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Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

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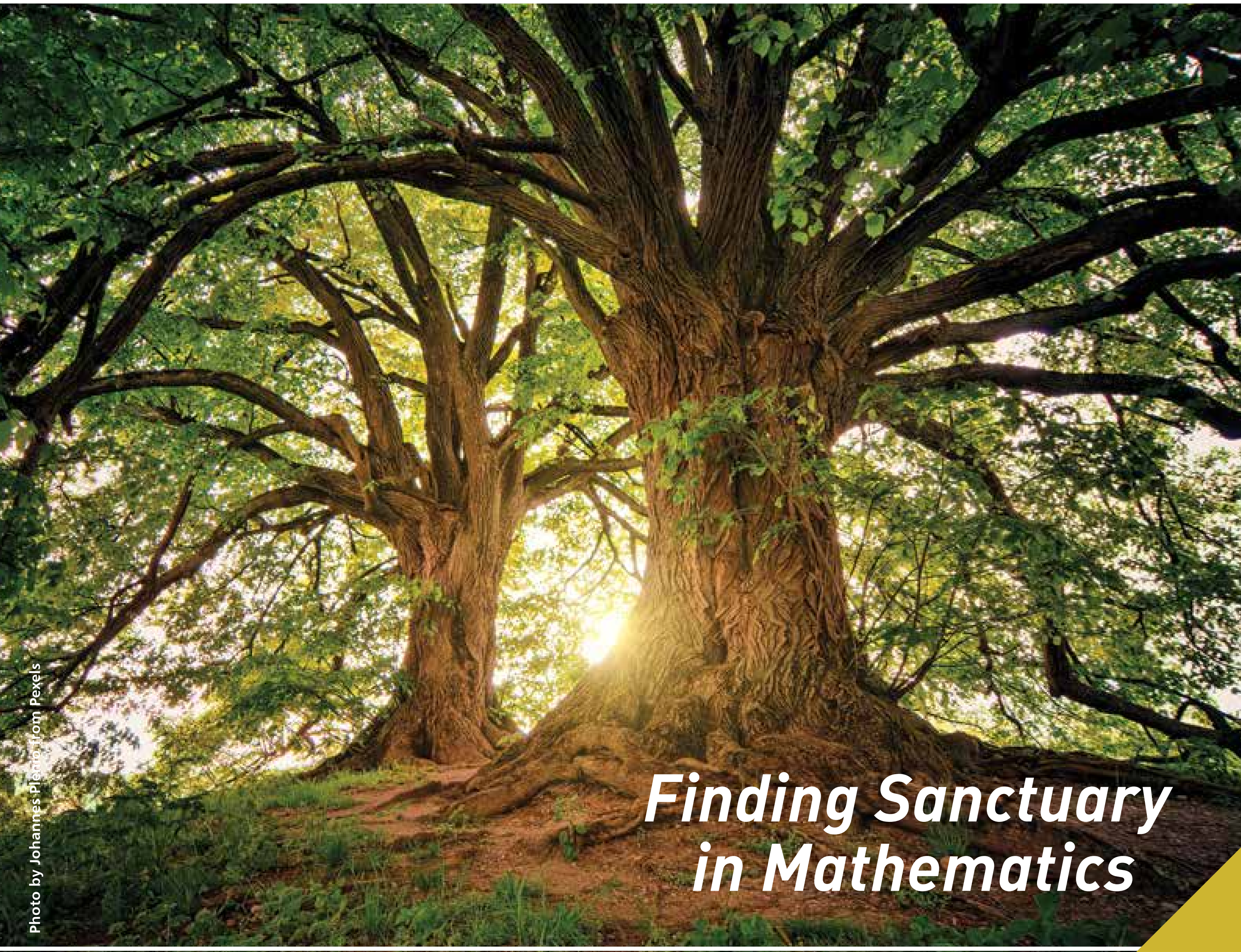

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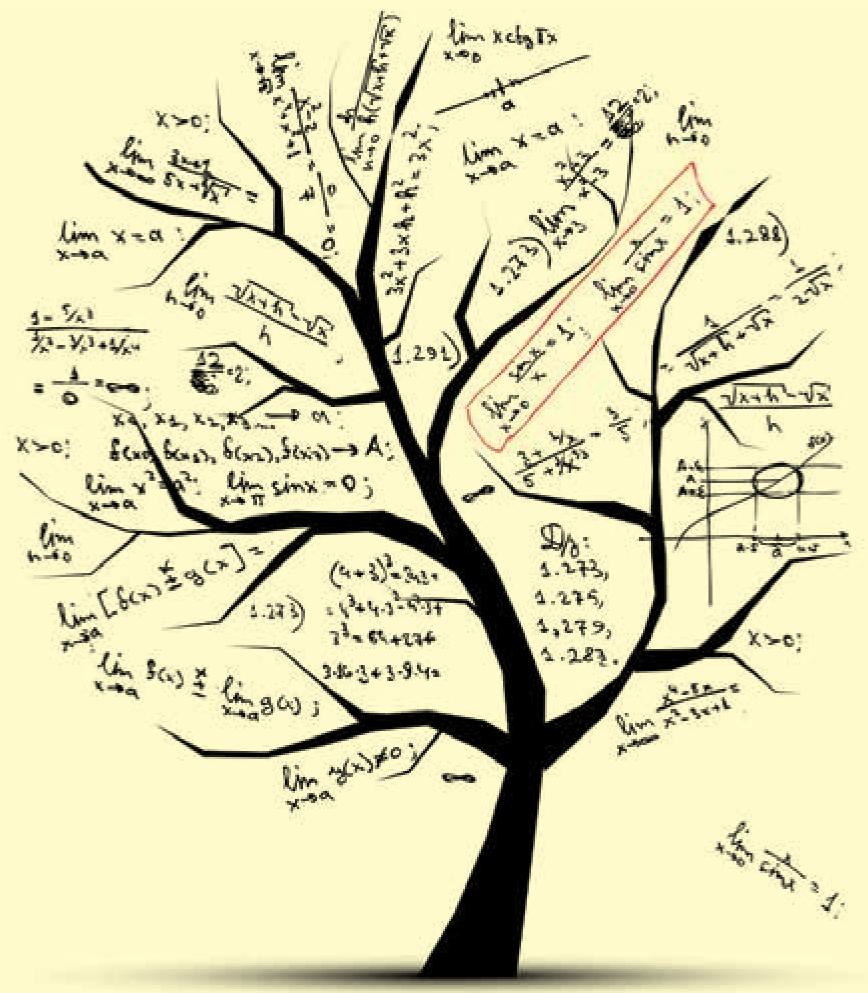
Finding Sanctuary in Mathematics

PULLOUT
ESTIMATION IN MATHEMATICS

"Trees are sanctuaries. Whoever knows how to speak to them, whoever knows how to listen to them, Can learn the truth"

Herman Hesse



Picture a much beloved tree in a garden, much like the tree in the cover image. Now imagine a child playing in its shade, learning to climb its branches, tasting and marvelling at the sweetness of its fruit, learning to go wide and go high, seeing the world from a different perspective… Isn't this what we want everyone's experience of mathematics to be? In a world ravaged by illness, can we find sanctuary in exploring the truths of mathematics?

From the Editor's Desk . . .

In these sombre times, one often wonders how we can engage with our normal routine when all around us, tragedies unfold and sorrow and loss multiply exponentially. Yet, it is precisely the routine that gives us a semblance of normality and enables us to keep going one day at a time. Our thoughts are with all our readers and we trust that you are holding on- this too shall pass.

The July issue carries Part 2 of *Geometry in the Sulvasutras* - read on to marvel anew at the level of mathematical thinking in ancient times. R Sivaraman's *Tremendous Tree* – which inspired our cover this time – is sure to keep you rooted to the spot and marvelling at all the mathematical connections emerging from one very simple mathematical operation on the unit fraction.

The ClassRoom section has a lot to offer this time - we have Mangal Pawar and Aaloka Kanhere's article on *Children's Algorithms*, Ankit Shukla's account of teaching the measurement of length and much more. What stands out in all of these is the value of good questions and the importance of listening to children making sense of the mathematics they practise and learn. And to celebrate this, we introduce what we hope will be a regular feature, a poster with graphic strips featuring some great student-teacher conversations.

Who would think that shuffling a pack of cards could lead to a TechSpace article? Riffle Shuffles is a fascinating account of the mathematics involved in shuffling a pack of cards and studying their rearrangements. Student contributor Keshav Lakshmi Narasimhan uses computational thinking to investigate how many riffle shuffles it would take to bring a pack back to its original order and even generates the Python code (provided in the online version of the article) needed to simulate the shuffling.

You will find more logical reasoning in the Problem Corner with problems and solutions that are elegant in their simplicity and originality. One wonders at the *niceness* of fractions or how a familiar shape like a triangle can have so many hidden features and connections.

Two endearing children's books are reviewed by children's author Arundhati Venkatesh, and the PullOut is all about developing that rather undervalued mathematical process skill of estimation.

Go online to read even more- we have not just one or two but seven additional articles - details are given on the Contents Page with the QR code which will take you directly to the online version.

Do send in your feedback to AtRiA.editor@apu.edu.in We can also be found on our FaceBook page AtRiuM. The magazine is available for free download on https://azimpremjiuniversity.edu.in/SitePages/resources-at-right-angles.aspx and if you would like to subscribe for a hard copy, send your complete postal address to AtRightAngles@apu.edu.in The same id holds if you would like to be notified of our monthly webinars – to view the latest, go to https://youtu.be/a2e8FjsLzDs

Sneha Titus
Associate Editor

The Opening Bracket . . .

As I write this, we are in the middle of the so-called 'second wave' of the Covid-19 pandemic in India. This catastrophe is as much the result of failure of governance as it is of the enormity of the disease. I would suggest that it is also the result of emotions, beliefs, value systems and ideologies playing themselves out.

These 'inner forces' that drive human behaviour seem to be immune to reason and education. I can't think of a more poignant time to recall what H G Wells wrote in **The Outline of History** a hundred years ago:

"Civilization is in a race between education and catastrophe. Let us learn the truth and spread it as far and wide as our circumstances allow. For the truth is the greatest weapon we have."

To win this race, I believe we need to explore how education can affect our inner world. The word 'truth' is rather complicated in today's post-fact world, filled as it is with disinformation. Perhaps as educators we would be better off exploring a unique faculty we have, **the ability to ask questions**. Questioning has been at the heart of many paradigm shifts: it has created beautiful mathematics, yielded a deeper understanding of the universe, society and human nature. The best educational models have flourished from good questions.

When we ask questions of the world outside of us, they lead to models which in turn become part of human knowledge, and the basis for future action. Even a very small fraction of humanity engaging in these questions will do the trick; the rest of us can ride on the knowledge and technology created by a few.

Questioning our inner world, we could ask ourselves what motivates us, what are our deep biases, why we believe in nations, why we have identities? These are important questions, yet there is resistance or lack of interest to ask them. The same brain that can ask deeply insightful questions about the external world is unable to question many assumptions that make up our inner world. For example, let's take identity. It comes with boundaries and territories and is tied to our instinct for self-preservation. Most of the time, we are probably trying to protect our psychological selves, not our physical selves. We are protecting our beliefs, our ideas, our culture, our nations. This is a large territory!

Why can't we use the same tools that we have used so successfully to model and control our environment, to understand the self-protective instinct? Because we are both the subject and the object when we explore our inner world, any model we have of ourselves becomes part of the content of ourselves, and will add to the territory we are trying to protect. The very thing that is being questioned ends up protecting itself!

If we play with it a bit, we see the absurdity of this need to protect our psychological selves. This seeing does not expand our territory, in fact it shrinks it! It happens when we are alert and attentive, and it has the same quality and energy as when we hold a question without looking for answers.

As mathematicians we are often told: *if you really understand a problem, then the answer lies in it*. It appears to me that the key to understanding ourselves lies in this ability to ask questions of our inner world, without judgment or defence, and without settling on answers.

Is this something we can explore with our students? Is this something that is accessible to all human beings through a good education?

Editorial Office

The Editor, Azim Premji University
Survey No. 66, Burugunte Village,
Bikkanahalli Main Road, Sarjapura,
Bengaluru – 562 125
Phone: 080-66144900
Fax: 080-66144900
Email: publications@apu.edu.in
Website: www.azimpremjiuniversity.edu.in

Publication Co-ordinator

Shahanaz Begum
Azim Premji University
Email: shahanaz.begum@azimpremjifoundation.org

Print

SCPL
Bengaluru 560 062
www.scpl.net

Design

Zinc & Broccoli
enquiry@zandb.in

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

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A Publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley and Sahyadri School, Pune

Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

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Continue . . .

Student Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Online Articles

Features

On some Geometric Constructions in the Sulvasutras from a Pedagogical Perspective – II

S.G. DANI
MEDHA LIMAYE

In the first part of this article we described briefly the setting of the sulvasutra geometry and construction of various basic rectilinear figures with a given area (or equivalently transformation of shapes into one another, with the same area). In this sequel we continue on the topic, branching out along the following themes: Firstly, using some arithmetic, we discuss conversion of multiple squares together into one, more efficiently than by simple repeated augmentation of squares as described in part I. In the second section we discuss the topic at hand with regard to the semicircles and circles. The last section is devoted to discussion of certain constructions which are not found explicitly in the sulvasutras, but could have been the basis of some of the knowledge that is propounded in them, specifically, the Pythagoras theorem and the value of $\sqrt{2}$.

I. Merging multiple squares into one

Merging a number of squares into one, irrespective of their sizes, can be done geometrically, in principle, by successively following the method described in §2.1 in part I of this article. In this section we discuss some further points on the theme. The first two subsections below concern possible simplifications in adjoining a number of squares of the same size. For simplicity of exposition we adopt here the modern

Keywords: History of mathematics, Vedic maths, geometry, constructions, area

symbolic notation, and refer to n squares of unit size, etc., though such symbolism is not found in the sulvasutras, and all statements are expressed in words. In the last subsection we discuss a specific way in which the process for joining squares was used.

I.1. Combining n squares through diagonal constructions. When $n = k^2 + l^2$, where k and l are two natural numbers, a square with area n can be constructed as the square on the diagonal of a rectangle with sides k and l units respectively.

Such an application is seen explicitly in Katyayana sulvasutra[1], in KSS 2.4 (2.8) (see the footnote below[2]), where a square with area 10 units is prescribed to be constructed as the diagonal of a rectangle with sides being 3 units and 1 unit.

More generally for n expressed as a sum of r square integers, a square with area n units can be produced geometrically through $r - 1$ iterations of the above procedure, as against $n - 1$ iterations if one follows only a simple-minded procedure of adding a unit square at a time. The process works irrespective of the ordering of the r squares, but a simple choice would be to take the first term to be the largest square not exceeding n, the next to be the largest square not exceeding the balance to be added, etc.; thus, for example, 15 would be expressed as $3^2 + 2^2 + 1 + 1$, and 59 as $7^2 + 3^2 + 1$. Incidentally, for constructing a square with area 15 units one could also use alternatively the fact that $15 = 4^2 - 1$ and use the process for the difference of two squares; in the same way, for 29 one could use $29 = 7^2 - 4^2 - 2^2$.

I.2. Combining n squares via altitudes of isosceles triangles. Katyayana sulvasutra describes an alternative method for producing a square with area n. KSS 6.7 (6.7) states the following:

> When it is desired to put together a number of squares (of unit size) construct an isosceles triangle whose base is one less than the number, and the other two (equal) sides add up to one more than the number; the altitude of the triangle then serves as the side for the desired square.[3]

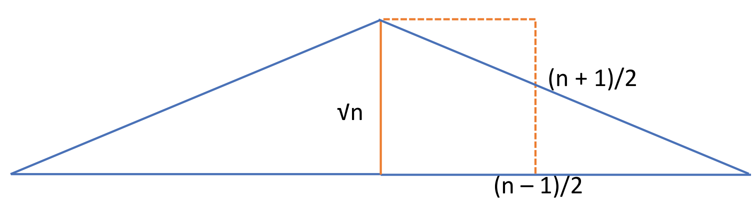

Figure 1. Construction of a square of area n, via isosceles triangles (for $n > 1$)

We note that for the given choices each of the slanted sides of the isosceles triangle is $\frac{1}{2}(n+1)$ and half the base is $\frac{1}{2}(n-1)$. Together with the altitude of the triangle these form a right angled triangle, and hence by the Pythagoras theorem the square of the altitude is $\frac{1}{4}(n+1)^2 - \frac{1}{4}(n-1)^2$. Thus the construction is based on (knowledge of) the identity

$$(n+1)^2 - (n-1)^2 = 4n.$$

1 As in part I we will not use diacritical marks to indicate pronunciation; a Glossary is included as a guide for pronunciation.

2 The first number is as per [7], and the parenthetical one corresponds to the numbering adopted in some of the older sources, including [5] and [6].

3 It may be of some interest to note here that the word used in the sutra for altitude is *iṣu*, which corresponds to "arrow", evidently derived from the analogy of an isosceles triangle with a stretched bow-string. Similar terminology occurs also in Jaina mathematics from the first millennium BCE; see [3].

The identity is straightforward to verify in terms of symbolic algebra. However, in their context it would have been derived differently, in absence of symbolic algebraic methods; one such possibility consistent with the ethos of the time is through counting of tiles (note that though for convenience we use algebraic notation here, the underlying statement can readily be formulated without it): in a grid of square tiles with $n+1$ rows and columns we see that there are $4n$ tiles along the edges, and together with the tiles in the middle square, which has $n-1$ rows and columns, they account for all tiles; hence $(n+1)^2-(n-1)^2=4n$.

I.3. Enhancement of units. The performance of certain yajnas was repeated periodically, and at each successive event the sizes of the vedis were to be increased in a stipulated manner; the first time the area would be $7\frac{1}{2}$ (square) units (the unit being *puruṣa*, measured as the height of a man, normally the yajamana, with uplifted arms), the next time it would be $8\frac{1}{2}$ units, and then $9\frac{1}{2}$ units, and so on, in the same shape (as scaled up figures). The shapes involved being quite intricate, it would be quite complicated to adjust each part; instead this was achieved by *enhancing the size of the unit* adopted for the construction, and following the same steps as before. Thus for instance for the first step as above where the net area is to be scaled up from $7\frac{1}{2}$ units to $8\frac{1}{2}$, namely by a factor of $1+\frac{2}{15}$ the measure of the unit length would be increased to $\sqrt{(1+\frac{2}{15})}$; this desired length for the new unit would be determined, geometrically, as the length of the side of the square obtained by joining to the unit square a square of size $\frac{2}{15}$; for the latter, one may start with a rectangle with sides $\frac{2}{5}$ and $\frac{1}{3}$, for example, and turn it into a square, by the procedure discussed earlier, in part I.

II. Conversions between circle, semicircle and square

There is a sutra in Manava sulvasutra (MSS 1.8 (10.1.1.8)) which concerns describing a circle, a semicircle and a square with the same area, they being the shapes of the three *agni*s (fireplaces), stipulated to have the same area. While the occurrence of the three of them together is unique to Manava sulvasutra, conversion from circle to square and vice versa is treated in all the four sulvasutras. The method for transforming a circle to semicircle is exact while the methods for inter-conversions between the circle and the square hold only approximately, as we shall see below.

II.1. Transforming a circle to a semicircle. According to the sutra mentioned above concerning this, to construct a semicircle with the same area as a given circle, the radius of the desired semicircle is to be taken as the size of the diagonal of a square whose side is the radius of the given circle; see Figure 2 for an illustration of the construction - a square is formed of two radial segments, OA and OB, perpendicular to each other, and its diagonal OP is used for the radius of the desired semicircle. The area of the circle with the new radius will then be twice that of the original circle, and hence the area of the semicircle is the same as the area of the given circle. The reader may observe the close connection of this with the problem of doubling of a square discussed in part 1 of this article.

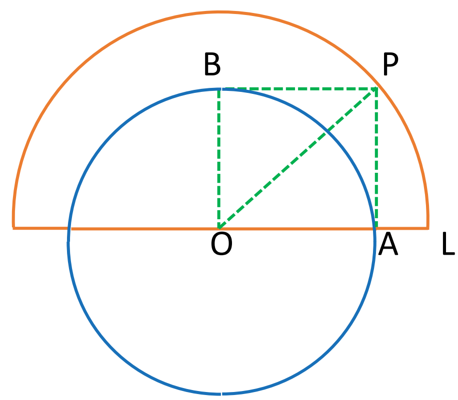

Figure 2. A semicircle drawn to have the same area as a given circle.

The description in the sutra does not specifically concern itself with the converse issue of starting with a semicircle and turning it into a circle. However, it is reasonable to suppose that if called upon to do so they would have retraced the steps: construct a square whose diagonal coincides with the radius of the given semicircle; the side of the desired square can be obtained by taking the endpoint of $\frac{1}{4}$th of the arc segment from one end, viz., LP as in Figure 2, and dropping the perpendicular on the diameter of the semicircle (to determine the point A).

II.2. Circling the square. The sutrakaras are also seen to have been interested in converting a square into a circle and the other way. The methods prescribed for this however happen to be approximate, and rather crude ones at that.

For converting a square to a circle (*maṇḍala*) BSS 2.9 (1.58) has the following instruction.

> Desiring to convert a square to a circle drop the semi-diagonal from the midpoint (of the square) along the line of symmetry perpendicular to a pair of sides, and draw a circle including one-third of the part remaining outside the square.

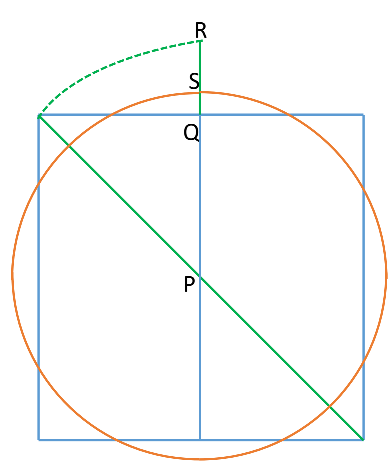

Figure 3. Circling the square

In place of "line of symmetry perpendicular to a pair of sides" that we have referred to, the sutra actually mentions the eastward direction, it being assumed (implicitly) that the sides of the squares are set along the cardinal directions. See Figure 3 for an illustration of the construction; here P is the midpoint of the square, PR is the semi-diagonal laid along a line of symmetry, Q is its point of intersection with the side, and S is the point on QR at a distance one-third the length of QR, from Q, which is prescribed as the point through which the desired circle is to pass.

For the unit square the area of the circle produced as above is seen to be $\frac{\pi}{4}(\frac{2+\sqrt{2}}{3})^2$ (with π as the area of the unit circle), in place of the desired value 1; this involves an error of about 1.7%. To get a sense about the error involved, it may be seen that for the value as above to be 1, corresponds to a value of π which is approximately 3.08, as against about 3.14, at 2 decimal places; it should be borne in mind however that it was not their purpose to determine the ratio π as we understand it, but only to produce a circle equivalent to the square in area to meet their requirement in the ritual context, and evidently the value was considered good enough for the purpose.[4]

Notwithstanding the degree of error the heuristic involved in the process is worth taking note of. Clearly the radius of the desired circle would have to be between half the side of the square and half the diagonal, and $\frac{1}{3}$rd of the extra part was added to the former in this light, presumably based on some visual intuition.[5] It turns out that if in place of "one third" of the jutting out part they were to add $\frac{3}{10}$th of the same, they would have got a much closer approximation for the area of the circle produced! However, though the latter may now seem a natural possibility to have been considered, in the context of the decimal place value system of representation of numbers, in India such representation remained confined to whole numbers and did not get extended to fractions until the modern era. At any rate, the fact that one had to look for an ad hoc choice, without any means of determining the factor mathematically, was a major handicap in dealing with the issue at that time.

II.3. Squaring the circle. The converse problem of finding a square equal in area to a given circle is also addressed.[6] However the resolution is not through a geometrical construction in the same spirit as seen so far. Instead, numerical values are assigned for the ratio of the side of the desired square to the diameter of the given circle. The relatively more accurate value given by Baudhayana in BSS 2.10 (I.59) (described in words in the original) is

$$\frac{7}{8}+\frac{1}{8\times 29}-\frac{1}{8\times 29\times 6}+\frac{1}{8\times 29\times 6\times 8}.$$

This involves the same degree of error as in the solution of circling the square as above, but in the opposite direction. This readily suggests that the ratio was obtained by inverting the ratio involved in the converse process of turning a square into a circle. How the computations went is unclear, but one thing that seems clear is that the fairly accurate formula for $\sqrt{2}$ as a fraction that they found (see the next section), was inspired by this problem; the value had no other ostensible purpose in their context, as the geometrical construction of $\sqrt{2}$ as *dvikaraṇī* would suffice for all their other requirements, and it would also be more accurate.

4 In [1] (followed up also in [2]) a case is made that one of the sutras in Manava sulvasutra gives an analogous construction with a better result, with an error less than $\frac{1}{2}$%. We shall however not go into the details here.

5 A preliminary option that would suggest itself in respect of such "interpolation" would have been to take the average, viz. a factor of $\frac{1}{2}$ in place of $\frac{1}{3}$, which they apparently ruled out as unsuitable.

6 It should be noted that the problem is considered here from a practical point of view and should not be confused with the Greek problem of "squaring the circle" with only a straight edge and compass (and calls for an exact solution), which was later realized to be impossible, on account of π being a transcendental number, as established by Ferdinand von Lindemann in 1882.

III. Other related constructions

As noted earlier the sulvasutras were composed with the objective of providing instruction related to construction of vedis and agnis, and did not concern the broader aspect of creating a repository of all knowledge of that time on any topic. In particular there are no proofs given for the principles enunciated or the constructions proposed for various purposes. While in case of many statements one can envision how the scholars of that time may have arrived at them, there are various instances where one would wonder about their reasoning, or the process by which they would have arrived at the conclusion in question. In this section we discuss two such instances, with surmises about how the conclusions would have been arrived at, via the knowledge or familiarity with constructions akin to those that we discussed so far.

III.1. A value for $\sqrt{2}$. BSS 2.12 (1.61) describes an approximate value for $\sqrt{2}$ which may be expressed as

$$1+\frac{1}{3}+\frac{1}{3\times 4}-\frac{1}{3\times 4\times 34};$$

the original is of course in words, and the form of presentation as above (rather than its simplified fractional form as $\frac{577}{408}$) conveys how the number is described, giving each successive term in the expression as a certain part of the preceding one. One may also expect that the way in which it is expressed may hold a clue as to how the formula was arrived at. The value as above turns out to be quite close to the actual value of $\sqrt{2}$, coinciding in its decimal expansion upto five decimal places, as 1.4142156 . . .[7] in place of 1.4142139 . . . .[8]

The numerical value for $\sqrt{2}$ would not have played a role in their constructions for practical purposes, since they could always construct the magnitude geometrically with much less effort. The endeavour of producing an expression as above for it in terms of fractions seems to have been inspired by the problem of squaring the circle, discussed in §II.3 above.

While it is not known how they arrived at the expression (see [1] for more discussion and references on this) one of the suggestions in this respect, due to Bibhutibhushan Datta [4] (see also [7]) would be worth recalling here, it being in the spirit of our present theme of conversion of figures, as treated in the sulvasutras. It may be described as follows.

The aim is to find the side of a square obtained by putting together two unit squares, in numerical terms, as against by a geometrical procedure, the latter being already described. Considering the problem as one of converting a rectangle with sides 2 and 1, the two unit squares being put side by side, the geometric procedure described earlier involves bisecting one of them and putting one part along the perpendicular side, so that we arrive at a figure which is a difference of two squares, which is subsequently adjusted for. Instead, we now divide the second unit square in three equal strips along a pair of parallel sides of the square. (The reader is advised to follow Figure 4 stepwise for the construction described here.)

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7 It may be appropriate to recall here that the Babylonians also had a similarly accurate value, around 1700 BCE, in fact with a slightly smaller error, which is on the opposite side.

8 The closeness of the value has led some authors to argue that the sutrakaras had the idea of irrational numbers, and recognized $\sqrt{2}$ as one such. Such arguments however bear no substance. Evidently they would have arrived at the formula by an iterative procedure (whether the one discussed here or some other one), and at the stage of arriving at the expression it would be apparent that the actual value is not reached exactly, and as such the formula involves an error; the process involved may also hold a clue on the degree of error involved. Realization of these, though remarkable in the given context, has little to do with realizing the number to be irrational, namely that there is *no way at all* to write it as a ratio of two integers. Nor does the issue of irrationality of numbers have any bearing to their context.

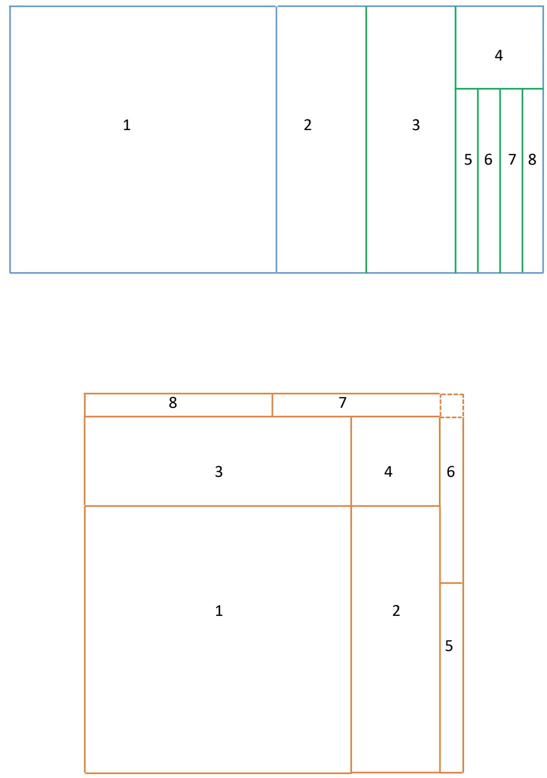

Figure 4. Part of Datta's explanation of Baudhayana's formula for $\sqrt{2}$.

Retain one of the strips on the side of the first square, place another along the perpendicular side in analogy with the geometric procedure. From the remaining strip a square is cut off and is used to fill in the square gap, thus producing a square of side $1 + \frac{1}{3}$ units. We are now left with a strip with sides $\frac{2}{3}$ and $\frac{1}{3}$. Dividing the strip into four equal strips along the longer side, two each are placed along the two sides of the square with side $1 + \frac{1}{3}$. This leads to a square with side $1 + \frac{1}{3} + \frac{1}{3 \times 4}$, whose area is in excess over the desired square, by a square of side $\frac{1}{12}$, as the strips do not cover the tiny square in the corner (on the side of strips 6 and 7 in Figure 4). To adjust for it one subtracts, from the side of the square constructed so far, $\frac{1}{34}$th of the preceding $\frac{1}{12}$th size (this is not shown in our diagram, it being too fine a proportion for the figure); we note that when the tiny square is cut into strips of $\frac{1}{34}$th of the size along one of its sides, by reassembling the pieces along the length we can get two rectangles whose other side is $\frac{17}{12}$, viz., the same length as the side of the last constructed square. This now gives the square with the side as described by Baudhayana. It may be noted that the square is short of the actual value by $\frac{1}{(12 \times 34)^2} \approx 0.000006\ldots$, which may be, and has been, neglected; incidentally, this also explains the matching of the expression with the value of $\sqrt{2}$ upto 5 decimal places.

III.2. The Pythagoras theorem. As noted earlier the Pythagoras theorem was known to the authors of the sulvasutras, the complete statement being found in all the four of them. Here again, one would be curious how they found out that the conclusion holds, and whether they "proved" it geometrically in a suitable sense (though of course they did not have any axiomatic scheme for placing it), as against knowing it simply as a fact about the physical world we live in. There have been a variety of perceptions and speculations in this respect in the literature, but this would not be a place to go into a discussion on it. The theorem is in one way about squares on the sides of the rectangle adding up to the square on the diagonal, and hence relates to the theme we have pursued. Here we content ourselves noting that the statement of the theorem is accessible through constructions akin to the ones we discussed; our formulation here is similar to, but not quite the same, as in [4], pages 115-117.

The argument that we describe is illustrated by Figure 5.[9] Wanting to determine the area of the square ACDE as in the figure one would be inclined to situate it in the larger square BFGH, and observe that the triangular pieces match with those with the corresponding numbers in the other division of the square shown alongside. The comparison then shows that the area of ACDE must coincide with the sum of the areas of the squares numbered 6 and 7, which are indeed the squares of the sides AB and BC respectively.

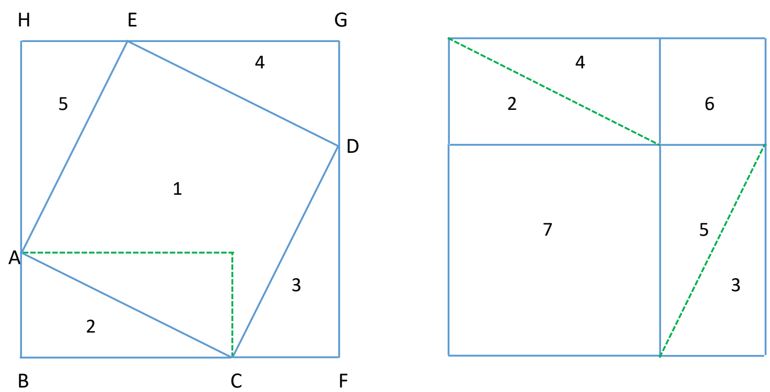

Figure 5. A diagramatic proof of the Pythagoras theorem.

IV. In Place of Conclusion

We have attempted to bring out in this two-part article some of the mathematical essence of the engagement of the authors of the sulvasutras with construction of various planar geometrical shapes with a given area, viz., that of a given square (there are of course other geometric constructions in the sulvasutras, but here we have chosen to focus on the theme as above). The constructions involve a variety of ideas, some at elementary visual level, others in the form of application of the Pythagoras theorem, and yet others certain elements of arithmetic. While in many instances their general scheme of reasoning is clear, in others inevitably the reconstruction of how they arrived at the result is conjectural or speculative, as no proofs are recorded in the sulvasutras. Notwithstanding the lack of certainty or finality the analyses have been instructive in aiding our comprehension of the development of mathematical ideas. It seems especially notable that in the problem of squaring the circle, presumably for want of success in finding a geometric procedure, they appealed to an arithmetical approach, and endeavoured to find a good enough value for $\sqrt{2}$. Unfortunately the full picture with regard to the results is still lacking in clarity, but the process of arriving at it has been instructive. We conclude with the hope that exposure to these ideas would help in enriching geometric understanding of our young readers.

9 It may be borne in mind, however, that the argument presented here is *not* a complete proof of the Pythagoras theorem. In a proof, from a modern perspective, the steps involved need to be based on axioms or known propositions, whereas here various "facts" about planar figures are taken for granted in the course of the argument. In terms of Euclidean geometry, for instance, it needs to be deduced from the basic axioms that the square BFGH involved in the argument can in fact be constructed, and that we can indeed have the partitions into pieces, as posited, that would actually match pairwise, justifying the desired conclusion. These details can indeed be filled in, as an exercise in Euclidean geometry, which we shall not go into here; the issues involved are, however, unlikely to have touched the thought process of the sutrakaras. The above demonstration concerns only the aspect of how the sutrakaras could have convinced themselves of the validity of the theorem, the steps involved being *visually* convincing.

