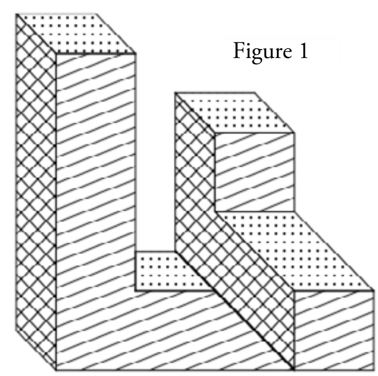


Figure 1

   c. The last task requires focussing on different views. It may require imagining and rotating the solid. It can help to imagine a lamp above the solid and colouring/shading the parts that will be lit. These will form the top view. Similarly, the non-distorted parts enclosed by horizontal and vertical lines in the oblique sketch can be coloured/shaded differently. These form the front view. The remaining sides generate the side view. It may help to colour/shade them in another way. For example, in Figure 1,
      - The dotted parts correspond to the top view (Figure 2)
      - The hatched parts are without distortion and they correspond to the front view (Figure 3)
      - The remaining cross-hatched parts correspond to the side view (Figure 4)

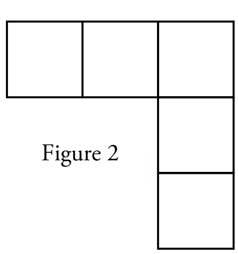


Figure 2

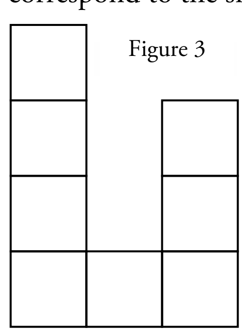


Figure 3

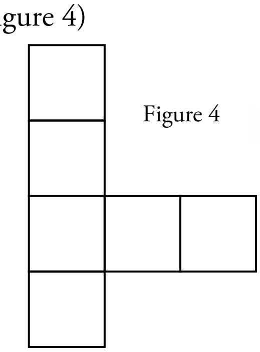


Figure 4

Similar colouring/shading can be done with the isometric sketch also.

It is easier to figure out the views from the solid. But it is a good idea to ask children to figure it out just from the sketch or sketches. This will help them visualize the solid being considered and develop stronger spatial sense.

The remaining two problems can be done as group activities with minimum three children sitting in a circle or in groups of three each. Each child can have, say, 10 interlocking cubes. They can create a solid using these cubes (not necessarily all) and draw the isometric sketch of their solid. Then they pass this sketch (and not the solid) to the child on their right. Now each child has just the isometric sketch of a solid. They draw the oblique sketch of the solid based on its isometric sketch. Then both sketches are passed to next child on the right. Now, each child makes the solid with the help of the sketches. Finally, this solid is passed again to the right and it goes to the child who made the original solid. The two solids are compared to see if the task was completed successfully. The sequence of isometric sketch followed by oblique can be changed. Also, children can be asked to make the solid based on only one of the sketches.

We encourage the teacher to collect the sketches generated by the above activity since they can be used to create similar worksheets in future.

Part B

4. Views and Sketches
    a. The first part is the most challenging where the solid has to be made based on the views. Each of the solids can be made with 10-12 cubes only.
    b. Once the solids are made, the isometric sketches will follow. It is important to note that the top view should be considered for the correct orientation of the isometric sketch. There should be more than one correct possibility.
    c. On the other hand the top and the front view fix the oblique sketch to a great extent. The only possible variation remains in choosing the oblique lines – they can be on the right or on the left of the non-distorted faces of the solid.

5. This last question allows children to try out a range of possibilities. Essentially, they create the views. Incidentally, these views will be independent of each other. Then, the solid is made based on them. Note that it is possible that some combination of views may not generate any solid. In such cases, one of the views can be modified so that a solid can be made. In some cases, a solid can be formed in theory e.g. if each view consists of 5 alternate squares including the corners and the centre. The solid in question is a collection of 9 unit cubes filling a space of a $3 \times 3 \times 3$ cube with 8 cubes at the 8 corners and the remaining one at the centre. But this solid can't be made with interlocking cubes. So, this activity can provide the opportunity to explore various possibilities.

    The restriction of making the views on 3×3 squares can be relaxed. Same goes for how many squares should be shaded to generate each view.

    We encourage the teacher to collect the sets of top, front and side views that do generate solids which can be made with interlocking cubes. These can be useful to create similar worksheets and do various similar activities.

Understanding solids is challenging since textbooks, notebooks, board and screens are all 2D and therefore can't provide the full flavour of 3D. The sketches and views are ways to map solids within 2D. This is crucial for designing and manufacturing 3D objects. These activities can provide some opportunity to engage with various solids, map them to 2D as well as create them based on the 2D representations – the sketches and the views. The optional tasks allow a wide range of possibilities to create solids and engage with them.

Exploring the Nine-point Circle: Conjecture making and proof using Dynamic Geometry Software

JONAKI GHOSH

Digital technologies provide many opportunities for learning mathematics through exploration and visualization. Dynamic Geometry Software (DGS) are a special class of digital tools that have made a significant impact in mathematics education across the world and have led to many research studies. In the article [1] in the July 2020 issue of At Right Angles, we demonstrated how an open-source DGS like GeoGebra functions as a conjecture making tool and can be used by students to explore mathematical concepts. In particular, the article described how grade VI students used GeoGebra to conjecture the angle sum property of a triangle.

In this article we shall describe how grade IX students visualised the Euler's circle (more popularly known as the nine-point circle) of a triangle, using GeoGebra. The journey undertaken by them as they constructed the figure, made conjectures and finally arrived at the proof will be discussed.

Background of the Study

The 34 students, who participated in the study had no prior experience in using GeoGebra or any other dynamic geometry software. They came from two schools in Mumbai and had volunteered for the study. In their regular mathematics classes in school, they had studied the properties of triangles, quadrilaterals and circles. They were also familiar with the proofs of certain theorems such as the midpoint theorem of a triangle and others related to circles and quadrilaterals. The exploration was conducted over two 2- hour sessions in a computer lab. The students were divided into pairs and each pair of students was assigned to a computer, which had GeoGebra. Before the

Keywords: Dynamic geometry software, exploration, visualization, conjecture, proof, triangles, circles.

sessions, students were informed (by their teacher) that they would explore a geometrical problem using a digital tool. The first 30 minutes (of the first session) was used by the teacher to familiarize the students with the basic construction tools of the GeoGebra toolbar, namely the **Move tool, Point Tool, Line tool, Polygon tool** and the **Circle tool**. It was interesting to observe how students enthusiastically explored the various features within each tool, on their own. Indeed, they asked for extra time to "play" with the tools.

After the students had gained some initial familiarity with the construction tools, the teacher posed the following tasks. The tasks were given to them in a worksheet in which they were required to record their observations and explain their reasoning.

Task 1: Draw a triangle ABC on your GeoGebra screen and do the following.

(i) Mark the midpoints D, E and F of the sides BC, CA and AB respectively.

(ii) Construct the three altitudes of the triangle and label the orthocenter as H.

(iii) Mark the feet of these altitudes. Let the foot of the altitude from A to BC be I, from B to CA be J and from C to AB be K.

(iv) Draw the segments connecting the orthocenter H to the vertices of the triangle A, B, C and mark their midpoints. Label the midpoints of HA, HB and HC as L, M and N respectively.

Task 2: Drag any vertex of the triangle ABC (you may make it an acute, obtuse or a right-angled triangle). What do you observe about the points D, E, F, I, J, K, L, M and N? Can you make a conjecture regarding these nine points? Can you prove your conjecture?

The GeoGebra Construction

The following steps will enable the reader to go through the construction process in GeoGebra (as given in Task 1).

Step 1: Use the **Polygon tool** to draw a triangle ABC in the **Graphics view**. Make sure that the **Algebra view** is on. The latter will highlight all elements of the geometrical objects constructed in the **Graphics view**.

Step 2: Using the **Midpoint or Centre tool** (available within the **point tool**), click on BC, CA and AB. The midpoints will automatically be marked D, E and F respectively. See Figure 1.

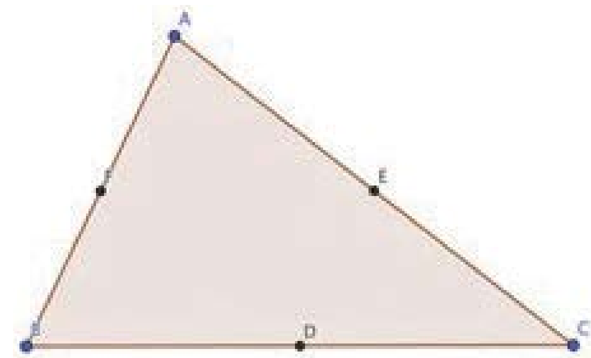


Figure 1. Triangle ABC and the midpoints of the sides.

Step 3: Now to construct the altitudes, select the **perpendicular line tool**. Click on A first, and then anywhere on the line BC. This will produce the altitude from A to BC. Similarly, draw the other two altitudes of the triangle.

Step 4: Select the **Intersect tool** (available within the **Point tool**) and mark the feet of the altitudes. (Clicking on the point of intersection produces a point, which can be renamed with a right-click.) Also, use the **Intersect tool** to mark the point of concurrency of the three altitudes as H. This is the orthocenter of triangle ABC. See Figure 2.

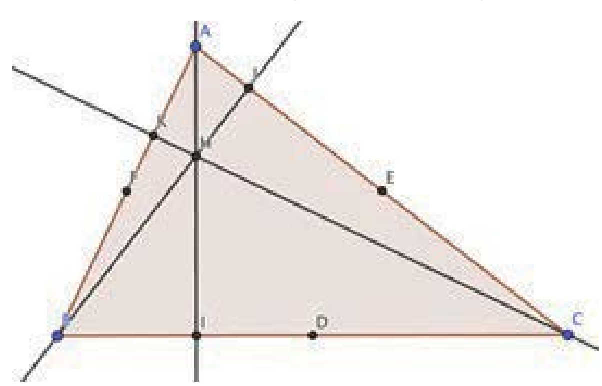


Figure 2: The Perpendicular line option is used to construct the three altitudes.

Step 5: All the elements of the constructed figure (points, lines and polygon) will appear in the **Algebra view**. Click on the blue dots alongside the lines (altitudes) to hide them.

Step 6: Using the **Segment tool** draw the segments HA, HB and HC. Mark their midpoints as L, M and N respectively using the **midpoint tool.** See Figure 3.

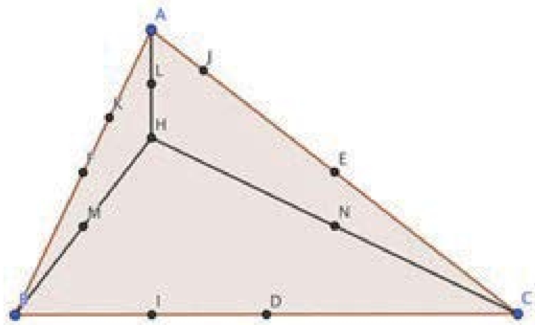


Figure 3. The segments HA, HB and HC and their midpoints L, M and N.

Step 7: Drag vertex A (or any other vertex of the triangle) and observe what happens to the nine points D, E, F, I, J, K, L, M and N.

Students' Explorations

After completing task 1, the students dragged the vertices of the triangle while at the same time observing the nine points as the triangle changed from being acute-angled to right-angled and then obtuse-angled. Dragging led many students to make the conjecture that the nine points lie on a circle. To many, this was more obvious when the triangle ABC was acute-angled. One student remarked, "The points appear to be swimming around in a circle. It looks like a circle at least when the triangle has acute angles." Some students were eager to verify their conjecture and enquired "but can we check if these points do lie on a circle?"

The teacher encouraged them to search the menu of the circle tool, which has a '**circle through 3 points**' option. This produces a circle passing through any three selected points. The option led to much excitement as the students quickly used this tool to select any three points (among D, E, F, I, J, K, L, M and N) to obtain the circle passing through them. Figure 4 shows the nine-point circle when triangle ABC is acute-angled. After further exploration another student observed, "Whichever three points I choose for a given triangle ABC, I always get the same circle. So this must be the nine-point circle we are looking for!" It was a high point for the teacher who realized that by now students were already engaged in reasoning and argumentation.

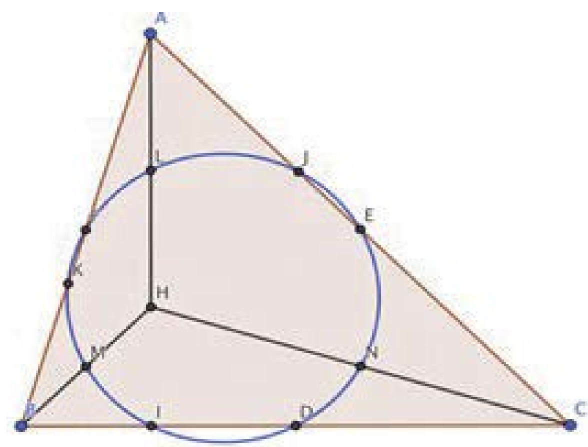


Figure 4. The nine-point circle of the triangle ABC, drawn using the Circle tool.

By dragging vertex A, the triangle ABC can be made right-angled at B as shown in Figure 5. In this case, the feet of the altitudes, I and K, the point M and the orthocenter H, all coincide with the vertex B. During the exploration, some students commented on how "some points have disappeared and only five points are visible." Others reinforced this with the argument "some points come on top of others (meaning that they

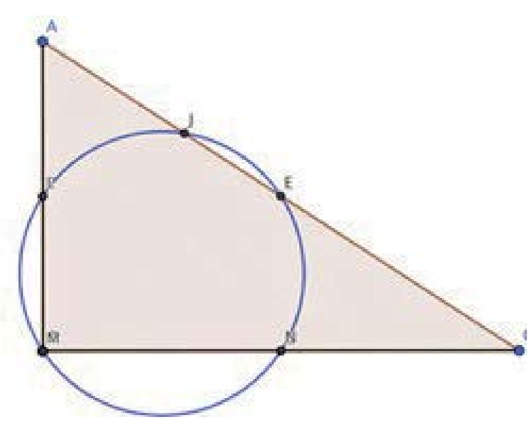


Figure 5. The nine-point circle when ABC is a right-angled triangle.

coincide).” Figure 5 shows the circle passing through the five points M, L, J, E and N. This may be considered as a degenerate case of the nine-point circle. It led to a discussion among the students as to which points coincide and why. For example, with some scaffolding by the teacher, the students concluded that the legs of a right-angled triangle are also its altitudes and hence the orthocenter (H) coincides with the vertex B.

When the triangle becomes obtuse-angled, the segments HA, HB and HC and the orthocenter H lie outside the triangle, but the nine-point circle remains. (All nine points are now visible as shown in Figure 6.)

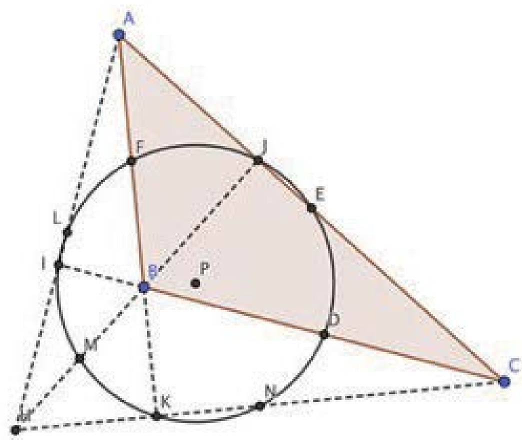


Figure 6. The nine-point circle when ABC is an obtuse-angled triangle.

Dragging as a tool leading to the conjecture

In GeoGebra and other DGS, dragging as a tool lends many opportunities to make and verify conjectures. In a dragging episode, the user observes invariant properties of the object and simultaneously experiences the varying aspects. Researchers have corroborated this unique feature of a DGS.

According to Leung [3]

….. one of DGS power is to equip us with the ability to retain (keep fixed) a background geometrical configuration while we can selectively bring to the fore (via dragging) those parts of the whole configuration that interested us in a mathematical thinking episode.

In the nine-point circle exploration, dragging a vertex of the triangle led to variations in the side lengths, angle measures and positions of the nine points. However, the existence of the circle passing through the nine points remained invariant. This contrasting experience (of the varying and the invariant) was instrumental in enabling the students to make the conjecture. While the students delighted in seeing the circle on their GeoGebra screen, the teacher asked: “how should we prove this conjecture?” One student pointed out that the nine-point circle is “always present for every kind of triangle” which is enough to substantiate the claim, while another felt “this is not a proof.” Finally, the students agreed that a formal proof of the fact that all the nine points were on a circle, was required.

Towards proof

Since the students were in grade IX, they were familiar with properties of rectangles, parallelograms and circles. They had also studied the midpoint theorem in their regular classes. Fortunately, this knowledge was enough to enable them to explore the proof of the existence of the nine-point circle. In this part of the session, the teacher had to play a more active role of facilitation and scaffolding.

The first step was to prove that the quadrilateral FENM (marked in green in Figure 7) forms a rectangle. Note that in triangle ABC, F and E are the midpoints of AB and AC respectively. Hence by midpoint theorem, FE is parallel to BC and equal to half of it. Similarly, in triangle HBC, M and N are the midpoints of HB and HC. Thus MN is parallel to BC and equal to half of it. With this line of argument, the students concluded that FE is equal and parallel to MN and hence FENM is a parallelogram. “But is it a special kind of a parallelogram?” the teacher asked. When one student remarked “it looks like a rectangle” another responded, “but we need to prove it.” Once again it was a high point for the teacher. The students were actively engaged in the process of proving their arguments. At the same time, they were also recording their observations and reasoning on their worksheets.

In triangle ABH, FM is parallel to AH (by midpoint theorem). Also since AH is perpendicular to BC (it is an altitude of triangle ABC), it implies that FM is perpendicular to MN. By similar arguments, students were able to show that EN is perpendicular to MN. Thus FENM is a rectangle. Since a rectangle is a cyclic quadrilateral, it was concluded that the points F, E, M and N lie on a circle with diameter FN (this is so because angle FEN is a right angle, as is angle FMN). See Figure 7.

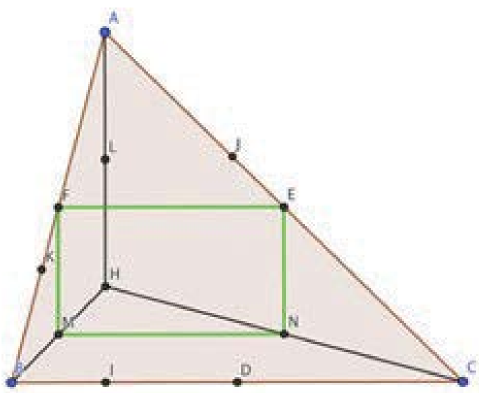


Figure 7. The quadrilateral FENM is a rectangle with diagonal FN.

Using similar arguments students proved that FLND (marked in blue in Figure 8) is also a rectangle. The circle passing through the points F, L, N, D also has FN as a diameter. As the two circles share a diameter, they are identical. Thus the points, D, E, F, L, N and M lie on the same circle. Students also observed that FN, a diameter of the circle, is also the common diagonal of the rectangles FENM and FLND. FN is marked as a

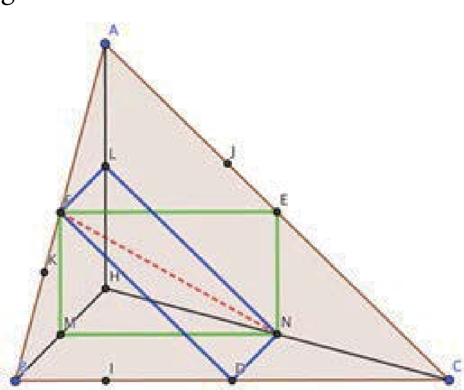


Figure 8. The quadrilateral FLND is a rectangle with diagonal FN.

red dotted line in Figure 9. Also, the midpoint of FN is the centre of the circle as shown in Figure 9.

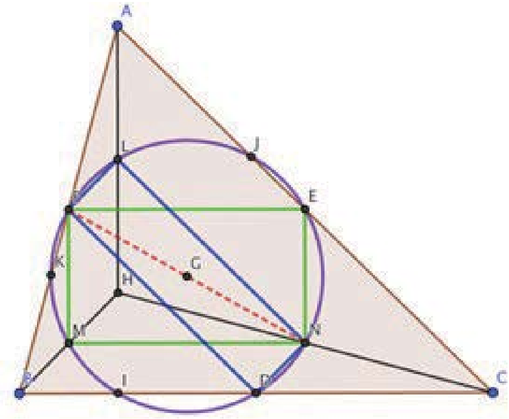


Figure 9. The nine-point circle has the diagonal FN or LD as a diameter.

Finally, it was required to prove that the feet of the altitudes, namely, I, J and K also lie on the circle (with diameter FN), which contains the six points. GeoGebra offers the option to 'hide' or 'unhide' parts of a figure. This useful feature is a great advantage over paper-pencil constructions. At this stage of the exploration, the rectangles and the line segments HA, HB and HC were 'hidden.' The diagonal LD of the rectangle FLND (which is also a diameter of the circle) was drawn. It was easy to see (as shown in Figure 10) that the angle LID is a right angle. This implies that I lies on the circle with diameter LD (the converse of the theorem which states that angle in a semicircle is a right angle) and is therefore also on the circle. Similarly, it was shown that J and K also lie on the circle. This proved the conjecture about the nine-point circle.

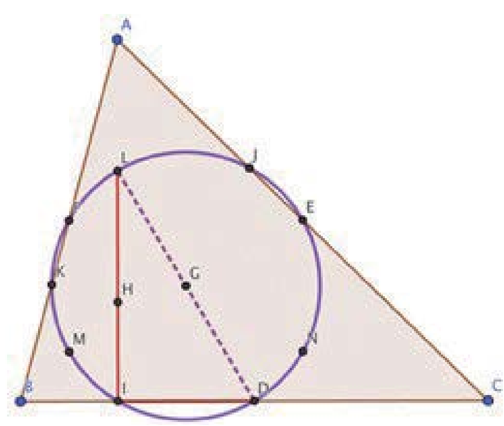


Figure 10. Angle LID is a right angle implying that point I lies on the circle with LD as diameter.

DGS as an amplifier and organiser

One of the important roles of the use of digital technology in mathematics learning is its use as an *amplifier* and *organiser* of mental activity [5]. When technology takes over time-consuming computations and operations (that might be done by hand), it assumes the role of an amplifier. For example, a digital tool can quickly and accurately produce graphs or generate a table of values leading the student (and teacher) to focus on observing patterns and developing insight. As an amplifier, technology supports students' mathematical thinking. However, when technology is used as a reorganizer, it enhances students' mathematical thinking by giving them access to higher-level processes that would be difficult to achieve in a paper-pencil mode.

In the exploration of the nine-point circle by grade IX students, GeoGebra played a pivotal role as an amplifier and reorganizer. When asked, how they viewed the GeoGebra exploration in comparison to a compass and ruler construction, many students felt that constructions in GeoGebra were 'faster', 'more accurate', 'dynamic' and 'easy to understand'. GeoGebra quickly produced constructions allowing students to focus on the properties and measurements of parts of the figure (such as lengths of segments), from the Algebra view. Here GeoGebra played the role of an amplifier. However, once the construction was completed, the students began to focus on changes in the figure brought about by dragging. Dragging enabled them to experience varying appearances of the triangle ABC and look for invariant properties, leading to the conjecture about the nine-point circle. Here, GeoGebra played the role of an organiser by extending student's mathematical thinking. For example, by exploring the acute angle case (figure 4), a majority of the students were able to conjecture that the nine points lie on a circle. Students asked many 'what if?' and 'why?' questions and tried to justify their arguments with reasoning. GeoGebra also motivated the need to find a proof of the invariant nine-point circle. According to a study by King and Schattschneider [2], a dynamic geometry software cannot produce proofs, but it can motivate the desire for proof in students. Appropriate facilitation by the teacher led students to write the proof using their knowledge of properties of quadrilaterals and the midpoint theorem.

Conclusion

The outcome of the study was illuminating as the grade IX students, with no prior knowledge of GeoGebra, were able to use it to explore a geometrical problem, make conjectures and finally arrive at a proof. Throughout the session, the students worked in pairs. Within each pair, they took turns at working with GeoGebra on their computer. While one student worked on the computer, the other recorded their common observations on their worksheet. They soon realized that GeoGebra would help them make observations but when it came to writing the proof, they needed to provide sound justifications for their arguments. GeoGebra empowered the students by extending their mathematical thinking and by enabling them to perform geometrical explorations beyond the scope of grade IX. The role of the teacher (who was also the researcher) was critical, as she had to steer the students' explorations without giving away answers, making sure that there was enough opportunity for collaboration and discussion. The teacher encouraged discussions among students throughout the two sessions. Overall GeoGebra led to a satisfying combination of technology use and engagement with argumentation and proof. Further, the positive feedback from students was highly encouraging as they expressed the desire to undergo similar sessions in other topics. Lingefjard and Ghosh [4] report their findings of a similar study where students of grade IX explored the Euler line (collinearity of the centroid, circumcentre and orthocenter of a triangle) using GeoGebra. The encouraging results of such studies point to the possibilities and affordances in integrating Dynamic Geometry Software for teaching and learning mathematics.

References

[1] Ghosh, J (2020). *Dynamic Geometry Software: A conjecture making tool.* At Right Angles, July 2020, pp 86-92.

[2] King, J., & Schattschneider, D. (1997). Preface: Making geometry dynamic. In J. R. King & D. Schattschneider (Eds.), *Geometry Turned On!: Dynamic Software in Learning, Teaching, and Research* (pp. ix-xiv). Washington, D.C.: The Mathematical Association of America.

[3] Leung, A. (2008). *Dragging in a dynamic geometry environment through the lens of variation*. International Journal of Computers for Mathematical Learning 13, 135-157.

[4] Lingefjärd, T & Ghosh, J. (2016). The Euler Line: Students' Exploration in a Dynamic Geometry Environment. The Electronic Journal of Mathematics and Technology, Volume 13 (3) pp. 220 – 236.

[5] Pea, R. D. (1987). *Cognitive technologies in mathematics education*. In A. H. Schoenfeld (Ed.), Cognitive science and mathematics education (pp. 89–122). Hilldale, NJ: Erlbaum.

JONAKI GHOSH is an Assistant Professor in the Dept. of Elementary Education, Lady Sri Ram College, University of Delhi where she teaches courses related to math education. She obtained her PhD in Applied Mathematics from Jamia Milia Islamia University, New Delhi, and her M.Sc. from IIT Kanpur. She has taught mathematics at the Delhi Public School, R K Puram, where she set up the Math Laboratory & Technology Centre. She has started a Foundation through which she conducts professional development programmes for math teachers. Her primary area of research interest is in the use of technology in mathematics instruction. She is a member of the Indo Swedish Working Group on Mathematics Education. She regularly participates in national and international conferences. She has published articles in journals and authored books for school students. She may be contacted at jonakibghosh@gmail.com.

VIEWPOINT

– A Ramachandran

A line segment can be divided internally and externally in the same ratio, with the exception of the ratio 1:1. Let A and B be two points, and P and Q the points that divide line segment AB internally and externally, respectively, in the ratio a:b. (That is, $\frac{AP}{PB} = \frac{a}{b}$ and $\frac{AQ}{QB} = \frac{a}{b}$.)

If $a \ll b$ (this means that a is very much smaller than b), then P lies close to A, as does Q on the other side of A. As the value of a approaches the value of b, i.e., the ratio approaches 1:1, P moves towards the midpoint of AB, while Q moves to extreme left. When the ratio is 1:1, the position of Q is not defined. If the value of a is slightly more than the value of b, P lies closer to B than to A, while Q appears at extreme right. When $a \gg b$ (this means that a is very much larger than b), then P and Q approach B from opposite sides.

We see that as P moves continuously from A to B, Q moves away to left and then reappears to right. If we grant that this too must be a continuous process, can we say that a straight line is actually part of an infinite circle? As a similar exercise can be done with line segment AB in any orientation, is the Euclidean plane actually an infinite spherical surface?

A Problem in Elementary Number Theory

SASIKUMAR K

In this short note, we look for all cases where the sum of a power of 2 and a power of 3 is a perfect square.

Problem

Find all pairs (m, n) of positive integers such that $2^m + 3^n$ is a perfect square.

Solution

Let $2^m + 3^n = k^2$; note that k is an odd number. We shall make use of the following easily proved number-theoretic facts.

(N1) Any perfect square is of one of the forms $3t$, $3t + 1$ (where t is a non-negative integer).

(N2) Any perfect square is of one of the forms $4t$, $4t + 1$ (where t is a non-negative integer).

(N3) An even power of 2 is of the form $3t + 1$, and an odd power of 2 is of the form $3t + 2$ (where t is a non-negative integer).

(N4) An even power of 3 is of the form $4t + 1$, and an odd power of 3 is of the form $4t + 3$ (where t is a non-negative integer).

In the analysis below, we consider separately the cases when m is odd and when m is even.

Case 1: m is odd. Consider the sum $2^m + 3^n$. Making use of (N3), we see that 2^m is of the form $3t + 2$. As 3^n is a multiple of 3, it follows that $2^m + 3^n$ is of the form $3t + 2$. But no perfect square has this form. Hence the stated equality is not possible. So there is no solution where m is odd.

Keywords: Perfect square, power, number theory

Case 2: m is even. We first show that in this case n itself is even.

Suppose that n is odd. Consider the sum $2^m + 3^n$. Making use of (N4), we see that 3^n is of the form $4t + 3$. Also, 2^m is a multiple of 4, as m is even. Hence $2^m + 3^n$ is of the form $4t + 3$. But no perfect square has this form. Hence the stated equality is not possible. So there is no solution where n is odd.

So we need to only consider the case when both m and n are even.

Let $m = 2a$, $n = 2b$, where a, b are positive integers. We write:

$$2^{2a} + 3^{2b} = k^2,$$
$$\therefore\ 2^{2a} = k^2 - 3^{2b},$$
$$\therefore\ 2^{2a} = \left(k - 3^b\right) \cdot \left(k + 3^b\right).$$

As the quantity on the left side is a power of 2, it follows that both the factors on the right side are powers of 2. Let

$$k - 3^b = 2^c, \quad k + 3^b = 2^d,$$

where c, d are non-negative integers ($c < d$, $c + d = 2a$). By subtraction we get:

$$2 \cdot 3^b = 2^d - 2^c,$$
$$\therefore\ 3^b = 2^{d-1} - 2^{c-1}.$$

If $c > 1$, the quantity on the right side would be even. However, the quantity on the left side (i.e., 3^b) is odd. It follows that $c = 1$. Hence we have:

$$3^b = 2^{d-1} - 1.$$

Since $c + d = 2a$ and $c = 1$, it follows that $d = 2a - 1$. This means that $d - 1 = 2a - 2$ is an even number.

We now have:

$$3^b = 2^{2a-2} - 1,$$
$$\therefore\ 3^b = \left(2^{a-1} - 1\right) \cdot \left(2^{a-1} + 1\right).$$

From the equality in the second line, it follows that both factors (i.e., $2^{a-1} - 1$ and $2^{a-1} + 1$) are powers of 3. *But these numbers are consecutive odd numbers.* The only consecutive odd numbers which are both powers of 3 are 1 and 3 (with $1 = 3^0$ and $3 = 3^1$). Hence $a - 1 = 1$, implying that $2a = 4$ and also $d - 1 = 2$, i.e., $d = 3$, which leads to $b = 1$.

It follows that $m = 4$ and $n = 2$.

We conclude that there is just one pair (m, n) of positive integers such that $2^m + 3^n$ is a perfect square; namely, $(m, n) = (4, 2)$. The associated equality in this case is

$$2^4 + 3^2 = 5^2.$$

SASIKUMAR K is presently working as a PG Teacher in Mathematics at Jawahar Navodaya Vidyalaya, North Goa. He completed his M Phil under the guidance of Dr.K.S.S. Nambooripad. Earlier, he worked as a maths Olympiad trainer for students of Navodaya Vidyalaya Samiti in Hyderabad. He has research interests in Real Analysis and Commutative Algebra. He may be contacted at 112358.ganitham@gmail.com.

A Problem concerning Prime Numbers

ANAND PRAKASH

In this paper, I explore the following problem:

Problem. Find instances where a product of distinct prime numbers is an integer multiple of the sum of the same prime numbers. Find as many such instances as possible.

Here is an example of such a situation:

$$\frac{2 \times 3 \times 5}{2+3+5} = 3, \quad \text{an integer.}$$

So the problem is to find prime numbers $p, q, r, \ldots$, with $p < q < r < \cdots$, such that

$$\frac{pqr\cdots}{p+q+r+\cdots} = \text{an integer.} \qquad (1)$$

Let k denote the number of primes in $\{p, q, r, \ldots\}$. The case $k = 1$ is trivial, as the fraction simplifies to 1. In this article, we study the cases $k = 2$, $k = 3$, and $k = 4$ (the last case only partially).

The case k = 2. Here the problem is to find primes p, q (with $p < q$) such that pq is an integer multiple of $p + q$. We shall show that it is not possible to find any such pair.

To start with, note that if both p, q are odd, then pq is odd whereas $p + q$ is even, and an even number obviously cannot be a divisor of an odd number. Hence this situation cannot occur.

The case left to be considered is when the smaller prime is 2. Suppose that $p = 2$. Then the condition is that $2q$ is an integer multiple of $q + 2$, with $q > 2$. However, if $q > 2$, then

$$q + 2 < 2q < 2(q+2).$$

Keywords: Primes, product, divisibility, parity, twin primes, arithmetic progression

So it is not possible for $2q$ to be an integer multiple of $q+2$ if $q>2$. Hence this situation too cannot occur.

This means that there are no solutions when $k=2$.

The case k = 3. Here the problem is to find primes p, q, r (with $p<q<r$) such that pqr is an integer multiple of $p+q+r$. We have seen one such instance, above. The analysis turns out to be quite rich in this case. We start by considering the cases $p=2$ and $p=3$.

The case $p=2$. Suppose that $p=2$. We need to find odd primes q, r, with $2<q<r$, such that

$$2qr = n(2+q+r), \quad \text{for some positive integer } n. \tag{2}$$

The prime factorisation of the quantity on the left side of (2) is $2 \times q \times r$. Hence $2+q+r$ is one of the following:

$$2,\ q,\ r,\ 2q,\ qr,\ 2r,\ 2qr.$$

All but two of these possibilities can be immediately eliminated. For example, suppose that $2+q+r=qr$. This may be written as

$$qr-q-r=2,$$
$$\therefore\ (q-1)(r-1)=3.$$

The last equality is not possible, since $r \geq 5$. Hence it is not possible that $2+q+r=qr$.

Continuing, we find that the only possibilities left are $2+q+r=2q$ and $2+q+r=2r$. We consider both these in turn.

- If $2+q+r=2q$, then $2+r=q$, or $r=q-2$. But this is not possible, since we have supposed that $r>q$.
- If $2+q+r=2r$, then $2r=2+q+r$, so $2+q=r$. **This means that q, r are a pair of twin primes!** The last case has yielded a very nice conclusion!

We see that the instance $(2,3,5)$ is not an isolated one (note that 3, 5 are a pair of twin primes); $(2,q,r)$ is a solution for any pair of twin primes q, r. We thus have the following instances:

- $(2,5,7)$, with $2\times5\times7 = 5\times(2+5+7)$;
- $(2,11,13)$, with $2\times11\times13 = 11\times(2+11+13)$;
- $(2,17,19)$, with $2\times17\times19 = 17\times(2+17+19)$;
- $(2,29,31)$, with $2\times29\times31 = 29\times(2+29+31)$; and so on.

The case $p=3$. Suppose that $p=3$. We need to find odd primes q, r, with $3<q<r$, such that

$$3qr = n(3+q+r), \quad \text{for some positive integer } n. \tag{3}$$

The prime factorisation of the quantity on the left side of (3) is $3\times q\times r$. Hence $3+q+r$ is one of the following:

$$3,\ q,\ r,\ 3q,\ qr,\ 3r,\ 3qr.$$

All but two of these possibilities can be immediately eliminated. For example, suppose that $3+q+r=qr$. This may be written as

$$qr-q-r=3,$$
$$\therefore\ (q-1)(r-1)=4.$$

The last equality is not possible, since $r \geq 7$. Hence it is not possible that $3 + q + r = qr$.

Continuing, we find that the only possibilities left are $3 + q + r = 3q$ and $3 + q + r = 3r$. We consider both these in turn.

- If $3 + q + r = 3r$, then $3 + q = 2r$. Now we have $r \geq q + 2$, so we get $3 + q \geq 2q + 4$, or $q \leq -1$, which is absurd. Hence this possibility does not work out.
- If $3 + q + r = 3q$, then $3 + r = 2q$. This means that q is the arithmetic mean of 3 and r. **Otherwise expressed, the primes 3, q, r form an arithmetic progression!** Once again, the last case has yielded a very nice conclusion.

  To list such instances, we need to list primes q such that $2q - 3$ is also prime. There are many primes with this property, for example:

  $$5,\ 7,\ 11,\ 13,\ 17,\ 23,\ 31,\ \ldots,$$

  resulting in the following triples which satisfy the given condition:

  $$(3,5,7),\quad (3,7,11),\quad (3,11,19),\quad (3,13,23),\quad (3,17,31),\quad \ldots.$$

The case $p = 5$. Suppose that $p = 5$. We need to find odd primes q, r, with $5 < q < r$, such that

$$5qr = n(5 + q + r), \quad \text{for some positive integer } n. \tag{4}$$

The prime factorisation of the quantity on the left side of (4) is $5 \times q \times r$. Hence $5 + q + r$ is one of the following:

$$5,\ q,\ r,\ 5q,\ qr,\ 5r,\ 5qr.$$

Arguing as earlier (we leave out the details; please try to fill in the details yourself), we find that only one possibility works out: $5 + q + r = 5q$. This leads to the following:

$$5q = 5 + q + r, \quad \therefore\ r = 4q - 5. \tag{5}$$

So we must look for odd primes $q > 5$ such that $4q - 5$ too is prime. Here are some primes that satisfy this condition:

$$7,\ 13,\ 19,\ 43,\ 61,\ 67,\ 79,\ 97,\ 109,\ 127,\ 151,\ 163,\ 181,\ \ldots,$$

resulting in the following triples which satisfy the given condition:

$$\begin{array}{llll} (5,7,23), & (5,13,47), & (5,19,71), & (5,43,167), \\ (5,61,239), & (5,67,263), & (5,79,311), & (5,97,383), \\ (5,109,431), & (5,127,503), & (5,151,599), & \ldots \end{array}$$

Other primes. The cases $p = 7, 11, 13, \ldots$ may be analysed in the same way; we leave the details to the reader. (The reasoning used is nearly the same in all these cases.) Many more solutions may be formed, all obeying similar relationships.