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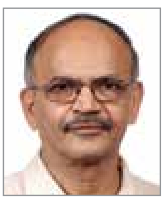
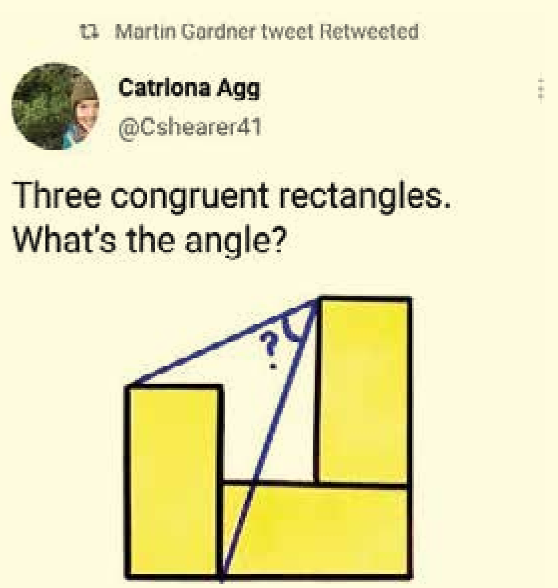


S.G. DANI was affiliated with the Tata Institute of Fundamental Research (TIFR), Mumbai until retirement. Subsequently, he was with IIT Bombay, and is currently affiliated with the UM-DAE Centre for Excellence in Basic Sciences, Mumbai. He was the President of the *Indian Society for History of Mathematics* for over a decade, and continues to be the Editor of *Ganita Bharati*, the Bulletin of ISHM. He can be contacted at shrigodani@cbs.ac.in.

MEDHA LIMAYE retired as Mathematics Teacher from the Arvind Gandbhir High School, Jogeshwari, Mumbai. She holds a Doctoral degree in Sanskrit, for her thesis on Cultural Aspects of Ancient Indian Mathematics. She has written a book entitled *Treasure of Interesting Numerical Problems*, and several articles on history of mathematics. She is currently serving as Joint Secretary of *Brihanmumbai Ganit Adhyapak Mandal*, Mumbai. She can be contacted at medhalimaye@gmail.com.

A Cute Problem



Glossary of names and terms

As in text	As in technical literature	In Devanagari script
Apastamba	Āpastamba	आपस्तम्ब
Baudhayana	Baudhāyana	बौधायन
Circle	Maṇḍala	मण्डल
Diagonal	Akṣaṇayā/ Akṣaṇayārajju	अक्ष्णया/अक्ष्णयारज्जु
Flank (longitudinal side)	Pārśvamānī	पार्श्वमानी
Isosceles triangle	Prauga	प्रउग
Katyayana	Kātyāyana	कात्यायन
Manava	Mānava	मानव
Pointed	Aṇimat	अणिमत्
Puruṣa (height of man with uplifted arms)	Puruṣa	पुरुष
Quadrilateral	Caturasra	चतुरस्र
Rectangle	Dīrghacaturasra	दीर्घचतुरस्र
Rhombus	Ubhayataḥ prauga	उभयतः प्रउग
Rope or cord	Rajju, Śulva/Śulba	रज्जु, शुल्व/शुल्ब
Semicircle	Ardhamaṇḍala	अर्धमण्डल
Square (1)	Caturasra	चतुरस्र
Square (2)	Samacaturasra	समचतुरस्र
Stretch	Vistāra	विस्तार
Sulvasutra	Śulva-sūtra/ Śulba-sūtra	शुल्बसूत्र
Sutra (statement in aphoristic style)	Sūtra	सूत्र
Sutrakara (composer of sutras)	Sūtrakāra	सूत्रकार
Transverse (lateral side)	Tiryaṅmānī	तिर्यङ्मानी
Width	Āyāma	आयाम
Yajamana (master of ceremony)	Yajamāna	यजमान
Yajna (fire worship/ritual)	Yajña	यज्ञ

Tremendous Tree

R SIVARAMAN

In this article we are primarily interested in constructing what are called "Fraction Trees" and introducing some mathematical concepts behind it. First, we start our exploration from the following construction rule:

Let $a, b > 0$ where a, b are coprime natural numbers. If $\frac{a}{b}$ is a given fraction (proper or mixed) then it generates two terms as shown below

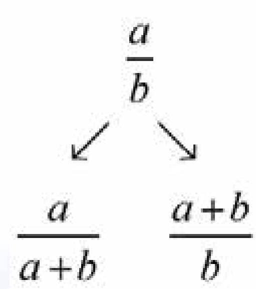


Figure 1.

The term $\frac{a}{a+b}$ is called left child and $\frac{a+b}{b}$ is called the right child of $\frac{a}{b}$. With this convention, we see that $\frac{a}{b}$ is viewed as the parent of both left and right child terms. Also, we see that the left child $\frac{a}{a+b} < 1$. The right child $\frac{a+b}{b} > 1$.

Thus, "Every fraction has two children: a left child, a fraction smaller than 1, and a right child larger than 1."

Keywords: Fractions, iteration

Beginning with the fraction $\frac{1}{1}$ we have the following fraction tree (in this article, we only consider trees beginning with 1):

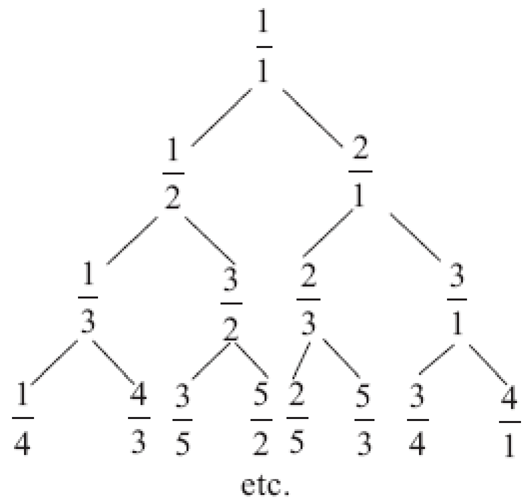


Figure 2.

If we continue drawing another two rows, we get the following tree:

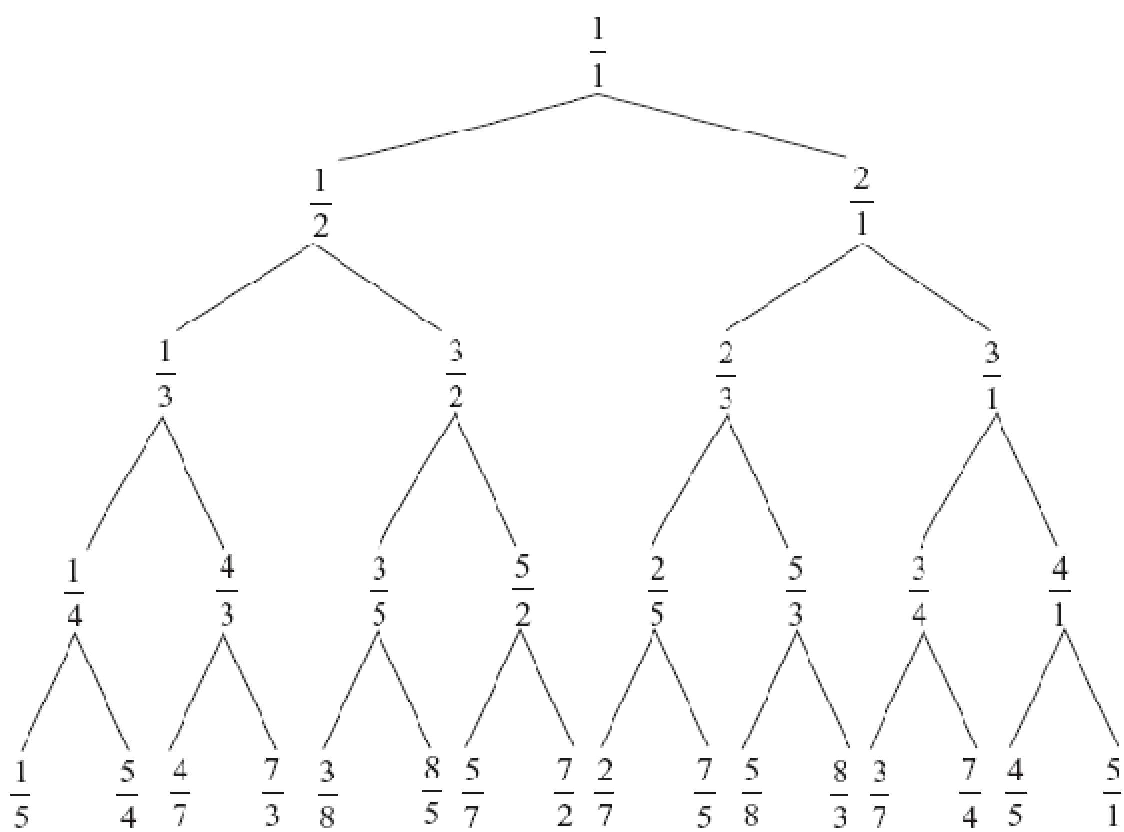


Figure 3.

We see that $\frac{1}{1}$ yields left child $\frac{1}{2}$ and right child $\frac{2}{1}$. Similarly, $\frac{1}{2}$ yields the left child $\frac{1}{3}$ and right child $\frac{3}{2}$. Hence, $\frac{1}{1}$ is considered as grandparent for both $\frac{1}{3}$ and $\frac{3}{2}$. Similarly, $\frac{2}{3}$ is the grandparent for both $\frac{2}{7}$ and $\frac{7}{5}$. In this view, we can consider $\frac{2}{1}$ as the great-grandparent of $\frac{2}{7}$ and $\frac{7}{5}$. (In each case, there may be other grandparents and other grandchildren.) Continuing in this manner and generating more rows in Figure 2, we get several numbers in each generation following the construction rule in Figure 1.

Observing, we will try to answer the following questions:

Challenge I

1. Does the number $\frac{22}{7}$ appear in the tree?
2. If so, what will its parent and grandparent be and how many times does it appear?
3. Did you see any pattern among numerators and denominators in each row of the fraction tree?
4. What is the sum of numerators and denominators of the fractions appearing in each row of the tree?

Many such interesting questions may be asked regarding the fraction tree constructed above. In this article, I will provide a hint for answering some questions regarding this tree and leave the rest to our enthusiastic readers.

Geometer's Algorithm. The ancient Greek mathematician Euclid (hailed as the 'Father of Geometry'), provided a wonderful technique which is now named in his honour as "Euclidean Algorithm" to find the greatest common divisor of two given integers.

He observed that if the greatest common divisor (GCD) of a and b is d where $a > b$, then the greatest common divisor of b and $a - b$ will also be d. By using this principle repeatedly, we arrive at the final pair (d, d) after finitely many steps as both a and b are finite. So, the Algorithm terminates if we get equal numbers in the pair and that number will actually be the GCD of the given numbers.

As an illustration, if we want to determine GCD of 48 and 132, we get

$$(48, 132) \to (48, 84) \to (48, 36) \to (12, 36) \to (12, 24) \to (12, 12).$$

Thus 12 is the GCD of 48 and 132. But what, if anything, has this technique to do with our fraction tree?

One of the most curious aspects of mathematics is its ability to connect completely different ideas unexpectedly. Here we are going to witness one such case.

We will try to determine the GCD of the numbers 8 and 5 using the Euclidean Algorithm.

$$(8, 5) \to (3, 5) \to (3, 2) \to (1, 2) \to (1, 1).$$

If we look at this, we find that the Greatest Common Divisor of 8 and 5 must be 1. But if we look at the pairs between (8, 5) and (1, 1) we notice that they are (3, 5), (3, 2), (1, 2). Rewriting the pair (h, k) as $\frac{h}{k}$, we observe that the pairs formed when the Euclidean Algorithm is applied may be rewritten as $\frac{8}{5} \to \frac{3}{5} \to \frac{3}{2} \to \frac{1}{2} \to \frac{1}{1}$. But this is precisely the path from $\frac{8}{5}$ to reach $\frac{1}{1}$ if we trace backwards in our fraction tree, as can be seen in Figure 3. Thus, the parent, grandparent, great-grandparent, great-great-grandparents of $\frac{8}{5}$ are $\frac{3}{5}, \frac{3}{2}, \frac{1}{2}, \frac{1}{1}$, respectively.

Thus, the Euclidean Algorithm helps us to trace the ancestral path in the fraction tree constructed through the rule provided in Figure 1. Using Euclidean Algorithm, we could gain much more information about the fraction tree.

Connections to the Fraction Tree. As just stated, starting at $\frac{a}{b}$ and following the path back to $\frac{1}{1}$ is precisely the path taken by Euclid's Algorithm when applied to the pair (a, b). Because the path yields the final pair (1, 1), the fraction $\frac{a}{b}$ is in its simplest form (i.e., its reduced form), and the greatest common divisor of a and b is 1. That is, every fraction of the form $\frac{a}{b}$ in the tree will be in its simplest form.

It can be shown that every reduced fraction $\frac{a}{b}$ must appear somewhere in the tree. By applying Euclidean Algorithm to a and b, we can locate such a fraction in the tree.

Further, we notice that no fraction appears twice in the tree, because if such a fraction occurs twice, then its parents would occur twice, as would its grandparents, great-grandparents, and so on, all the way up to $\frac{1}{1}$ appearing twice, which is not so since we have only one fraction $\frac{1}{1}$ at the apex of the tree. Hence every reduced fraction appears just once in the tree.

These observations probably help you to find answers for questions 1 and 2 of **Challenge I**.

Extra Mile. If we list the fractions of the tree in Figure 2, from left to right across the rows of the tree we obtain the sequence $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \cdots$ We call this sequence as fraction tree sequence. This list contains all positive fractions. The fact that one can list the rational numbers was first discovered by the great German mathematician Georg Cantor in the 19^{th} century. He accomplished this fact in a different way though. (See references [1], [2].)

Further Explorations in the Fraction Tree. If we list the numerators of fractions appearing in each row of the fraction tree in Figure 2, we get 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, . . . If we try to list the denominators then we get 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, . . . which is the same sequence as that of the numerators offset by one. To explore further properties we try to list the numerators reading from left to right, row wise in the fraction tree in Figure 2.

Row 0: 1

Row 1: 1, 2

Row 2: 1, 3, 2, 3

Row 3: 1, 4, 3, 5, 2, 5, 3, 4

. . . .

We observe that **Row *n*** contains 2^n numbers whose sum is 3^n.

If we now list the denominators row wise reading from left to right in the fraction tree in Figure 2, we get

Row 0: 1

Row 1: 2, 1

Row 2: 3, 2, 3, 1

Row 3: 4, 3, 5, 2, 5, 3, 4, 1

. . . .

We notice that each row of denominators is just a mirror image of the corresponding row of numerators. Hence here too, **Row *n*** contains 2^n numbers whose sum is 3^n.

These observations will help you to answer questions 3 and 4 of **Challenge I**.

Considering the list of numerators 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2 . . ., we observe the following properties:

1. Consider every second term, that is the numbers in the even positions. This gives 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2. . ., the original sequence reappears magically.
2. Each term in the odd position (from the third term onwards) is the sum of its two neighbors. That is, $2 = 1 + 1$, $3 = 1 + 2$, $3 = 2 + 1$, $4 = 1 + 3$, $5 = 3 + 2$. . .
3. Every third term is even and all other terms are odd.
4. The terms between the ones are palindromes given by 2, 323, 4352534, 547385727583745. . . The digits of these palindrome numbers sum to one less than a power of three.

Challenge II

1. Try to prove the first three properties described above.
2. Can you find the 100^{th} term of the fraction tree sequence?
3. Can you find the 1001^{th} term of the fraction tree sequence?
4. In which row does the number $\frac{22}{7}$ appear in the fraction tree?

Binary Representation and Fraction Tree. We know that every number N can be written in binary representation (base 2) containing only 0's and 1's. For example the number 100 in binary (base 2) is 1100100. Similarly the number 17 in binary is 10001. We notice that if N is even, then the binary representation always ends in 0 and if N is odd, it ends with 1.

We try to write N in base two and count the digits that appear in each block of 1s and 0s in its representation. Suppose N is of the form shown below

$$N = \overbrace{1\cdots1}^{a_k}\ \overbrace{0\cdots0}^{a_{k-1}}\ \overbrace{1\cdots1}^{a_{k-2}}\ \cdots\ \overbrace{0\cdots0}^{a_1}\ \overbrace{1\cdots1}^{a_0}$$

where $a_0 = 0$ if N is even. With this representation we observe the following result.

The N th fraction f_N in the fraction tree sequence is the continued fraction given by
$$f_N = [a_0; a_1, a_2, \cdots, a_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_k}}}$$

The expression provided above for f_N is called a Continued Fraction. Thus, the Nth fraction f_N in the fraction tree sequence is the finite continued fraction given above.

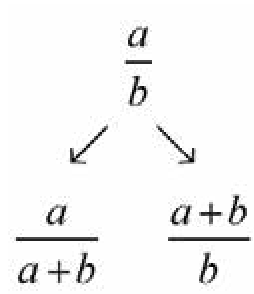
For example, the 25^{th} term of the fraction tree sequence can be calculated using the above continued fraction expression. First, we find that 25 in base two is 11001. Hence $a_0 = 1, a_1 = 2, a_2 = 2$. Thus, $f_{25} = 1 + \cfrac{1}{2 + \cfrac{1}{2}} = 1 + \cfrac{1}{\frac{5}{2}} = 1 + \frac{2}{5} = \frac{7}{5}$. We can verify from the last row of the fraction tree in Figure 3, that the 25^{th} term of the fraction tree sequence is indeed $\frac{7}{5}$.

This concept will help you to resolve questions 2 and 3 presented in Challenge II.

By making further investigation, you can discover many properties of this amazing fraction tree. In fact, the fraction tree that we are discussing in this article is called the Stern-Brocot tree in mathematics literature. By framing simple rules of creation as in Figure 1, we could generate as many mathematical properties as possible. This is the real charm in doing mathematics.

Suggestions for Further Exploration. For curious readers, I present the following suggestions through which you can learn much more about the Stern-Brocot tree and its associated properties.

1. By creating a construction rule similar to that given in Figure 1, try to construct a tree beginning with $\frac{1}{1}$. This tree is called the Calkin-Wilf tree.



Try to explore similar properties to those we discussed with respect to the Stern-Brocot tree. See if these two trees share anything common between them.

2. The concept of Stern-Brocot tree is applied in various branches of Science and Technology. In particular, it has profound applications in Graph Theory which concerns the structure of networks and connectivity problems. It is also applied in the study of Binary Trees in Computer Science and Algorithms. Try to know these applications and discover new ideas for the future.

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DR. R. SIVARAMAN has more than two decades of teaching experience and had been working at D.G. Vaishnav College since 1999. He has published more than 80 research papers in mathematics and has received more than 50 illustrious awards including Best Teacher Award from Indian National Science Academy and National Award for popularizing mathematics among masses from Department of Science and Technology, Government of India. He has written more than 20 books on popularizing mathematics. He has been taking free classes for college students from deprived backgrounds for many years. Propagating the beauty and applications of mathematics to everyone is his life mission. He may be contacted at rsivaraman1729@yahoo.co.in.

Children's Algorithms and the Mathematics behind them

MANGAL PAWAR & AALOKA KANHERE

The first author of this article is a mathematics teacher who observes students' answers, engages with them and is fascinated by them. She approached the second author who works in the field of mathematics education but is not a teacher herself. This article is a collaboration between them.

Due to constraints of time and several other hurdles, many of us do not pause to think about the methods that students use. Why did they think of 'that' particular method? Where did they see it? Will it work for all cases or only special cases?

Most teachers would agree that many of their students have alternative ways of solving problems. Sometimes, as teachers we are not even sure why these methods work but they seem to work.

These alternative methods and discussions on them make the classroom space richer and a more democratic one, as a student who comes up with an interesting method of his or her own may not, usually, be a mathematics enthusiast. She could be the one who hardly participated or was otherwise a mathematical introvert. Giving spaces to students' own methods also offers opportunities to explore new ideas with the reins of discussions in the students' hands.

Keywords: Addition, Subtraction, Algorithms, Children's mathematics

Both authors find studying students' own ways of solving problems very exciting. That is the main reason behind this collaboration. In this article, we would be talking about different methods of solving problems that students from the first author's class gave her.

Let us peep into one of the classrooms:

The teacher asked the students to solve 45 – 19. She knew that the students were comfortable in subtracting numbers when the units digit of the subtrahend was smaller than the units digit of the minuend, that is when no borrowing was needed. This particular question was given to see how students deal with problems where the 'borrowing' technique is typically used.

Sai was seen solving this method using 'number completion' method.

```
     20
   45
 – 19↗
 -----
   26
```

Teacher: Sai, why have you put an arrow and written 20 here?

Sai: The closest 'tens number (दशक संख्या)' (multiple of 10) to 19 is 20. I found subtracting 20 from 40 more convenient.

Teacher: Where did you see this way of calculating?

Sai: Tai[1], The bus conductor also calculates like this.

Teacher: I would like to understand your method.

Sai: The closest 'tens number (दशक संख्या)' (multiple of 10) to 19 is 20. So, I subtracted 20 from 40 and got 20. Then added 5 ones to 20.

Teacher: But then the answer is 25. But you have written 26 instead.

Sai: The closest 'tens number (दशक संख्या)' (multiple of 10) to 19 is 20, which is 1 more than 19. While subtracting, I subtracted 20 from 45 and I got 25. So then, I added that 1 to 25 and got 26.

In the given transcript, one can see Sai using a 'number completion' method wherein the number closer to a multiple of 10 (here, 19) was completed to its nearest ten, followed by subtraction and then compensating for 'completion'. This is not done traditionally in a usual class. The teacher is seen asking questions to understand the method from Sai. This approach would not only encourage Sai and her classmates to use their own methods to solve problems but would also help Sai in articulating her method better. Sai had observed her bus conductor using this strategy and she applied it to her class problem.

Let us look at some more examples of how Sai and her classmates solved problems by their own methods in an atmosphere which not only allows but also encourages them to use their own algorithms.

Teacher asked Sai another question:

```
   47
 – 29
 ----
```

Sai clearly ignored the vertical arrangement of the numbers which was written to facilitate the *'borrowing'* method and solved the problem in the following way.

"I subtracted 20 from 40, then I subtracted 9 from the difference (20), 11 remains.

Then I added 7 to the 11 so that the answer is 18."

A few things that one notices about Sai's method are:

Sai is looking at the numbers 47 and 29 as a whole not as individual digits 4, 7 and 2, 9. For her, she decomposes the numbers 47 and 29 as 40 + 7 and 20 + 9 respectively.

1 Tai means elder sister in Marathi but here the students are using it to address their teacher.

Let us solve another example using Sai's number decomposition method.

$$\begin{array}{r} 327 \\ -\ 258 \\ \hline \end{array}$$

We can imagine that Sai would solve this in the following way.

$$\begin{array}{r} 327 \\ -\ 258 \\ \hline 69 \end{array}$$

She would subtract 250 from 300, followed by subtracting 8 from the difference (50), 42 remains.

Finally, Sai would add 27 to the 42 to get the answer 69.

Or, she may do the following:

Subtracting 200 from 300, followed by subtracting 50 from the difference (100), 50 remains. And, then subtracting 8 from 50 to get 42. Finally, adding 27 to get 69.

Let us understand the decomposition strategy used by Sai.

Let us think of two 2-digit numbers, $N = a_1 \times 10 + a_0$ and $M = b_1 \times 10 + b_0$

(Note that all a's and b's stated here are digits, that is, each of them is a numeral from the set 0 to 9.)

First subtract the digits at the largest place. Here, since the largest place holders are at 'tens' place, so perform $10a_1 - 10b_1 = 10\ (a_1 - b_1)$

Then, subtract the units digit of the subtrahend from the difference obtained, $10(a_1 - b_1) - b_0$. This is fairly simple because of the multiple of 10 present in the subtraction.

Finally, add the units digit of the minuend $(10(a_1 - b_1) - b_0 + a_0 = N - M)$

By using her own algorithm, Sai has completely circumvented the problem of borrowing, a process which can be very troublesome for some children.

We could see that Sai used her indigenous algorithm to subtract numbers that involved borrowing. She found this algorithm more comfortable than the traditional borrowing method. Her algorithm could be used for any subtraction problem involving whole numbers. One may ask questions like "Can this be done for all numbers?", "Would the strategy change from number to number?", "Can this be an alternative algorithm for subtraction of numbers?"

Over the years, many people working in mathematics education have talked about children's methods of solving mathematical problems. Whenever questions such as, "Is it applicable for all numbers?" are asked, these are countered by saying, "Why should it work for all numbers for it to be interesting?" There is some truth in what they say. But if a child sees a pattern and hence develops a method to solve a particular type of problem, it would be interesting to look at the underlying mathematical principles that she used. If one is convinced by the underlying mathematical principles that were adopted, one may decide if the developed algorithm may be used only in particular situations or these can be generalized.

Though Sai's algorithm is fairly simple, we will see that sometimes children may come up with rather complicated algorithms. When Sai's classmates and schoolmates were asked the same problems, they came up with more interesting methods. Let us look at some of them:

When Poornima was asked to solve 47 – 29, she first subtracted 30 from 40 saying that it was the closest tens-number (दशक-संख्या), and she got 10. She then added 7 to the 10 and got 17. Then, she reminded her teacher that she had subtracted one extra from 40 (instead of 29 she had subtracted 30) and so, 1 is to be added to 17, thus making the answer 18.

Another student, Kajal, used a different method and following is the conversation between Kajal and her teacher.

Teacher: Kajal, what is your answer to 47 – 29?

Kajal: 18, because 4 tens means 40. From these, I subtract 2 tens or 20. So I have 20 with me. While subtracting 9 from 7, I have 2 less. These two I subtract from 20, so that the answer is 18.

Poornima and Kajal are both Sai's classmates. Looking at the various methods children use to solve these problems one wonders, "Who taught them these alternative methods?"

When Sai was asked from where she picked up her method, she said that she sometimes accompanies her mother who serves as home help in many houses. She has often seen her mother doing *hisaab* (calculations) that way. Similarly, Kajal had also seen her mother doing calculations when she sells corn. Both children had adopted the day-to-day techniques learnt by accompanying their elders.

For both of them, their lives outside school have helped them develop their mathematics within the classroom. Both of them seem very comfortable in using non-traditional algorithms adopting them in formal school problems. Although they were familiar with the techniques informally, they developed them into formal techniques by using the terminology used in school, like tens and units (ones) which is never used in out-of-school calculations.

All of us have seen such algorithms emerging in our classrooms. One such example was also observed by the second author with another student, Aman. Aman was very comfortable with multiplication facts of 12, possibly because he used to help his brother deliver eggs to shops and hotels. He solved problems like 13 × 11 as follows:

13 × 11:

13 × 12 = 156

13 × 11 = 156 – 13 = 143.

His daily activities helped Aman remember the multiplication facts of 12 but he figured out the relationship between 13 × 12 and 13 × 11 himself.

Children who, as part of their routine, take various responsibilities in their family are more likely to design their own approaches to learning. These approaches draw on what they have learnt while fulfilling their work or other responsibilities. These instances indicate that students notice mathematical patterns outside the classroom. It is another matter whether they get a chance to share what they notice. Some of the examples above, also illustrate how their noticing is coupled with formal terminology of mathematics, either to make it as an algorithm for a classroom or to take advantage of the compactness of the terminology.

Another student from Sai's school was asked

$$\begin{array}{r} 53740 \\ -\ 38999 \\ \hline \end{array}$$

This was the algorithm he used:

5 TT (ten thousands) + 3 Th (thousands) + 7 H (hundreds) + 4 T (tens) + 0 O(ones)

4 TT + 12 Th + 16 H + 13 T + 10 O

– 3 TT + 8 Th + 9 H + 9 T + 9 O

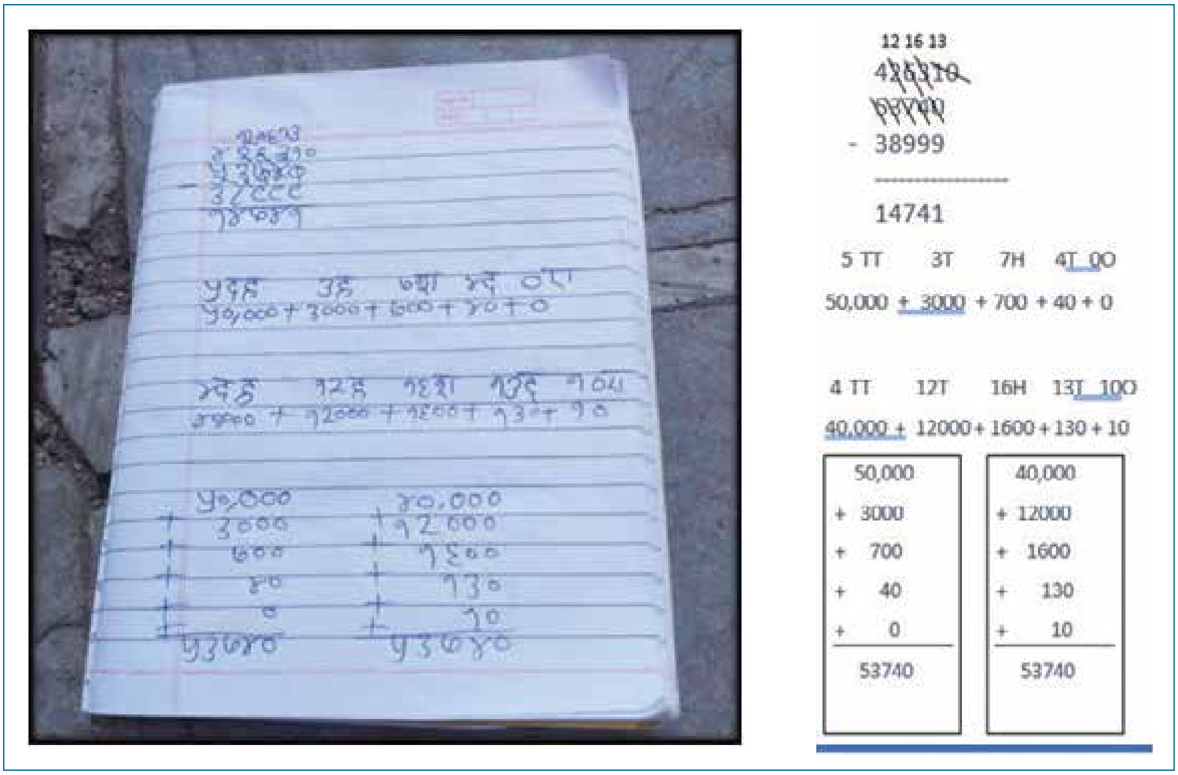
= 1 TT + 4 Th + 7H + 4 T + 1 O

= 14741

When asked how he was sure of his answer, he went on to demonstrate his method:

5 TT (ten thousands) + 3 Th (thousands) + 7 H (hundreds) + 4 T (tens) + 0 O(ones) is equal to 4 TT + 12 Th + 16 H + 13 T +10 O by actually adding each of them.

Look at the image attached.



Let us solve another problem using this student's method.

$$\begin{array}{r} {\scriptstyle 4\;16\;3\;13} \\ \not{5}\,\not{6}\,\not{4}\,\not{3}\,\not{2} \\ -\,4\,8\,3\,5\,1 \\ \hline 0\,8\,0\,8\,1 \end{array}$$

Let us try to see the *'mathematics'* behind his answer.

Let us think of a five-digit number,
N = $a_4\ a_3\ a_2\ a_1\ a_0$ so the expanded form of this number is $a_4 \times 10000 + a_3 \times 1000 + a_2 \times 100 + a_1 \text{x}\ 10 + a_0$.

Now we can rewrite this as,

$(a_4 - 1) \times 10000 + (9 + a_3\ ) \times 1000 + (9 + a_2\ ) \times 100 + (9 + a_1) \times 10 + (10 + a_0\ )$.

So we have to find the final form of the answer to: $a_4\ a_3\ a_2\ a_1\ a_0 - b_4\ b_3\ b_2\ b_1\ b_0$

(Note that if $a_4 = 1$, $a_3 = 2$, $a_2 = 3$, $a_1 = 4$, $a_0 = 5$, then $a_4\ a_3\ a_2\ a_1\ a_0 = 12345$ and not the product of 1, 2, 3, 4 and 5.)

Case 1: $a_4 > b_4$

Then,

$a_4\ a_3\ a_2\ a_1\ a_0 - b_4\ b_3\ b_2\ b_1\ b_0$
$= (a_4 - 1 - b_4) \times 10000 + (9 + a_3 - b_3) \times 1000 + (9 + a_2 - b_2) \times 100 + (9 + a_1 - b_1) \times 10 + (10 + a_0 - b_0)$

You can check that this when added to $b_4\ b_3\ b_2\ b_1\ b_0$ gives you $a_4\ a_3\ a_2\ a_1\ a_0$

Case 2: $a_4 = b_4$

Then the answer will be: $(a_3 - 1 - b_3) \times 1000 + (9 + a_2 - b_2) \times 100 + (9 + a_1 - b_1) \times 10 + (10 + a_0 - b_0)$

Case 3: $a_3 < b_3$

This will lead to an answer in integers which is beyond the scope of this article.

These algorithms work on all subtraction sums, even those with more than 5 digits! You have to try writing the same algorithm for numbers with more digits. Additionally, we have here algorithms where you can do subtraction from left to right, something which is not followed in ritualistic school-taught procedures.

Looking at these different methods used by the students, one can't help but wonder about their school and their society. All these students were from a government school in the district of Ahmednagar, in Maharashtra. The school is run by the local government, Nagar Palika.

Most of these children belong to Nomadic Tribe communities. Many of them are also first -generation school goers. Most parents leave home early in the morning and engage in small businesses such as selling corn. These students come from extremely marginalized communities.

If given a chance, where rote learning is not given a premium and independent thinking not penalized, it is likely that children come up with very different and novel methods of solving problems, that are more comprehensible to them and not a mere mechanical application of formulae or the use of algorithms which are not transparent such as the division algorithm.

Most students in these classes were seen not being scared of mathematics as their engagement in mathematics was always through their own contexts. They were seen attaching different meanings to the same operation. Often, unfamiliar problems asked in the classroom, with minimal instructions motivate students to find newer ways, apply their prior understanding, use methods which are familiar to them from their outside the classroom experiences. Their success in solving these unfamiliar problems increases their confidence in doing mathematics. They also tend to be more open to others' methods and they try to understand the solutions given by their classmates. The reason the students are so articulate in expressing their way of solving might be because giving reasons for their own methods is encouraged in their class and is an important part of their classroom culture.

It is very heartening to see more and more teachers encouraging children to think and devise methods to solve problems differently and innovatively without being bogged down by rote learning or mechanically following the formulae or algorithms. Encouraging students to devise their own methods is a powerful approach in bringing out a creative, thinking and rational student body.

MANGAL is a primary school teacher in the Nagar Palika School of Kopargaon, in the Ahmednagar district in the state of Maharashtra. She has served on various committees for textbook writing and curriculum designing for the Balbharati and the SCERT, Maharashtra. She has also been selected as a state resource person for teacher training across Maharashtra.

AALOKA KANHERE is a mathematics educator, currently working as a freelancer. She has been associated with the Homi Bhabha Centre for Science Education, Mumbai and Eklavya Foundation, Madhya Pradesh in the past. She may be contacted on aalokakanhere@gmail.com

Lessons from Proofs both False and True

ABHRONEEL GHOSH
MAHIT WARHADPANDE

Typically, it is only after a false proof has reached some absurd conclusion that one backtracks to see what went wrong. Often one learns something of interest. We wondered if we routinely miss such lessons by not analysing 'correct' proofs just as diligently. We decided to investigate.

We first summarise a popular false proof [1] showing that all triangles are isosceles. We then analyse what we would have questioned in the resolution of this false proof had it too given an absurd result. Finally, we develop our own counter to the false proof.