The case k = 4. Here the problem is to find primes p, q, r, s (with $p < q < r < s$) such that $pqrs$ is an integer multiple of $p + q + r + s$. We see right away that $p = 2$; for, if all four primes are odd, then their sum is even but their product is odd, so it is not possible for the product to be a multiple of the sum. Therefore the task reduces to the following: find odd primes q, r, s (with $q < r < s$) such that

$$2qrs = n(2 + q + r + s), \quad \text{for some positive integer } n. \tag{6}$$

Arguing as earlier, we see that $2 + q + r + s$ must be one of the numbers

$$2,\ q,\ r,\ s,\ 2q,\ 2r,\ 2s,\ qr,\ qs,\ rs,\ 2qr,\ 2qs,\ 2rs,\ qrs,\ 2qrs.$$

Many of these possibilities can be eliminated immediately, but some of them do yield results. We will not analyse this case in detail, but only consider one possibility, namely that $2 + q + r + s = qr$. This equality leads to

$$qr - q - r + 1 = s + 3, \quad \text{i.e.,}\ \ s = (q-1)(r-1) - 3. \tag{7}$$

So we must look for primes q, r such that $(q-1)\ (r-1) - 3$ is a prime number. Here are some possibilities:

$$(q, r) = (3, 11),\ (3, 17),\ (3, 23),\ (3, 29),\ \ldots,\ (5, 11),\ (5, 17),\ (5, 29),\ \ldots.$$

We see that there are many solutions. The above possibilities lead to the following quadruples which are all solutions of the original problem:

$$\begin{array}{llll}
(2,3,11,17), & (2,3,17,29), & (2,3,23,41), & (2,3,29,53), \\
(2,3,47,89), & (2,5,11,37), & (2,5,17,61), & (2,5,29,109), \\
(2,5,41,157), & (2,5,47,181), & (2,11,17,157), & (2,11,29,277), \\
(2,11,41,397), & (2,11,47,457), & (2,17,23,349), & (2,17,47,733), \\
(2,23,29,613), & (2,23,41,877), & (2,23,47,1009), & \ldots
\end{array}$$

There are clearly lots of solutions, and further exploration is surely possible. The problem seems rich, revealing many interesting relationships among prime numbers.

ANAND PRAKASH runs a small garment shop at Kesariya village in the state of Bihar. He has a keen interest in number theory and recreational mathematics and has published many papers in international journals in these fields. He also has a deep interest in classical Indian music as well as cooking. In addition, he has written a large number of poems in Hindi. He may be contacted at prakashanand805@gmail.com.

Middle School Problems on Polygons

A. RAMACHANDRAN

Polygons are closed 2-D figures whose sides are straight line segments.

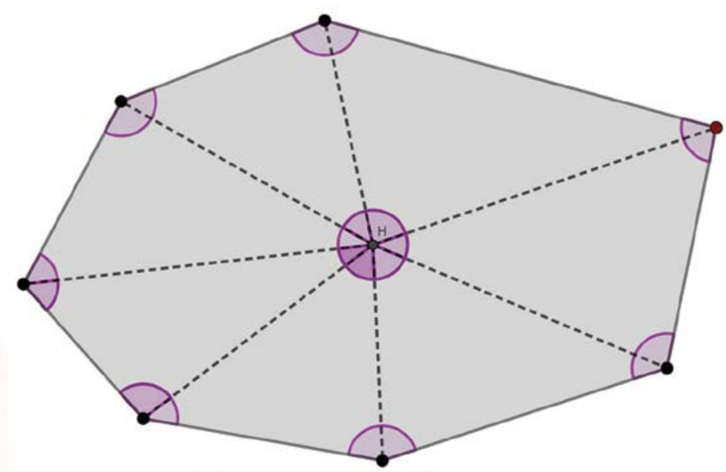


Figure 1

Figure 1 is an example of a heptagon - a polygon with 7 sides.

Here are some generalisations about polygons.

- A polygon must have at least 3 sides.
- A polygon has as many vertices as sides.
- The sum of the interior angles of a polygon of n sides is given by $(n-2) \times 180°$. (To understand this formula, look at the heptagon - in which H is any point in the interior. If you add up the angles of the 7 triangles you see in the heptagon, you would see

Keywords: Polygons, properties, ratios, visualisation, reasoning, algebra.

that the sum is $7 \times 180°$. To get the sum of the interior angles of the polygon, you would have to subtract the angles at the centre, which sum up to $360°$ or $2 \times 180°$. So the sum of the interior angles of the heptagon is $5 \times 180°$ or $(7-2) \times 180°$.)

- The sum of the exterior angles of any polygon is always $360°$. (An exterior angle is the angle formed when a side is extended at a vertex. See if you can reason why the exterior angles add up to $360°$.)
- If all the sides and angles of a polygon are equal, we call it a regular polygon. An equilateral triangle and a square are regular polygons of three and four sides, respectively. Each angle of a regular polygon is $\frac{(n-2) \times 180}{n}$ degrees.

Problem X-1-M-.1

It is possible to have a triangle whose angles are in the ratio 1:2:3 or a quadrilateral with angles in the ratio 1:2:3:4. It is not possible to have a pentagon with angles in the ratio 1:2:3:4:5 or an octagon with angles in the ratio 1:2:3:4:5:6:7:8. Why?

Problem X-1-M-.2

Find expressions in terms of n for the least and greatest angles of a polygon of n sides with angles in the ratio 1:2:3 . . . . :n.

The remaining problems are all concerned with regular polygons.

Problem X-1-M-.3

Regular polygons A and B have number of sides in the ratio 1 : 2, and interior angles in the ratio 3 : 4. Find the number of sides of the regular polygons.

Problem X-1-M-.4

Regular polygon A has 3 more sides than regular polygon B, while the interior angle of the former is $6°$ more than that of the latter. Find the number of sides of the polygons.

Problem X-1-M-.5

Regular polygons A, B and C have m, n and $2mn$ sides, respectively. If the sum of an interior angle of polygon A and an interior angle of polygon B equals an interior angle of polygon C, find the values of m and n.

Pedagogical Note: These problems are an excellent means to combine a student's knowledge of arithmetic (ratio, fractions), geometry and algebra. By getting them to draw their solutions, they will be able to visualise their meaning. Drill and practice become easier with such problems.

Solutions to the Problems

Solution to Problem X-1-M-.1

The sum of the interior angles of a pentagon equals (5– 2) × 180° = 540°.This has to be divided into 5 parts in the ratio 1:2:3:4:5. Let these angles be k, 2 k, 3 k, 4 k and 5 k.

Adding, we get $k + 2k + 3k + 4k + 5k = 15k = 540°$. This gives $k = 36°$. But then one of the angles is $5k$ which is 180° which is not possible. Try proving this for an octagon now!

Solution to Problem X-1-M-.2

So the angles can be taken to be k, $2k$, $3k$,....nk. Then

$$\frac{kn(n+1)}{2} = (n-2) \times 180°$$

Or, $k = \dfrac{(n-2)\,360°}{n(n+1)}$. This is the least angle, while the greatest angle, which is n times this, is $\dfrac{(n-2)\,360°}{(n+1)}$.

Pedagogical Note: It is a good idea for students to try getting the angles and drawing different polygons with different values of n.

Solution to Problem X-1-M-.3

	Polygon A	Polygon B
Sides	n	$2n$
Each interior angle	3ϑ $\dfrac{(n-2) \times 180}{n}$	4ϑ $\dfrac{(2n-2) \times 180}{2n}$

Equating both expressions for ϑ we get:

$$\frac{(n-2) \times 180}{3n} = \frac{(n-1) \times 180}{4n}$$

Solving this we get $n = 5$. So, the number of sides of the polygons are 5 and 10.

Pedagogical Note: Confusing statements in word problems become much easier when students use tables.

Solution to Problem X-1-M-.4

	Polygon A	Polygon B
Sides	$n + 3$	n
Each interior angle	$\vartheta + 6$ $\dfrac{(n+1) \times 180}{n+3}$	ϑ $\dfrac{(n-2) \times 180}{n}$

Equating both expressions for ϑ we get,

$$\frac{(n+1) \times 180}{n+3} - 6 = \frac{(n-2) \times 180}{n}$$

From this we get $n^2 + 3n - 180 = 0$.

Solving the quadratic equation we get the only positive value of $n = 12$.

So, the number of sides of regular polygons A and B are 15 and 12, respectively.

Pedagogical Note: This problem requires the solution of a quadratic equation. While this is beyond the middle school math syllabus, students can understand the meaning of this expression by substituting different values of n and they may even find the solution by trial and error.

Solution to Problem X-1-M-.5

	Polygon A	Polygon B	Polygon C
Sides	m	n	$2mn$
Each interior angle	ϑ_1 $\frac{(m-2) \times 180}{m}$	ϑ_2 $\frac{(n-2) \times 180}{n}$	ϑ_3 $\frac{(2mn-2) \times 180}{2mn}$

Since $\vartheta_1 + \vartheta_2 = \vartheta_3$

We form the equation

$$\frac{(m-2) \times 180}{m} + \frac{(n-2) \times 180}{n} = \frac{(mn-1) \times 180}{mn}$$

This simplifies to $mn - 2m - 2n + 1 = 0$. This expression is not factorisable. Adding 3 to both sides, we get $mn - 2m - 2n + 4 = 3$, which can be factorised as $(m-2)(n-2) = 3$. The only factorisation of 3 in positive integers is 1×3 and so we could say

$$m - 2 = 1 \text{ or } m = 3 \text{ and } n - 2 = 3 \text{ or } n = 5.$$

The number of sides of polygons A, B and C are then 3, 5 and 30.

Pedagogical Note: An interesting way to solve an equation and a great way for students to see the uses of factorisation.

A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

On the Tens Digit of a Prime Power

SIDDHARTHA SANKAR CHATTOPADHYAY

In this note, we prove that the tens digit of any power of an infinite number of prime numbers is even. This is a generalization of a problem that appeared in the Regional Mathematics Olympiad in 1993.

The study of the digits appearing in the decimal expansion of a real number plays an important role in the study of various number theoretic problems. In particular, the parity of the digits in the powers of a prime number is an interesting object to study. In RMO 1993, the following problem was posed.

Problem 1 (RMO 1993)**.** Prove that the tens digit of any power of 3 is even.

In this article, we take a close look at the RMO problem and prove that the conclusion holds for an infinite number of prime numbers p. More precisely, we prove the following theorem.

Theorem 1. Let p be a prime number such that $p \equiv 3 \text{ or } 7 \pmod{20}$. Then for any integer $r \geq 1$, the tens digit of p^r is an even number.

Remark 1. Note that 3 is a prime number congruent to 3 modulo 20. Therefore, Theorem 1 is indeed a generalization of Problem 1.

Remark 2. Dirichlet's theorem for primes in arithmetic progressions asserts that if a and m are integers such that $\gcd(a, m) = 1$, then there exist infinitely many prime numbers q such that $q \equiv a \pmod{m}$. Since $\gcd(3, 20) = 1 = \gcd(7, 20)$, the theorem tells us that there exist infinitely many prime numbers of the forms $3 \pmod{20}$ and $7 \pmod{20}$. Theorem 1 now assures us that the tens digit of any power of any such prime number is even.

Keywords: Prime number, Regional Mathematical Olympiad (RMO)

Proof of Theorem 1

We give a detailed proof for $p \equiv 3 \pmod{20}$. For the residue class $7 \pmod{20}$, the proof follows almost the same line of argument.

Let $r \geq 1$ be an integer and let $p \equiv 3 \pmod{20}$ be a prime number. Then $p = 20m + 3$ for some integer m. Since we are dealing with the tens digit of p^r, we shall be concerned with $p^r \pmod{100}$. Therefore, it is convenient to put $k = 2m$ and use the fact that k is an even integer.

The proof is by induction on r. For $r = 1$, the tens digit of p^r is even because $p = 10k + 3$ with k even. Therefore, the theorem holds true for $r = 1$. Now, suppose that the tens digit of p^r is even for some integer $r \geq 1$.

Let $p^r = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_s \cdot 10^s$ be the decimal expansion of p^r (where s is some positive integer). Then we have

$$\begin{aligned} p^{r+1} &= (10k+3)(a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \ldots + a_s \cdot 10^s) \\ &\equiv 3a_0 + 30a_1 + 10k \cdot a_0 \pmod{100}. \end{aligned} \tag{1}$$

By the induction hypothesis, a_1 is even. Also, we note that since $p \equiv 3 \pmod{10}$, we have $p^{r+1} \equiv a_0 \equiv 1, 3, 7, 9 \pmod{10}$. We consider the four cases separately.

Case 1, $\mathbf{a_0} = 1$. Then $p^{r+1} \equiv 3 + 30a_1 + 10k \pmod{100}$. Since $a_1 \in \{0, 2, 4, 6, 8\}$, we have

$$3 + 30a_1 + 10k = \begin{cases} 10k + 3 & \text{if } a_1 = 0, \\ 10k + 63 & \text{if } a_1 = 2, \\ 10k + 123 & \text{if } a_1 = 4, \\ 10k + 183 & \text{if } a_1 = 6, \\ 10k + 243 & \text{if } a_1 = 8. \end{cases}$$

Since k is even and the tens digits of 3, 63, 123, 183 and 243 are all even, we conclude that the tens digit of p^{r+1} is even.

Case 2, $\mathbf{a_0} = 3$. Then $p^{r+1} \equiv 9 + 30a_1 + 30k \pmod{100}$. Since $a_1 \in \{0, 2, 4, 6, 8\}$, we have

$$9 + 30a_1 + 30k = \begin{cases} 30k + 9 & \text{if } a_1 = 0, \\ 30k + 69 & \text{if } a_1 = 2, \\ 30k + 129 & \text{if } a_1 = 4, \\ 30k + 189 & \text{if } a_1 = 6, \\ 30k + 249 & \text{if } a_1 = 8. \end{cases}$$

Again we note that the tens digits of 9, 69, 129, 189 and 249 are all even. Hence the tens digit of p^{r+1} is even.

Case 3, $\mathbf{a_0} = 7$. Then $p^{r+1} \equiv 21 + 30a_1 + 70k \pmod{100}$. Since $a_1 \in \{0, 2, 4, 6, 8\}$, we have

$$21 + 30a_1 + 70k = \begin{cases} 70k + 21 & \text{if } a_1 = 0, \\ 70k + 81 & \text{if } a_1 = 2, \\ 70k + 141 & \text{if } a_1 = 4, \\ 70k + 201 & \text{if } a_1 = 6, \\ 70k + 261 & \text{if } a_1 = 8. \end{cases}$$

Since the tens digits of $21, 81, 141, 201$ and 261 are all even, we conclude that the tens digit of p^{r+1} is also even.

Case 4, $\mathbf{a_0 = 9}$. Then $p^{r+1} \equiv 27 + 30a_1 + 90k \pmod{100}$. Since $a_1 \in \{0, 2, 4, 6, 8\}$, we have

$$27 + 30a_1 + 90k = \begin{cases} 90k + 27 & \text{if } a_1 = 0, \\ 90k + 87 & \text{if } a_1 = 2, \\ 90k + 147 & \text{if } a_1 = 4, \\ 90k + 207 & \text{if } a_1 = 6, \\ 90k + 267 & \text{if } a_1 = 8. \end{cases}$$

Since the tens digits of $27, 87, 147, 207$ and 267 are all even, we conclude that the tens digit of p^{r+1} is also even.

Therefore, by the method of mathematical induction, we conclude that the tens digit of any power of p is even. This completes the proof of Theorem 1. $\square$

The interested reader can enquire for which prime powers the tens digit is divisible by 4, by 8, and so on.

SIDDHARTHA SANKAR CHATTOPADHYAY is a retired Mathematics teacher from Bidhannagar Govt. High School, Salt Lake, Kolkata. He is actively associated with the Kolkata based `Association for Improvement of Mathematics Teaching' for 30 years. He has participated in seminars and workshops for school students organized by Jagadis Bose National Science Talent Search over the past 15 years as a resource person, and has been engaged in spreading Mathematics Education across West Bengal by organizing seminars and workshops. He may be contacted at 1959ssc@gmail.com.

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Read the solution to one of the geometry problems in our online edition at https://azimpremjiuniversity.edu.in/geometry-problem-putnam. Why not send us other solutions to the same problem or solutions for other problems?

Upper Primary

Problem Corner

Some Results concerning Regular Polygons

UTPAL MUKHOPADHYAY

Consider a regular n-sided polygon with centre O and side length a. We first find an expression for its area in terms of a and n. The angle subtended by each side at the centre O of the circumscribing circle is (in radian measure) $2\pi/n$. For convenience, we denote this angle by 2θ.

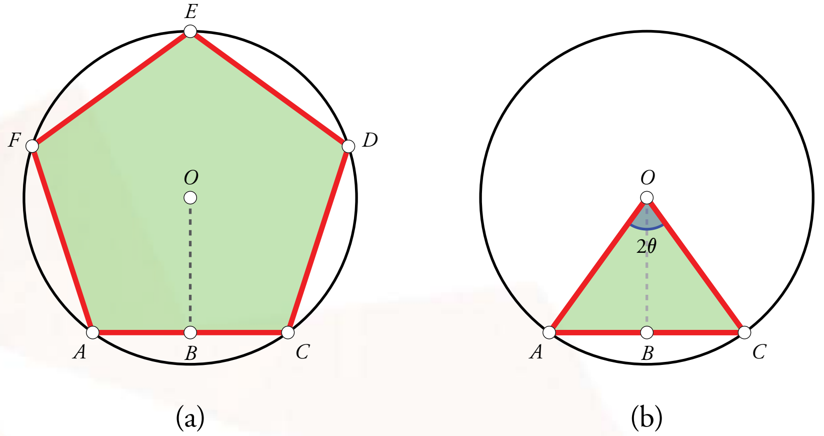


Figure 1. Regular n-sided polygon: $AC = a$, $\measuredangle AOC = 2\pi/n$, $\theta = \pi/n$

If AC is any side of the polygon (see Figure 1), and B is the foot of the perpendicular from O to AC, then $OB = a/2 \cdot \cot\theta$, so:

$$\text{Area of } \triangle OAC = \frac{a^2}{4} \cdot \cot\theta,$$

$$\therefore \text{ Area of polygon } = \frac{n\,a^2}{4} \cdot \cot\theta.$$

Keywords: Regular polygon, incircle, circumcircle, area, perimeter

Incircle and circumcircle of a regular polygon

The incircle. Consider first the incircle of a regular n-sided polygon with side length a. If AC is a side of the polygon (Figure 2a), its point of contact with the incircle being B, then $\angle AOB = \theta$ and $AB = a/2$. A study of $\triangle OAB$ shows that the radius of the incircle is $a/2 \cdot \cot\theta$, so

$$\text{Area of incircle} = \frac{\pi a^2}{4} \cdot \cot^2\theta,$$
$$\text{Circumference of incircle} = \pi a \cdot \cot\theta.$$

Special case. If the numerical values of the circumference and area of the incircle are equal, then

$$\frac{\pi a^2}{4} \cdot \cot^2\theta = \pi a \cdot \cot\theta,$$
$$\therefore \quad a = 4\tan\theta.$$

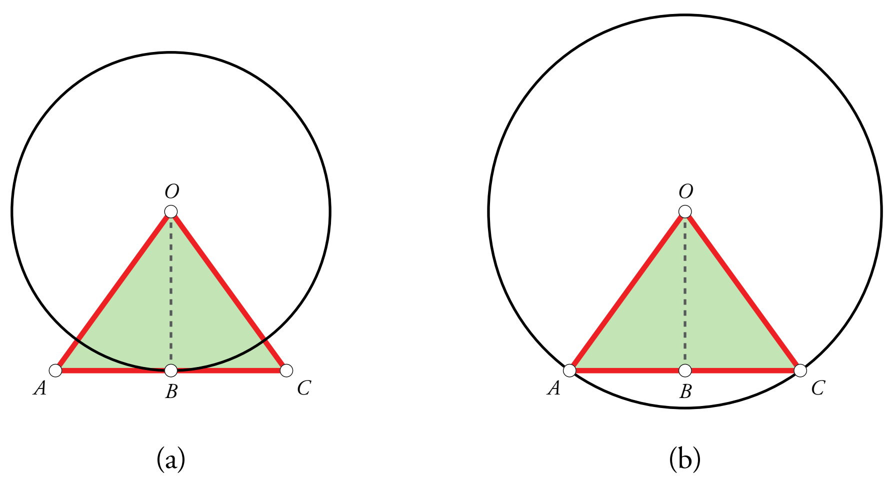


Figure 2.

The circumcircle. Now consider the circumcircle of a regular n-sided polygon with side length a. If AC is a side of the polygon (see Figure 2b), and B is its midpoint, then $BC = OC \cdot \sin\theta$, so

$$OC = \frac{a}{2} \cdot \csc\theta,$$
$$\therefore \quad \text{Circumference of circumcircle} = \pi a \cdot \csc\theta,$$
$$\text{Area of circumcircle} = \frac{\pi a^2}{4} \cdot \csc^2\theta.$$

Special case. If the numerical values of the circumference and area of the circle are equal, then

$$\pi a \cdot \csc\theta = \frac{\pi a^2}{4} \cdot \csc^2\theta,$$
$$\therefore \quad a = 4\sin\theta.$$

Observe that in this situation, the length of the side of such a polygon cannot exceed 4, as the sine of an angle cannot exceed 1.

A remarkable finding. From the above relations, we see that the difference in the areas of the circumcircle and the incircle of a regular n-sided polygon with side length a is equal to

$$\frac{\pi a^2}{4}\cdot\csc^2\theta-\frac{\pi a^2}{4}\cdot\cot^2\theta=\frac{\pi a^2}{4}.$$

So the difference in the areas of the circumcircle and the incircle of a regular polygon depends only on the length of its side and not on the number of sides of the polygon. A striking result!

Inscribed and circumscribed regular polygon in a circle

We now consider the reverse situation: we are given a circle with centre O and radius r, and we inscribe in it, and circumscribe about it, regular n-sided polygons.

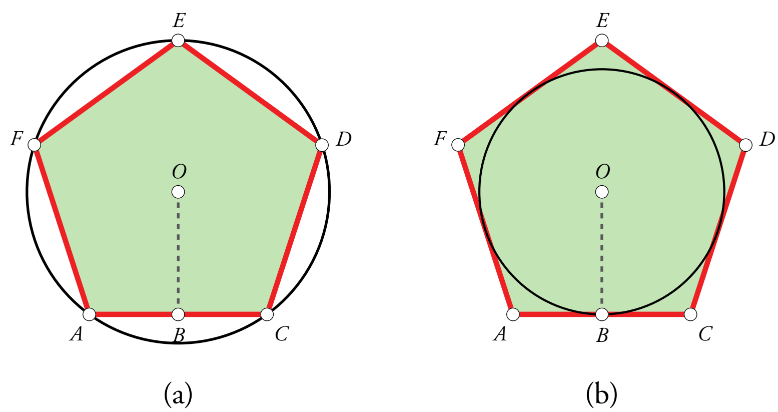


Figure 3.

Inscribed regular polygon. Consider first the inscribed regular polygon. (Recall that 'inscribed' means that the polygon is inside the circle, with all its vertices on the circumference of the circle.) If AC is any side of the polygon, and B is its midpoint (Figure 3a), then $\measuredangle AOB=\theta$, so $AB=r\cdot\sin\theta$. Hence the length of each side of the polygon is $2r\cdot\sin\theta$. Therefore:

$$\text{Perimeter of inscribed polygon} = 2nr\cdot\sin\theta,$$

$$\text{Area of inscribed polygon} = \frac{nr^2}{2}\cdot\sin 2\theta = nr^2\cdot\sin\theta\cdot\cos\theta.$$

Special case. If the numerical values of the perimeter and the area of the inscribed polygon are equal, then

$$2nr\cdot\sin\theta = nr^2\cdot\sin\theta\cdot\cos\theta,$$

$$\therefore\quad r = 2\sec\theta.$$

Observe that in this situation, the length of the side of the polygon cannot be less than 2, as the secant of an acute angle cannot be less than 1.

Circumscribed regular polygon. Consider next a circumscribed regular polygon. (Recall that 'circumscribed' means that the circle lies inside the polygon, all of its sides being tangents to the circle.) If AC is a side of the polygon, its point of contact with the circle being B (Figure 3b), then $\measuredangle AOB=\theta$. A study of $\triangle OAB$ yields $AB=r\cdot\tan\theta$. Hence the side of the polygon is $2r\cdot\tan\theta$. It follows that:

$$\text{Perimeter of the circumscribed polygon} = 2nr\cdot\tan\theta,$$

$$\text{Area of the circumscribed polygon} = nr^2\cdot\tan\theta.$$

Special case. If the numerical values of the perimeter and the area of the circumscribed polygon are equal, then

$$2nr \cdot \tan\theta = nr^2 \cdot \tan\theta,$$
$$\therefore \quad r = 2.$$

Since r is independent of n, it implies that for a circle of radius 2, the area and perimeter of a circumscribed polygon are numerically equal irrespective of the number of sides of the polygon.

Convergence patterns

Consider a regular n-sided polygon with side length a. Let circles be inscribed in it and circumscribed about it. Suppose that we regard the area of the incircle as an approximation for the area of the polygon. In this case, the relative error is

$$\frac{na^2/4 \cdot \cot\theta \;-\; \pi a^2/4 \cdot \cot^2\theta}{na^2/4 \cdot \cot\theta} = \frac{n - \pi \cdot \cot\theta}{n} = 1 - \frac{\pi}{n} \cdot \cot\frac{\pi}{n}.$$

We display below values of the relative error for a few values of n:

n	4	5	6	10	100	1000
Relative error	0.21	0.145	0.093	3.3×10^{-2}	3.3×10^{-4}	3.3×10^{-6}

As expected, the relative error decreases with n. (Observe that for large values of n, when n grows by a factor of 10, the relative error shrinks by a factor of 100. This is a very striking pattern.)

Similarly, if we regard the area of the circumcircle as an approximation for the area of the polygon, then the relative error is

$$\frac{\pi a^2/4 \cdot \csc^2\theta \;-\; na^2/4 \cdot \cot\theta}{na^2/4 \cdot \cot\theta} = \frac{\pi \cdot \csc^2\theta - n \cdot \cot\theta}{n \cdot \cot\theta} = \frac{2\pi/n}{\sin 2\pi/n} - 1.$$

We display below values of the relative error for a few values of n:

n	4	5	6	10	100	1000
Relative error	0.57	0.32	0.21	0.069	6.6×10^{-4}	6.6×10^{-6}

The relative error decreases with n. (Some interesting patterns may be seen. Once again we note that when n grows by a factor of 10, the relative error shrinks by a factor of 100. Moreover, for each fixed large value of n, the relative error here appears to be twice the relative error in the previous case. This certainly merits further exploration.)

We now invert the situation and regard the area of the polygon as an approximation for the area of the circle. Let us start with the case of the inscribed regular polygon. Here the relative error is

$$\frac{\pi r^2 - nr^2 \cdot \sin\theta \cdot \cos\theta}{\pi r^2} = 1 - \frac{\sin\pi/n \cdot \cos\pi/n}{\pi/n}.$$

We display below values of the relative error for a few values of n:

n	4	5	6	10	100	1000
Relative error	0.36	0.24	0.17	0.065	6.6×10^{-4}	6.6×10^{-6}

The relative error decreases with n. Yet again, we see some interesting patterns.

Finally, if we regard the area of the circumscribed polygon as an approximation for the area of the circle, then the relative error is

$$\frac{nr^2 \cdot \tan\theta - \pi r^2}{\pi r^2} = \frac{\tan \pi/n}{\pi/n} - 1.$$

We display below values of the relative error for a few values of n:

n	4	5	6	10	100	1000
Relative error	0.27	0.16	0.10	0.034	3.3×10^{-4}	3.3×10^{-6}

Interesting patterns yet again. The student should take note of these patterns and try to justify them analytically.

Concluding remarks. In this article, a few features of regular polygons inscribed in and circumscribed about circles have been explored. A few striking results have been uncovered, and the idea of using one quantity to approximate another has yielded some interesting patterns which may be explored further by the student.

UTPAL MUKHOPADHYAY is a retired teacher of Satyabharati Vidyapith, Barasat, West Bengal. He received his PhD from Jadavpur University, Kolkata, by working on accelerating universe and lambda-dark energy. He has been actively engaged in science popularisation programme for more than 25 years through his writings and other outreach activities. He has nearly 250 technical, semi-popular and popular publications in various journals and magazines. He may be contacted at utpalsbv@gmail.com.

Solutions to Two Problems

PRANEETHA KALBHAVI

We present solutions to two of the problems which appeared in the Senior Problem Set of the November 2020 issue of *At Right Angles*. Both have been submitted by Praneetha Kalbavi of Class XI, The Learning Center PU College, Mangalore.

Problem IX-3-1

Consider the quadratic function $f(x) = x^2 + bx + c$ defined on the set of real numbers. Given that the zeros of f are two distinct prime numbers p and q, and $f(p-q) = 6pq$, determine the primes p and q, and the function f.

Solution. As p, q are the roots of $x^2 + bx + c = 0$, we have $p^2 + bp + c = 0$ and $q^2 + bq + c = 0$. Hence by subtraction,

$$p^2 - q^2 + b(p-q) = 0, \quad \therefore \; p^2 + b(p-q) = q^2.$$

Combining this with the given fact that $6pq = f(p-q) = p^2 + q^2 - 2pq + b(p-q) + c$, we get

$$8pq = 2q^2 + c,$$

giving $c = 2q(4p - q)$. Now, using the familiar equalities for sum and product of the roots of a quadratic equation, we get:

$$c = pq, \quad \therefore \; 7pq = 2q^2, \quad \therefore \; 7p = 2q.$$

Since p and q are prime numbers, the last equality can only be satisfied if $p = 2$ and $q = 7$. This gives $c = 14$. Hence $f(x) = x^2 - 9x + 14$. □

Keywords: Quadratic function, prime number, equation, solution

Problem IX-3-5

Solve for real x:

$$4^x + 9^x + 36^x + \sqrt{\frac{1}{2} - 2x^2} = 1.$$

Solution. To start with, note that we must have $\frac{1}{2} - 2x^2 \geq 0$, and therefore $x^2 \leq \frac{1}{4}$, i.e., $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

Now if $x \geq 0$, then $a^x \geq 1$ for any $a > 1$, hence $4^x + 9^x + 36^x + \sqrt{\frac{1}{2} - 2x^2} > 1$. So the given equation has no solution with $0 \leq x$.

Next, note that $4^{-1/2} + 9^{-1/2} + 36^{-1/2} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$, and that $\sqrt{\frac{1}{2} - 2x^2} = 0$ when $x = -\frac{1}{2}$. This means that $x = -\frac{1}{2}$ solves the given equation. So $x = -\frac{1}{2}$ is a solution of the equation.

Finally, suppose that $-\frac{1}{2} < x < 0$. Then $4^x + 9^x + 36^x > \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$, i.e., $4^x + 9^x + 36^x > 1$, and therefore $4^x + 9^x + 36^x + \sqrt{\frac{1}{2} - 2x^2} > 1$. So the given equation has no solution with $-\frac{1}{2} < x < 0$.

It follows that the only solution to the given equation is $x = -\frac{1}{2}$. □

PRANEETHA KALBHAVI is currently a student studying in Class 11 at The Learning Center PU College, Mangalore, Karnataka. She has deep love for Mathematics and Science. She participated in INMO 2020. Her hobbies include reading and badminton. She wishes to pursue a career in mathematics or computer science. She may be contacted at praneethakalbhavi@gmail.com.

Student Uttkarsh Kohli described the mystical Kohli number in the November 2020 issue, you can find it at https://azimpremjiuniversity.edu.in/SitePages/resources-ara-issue-no-8-november-2020-mystical-number.aspx

In the online issue of At Right Angles, you will find Dr. Shailesh Shirali's proof of the genuineness of this constant. Access this article at https://azimpremjiuniversity.edu.in/kohlis-number to see how mathematics evolves.

On Some Questions Related to a Triangle

PRITHWIJIT DE

A case of two coincident centroids

Consider an acute-angled triangle ABC and let Ω be its circumcircle (Figure 1). Let G be the centroid of ABC. Let the lines AG, BG, and CG meet Ω again at A_1, B_1, and C_1, respectively. Note that if ABC is equilateral, then its centroid G is also the centroid of $A_1B_1C_1$. Suppose it happens that G is the centroid of $\triangle A_1B_1C_1$. Can we then conclude that triangle ABC is equilateral? This is one of the questions we explore in this article.

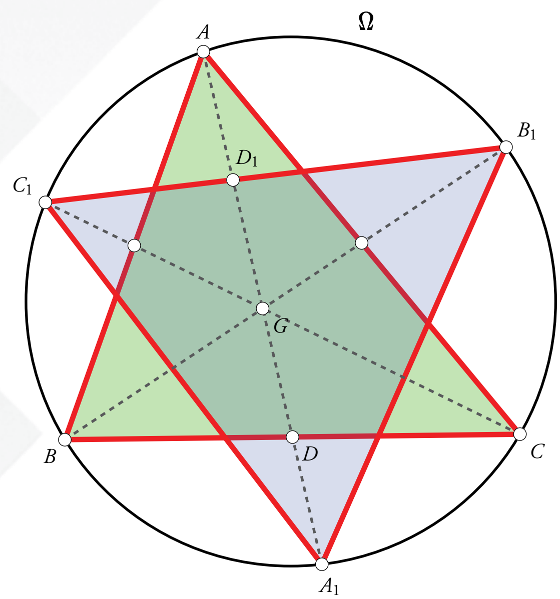


Figure 1. What can be said if G is the centroid of both ABC and $A_1B_1C_1$?

Keywords: Triangle, circumcircle, centroid, incentre, orthocentre, circumcentre, equilateral

Observe that in the triangles BGC and B_1GC_1, $\measuredangle CBG = \measuredangle B_1C_1G$ and $\measuredangle BGC = \measuredangle B_1GC_1$. Therefore they are similar and

$$\frac{BC}{B_1C_1} = \frac{BG}{C_1G}.$$

Let AG meet BC at D, and let A_1G meet B_1C_1 at D_1. Then since D and D_1 are midpoints of BC and B_1C_1, respectively it follows that

$$\frac{BC}{B_1C_1} = \frac{DB}{D_1C_1} = \frac{BG}{C_1G}.$$

Therefore, in triangles DBG and D_1C_1G, we have

$$\measuredangle DBG = \measuredangle D_1C_1G, \quad \frac{DB}{BG} = \frac{D_1C_1}{C_1G},$$

hence they are similar. Hence $\measuredangle DGB = \measuredangle D_1GC_1$. But

$$\measuredangle DGB = \measuredangle B_1GD_1, \qquad \measuredangle D_1GC_1 = \measuredangle DGC.$$

Therefore, we have

$$\measuredangle DGB = \measuredangle D_1GC_1 = \measuredangle DGC = \measuredangle B_1GD_1.$$

This shows that in triangle BGC, GD bisects $\measuredangle BGC$, and in triangle B_1GC_1, GD_1 bisects $\measuredangle B_1GC_1$. Therefore,

$$\frac{BG}{CG} = \frac{BD}{CD} = 1; \quad \frac{B_1G}{C_1G} = \frac{B_1D_1}{C_1D_1} = 1.$$

Hence $BG = CG$ and $B_1G = C_1G$. Similarly, we can prove that $CG = AG$ and $C_1G = A_1G$. Thus, $AG = BG = CG$ and $A_1G = B_1G = C_1G$, implying that the medians of ABC are equal, and so are the medians of $A_1B_1C_1$. Therefore, ABC and $A_1B_1C_1$ are equilateral triangles, and as they have the same circumcircle, they are congruent to each other.

Variations on the theme

One can also explore the cases where instead of the centroids of the two triangles being coincident, the incentres and the orthocentres coincide.

Coincident incentres. Suppose the incentres coincide. Let I be the common incentre of ABC and $A_1B_1C_1$. Then

$$\measuredangle BIC = 90^\circ + \frac{A}{2}, \qquad \measuredangle B_1IC_1 = 90^\circ + \frac{90^\circ - \frac{A}{2}}{2} = 135^\circ - \frac{A}{4},$$

and $\measuredangle BIC = \measuredangle B_1IC_1$ which readily yields $A = 60^\circ$. Similarly, $B = C = 60^\circ$ and ABC is equilateral. Also, $A_1 = 90^\circ - \frac{A}{2} = 60^\circ$ and similarly $B_1 = C_1 = 60^\circ$ showing that $A_1B_1C_1$ is equilateral.

Here we have used the fact that the angles of $A_1B_1C_1$ are $A_1 = 90° - \frac{A}{2}$, $B_1 = 90° - \frac{B}{2}$ and $C_1 = 90° - \frac{C}{2}$. These relations can readily be deduced by angle-chasing.

Coincident orthocentres. Suppose the orthocentres coincide. Let H be the common orthocentre of ABC and $A_1B_1C_1$. Then

$$\measuredangle B_1HC_1 = \measuredangle BHC.$$

But $\measuredangle BHC = 180° - A$ and $\measuredangle B_1HC_1 = 180° - A_1 = 180° - (180° - 2A) = 2A$. Hence

$$2A = 180° - A,$$

and $A = 60°$. Similarly, it follows that $B = C = 60°$ and that ABC is equilateral. So H is also the circumcentre of ABC. Since both ABC and $A_1B_1C_1$ have the same circumcircle, H must also be the circumcentre of $A_1B_1C_1$ as well. But, then the circumcentre and orthocentre of $A_1B_1C_1$ are coincident points implying that $A_1B_1C_1$ is equilateral.

One could have also reached this conclusion by computing the angles of $A_1B_1C_1$ with the help of the expressions

$$A_1 = 180° - 2A, \qquad B_1 = 180° - 2B, \qquad C_1 = 180° - 2C.$$

What if the centroid G of ABC is the incentre of $A_1B_1C_1$? Are both ABC and $A_1B_1C_1$ equilateral? Yes. To prove this, we use the fact that the incentre of $A_1B_1C_1$ is the orthocentre of ABC (a simple exercise for the reader). This shows that the centroid and the orthocentre of ABC are coincident, forcing it to be an equilateral triangle. Since $A = 90° - \frac{A_1}{2}$ we obtain $A_1 = 60°$ and similarly $B_1 = C_1 = 60°$, making $A_1B_1C_1$ equilateral too.

PRITHWIJIT DE is a member of the Mathematical Olympiad Cell at Homi Bhabha Centre for Science Education (HBCSE), TIFR. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. His other interests include puzzles, cricket, reading and music. He may be contacted at de.prithwijit@gmail.com.