The False Proof: All Triangles are Isosceles

With reference to Figure 1, which shows an arbitrary $\triangle ABC$:

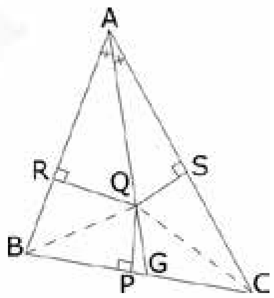


Figure 1.

Step 1. Let the angle bisector AG of $\measuredangle A$ and perpendicular bisector PQ of BC intersect at Q.

Step 2. Draw QR and QS perpendicular respectively to AB and AC.

Step 3. By Angle-Angle-Side congruence (AAS), $\triangle RAQ \cong \triangle SAQ$ so $AR = AS$. Also, $RQ = SQ$.

Step 4. Since PQ is the perpendicular bisector of BC, by Side-Angle-Side congruence (SAS), $\triangle BPQ \cong \triangle CPQ$. So $QB = QC$.

Step 5. From Steps 3 and 4, by Right-Hypotenuse-Side congruence (RHS), $\triangle RQB \cong \triangle SQC$ so $RB = SC$.

Step 6. Thus, $AR + RB = AS + SC$, i.e., $AB = AC$, i.e., $\triangle ABC$ is isosceles.

Keywords: False proof, isosceles triangle, all triangles isosceles, Euclid's Elements, Wikipedia

If Q lies outside the triangle as shown in Figure 2, the proof is identical till Step 5.

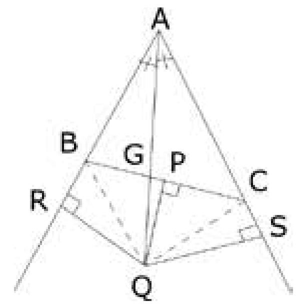


Figure 2.

Then, we say: $AR - RB = AS - SC$, thus, $AB = AC$.

The Resolution

The false proof misleads us in two places [4]:

Flaw #1: The point of intersection Q cannot lie inside the triangle as suggested by Figure 1.

But the proof works for Figure 2 (with Q outside the triangle) as well!

Flaw #2: In Figure 2, the perpendiculars QR and QS cannot both meet AB and AC respectively outside the triangle.

Figure 3 depicts how the diagram will look when drawn accurately for $AB \neq AC$. Now, $AR = AS$ and $RB = SC$ still hold, but $AB = AR - RB$ while $AC = AS + SC$ and thus, $AB \neq AC$.

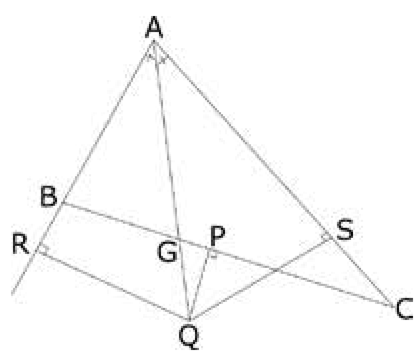


Figure 3.

Both the flaws 'worked' by making an impossible construction look feasible.

To prove that these constructions are impossible, [4] states that Q must lie on the circumcircle of $\triangle ABC$ and so must be outside the triangle. For Flaw #2, a proof is not outlined in [4] (as of this writing).

Our Journey Begins

Treating the resolution as warily as we would a false proof, we asked:

Q1. Let's analyse some available proofs of Q being on the circumcircle.

Q2. Does Q being on the circumcircle guarantee that it lies outside the triangle?

Q3. Can we approach a resolution differently? What about Flaw #2?

Before answering these questions, we need a baseline set of results so that we avoid re-inventing all of Geometry. We choose to use Euclid's *Elements* [2] as our database of 'given' results.

Thoughts on Q1

Several proofs of Q being on the circumcircle assume that Q exists, i.e., the angle bisector and the perpendicular bisector do intersect. In fact, even the false proof assumes this!

Here's a proof that does not make this assumption [3]:

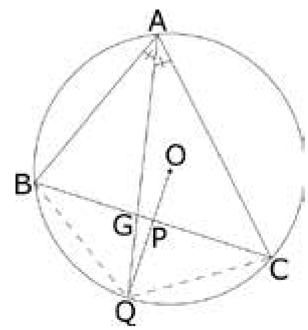


Figure 4.

(1) In Figure 4, extend AG, the bisector of $\measuredangle A$, to cut the circumcircle at Q.

(2) Join QO, where O is the centre of the circumcircle. Let QO cut BC at P.

(3) It is then shown that $\triangle BQP \cong \triangle CQP$ and hence, PQ must be the perpendicular bisector of BC.

Again, the construction looks feasible. What, if anything, would we have questioned had the end result been absurd?

From *Elements*, we know that $\triangle ABC$ and its circumcircle can be constructed, $\measuredangle A$ can be bisected and this bisector AG extended to cut the circumcircle in Q. Finally, given Q and O, we can always construct the segment QO.

But what if QO did not intersect BC?

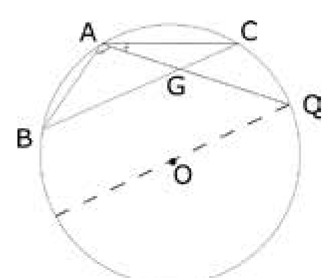


Figure 5.

Exercise 1. Is it possible that for some $\triangle ABC$, $QO \parallel BC$ as indicated in Figure 5?

Thoughts on Q2

Suggesting that a point on the circumcircle could lie inside the triangle seems absurd. But could a circumcircle behave as shown in Figure 6? If yes, then Q could be on the circumcircle AND inside (or on) the triangle! Now, the shape in Figure 6 clearly doesn't look like a circle at all, but how can we prove that it is not?

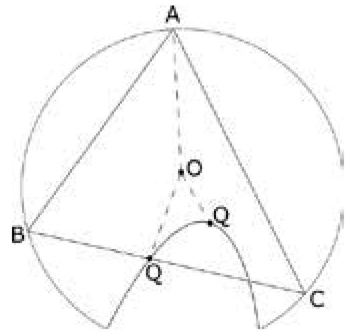


Figure 6.

Actually, for our purposes, proving a somewhat different result is sufficient:

Exercise 2. Show that, for any point Q inside or on $\triangle ABC$, not coincident with the vertices, $OQ < OA$, O being the centre of the circumcircle of $\triangle ABC$. Note: $OA (= OB = OC)$ is the radius of the circumcircle.

Or, we could make it a little more challenging:

Exercise 3. Show that $SQ < \max(SA, SB, SC)$ if S is any point inside the circumcircle.

Exercise 2 is a special case of Exercise 3 and guarantees that any point inside or on the triangle (except the vertices) can't be on the circumcircle.

Thoughts on Q3

Our own examination of the false proof went back to questioning the very existence of Q.

In Figure 7, AG is the angle bisector of $\measuredangle A$ in $\triangle ABC$ and PN is the perpendicular bisector of side BC. If $AB \neq AC$, must AG and PN intersect?

Let's assume $AC > AB$ (the case of $AB > AC$ can be similarly handled). Then, in $\triangle ABC$, $\measuredangle B > \measuredangle C$ (angle opposite larger side is larger).

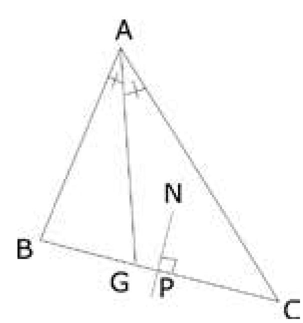


Figure 7.

By simple angle computations, we get: $\measuredangle AGC = B + A/2 > (A + B + C)/2$ since $B > C$. Hence $\measuredangle AGC > 90^\circ$. Thus, $AG \nparallel PN$. Hence AG and PN intersect, meaning, Q exists. But where?

We felt that Euclid's (in)famous Postulate 5 might help answer that question:

Euclid's Postulate 5. *If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines (produced), meet on that side.*

In Figure 8, the angle bisector AG is extended to H. We have just seen that $\measuredangle AGC > 90°$; thus $\measuredangle PGH < 90°$. Thus, $\measuredangle PGH + \measuredangle GPM < 180°$. So BC falling on AG and NP is making the interior angles below BC less than two right angles, and by Postulate 5, AG and NP must meet below BC.

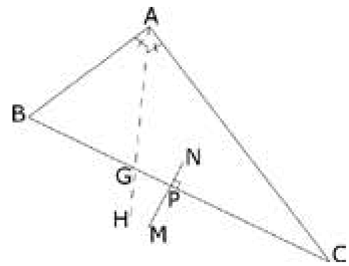


Figure 8.

And now for Flaw #2. We found in [5] a proof addressing Flaw #2. This proof uses 'reflection' and 'symmetry' which we felt, given our 'Element'ary framework, should be simplified a little.That done, we couldn't find anything we would've questioned had this too yielded an absurd result. Can you? The 'simplified' proof is outlined below:

In Figure 9, with $AB < AC$, angle bisector AG of $\measuredangle BAC$ and perpendicular bisector PQ of side BC meet at Q as shown. Since $AB < AC$, we can cut $AE = AB$ on AC and $AD = AC$ on AB-extended.

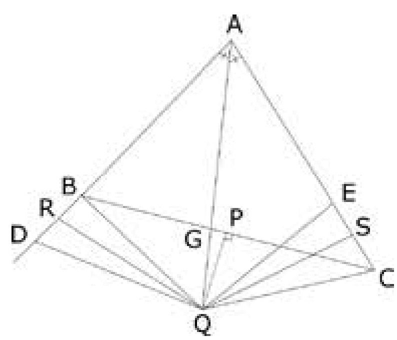


Figure 9.

Now, $\triangle BAQ \cong \triangle EAQ$ by SAS congruence [$AB = AE$ (by construction), $\measuredangle BAQ = \measuredangle EAQ$ (angle bisector) and $QA = QA$ (common)]. Thus, $QB = QE$.

Similarly, $\triangle CAQ \cong \triangle DAQ$ and $QC = QD$.

Further, $\triangle BPQ \cong \triangle CPQ$ by SAS congruence [$BP = CP$, $\measuredangle BPQ = \measuredangle CPQ = 90°$ (perpendicular bisector) and $QP = QP$ (common)]. Thus, $QB = QC$. Thus, we get $QB = QC = QD = QE$.

Thus, $\triangle CQE$ and $\triangle BQD$ are isosceles. From this point, we can continue with the proof exactly as given in [5] and outlined here. The angle bisector of $\measuredangle BQD$ must cut AD in R between B and D, i.e., outside $\triangle ABC$ while the angle bisector of $\measuredangle CQE$ must cut AC in S between E and C, i.e., inside $\triangle ABC$. Finally, we can show that QR and QS are respectively perpendicular to AB and AC.

Conclusion

This exploration taught us that the need for brevity may have hidden interesting results even within correct proofs. Perhaps more importantly, it taught us to watch out for potential oversights when constructing our own proofs.

We invite readers to examine if the proofs we have given have any gaps and how these can be resolved.

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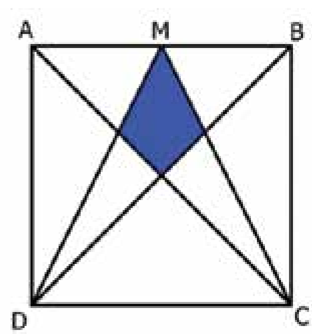
ABHRONEEL GHOSH is a student of class 12 in Delhi Public School Bangalore East. He is interested in exploring various topics in mathematics and science. He has qualified for the INMO 2021. His other hobbies include the piano, juggling, and writing. He may be contacted at abhroneel.ghosh@gmail.com.

MAHIT WARHADPANDE, a.k.a. the Jigyasu Juggler, retired after a 16-year career at Texas Instruments, Bangalore, to pursue his interests at leisure. These include Mathematics and Juggling, often in combination (see http://jigyasujuggler.com/blog/). He may be contacted at jigyasujuggler@gmail.com.

FLYING HIGH WITH MATH



ABCD is a square with edge 1 unit. AM = MB.

Find the area of the kite.

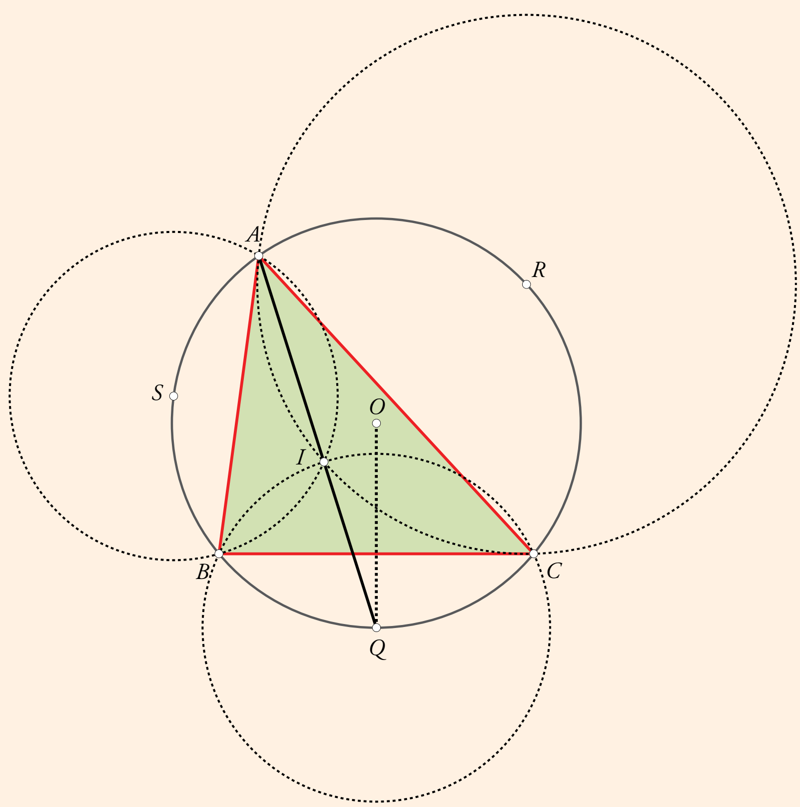
Can you find this area by more than one method?

Hint: look for symmetry and similar triangles
https://aiminghigh.aimssec.ac.za/kite-in-a-square/

Addendum to "Lessons from Proofs . . . "

In connection with the above article which makes an in-depth study of a popular fallacy in which it is proved that every triangle is isosceles, readers may be interested in the following result.

The figure shows a $\triangle ABC$ inscribed in its circumcircle (centre O); the bisector of $\measuredangle A$ cuts the circumcircle again at point Q. Let I be the incentre of $\triangle ABC$. Then it turns out that Q is the circumcentre of $\triangle IBC$. This may be proved by computing the angles of $\triangle QBI$ and verifying that $QB = QI$ (and thus by symmetry that $QC = QI$ as well). After drawing the circumcircle of $\triangle IBC$, we see that the circumcircles of $\triangle ABC$ and $\triangle IBC$ share the common chord BC.



Two more such circles can be drawn: the circumcircles of $\triangle ICA$ and $\triangle IAB$ (centres R and S respectively). The configuration makes for an interesting composition.

How Long, How High, How Wide?

ANKIT SHUKLA

This article describes and reflects upon the classroom experience of helping a group of children understand the concept of length. The activities are inspired by the Measurement PullOut published with the July 2015 issue of At Right Angles available at https://azimpremjiuniversity.edu.in/SitePages/pdf/Publications/At-Right-Angles/At-Right-Angles-Vol-4-No-2-july-2015.pdf.

During the period when I did my Fellowship with Azim Premji Foundation, I had to visit schools regularly as part of my School Understanding Process. On one of these visits, a teacher who taught mathematics to class 5 students requested that I have a discussion with the students on the measurement of length as he felt that the students had just crammed the units used to measure length along with the process of inter-conversion of these units. He felt that they lacked conceptual understanding of the topic as they didn't know why these units were required, when to use which units, how to use a metre tape, etc. Accordingly, my colleague from the Foundation and I designed a plan with the teacher.

The plan was woven around some activities which were designed in a sequence so that they would allow students to build an understanding of the related concepts. On day 1, we would do some activities which would help them to conceptualize and estimate length. Then on day 2, we would move to units of length after which we would provide them with some hands-on experience of measuring the lengths of different objects.

Understanding the Concept of Length

To get an understanding of length we did the following activities:

In the **first activity**, we divided the class into pairs. Each pair had to find out who was the taller of the two. They compared their heights by standing next to each other. Children participated very actively and figured this out easily.

Keywords: Measurement, Length, Context, Conceptual Understanding

In the **second activity**, they were asked to compare the height of the door of the classroom to the height of the shutter of the main door of the single-storey school building. In this case, they couldn't compare the heights of the objects by keeping them side by side. Some children just guessed that the shutter was longer than the door. But some students held a broomstick and extended their arm until the top of the broomstick reached the top of the shutter. Then they moved forward, holding the broomstick in the same position until they reached the door of the classroom. The top of the broomstick was lower than the top edge of the door. So, they claimed that the door was slightly higher than the shutter.

The **third activity** was about thickness – a related concept. They were divided into pairs again, each pair was given a thread and asked to compare the circumference of each other's heads **using the reasoning that the larger the circumference, the greater the thickness (diameter of head).** In this case, whoever's thread was longer, had the larger circumference.

After this, they chose to revisit the door and the shutter. The thread was used to measure the height of each and then both were found to be of the same height! This was followed by a discussion on length and its properties.

We discussed that length is the characteristic of the object in which we define how long (or high, or wide) it is. Here, we assume that the measure is in continuum, that is, the object is not in fragmented sections. We also discussed the difference between length and distance. For example, we usually say that distance between place A and B is so and so. We never say that length between A and B is so and so. However, we may speak of the length of the road which connects places A and B.

Then we discussed about the basic properties of length measurement which are as follows:

- Length of an object does not depend on the orientation of the object.
- Length never changes no matter which scale/ unit you use.
- It is independent of the unit of measurement i.e. in whichever unit you measure, the length remains the same. However, its representation may differ in different units.

Then as a part of homework, I asked them to find out the longest object in their home. On the next day, the students came up with answers as follows:

Monika: Wall of the house

Renuka: Wooden Beam of the roof

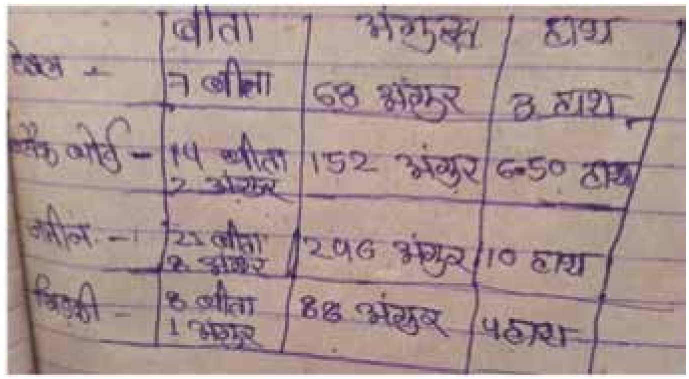
Neeraj: Electric Wire

Ghanshyam: Bamboo Stick

Leman: Rope used for drying clothes

Units: Need for and Conversions

On day 2, we got into units. After a brief revision of the previous day's discussion, we discussed some ancient measurement techniques which we still use sometimes. We talked about measuring length by using our body parts and some students elaborated what they use viz. hands as 'haath', fingers as 'anguls', full stretch from thumb to little finger as 'beeta'. Then I asked them to measure some objects with each of these three units and record the data in the given format.



Object	Handspan (Beeta)	Fingers (Angul)	Hands (Haath)
Table			
Window			

They diligently measured each object, and it took a long time. Then they recorded the readings in their notebooks. Then I asked them to compare the readings with each other. It was observed that no two readings were the same. Therefore, using body parts as unit of measurement is not good as it differs from person to person. The following story helped the students to internalise this idea.

Once upon a time, a king thought of giving his queen a new bed on her birthday. The carpenter needed the measurements for the bed, so that it would be both long enough and wide enough to fit the queen comfortably. So, the queen was asked to lie on the floor while the king measured with his foot – 3 feet wide and 6 feet long. But then the bed was made based on the little feet of the apprentice of the carpenter. Naturally, though beautiful, it was too small for the queen. The king was angry and as a result the apprentice was put in jail. But there the latter pondered over what went wrong and realized that he needed to know the size of the king's foot to make the bed. So, a marble copy of the king's foot was made and sent to the apprentice. Another bed was made with the help of the marble foot. Of course, it was a perfect fit for the queen and just in time for her birthday! Since then whenever anything needed to be made based on the size of the king's foot, the marble foot was used.

The **fourth activity** was to use a measuring tape (like the one a tailor uses or the type we get in a hardware store). We divided the class into pairs and asked them to measure each other's height with the tape. Some students were able to measure with the tape while some could not and were helped by the other students. The heights of some of the children were as follows:

Ritik: 1m 40cm	Ritik: 1m 40cm	Renuka: 1m 45cm
Tanya: 1m 41cm	Niraj: 1m 31cm	Ghanshyam: 1m 38cm

They were then asked to arrange themselves in the ascending order of their heights. This enabled them to arrange the numerical values of the heights in ascending or descending order. This in turn gave them some idea on how to compare lengths/heights expressed in such mixed units.

The students already knew a little bit about unit conversions. They were aware that there are 100 centimetres in 1 metre, and 1000 metres in 1 kilometre. They knew that a metre is longer than a centimetre. We had a discussion with the students on the different units which are used nowadays. I told them that to convert from a larger unit to a smaller one, one has to multiply and to convert from a smaller unit to a larger one, one has to divide. For example, in the question,

5 m = ________cm, metre is the larger unit compared to centimetre, so you must multiply it with 100 while in another one,

14 cm = ________________m, centimetre is the smaller unit compared to metre so you have to divide by 100 (they had completed the topic of decimals by this time).

They figured out the conversion between kilometre and metre on their own. Some students were facing difficulties with the conversions. So, we got other students to help them as follows:

Let the child who understood be C1 and one who didn't understand be C2.

C1: How many metres are there in 1 kilometre?

C2: It's simple. 1000 metres

C1: Then in 2 kilometres?

C2: 2000 metres

C1: Then in 3 kilometres?

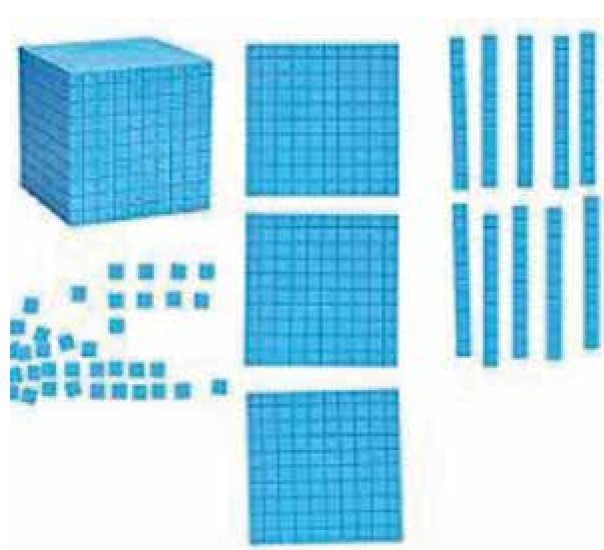
C2: 3000 metres

C1: Then in 14 kilometres?

And C2 got it and asked for more sums!!

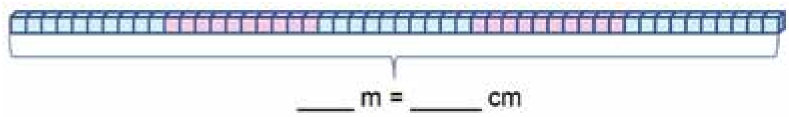
However, on hindsight, I realized how to give a sense of metre as a length.

On day 3, we had a recap of the previous day's discussion. I decided to utilize the Dienes blocks (which were available in the school) to give them a sense of centimetre and metre. Alternatively, the kit (flats, longs, units) shown in the figure can also be used. Each unit is 1cm along each dimension. Each of the longs of tens are 10 cm × 1 cm × 1 cm. So, if we line up 10 longs then the total length is 100 cm or 1 m. However, it would be difficult to create a length of 1 km using just one set since that would require 10,000 longs! It is tempting to use the 10 cm × 10 cm × 1 cm flat to show a metre or ten 10 cm × 10 cm × 10 cm

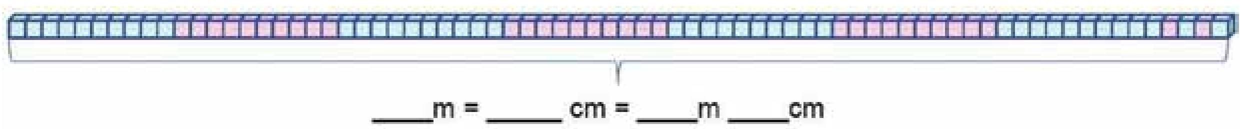


cubes to show a kilometre, but that does not convey the magnitude of the length to the students. In other words, a lot of imagination is needed to connect 100 cm, a length, to the 100 unit-cubes that make a flat though they are numerically equal. However, it is possible for students to visualise the following:

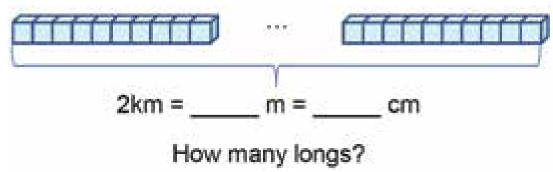
- What would be the total length of 5 longs arranged end-to-end? 50 cm = half metre = 0.5 m



- What would be the total length of 7 longs and 4 units? 74 cm = 0.74 m

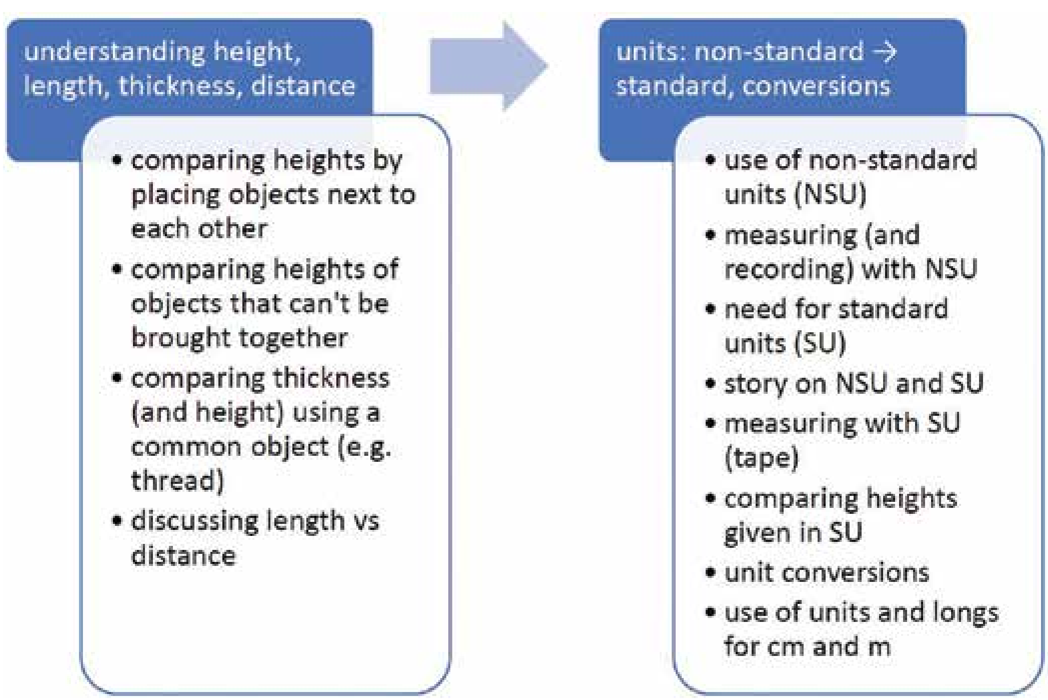


- How many longs would be needed to make a 2km long line? 2km = 2000m = 2000 × 100cm, so 20000 longs.



Thus, we discussed length in a child-centric way that I found to be effective because the students understood the concept and were able to correlate it with their daily experiences. Also, they were able to answer the questions we posed.

The flow was as follows:



The students took active part in all the activities with a lot of enthusiasm and sincerity. The activities were designed to build up their understanding gradually. Children got the scope to explore and measure, and this active engagement made their learning deeper. The discussions after each activity helped in consolidating their understanding. Peer learning was evident as they helped each other. A child, who was shy about asking the teacher, did not hesitate to ask a friend.

Thus, in two or three classes, the children went from knowing just the units and unit conversions to a deeper conceptual understanding of length, measuring things with various units and their conversions in an experiential way.

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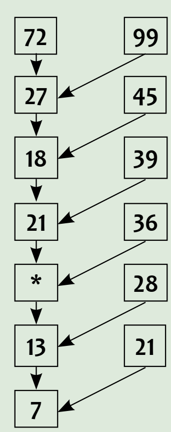
ANKIT SHUKLA did his MBA and B. Tech from Uttar Pradesh Technical University, Lucknow and joined the Fellowship Programme of Azim Premji Foundation in March 2017. After completing his fellowship, he was posted to Raigarh district and now works in the field of mathematics pedagogy. The major part of his engagement includes the capacity enhancement of government school mathematics teachers with regard to content, perspectives and pedagogy of mathematics. He also interacts with children of government schools. He has a keen interest in exploring different contexts and resources which clears misconceptions in mathematics.

WHAT APPEARS SIMPLE MAY NOT BE THAT OBVIOUS

Contributed by Wallace Jacob

Question 1: In a 10 km race, Akshay beats Harvinder by 1 km and Harvinder beats Amit by 1 km. By how many metres did Akshay beat Amit? Assume that Akshay, Harvinder and Amit run at uniform speeds throughout the 10 km race. (No, the answer is not 2000 metres. Check page **59**)

Question 2: Which number comes in place of the *? (No, the problem is not wrong. Check page **61**)



Which is the Best Deal?

AAVISHKAAR

This article presents a task in which students explore how to help a shopkeeper decide the best deal for purchasing soaps from three different wholesalers. The students express the options in the language of mathematics to find the cost for a certain number of soaps. They further write linear equations to find the changeover point at which a different shop becomes a better deal. The exploration task lets students "get inside" the mathematics and appreciate how algebra can be used in decision making. This task is suitable for students of classes 8 to 10. It covers the content domains of Arithmetic, Algebraic Expressions, Linear Equations and Coordinate Geometry.

In the current education system, students do not have a positive relationship with Mathematics. There is a lot of fear and anxiety connected to the subject as they have a very procedural relationship with it, and they are not able to connect it with the world around them. When it comes to algebra, the relationship gets further broken. Students who had learnt procedurally are now expected to solve linear equations with one or two variables, which do not make sense to them. This takes them further away from conceptual understanding. The result is that many students drop mathematics as soon as they can.

When middle-schoolers see an expression like $2x + 3$, they think of a polynomial with two terms and one variable. It does not conjure up any image of a function, nor do they think it could be a code for a pattern. When this expression becomes an equation $2x + 3 = 11$, they start thinking about how to manipulate this equation to get the value of x. They miss the **story** of this function.

The situation becomes further challenging when they see x on both sides of the equation $2x + 3 = x + 10$. Some students may know what steps to take to find the value of x. They think of it as a procedure that they have to learn to clear the exam which has no connection to their daily life. The appreciation for algebra is lost for them.

This paper presents an Exploration Task where students help a shopkeeper in deciding from where to source a

Keywords: Arithmetic, Algebraic Expressions, Linear Equations, Coordinate Geometry, Graphs, Line graph, Classroom Engagement

product and how to justify their reasoning for the decision. It is designed for students to work on making and solving linear equations with variables on both sides of the equal to sign. It may take two class sessions of 45 minutes each. This task was implemented in a 1.5 hr online session with teachers and students. The students who participated in the online session were from government schools of Delhi. The presentation of the online session was done using Power Point.

This exploration problem uses the following lesson framework:

1. **Engage:** Group warm-up through:
   - *Ganit Charchaa*, short discussion around a mathematical problem and
   - Mindset message
2. **Explore:** Work on a real-life problem in small groups.
3. **Explain:** Individuals discuss and justify their selection within their group through mathematical reasoning.
4. **Extend:** Work on extensions of the problem which may either be suggested by the facilitator or may surface as questions in the student's mind.
5. **Essence:** Understand the essence of mathematics through individual and group reflections.

Engage

The classroom engagement begins with a *GanitCharchaa* [1], a 5-7 min discussion around a mathematical problem for the whole class. The aim of the discussion is to reinforce that the mathematics classroom is a safe space to express opinions. The participants should be able to see that there are multiple ways of solving a problem, all answers are accepted and participants are encouraged to justify their thinking and to listen respectfully to others' opinions. A *GanitCharchaa* is usually connected to the topic which is to be studied and is a good way to recap the previous lesson or to informally gauge students' level of preparedness for the upcoming lesson.

GanitCharchaa of the relevant topic should be a part of every classroom irrespective of what is planned for the rest of the class. Of course, it need not only be an introductory activity, it should be part of the mathematics classroom culture.

If the sessions are being conducted online, it's best to take the answers in private chat, and then callout participants to explain their reasoning.

GanitCharchaa 1

The participants are shown Figure 1 and asked the following questions:

1. How do you see the Star-Sets grow?
2. If a picture has 28 stars what would be its Star-Set number?

Figure 1: GanitCharchaa about growing Star Sets

In the classroom sessions, when the students were asked about how they saw a pattern grow, they described what they saw rather than just count the stars, and calculate. They described their thinking, and were able to write the mathematical expression for it. They were able to create the equation for which the stars would be 28. The following Figure 2 shows some of the growth patterns that students described.

GanitCharchaa 2

Make a story problem that can be expressed in the form $25x = 300$. Change your story in such a way that it can be extended to express equations of the form $25x + 100 = 300$.

Several different stories came out e.g. One pen costs ₹ 25. If I paid ₹ 300, how many pens did I buy? Extending this further to the second

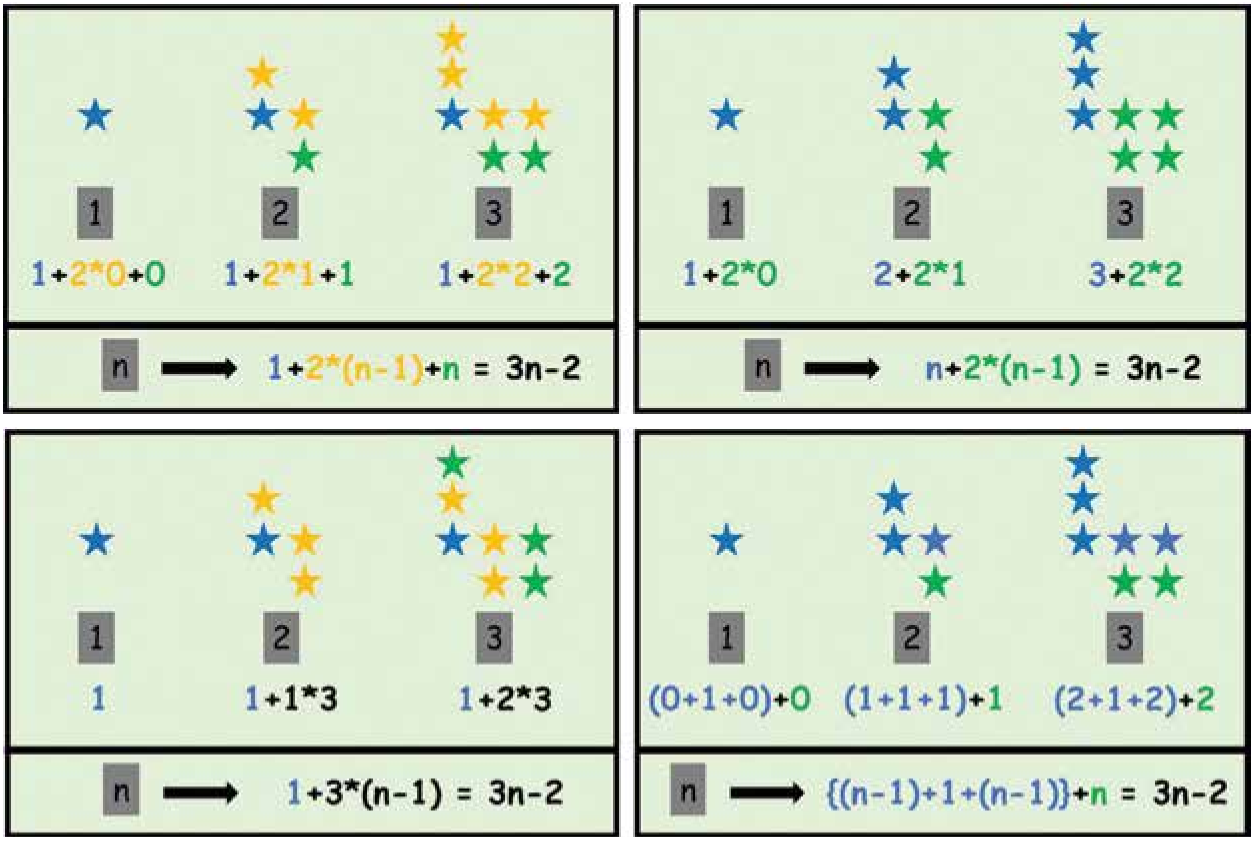


Figure 2: Some of the ways that students see the Star Set pattern growth

question: One pen costs ₹ 25. If I paid ₹ 100 in travelling to the store from home and spent a total of ₹ 300, how many pens did I buy?