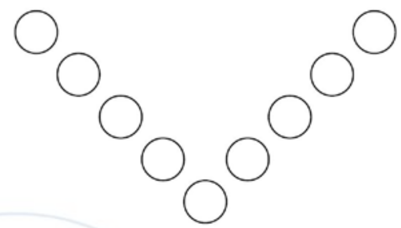
Can a problem in mathematics have a history? Can it be crowd-solved? The Klingens Lux problem – and its solution is a brilliant example of both these features. Read the account at https://azimpremjiuniversity.edu.in/ghosts-of-a-problem, try the dynamic geometry interactivity at http://dynamicmathematicslearning.com/Klingens-Lux-dynamic-proof.html (footnote 3 in the article) and get caught in the fascination of problem solving.

Summing V
a correction

HARAN MOULI

In the March 2020 issue of *At Right Angles*, as part of the 'Low Floor High Ceiling Tasks' series [1], the following problem was studied.

In how many ways can nine given numbers in arithmetic progression be arranged in a V shape such that the sums of the numbers on both the arms of the V are equal?



In our exploration of the problem, we noticed that the solution given in the article had an error. Here we provide a corrected count for the number.

Without loss of generality, we assume the 9 numbers in arithmetic progression to be the integers 1, 2, . . . , 9. We have:

$$1+2+\cdots+9=\frac{9\cdot 10}{2}=45.$$

If x is the number at the bottom of the V, then the sum of the numbers in each arm (excluding the centre) must be $(45-x)/2$. Clearly, x must be odd. So $x \in \{1,3,5,7,9\}$.

To start with, we ignore the actions of rearranging the numbers in the arms and mirroring the arms. To account for this, at the end we multiply by $2\cdot(4!)^2$.

The cases when 1 and 9 are at the centre may be matched 1 – 1 with each other, by replacing each number k by $10-k$, uniformly through the V. These two cases must therefore have the same number of possibilities. The same applies to 3 and 7. So we only need to focus on the cases when the central number is 5, 7 or 9.

When the central number is 9, the sum in each arm is $(45-9)/2 = 18$. Consider the pairs {1, 8}, {2, 7}, {3, 6} and {4, 5}. Each pair has the same sum, 9, which is half of 18. It follows that if

Keywords: Summing V

even one of these pairs stays intact (i.e., has both numbers on the same arm of the V), then all the pairs must stay intact. By fixing {1,8} on one arm, we have 3 choices for the other pair which must accompany it, thus making for 3 possibilities. Once this is done, there are no more choices possible. Thus there are 3 possibilities in which all the pairs stay intact.

Next, consider the case where 1 and 8 lie on different arms. The three other numbers on the same arm as 8 must add to 10. Since 3 + 4 + 5 > 10, the smallest number on that arm must be 2. The two remaining numbers on that arm must now add to 8. There is just one possible way of achieving this: 3 + 5. It follows that the numbers on the same arm as 8 are {2, 3, 5, 8}, which means that the numbers on the other arm are {1, 4, 6, 7}. Observe that with 1 and 8 on different arms, there is just one way of assigning the remaining numbers to the two arms.

So, with 9 at the centre of the V, there are a total of 3 + 1 = 4 possibilities. So there are also 4 possibilities with 1 at the centre of the V.

Next, consider the case when the central number is 7. The sum of the numbers on each arm is (45 – 7)/2 = 19. As the sum is odd, each arm must have an odd number of odd numbers. As there are four odd numbers available (namely, {1, 3, 5, 9}), one arm must have three odd numbers and the other arm must have one odd number. There are 4 ways of choosing 3 odd numbers from the collection {1, 3, 5, 9}. The choice 1, 3, 5 forces the fourth number on that arm to be 10, which is not admissible; so this choice is not available. The other three choices all lead to valid solutions:

Choice of 3 numbers	Solution for the V
{1,3,9}	{1,3,9,6} | {7} | {2,4,5,8}
{1,5,9}	{1,5,9,4} | {7} | {2,3,6,8}
{3,5,9}	{3,5,9,2} | {7} | {1,4,6,8}

This gives 3 possibilities each for 3 and 7 at the centre.

Finally, we consider the case when 5 is at the centre. The sum of the numbers on each arm is (45 – 5)/2 = 20. As the sum is even, each arm must have an even number of odd numbers. The odd numbers available are {1, 3, 7, 9}. We could have all the odd numbers on the same arm; this leads to a solution since 1 + 3 + 7 + 9 = 20. Else, we must have two odd numbers on each arm. If {1, 3} are on the same arm, then the other two numbers can only be {7, 9}, which leads to the solution already listed; so we do not consider this possibility. There are two other possibilities, and both lead to valid solutions. If {1, 7} are on the same arm, then the other two numbers can only be {4, 8}; the numbers on the other arm are then {3, 9, 2, 6}. Finally, if {1, 9} are on the same arm, the other two numbers being even, then the other two numbers can be {4, 6} or {2, 8}. This possibility thus leads to two valid solutions. It follows that with 5 at the centre, there are a total of 1 + 1 + 2 = 4 possibilities.

Our analysis thus yields a total of 4 + 3 + 4 + 3 + 4 = 18 possibilities.

Taking rearrangements into account, it follows that the number of ways of filling the V according to the required conditions is

$$2\cdot(4!)^2\cdot 18 = 20736.$$

References

[1] Math Space, "Summing V" from https://azimpremjiuniversity.edu.in/SitePages/resources-ara-vol-9-no-6-march-2020-summing-V.aspx

HARAN MOULI is a 11th grade student of the PSBB group of schools. A fervent math enthusiast, he has an keen interest in problem solving and the learning and discussion of mathematical concepts, with a particular fondness for number theory. He loves teaching and volunteers at Raising a Mathematician Foundation to guide high school students passionate about Mathematics. He may be contacted at mouliharan@gmail.com.

An Approach to Cubic Equations

GAURAV CHAURASIA

Suppose we have come to know one root of a cubic equation. What is the quickest way to find the other two roots? In this note, we present a formula for the other two roots. Let the given cubic equation be

$$ax^3 + bx^2 + cx + d = 0, \qquad (1)$$

where $a \neq 0$. Let its roots be u, v, w, and suppose that we have come to know one of them, say w. We derive here a formula for u and v in terms of w and the coefficients a, b, c, d.

Since u, v, w are the roots of the equation, we have

$$ax^3 + bx^2 + cx + d = k(x-u)(x-v)(x-w)$$

for some $k \neq 0$. Expanding the expression on the right and equating coefficients of like powers of x, we get:

$$ax^3+bx^2+cx+d = k\left(x^3 - (u+v+w)x^2 + (uv+vw+wu)x - uvw\right),$$

giving $k = a,\ -k(u+v+w) = b,\ k(uv+vw+wu) = c,$ $-k(uvw) = d$. Hence:

$$u+v+w = -\frac{b}{a}, \quad uv+vw+wu = \frac{c}{a}, \quad uvw = -\frac{d}{a}, \qquad (2)$$

so:

$$u+v = -\frac{b}{a} - w, \quad uv = -\frac{d}{aw}.$$

From these we get:

$$(u-v)^2 = (u+v)^2 - 4uv = \left(\frac{b}{a} + w\right)^2 + \frac{4d}{aw},$$

giving

$$u - v = \pm\sqrt{\left(\frac{b}{a} + w\right)^2 + \frac{4d}{aw}}.$$

Keywords: Cubic equations

From the expressions for $u+v$ and $u-v$, we get by addition and subtraction,

$$u, v = \frac{1}{2}\left(-\frac{b}{a} - w \pm \sqrt{\left(\frac{b}{a}+w\right)^2 + \frac{4d}{aw}}\right). \qquad (3)$$

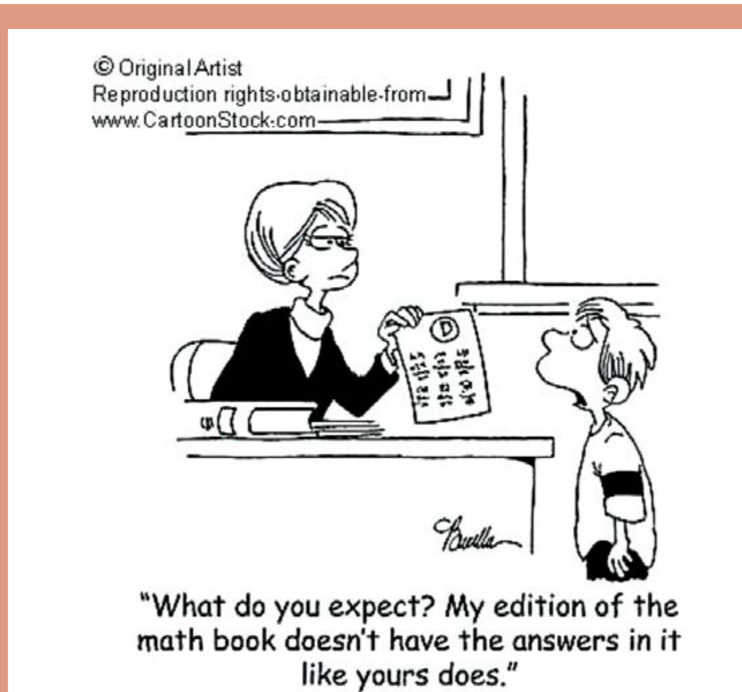
Thus we obtain the other two roots in terms of w and the coefficients of the equation.

Example. Consider the equation $x^3 + 2x^2 - 4x + 1 = 0$. One of its roots is $w = 1$ (check: $1 + 2 - 4 + 1 = 0$). Here we have $a = 1,\ b = 2,\ c = -4,\ d = 1,\ w = 1$. Therefore:

$$u, v = \frac{1}{2}\left(-2 - 1 \pm \sqrt{3^2+4}\right) = \frac{1}{2}\left(-3 \pm \sqrt{13}\right).$$

Acknowledgment: Thanks to my parents and Prof. Dr. Shailesh Shirali for the illuminating discussions on this topic and for inspiring me to work furthermore in this field.

GAURAV CHAURASIA is currently a student in class 10 in RPS Academy School, Deoria, Uttar Pradesh. He has a particular love for algebra, geometry, trigonometry, and number theory. He may be contacted at gauravvchaurasia@gmail.com.



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What do you think?
Should student textbooks have answers in them?

Two Problems in Number Theory - Part I

RAKSHITHA

In this two-part article, we study two number theory problems from the UK Math Olympiad, Round 2, years 2006 and 2003 respectively. We consider the first of these problems in Part I. Both problems were discussed during meetings of the problem-solving group of our school.

Problem 1. Let x and y be positive integers with no prime factors larger than 5. Find all such x and y such that

$$x^2 - y^2 = 2^k \tag{1}$$

for some positive integer k.

Solution 1. The problem asks us to find all x and y with no prime factors larger than 5 such that the difference of their squares is some power of 2. There are two things to be dealt with. The first is to find x and y. After that, we must select x and y such that they do not contain prime factors larger than 5. Let us start with the first task.

First, observe that $k \geq 2$. For, as k is a positive integer, 2^k is even, so x, y are both odd or both even. But in this case both $x + y$ and $x - y$ are even numbers, so their product is a multiple of 4.

In the analysis below, we will assume that both x and y are odd. There is no loss of generality in doing so, because if x and y are both even, we can replace them by $x/2$ and $y/2$ respectively (and replace k by $k - 2$), and we can continue doing this until both x and y are odd.

We shall now make use of the fact that a divisor of a power of 2 must itself be a power of 2; no other prime can enter into the factorisation. We have:

$$x^2 - y^2 = 2^k,$$
$$\therefore \quad (x + y)(x - y) = 2^k.$$

Keywords: Diophantine problem, prime number, greatest common divisor

Therefore we must have

$$x + y = 2^a, \quad x - y = 2^b, \tag{2}$$

for some integers a, b, where $a > b \geq 1$. (Note that $a > 1$.)

Solving this pair of simultaneous equations for x and y, we get:

$$x = 2^{a-1} + 2^{b-1},$$
$$y = 2^{a-1} - 2^{b-1}.$$

Since x, y are odd, we must have $b = 1$. Hence:

$$\begin{cases} x = 2^{a-1} + 1, \\ y = 2^{a-1} - 1. \end{cases} \tag{3}$$

From the above, we see that x and y are a pair of consecutive odd numbers.

Having obtained expressions for x and y, we now focus on the requirement that x and y should not be divisible by primes larger than 5.

The only prime factors available are 3 and 5, as x and y are odd. But as $x - y = 2$, one out of x and y must have only 3 as a prime factor, and the other one must have only 5 as a prime factor. That is, one out of x and y is a power of 3, and the other is a power of 5. So the following two cases arise.

Case 1: $x = 3^m$ and $y = 5^n$ for some positive integers m, n. This means that

$$3^m = 2^{a-1} + 1,$$
$$5^n = 2^{a-1} - 1.$$

The second relation is not possible for $a > 2$, as the quantity on the right side is $-1 \pmod 4$ for $a > 2$, whereas the quantity on the left side is $1 \pmod 4$ for any positive integer n.

Hence $a \leq 2$. Since $a > b \geq 1$, we get $a = 2$, and therefore $b = 1$.

Thus, in this case we get $x = 2^1 + 1 = 3$ and $y = 2^1 - 1 = 1$. Note that this corresponds to the solution $3^2 - 1^2 = 2^3$.

Case 2: $x = 5^m$ and $y = 3^n$ for some positive integers m, n. This means that

$$5^m = 2^{a-1} + 1,$$
$$3^n = 2^{a-1} - 1.$$

The second relation is not possible for $a > 3$, as the quantity on the right side is $-1 \pmod 8$ for $a > 3$, whereas the quantity on the left side is $1 \pmod 8$ for $n \geq 2$. (The possibility of $n < 2$ does not arise if $a > 3$, as we would then have $3^n < 2^{a-1}$.) Hence $a = 2$ or $a = 3$.

If $a = 2$, then we get $x = 3$, $y = 1$; but this is the same as the solution obtained above.

If $a = 3$, then we get $x = 5$, $y = 3$. Note that this corresponds to the solution $5^2 - 3^2 = 2^4$.

Recall that we had taken x and y to be odd, by dividing by 2 as often as needed. Replacing these factors, we see that the solutions of the given equation are of the following forms:

$$\begin{cases} x = 3 \cdot 2^k, & y = 2^k, \\ x = 5 \cdot 2^k, & y = 3 \cdot 2^k, \end{cases} \tag{4}$$

where k is any non-negative integer.

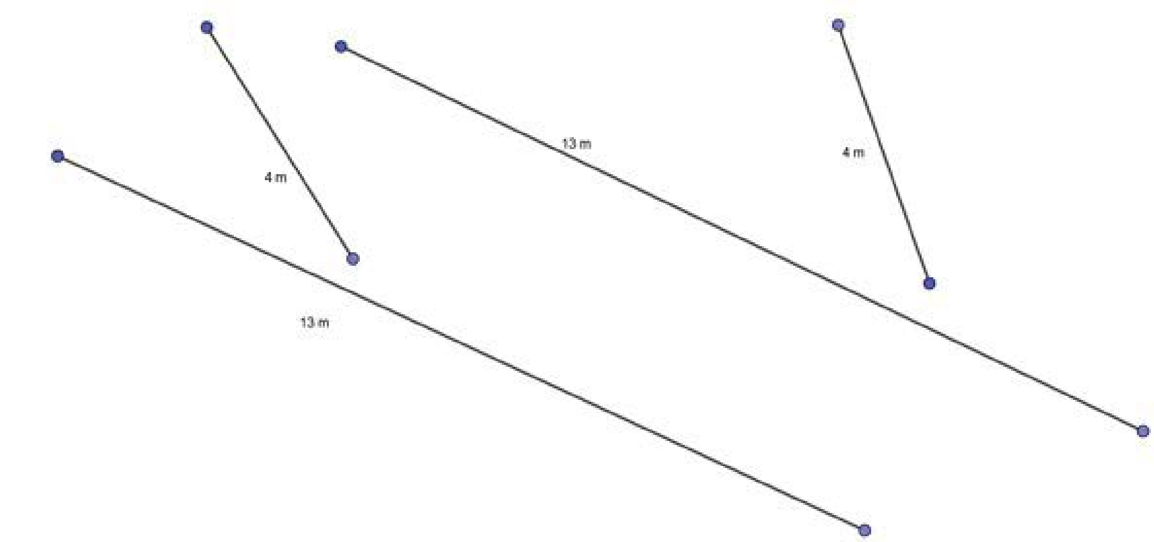
References

1. https://www.ukmt.org.uk

RAKSHITHA is a math enthusiast studying in class 12 at The Learning Centre, PU College, Mangalore. She has a deep interest in Analysis, Number Theory and Combinatorics. Other than Mathematics, she is fascinated by the sciences and wishes to pursue pure research in the future. She participated in the Indian National Mathematics Olympiad in 2019 and 2020. She may be contacted at krraja1974@rediffmail.com.

More Space!

What is the maximum area you can cover if you have four fences: two of them 4m long and two of them 13m long? Is there a unique quadrilateral? Justify.

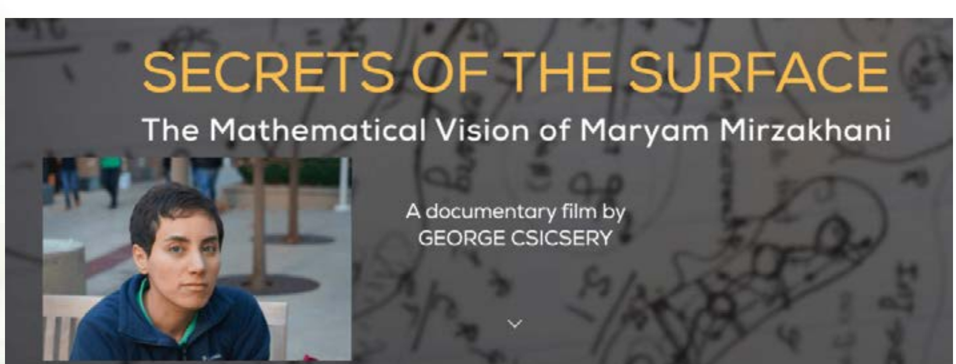


Review

Review of the Film *Secrets of the Surface* by George Csicsery, 2020

Keerthi Mukunda & Kamala Mukunda

If you are a young girl who enjoys mathematics, what are the chances that you will choose to study this subject beyond school, beyond college? What are the chances that you will end up as a woman mathematician? Sadly, very low. There may be a few different reasons for this, one of which is also true for boys: most young people are advised to choose subjects based on career paths, not on whether the subject excites and enthralls them! But there is another very unfortunate reason: many of us, including girls and women, believe that female brains are less capable of learning mathematics than male brains! This deep-seated stereotype is widespread in the world, and we call it a stereotype because there is simply no convincing evidence that the female brain is inherently less capable of some kinds of learning and thinking. On the contrary, plenty of research with babies shows that girls and boys are indistinguishable in their cognitive capacities. But the problematic data is that at various levels of education, girls perform less well than boys on high-stakes math tests, and of course there are the enrolment numbers that tell the story of the 'leaky pipeline': as you go higher and higher in mathematics degrees, you find fewer and fewer women. Studies suggest that the main reason for females dropping out along the way is that most of



us have this picture that *math is for men*. It seems that socialisation is the source of the problem, not innate ability.

Several articles and books have been written on this topic, and a few resources are listed at the end of this piece. It is very important for us teachers of mathematics to realise that we may implicitly have this belief ourselves, and to watch for signs of it guiding our actions. We should also think of ways to offer counter evidence for the stereotype, and this is why the film *Secrets of the Surface* is a good one to share with your students.

When you and your students watch *Secrets of the Surface*, the 2020 documentary film by George Csicsery about the renowned Iranian mathematician Maryam Mirzakhani, who lived from 1977-2017, you will be lifted off into another world: a world of culture, mathematics, and excitement around discoveries in geometry. The film traces her story of growing up in Tehran, going to school, developing an interest in math, shooting to fame and recognition, journeying to the USA, working at Harvard, Princeton and Stanford Universities, culminating in her being the first woman ever to win the prestigious Fields medal for mathematics. She died tragically from cancer at the young age of 40.

One will be amused to hear in the film's voiceover (in Maryam's own voice), that she was initially not interested in math! She enjoyed reading and writing novels and wanted to become a writer. Through middle school and high school though, she learned to think mathematically, succeed at many problems, and was chosen to study in a special school for talented girl students. Interestingly, the film explains how in Iran, promising students of both sexes are encouraged to pursue math. Studying in a girls-only school would have avoided the comparisons between boys and girls that feed into gender-biased behaviour and stereotypes. Indeed, the film makes clear that in Iran there was no stereotype that girls couldn't do math or that women were discouraged to pursue it at higher study. It says in the film that 50% of those students going into mathematics at higher levels, are girls!

Through the film, it is a compelling experience to watch and listen to many young Iranian women or now-grown classmates of Maryam's talk passionately about the subject and the inspiration Maryam had given them in their lives. Thus it is a film also about women in the field of mathematics and gives a glimpse of girls being educated in Tehran and Isfahan. Maryam weaves her way through awards, competitions, and even won the gold medal at the national level Olympiad. No girls had won this before her.

The film has frequent images of beautiful buildings, artistic walls, intricate carvings, courtyards and spaces from Iran; in a sense math is everywhere. You will be intrigued by the concepts and geometric theorems being proposed. Through the many voices carrying you in the film, you feel the excitement of those embedded in a discovery. The film does a good job of explaining something very complex in such a way that you feel you got a glimpse of its depth. Students of mathematics may baulk at the idea of giving a hard problem more than an hour or so of their effort, but Maryam had to persevere for nearly two years at a beautiful and challenging problem, before the solution came to her! It was well worth the wait.

In her country, Maryam was a symbol of hope and inspiration to many young school-going children especially girls, and was even solicited for a political role! On large posters she was represented without a scarf because that was how she came to be known beyond her country. It is interesting that someone who excelled in a particular discipline was treated as a national hero in the general context as well. That such a person could come from the heart of a war-torn country will always be a source of national pride for Iran.

After her passing, people felt such a loss and sadness as if they had lost a good friend, a colleague and an incredibly inspiring person. Despite her genius and vision, her friends and family describe her as a modest, collaborative, warm person, and quiet about her achievements. They felt she had a whole lifetime of mathematical work that could have been accomplished if she had lived on. However, some voices in the film do say that what she did contribute was already immense, and that there are gems in her papers that will yield mathematical problems for years to come. Maryam had an instinct for framing deep and profound mathematical questions that will long outlive her.

Secrets of the Surface is a memorable film and one that opens doors in our minds: mathematics is an exciting vocation, and girls *can* become mathematicians. After watching the film, you can have a discussion in class around these points, including the question of whether one must be a genius to enjoy and fruitfully study higher math! On the contrary, at all levels, students of mathematics can partake of the kind of excitement that Maryam had, the way patterns fall into place and the beauty of numbers and shapes. In the end, maybe we could think of Maryam less as a 'prodigy' and more as a bright spark in the world of math, a spark that was encouraged to burst into a beautiful, light-giving flame.

1. Nora S Newcombe (2010). Picture This: Increasing Math and Science Learning by Improving Spatial Thinking. *American Educator*, Summer 2010 (29-43)
2. Kamala V Mukunda (2019). Ch 6: Untapped Potential. From *What Did You Ask at School Today? Book Two*: Harper Collins India.

The authors are both teachers at Centre For Learning. Keerthi teaches English and social science for middle and high school age groups, and is interested in curriculum development and classroom practices. Kamala teaches mathematics and psychology, and has written two books—*What Did You Ask at School Today* (Books One and Two: HarperCollins India)—summarizing psychological findings of relevance to educators and parents.

Review: Ganitmala

By Swati Sircar

Ganitmala (Figure 1) is a powerful manipulative to develop number sense. It was introduced in India by Jodo Gyan[1] and later picked up by many resource organisations working in primary math. It can be easily made by threading 100 beads as follows: get 50 beads in one color (say white) and 50 more beads in a contrasting color (say blue); thread the beads in groups of 10 in alternating colours, i.e., 10 white, 10 blue, 10 white, 10 blue, etc. Check reference for more details especially for the accessories. It models the number line and allows a lot of exploration with the numbers 0-100 including comparing numbers and all four operations – addition, subtraction, multiplication and division.

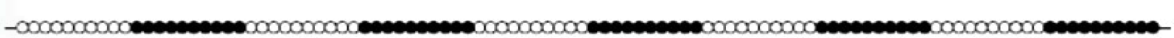


Figure 1

It is a proportional manipulative since 10 beads represent a 'ten' while a single bead represents a 'one'. In a sense, it is pre-grouped[2], since the colours alternate for every 10 beads. On the other hand, each bead can be part of a 'ten' or be considered a 'one' depending on the number. So, it also has the advantages of groupable[3] materials. This makes ganitmala a powerful manipulative with some unique features. But before that, we need to discuss some dos and don'ts.

Since it models the number line, the zero should be on the left. So, both the teacher and the students should be on the same side of the mala to avoid left-right confusion. Second, each number is placed in between beads and it indicates how many beads are on the left (Figure 2). So, in a way each bead represents an interval

2 Other pre-grouped materials include static beads, Diene's blocks and Flat-Longs-Units (2D base-10 blocks)

3 Other groupable materials include bundle and stick where each stick can be used as a 'one' or be included in a bundle or 'ten'.

(0, 1), (1, 2), etc. Now for a 2-digit number, this automatically puts the 'tens' to the left and the 'ones' to the right. For example, the three tens of 31 are on the left and the one is on the right (Figure 3). This directly correlates with how we write a 2-digit number, i.e., TU and can help young children learn that 31 is 3 tens and one (and not 3 ones and ten). Moreover, it helps integrate the ordinal and the cardinal aspects of numbers.

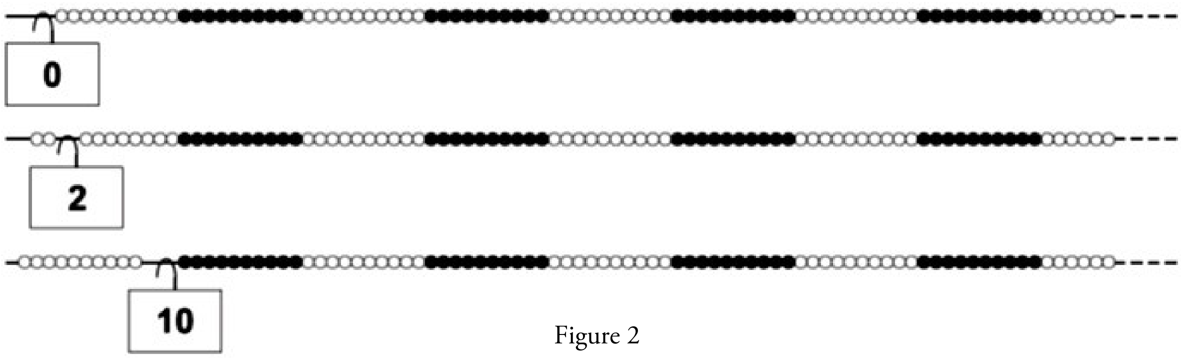


Figure 2

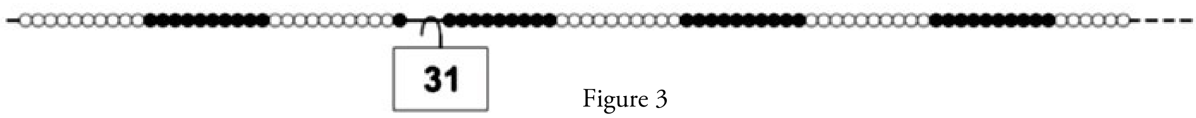


Figure 3

Ganitmala helps to transition to the open or empty number line where the order of the numbers is maintained but the distances between them are not scaled. It helps students find multiple strategies to add and subtract numbers < 100. Thus, it provides a lot of opportunity to play with numbers and develop number sense before getting into standard algorithm.

For multiplication and division (Figure 4), it is a good idea to use catchers, which can also be made locally. The reference includes details and diagrams. The mala can be used even for the division algorithm for HCF! This was discovered by a govt school teacher whose imagination was sparked by this manipulative.

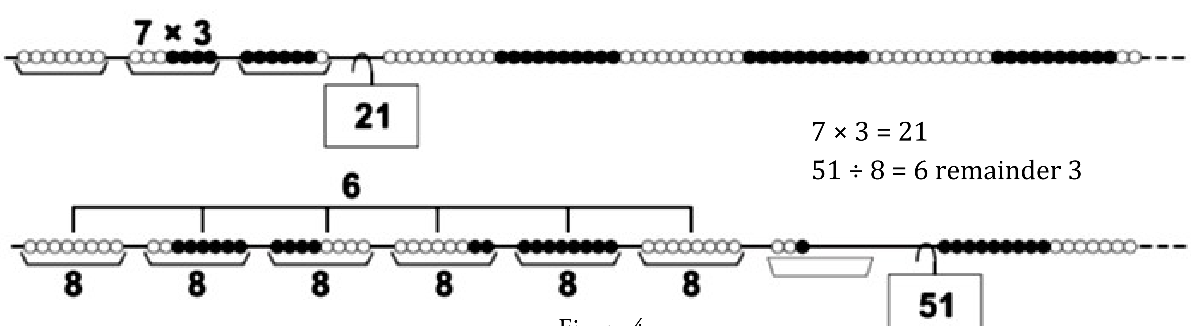


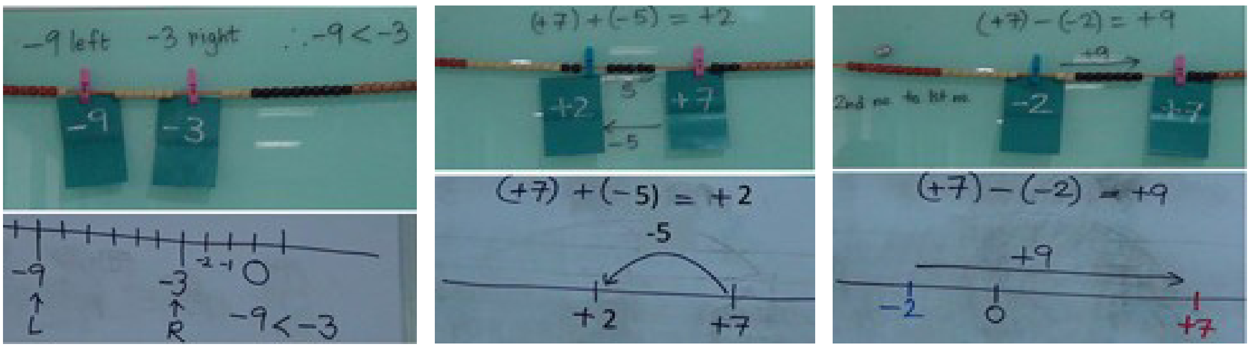
Figure 4

However, the standard ganitmala is limited to 100. Some can use a 200-bead mala with 4 colours (see p.2 of reference – double Ganitmala). There are 1000 bead malas also. But most classrooms won't have adequate space to hang such a long mala. Instead it is a better idea to transition to open number line for numbers > 100.

Just as the number line stretches to the negative side, similarly, the ganitmala also 'doubles' for integers. A 200-bead mala is used for that. It is essentially two 100-bead malas (with different colours) joined.

The 100 beads on the left represent the negative part of the number line, while the remaining model the positive part as before.

This is very good for introducing integers (ideally with a story), comparison and addition-subtraction of integers (Figure 5).



Comparing integers | Adding integers | Subtracting integers

Figure 5

Moreover, it can be used to solve a wide range of linear equations in single variable, eg. $(4 - x)/3 = 5$. In fact, it is safe to say that it can be used to solve any such equation as long as the variable appears only once, and the solution is an integer! The reference includes links to such details.

Reference: http://teachersofindia.org/en/article/making-ganitmala

SWATI SIRCAR is Assistant Professor at the School of Continuing Education and University Resource Centre of Azim Premji University. Math is the 2nd love of her life (1st being drawing). She is a B.Stat-M.Stat from Indian Statistical Institute and an MS in math from University of Washington, Seattle. She has been doing mathematics with children and teachers for more than a decade and is deeply interested in anything hands on - origami in particular.

A Geometry Problem from the Putnam 2019 Competition

RAKSHITHA &
P N SUBRAHMANYA

In this article, we study a geometry problem adapted from the Putnam exam of 2019. (The William Lowell Putnam Mathematical Competition or the 'Putnam Competition' is an annual mathematics competition for undergraduate college students enrolled at institutions of higher learning in the United States and Canada.)

Problem. In triangle ABC, let G be the centroid and I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB, and $\tan \beta/2 = 1/3$. Find α.

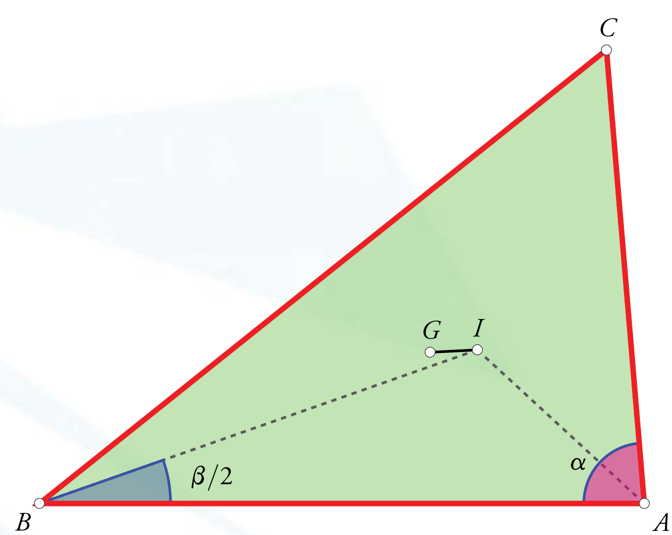


Figure 1.

Keywords: Putnam, triangle centres, angles, similar triangles, trigonometry, ratios

Solution. We use approaches from coordinate geometry, trigonometry and pure geometry and argue as follows.

- We are told (see Figure 1) that $\tan \beta/2 = 1/3$. From this it follows that
$$\tan \beta = \frac{2 \tan \beta/2}{1 - \tan^2 \beta/2} = \frac{2/3}{1 - 1/9} = \frac{3}{4}.$$
- We start by assigning coordinates as follows: $B = (0,0)$, $A = (a,0)$. Note that AB lies on the x-axis.
- As $\tan \measuredangle CBA = 3/4$, we may fix the scale of the coordinate axes so that $C = (4,3)$.
- Using this, we find that the y-coordinate of the centroid G is 1.
- Since $IG \parallel AB$, it follows that the y-coordinate of I too is 1.
- Since $\tan \measuredangle IBA = 1/3$, it follows that the x-coordinate of I is 3; so $I = (3,1)$.
- Since the distance from I to AB is 1, it follows that the radius of the incircle is 1.

We may now follow two possible approaches.

First approach: We use the formula connecting radius of the incircle and area of the triangle:
$$\text{Radius of incircle} = \frac{\text{Area of triangle}}{\text{Semi-perimeter of triangle}}.$$

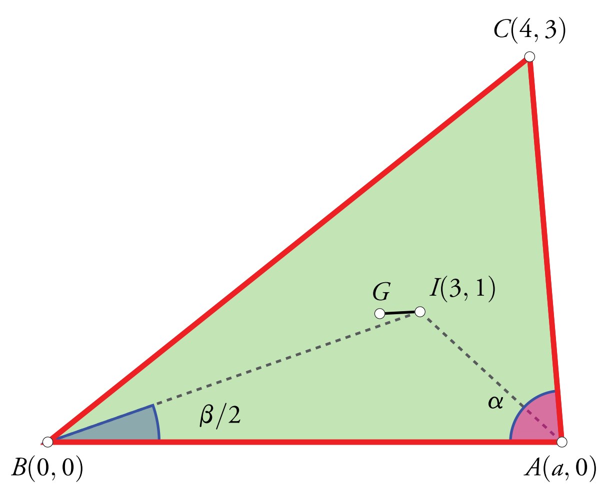


Figure 2.

Here we have (see Figure 2):
$$\text{Area} = \frac{3a}{2},$$
$$\text{Perimeter} = a + 5 + \sqrt{(a-4)^2 + 3^2},$$
$$\text{Radius} = 1.$$

Hence:

$$a + 5 + \sqrt{(a-4)^2 + 3^2} = 3a,$$
$$\therefore \quad (2a-5)^2 = (a-4)^2 + 3^2,$$
$$\therefore \quad 3a^2 = 12a,$$

giving $a = 4$. (The solution $a = 0$ is not meaningful.) Hence $A = (4, 0)$. Since $C = (4, 3)$, it follows that $CA \perp AB$. Thus $\alpha = 90°$. □

Second approach: Here we think geometrically rather than algebraically.

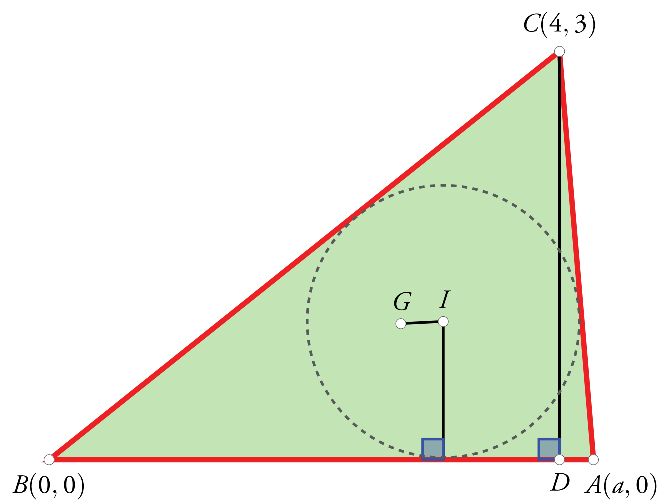


Figure 3.

Draw a perpendicular CD from C to AB. Also draw the incircle (centre I, radius 1) of triangle ABC. Since the x-coordinate of I is 3, and line CD has equation $x = 4$, it follows that the incircle touches CD. But it also touches line CA, by definition of an incircle. This means that both CD and CA are tangent to the incircle. Therefore they coincide, which means that D coincides with A. This implies that $\measuredangle CAB$ is a right-angle, i.e., $\alpha = 90°$. □

References

1. 2019 William Lowell Putnam Mathematical Competition Problems, https://www.maa.org/sites/default/files/pdf/Putnam/2019/2019PutnamProblems.pdf

RAKSHITHA is a math enthusiast studying in class 12 at The Learning Centre, PU College, Mangalore. She has a deep interest in Analysis, Number Theory and Combinatorics. Other than Mathematics, she is fascinated by the sciences and wishes to pursue pure research in the future. She participated in the Indian National Mathematics Olympiad in 2019 and 2020. She may be contacted at krraja1974@rediffmail.com.

P N SUBRAHMANYA has been teaching mathematics at the high school and pre-college level at The Center for Advanced Learning, Mangalore for the past 10 years. He is keenly interested in promoting non-routine problem solving among his students, particularly students of rural areas. His other interests include history and Kannada literature, especially poetry. He may be contacted at subrapadekal17@gmail.com.