Mindset Message: In our society, there are many fears and biases set against mathematics. Our society is willing to accept that some people get math while many don't. Such statements send a message to our children that math is difficult, and it's okay to not know it. It is very important for us to consistently work on breaking these biases. And our classrooms are the best place for us to start working on it.

Mindset messages [see 2] are the underlying themes of the classroom that help set the classroom culture and are intended to break the biases against mathematics that could be in the mind of students. This is also the space where we celebrate the mathematicians of today to show our students that there is still work going on in the field of mathematics. One such message is given in Figure 3.

In this class, the students teased out the problem and did not go directly to the solution; they were able to find the beauty and relevance of mathematics by this patient exploration and the mindset message certainly spoke to them.

The message celebrates Mariam Mirzakhani and the mindset message through her quote sets a tone for the session that exploring mathematics and understanding it needs consistent work and patience. There is no shortcut to it.

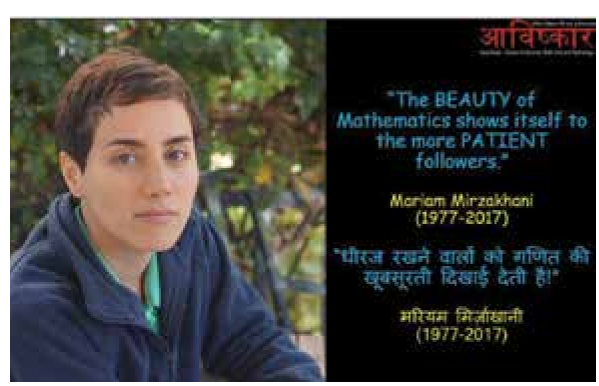


Figure 3: Mindset Message

Explore

The exploration problem presented here is aimed at delving deeper into the concept of linear expressions. This problem is set in a small village of Himachal. Aanchal runs a general store in the small village of Kandbari in Kangra Valley. She is running out of soaps at the store and needs to stock up some. She made some phone calls and collected information about the cost involved in soap purchase.

Since her village is rather small, she plans to stock 25 to 35 soaps of a particular brand. She called 3 different wholesalers in nearby towns and found out the price of the same brand of soap. Everyone gave the price of a single soap, but they all only sell the soap in packs of 5. Based on the information that she has collected, the class has to help her decide the best deal. Every group also needs to justify their thinking.

Based on Table 1, the students make initial predictions on which dealer they think is giving the best deal. After collecting responses, they work on the problem and fill out the table below. The problem is divided into four parts for students to work on. There are some key guiding questions written for teachers to help students in their thinking process.

1. In the first question students need to answer: Aanchal plans to buy 25 to 35 soaps. Which wholesaler is giving the cheapest deal? To help them answer this question and help them organize their thoughts, the following Table 2 may be provided.
2. Write a general expression for finding the cost of any number of soaps from a given wholesaler (including the travel cost).
3. Plot the general cost expression for each shop on a single graph.
4. Is there a particular number of soaps for which it will not matter which shop you bought the soap from? If yes, what is that number? How will you find out? Did you see this in the graph you drew? Justify your answer for whatever choice you make.

The students will work in their respective groups for this task and explain their reasoning in the larger group. If doing it in the same physical space, it would be best for every group to do a poster presentation of their work. In an online setting, they should be encouraged to make a slide presentation. Here are some guiding questions to ask the students in their small groups to help encourage their thinking.

Guiding questions for Part 1:

A. What will be the cost of soaps from each shop in case she decided to buy 25 soaps? How did you calculate?

B. Which is the best deal in this case (25 soaps)?

C. What will be the cost of soaps from each shop in case she decided to buy 35 soaps? How did you calculate?

D. Which is the best deal in this case (35 soaps)?[1]

#	विक्रेता Dealer	आना-जाना (रु) Travel (Rs)	दाम/साबुन (रु) Cost per soap (Rs)
1	तुलसीराम-मानचंद, पालमपुर Tulsiram-Manchand, Palampur	200	20
2	जनरल सेल्स एजेंसी, धर्मशाला General Sales Agency, Dharamsala	500	10
3	गणेश थोक विक्रेता, बीड़ Ganesh Wholesale, Bir	320	16

Table 1: Information collected by Aanchal from three different wholesalers.

#	विक्रेता Dealer	आना-जाना (रु) Travel (Rs)	कुल दाम Total Cost		
			1 साबुन (रु) 1 soap (Rs)	25 साबुन 25 soaps	35 साबुन 35 soaps
1	तुलसीराम-मानचंद, पालमपुर Tulsiram-Manchand, Palampur	200	20		
2	जनरल सेल्स एजेंसी, धर्मशाला General Sales Agency, Dharamsala	500	10		
3	गणेश थोक विक्रेता, बीड़ Ganesh Wholesale, Bir	320	16		

Table 2: Table with wholesaler data and additional columns provided to help students organize their thoughts.

[1] **Note for Teachers:** Ask the students to write the arithmetic expression for questions A and C rather than write the answer directly. It will help them see the pattern and write the general expression.

Guiding questions for Part 2 and Part 3:

A. What will be the price for purchasing "x" soaps from each shop?

Note for Teachers: Ask students to review the expression written in Part 1 for responding to this question. During the online session, some students had difficulty in this question and needed to be guided to observe their response to part 1 of the question. It seemed to happen due to their fear of "x" and the way they have interacted with algebra.

B. Is there a condition when it will not matter where she buys the soaps from?

C. If yes, find the value of "x" for which it will not matter where she buys soaps from? How would you find it? Justify your answer.

D. If not, how would you justify it? Show your work.[2]

Assuming x = number of soaps, the students should come up with the following three expressions:

Purchase cost from Palampur = $200 + 20x$

Purchase cost from Dharamsala = $500 + 10x$

Purchase cost from Bir = $320 + 16x$

The students should be able to write the equation equating the cost of two shops to find the value of x for Part C. And then calculate the price at the third wholesaler for that value of x. Some students check for all three sets of equations to justify their answer. It's important to encourage them to justify their answer in the way they see fit.

Guiding question for Part 4 before plotting the graph:

A. What will the graphs of these three equations look like? Can you predict for each wholesaler where the line would intersect with the y-axis? (They should see the fixed travel cost as the y-intercept.)

B. Where will the three lines intersect? [3]

C. Describe a situation where the line does not pass through the common intersection point in the three situations mentioned in the problem.

The students should be able to make a graph as shown in Figure 4.

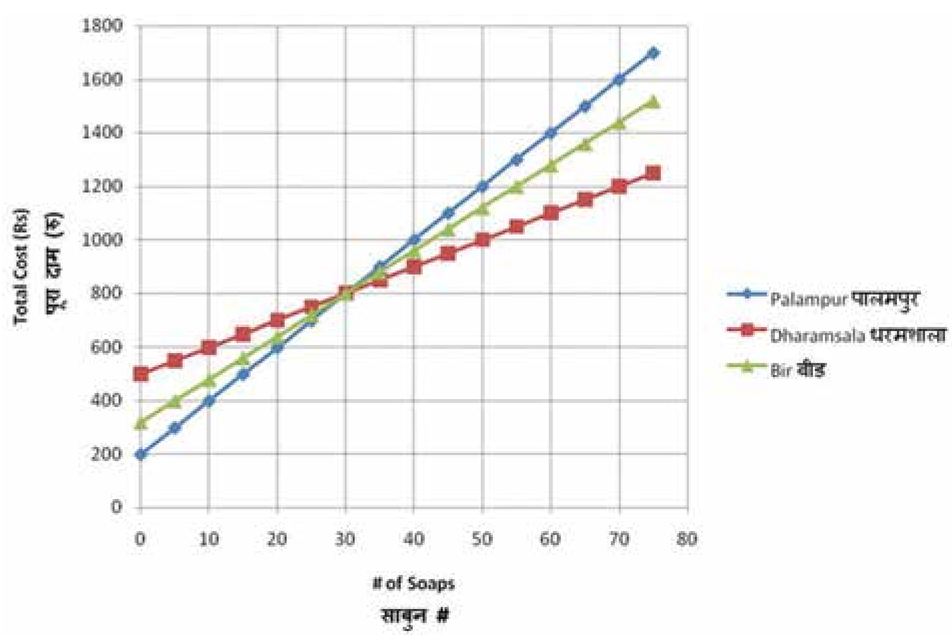


Figure 4: The cost vs # of soaps plot for the three wholesalers showing the point of intersection

2 **Note for Teachers:** During the online sessions, it was important to check for understanding of questions with the students. Some students thought that it would never be possible. Teachers encouraged students to justify their answer.

3 **Note for Teachers:** The data in this example is chosen so that the three lines are concurrent, this may not always be the case.

Explain

When students returned to the large group, they shared their responses to each question and gave a justification for it. For most of the students, the prediction of best deal without any calculation was different from what they arrived at after calculations. They were surprised by the fact that best deal for different number of soaps was different. The best deal depended on the number of soaps that were being purchased.

To find the number of soaps for which it wouldn't matter which shop Aanchal went to, several students used the trial-and-error method initially. Using the concept of equality of expressions was not intuitive to them.

For example, when students were asked to do a quick prediction of the best deal, students chose different options for different reasons.

Option 1 – Travel cost is very high compared to cost of soap, so that will decide the final price.

Option 2 – The cost of soap is least, so the high cost of travel won't matter.

Option 3 – Option 1 has highest soap cost while Option 2 has highest travel cost. Option 3 has the best of both worlds.

Filling out Table 2 made students realize that best deal will be different for different number of soaps. It was more like an "Aha!" moment as they had to do a prediction first.

Extension

Here are some possible extensions to the problem that students can work on individually or in small groups. [Note: These extensions were given in one of the online sessions when we were conducting the session with a group of teachers. The extension has not been tried with students.]

1. Aanchal wants to make a profit of Rs 400 by selling the soaps, she also knows that people will not pay more than Rs 30 for a soap. How many soaps does she need to sell to make this money? Which shop would you recommend her to buy the soap from? Why?

2. Immediately after her call to the shops, Ganesh Wholesaler calls back. He is giving one free face-mask with a pack of 5 soaps. It's Corona time, and Aanchal also sells face-masks. It costs her Rs 10 to buy a face-mask. Now which deal is the cheapest? Why? Describe your reasoning.

Essence

Appreciating its value in decision making: Many students had predicted that the Bir shop would be the best deal as the cost of the shop was in the middle and so was the travel cost. They were surprised that the best deal depended on how many soaps were being purchased.

Students were able to connect the problem to the real-life situation of shopkeepers. After completing the task one participant reacted, "Now I see why wholesalers go to faraway places to buy while the retailer buys from the local wholesaler."

There was a lot more appreciation for the fixed travel cost in any purchase. Students were able to connect it to their life. Some students questioned the value of going far away for an item if it can be purchased nearby.

The students were surprised that there was a value for which all expressions resulted in the same cost. Initially the students solved the problem of finding that value by trial-and-error. It was also insightful for them that linear equations can be used to solve for such situations and help in decision making. Making the justification visible through the use of graphs was revealing to the students.

Acknowledgement

The author is thankful to Tarun Agrawal of Aavishkaar for his support in creating this session. She would also like to thank the Trikon team of Teach For India, Delhi Fellows for implementing this lesson with their students and for sharing their constructive feedback.

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SANDHYA GUPTA is the founder of Aavishkaar: Center for Science, Maths, Arts & Technology which is located just outside Palampur, Himachal Pradesh. Aavishkaar aims to nurture curiosity, creativity and critical thinking through Science & Mathematics education. She is passionate about researching and developing creative ways of engaging with mathematics and making it visual, relevant and real for students and teachers. Sandhya earned her Ph.D. in Electrical Engineering, Iowa State University, Iowa, USA. She may be contacted at sandhya@aavishkaar-palampur.org

The Art in Numerals

Contributed by Carelin Christopher

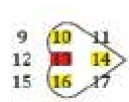
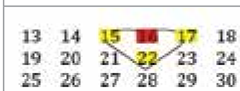
currently doing her MSc.Statistics in Viswhakarma University in Pune

Write the natural numbers in some ordered groups of 3's or 4's or 5's, for example:

1	2	3
4	5	6
7	8	9

12	13	14	15
16	17	18	19
20	21	22	23
24	25	26	27

10	11	12	13	14	15
16	17	18	19	20	21
22	23	24	25	26	27
28	29	30	31	32	33

Pattern	Chosen numbers	Operation	Answer	Finding
	12, 14, 18, 16 Vertices of the diamond	Add these four numbers and divide by 4.	$\frac{12 + 14 + 16 + 18}{4}$ = 15	This is the number at the centre of the diamond
	10, 16, 14 Corners of the left facing cardioid.	Add those 3 chosen numbers and subtract 1 from the sum. Then, divide the difference by 3	10 + 14 + 16 = 40 40 – 1 = 39 And 39/3 = 13	This is the number at the dent in the cardioid. Can you see a pattern in the right facing cardioids?
	15, 17, 22 Corners of the upward facing cardioid	Add these 3 numbers and subtract the number of numbers in each row of the array. Then, divide by 3.	15 + 17+ 22 = 54 54 – 6 = 48 (this is the array with 6 numbers in each row) And 48/3 = 16	This is the number at the dent in the cardioid. Can you see a pattern in the downward facing cardioids?

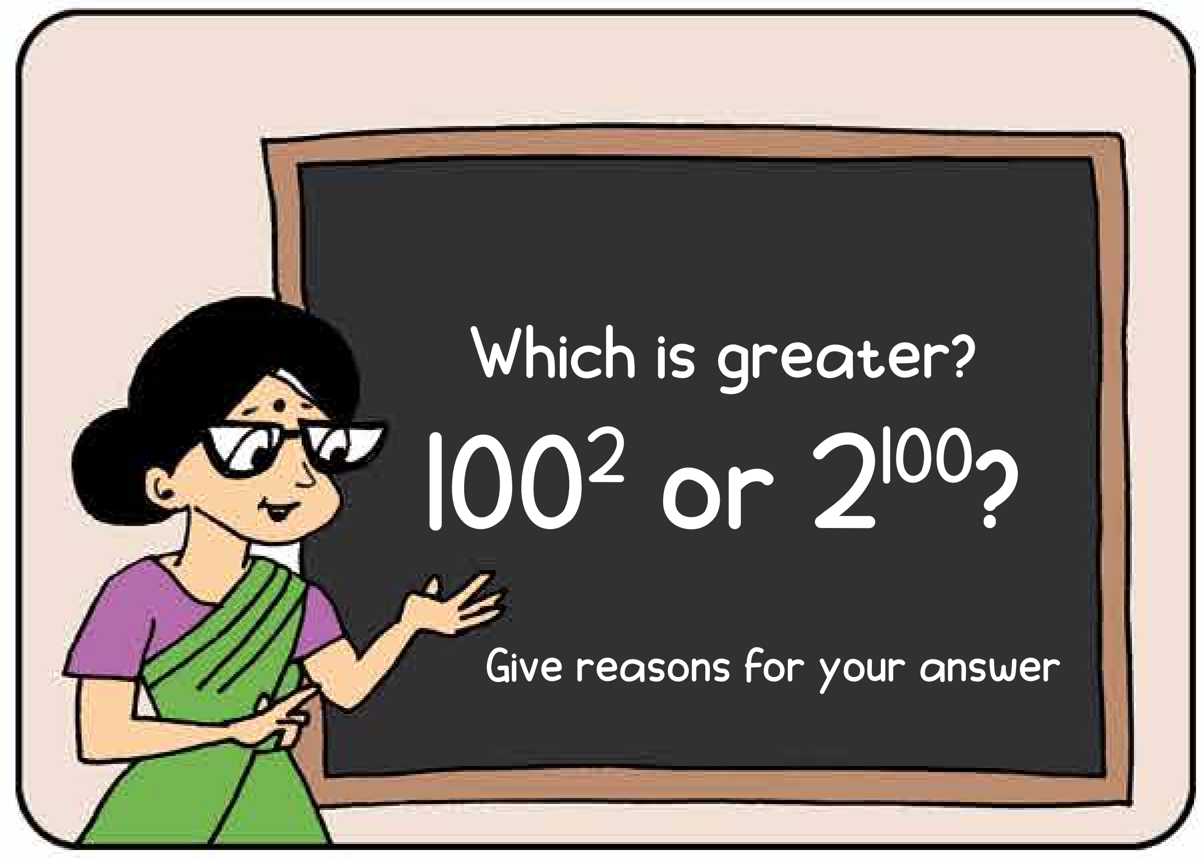
LITTLE MATHEMATICIANS

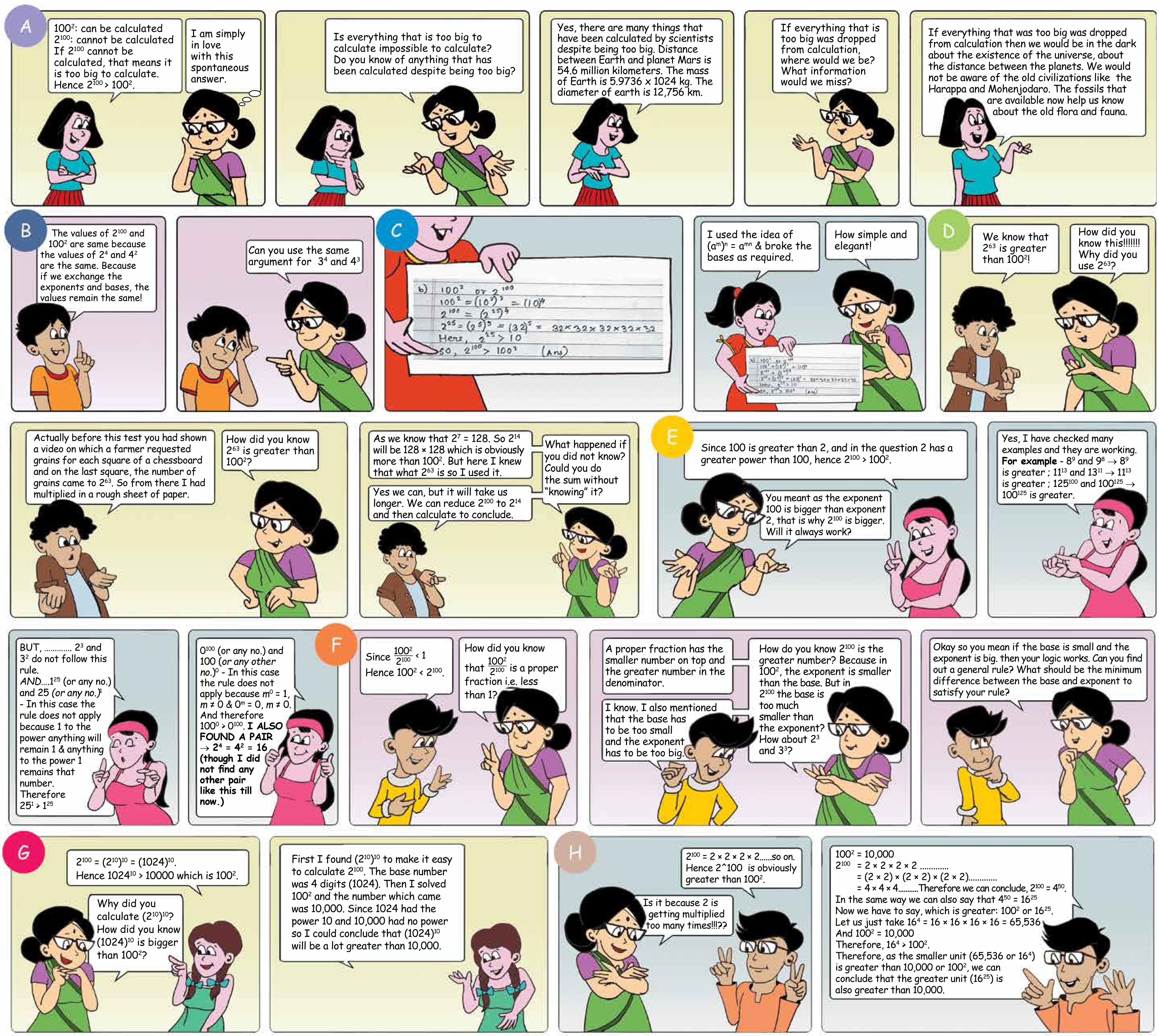
In the previous article, you read about children's algorithms for problem solving. Here, teacher Krittika Hazra shares her experience of the same. You are sure to be blown away by how these children reasoned. Note her probing questions which stimulate the child to reason, defend or backtrack on the argument presented.

Listen to Krittika: Has this ever happened to you? You have posed a question to the class and are moving around checking their answers. And then suddenly something happens. You pause, turn back to a page already checked and look at the answer once again. You wonder - how has this child arrived at this answer? This does not quite match the method you had used! And yet, it is flawless!

This is exactly what happened to me a few days back. I was amazed by the originality of the methods taken by the children and their responses to my questioning. Take a peek into my classroom.

This was my question · · · >





Mathematical Investigations

Restoring Order through Repeated Riffle Shuffles

KESHAV LAKSHMI NARASIMHAN

1. Overview

We are all familiar with the riffle shuffle, one of the most popular ways to shuffle cards. To perform this shuffle, divide the deck into two halves, riffle them together with your thumb, interlocking them, and then push the halves together.

This may seem a good way of shuffling; intuition suggests that the cards will get 'more mixed up' every time we shuffle. But is this so? Of course, we humans aren't flawless; we may not make accurate divisions or riffle the same number of cards each time. But what if the shuffle is performed perfectly? This requires dividing the deck exactly in half, riffling the cards so that the cards alternate exactly each time, and always starting the riffle with the same half, either the top (held in the right hand) or the bottom (in the left hand). A question that arises is: *If a deck of cards is repeatedly shuffled using a perfect riffle shuffle, will the cards ever come back to their original order?*

Keywords: Riffle shuffle, card deck, pseudocode, Python code

2. Solution and Proof

The answer turns out to be: **Yes!** If we do the riffle shuffle perfectly, the effect will eventually be undone. To demonstrate this, I wrote a Python code that generates a deck of cards and simulates the perfect riffle shuffle until the deck returns to its original order. The deck is displayed as a list. The pseudocode and the Python code are given in Appendix (**A.1**) available online.

3. How does a Riffle Shuffle Work?

What are the exact mechanics behind such a shuffle? Figure 1 shows how a riffle shuffle works, taking a deck of 8 cards as an example.

As you can see, we first split the deck exactly in half, with the top half in the right hand and bottom half in the left. Next, we riffle them together (release one card at a time, alternating between the two hands), starting with the left hand. Finally, we push the halves together. Note that the top and bottom cards always remain as the top and bottom cards! This method of riffle shuffling is called “out shuffling” and is the method used throughout this article.

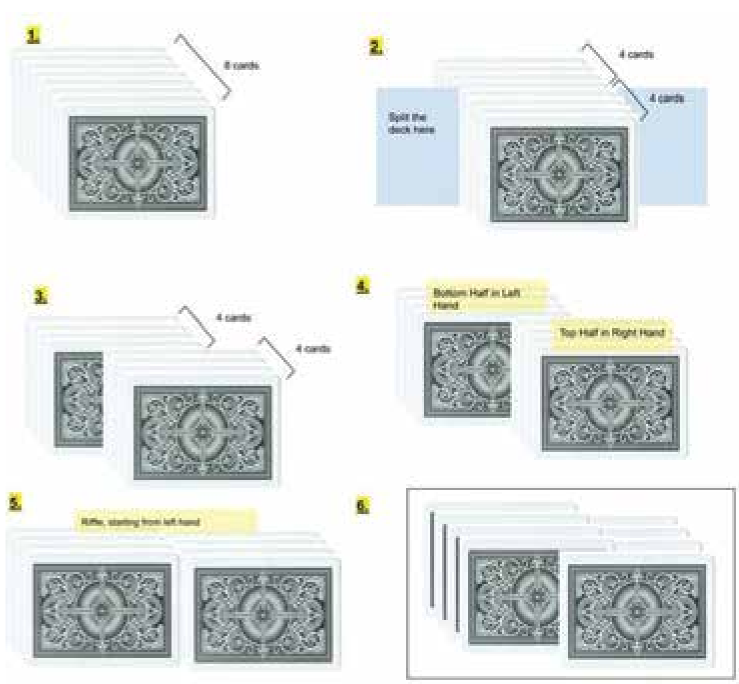


Figure 1: The process of riffle shuffling

To better understand riffle shuffles, here are a few more examples of decks with different numbers of cards that are powers of 2, the number of shuffles required to restore the original order, and the shuffling process.

Each number represents a card and each step in the shuffling process is denoted by <Cards to be held in right hand> : <Cards to be held in left hand>. Figure 2 illustrates this syntax more clearly, taking a deck of 4 cards as an example.

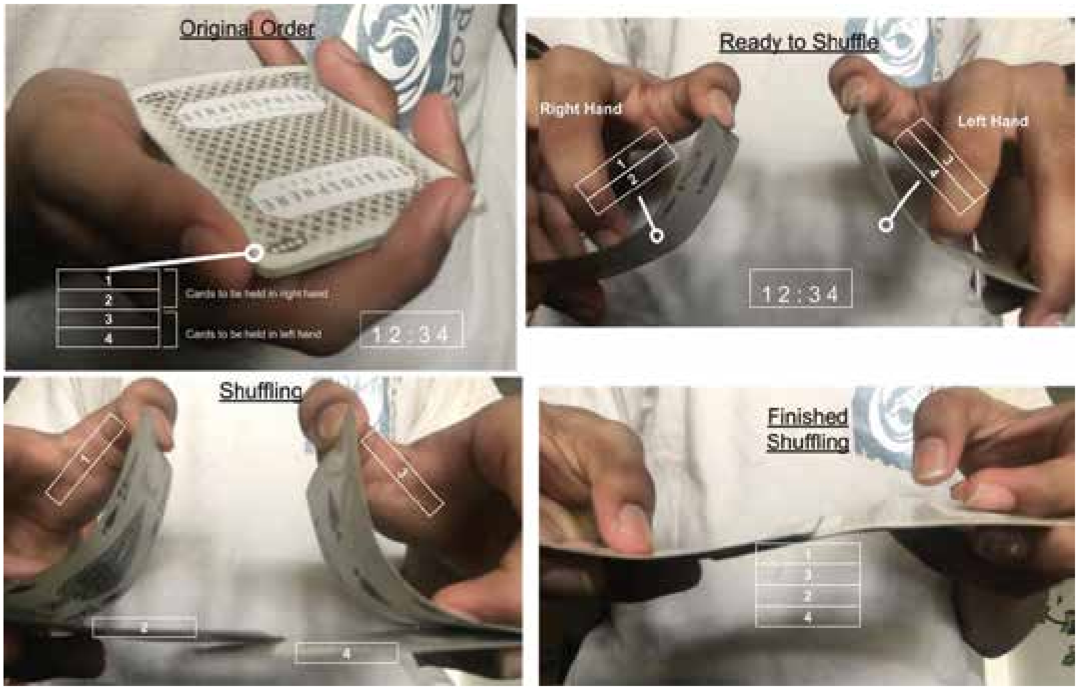


Figure 2: Explaining the deck configuration syntax for four cards.

Now, let us take a look at the table of shuffles.

No. Cards	No. shuffles required	Shuffling process	
4	2	Original	1 2 : 3 4
		After Shuffle One	1 3 : 2 4
		After Shuffle Two	1 2 : 3 4
8	3	Original	1 2 3 4 : 5 6 7 8
		After Shuffle One	1 5 2 6 : 3 7 4 8
		After Shuffle Two	1 3 5 7 : 2 4 6 8
		After Shuffle Three	1 2 3 4 : 5 6 7 8
16	4	Original	1 2 3 4 5 6 7 8 : 9 10 11 12 13 14 15 16
		After Shuffle One	1 9 2 10 3 11 4 12 : 5 13 6 14 7 15 8 16
		After Shuffle Two	1 5 9 13 2 6 10 14 : 3 7 11 15 4 8 12 16
		After Shuffle Three	1 3 5 7 9 11 13 : 15 2 4 6 8 10 12 14 16
		After Shuffle Four	1 2 3 4 5 6 7 8 : 9 10 11 12 13 14 15 16

Table 1: Shuffling configurations when the number of cards in the deck is a power of 2.

The code in section 2 generates a deck with 32 cards, numbered 1-8 for each suit. When this code was run, the deck was restored to the original order after five shuffles. Figure 3 shows the output (s represents spade, c represents clubs and so on).

```
['1s', '2s', '3s', '4s', '5s', '6s', '7s', '8s', '1c', '2c', '3c', '4c', '5c', '6c', '7c', '8c', '1d', '2d', '3d', '4d', '5d', '6d', '7d', '8d', '1h', '2h', '3h', '4h', '5h', '6h', '7h',
'8h']
['1s', '1d', '2s', '2d', '3s', '3d', '4s', '4d', '5s', '5d', '6s', '6d', '7s', '7d', '8s', '8d', '1c', '1h', '2c', '2h', '3c', '3h', '4c', '4h', '5c', '5h', '6c', '6h', '7c', '7h', '8c',
'8h']
['1s', '1c', '1d', '1h', '2s', '2c', '2d', '2h', '3s', '3c', '3d', '3h', '4s', '4c', '4d', '4h', '5s', '5c', '5d', '5h', '6s', '6c', '6d', '6h', '7s', '7c', '7d', '7h', '8s', '8c', '8d',
'8h']
['1s', '5s', '1c', '5c', '1d', '5d', '1h', '5h', '2s', '6s', '2c', '6c', '2d', '6d', '2h', '6h', '3s', '7s', '3c', '7c', '3d', '7d', '3h', '7h', '4s', '8s', '4c', '8c', '4d', '8d', '4h',
'8h']
['1s', '3s', '5s', '7s', '1c', '3c', '5c', '7c', '1d', '3d', '5d', '7d', '1h', '3h', '5h', '7h', '2s', '4s', '6s', '8s', '2c', '4c', '6c', '8c', '2d', '4d', '6d', '8d', '2h', '4h', '6h',
'8h']
['1s', '2s', '3s', '4s', '5s', '6s', '7s', '8s', '1c', '2c', '3c', '4c', '5c', '6c', '7c', '8c', '1d', '2d', '3d', '4d', '5d', '6d', '7d', '8d', '1h', '2h', '3h', '4h', '5h', '6h', '7h',
'8h']
>>>
```

Figure 3: Shuffling process output of the program for a deck of 32 cards

As you can see, the number of cards and the number of shuffles required to restore the original order has a pattern. **We observe that if the number of cards is a power of 2, say 2^n, then the number of shuffles required is n.** (Note that at this stage, it is only an observation. We have yet to justify the statement.)

Now, let us explore why this formula works.

4. Explanation

We consider the case when the number of cards in the deck is a power of 2. The general case will be dealt with later.

To understand this, let us define something called a 'slot'. A slot is merely a gap between any two consecutive cards, i.e., a place where another card can fit in.

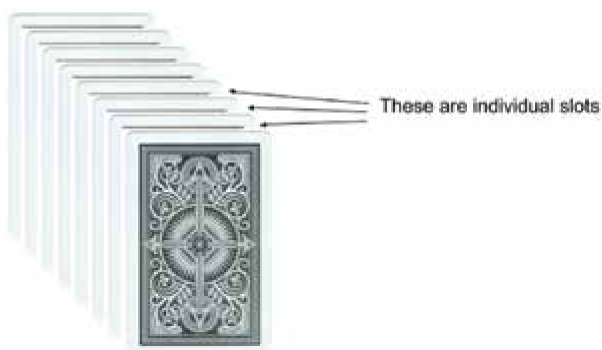


Figure 4: What is a slot?

In the riffle shuffle, since we are perfectly interlocking the cards, after each shuffle, exactly one card gets placed in each slot; that is, exactly one card gets placed between every two consecutive cards.

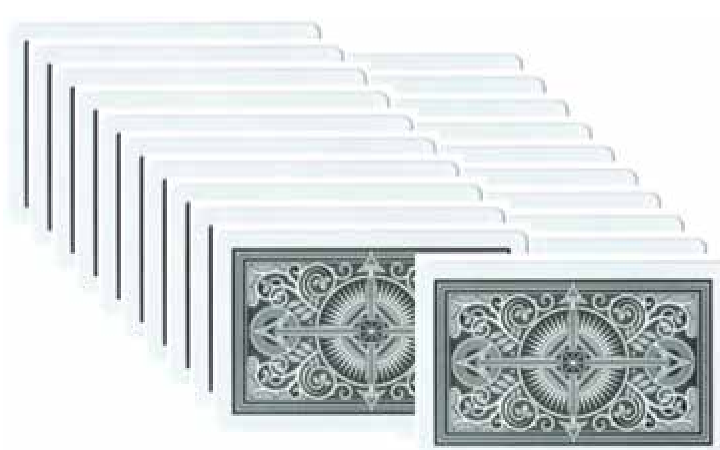


Figure 5: A riffle shuffled deck

Let us look at a deck of eight cards (as in Figure 1). The top half of the deck is held in the right hand and the bottom half, in the left hand. Let us look at the first two cards in the original order of the deck with eight cards: 1s and 2s.

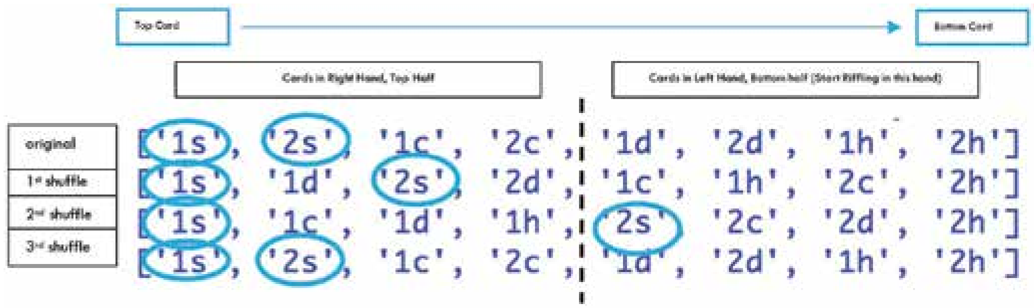


Figure 6: Focusing on the slots between 1s and 2s throughout the shuffling process

In the original order, there is one slot between 1s and 2s. After one shuffle, an additional card fills the slot and there are now 3 cards including 1s and 2s, with 2 slots between them. After the second shuffle, 2 cards fill the 2 slots, and there will now be 5 cards with 4 slots between them. After the third shuffle, there will be 5 + 4 = 9 cards with 9 – 1 = 8 slots. However, the number of cards in the deck itself is 8, which also means that there can be at most 7 slots between any two cards. So, what does this mean? 7 slots can only exist between the first and last cards, that is, 1s and 2h in the original deck order. If there are 8 slots between 1s and another card, the number of slots gets reset to 1 slot, as this is a card deck and is cyclic in shuffling. So, after three shuffles there will be 8 slots between 1s and 2s; equivalently, 1 slot. But if there is 1 slot, this obviously means that 1s and 2s have to be consecutive. Hence, after three shuffles, 1s and 2s will be consecutive once again. This idea can be applied to all consecutive cards of the deck, which will give the original order of the deck.

Now, let us take the case with 32 cards. In the original order, there is 1 slot between 1s and 2s. After the first shuffle, there will be 2 slots. After the second shuffle, there will be 4 slots. After the third shuffle, there will be 8. After the fourth, there will be 16. After the fifth, there will be 32 slots. However, the number of cards in the deck itself is 32. This means the number of slots will be reset to 1 and hence, 1s and 2s will be together again after 5 shuffles. This means that for a deck of 32 cards, it will take 5 riffle shuffles to get back to the original order.

Using this idea, we can prove the formula mentioned in Section 3. The recurring pattern seen here is that the number of slots gets multiplied by 2 with each shuffle, starting from 1, until the number of cards in the deck is reached. The number of shuffles needed for this to happen is that which will restore the original order. Therefore, 2 raised to the number of shuffles needed to restore the original order is equal to x, the number of cards in the deck. That is:

$$2^n = x.$$

Why is 2 used here instead of any other number? This is because while doing this shuffle, we split the deck into two packs, causing the number of slots to increase in powers of 2. Riffling together two packs inserts one card between any two consecutive cards, increasing the number of slots by 1. Theoretically, if we could do a riffle shuffle with three packs, the number of slots would increase in powers of 3 and we would use 3 in the formula instead.

The above logic works regardless of the initial arrangement of cards. In other words, after n shuffles, the original order of the cards will get restored.

5. General Case

So far, we have looked at the case where the number of cards (x) is a power of 2. This is why the 'slot-counting' is reset to 1 so easily. What if x is not a power of 2? The standard card deck has 52 cards, 13 in each suit. Does this reasoning still work? Yes. We consider a few cases to understand the underlying pattern. Note that x must be an even number (otherwise we would not be able to perform a riffle shuffle at all).

For example, take a deck of 12 cards and go through the shuffling process step by step. In the original deck there is 1 slot between 1s and 2s.

- After 1st shuffle: there will be a total of three cards (so 1 card between 1s and 2s) and two slots.
- 2nd shuffle: there will be 5 cards and 4 slots.
- 3rd Shuffle: there will be 9 cards and 8 slots.
- 4th Shuffle: there will be 16 slots. However, the maximum number of slots that can exist between any two cards, in our case, is 11. So, this gets reset to $16 - 11 = 5$ slots. Now, we continue as usual.
- 5th Shuffle: $5 \times 2 = 10$ slots.
- 6th Shuffle: 20 slots. This gets reset to $20 - 11 = 9$ slots.
- 7th Shuffle: $9 \times 2 = 18$ slots. This gets reset to $18 - 11 = 7$ slots.
- 8th Shuffle: 14 slots, which gets reset to 3 slots.
- 9th Shuffle: 6 slots.
- 10th Shuffle: 12 slots, which is reset to 1 slot.

Hence, a deck of 12 cards needs to be shuffled ten times to get it back into the original order. Figure 7 is a screenshot of the output when shuffling a deck of 12 cards. A slight modification was made that displays the number of slots 1s is away from 2s each time.

(Refer to **A.2** in Appendix)

```
['1s', '2s', '3s', '1c', '2c', '3c', '1d', '2d', '3d', '1h', '2h', '3h']
2s is 1 slot away.
['1s', '1d', '2s', '2d', '3s', '3d', '1c', '1h', '2c', '2h', '3c', '3h']
2s is 2 slots away.
['1s', '1c', '1d', '1h', '2s', '2c', '2d', '2h', '3s', '3c', '3d', '3h']
2s is 4 slots away.
['1s', '2d', '1c', '2h', '1d', '3s', '1h', '3c', '2s', '3d', '2c', '3h']
2s is 8 slots away.
['1s', '1h', '2d', '3c', '1c', '2s', '2h', '3d', '1d', '2c', '3s', '3h']
2s is 5 slots away.
['1s', '2h', '1h', '3d', '2d', '1d', '3c', '2c', '1c', '3s', '2s', '3h']
2s is 10 slots away.
['1s', '3c', '2h', '2c', '1h', '1c', '3d', '3s', '2d', '2s', '1d', '3h']
2s is 9 slots away.
['1s', '3d', '3c', '3s', '2h', '2d', '2c', '2s', '1h', '1d', '1c', '3h']
2s is 7 slots away.
['1s', '2c', '3d', '2s', '3c', '1h', '3s', '1d', '2h', '1c', '2d', '3h']
2s is 3 slots away.
['1s', '3s', '2c', '1d', '3d', '2h', '2s', '1c', '3c', '2d', '1h', '3h']
2s is 6 slots away.
['1s', '2s', '3s', '1c', '2c', '3c', '1d', '2d', '3d', '1h', '2h', '3h']
2s is 1 slot away.
>>>
```

Figure 7: Shuffling process output for a deck of 12 cards, along with the number of slots after each shuffle

Here are a few more examples of shuffling decks where x is not a power of 2 (the notation is the same as that in Table 1, but with a new column containing the number of slots between the first and second cards).

A python code was used to generate the following shuffling steps (Refer to **A.3** in Appendix).

No. Cards	No. shuffles required	Shuffling process		
10	6	Original	1 2 3 4 5 : 6 7 8 9 10	1
		After Shuffle One	1 6 2 7 3 : 8 4 9 5 10	2
		After Shuffle Two	1 8 6 4 2 : 9 7 5 3 10	4
		After Shuffle Three	1 9 8 7 6 : 5 4 3 2 10	8
		After Shuffle Four	1 5 9 4 8 : 3 7 2 6 10	7
		After Shuffle Five	1 3 5 7 9 : 2 4 6 8 10	5
		After Shuffle Six	1 2 3 4 5 : 6 7 8 9 10	1
14	12	Original	1 2 3 4 5 6 7 : 8 9 10 11 12 13 14	1
		After Shuffle One	1 8 2 9 3 10 4 : 11 5 12 6 13 7 14	2
		After Shuffle Two	1 11 8 5 2 12 9 : 6 3 13 10 7 4 14	4
		After Shuffle Three	1 6 11 3 8 13 5 : 10 2 7 12 4 9 14	8
		After Shuffle Four	1 10 6 2 11 7 3 : 12 8 4 13 9 5 14	3
		After Shuffle Five	1 12 10 8 6 4 2 : 13 11 9 7 5 3 14	6
		After Shuffle Six	1 13 12 11 10 9 8 : 7 6 5 4 3 2 14	12
		After Shuffle Seven	1 7 13 6 12 5 11 : 4 10 3 9 2 8 14	11
		After Shuffle Eight	1 4 7 10 13 3 6 : 9 12 2 5 8 11 14	9
		After Shuffle Nine	1 9 4 12 7 2 10 : 5 13 8 3 11 6 14	5
		After Shuffle Ten	1 5 9 13 4 8 12 : 3 7 11 2 6 10 14	10
		After Shuffle Eleven	1 3 5 7 9 11 13 : 2 4 6 8 10 12 14	7
		After Shuffle Twelve	1 2 3 4 5 6 7 : 8 9 10 11 12 13 14	1
24	11	Original	1 2 3 4 5 6 7 8 9 10 11 12 : 13 14 15 16 17 18 19 20 21 22 23 24	1
		After Shuffle One	1 13 2 14 3 15 4 16 5 17 6 18 : 7 19 8 20 9 21 10 22 11 23 12 24	2
		After Shuffle Two	1 7 13 19 2 8 14 20 3 9 15 21 : 4 10 16 22 5 11 17 23 6 12 18 24	4
		After Shuffle Three	1 4 7 10 13 16 19 22 2 5 8 11 : 14 17 20 23 3 6 9 12 15 18 21 24	8
		After Shuffle Four	1 14 4 17 7 20 10 23 13 3 16 6 : 19 9 22 12 2 15 5 18 8 21 11 24	16
		After Shuffle Five	1 19 14 9 4 22 17 12 7 2 20 15 : 10 5 23 18 13 8 3 21 16 11 6 24	9
		After Shuffle Six	1 10 19 5 14 23 9 18 4 13 22 8 : 17 3 12 21 7 16 2 11 20 6 15 24	18
		After Shuffle Seven	1 17 10 3 19 12 5 21 14 7 23 16 : 9 2 18 11 4 20 13 6 22 15 8 24	13
		After Shuffle Eight	1 9 17 2 10 18 3 11 19 4 12 20 : 5 13 21 6 14 22 7 15 23 8 16 24	3
		After Shuffle Nine	1 5 9 13 17 21 2 6 10 14 18 22 : 3 7 11 15 19 23 4 8 12 16 20 24	6
		After Shuffle Ten	1 3 5 7 9 11 13 15 17 19 21 23 : 2 4 6 8 10 12 14 16 18 20 22 24	12
		After Shuffle Eleven	1 2 3 4 5 6 7 8 9 10 11 12 : 13 14 15 16 17 18 19 20 21 22 23 24	1

Table 2: Shuffling processes of different generic decks

The same approach can be applied to a deck of 52 cards. Here is a table of the shuffle no. and the number of slots:

Shuffle no.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Slots	2	4	8	16	32	13	26	52 (1)

Table 3: Number of slots for each shuffle in a standard deck of 52 cards

As we can see, the pattern repeats. After each shuffle, the number of slots between 1s and 2s gets doubled. If it is greater than the maximum number of slots (total # of cards -1), the maximum number of slots is subtracted from it. This continues until the number of slots reaches 1. Using this reasoning, a general formula can be derived, for any given number of cards. If s = the number of slots between 1s and 2s, t = the total number of cards, and n = number of shuffles, then

$$2^n \equiv s \bmod (t-1).$$ **This is generally written in computer languages as s = 2n % (t – 1).**

This formula works for any number of cards. In order to find out how many shuffles it takes to bring the deck back into original order, plug in s as 1, plug in the value of t and find n. This may take time as n cannot be made the 'subject' of the formula so easily. For a given number of cards in the deck (t), and s = 1, there are infinitely many pairs (n, s) which will satisfy the above equation. However, our interest is in the *least* possible integer value of n. This value is best found using computer-based enumeration. For example, take the case of an ordinary deck of 52 cards. We must solve the equation

$$2^n \equiv 1 \bmod 51.$$

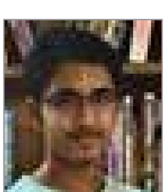
Starting with $n = 1$, we compute the remainders in the divisions $2^n \div 51$ by starting with 2, doubling repeatedly and throwing away multiples of 51, and continuing till we reach remainder 1. We get the remainders shown in Table 3, i.e., the following numbers: 2, 4, 8, 16, 32, 13, 26, 1… Since a remainder of 1 is first reached when $n = 8$, it means that for a deck of 52 cards, 8 riffle shuffles are required to return to the original order.

I wrote a python program (**A.4** in Appendix) that finds the required number of shuffles. Using this, the number of times needed to riffle shuffle a deck to restore it to its original order can be found with ease. While the program mentioned in the beginning of the paper simulates the riffle shuffle, this program uses a formula-based approach.

6. Conclusion

To sum up, though it may seem at first sight that using a perfect riffle shuffle repeatedly will mix the deck more and more, we find in fact that it must necessarily return to the original order after some shuffles. This is counter-intuitive but true.

But don't worry, because of our imperfect nature, I guarantee that you are safe using this way to shuffle your cards!



KESHAV LAKSHMI NARASIMHAN is a 10th standard student living in Chennai. He has been an avid learner of mathematics since his elementary grades and is fond of solving challenging problems. He sees problem-solving as an opportunity to understand the universe. He takes inspiration from his other hobbies such as art, python programming and cardistry to ask questions which challenge his reasoning skills. He may be contacted at kl77.airsim@gmail.com.

Tom and Jerry Play with Fractions

PRITHWIJIT DE

In the current school term in Cat and Mouse Academy, Tom, the cat, is teaching algebra to Jerry, the mouse. Having covered the basics Tom is now in a mischievous mood and is challenging Jerry with brainteasers. Instead of revealing the challenge at once he builds it up gradually in his characteristic style.

So he writes on the board:

$$\frac{1}{4} = \frac{16}{64} = \frac{166}{664} = \cdots = \frac{1\overbrace{666\ldots6}^{n}}{\underbrace{666\ldots6}_{n}4} = \cdots,$$

turns to Jerry and asks him if the relationship is true for all integers $n \geq 0$.

Little Jerry quickly checks that

$$\frac{1}{4} = \frac{16}{64} = \frac{166}{664} = \frac{1666}{6664}$$

and wonders how to tackle the general term. After some time, he figures out that

$$\underbrace{666\ldots6}_{n}4 = 60\left(\frac{10^n - 1}{9}\right) + 4 = 4\left(\frac{15 \cdot 10^n - 6}{9}\right)$$

and

$$1\underbrace{666\ldots6}_{n} = 10^n + 6\left(\frac{10^n - 1}{9}\right) = \frac{15 \cdot 10^n - 6}{9}$$

to observe to his delight that indeed

$$\frac{1}{4} = \frac{1\overbrace{666\ldots6}^{n}}{\underbrace{666\ldots6}_{n}4}.$$

Keywords: Decimals, recurrence, fractions, patterns

Moving on. Impressed by Jerry's response, Tom moves a step ahead. He asks Jerry to find out if there are other proper fractions with the same property. To help Jerry understand the problem, Tom spells it out in clear terms as stated below.

Challenge problem. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find all $a, b \in S$ with $a < b$ such that given a and b, there exists $c \in S$ (with $c \neq a,\ c \neq b$) for which

$$\frac{a}{b} = \frac{a\,\overbrace{ccc\ldots c}^{n}}{\underbrace{ccc\ldots c}_{n}\,b}$$

for all integers $n \geq 0$.

With his task cut out, Jerry summons up courage and begins his investigation. He immediately observes that he had already found $(a, b, c) = (1, 4, 6)$ to be a solution and decides to mimic the steps that he followed to find that solution. After a brief bout with algebra, he obtains

$$\frac{a}{b} = \frac{(9a + c)10^n - c}{c.10^{n+1} + 9b - 10c}$$

which leads to

$$9a(c - b) = c(b - a).$$

He observes that 9 divides $c(b - a)$ and $(b - a) \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. Now Jerry makes a clever move as he often does to heckle Tom. He considers two cases:

(I) 3 does not divide $b - a$;

(II) 3 divides $b - a$.

In case (I) he readily obtains $c = 9$ and

$$b = \frac{10a}{a + 1} = 10 - \frac{10}{a + 1}$$

to conclude that $a + 1$ must divide 10. So $a \in \{1, 4\}$ and $(a, b, c) = (1, 5, 9)$ or $(4, 8, 9)$.

In case (II) he observes that $b - a \in \{3, 6\}$ and decides to treat the two sub-cases separately. If $b - a = 3$ then he obtains

$$c = \frac{3ab}{3a - 1} = \frac{3a(a + 3)}{3a - 1} = a + 3 + \frac{a + 3}{3a - 1}$$

and asserts that for c to be an integer $3a - 1$ must divide $a + 3$ and since both are positive integers, $3a - 1 \leq a + 3$ or $a \leq 2$. He quickly disposes of the two possibilities $a = 1$ and $a = 2$ to obtain $(a, b, c) = (1, 4, 6)$ or $(2, 5, 6)$.

If $b - a = 6$, he finds

$$c = \frac{3a(a + 6)}{3a - 2}$$

and uses the fact $c \leq 9$ to obtain

$$9 - \frac{3a(a + 6)}{3a - 2} \geq 0,$$
$$\therefore\ a^2 - 3a + 6 \leq 0 \quad \text{(after simplification).}$$

But

$$a^2 - 3a + 6 = \left(a - \frac{3}{2}\right)^2 + \frac{15}{4} \leq 0,$$

an absurd result. So he rules out this possibility.

Having exhausted all possibilities he jumps in joy and claims that the only possible fractions are

$$\frac{1}{5} = \frac{1\overbrace{999\ldots9}^{n}}{\underbrace{999\ldots9}_{n}5}, \qquad \frac{1}{4} = \frac{1\overbrace{666\ldots6}^{n}}{\underbrace{666\ldots6}_{n}4},$$

$$\frac{2}{5} = \frac{2\overbrace{666\ldots6}^{n}}{\underbrace{666\ldots6}_{n}5}, \qquad \frac{4}{8} = \frac{4\overbrace{999\ldots9}^{n}}{\underbrace{999\ldots9}_{n}8}.$$

The show is not over. Highly encouraged by this success Jerry asks the following question.

Question. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find all $a, b \in S$ with $a < b$ such that given a and b, there exists $c \in S$ (with $c \neq a$, $c \neq b$) for which

$$\frac{a}{b} = \frac{\overbrace{ccc\ldots c}^{n}a}{b\underbrace{ccc\ldots c}_{n}}$$

for all integers $n \geq 0$.

Tom is still searching for the answer. Can you help Tom?

PRITHWIJIT DE is a member of the Mathematical Olympiad Cell at Homi Bhabha Centre for Science Education (HBCSE), TIFR. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. His other interests include puzzles, cricket, reading and music. He may be contacted at de.prithwijit@gmail.com.

Solution to Question on Page 41

The method which I used for answering the question is detailed below:

If Akshay beat Harvinder by 1 km in a 10 km race, then it implies that in the time interval in which Akshay covers 10 km, Harvinder covers 09 km. Thus, the ratio of the speeds of Akshay and Harvinder is 10 : 9. Similarly, the ratio of the speeds of Harvinder and Amit is 10 : 9.

Akshay	:	Harvinder	:	Amit
10	:	9		
		10	:	9

Thus, the ratio of their speeds is 100 : 90 : 81

If Akshay covers 100 metres in a certain time period, then Amit covers 81 metres in the same time period. If Akshay covers 10000 metres in a certain time period, then Amit covers 8100 metres in the same time period. Akshay beat Amit by 1900 metres.

A Property of the Centroid of a Triangle

UJJWAL RANE

In this note, we establish an unexpected property of the centroid of a triangle.

Given any triangle ABC, let P be an arbitrary point lying within the triangle. Drop perpendiculars PD, PE, PF to the sides BC, CA, AB respectively. We ask: For which point P does the product $PD \cdot PE \cdot PF$ take its largest possible value?

We shall show that for any triangle, $PD \cdot PE \cdot PF$ *takes its maximum value when* P *lies at the centroid of the triangle*.

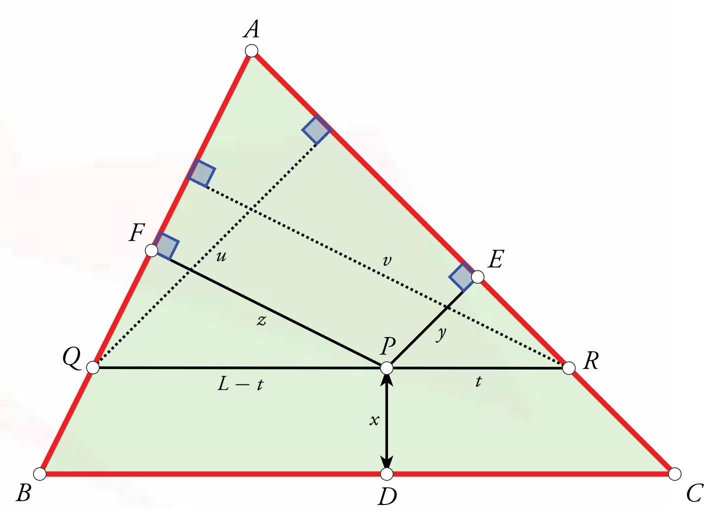


Figure 1.

Keywords: Centroid, perpendicular, product, similar triangles, quadratic, maximum

For any point P within the triangle, let x, y, z be the distances from P to BC, CA, AB respectively ($x = PD$, $y = PE$, $z = PF$). To start with, let x be kept constant. The set of points P such that x has any given fixed value is a segment QR parallel to side BC and at distance x from it (see Figure 1). Let u and v be the perpendicular distances from Q to AC and R to AB, respectively. Let L be the length of QR. Let $PR = t$; then $QP = L - t$.

Using similarity we have:

$$\frac{y}{u} = \frac{PR}{QR}, \quad \therefore\ y = \frac{tu}{L},$$
$$\frac{z}{v} = \frac{QP}{QR}, \quad \therefore\ z = \frac{(L-t)v}{L}.$$

For any given fixed value of x, both u and v are constants, as is L. Hence:

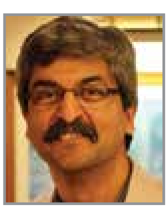
$$yz = t(L-t) \times \text{some constant which depends only on } x.$$

The variable component on the right side is the quadratic expression $t(L - t)$, which achieves its maximum value when $t = L/2$, i.e., when P lies at the midpoint of QR. (Recall that if the sum of two non-negative numbers is a positive constant, then their product takes its largest value when the two numbers are equal.)

So, for each value of x, the optimal location of P is the midpoint of QR. That is, for each value of x, the optimal location of P lies on the median of the triangle through vertex A.

By symmetry, the optimal location of P must also lie on the medians through vertices B and C. This implies that the optimal location is the centroid of the triangle.

Since the perpendicular distance of the centroid from each side is $1/3$ of the corresponding altitude, it follows that the largest possible value of the product of perpendicular distances from the sides is equal to $1/27$ of the product of the three altitudes. □



UJJWAL RANE is a Mechanical engineer with a M.Tech. (Machine Dynamics) from IIT Madras and a M.S. from Arizona State University in Computer Aided Geometric Design. He likes to find visual representations and solutions of mathematical and physical phenomena, and uses this approach in teaching Engineering, Physics and Math and also online via his channel on YouTube (https://www.youtube.com/user/UjjwalRane). He may be contacted at ujjukaka@gmail.com.

Solution to Question on Page 41

Several of my students have answered that 15 comes in place of the asterisk and there is an error in the question: The number 8 should appear in the shaded box instead of 7.

The answer is NOT 15. The question is NOT wrong. The answer is 12.

7 + 2 + 9 + 9 = 27
2 + 7 + 4 + 5 = 18
1 + 8 + 3 + 9 = 21
2 + 1 + 3 + 6 = 12
1 + 2 + 2 + 8 = 13
1 + 3 + 2 + 1 = 7

Therefore, 12 comes in place of *.

Middle School Problems on the Weighted Average

A. RAMACHANDRAN

A bit of theory before we present the problems.

The average (or arithmetic mean) of two numbers a and b is given by $\frac{a+b}{2}$. The average of three numbers p, q and r is given by $\frac{p+q+r}{3}$. These expressions could be rewritten as

$$\frac{1}{2}a + \frac{1}{2}b \text{ and } \frac{1}{3}p + \frac{1}{3}q + \frac{1}{3}r,$$

showing that the various components contribute similarly to the average value. However, in some circumstances one component may have a greater bearing on the average value. For instance, if 30 litres of water at a temperature of 80°C and 20 litres of water at 20°C are mixed together (without loss or gain of heat), the temperature of the mixture is not 50°C (arithmetic mean of 80° and 20°), but is given by

$$80^\circ \times \frac{30}{50} + 20^\circ \times \frac{20}{50} = 56^\circ\text{C}.$$

Each temperature (the variable in this case) is multiplied by the volume fraction of that component and added to get the 'weighted average.' The volume fractions are the 'weights,' which always add up to unity. Use this idea to solve the following problems.

1. The average monthly income of the 28 men working in an office is Rs.80,000, while that of the 22 women working there is Rs.72,000. What is the overall average income of an employee?
2. A trader sells seven-tenths of his stock of grains at 60% profit, one fifth at 20% profit and the rest at 10% loss. What is his overall profit percentage?
3. A trader acquires a stock of electric heaters at a certain wholesale price. He sells four-fifths of his stock at 80% profit, but the rest are completely destroyed in a fire accident. What is the overall profit percentage achieved by him in the deal?
4. Nine litres of a 6% solution of a salt in water is mixed with 15 litres of another solution of the same salt in water. If the resulting mixture has a salt content of 8.5%, what was the percentage of salt in the latter? By 'percentage solution' we mean the number of grams of solute in 100 ml. of solution.
5. From stocks of a 9% solution and a 16% solution of pesticide, a farmer wants to obtain a 10.5% solution. In what ratio should he mix the two stock solutions?
6. How much pure water should be added to 10 litres of a 7% solution of a salt to obtain a 3% solution?
7. Masses of 5 units, 7 units and 8 units are placed at locations −8, 0 and +15, respectively, on the X-axis. Where would their centre of mass be located?

Solutions

1. The weighted average of the given incomes (which is the variable in this case) is given by

$$80,000\times\frac{28}{50}+72,000\times\frac{22}{50}=\text{Rs.}\,76,480.$$

2. We can consider loss to be negative profit, and form the expression

$$60\%\times\frac{7}{10}+20\%\times\frac{1}{5}-10\%\times\frac{1}{10}$$

which equals 45% profit on the whole. It is not necessary in such situations to know the actual cost price, selling price or the quantity sold.

3. Destroyed goods indicate a 100% loss. So we can form the expression

$$80\%\times\frac{4}{5}-100\%\times\frac{1}{5}$$

which leads to 64% – 20% = 44% profit.

4. Here, the solution strength is the variable. Let the unknown solution percentage be P%. Then $6\%\times\frac{9}{24}+\text{P}\%\times\frac{15}{24}=8.5\%$. On simplification, this yields

$$15\text{P}=8.5\times 24-54 \text{ or } P\%=10\%.$$

5. Let the required ratio be a:b. Then $9\%\times\frac{a}{a+b}+16\%\times\frac{b}{a+b}=10.5\%$. This equation leads to $9a+16b=10.5(a+b)$ or $1.5a=5.5b$ or $a:b=11:3$.

6. Pure water can be considered a 0% solution. If we take the volume of pure water required to be x litres, then we can form the equation

$$7\%\times\frac{10}{10+x}+0\%\times\frac{x}{10+x}=3\%.$$

This leads to $70=3(10+x)$ or $x=\frac{40}{3}$ litres.

7. Here the position on the X-axis is the variable. So we form the expression

$$-8\times\frac{5}{20}+0\times\frac{7}{20}+15\times\frac{8}{20}$$

which yields +4 as the location of the centre of mass.

Secondary

Problem Corner

The Bevan Point and Associated Points and Circles

HANS HUMENBERGER

The following note combines aspects of mathematics (geometry), history, heuristics (problem-solving), dynamic geometry software and pedagogy. Although it may go a bit beyond the curriculum at school, the mathematics and arguments stay elementary. In the second part it should be a vivid demonstration that simply trying things and seeing what happens – i.e., fostering the process of 'doing mathematics' – can lead to nice results that can also be useful for teaching situations.

In 1804 the British engineer Benjamin Bevan posed a problem [1] of Euclidean geometry (it was solved in the same year by John Butterworth):

Problem. Given a triangle $\triangle ABC$ with incentre I, circumcentre O, and excentres I_a, I_b, I_c. Let V be the circumcentre of the excentral triangle $\triangle I_aI_bI_c$ (see Figure 1). Prove the following:

(1) V is the reflection of I in O;
(2) The circumradius of $\triangle I_aI_bI_c$ is twice the circumradius of $\triangle ABC$.

General remarks. Owing to the history of the problem, the circumcircle of the excentral triangle $\triangle I_aI_bI_c$ is called the *Bevan circle* and its centre the *Bevan point* (point X(40) in Clark Kimberling's *Encyclopedia of Triangle Centers* or 'ETC'). In some approaches the nine-point-circle is involved (especially when other properties also need to be proved), but if one restricts to (1) and (2), a weaker result suffices, which is accessible also to students, e.g., in a problem-solving class. Let us have a closer look at the above. If we can prove that the circumcircle of $\triangle ABC$ bisects the segments joining the incentre I to the excentres (points of bisection being D, E, F, see Figure 2; let us call this Lemma 1), then we are done, because then we can conclude: Under an enlargement (also known as a 'homothety') with centre I and scale factor 2, the circumcircle of $\triangle ABC$ is mapped to the circumcircle of $\triangle I_aI_bI_c$, and this proves both (1) and (2).

Keywords: Bevan circle, Bevan point, incentre, circumcentre, excentre, circumradius, dynamic geometry, enlargement, homothety

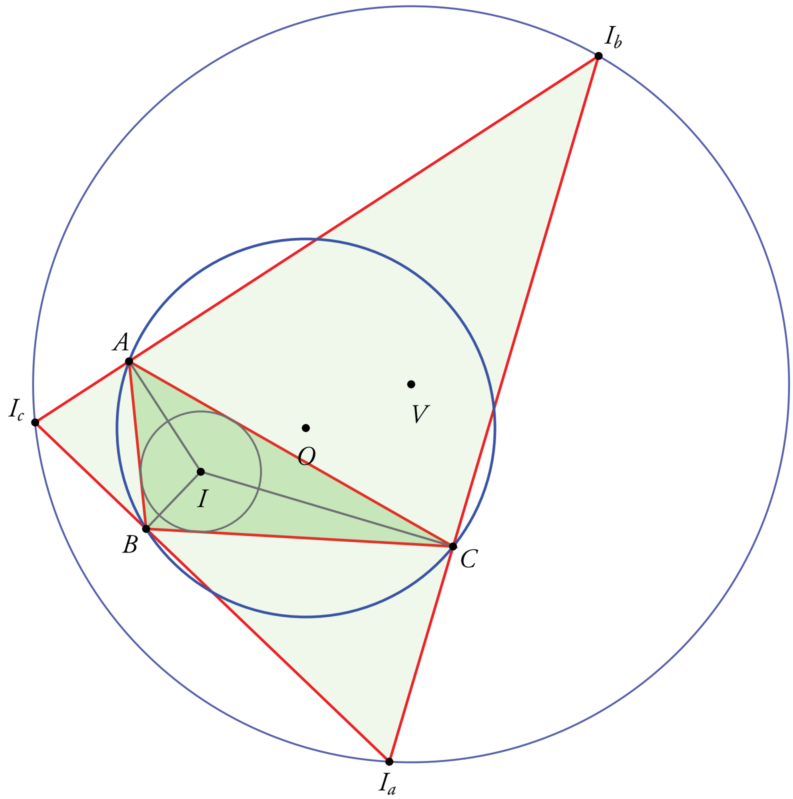


Figure 1. The Bevan circle, centre V

Proof of Lemma 1. We consider the connector II_a (see Figure 2; the others work the same way). Let D be the intersection point of the line segment II_a and the arc BC of the circumcircle of $\triangle ABC$; this must be the midpoint of arc BC, because AD is the bisector of angle BAC; this also implies that $DB = DC$. Quadrilateral BI_aCI is cyclic (note the right angles at B and C), and II_a is a diameter of its circumcircle, so the centre of the circumcircle lies at the midpoint of II_a. Angle computations show that $DB = DI = DC$ (for $\measuredangle DBI = \measuredangle DBC + \measuredangle CBI = (A + B)/2$, and $\measuredangle DIB = (A + B)/2$ too); so D is the centre of the circumcircle. Therefore D bisects II_a (see [3], p. 185 and p. 192).

Many references can be cited for the results quoted above; the facts are quite well known.

Exploration. Now we come to the second part of the article, which is exploratory in nature. We interchange one of the excentres with the incentre in the process described above and see what happens (thereby fostering the process of 'doing mathematics'). For instance, take I instead of I_c and find the circumcentre of $\triangle I_aI_bI$ and label it V_c. Do the same with I instead of I_a (giving V_a), and I instead of I_b (giving V_b), respectively. What we get is a triangle $V_aV_bV_c$ that is nothing but triangle $I_aI_bI_c$ reflected in point O (see Figure 3). Initially we could not find any reference concerning this fact, but then colleagues pointed us to [5] (maybe the above phenomenon is described there for the first time? This approach is completely different from the one presented here) and also to [6] (p. 110, Ex. 10) ([5] is cited here too). Overall, we have the impression that this phenomenon is not so well known, In addition, this topic may be used in a geometry class, thus we wrote this short note.

Let us initially consider the first case (I instead of I_c, giving V_c). The new circle passing through I_a, I_b, I has properties similar to the Bevan circle above:

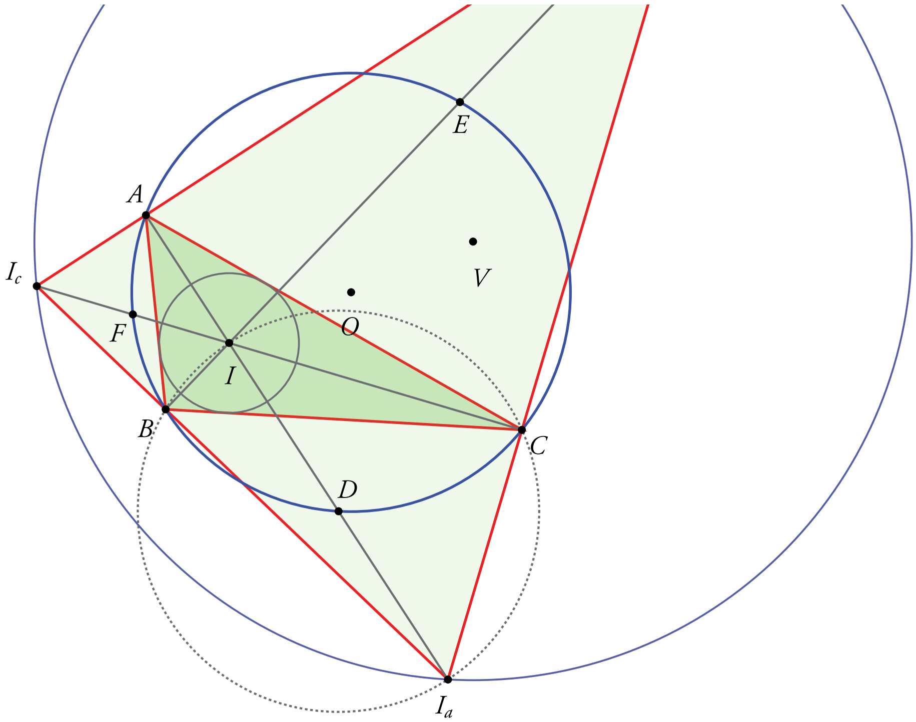


Figure 2. Why is segment II_a bisected by D?

(a) Its radius is twice the circumradius of $\triangle ABC$;

(b) Its centre V_c is the reflection of I_c in O.

The two other cases behave analogously. Altogether, we can think of five equally large circles:

(1) Bevan circle through I_a, I_b, I_c (centre: V)

(2) Circle through I_a, I_b, I (centre: V_c)

(3) Circle through I_a, I, I_c (centre: V_b)

(4) Circle through I, I_b, I_c (centre: V_a)

(5) Circle through V_a, V_b, V_c (centre: I)

In some sense, we could call all these circles 'Bevan circles' and all the points V_a, V_b, V_c 'Bevan points.'