Secondary

Problem Corner

Ghosts of a Problem Past

MICHAEL DE VILLIERS (MdV) & HANS HUMENBERGER (HH)

Recently a student of one of us (HH) with the surname Lux[1] brought the following interesting problem to class:

Let c_1 and c_2 be two circles intersecting in A and B. Let a straight line be drawn through A, different from AB, intersecting the two circles in M and N (these being the intersection points different from A). Let K be the midpoint of MN, P the intersection point of the angle bisector of ∠MAB with c_1, and R the intersection point of the angle bisector of ∠BAN with c_2. (We take angles to be 'non-oriented.' That is, they lie between 0° and 180°.) Prove that ∠PKR = 90° (see Figure 1).

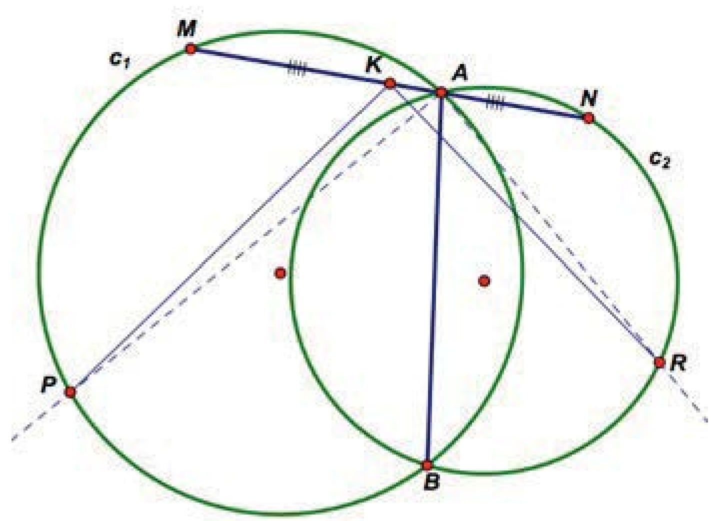


Figure 1: Lux problem

1 The student got the problem from his grandfather, a retired mathematics teacher from France, but without a solution.

Keywords: Intersecting circles, angle bisectors, visualisation, proof

A dynamic, interactive sketch of the Lux problem is available at: http://dynamicmathematicslearning.com/lux-problem.html

The reader is invited to first explore the problem and attempt to prove it before continuing.

Though the problem may be solved using inversion or coordinate geometry, a pure geometry solution proved elusive to find. Despite its elementary appearance, the problem was deceptively hard and resisted several different approaches.

The problem was then shared with MdV who first attempted to prove it using theorems from circle geometry (e.g. trying to prove that quadrilateral *KPRA* is cyclic, etc.), but with no success. It should also be mentioned that though the problem appeared vaguely familiar, MdV was unable initially to make a connection with a past problem, to which we'll come back later. The Lux problem was subsequently shared with several others including a colleague, Waldemar Pompe (WP), from the University of Warsaw, Poland.

After a while, WP came back with a straightforward solution, pointing out that the Lux problem was merely a special case of the following little known but interesting hexagon theorem (see Pompe, 2016, p. 28-29)[2]:

Given a hexagon ABCDEF with AB = BC, CD = DE and EF = FA, and angles α, β, γ *such that* $\alpha + \beta + \gamma = 360°$, *then the respective angles of* $\triangle BDF$ *are* $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ *(see Figure 2).*

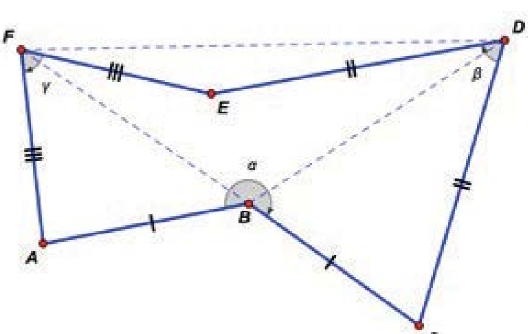


Figure 2: Pompe's Hexagon Theorem

Proof 1 of the Lux problem, using the hexagon theorem

In Figure 1, *AP* and *AR* are the respective angle bisectors of ∠*MAB* and ∠*NAB*, so the points *P* and *R* respectively bisect the arcs *MPB* and *NRB*; hence *MP* = *PB* and *BR* = *NR* (see Figure 3). Furthermore, it follows from the given that ∠*MPB* + ∠*BRN* + ∠*MKN* = 360°. Therefore, the conditions of Pompe's hexagon theorem are met for hexagon *PBRNKM* (at *K* there is a 180° angle!), and it follows that ∠*PKR* = 90°.

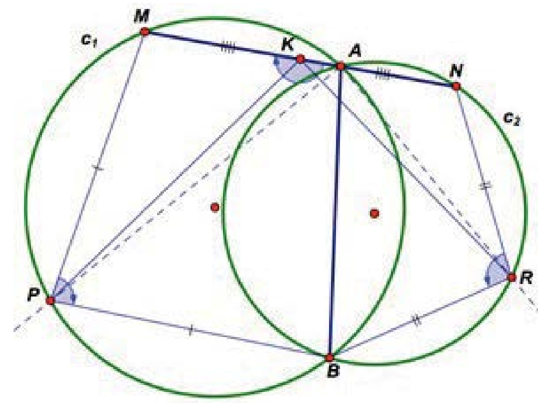


Figure 3: Pompe's Hexagon Proof

Further reflection on Pompe's hexagon theorem reminded MdV of an earlier paper (De Villiers, 2017) involving the sum of rotations, and led to the following proof of the Lux problem.

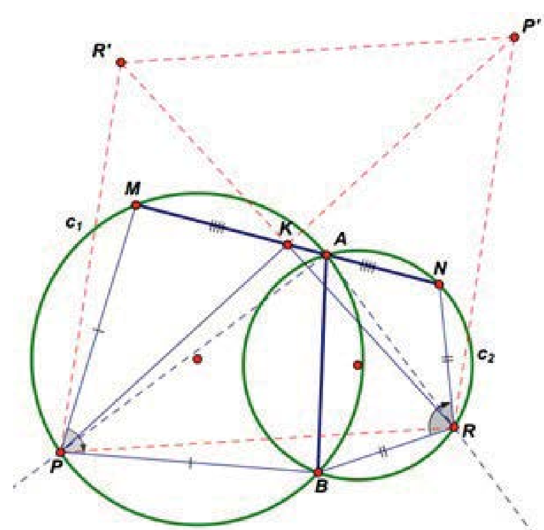


Figure 4: Sum of Two Rotations Proof

2 A proof of Pompe's Hexagon theorem is given in the Appendix. The theorem can also be interactively explored by the reader at: http://dynamicmathematicslearning.com/pompe-hexagon-theorem.html

Proof 2 of the Lux problem, using rotations

Consider Figure 4. Note that a clockwise rotation through $\angle MPB$ of M around P, maps M onto B, and that a clockwise rotation of B through $\angle BRN = 180° - \angle MPB$ around R, maps B onto N. Therefore, the sum of these two rotations is equivalent to a rotation of 180° around the midpoint K of MN.

But a counter-clockwise rotation of $\triangle PBR$ through $\angle BPM$ around P and a clockwise rotation of $\triangle PBR$ through $\angle BRN = 180° - \angle BPM$ around R, produces a quadrilateral $R'PRP'$. But since angles BPM and BRN are supplementary, and $R'P = RP = RP'$ from the construction, it follows that $R'PRP'$ is a rhombus.

Since $\triangle PMR'$ is congruent to $\triangle P'NR$ from the earlier rotation of $\triangle PBR$, we now rotate $\triangle PMR'$ through a half-turn (180°) around the midpoint of MN, namely, K, to map onto $\triangle P'NR$ with $M \to N$, $P \to P'$ and $R' \to R$. But since $R'PRP'$ is a rhombus, the only half-turn which will map $P \to P'$ and $R' \to R$ is the one around the "centre" of the rhombus (i.e. intersection point of its diagonals). Therefore K must be this centre of the rhombus, and it follows that $\angle PKR = 90°$. (Comment: The proof by Sjoerd Zondervan given in Lecluse (2012) also utilizes the construction of a rhombus, and is very similar to the two rotations proof given here, though not identical.)

Having left the Lux problem for a while before coming back to it later, MdV was reminded of a problem posed by Dick Klingens from the Netherlands at the NVvW annual meeting in November 2011. The problem and several solutions to it were published in the March 2012 issue of *Euclides* (Lecluse, 2012). To our (MdV & HH) surprise, this Klingens problem was identical to the Lux problem!

Ironically, when MdV came across the article by Lecluse during 2012, MdV managed to rather quickly produce an alternative proof involving the nine-point circle, and showing that the problem was really just a special case of a generalization of Van Aubel's theorem involving similar rectangles on the sides (De Villiers, 2013). An interactive, dynamic sketch was also created by MdV in 2013, and was posed as a challenge to mathematically talented students at (with links to relevant papers): http://dynamicmathematicslearning.com/vanaubel-application.html

However, despite this, MdV had completely forgotten about this and did not make the connection until much later. After all, in the process of re-investigating the Klingens-Lux problem, another alternative proof was produced, and it was therefore nonetheless quite productive to revisit these 'ghosts of a problem past.'

Proof 3 of the Lux problem, using angular motion

What follows is a *dynamic* proof because it relies on the *motion of points* (a circle as their orbit and their angular velocity). We found related ideas also in Goddijn (2012) and now we will present a sort of mixture of our and Goddijn's ideas, so that the proof is as short and clear as possible. Our own way had more steps and was more complicated to communicate, but with the help of Goddijn things become more straightforward. This proof will be longer than proofs 1 and 2, but the beauty of this proof lies in its use of *dynamic* issues.

To prepare for the proof we need two lemmas.

Lemma 1: As the point M moves on c_1 the point N moves on c_2 with the same angular velocity.

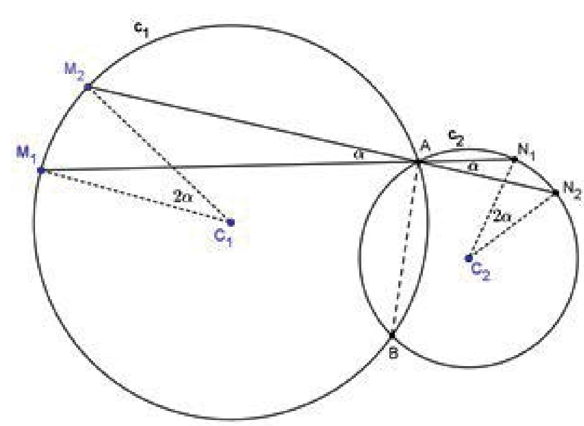


Figure 5: Equal angular velocities of M and N

The proof follows immediately from Figure 5 and the inscribed angle theorem.

Lemma 2: If two points M, N are moving on circles (centres C_1 and C_2) with equal angular velocities, then also their midpoint K moves on a circle with the same angular velocity, and the centre of this circle is the midpoint D of C_1 and C_2 (see Figure 6).

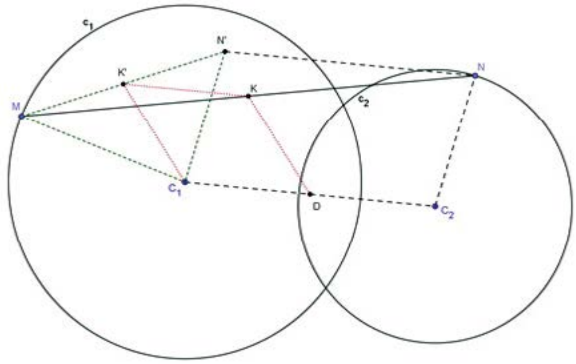


Figure 6: The orbit of *K* is a circle centred at *D*

In Figure 6 the point N' is chosen such that C_1C_2NN' is a parallelogram and K' is the midpoint of MN'. Since M and N have the same angular velocity, the angle MC_1N' in the triangle MC_1N' is fixed; in other words the motion of the triangle MC_1N' is a rotation around C_1 with the angular velocity of M on c_1. Because DC_1 and KK' are parallel and equal, we can conclude that C_1DKK' is a parallelogram, too. And from the fact that K' makes a rotation around C_1 it follows that K performs a rotation around D with the same angular velocity, and this completes the proof of Lemma 2.

Now, we know that P and R are moving on their circles with *equal* angular velocity, because their angular velocity is *half* the angular velocity of M and N (P and R come from the angle bisectors!) and M and N do have equal angular velocity (see above). And with Lemma 2 it follows that also the midpoint L of P and R executes a rotation around D with the same angular velocity as P and R (half the one of K; see Figure 7). It is clear that the circle of K passes through A and B, because A is a possible position of K, and we will shortly show that the circle of L passes through C_1 and C_2. Altogether this means in the end: When M, N, K make a full turn around their respective centres, the points P, R, L make a half turn. And for a full turn of P, R, L the points M, N, K need two full turns.

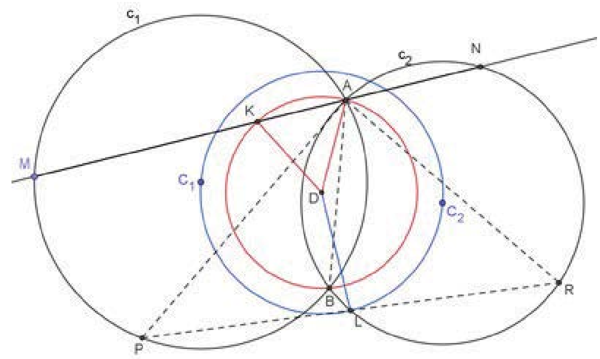


Figure 7: *K* and *L* lie on concentric circles centred at *D*

What happens if M is rotated counter clockwise towards B? Then M, P, N, and K coincide in B (Figure 8[3]).

In the situation $M = B$[4] it is clear that L coincides with C_2 because $\angle PAR = 90°$. In Figure 8b we see that $\angle KDA = 2 \cdot \angle LDA$ (LD is perpendicular to the chord AB and an angle bisector of $\angle BDA = \angle KDA$) and the line segment AD stays fixed during all the motions of the points M, N, K, L, P, R. Since the point K always has the double angular velocity of L, it is clear that during the motions the relation $\angle KDA = 2 \cdot \angle LDA$ always holds. For instance, look again at Figure 7 (so

3 An interactive webpage illustrating the corresponding motions of this dynamic proof is available at: http://dynamicmathematicslearning.com/Klingens-Lux-dynamic-proof.html

As it is really instructive to see the dynamic movement, applets in *GeoGebra* & *SketchPad* can also be downloaded at the above page, or directly from: http://dynamicmathematicslearning.com/klingens-lux-dynamic-proof.zip

4 In this particular situation–which is actually excluded, see page 1–the points K and P coincide, therefore the angle PKR is not defined (and, of course, not right). Another exceptional position of M is $M = A$, because in that situation the line from M to A is not uniquely defined. In our *dynamic* approach it is quite natural to take the tangent of c_1 at A in this situation, so to speak we take the limit of MA (secant) as $M \to A$ (tangent).

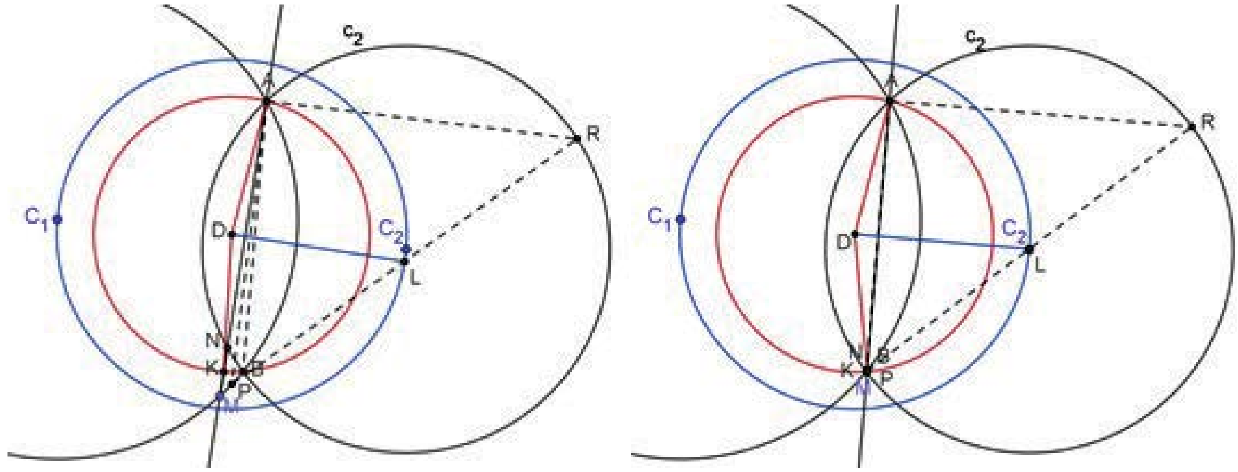


Figure 8: Situation of $M \to B$, shortly before and M = B

to speak we reverse the motions that led from Figure 7 to Figure 8): $\angle LDA \approx 155^\circ$ and $\angle KDA$ is exactly the double. And therefore, *DL* lies on the perpendicular bisector of *AK*, and we can conclude that $KL = AL$ and with Thales' theorem $\angle PKR = 90^\circ$ follows.

Proof 4 of the Lux problem, using similar triangles

Here we present a proof for the Klingens-Lux problem based on the idea by Just Bent (2012). It cleverly makes use of similar triangles in a short and elegant way. To prepare that proof we first formulate the following.

Lemma 3: Let *ABC* be a right triangle, and *CDE* and *BFG* congruent right triangles similar to *ABC* such that $\triangle CDE$ and $\triangle BFG$ are *translations* from each other, i.e., their sides are pairwise parallel. Then the triangle *AFE* is also a right triangle similar to *ABC* (see Figure 9).

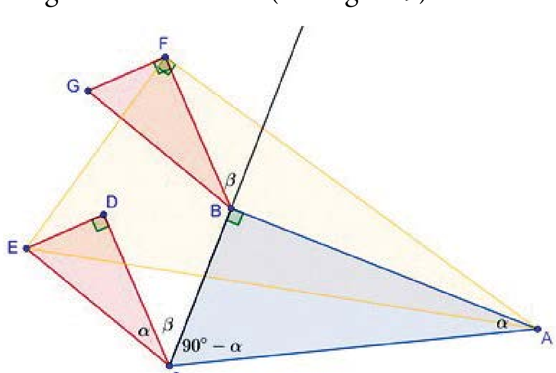


Figure 9: Four similar right triangles

For a proof of Lemma 3, let $\angle CAD = \alpha$ and observe that $\triangle AFB$ and $\triangle AEC$ are similar (they have equal angles at *B* and *C*, namely, $90^\circ + \beta$, and the ratio of the adjacent side lengths is equal: $AB : AC = k = BF : CE$). Therefore, also $AF : AE = k$ and $\angle EAF = \alpha$ hold, and thus the claimed similarity is proven.

And in the Klingens-Lux problem one just has to use this lemma a single time (see Figure 10). In the retrospect things often seem to be very simple, but to *find* these simple relations is sometimes not simple at all; the Klingens-Lux problem is definitely a really hard problem from the perspective of a solver!

Now, for the proof consider Figure 10. Let *H* be the midpoint of *BM, J* the midpoint of *BN*, and $\alpha = \angle BRJ$. Then *KJ* is parallel to *BH* and equal (intercept theorem), all the other angles marked with α, $90^\circ - \alpha$, $180^\circ - 2\alpha$, 2α can easily be derived (cyclic quadrilaterals). And using the above Lemma 3 it follows immediately that $\angle PKR = 90^\circ$.