In order to prove that $V_aV_bV_c$ is the triangle $I_aI_bI_c$ reflected in the point O, it suffices to show a) and b) from above (the proofs for the other points V_a, V_b work the same way). As before we are done if we can prove that the circumcircle of $\triangle ABC$ bisects the segments joining I_c to I_a, I_b, I (see Figure 4). We proved already that D bisects the line segment II_c. As above we see (note the right angles at C and B) that I_bI_cBC is a cyclic quadrilateral. Let G be the intersection point (other than A) of line segment I_bI_c and arc CB (in other words, the midpoint of the arc CB). Then we know that $GC = GB$ and because of the cyclic quadrilateral I_bI_cBC (for which I_bI_c is a diameter) we know that $GI_b = GC = GI_c$, and this means that G bisects I_bI_c (and similarly, H bisects I_aI_c).

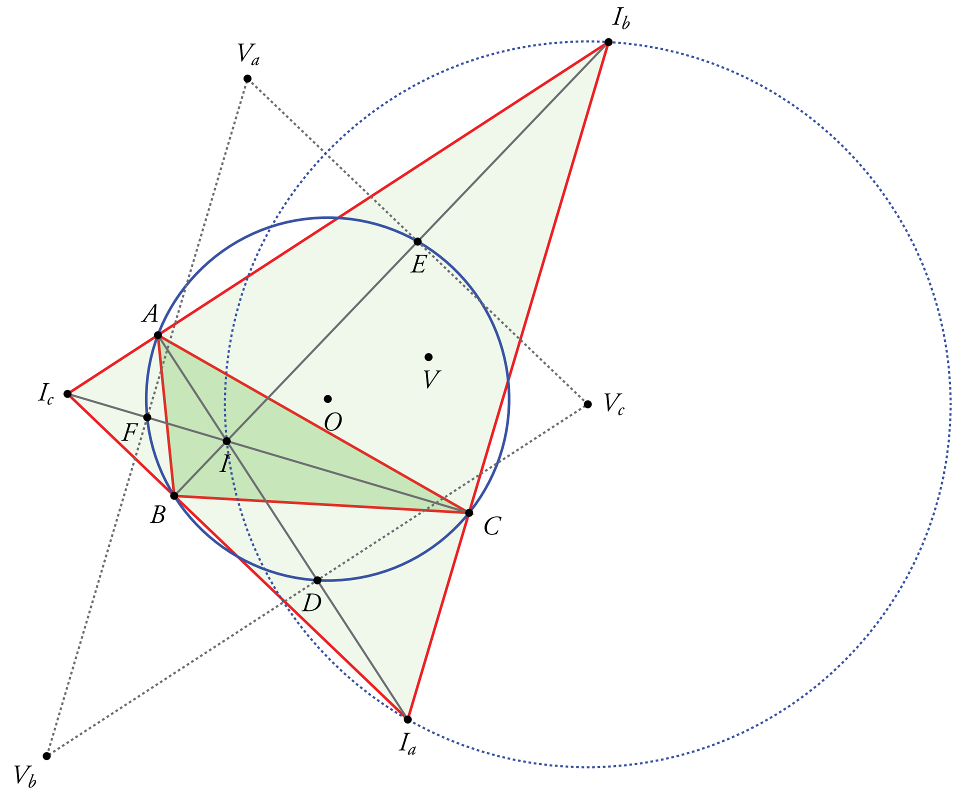


Figure 3. The three points V_a, V_b, V_c and the circumcircle of $\triangle I_a I_b I$

Teachable points. Possible teaching situations concerning this topic could be the following.

(I) Problems (1) and (2) posed at the start can be an occasion for autonomous problem solving by gifted students. The teacher may need to give several hints (dealing with excentres is not self-evident for students in most countries):

(a) Why do segments AI_a, BI_b, CI_c all pass through I (Figure 2)?

(b) Why does it suffice to show that the circumcircle of $\triangle ABC$ bisects the segments connecting the incentre I and the excentres?

(c) How can we be sure that there are right angles at points B and C (Figure 2)?

(d) Why is D the centre of the circumcircle of cyclic quadrilateral IBI_aC (Figure 2)?

By thus dividing the problem into smaller pieces, problem solving and proving may become more manageable for the students, enabling them to find correct arguments.

(II) (1) and (2) can also be demonstrated by a teacher, after which (a) and (b) can serve as exercises for students (not just gifted students) for practising techniques they have learnt.

In both cases dynamic geometry software can be used for setting up conjectures and seeing all the results (which gives a strong hint that the claims are true). Of course, the proof must be provided by the learners; here the computer cannot help. In teaching, to see *why* something is true (explanation function of proof) is even more important than to see that something is true (verification function of proof, [2]).

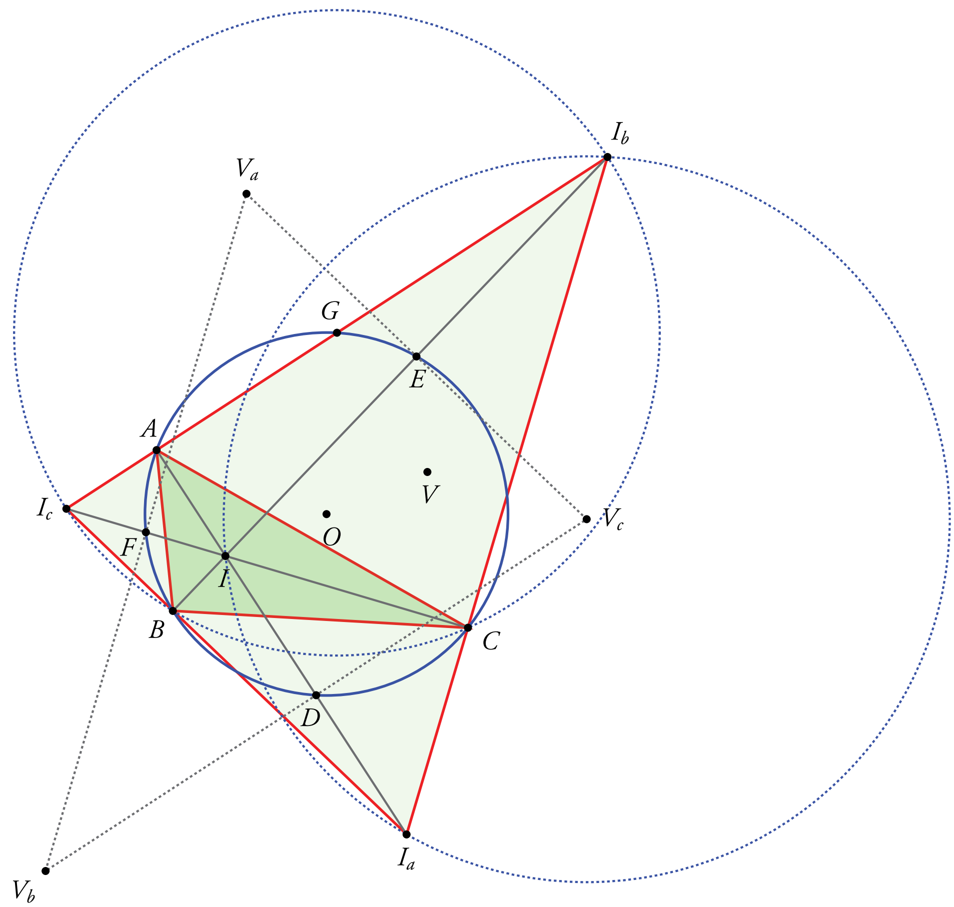


Figure 4. Why is I_bI_c bisected by G?

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HANS HUMENBERGER has been a professor of mathematics education at the University of Vienna since 2005. In this position, he is responsible for the education of students who want to become teachers at secondary schools or high schools (grades 5-12, 11-18-year-old students). His primary interests include teaching mathematics as a process, teaching mathematical modeling, problem solving, geometry, and elementary mathematics. He may be contacted at hans.humenberger@univie.ac.at.

Addendum: The nine-point circle of a triangle

Reference has been made in the above article to the nine-point circle of a triangle. Not all readers may be familiar with this notion, so the editors have included this addendum.

References to the nine-point circle are easily found online. See for example:

- "Exploring the Nine-point Circle: Conjecture making and proof using Dynamic Geometry Software" by Jonaki Ghosh in the March 2021 issue of *At Right Angles*
- https://en.wikipedia.org/wiki/Nine-point_circle
- https://artofproblemsolving.com/wiki/index.php/Nine_point_circle

Let ABC be any triangle. Let D, E, F be the midpoints of sides BC, CA, AB, respectively. (See Figure 5.) Let the altitudes of the triangle be AP, BQ, CR, with P, Q, R lying on BC, CA, AB, respectively. Let H be the orthocentre of the triangle (i.e., the common point of the three altitudes). Let U, V, W be the midpoints of HA, HB, HC, respectively. Let O be the circumcentre of $\triangle ABC$. Then it turns out that the nine points $D, E, F, P, Q, R, U, V, W$ lie on a circle. This circle is known as the **nine-point circle** of $\triangle ABC$. Let N be its centre. It turns out that N lies at the midpoint of segment OH. (The centroid G of $\triangle ABC$ also lies on segment OH.)

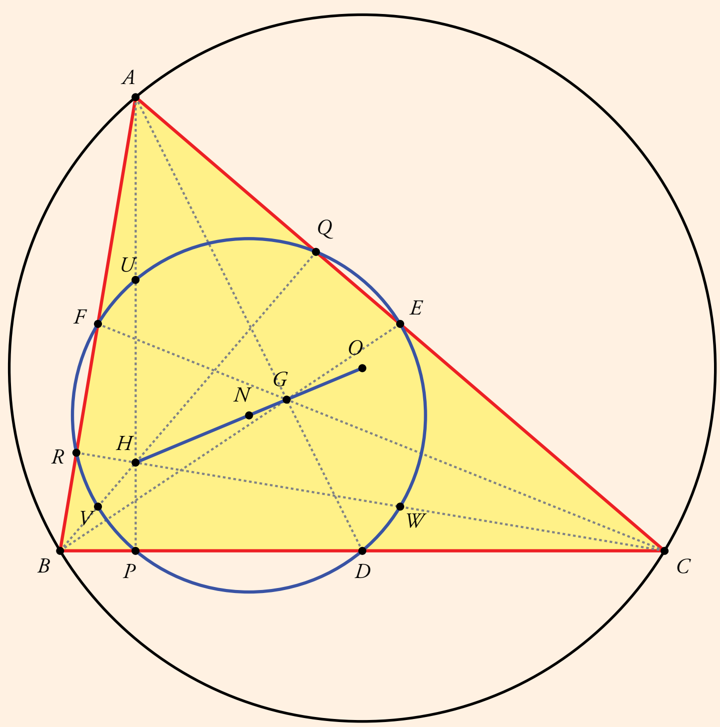

Figure 5

Infinite Kepler Triangles

PRANAV VERMA

In this article, we shall explain what a Kepler triangle is and describe an infinite nested array of such triangles.

Introduction

The Fibonacci numbers form one of the most intriguing sequences of natural numbers. The sequence goes $1, 1, 2, 3, 5, 8, \ldots$. Here the n-th Fibonacci number F_n (for $n \geq 3$) can be expressed as the sum of the previous two Fibonacci numbers, i.e.,

$$F_{n+2} = F_{n+1} + F_n \quad (n \geq 1).$$

It is well-known that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi,$$

where $\varphi = (1+\sqrt{5})/2 \approx 1.618$ is the **golden ratio**. Note that φ is the positive root of the quadratic equation $x^2 - x - 1 = 0$. So the golden ratio satisfies the relation $\varphi^2 = \varphi + 1$. A consequence of this basic relation is the following:

$$\varphi^{n+1} = F_{n+1}\varphi + F_n, \quad n \geq 1.$$

The Kepler triangle

An interesting occurrence of the golden ratio in Euclidean geometry is in the Kepler triangle. A Kepler triangle is a right-angled triangle whose sides are in geometric progression. If the common ratio of this geometric progression is $\sqrt{x}$, then the sides are in the ratio $1 : \sqrt{x} : x$, so we have $1 + x = x^2$, which shows that $x = \varphi$. So the sides of the Kepler triangle are in the ratio

$$1 : \sqrt{\varphi} : \varphi \approx 1 : 1.272 : 1.618.$$

Keywords: Kepler triangle, Fibonacci numbers, Golden ratio

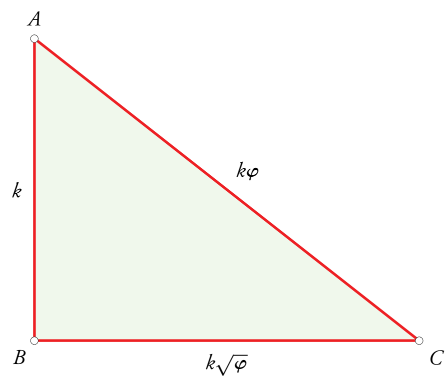

Figure 1. Kepler Triangle

See Figure 1. Note that a triangle with sides $k,\ k\sqrt{\varphi},\ k\varphi$ $(k > 0)$ is also a Kepler triangle since

$$(k\varphi)^2 = (k\sqrt{\varphi})^2 + k^2.$$

Clearly, a triangle that is similar to a Kepler triangle is itself a Kepler triangle.

The problem

Let us now do something interesting. Let $\triangle ABC$ be a Kepler triangle with hypotenuse AC; let its sides AB, BC, CA be $k,\ k\sqrt{\varphi},\ k\varphi$. Let the sides be divided by points F, D, E, respectively, such that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{BF}{FA} = \frac{1}{\varphi}. \qquad (1)$$

That is, each side is divided in the 'golden section' (see Appendix for details). See Figure 2. The question now is: What can be said about the four triangles thus created: $\triangle DEF$, $\triangle AFE$, $\triangle BDF$, $\triangle CED$? In general, this process can be continued to get more nested triangles; what can be said about them?

Let $BF = x$, $AF = \varphi x$; $BD = y$, $DC = \varphi y$; $CE = z$, $AE = \varphi z$ for some x, y, z. Then:

$$x(\varphi + 1) = k, \quad \therefore\ x = \frac{k}{\varphi + 1}. \qquad (2)$$

Similarly:

$$y(\varphi + 1) = k\sqrt{\varphi}, \quad \therefore\ y = \frac{k\sqrt{\varphi}}{\varphi + 1}, \qquad (3)$$

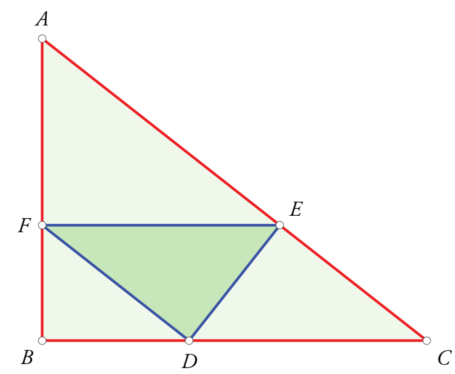

Figure 2. Constructing $\triangle DEF$ from $\triangle ABC$

and

$$z(\varphi + 1) = k\varphi \quad \therefore\ z = \frac{k\varphi}{\varphi + 1}. \qquad (4)$$

Observe that

$$x : y : z = 1 : \sqrt{\varphi} : \varphi. \qquad (5)$$

Now note that FE is parallel to BC (because $AF/FB = AE/EC$); hence $\triangle AFE$ is similar to $\triangle ABC$. Consequently, $\triangle AFE$ is a Kepler triangle.

In the same way, since DF is parallel to CA, it follows that $\triangle BDF$ is similar to $\triangle BCA$, and therefore that $\triangle BDF$ too is a Kepler triangle.

It remains to check $\triangle CED$ and $\triangle DEF$. We shall show that they too are Kepler triangles.

Consider $\triangle CED$ first. Noting that $\measuredangle C$ is shared by the two triangles, we must prove that $\triangle CED \sim \triangle CBA$. For this we must prove that

$$\frac{CE}{CB} = \frac{CD}{CA}, \quad \text{i.e.,} \quad CE \cdot CA = CD \cdot CB.$$

Now observe that

$$CE \cdot CA = z \cdot z(1 + \varphi) = z^2 \cdot (1 + \varphi),$$
$$CD \cdot CB = y\varphi \cdot y(1 + \varphi) = y^2 \cdot \varphi \cdot (1 + \varphi).$$

Equality follows since $z = y\sqrt{\varphi}$. It follows that $\triangle CED$ is a Kepler triangle.

To show that $\triangle DEF$ is a Kepler triangle, we only have to note that $DFEC$ is a parallelogram, hence $\triangle DEF$ is congruent to $\triangle EDC$; therefore $\triangle DEF$ too is a Kepler triangle.

It follows that by starting with a Kepler triangle and inscribing a triangle in it as described above, the inscribed triangles are also Kepler triangles. By repeating this process, we get an infinite array of nested Kepler triangles (see Figure 3). We may call them 'hidden Kepler triangles'.

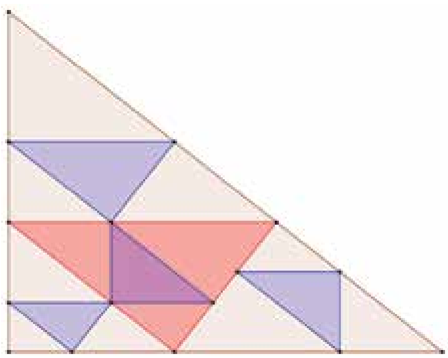

Figure 3.

Appendix: The golden section

Consider a line segment AB. We say that a point E on AB divides it in the **golden section** if $AE : EB = \varphi : 1$ or $1 : \varphi$.

Construction of point E. Suppose $AB = x$. Using a compass, construct AC perpendicular to AB, with $AC = x/2$. Join BC. With C as centre and radius AC, draw an arc intersecting BC at point D. With B as centre and radius BD, draw another arc intersecting AB at point E. Since $\triangle ABC$ is right-angled, $BC = \sqrt{5}x/2$, so $BD = BE = (\sqrt{5} - 1)x/2$. Hence $AE = (3 - \sqrt{5})x/2$. Taking ratios, we get $BE/AE = \varphi$, i.e., E divides AB in the ratio $1 : \varphi$ (Figure 4).

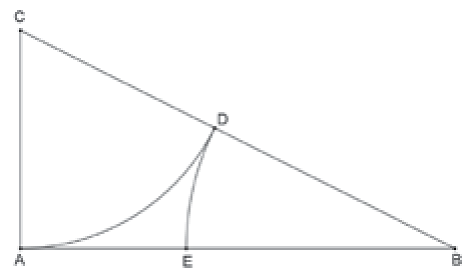

Figure 4.

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PRANAV VERMA is a student of class 12, Kensri School, Bangalore. He enjoys solving mathematical puzzles and painting watercolor landscapes. He recently held an art exhibition at the Chitrakala Parishath, Bangalore. Reading is another hobby; he particularly likes reading books written by Satyajit Ray. He may be contacted at adityapranav2016@gmail.com.

Two Problems in Number Theory - Part II

RAKSHITHA

In this two-part article, we study two number theory problems from the UK Math Olympiad, Round 2, years 2006 and 2003 respectively. We had studied the first problem in the March 2021 issue, in Part I, and now we study the second problem in Part II. Both problems were discussed during meetings of the problem-solving group of our school.

Three-term arithmetic progressions with small prime divisors

Problem. For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Find all triples (x, y, z) of distinct positive integers satisfying the following two conditions: (i) x, y, z are in an arithmetic progression; and (ii) $p(xyz) \leq 3$.

Solution. The condition $p(xyz) \leq 3$ implies that x, y, z are not divisible by any prime exceeding 3. This means that their only prime factors are 2 and 3.

Without loss of generality, we assume that the greatest common divisor of x, y, z is 1. For if $\gcd(x, y, z) = d$, where $d > 1$, then we may divide throughout by d and obtain a new set of triples satisfying the given conditions. On the other hand, starting with any triple x, y, z satisfying the conditions and with $\gcd(x, y, z) = 1$, we can obtain a new triple by multiplying each quantity by any fixed positive integer which does not have a prime factor greater than 3.

We may further assume that $x < y < z$.

Since $2y = x + z$, any common divisor of y with either x or z will be a divisor of the other. Hence $\gcd(x, y) = 1$ and $\gcd(y, z) = 1$. Combining this observation with the fact that x, y, z are divisible only by the primes 2 and 3, we are led to consider the following cases.

Keywords: Arithmetic progression, prime factor, greatest common divisor

Case 1: $x = 2^a, \quad y = 3^b, \quad z = 2^c$ for some non-negative integers a, b, c

Since x, y, z are in an A.P., $2^a + 2^c = 2 \cdot 3^b$, so

$$2^{a-1} + 2^{c-1} = 3^b.$$

If $a \geq 2$ and $c \geq 2$, then the left side is even whereas the right side is odd; so equality cannot hold. Therefore, one out of a and c must be 1. As we have assumed that $x < z$, it must be that $a = 1$. This gives $x = 2$. Substituting we get

$$1 + 2^{c-1} = 3^b.$$

Now we consider the following two sub-cases.

Sub-case 1a: $c - 1 = 1$

In this case we have $c = 2$ and $b = 1$, so $z = 4$ and $y = 3$. The triple in this case is $(2, 3, 4)$.

Sub-case 1b: $c - 1 \geq 2$

In this case we have

$$1 + 2^{c-1} = 3^b.$$

Since $c - 1 \geq 2$, it follows that 2^{c-1} is a multiple of 4. Also, $3 \equiv -1 \pmod 4$. Hence, reading both sides modulo 4, we get $1 \equiv (-1)^b \pmod 4$. Therefore b is even, i.e., $b = 2k$ for some integer $k \geq 0$.

Substituting in the original equation, we get $1 + 2^{c-1} = 3^{2k}$. Hence:

$$2^{c-1} = 3^{2k} - 1 = (3^k + 1) \cdot (3^k - 1).$$

Since the quantity on the left side is a power of 2, both factors on the right side (i.e., $3^k - 1$ and $3^k + 1$) must be powers of 2. As they are also consecutive even numbers, it follows that $3^k - 1 = 2$ and $3^k + 1 = 4$. (There is no other possibility.) Hence $k = 1$ and $b = 2$, giving $y = 3^b = 9$ and $z = 16$.

Thus in Case 1, we get the following two triples: $(2, 3, 4)$, $(2, 9, 16)$.

Case 2: $x = 3^a, \quad y = 2^b, \quad z = 3^c$ for some non-negative integers a, b, c

From $x + z = 2y$ we get

$$3^a + 3^c = 2^{b+1}.$$

If both $a, c > 0$, then the quantity on the left is a multiple of 3, but the quantity on the right cannot be a multiple of 3. Hence at least one out of a, c must be 0. Since $x < z$, we obtain $a = 0$. Hence $x = 1$. The equation now reads:

$$1 + 3^c = 2^{b+1}.$$

If $c = 0$, then we get $b = 0$ as well, and the triple in question is $(1, 1, 1)$. This obviously satisfies the given conditions.

If $c > 0$, then reading the equation modulo 3, we get:

$$1 \equiv (-1)^{b+1} \pmod 3.$$

This implies that $b+1$ is even, so $b = 2k-1$ for some positive integer k. From this we get:

$$1 + 3^c = 2^{2k},$$
$$\therefore\ 3^c = 2^{2k} - 1 = (2^k+1)\cdot(2^k-1),$$
$$\therefore\ 2^k + 1 = 3 \quad \text{and} \quad 2^k - 1 = 1.$$

This gives $k = 1$ and hence $b = 1, y = 2, z = 3$. The only solution in this case is $(1, 2, 3)$.

We had said at the start that we would be taking x, y, z to be coprime. Under this restriction, the only solutions to the problem are the triples $(1, 2, 3)$, $(2, 3, 4)$ and $(2, 9, 16)$. If we remove this restriction, we see that the solutions to the given problem are the triples $(d, 2d, 3d)$, $(2d, 3d, 4d)$ and $(2d, 9d, 16d)$, where d is any positive integer whose only prime factors are 2 and 3 (i.e., d is of the form $2^u \cdot 3^v$ for some non-negative integers u, v).

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RAKSHITHA is a math enthusiast studying in class 12 at The Learning Centre, PU College, Mangalore. She has a deep interest in Analysis, Number Theory and Combinatorics. Other than Mathematics, she is fascinated by the sciences and wishes to pursue pure research in the future. She participated in the Indian National Mathematics Olympiad in 2019 and 2020. She may be contacted at krraja1974@rediffmail.com.

Story of the Hemachandra-Fibonacci Number

Contributed by Suvam Mukherjee

Department of Basic Science and Humanities Regent Institute of Science and Technology Kolkata - 700121

We are very familiar with Fibonacci numbers and with various manifestations of the Golden Ratio. But have you heard of Acharya Hemachandra and do you know why these numbers are now called the Hemachandra-Fibonacci numbers? We feature one connection here.

Hemachandra was born in 1089 in Dhanduka, Gujarat. An ordained Jain muni and a great influence on the Solanki rulers, he was a polymath with knowledge of religion, philosophy, logic and more.

In Sanskrit prosody (patterns of rhythm and sound in poetry), *varnas* (letters) are the fundamental units. A letter having one *matra* (syllable) is called *laghu* (light) and that having two *matras* is called *guru* (heavy) and this takes twice the time taken to articulate a *laghu*. In the text *Chandonusasana*, Hemachandra formulated the number of ways to formulate an 8 beat *taal*.

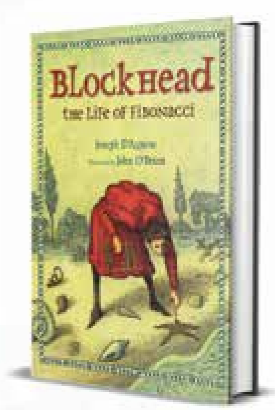
n	Laghu (short)	Guru (long)	Number of permutations	a_n
1	1	0	1	1
2	2	0	1	2
	0	1	1	
3	3	0	1	3
	1	1	2	
4	4	0	1	5
	2	1	3	
	0	2	1	
5	5	0	1	8
	3	1	4	
	1	2	3	

Look at the following table (here a_n represents the total number of combinations having n beats):

We can look at it like this: If the line of n units contains a short *(laghu)* syllable at the end, then we have the remaining portion of $n-1$ units which can be composed in a_{n-1} ways. If there is a long *(guru)* syllable, then there will be $n-2$ units remaining which can be composed in a_{n-2} ways. So clearly, we have the relation $a_n = a_{n-1} + a_{n-2}$. This is the recursive relation of Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34..., except that the first term is missing. This means that a_n is equal to F_{n+1}, the $(n+1)$-th Fibonacci number.

Blockhead
The Life of Fibonacci

Reviewed by Arundhati Venkatesh



Author: Joseph D'Agnese
Illustrator: John O' Brien

It is evident from the cover illustration that the book is set in a different period. A note before the story begins informs the reader that modern names have been used for places and that Hindu-Arabic numerals have been altered to appear the way they do today; they would have looked slightly different in Fibonacci's day.

Everyone calls Leonardo a blockhead, but he has always had a way with numbers. He solves math problems in seconds, much faster than the others in his class. The illustration — of Maestro timing the children with an hour-glass and the schoolchildren sprawled on the floor of the classroom dressed in medieval clothes — sets the time period of the book firmly. Once Leonardo has finished, he starts counting the birds outside the window, dreaming about numbers, coming up with questions and patterns. But thinking is prohibited in Maestro's classroom. Maestro scolds, the others laugh, Leonardo runs out of school and onto the streets of Pisa. There is a joke here about the people building the tower of Pisa getting the math wrong. Could this mislead young readers? The real reason for the tower leaning is soft ground and an unstable foundation.

Leonardo's merchant father hears of his son being called a blockhead. Humiliated and enraged, he takes Leonardo with him on his travels to Africa. Leonardo's father's adviser, Alfredo, helps lift his spirits by reminding him that people are happiest when they know what pleases them. Leonardo now knows what to do. He learns all about numbers wherever he goes. In the city of Bugia in Northern Africa, he learns the numerals the Arab merchants use, which they had borrowed from the "Hindu people of India."

However, referring to them as Hindus isn't strictly accurate. Leonardo lived in the 12th century, at which time the people who lived in the Indian subcontinent were not all Hindus.

Leonardo travels to faraway places as he grows older, learning about numbers everywhere. He begins writing a book about Hindu-Arabic numerals and includes a riddle to do with rabbits. He is invited as a guest to the palace of Frederick II, ruler of the Holy Roman Empire. Impressed by Leonardo's mathematical prowess, Frederick II calls him "one smart cookie." Neither Frederick II nor Leonardo could possibly have known the expression though, or heard of a cookie. Earlier in the book, a young Leonardo says "yuck" on learning that his father wants to make a merchant of him. Dialogues like these stick out.

On his return to Pisa, Leonardo tries to popularise the numerals from India, but faces opposition. Leonardo remembers his old friend Alfredo, thinks what he would have said, and doesn't despair. He looks around in nature and finds the same numbers everywhere — the numbers from his rabbit problem. Leonardo shows how the same numbers can be depicted in different ways, drawing squares and a spiral in the sand. Like the rabbit problem, these too are pictorially depicted for the benefit of budding mathematicians. The story ends with lines that will inspire readers to hunt for Fibonacci numbers not just in the pages of the book, but also in real life.

A note at the end tells us that the book is a bit of make-believe, and that it is based on the few things we do know about Leonardo, like his nickname, Bigollo, which translates to wanderer, traveller, but that could also mean idler or dreamer; in other words, a blockhead.

On the last page are activities that will encourage young readers to think for themselves. Why animal horns spiral as they grow, why nature prefers spiral shapes and other such interesting questions.

At just twenty-four pages, this slim book is a great classroom resource for math teachers, with some history, geography, art and nature observation thrown in.

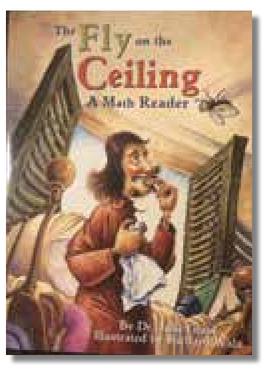
There are little treats tucked away in the illustrations. After being introduced to Leonardo and the Fibonacci numbers, kids will enjoy re-reading the book and hunting for hidden spirals.

A few minor quibbles:

- The book is written in first person and flits between time periods, with sentences like "That's how we did our math back then." Surely, Leonardo could not have known that we do our math differently in the 21st century. Third person may have been a better choice.
- When Leonardo stares at birds, the questions he wonders about are how many legs they have, how many eyes, and how many wings, the answers to which are all the same and make him appear daft, really a blockhead! But Leonardo goes on to say they are "beautiful questions."
- What I also found problematic are references to "Mother Nature." While this personification of nature and the assigning of gender may have been prevalent in medieval Rome (was it?) it could perhaps have been avoided here.

The Fly on the Ceiling
A Math Myth

Reviewed by Arundhati Venkatesh



Author: Dr. Julie Glass
Illustrator: Richard Walz

The book starts off with a lovely note to parents from a primary grades mathematics teacher about developing mathematical thinking and a lifelong love for math by not seeing math as an isolated phenomenon, but by connecting math to familiar experiences and by supporting children's natural affinity.

We are introduced to the philosopher René Descartes (and his poodle, who resembles him and can be seen in all the visuals; kids will enjoy the poodle-spotting). René is portrayed in the book as messy and rather absent-minded. He has a problem but does not even realise it! Until he begins to lose things…

Heavily illustrated, with sparse text, the book can be read and enjoyed even by a very young audience. Both the text and the illustrations are infused with humour, always a hit with children. The wry, tongue-in-cheek humour is done really well too. Sample this: When René realises he has a problem (finding things) he decides to put an end to it.

'This must stop!' René said to himself. He decided to take a walk and think of a solution to his problem. It took him a moment to find his coat, his hat, and the front door.'

The character and the situation provide plenty of opportunity, and the illustrator makes good use of it, adding to the humour quotient.

We also get a glimpse of 17th century France, with its cobbled streets and bakeries, the river Seine… The absent-minded philosopher soon finds himself at the bottom of the river. The visual depiction of the mishap and the rescue operation are sure to draw giggles from young readers and chuckles from older ones. René ends up as wet as the bread he had been eating while walking.

The next morning, René is still looking for things — his handkerchief, an extra blanket, logs to make a fire, his soggy bread. René lies in bed, staring at the ceiling, the only part of the room that isn't messy. He sees a fly landing in different places. That gets him thinking. He finds a way to record where the fly lands: drawing criss-crossing lines on the ceiling using charcoal, numbering two of them, and using the grid thus formed to come up with 'coordinates'. Amazingly, the pictures and the words are amusing even while going into this explanation of Cartesian Coordinates. Oh, and we now have a fly, in addition to the poodle!

Next, René paints the floor of his room. Now, he knows where every object is located. He comes up with a chart that indicates (using Cartesian Coordinates) where everything is kept. The system becomes popular around the world, and is named after him — the Cartesian Coordinate System.

The author's note helps separate fact from fiction. Not much is known about René, except that he was a philosopher and that he helped popularise the Cartesian Coordinate System. The rest is made up. When it comes to historical fiction, a common problem is that character traits and events in the story have been wildly imagined and have no basis in history. That isn't a complaint I have here. Historical fiction is fiction based on fact, and does entail intelligent deduction. Something like this could well have happened. Maybe there is some truth to it, and even if there isn't, it is a delightful book, and a wonderful way to learn about Cartesian Coordinates. My only grouse is that in the first page of the book, Descartes is referred to as a "guy", which is a tad too casual for my taste.

The story would lend itself well to a stage enactment. The author does a splendid job of converting what could otherwise have been a dull lesson into a fun experience, and something children will never forget. Books like this one can get students thinking about how we know what we know, who came up with the systems and inventions we use, what triggers innovation, and how what is learnt can be applied. While the book is targeted at grades two and three, it can be used when introducing graph theory to higher grades too.

Both the books deal with topics that require visual learning, and hence the picture book format is ideal.

ARUNDHATI VENKATESH is the author of *Bookasura* and the *Petu Pumpkin* series (for children 6 to 9 years old). Her books have won several awards, including the SCBWI Crystal Kite Award 2015 for India, Middle East and Asia and the Comic Con India 2015 Best Publication for Children award.

Two Famous Series for π

MAYADHAR SWAIN

π is an interesting and mysterious number in mathematics. Mathematicians have been interested in this number for the last 4000 years, and work on it is still going on. Its definition is very simple: it is defined as the ratio of the circumference of a circle to its diameter. It does not matter how big or small the circle is, the ratio stays the same. Its approximate value is 3.14159. In this article, we discuss two famous infinite series for π

Although π is defined from the circle in geometry, it mysteriously appears in many formulas in physics and engineering. Surprisingly, it cannot be expressed as a ratio of two whole numbers; i.e., it is an irrational number. In addition to being irrational, it is also transcendental, which means that it is not a solution of any non-constant polynomial equation with rational coefficients.

Mathematicians have been interested for centuries in finding good estimates for π. Archimedes of ancient Greece computed an extremely accurate value around 250 BC. Beginning with regular hexagons inscribed in a circle and circumscribed about the circle, and doubling the number of sides repeatedly till he obtained a regular polygon with 96 sides, he succeeded in showing that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}, \quad \text{i.e.,} \quad \frac{223}{71} < \pi < \frac{22}{7}.$$

The upper bound in this double inequality (i.e., $\frac{22}{7}$) is widely used in schools as the value of π.

Around 150 AD, the Greek scientist Ptolemy gave the value of π as 3.1416 in his book *Almagest*. The Indian astronomer Aryabhata gave the same value in his book *Aryabhatiya* (499 AD).

Today we know that the value of π is 3.141592653589793238 . . . For practical calculations, we need a maximum of 5 places after the decimal point. But in their enthusiasm, mathematicians have calculated its value correct up to many trillions of places using modern computers. Even before the invention of the modern computer, mathematicians were able to calculate its value correct up to many places. For example, the English

Keywords: Approximations to pi, infinite series, definite integrals, inverse trigonometric functions, Ptolemy, Aryabhata, Aryabhatiya, Madhava, Shanks, irrational number, transcendental number

mathematician William Shanks (1812-1882) calculated the value of π up to 707 places in 1874 and he took 15 years to find this value ([4]). Later it was found that this value is correct only up to 527 places. The error was found by another English mathematician D.F. Ferguson in 1944, and Ferguson himself calculated the value of π correct up to 710 places in 1947.

How could the value of π be calculated to so many places correctly without the use of computers? It was because of the discovery of some infinite series for π. Here we will discuss two classical series for π, both found by the Indian mathematician Madhava [5] of Sangamagrama (1350-1425).

The first result is the following:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right).$$

Although Madhava's books are lost, the series is found in the book *Tantrasangraha* of Nilakantha Somayaji, written around 1500 AD. The author has attributed it to Madhava.

Madhava also discovered the infinite series for the sine, cosine and inverse tangent functions. Later these were discovered in Europe by James Gregory (1671) and Gottfried Wilhelm Leibniz (1674). The inverse tangent series is:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \qquad (\text{valid for } -1 < x \leq 1).$$

This formula was known as the *Gregory-Leibniz series* and is now referred to as the *Madhava series*.

In the above formula, if we put $x = 1$, we get;

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots,$$

which yields the result quoted above.

Proof. From elementary calculus, we know that

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^1 = \tan^{-1} - \tan^{-1} 0 = \frac{\pi}{4}.$$

We also know that for $-1 < x < 1$,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots.$$

It can be derived by doing the implied long division or simply by cross multiplying, or by summing the geometric progression on the right side (its common ratio is $-x^2$). Then,

$$\begin{aligned}
\frac{\pi}{4} &= \int_0^1 \frac{dx}{1+x^2} \\
&= \int_0^1 \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) dx \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\Big|_0^1 \\
&= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.
\end{aligned}$$

Hence

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

This is the first series developed for π. But it is not useful for calculation purposes, as it converges very slowly. Leonhard Euler (1707-1783) wrote in 1737 that to get just 50 digits from this series, it would require us to "labor fere in aeternum" (work almost forever).

The second result is the following:

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \cdots\right).$$

We may obtain this result by putting $x = \frac{1}{\sqrt{3}}$ in Madhava's series for $\tan^{-1} x$.

We can also find the series as follows:

$$\int_0^{1/\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^{1/\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0 = \frac{\pi}{6}.$$

Hence,

$$\begin{aligned}\frac{\pi}{6} &= \int_0^{1/\sqrt{3}} \frac{dx}{1+x^2}\\ &= \int_0^{1/\sqrt{3}} \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) dx\\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\Big|_0^{1/\sqrt{3}}\\ &= \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \cdots\right).\end{aligned}$$

So

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \cdots\right).$$

This series converges quickly, and the sum of just the first 10 terms correctly gives the first five digits of π. The English mathematician Abraham Sharp (1651-1699) used the first 150 terms of the series in 1699 to calculate the first 72 digits of π.

The second series (above) is found in the Malayalam book *Yuktibhasa* written by Jyesthadeva of the Kerala School of Mathematics in about AD 1530. The author has attributed it to earlier mathematicians Madhava and Nilakantha Somayaji. Madhava used this series to calculate the value of π to 11 digits.

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MAYADHAR SWAIN (B.Sc. M.Tech IIT Roorkee) retired from National Thermal Power Corporation and worked as Director in KIIT University, Bhubaneswar for 3 years. He is a popular science writer and has written 52 books on science, engineering and mathematics. He is deeply interested in Number Theory and History of Mathematics, particularly Ancient Indian Mathematics. He is currently working as Editor of Science Horizon, a monthly science magazine published by Odisha Bigyan Academy, Govt. of Odisha, Bhubaneswar. He may be contacted at mayadhar2002@yahoo.co.in.

Convergence issues

When we deal with infinite series, it is in general wise to check the matter of convergence, else we sometimes obtain absurd results. Accordingly we shall do so for the Madhava-Gregory-Leibniz series for π. Fortunately, this is easy to do, as we show below. The analysis also gives an estimate of the error term if we cut short the computation after a certain number of terms.

By summing the GP, we may verify that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2} + \frac{(-1)^n x^{2n}}{1+x^2}.$$

Therefore, integrating from 0 to 1, we get:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1} + (-1)^n \int_0^1 \frac{x^{2n}\,dx}{1+x^2}.$$

This relation is exact—it is an identity for all positive integers n. Let us now estimate the value of the integral on the right side. Since $0 \le x^2 \le 1$ for $0 \le x \le 1$, it follows that $1 \le 1 + x^2 \le 2$ for $0 \le x \le 1$, and therefore that

$$\frac{x^{2n}}{2} \le \frac{x^{2n}}{1+x^2} \le x^{2n} \quad \text{for } 0 \le x \le 1.$$

Hence

$$\int_0^1 \frac{x^{2n}\,dx}{2} \le \int_0^1 \frac{x^{2n}\,dx}{1+x^2} \le \int_0^1 x^{2n}\,dx,$$

and so

$$\frac{1}{2(2n+1)} \le \int_0^1 \frac{x^{2n}\,dx}{1+x^2} \le \frac{1}{2n+1}.$$

Hence the error obtained by cutting short the computation at the $\dfrac{1}{2n-1}$ term lies between $\dfrac{1}{2(2n+1)}$ and $\dfrac{1}{2n+1}$. This implies in particular that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \to \frac{\pi}{4}.$$

The other series may be handled in the same manner.

Box 1. A note on convergence (added by the editors)

Computation of a Surd

$\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$

Problem. Compute the value of the following surd:

$$\left(8+3\sqrt{21}\right)^{1/3}+\left(8-3\sqrt{21}\right)^{1/3},$$

only real values being considered.

Solution. All the quantities involved are irrational, yet if we do the computations using a calculator, we find something remarkable:

$$\begin{aligned} 8+3\sqrt{21} &\approx \quad 21.74772708,\\ \left(8+3\sqrt{21}\right)^{1/3} &\approx \quad 2.791287847,\\ 8-3\sqrt{21} &\approx -5.747727084,\\ \left(8-3\sqrt{21}\right)^{1/3} &\approx -1.791287847, \end{aligned}$$

and:

$$\left(8+3\sqrt{21}\right)^{1/3}+\left(8-3\sqrt{21}\right)^{1/3}=1.000\ldots.$$

Well! The answer appears to be 1. Is it *exactly* equal to 1? — or so close that we cannot tell the difference? We shall show that the answer is exactly 1.

Let $u=\left(8+3\sqrt{21}\right)^{1/3}$ and $v=\left(8-3\sqrt{21}\right)^{1/3}$. We then have:

$$u^3+v^3=\left(8+3\sqrt{21}\right)+\left(8-3\sqrt{21}\right)=16,$$

and

$$\begin{aligned} uv &= \left(8+3\sqrt{21}\right)^{1/3}\cdot\left(8-3\sqrt{21}\right)^{1/3}\\ &= (64-189)^{1/3}=(-125)^{1/3}=-5. \end{aligned}$$

Keywords: Irrational numbers, cube roots, quadratic, discriminant, exact value

Next, we have:

$$u^3 + v^3 = (u+v)\cdot\left(u^2 - uv + v^2\right)$$
$$= (u+v)\cdot\left((u+v)^2 - 3uv\right).$$

Let $x = u + v$; we need to find x. (Note that x is a real number.) Since $u^3 + v^3 = 16$ and $uv = -5$, we obtain

$$16 = x\left(x^2 + 15\right).$$

Hence x is a root of the cubic equation

$$x^3 + 15x - 16 = 0.$$

Using the factor theorem, we are able to factorize the cubic; we obtain:

$$(x-1)\cdot\left(x^2 + x + 16\right) = 0.$$

The only real root of this equation is 1 (the quadratic component yields non-real roots, as its discriminant is $1^2 - (4\times 1\times 16) = -63$, which is negative). Hence $x = 1$.

So the value of the given expression is 1. □

A direct approach. Might there be another approach to solving this problem? Could we actually evaluate the two cube roots and thereby find their sum?

Let us assume that $\left(8 + 3\sqrt{21}\right)^{1/3}$ can be expressed in the form $a + b\sqrt{21}$, where a, b are rational numbers, and let us try to find a, b under this assumption. Taking the cubes of both the quantities, we obtain:

$$8 + 3\sqrt{21} = a^3 + 3\sqrt{21}a^2b + 63ab^2 + 21\sqrt{21}b^3,$$

from which it follows, by equating the rational and irrational parts on both sides,

$$a^3 + 63ab^2 = 8,$$
$$a^2b + 7b^3 = 1.$$

The equations imply that $a \neq 0$, $b \neq 0$. Let $k = b/a$; then, by assumption, k is a rational number. Substituting, we get:

$$a^3\left(1 + 63k^2\right) = 8,$$
$$a^3\left(k + 7k^3\right) = 1.$$

These equations imply that $1 + 63k^2 = 8\left(k + 7k^3\right)$, i.e.,

$$56k^3 - 63k^2 + 8k - 1 = 0.$$

The cubic expression on the left is readily factorized, as the factor theorem tells us that $k - 1$ is a factor. We thus obtain:

$$(k-1)\cdot\left(56k^2 - 7k + 1\right) = 0.$$

The quadratic component has discriminant $7^2 - (4\times 1\times 56) = -175$, which is negative; so it does not yield any real roots (and therefore no rational roots either). The only real root is $k = 1$, a rational number. It follows that $a = b$.

From this, it follows that $8a^3 = 1$, and hence that

$$a = \frac{1}{2}, \quad b = \frac{1}{2}.$$

Therefore, the real cube root of $8 + 3\sqrt{21}$ is

$$\frac{1}{2} + \frac{\sqrt{21}}{2},$$

and the real cube root of $8 - 3\sqrt{21}$ is

$$\frac{1}{2} - \frac{\sqrt{21}}{2},$$

implying that

$$\left(8 + 3\sqrt{21}\right)^{1/3} + \left(8 - 3\sqrt{21}\right)^{1/3} = 1,$$

in agreement with what we had obtained earlier.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

A Problem Related to an Equilateral Triangle

KOUSIK SETT

In this article, we offer a geometric solution to a well-known problem which is generally solved using trigonometry.

The following problem can be found in the literature [1]; people generally use trigonometry to solve it. The statement of the problem and a typical solution are given below.

Problem. *If P is any point in the plane of an equilateral triangle ABC with side a and $PA = x$, $PB = y$ and $PC = z$, show that*

$$a^4 + x^4 + y^4 + z^4 = a^2\left(x^2 + y^2 + z^2\right) + x^2y^2 + y^2z^2 + z^2x^2.$$

Solution. The following solution is based on trigonometry.

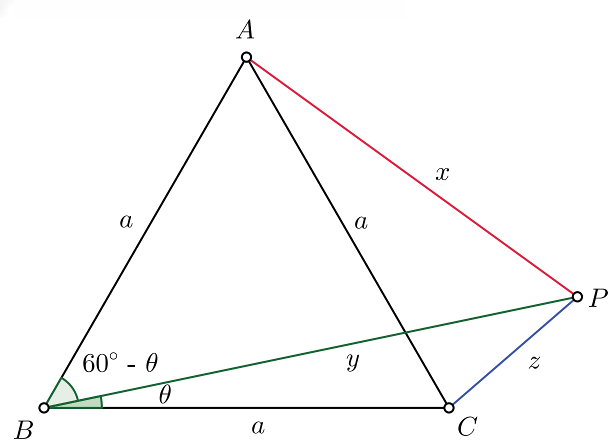

Figure 1.

Let $\measuredangle PBC = \theta$. Since $\triangle ABC$ is equilateral we have $\measuredangle ABP = 60^\circ - \theta$.

Keywords: Equilateral triangle, cosine rule, rotation, Stewart's theorem

From $\triangle BCP$ we get,

$$\cos\theta = \frac{a^2+y^2-z^2}{2ay}. \tag{1}$$

From $\triangle ABP$ we get,

$$\cos(60^\circ-\theta) = \frac{a^2+y^2-x^2}{2ay} \tag{2}$$

From (1) and (2) we obtain,

$$\sin\theta = \frac{a^2+y^2-2x^2+z^2}{2\sqrt{3}ay} \tag{3}$$

Eliminating θ from (1) and (3), we get,

$$\left(\frac{a^2+y^2-z^2}{2ay}\right)^2 + \left(\frac{a^2+y^2-2x^2+z^2}{2\sqrt{3}ay}\right)^2 = 1. \tag{4}$$

On simplification this gives ,

$$a^4+x^4+y^4+z^4 = a^2(x^2+y^2+z^2)+x^2y^2+y^2z^2+z^2x^2. \tag{5}$$

Note that (5) can also be written as

$$3\left(a^4+x^4+y^4+z^4\right) = \left(a^2+x^2+y^2+z^2\right)^2. \tag{6}$$

Can we solve the problem geometrically?

The answer is Yes! However, the task is quite challenging. We will investigate a few possible cases depending on the location of P, i.e., on whether P is inside or outside or on the triangle.

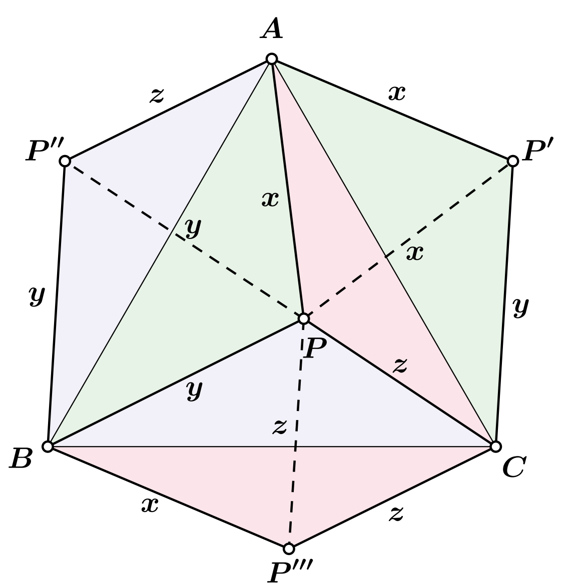

Figure 2.

Case 1: When P is inside the triangle. Let P be a point inside the triangle such that $PA = x$, $PB = y$ and $PC = z$. Now we will perform some rotation operations (see Figure 2).

- Rotate $\triangle ABP$ anticlockwise through an angle of 60° with respect to point A. Let P move to P'; then $\triangle ABP \cong \triangle ACP'$.
- Rotate $\triangle BCP$ anticlockwise through an angle of 60° with respect to point B. Let P move to P''; then $\triangle BCP \cong \triangle BAP''$.
- Rotate $\triangle CAP$ anticlockwise through an angle of 60° with respect to point C. Let P move to P'''; then $\triangle CAP \cong \triangle CBP'''$.

Observe that $\triangle APP'$, $\triangle BPP''$, $\triangle CPP'''$ are equilateral with side lengths x, y, z respectively. Also, $\triangle APP''$, $\triangle BPP'''$ and $\triangle CPP'$ have side lengths x, y, z. Note also that the area of hexagon $AP''BP'''CP'$ is twice the area of $\triangle ABC$.

Let each side of $\triangle ABC$ be a; let $2s = x + y + z$. Then:

$$2 \times \triangle ABC = (\triangle APP' + \triangle BPP'' + \triangle CPP''') + (\triangle APP'' + \triangle BPP''' + \triangle CPP')$$

$$\implies 2 \cdot \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{3}}{4}y^2 + \frac{\sqrt{3}}{4}z^2 + 3 \cdot \sqrt{s(s-x)(s-y)(s-z)}$$

$$\implies a^2 = \frac{1}{2}(x^2 + y^2 + z^2) + 2\sqrt{3} \cdot \sqrt{s(s-x)(s-y)(s-z)}$$

$$\implies \left(2a^2 - (x^2 + y^2 + z^2)\right)^2 = 3\left(2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4\right).$$

Simplifying we get,

$$a^4 + x^4 + y^4 + z^4 = a^2(x^2 + y^2 + z^2) + x^2y^2 + y^2z^2 + z^2x^2 \tag{7}$$

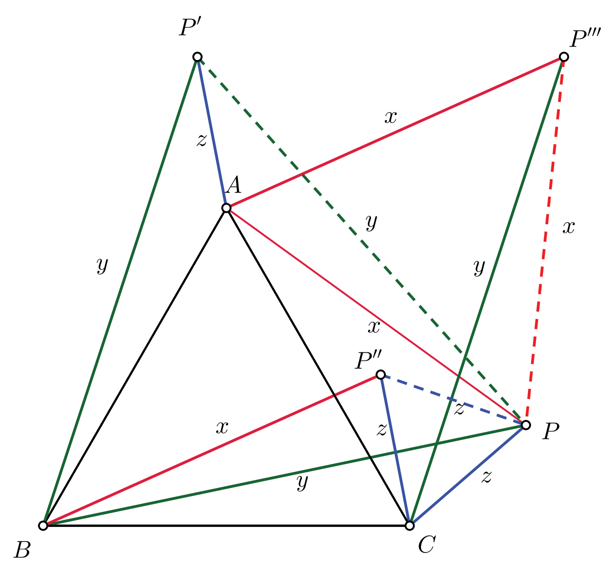

Figure 3.

Case 2: When P is outside the triangle.

- Rotate $\triangle BCP$ anticlockwise through an angle of 60° with respect to the point B. Let P move to P'; then $\triangle BAP' \cong \triangle BCP$.
- Rotate $\triangle ACP$ anticlockwise through an angle of 60° with respect to the point C. Let P move to P''; then $\triangle BCP'' \cong \triangle ACP$.
- Rotate $\triangle ABP$ anticlockwise through an angle of 60° with respect to the point A. Let P move to P'''; then $\triangle ACP''' \cong \triangle ABP$.

Let X denote the area of $\triangle BCP$ (equivalently, of $\triangle BAP'$); let Y denote the area of $\triangle ACP$ (equivalently, of $\triangle BCP''$); and let Z denote the area of $\triangle ABP$ (equivalently, of $\triangle ACP'''$). Observe that

$$\triangle ABC = \triangle ABP + \triangle BCP - \triangle ACP = Z + X - Y. \tag{8}$$

Also note that $\triangle APP'''$, $\triangle BPP'$, $\triangle CPP''$ are equilateral with side lengths x, y, z respectively; and that $\triangle APP'$, $\triangle BPP''$, $\triangle CPP'''$ have side lengths x, y, z. Let k denote the area of $\triangle APP'$; then $\triangle BPP''$ and $\triangle CPP'''$ also have area k.

So we have,

$$\triangle BCP + \triangle BPP'' = \triangle BCP'' + \triangle CPP'' \implies X + k = Y + \frac{\sqrt{3}}{4}z^2$$

$$\triangle ACP''' + \triangle CPP''' = \triangle ACP + \triangle APP''' \implies Z + k = Y + \frac{\sqrt{3}}{4}x^2$$

$$\triangle ABP' + \triangle APP' + \triangle ABP = \triangle BPP' \implies X + k + Z = \frac{\sqrt{3}}{4}y^2$$

Adding the above three equalities, we get,

$$2X + 2Z + 3k = 2Y + \frac{\sqrt{3}}{4}(x^2 + y^2 + z^2)$$

$$\implies 2(X + Z - Y) = \frac{\sqrt{3}}{4}(x^2 + y^2 + z^2) - 3k$$

$$\implies 2\triangle ABC = \frac{\sqrt{3}}{4}(x^2 + y^2 + z^2) - 3\sqrt{s(s-x)(s-y)(s-z)}$$

$$\implies 2 \times \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}(x^2 + y^2 + z^2) - 3\sqrt{s(s-x)(s-y)(s-z)}$$

$$\implies a^2 = \frac{1}{2}(x^2 + y^2 + z^2) - 2\sqrt{3}\sqrt{s(s-x)(s-y)(s-z)}$$

$$\implies \left\{2a^2 - (x^2 + y^2 + z^2)\right\}^2 = 3(2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4).$$

Simplifying, we get,

$$a^4 + x^4 + y^4 + z^4 = a^2(x^2 + y^2 + z^2) + x^2y^2 + y^2z^2 + z^2x^2.$$

Case 3: When P lies on a side of the triangle. This special case can easily be tackled by Stewart's theorem [3]. (A statement of the theorem is given in Box 1 at the end of the article.)

Suppose that P lies on side BC. Then $a = y + z$. In this case the relation to be proved reduces to $x^2 = y^2 + yz + z^2$ or equivalently, $a^2 = x^2 + yz$.

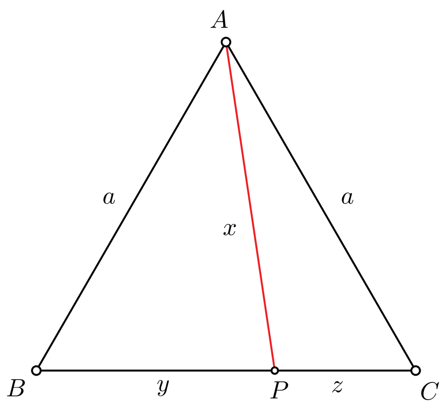

Figure 4.

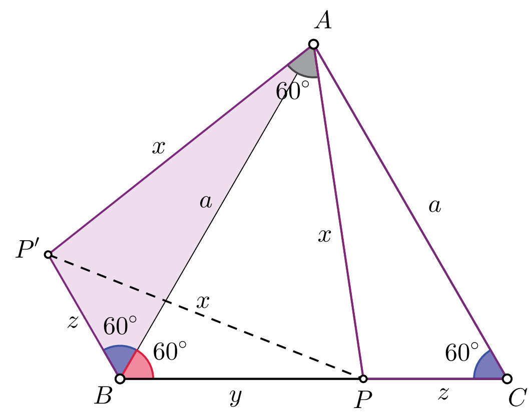

Figure 5.

Applying Stewart's theorem in $\triangle ABC$ (Figure 4) we get,

$$\begin{aligned} a^2y + a^2z = a(yz + x^2) &\implies a(y+z) = x^2 + yz \\ &\implies a^2 = x^2 + yz \qquad [\text{since } a = y + z] \\ &\implies (y+z)^2 = x^2 + yz \implies y^2 + yz + z^2 = x^2. \end{aligned}$$

Proof using rotation. The above relation can also be deduced from Figure 5.

If we rotate $\triangle APC$ clockwise through an angle of 60° with respect to A, then C goes to B and P goes to P' and $\triangle APC \cong \triangle ABP'$. Hence $\measuredangle PBP' = 60^\circ + 60^\circ = 120^\circ$. Also $\measuredangle PAP' = 60^\circ$ and $AP = AP'$. Hence $\triangle APP'$ is equilateral, so $PP' = x$. Now from $\triangle BPP'$ we easily get, $x^2 = y^2 + yz + z^2$. (We invite the reader to find a proof without the use of the cosine rule.)

A different interpretation

It is interesting to note that the relation to be proved can be interpreted differently. If we let x, y, z be three given distances from a fixed point and treat a as variable, then we get an interesting outcome: two possible equilateral triangles. If the side lengths of these two equilateral triangles are a_1 and a_2 then solving (6) as a quadratic in a^2 we get,

$$a_1^2 = \frac{1}{2}(x^2 + y^2 + z^2) + 2\sqrt{3} \cdot \sqrt{s(s-x)(s-y)(s-z)},$$
$$a_2^2 = \frac{1}{2}(x^2 + y^2 + z^2) - 2\sqrt{3} \cdot \sqrt{s(s-x)(s-y)(s-z)},$$

which give

$$a_1^2 + a_2^2 = x^2 + y^2 + z^2 \tag{9}$$

$$\implies \frac{\sqrt{3}}{4}a_1^2 + \frac{\sqrt{3}}{4}a_2^2 = \frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{3}}{4}y^2 + \frac{\sqrt{3}}{4}z^2. \tag{10}$$

Hence we can state the following result.

The sum of areas of two equilateral triangles each of which has its vertices at three given distances from a fixed point is equal to the sum of the areas of the equilateral triangles described on these distances.

An equivalent problem. The following problem, due to Francisco Javier García Capitán [2] of Spain, provides an equivalent interpretation to the above problem: *Given three concentric circles with radii* r_1, r_2, r_3, *find the lengths of the sides of equilateral triangles with one vertex on each circle.*

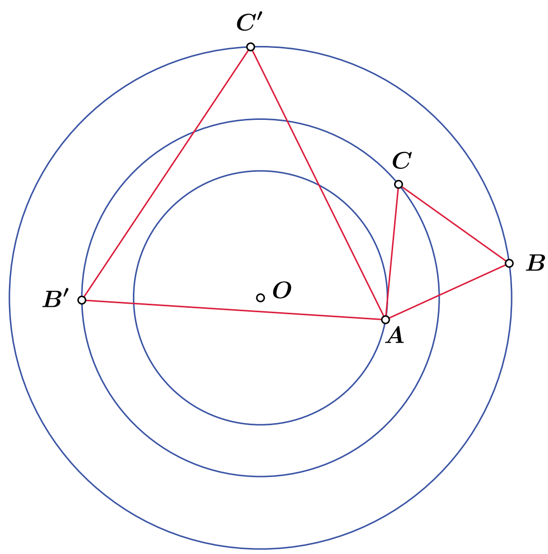

Figure 6.

Figure 6 depicts the problem. It turns out that if L is the length of a side of any such equilateral triangle, then

$$L^4 - \left(r_1^2 + r_2^2 + r_3^2\right) L^2 + r_1^4 + r_2^4 + r_3^4 - r_1^2 r_2^2 - r_2^2 r_3^2 - r_3^2 r_1^2 = 0.$$

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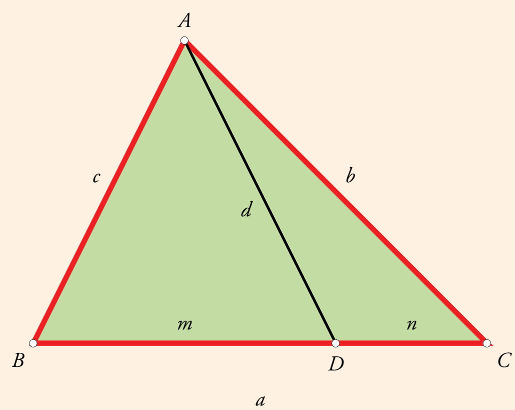
KOUSIK SETT works as an assistant teacher in mathematics at a government sponsored school in West Bengal. He did his MCA from North Bengal University, and taught computer science at Shibpur Dinobundhoo College, Howrah. He also did an MSc in mathematics. He currently serves as Assistant Editor of the journal *Maths, Physics, Technology Quest*, published from Singapore. He has a deep interest in Ramanujan's Mathematics, Sangaku Geometry, and Mathematical Olympiad Problems. He can be contacted at kousik.sett@gmail.com.

Statement of Stewart's theorem

Let ABC be any triangle, and let D be any point on side BC. Let a, b, c be the lengths of BC, CA, AB, respectively, and let d be the length of AD. Finally, let $BD : DC = m : n$. (See the figure below. Note that m, n could also represent the lengths of BD, DC, respectively.)



Then Stewart's theorem [3] states that

$$a\left(d^2 + mn\right) = b^2 m + c^2 n.$$

Box 1. Stewart's theorem

Inequalities - Part 1

SOURANGSHU GHOSH

Inequalities are encountered in almost every branch of mathematics. They have fascinating properties and many applications. In this series of articles, we will consider how different forms may be reduced or converted into known inequalities. We will study various problems taken from mathematical Olympiads across the world. The motivation and structure of the text is due to the wonderful resources [1], [5] and [2]. In the first part of the article, we study the most fundamental and basic inequality of all—the arithmetic mean-geometric mean inequality.

The AM-GM Inequality

As already noted, we start with the AM-GM inequality. It is regarded as one of the most fundamental and basic inequalities of all, as it implies so many other results. The simplest form of the AM-GM inequality is:

Theorem 1 (AM-GM inequality)**.** *For positive real numbers a, b, the following inequality holds:*

$$\frac{a+b}{2} \geq \sqrt{ab}. \tag{1}$$

Moreover, equality holds if and only if $a = b$.

Proof. As all the quantities involved are positive, the inequality is equivalent to the one obtained by squaring it. That is, it is equivalent to $(a+b)^2 \geq 4ab$. This in turn is equivalent to $(a+b)^2 - 4ab \geq 0$, i.e., to $(a-b)^2 \geq 0$. The last statement is clearly true; hence the AM-GM inequality follows. Moreover, equality holds in the last statement precisely when $a = b$. Hence equality holds in the AM-GM inequality precisely when $a = b$. □

The inequality can be further generalized in the form below.

Theorem 2 (AM-GM inequality for n numbers)**.** *For any n positive real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds true:*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}. \tag{2}$$

Proof. There are many ways to prove this inequality. Most often, it is proved by induction on n, starting with the case $n = 2$. We present a different and very elegant proof using the exponential function; it was first given by Pólya. The proof goes as follows.

Keywords: Inequality, arithmetic mean, geometric mean, harmonic mean, AM-GM inequality.

We consider the function $g(x) = e^{x-1} - x$. Its first and second derivatives are $g'(x) = e^{x-1} - 1$ and $g''(x) = e^{x-1}$. Observe that $g(1) = 0$ and $g'(1) = 0$. The second derivative is always positive, hence g is a strictly convex function with an absolute minimum value of 0 at $x = 1$. It therefore follows that:

$$x \le e^{x-1} \quad \text{for all real values of } x. \tag{3}$$

Note that equality holds precisely when $x = 1$.

Now consider a collection of n real non-negative numbers $a_1, a_2, \ldots, a_n$. Let b be their arithmetic mean. Then we have, using the above result:

$$\begin{aligned}
\frac{a_1}{b} \cdot \frac{a_2}{b} \cdots \frac{a_n}{b} &\le e^{a_1/b-1} \cdot e^{a_2/b-1} \cdots e^{a_n/b-1} \\
&= e^{(a_1+a_2+\cdots+a_n)/b-n} = e^{n-n} = 1, \\
\therefore\ a_1 \cdot a_2 \cdots a_n &\le b^n, \\
\therefore\ (a_1 \cdot a_2 \cdots a_n)^{1/n} &\le b, \qquad (4)
\end{aligned}$$

which proves the AM-GM inequality. Moreover, equality will hold precisely when all of the values a_1/b, $a_2/b, \ldots, a_n/b$ are equal to 1, which means that the n numbers $a_1, a_2, \ldots, a_n$ are all equal to each other. □

Weighted form of AM-GM inequality. It is possible to generalise the AM-GM inequality still further, into a weighted form. Let $a_1, a_2, \ldots, a_n$ be n real positive numbers, and let $w_1, w_2, \ldots, w_n$ be n positive weights with sum 1. The weighted form of the AM-GM inequality is the following statement:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \ge a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}. \tag{5}$$

The particular case when all the weights are equal to $1/n$ is Theorem 2. The proof runs along exactly the same lines as the one given above. For details, please see [4].

Some applications of the AM-GM inequality

We now look at some inequalities that make use of the AM-GM inequality.

Example 1: Nesbitt's Inequality. This states that for any three positive real numbers a, b, c,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}. \tag{6}$$

Proof. The solution comes from [1] and [2].

Let k denote the quantity on the left side. Then we have:

$$k = \frac{1}{2}\Big((a+b) + (b+c) + (c+a)\Big) \cdot \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} - 6\right). \tag{7}$$

Let us now substitute new variables $x = a + b$, $y = b + c$, $z = c + a$ in (7). Then:

$$\begin{aligned}
k &= \frac{1}{2}(x+y+z) \cdot \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 3 \\
&= \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z}\right) - \frac{3}{2}.
\end{aligned}$$

Now for any positive real number a, we have

$$a+\frac{1}{a}\geq 2,$$

by the AM-GM inequality. Therefore we can say that

$$k\geq\frac{1}{2}(2+2+2)-\frac{3}{2},\quad \text{i.e.,}\quad k\geq\frac{3}{2}.$$

This proves the stated inequality. □

Example 2. For any three positive real numbers a, b, c,

$$\left(a^2b+b^2c+c^2a\right)\cdot\left(b^2a+c^2b+a^2c\right)\geq 9a^2b^2c^2. \tag{8}$$

Proof. The solution comes from [2]. We apply the AM-GM inequality to the bracketed terms on the left side:

$$a^2b+b^2c+c^2a\geq 3\left(a^3b^3c^3\right)^{1/3}=3abc,$$
$$b^2a+c^2b+a^2c\geq 3\left(a^3b^3c^3\right)^{1/3}=3abc.$$

Therefore by multiplication we get:

$$\left(a^2b+b^2c+c^2a\right)\cdot\left(b^2a+c^2b+a^2c\right)\geq 9a^2b^2c^2.$$

Example 3: A problem from IMO 1964. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\leq 3abc. \tag{9}$$

Proof. The solution comes from [3]. We use the substitution

$$a=y+z,\quad b=z+x,\quad c=x+y.$$

If we solve for x, y, z in terms of a, b, c we get:

$$x=\frac{b+c-a}{2},\quad y=\frac{c+a-b}{2},\quad z=\frac{a+b-c}{2}.$$

Therefore, by the triangle inequality, x, y, z are positive numbers. We now rewrite the inequality in the following equivalent form:

$$2x(y+z)^2+2y(z+x)^2+2z(x+y)^2\leq 3(x+y)(y+z)(z+x). \tag{10}$$

On expanding the squared terms and simplifying, this assumes the following equivalent form:

$$x^2y+y^2z+z^2x+x^2z+y^2x+z^2y\geq 6xyz.$$

But this easily follows from the AM-GM inequality:

$$x^2y+y^2z+z^2x+x^2z+y^2x+z^2y\geq 6\left(x^2y\cdot y^2z\cdot z^2x\cdot x^2z\cdot y^2x\cdot z^2y\right)^{1/6},$$
$$\text{i.e.,}\quad x^2y+y^2z+z^2x+x^2z+y^2x+z^2y\geq 6xyz.$$

Example 4. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \le 1. \tag{11}$$

Proof. The solution comes from Nguyen Manh Dung in [1].

We shall use the weighted AM-GM this time.

The following three inequalities all follow from the weighted AM-GM inequality:

$$\frac{a^2+b^2+c^2}{a+b+c} \ge \left(a^a b^b c^c\right)^{1/(a+b+c)},$$
$$\therefore\ a^2+b^2+c^2 \ge a^a b^b c^c \quad (\text{since } a+b+c=1).$$

Next:

$$\frac{ab+bc+ca}{a+b+c} \ge \left(a^b b^c c^a\right)^{1/(a+b+c)},$$
$$\therefore\ ab+bc+ca \ge a^b b^c c^a \quad (\text{since } a+b+c=1),$$

and:

$$\frac{ac+ba+cb}{a+b+c} \ge \left(a^c b^a c^b\right)^{1/(a+b+c)},$$
$$\therefore\ ac+ba+cb \ge a^c b^a c^b \quad (\text{since } a+b+c=1).$$

Summing the three inequalities we get

$$(a+b+c)^2 \ge a^a b^b c^c + a^b b^c c^a + a^c b^a c^b,$$
$$\therefore\ 1 \ge a^a b^b c^c + a^b b^c c^a + a^c b^a c^b,$$

as required. □

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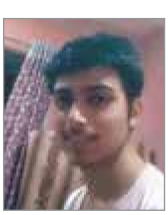

SOURANGSHU GHOSH is a fourth-year undergraduate student of the Department of Civil Engineering at the Indian Institute of Technology Kharagpur. He is also pursuing Mathematics as a minor degree. He is interested in Discrete Mathematics and its applications. He has written several articles in mathematics, many of which have been published in international Journals. He has also worked on the Reliability of Weld Joints of Nuclear Power Plants in collaboration with IGCAR Kalpakkam, and Structural reliability of coherent systems, as part of his undergraduate thesis. He enjoys playing the violin in the Indian Classical Music Style. He may be contacted at sourangshug123@gmail.com.