We have presented here several purely geometric proofs that use completely different means. Proof 1 uses a fairly unknown hexagon theorem, proof 2 uses the fact that the sum of two rotations is a rotation again (the rotation angles add up!)–therefore, this proof could be called a *transformation proof*, proof 3 uses dynamic arguments of motion and could be called a *dynamic proof*, proof 4 uses similar triangles in

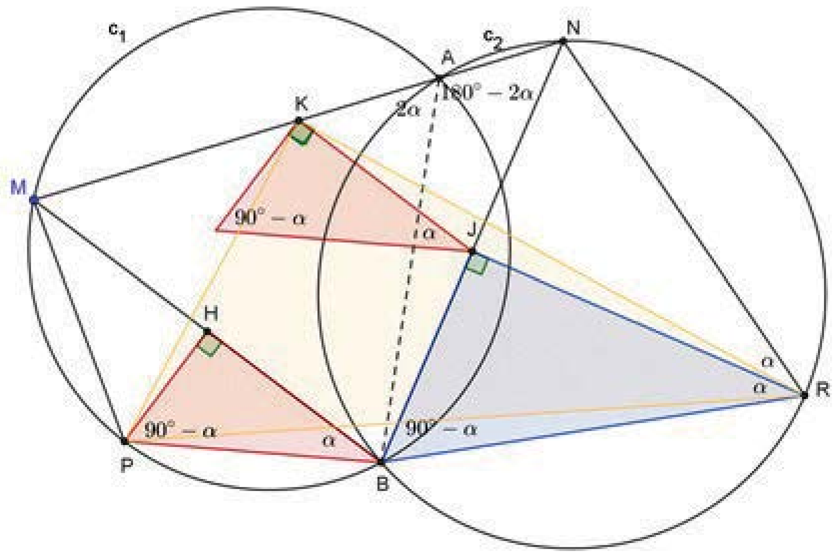


Figure 10: Proof of the Klingens-Lux problem with similar triangles

a smart way, it could be called a *similarity proof*. And still there are many other proofs (see Lecluse 2012 and the references below), so that this problem is a really "rich" one–but one has to admit: quite hard to solve!

Moreover, each of the proofs sheds light in a different way on *why* the result is true; i.e. *explaining* it in a different way. In case of Proof 3 (dynamic) one could also mention the *discovery* function of proof; it was discovered that the points K and L always lie on special circles. This not only illustrates the value of having different proofs for the same result, but also once again, that the value of proof goes far beyond merely that of *verification/conviction*, and that ultimately in mathematics, understanding and insight count for much more.

References

1. Bent, J. (2012). Accessed 29 March 2020 at: http://dynamicmathematicslearning.com/further-proofs.zip
2. De Villiers, M. (2012). Another proof of De Opgave 2011: An application of a Van Aubel generalization. *Euclides* (Journal of the Nederlandse Vereniging van Wiskundeleraars), Desember, 88(3), p. 133.
3. De Villiers, M. (2017). An explanatory, transformation geometry proof of a classic treasure-hunt problem and its generalization. *International Journal of Mathematical Education in Science and Technology*, Volume 48, No. 2, pp. 260-267. DOI: 10.1080/0020739X.2016.
4. Goddijn, A. (2012). Accessed 29 March at: http://dynamicmathematicslearning.com/further-proofs.zip
5. Lecluse, T. (2012). Vanuit de oude doos: De Opgave 2011, uitgedeeld op de Jaarvergadering. *Euclides* (Journal of the Nederlandse Vereniging van Wiskundeleraars), 87(5), Maart, pp. 215-217. Accessed 29 March 2020 at: http://dynamicmathematicslearning.com/lecluse-opgave2011.pdf
6. Pompe, W. (2016). *Wokół obrotów: przewodnik po geometrii elementarnej*. Wydawnictwo Szkolne Omega, Kraków, Poland.

Glossary of acronyms/abbreviations

- MdV = Michael de Villiers
- HH = Hans Humenberger
- WP = Waldemar Pompe
- NVvW = Nederlandse Vereniging van Wiskundeleraren

Appendix: Proof of Pompe's Hexagon theorem

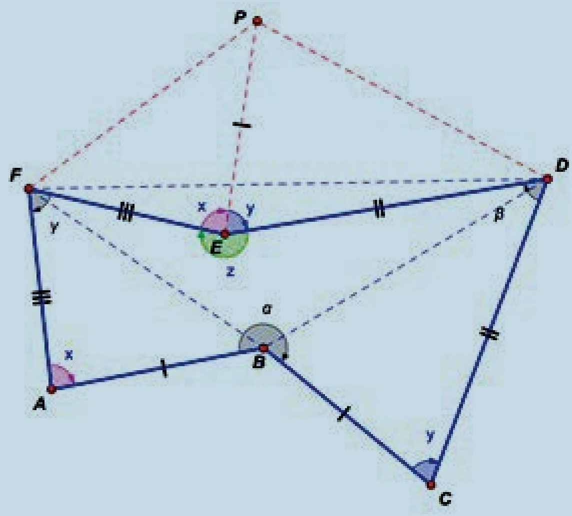


Figure 11: Proof of Pompe's Hexagon Theorem

Proof: Consider Figure 11. Since $\alpha + \beta + \gamma = 360°$ is given, it follows that $x + y + z = 360°$. We now have the following possible cases, (a) all x, y and z are less than 180°, or (b) exactly one angle of x, y, z is at least 180°. In (b) we can, without loss of generality, assume $z \geq 180°$. We shall here prove case (b) as the convex case is similar and is left as an exercise to the reader.

Rotate $\triangle FAB$ counter-clockwise around centre F through angle γ and $\triangle DCB$ clockwise around centre D through angle β. Both A and C map to point E, and since $x + y + z = 360°$, it follows that B' and C' coincide in point P.

Since $\triangle$'s FEP and FAB are congruent from the rotation, we have $\angle EFP = \angle AFB$. Hence, $\angle BFP = \angle AFE = \gamma$. Similarly, it follows that $\angle BDP = \angle CDE = \beta$.

Further, since $FP = FB$ and $DP = DB$, triangles BDF and PDF are congruent (s, s, s). Therefore, $\angle BFD = \frac{1}{2} \angle BFP = \frac{1}{2}\gamma$ and $\angle BDF = \frac{1}{2} \angle BDP = \frac{1}{2}\beta$.

But since $\frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma = 180°$, it follows from the sum of the angles of a triangle in $\triangle BDF$ that $\angle FBD = \frac{1}{2}\alpha$. This completes the proof.

Comment: The hexagon theorem of Pompe certainly deserves to be better known as it not only easily proves the Klingens/Lux problem as shown earlier, but also directly applies to 1) proving Napoleon's theorem (the centres of equilateral triangles on the sides of any triangle form another equilateral triangle) as well as 2) immediately showing that in Van Aubel's quadrilateral theorem, the angle formed by the centres of two squares on adjacent sides, say AB and BC, and the midpoint of the diagonal AC, is a right angle.

MICHAEL DE VILLIERS has worked as researcher, mathematics and science teacher at institutions across the world. From 1991-2016 he was part of the University of KwaZulu-Natal, and since 2016, Honorary Professor in Mathematics Education at the University of Stellenbosch. He was editor of Pythagoras, the research journal of the Association of Mathematics Education of South Africa, and is currently chair of the Senior South African Mathematics Olympiad problems committee. His main research interests are Geometry, Proof, Applications and Modeling, Problem Solving, and the History of Mathematics. His home page is http://dynamicmathematics-learning.com/homepage4.html. He maintains a web page for dynamic geometry sketches at http://dynamic-mathematicslearning.com/JavaGSPLinks.htm. He may be contacted at profmd1@mweb.co.za.

HANS HUMENBERGER has been a professor of mathematics education at the University of Vienna since 2005. In this position, he is responsible for the education of students who want to become teachers at secondary schools or high schools (grade 5–12, 11–18-year-old students). His primary interests include teaching mathematics as a process, teaching mathematical modeling, problem solving, geometry, and elementary mathematics. He may be contacted at hans.humenberger@univie.ac.at.

Kohli's Number 2997

SHAILESH SHIRALI

Consider the following function f defined from the set of positive integers $\mathbb{N}$ into itself:

$$f(n) = 111 \times \text{the sum of the digits of } n. \qquad (1)$$

(Note that we work throughout in base 10.)

For example,

$$f(23) = 111 \times 5 = 555, \quad f(2345) = 111 \times 14 = 1554.$$

In the article [1], Uttkarsh Kohli describes a curious property of this function when it is iterated. Namely, if we start with any positive integer n and compute the sequence

$$n, \quad f(n), \quad f(f(n)), \quad f(f(f(n))), \quad \ldots, \qquad (2)$$

then after just a few steps we will reach the number 2997. Moreover, once we reach that number (2997), we stay there.

> A comment is needed here regarding the notation. The expression $f(f(f(n)))$ looks quite awkward, and the succeeding terms, $f(f(f(f(n))))$, $f(f(f(f(f(n)))))$, ... look more awkward still, with more and more closing brackets that start to resemble the layers of an onion. Some mathematicians prefer to write $f \circ f(n)$ in place of $f(f(n))$, $f \circ f \circ f(n)$ in place of $f(f(f(n)))$, $f \circ f \circ f \circ f(n)$ in place of $f(f(f(f(n))))$, and so on, with '$\circ$' denoting the function composition symbol. This is certainly far more pleasant to the eye!

We give a proof of Kohli's assertion here. To start with, we claim that:

$$\textit{If } n < 10^k, \textit{ then } f(n) \leq 999k. \qquad (3)$$

To see why this is true, note that among all numbers less than 10^k, the number with the largest sum of digits is $10^k - 1$, which is made up entirely of 9's. The sum of the digits of this number is $9k$. Hence the claim.

Keywords: Iteration, base 10, function, logarithmic function, fixed point, Kohli's number

Since $999 < 1000$, the above claim implies the following.

$$\textit{If } n < 10^k, \textit{ then } f(n) < 1000\,k. \tag{4}$$

This may be stated in another way as follows.

$$\textit{For all positive integers } n, \quad f(n) < 1000 \log_{10} n. \tag{5}$$

Next, we would like to find a number M such that

$$\textit{If } n > M, \textit{ then } n > 1000 \log_{10} n. \tag{6}$$

(For, if we find such a number, then we have: if $n > M$, then $1000 \log_{10} n < n$ and also $f(n) < 1000 \log_{10} n$, which means that if $n > M$, then $f(n) < n$.) To find M, we must study the behaviour of the following function g (defined for $x > 1$) as x grows indefinitely large:

$$g(x) = \frac{x}{\log_{10} x}. \tag{7}$$

It is easy to verify via differentiation that $g(x)$ decreases for $1 < x < e$, takes its minimum value at $x = e$, and then steadily rises for $x > e$. (The 'steady rise' should not come as a surprise, considering the behaviour of the logarithmic function, which cuts even extremely large numbers down to manageable size.) The following table of values illustrates this assertion.

x	3	10	10^2	10^3	10^4	10^5
$x/\log_{10} x$	6.3	10	50	333.3	2500	20000

Computations reveal that $g(x)$ crosses the value 1000 roughly around $x = 3555$. As this number is less than 4000, we can safely take $M = 4000$ and thus state the following:

$$\textit{If } n > 4000, \textit{ then } 1000 \log_{10} n < n. \tag{8}$$

Next, observe that the number below 4000 with the largest f-value is 3999, whose f-value is $30 \times 111 = 3330$, and note that this number itself is below 4000. Combining this observation with (8), we obtain the following two important results:

$$\left.\begin{array}{l} \bullet \quad \textit{If } n > 4000, \textit{ then } f(n) < n. \\ \bullet \quad \textit{If } n \leq 4000, \textit{ then } f(n) \leq 4000. \end{array}\right\} \tag{9}$$

Why are these two results important? They imply that even if we start with extremely large values of n, the sequence of iterates

$$n, \quad f(n), \quad f(f(n)), \quad f(f(f(n))), \quad \ldots, \tag{10}$$

is *strictly decreasing* till we reach a number below 4000. (The first result in (9) guarantees this.) Once we do reach a number below 4000, the sequence of iterates is no longer strictly decreasing or strictly increasing, but the numbers stay below 4000. (This is guaranteed by the second result in (9).)

This means that if we wish to study the behaviour of iterates of the function f, it suffices to restrict our attention to the set S_0 of integers between 1 and 4000. Results (9) imply the following important result:

$$\textit{If } n \in S_0, \textit{ then } f(n) \in S_0. \tag{11}$$

Combining the assertions in (10) and (11), we have the following claim:

For any positive integer n, however large, repeated applications of f will ultimately yield numbers in S_0, and once we reach S_0, we never leave it.

Next, note that the definition of f implies that for any n, $f(n)$ is a multiple of 111 (and therefore also a multiple of 3). This means that by applying f to all the numbers in S_0, the resulting set will be a subset of the set of multiples of 111 within S_0, i.e., a subset of the following set:

$$S_1 = \{111,\ 222,\ 333,\ 444,\ \ldots,\ 3663,\ 3774,\ 3885,\ 3996\}\,. \tag{12}$$

Set S_1 has 36 elements, but we have listed only the first 4 and the last 4 elements. The following claim should now be clear:

> For any positive integer n, however large, repeated applications of f will ultimately yield numbers in S_1, and once we reach S_1, we never leave it.

Now consider the second iterate $f(f(n))$. As there is again a multiplication by the factor 111, it follows that:

$$\textit{For any } n,\ f(f(n)) \textit{ is a multiple of } 9. \tag{13}$$

This implies that by applying f to all the numbers in S_1, the resulting set will be a subset of the set of multiples of 333 within S_1, i.e., a subset of the following set:

$$S_2 = \{333,\ 666,\ 999,\ 1332,\ 1665,\ 1998,\ 2331,\ 2664,\ 2997,\ 3330,\ 3663,\ 3996\}\,. \tag{14}$$

Consequently, we can now claim the following:

> For any positive integer n, however large, repeated applications of f will ultimately yield numbers in S_2, and once we reach S_2, we never leave it.

The progression should now be clear. The f-values of the numbers in S_2 form the following set:

$$S_3 = \{999,\ 1998,\ 2997\}\,. \tag{15}$$

We can now claim the following:

> For any positive integer n, however large, repeated applications of f will ultimately yield numbers in S_3, and once we reach S_3, we never leave it.

A quick check shows that the f-values of the numbers in S_3 are all the same, because all three numbers have the same sum of digits (namely, 27). This common f-value is 2997. We can therefore claim the following:

> For any positive integer n, however large, repeated applications of f will ultimately yield the number 2997. Once we reach this number, no further changes take place.

The claim that "no further changes take place" is true because $f(2997) = 2997$. This is sometimes expressed by saying that 2997 is a *fixed point* of the function f. (A 'fixed point' of a function h is any number x such that $h(x) = x$.)

The claim made at the start of the article thus stands proved. Kohli's number is a genuine constant!

References

1. Uttkarsh Kohli, "The Mystical Number 2997 (Kohli's Number)" from https://azimpremjiuniversity.edu.in/SitePages/resources-ara-issue-no-8-november-2020-mystical-number.aspx

SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

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A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

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We invite articles from teachers, teacher educators and others that are helpful in designing and implementing effective instruction. We strongly encourage submissions that draw directly on experiences of teaching. This is an opportunity to share your successful teaching episodes with AtRiA readers, and to reflect on what might have made them successful. We are also looking for articles that strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

Articles in this section may address key questions such as -

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The Closing Bracket . . .

Where We Celebrate Teachers

Prebeesh Kumar K is a higher secondary school mathematics teacher at Palora Higher Secondary School in Ulliyery, Kozhikode district, Kerala. He has a deep interest in teaching mathematics using models and is a 9- time winner in the Kerala state school *sasthrolsavam* (celebration of teaching). AtRiA spoke to him about teaching in COVID times, particularly about his passion for meaning making while teaching and how he was able to teach online without compromising on his pedagogical style. What started off as highlighting the work of one person unraveled a story of a problem-solving team.

Prebeesh Sir used an interesting analogy when asked the question. He spoke about the trapping of a junior football team in the 2018 flooding of the Tham Luang Nang Non cave in north Thailand. Many people were injured and the need of the hour was medical aid: delivered by doctors who needed an additional skill – they needed to be able to swim too. Teachers who wanted to reach their students during the pandemic needed to be teachers with the additional skill of teaching online. The Kerala government realised this very quickly and in the first two months of the pandemic, the DigiFit programme was launched in which teachers from primary to higher secondary were trained in presentation software, video recording, sound mixing and cyber law. At the higher secondary level, this was extremely useful value add for the stated intention to change the weightage for the class 12 examination to Theory 60%, Lab Activity 20% and Continuous Evaluation 20%. Teachers like Prebeesh sir immediately saw the opportunity to implement their approach through online classes using software such as GeoGebra (dynamic geometry software). What was impressive was the level of team playing that the teachers demonstrated as they shored up the teaching community's ability to handle this challenge.

The effort was multi-pronged: Prebeesh Sir and others like him who were already adept with GeoGebra set up training sessions for their colleagues using Google Meet and Zoom. Their training enabled teachers to use Learning Management Systems and gain confidence in the use of Google Classroom, Google Forms, editing software – their one criterion was that all software would be open source. While lab activities were designed by the experts, the pedagogical intent and the ability to design more such activities enabled those who were trained to gain confidence. At the same time, since the class 12 State Board examinations had been postponed, the first two months saw the teachers set up online tutorials for their students who had access to smart phones. Links to worksheets were provided in the student WhatsApp groups, students were able to access these and interestingly, complete a self-assessment after doing their work. Through the pandemic, this was the pedagogy and assessment style adopted. Teacher and student worked as a team to address difficulties and students who understood this approach benefited from functioning at this metacognitive level. Once the class 12 examination was over, the focus shifted to the new batch.

One of the greatest supports that teachers and students received was through the Victers channel which delivered lessons through the television. Issues of access were partly resolved by this. In addition, philanthropic initiatives such as the 'Phone Challenge' and the 'TV Challenge' helped raise money for those who could not access the lessons for economic reasons. And for those who had problems with connectivity (Kozhikode district has many hilly regions), the same lesson would be posted on WhatsApp to enable them to download at times or in places when the range was good enough. Between June 1, 2020 and January 1 2021, over a hundred classes in mathematics were delivered through the Victers channel. In all of these, students followed the self-assessment model for understanding.

This write up is a salute to all the teachers who were true problem solvers in an unexpected and unprecedented crisis situation. Putting the students at the heart of the problem, understanding the shortcomings which needed to be addressed, sharing needs and sharing resources, harnessing the government support systems productively…… what a lot of lessons for all of us. There may be many more such unsung heroes and unheard tales, we hope to highlight more such in forthcoming issues.

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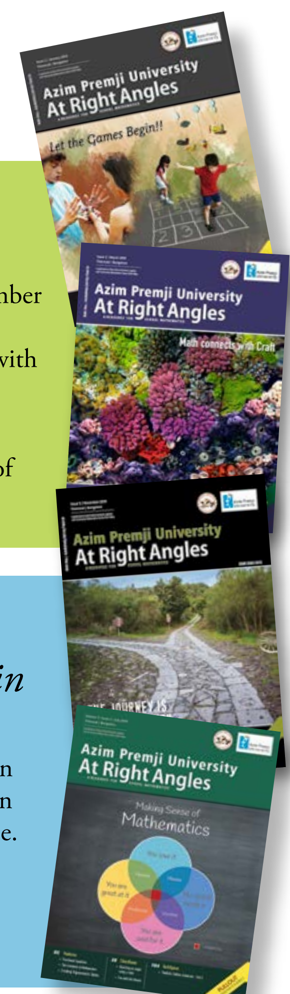

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