Solution to a Functional Equation

K SASIKUMAR

Functional equations occur quite frequently in problem contests. Typically they specify certain properties of an unknown function, on the basis of which we are supposed to find that function. If the answer is unique, it means that those properties characterise that particular function. We explore one such problem in this article.

Problem. To find all continuous functions f defined on the set of real numbers and taking real values, with the following two properties:

- $f(0) = 1$;
- $f(u + v + 1) = f(u) + f(v)$ for all real numbers u, v.

By simple experimentation, we find that the function $f(x) = x + 1$ satisfies all the given conditions. Could it be the only solution? Let us explore. We consider various classes of numbers.

Case 1: x is a positive integer. We make repeated use of the property $f(u + v + 1) = f(u) + f(v)$ for all real numbers u, v. The substitution $u = 0, v = 0$ gives:

$$f(1) = f(0 + 0 + 1) = f(0) + f(0) = 1 + 1 = 2.$$

We see that $f(1) = 1 + 1$. Next, the substitution $u = 1, v = 0$ gives:

$$f(2) = f(1 + 0 + 1) = f(1) + f(0) = 2 + 1 = 3.$$

We see that $f(2) = 2 + 1$.

Now assume that $f(k) = k + 1$ for some positive integer k. The substitution $u = k, v = 0$ gives:

$$f(k + 1) = f(k + 0 + 1) = f(k) + f(0)$$
$$= (k + 1) + 1 = k + 2.$$

Using the principle of induction, it follows that $f(x) = x + 1$ for every positive integer x.

Case 2: x is a negative integer. Let x be a negative integer, and let $y = -x$. Since y is a positive integer, we have $f(y) = y + 1$. The substitution $u = x, v = y$ gives:

$$f(1) = f(x + y + 1) = f(x) + f(y)$$
$$= f(x) + (y + 1) = f(x) - x + 1,$$

hence $2 = f(x) - x + 1$, i.e., $f(x) = x + 1$. Therefore $f(x) = x + 1$ for every negative integer x.

Keywords: Functional equation, continuous function, real number, rational number, integer

Case 3: x is a non-integral rational number. We first show that for any rational number x, the following are true:

(1) $f(x+1) = f(x) + 1$. Equivalently: $f(x-1) = f(x) - 1$.

(2) $f(-x) = 2 - f(x)$.

(3) For any positive integer m,

$$f(mx) = mf(x) - (m-1).$$

We justify these as follows.

$$\begin{aligned} f(x+1) &= f(x+0+1) = f(x) + f(0) \\ &= f(x) + 1. \end{aligned}$$

Next:

$$2 = f(x + (-x) + 1) = f(x) + f(-x),$$
$$\therefore\ f(-x) = 2 - f(x).$$

Finally, the relation $f(mx) = mf(x) - (m-1)$ is certainly true for $m = 1$:

$$f(x) = 1 \cdot f(x) - (1-1).$$

Assume that the relation $f(mx) = mf(x) - (m-1)$ is true for $m = k$, where k is some positive integer. Then

$$f(kx) = kf(x) - (k-1).$$

Therefore:

$$\begin{aligned} f((k+1)x) &= f(kx + x) \\ &= f(kx + (x-1) + 1) \\ &= f(kx) + f(x-1) \\ &= kf(x) - k + 1 + f(x) - 1 \\ &= (k+1)f(x) - k \\ &= (k+1)f(x) - (k+1-1). \end{aligned}$$

By the principle of induction, we conclude that $f(mx) = mf(x) - m + 1$ for all positive integers m.

Now let $x = p/q$ where p and q are integers, $q > 0$. Then $qx = p$ is an integer, therefore

$$f(qx) = qx + 1.$$

But $f(qx) = qf(x) - q + 1$. Hence:

$$qf(x) - q + 1 = qx + 1.$$

Solving for $f(x)$, we obtain:

$$f(x) = x + 1.$$

It follows that $f(x) = x + 1$ for all rational numbers x.

Case 4: x is a real but irrational number. Now let x be a real number. Then there exists a sequence $x_1, x_2, x_3, \ldots$ of rational numbers such that $\lim_{n\to\infty} x_n = x$. Since (by supposition) f is a continuous function,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

But $f(x_n) = x_n + 1$ for all n. It follows that

$$\lim_{n\to\infty} f(x_n) = x + 1.$$

Therefore $f(x) = x + 1$ for all real numbers x.

Conclusion. There is precisely one function f satisfying the given conditions: $f(x) = x + 1$ for all real numbers x.

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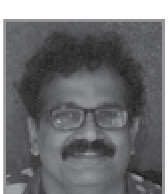

SASIKUMAR K is presently working as a PG Teacher in Mathematics at Jawahar Navodaya Vidyalaya, North Goa. He completed his M Phil under the guidance of Dr.K.S.S. Namboodiripad. Earlier, he worked as a maths Olympiad trainer for students of Navodaya Vidyalaya Samiti in Hyderabad. He has research interests in Real Analysis and Commutative Algebra. He may be contacted at 112358.ganitham@gmail.com.

An Olympiad Challenge

RAKSHITHA

Presenting a solution to a problem from Crux Mathematicorum

Problem

In a square $ABCD$, let M be the midpoint of AB, let P be the projection of point B onto line CM, and let N be the midpoint of segment CP. Let the angle bisector of $\measuredangle DAN$ intersect line DP at point Q. Prove that quadrilateral $BMQN$ is a parallelogram.

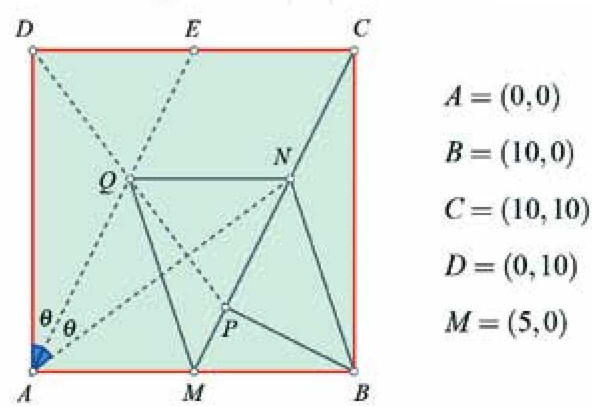

Figure 1

Solution

We use an approach combining methods of coordinate geometry and trigonometry. Assign coordinates so that A lies at the origin, and the sides AB and AD lie along the x-axis and the y-axis respectively. Choose the scale so that the side of the square is 10

Keywords: Quadrilateral, angle bisector, projection, parallelogram, square, Olympiad

units (this is merely to avoid fractions). Then the coordinates of A, B, C, D, M are as shown in the figure. Let AQ extended meet side CD at E. Let $\measuredangle DAE = \theta = \measuredangle EAN$.

The coordinates of P may be found by using similarity (triangles MPB, BPC and MBC are similar, so $MP/PB = 1/2 = BP/PC$, which means that $MP/PC = 1/4$; so P divides MC in the ratio 1 : 4); or by solving a pair of simultaneous equations (the equation of CM is $y = 2(x - 5)$, and the equation of BP is $y = -(x - 10)/2$; solving these simultaneously we get the coordinates of P). Using either approach we get $P = (6, 2)$ and hence $N = (8, 6)$. Therefore the slope of AN is 3/4.

Since $\measuredangle NAB = 90^\circ - 2\theta$, we get $\tan(90^\circ - 2\theta) = 3/4$, hence:

$$\tan 2\theta = \frac{4}{3}, \quad \therefore \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{4}{3}.$$

Solving the resulting quadratic equation for tanθ, we find that

$$\tan\theta = \frac{1}{2}.$$

This means that the equation of AE is $y = 2x$. Also, the equation of DP is

$$y - 10 = \frac{10 - 2}{0 - 6}(x - 0), \quad \text{i.e., } y = -\frac{4}{3}x + 10.$$

Solving the two equations simultaneously, we get $Q = (3, 6)$. We see that QN is parallel to the x-axis and has length 5 units. That is, QN is equal and parallel to AM and to MB. It follows that quadrilateral $AQNM$ is a parallelogram, and so is quadrilateral $BMQN$. □

RAKSHITHA is a math enthusiast studying in class 12 at The Learning Centre, PU College, Mangalore. She has a deep interest in Analysis, Number Theory and Combinatorics. Other than Mathematics, she is fascinated by the sciences and wishes to pursue pure research in the future. She participated in the Indian National Mathematics Olympiad in 2019 and 2020. She may be contactedat krraja1974@rediffmail.com

Extended version of Riffle Shuffles

Appendix A: pseudocode and Python Codes

A.1: Define a Function shuffle(deck):

    Half1 = [top half of deck]
    Half2 = [bottom half of deck]
    shuffled_deck = []
    Iterate **x** from {0} to {(length of **deck**) ÷ 2 – 1}:
        Add(item **x** of **Half1**) to **shuffled_deck**
        Add(item **x** of **Half2**) to **shuffled_deck**
    return **shuffled_deck**

carddeck = deck of cards with (8) cards in each suit
output: **carddeck**
original_deck = carddeck
carddeck = shuffle(carddeck)
output: **carddeck**
As long as **original_deck != carddeck:**
    carddeck = shuffle(carddeck)
    output: **carddeck**

The purpose of the function shuffle(d) is to simulate a riffle shuffle on a deck of cards d, which is in the form of a list. As this function actually shuffles the deck, it does not consider the number of cards. Hence, it will work regardless of whether or not the deck is a power of 2.

```
#Define a function to shuffle a given deck of cards

def shuffle(d):
    d1 = d[:len(d)//2]
    d2 = d[len(d)//2:]
    fd = []
    for a in zip(d1, d2):
        fd += list(a)
    return fd

#Code to create a deck

deck = []
for i in ['s', 'c', 'd', 'h']:
    deck += [str(x) + i for x in range(1, 5)]

#This last part of the program shuffles the deck until it has come back to the original order.

print(deck)
ordeck = deck
deck = shuffle(deck)
print(deck)
while ordeck != deck:
    deck = shuffle(deck)
    print(deck)
```

This number is the number of cards in each suit plus one, and can be altered as per the need.

A.2: This program is the same as A.1, only with a small tweak that displays the number of slots between the cards 1s and 2s for every shuffle (the changes were applied only to the last part of the program that prints each shuffle).

```
#Define a function to shuffle a given deck of cards

def shuffle(d):
    d1 = d[:len(d)//2]
    d2 = d[len(d)//2:]
    fd = []
    for a in zip(d1, d2):
        fd += list(a)
    if len(d2) % 2 == 1: fd.append(d2[-1])
    return fd

#Code to create a deck

deck = []
for i in ['s', 'c', 'd', 'h']:
    deck += [str(x) + i for x in range(1, 5)]

#This last part of the program shuffles the deck until it has come back to the original order.

print(deck)
print("2s is 1 slot away.")
ordeck = deck
deck = shuffle(deck)
print(deck)
print("2s is", deck.index('2s'), "slots away.")
while ordeck != deck:
    deck = shuffle(deck)
    print(deck)
    if deck.index('2s') == 1:
        print("2s is 1 slot away.")
    else:
        print("2s is", deck.index('2s'), "slots away.")
```

A.3: This program is very similar to A.1, except the cards are in the form of plain integers. They are also separated by a colon, dividing the cards held in the right hand and the cards held in the left hand. In order to be displayed in this manner, an additional function, display(l) is used.

```
#Define a function to shuffle a given deck of cards

def shuffle(d):
    d1 = d[:len(d)//2]
    d2 = d[len(d)//2:]
    fd = []
    for a in zip(d1, d2):
        fd += list(a)
    if len(d2) % 2 == 1: fd.append(d2[-1])
    return fd
def display(l):
    for x in l[:len(l)//2]:
        print(x, end = " ")
    print(":", end = " ")
    for x in l[len(l)//2:]:
        print(x, end = " ")
    print("")

#Code to create a deck

deck = [i for i in range(1, 25)]

#This last part of the program shuffles the deck until it has come back to the original order.

display(deck)
ordeck = deck
deck = shuffle(deck)
display(deck)
while ordeck != deck:
    deck = shuffle(deck)
    display(deck)
```

A.4: This program inputs the number of cards in a deck and outputs the number of shuffles needed to restore its original order, along with the value of k. n is calculated using the formula derived in Section 5 and the while loop iterates through integer values of k and calculates the corresponding value of n until n is an integer.

```
import math
t = int(input("Enter no. of cards"))
k = 1
n = math.log(1 + k*(t - 1), 2)
while n % 1 != 0:
    k += 1
    n = math.log(1 + k*(t - 1), 2)
print(f'This deck needs {int(n)} shuffles. k is {k}.')
```

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

Foundational topics include, but are not limited to, the following:

- Number systems, patterns and operations
- Fractions, ratios and decimals
- Proportional reasoning
- Integers
- Bridging Arithmetic-Algebra
- Geometry
- Measurement and Mensuration
- Data Handling
- Probability

We invite articles from teachers, teacher educators and others that are helpful in designing and implementing effective instruction. We strongly encourage submissions that draw directly on experiences of teaching. This is an opportunity to share your successful teaching episodes with AtRiA readers, and to reflect on what might have made them successful. We are also looking for articles that strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

Articles in this section may address key questions such as -

- What challenges did your students face while learning these fundamental mathematical topics?
- What approaches that you used were successful?
- What preparations, in terms of knowing mathematics, enacting the tasks and analysing students work were needed for effective instruction?
- What contexts, representations, models did you use that facilitated meaning making by your students?

Send in your articles to
AtRiA.editor@apu.edu.in

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'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

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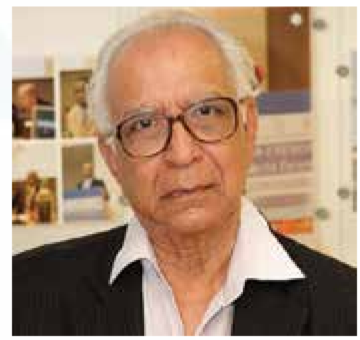
The Closing Bracket . . .

Breaking New Ground for Others: A Tribute to M S Narasimhan (7 June 1932 – 15 May 2021)



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The phrase 'end of an era' is so often used that it may even be considered to be clichéd, but the phrase is apt to describe the passing of M S Narasimhan (MSN), one of India's renowned mathematicians. Narasimhan did path-breaking work in algebraic and differential geometry, and made important contributions to number theory, representation theory and partial differential equations. He inspired an entire generation of mathematicians in India and abroad. Along with his collaborators and students, MSN carried out mathematical work of such high quality that it contributed immensely in the School of Mathematics at the Tata Institute of Fundamental Research, Mumbai (TIFR), becoming respected as one of the best centres of mathematics around the world.

The collaboration between Narasimhan and Seshadri (another mathematician we lost less than a year ago) in 1965 showed a deep and unexpected connection between topology and algebraic geometry, two different areas of modern mathematics. This result is so celebrated that in 2015 there was even a conference on *Fifty years of the Narasimhan-Seshadri theorem*.

MSN came from an agricultural family in Tiruvannamalai district in Tamil Nadu. He was born in Thandarai, a village that did not have a high school. He used to go by bullock cart to a town nearby for his high school studies. It was as an undergraduate student in Loyola College, Madras, that MSN discovered his deep love for mathematics. Seshadri was a student there as well, and they were taught by Reverend Father Racine, a Jesuit who had come from France. Racine had done his doctoral research under the mathematician Elie Cartan, and was eager to introduce contemporary mathematics to bright students. Racine had Modern Algebra introduced in the syllabus, when it was being taught only in two places in the country (by him, and in Presidency College in Calcutta). Racine recommended to Seshadri and Narasimhan that they study mathematics at TIFR.

Narasimhan played a crucial role in setting up the National Board for Higher Mathematics. Among other things, the Board funds regional mathematics libraries all over the country. This helps them acquire not only texts and monographs in mathematics but also subscribe to journals, which are indispensable for research.

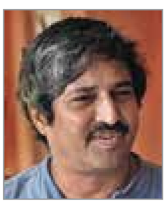
I was fortunate to have an (online) interview with MSN as recently as in October 2020. When asked what textbooks they used, Narasimhan laughed: "There were some textbooks, but they were very bad!" They had three separate books for Mechanics (all bad according to MSN), and an excellent one for Analysis. Surprisingly, despite Madras having some excellent number theorists (such as Ananda Rao, Vaidhyanathaswamy and Meenakshi Sundaram), they had no number theory. MSN reminisced that though he had learnt very complicated theorems in Analysis and could "use them well," he learnt even the Fundamental Theorem of Arithmetic only much later.

Racine used to consult his own copy of van der Waarden's celebrated text in German and supply notes to students. MSN heard of a new English translation of the text and ordered it from Higginbothams. Apparently the translation was so bad that the definition of finite set was wrong! (This was all fixed later.) MSN talked of the crucial role mathematics teachers could play in introducing the 'right material' to students.

Narasimhan received many honours for his work: the Padma Bhushan, the Shanti Swarup Bhatnagar Prize, the Fellowship of the Royal Society of London, the French Order of Merit, the King Faisal International Prize for Science, the Third World Academy of Sciences Prize for Mathematics and the Spirit of Salam award.

MSN was working on mathematical problems even a few months before the end. According to Ramanan, a long time collaborator of MSN, "India has lost one of its most versatile mathematicians."

Keywords: *M S Narasimhan, MSN, algebraic geometry, differential geometry, number theory, representation theory, partial differential equations, TIFR, Father Racine, Loyola College, NBHM*



R RAMANUJAM is a researcher in mathematical logic and theoretical computer science at the Institute of Mathematical Sciences, Chennai. He has an active interest in science and mathematics popularization and education, through his association with the Tamil Nadu Science Forum. He was awarded the Indira Gandhi Prize for Popularisation of Science for the year 2020. He may be contacted at jam@imsc.res.in.

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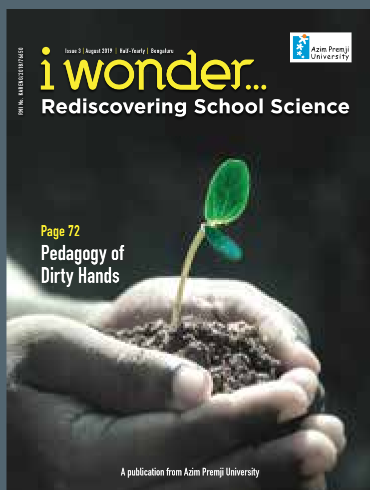

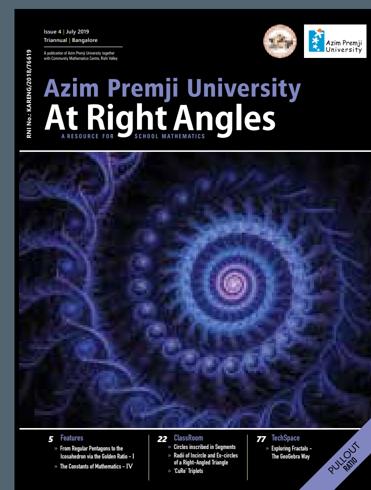

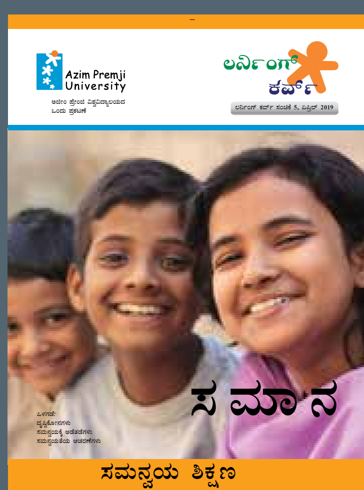

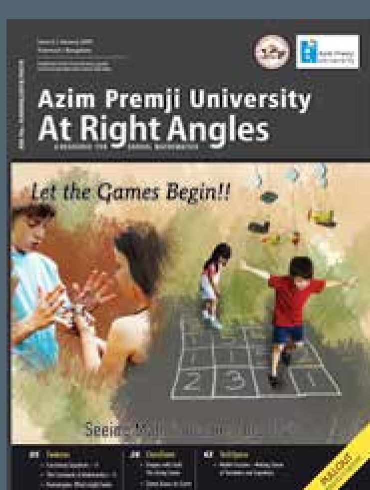

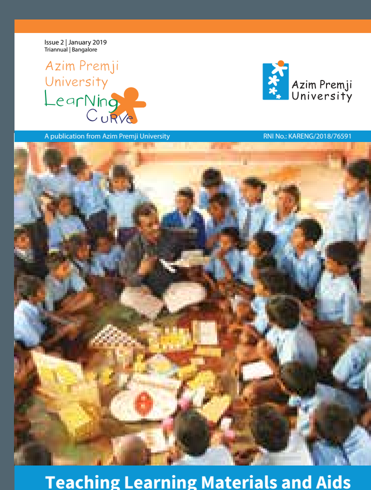

At Right Angles

A Resource for School Mathematics

ESTIMATION IN MATHEMATICS

PADMAPRIYA SHIRALI

Azim Premji University

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

ESTIMATION IN MATHEMATICS

"It is the mark of an instructed mind to rest assured with that degree of precision that the nature of the subject admits, and not to seek exactness when only an approximation of the truth is possible." – Aristotle

Is mathematics always about exactness and accuracy? Does estimation have a role to play in mathematics?

Estimated answers are realistic enough to serve our purposes in most situations of daily life. There are clear benefits with regard to quickness in obtaining an adequate answer, for example in making a time plan for homework, in ordering food items at a restaurant, in planning a hike or a drive from one city to another, in the kitchen (judging the size of a jar needed for the cookies!).

There is no denial of the fact that estimation is a time-saving approach. It is useful in building physical intuition. It is also a valuable way of spotting errors and cross-checking answers arrived at by other means.

Does that mean that estimation is of use mainly in daily living? How do mathematicians and scientists use estimation in their work?

"No scientist ever treats a real-world problem exactly – just 'good enough' for the accuracy of the result they need." Astronomers attempting to determine movements of celestial objects cannot obtain precise measurements. Similarly for geologists attempting to find the mass and size of some underground object. Scientists have to make estimates of many measures using available data and modelling techniques.

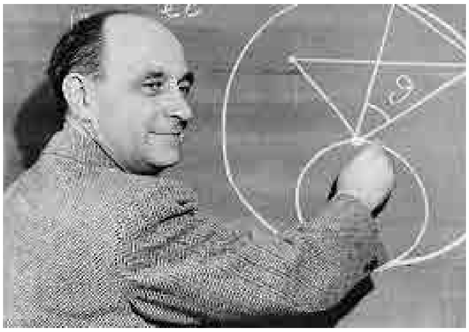

Figure 1

The physicist Enrico Fermi was famous for his ability to estimate various kinds of data with surprising precision.

WHAT IS ESTIMATION?

It is having a good guess at the size of something. But it is not a random guess. It makes use of prior experience and sound physical reasoning.

This reasoning involves usage of approximation, rounding, significant figures, scale factors, etc. And a good deal of common sense! It results in the computation of a value that is fairly close to the right answer.

Is there a right way to estimate? Does estimation need to be taught?

All of us have an innate capacity for estimation. If someone tosses a ball to us, we probably would know where to put our hands to catch the ball without thinking about it. If we had to select a line to stand in, we would estimate the number of people in each line and join the line with the least number of people. Or we may decide by the speed at which a line is moving. Yet, in many contexts of greater complexity, we need to practise the estimation skill to become good at it

Questions of estimation arise practically every day in our lives. Will this suitcase weigh more than 15 kg? How long will this download take? How many glasses can be filled by this jug of water?

Question: Seek more such examples from students from their daily life.

If you were a civil engineer you would make elaborate estimates of the quantity of the work and the expenditure on different works.

Estimations are context-based and the approaches used vary in different situations. However, there are some common techniques that are used in any estimation process. The estimation process involves an appreciation of the crucial variables and the way these variables interact.

Discussion: It is an interesting exercise to brainstorm with the students on examples of situations where exact answers are necessitated and where estimation would serve the purpose. A hospital nurse has to be precise in her measurement of a medicine.

Newspaper activity: Students can bring some newspaper cuttings of numerical information to class to discuss which ones involve exact numbers and which ones might be estimates.

Estimation is a highly neglected part of school mathematics. The ability to estimate can serve as a window into students' mathematical thinking and problem-solving. It is a high-level skill that requires students to be able to conceptualize and mentally manipulate numbers. Teachers should make use of estimation frequently in teaching and problem-solving sessions to help students build estimation skills. One needs to make a special effort to bring such problems into the classroom as there is not adequate focus given to this aspect in our textbooks.

Estimation skills have wide application in many subjects and it will be good to integrate mathematics with sciences and social sciences by choosing problems from all these subject areas.

Math talk: One can bring estimation questions into any topic in mathematics. It will be doubly good if we can ask the students to justify the reasons for their estimates. This will bring about math talk in the classroom and creates a space where peer learning can happen.

Figure 2

When teaching estimation teachers should use examples from life so that the children could try estimation based on the needs of the problem. They should also be ready to accept different solutions from students as long as they are reasonable. Also, students may occasionally know the answers to certain questions out of prior experiences. Teachers should be open and accept those answers. Allow students to share their personal experiences of estimation in daily life. Ultimately students need to know when to use which method, to get close to the precise value.

It is not necessary to give estimation problems as separate exercises. Often we require students to do some calculations. We can encourage students to make rough estimations before doing actual computations to check for the reasonableness of the answer.

Note: Teachers can point out to students that the guesswork that they do while dividing with double-digit or bigger numbers is estimation. Similarly, the square-root algorithm (which students may not be familiar with in upper primary school) also uses estimation.

Good estimation skills can be used as a tool in problem-solving and sensible approximations can simplify a problem to aid in focussing on the structure of the problem or patterns emerging from the given data of a problem.

Over or underestimation in crucial areas has a tremendous impact on any situation. India's overestimation of its production capacity for the Covid vaccine has created scarcity in the availability of vaccines. A lot of planning either at macro or micro levels is based on estimations of various things.

In this article, I have shared some crucial skills of estimation and varied estimation contexts which require their application. I have also shared questions framed by the famous Nobel Prize winner Fermi who posed open-ended estimation questions which challenge students' thinking.

SKILLS OF ESTIMATION

1. Estimation by comparison (with bodily measures, contrast with another object)
2. Using the prior sense of standard length, weight, volume which can serve as benchmarks for measurement
3. Scaling factor or ratios
4. Rounding
5. Usage of the highest place value
6. Choosing simpler fractions or decimals, estimating by halving or quartering
7. Modelling with simple shapes
8. Usage of averages
9. Adjustment of the answer based on rounding down or up
10. Estimation by sampling

ESTIMATION APPROACHES TO NON-ROUTINE QUESTIONS

The Nobel Prize winner Enrico Fermi posed questions that were open-ended, required numerical reasoning and generally came with limited information. His questions necessitated students to raise further questions and focus on the *process* of finding the solution rather than the answer itself.

They provoke curiosity and get students to think more creatively. The process of determining the answers requires students to brainstorm, discuss different approaches, debate them, find loopholes and refine them.

Here are some of his sample questions: "How many grains of sand are there on earth's beaches?" "How many atoms are there in the human body?"

How do we attempt questions of such a nature? We first write down any facts that we know related to the question. We look at possible procedures for determining the answer. We check the reasonableness of our answer

Example: How many hairs on a human head?

A question like this raises further questions. To start with what does a head resemble? A sphere. Assuming the head to be a sphere what is its radius? How much of it is covered with hair? What is the density of hair per square centimetre?

Here is another question. *"How many hours will you sleep in a lifetime?"*

The students begin to identify the variables in these problems as the first step.

How will they obtain the information that they need? Will they conduct an exhaustive survey? Will they test a sample group? Will they look for the relevant data on the internet?

They can then put down the formula or an algebraic expression that they have used while solving the problem.

Can they form some conclusions based on the results that they have got? Can they identify possible sources of error?

To attempt Fermi problems the class can be divided into groups so that each individual can participate actively in the discussion. The teacher plays the role of a guide, giving support as and when necessary.

1. ESTIMATION BY COMPARISON

1.1 Estimating by comparing with bodily measures

Children express quantities often in bodily terms though frequently exaggerated. In primary school, students can practise estimation of the lengths of various objects in the classroom by describing it as "The doormat is longer than my foot, it is twice my foot." Or: "The desk is shorter than me, it is half of my height."

Students of the upper primary school will be able to quantify the earlier statements with their prior awareness of the length of a ruler or their actual height. The descriptions would now be with estimated figures.

The ceiling must be 4 metres and the reasoning could be "If two of us students stand one over the other, there will be space for another half student. Each student is about one and a half metres."

Figure 3

1.2 Estimate by contrasting with another object

This is a common method that is frequently used by comparing the object to be measured with another object about which one has prior knowledge.

"The doormat must be about 60 cm long as two long rulers can fit on it."

In the same example of estimating the height of the ceiling, another student may estimate the ceiling to be four metres high. "The door is about two and a half metres and above the door, we can fit half of another door."

Activity with picture cards: Give students picture cards that show people and buildings, a group of people, people and tall structures like buildings, poles, etc.

Let students make intelligent guesses about the heights of various people in the picture and other structures. Students should be encouraged to justify their guesses.

Ensure that they keep perspective in mind and do not make the mistake of comparing nearby objects with objects that are at a distance, say a hill or trees in the background.

Question: Pose problems: "If this table is 2 feet long, how long is the bench?" "If this book is 4 cm thick, how thick will the dictionary be?"

If a table is 8 feet by 7 feet, what would it look like? Let students draw two or three different rectangles on the board and discuss the best solution.

In Figure 4 can you estimate the height of the tree?

Figure 4

Game: Expert estimate!!

Take a wound up nylon rope roll and ask the students to estimate the rope's total length. The student who makes the best estimate is the winner.

Figure 5

2. USING THE PRIOR SENSE OF STANDARD LENGTH, WEIGHT AND VOLUME WHICH CAN SERVE AS BENCHMARKS FOR MEASUREMENT

All of us have an intuitive sense of certain measures that are frequently used like 10 cm, 30 cm, 1 metre, 5 ml, 200 ml, 1 litre and 1 kg perhaps.

Give students pictures of various objects which are commonly used for estimation of capacity and weight.

Create a table to record the information. At the end students can measure actual capacity, lengths, and weights and compare their estimates with the actual figures.

Object	Estimate	Actual	Difference
Flag pole	6 metres	6.25 metres	.25 metre

Let's take the context of a picnic outing. We will need to carry food, fruits, water and some balls.

How many water bottles (standard bottles) can this crate hold?

Figure 6

Will 6 melons fit in the crate?

Figure 7

How many apples can it hold?

How many ice cubes can this bucket hold?

Figure 8

How many tennis balls can this bucket hold?

Pose questions: "If a banana weighs 45 grams what would be the weight of a guava fruit?"

How heavy would the rucksack be if I filled it with apples?

Figure 9

How many notebooks can my school bag hold?

If I fill it up completely, how heavy will my bag be?

Figure 10

3. SCALING FACTOR OR RATIOS

Solving an estimation problem requires enlargement and reduction.

It makes use of proportion and a scaling factor. Students will attempt to solve an equivalent problem with smaller values in order to obtain an answer to the original problem.

Example: How many books are there on these bookshelves?

Students would observe that the shelves hold roughly the same number of books. They would estimate or make an exhaustive count for one row of books or for the one-foot length of the shelf.

They would then multiply it by the number of such units to estimate the total number of books in the library.

This is an enlargement process.

Figure 11

Estimating bigger lengths using smaller lengths:

While measuring lengths or heights the unit varies from situation to situation. For instance, if we had to estimate the height of a multi-storeyed building, we would count the number of floors and multiply by the usual height of a single floor.

The height of a floor of the building can again be estimated from the height of a person standing near it or a doorway.

Here is a reduction process.

Estimating smaller length from a bigger length

At times we measure the total length of a stack or pile and divide by the appropriate number to arrive at the length of a unit.

Figure 12

Say, if we need to know the thickness of a wooden sheet we could measure the height of the stack and divide it by the number of sheets.

Example problems

1. A train terminal has 12 platforms. Each platform has about 115 trains departing each day. Find an estimate of the number of trains leaving the terminal every day.
2. If the current growth of population continues (get the latest data from the internet), what would be the population in India by 2030? Can we assume that this growth rate will stay the same? What can alter it?
3. At school, we are served milk and each one of us takes 2 full spoons of sugar with it. About how many kilograms of sugar do we use in the school every month?
4. Rahim can run 9.2 m per second. If he runs a 500m track what is your estimate of his timing?

Teachers can point out other mathematical concepts like factors and multiples which arise while doing these calculations.

4. ROUNDING

Rounding to the nearest tens, hundreds or powers of ten is employed to simplify the problems.

Students would probably have been taught by now the basics of rounding. In whole numbers, say 235, 236, 237, 238, 239 are rounded up to 240 while 231, 232, 233, 234 are rounded down to 230 (to the nearest tens). This can be shown on a number line for greater clarity.

Students should be taught to use rounding to estimate answers. They can also use rough answers to check calculations that they do either with paper-and-pencil or calculator. This would reveal any absurd errors that they might have made.

But students must also understand that rounding makes numbers easier to use – but at the cost of loss of precision.

If we are working with large numbers say in thousands it is easier to round the numbers to the nearest thousand for estimating answers. Choice of rounding is dependent on the level of precision needed. Close rounding makes calculations more precise. 2435 can be rounded to 2400 if we wish to retain hundreds or to 2000 if we don't.

In general, estimating involves rounding key quantities so that they can be manipulated easily. Rounding makes calculations easier. If we had to compute 31 x 49 it is easier to round it to 30 x 50 and get an approximate answer of 1500. The actual answer is 1519.

The same type of rounding can be applied to adding, subtracting, or dividing numbers as well.

- 83 × 31 = ?
- 39 + 97 = ?
- 83 – 57 = ?

Example: Sufi said that 523×34 gives 177,820. Is that a reasonable answer?

If students use the estimation of 500×30 to arrive at 15,000, they will quickly realize that the place value is way off.

If we have to work with decimal numbers we start by rounding the numbers to the nearest whole number. A number with a decimal value of 0.5 or higher is rounded up (for example, 1.8 becomes 2). A number with a decimal value lower than 0.5 is rounded down. (For example, 4.3 becomes 4).

Example: If a big parcel weighs 1.89 kg and a smaller parcel weighs 0.99 kg, what is the estimate for the total weight of 10 big parcels and 5 small parcels?

Rounding the big parcel to 2 kg and the small parcel to 1 kg would make the total 25 kg.

Intervals

Students can also use their understanding of the principles of rounding to determine the interval of the answer. That determines the lower and the upper-level figures for a given number.

Examples

1. "The number of pins in a box is rounded to the nearest fifty. If the number of pins is given as 2650 find the smallest and largest number of pins that could be in the box."

   The lower limit of this range would be 2625 and the higher limit will be 2674.
2. The number 11,200 has been rounded to the nearest one hundred. In what range does the actual number lie?
3. A number, when rounded to the nearest ten, becomes 400. In what range does the actual number lie?
4. Rohan makes a quick estimate of how many people weighing 72 kg can safely fit into a lift with a maximum load warning of 940 kg. He does this by rounding the values used in the calculation. What was his answer?

5. USAGE OF THE HIGHEST PLACE VALUE

When we sum amounts or evaluate products we use the highest place value as that gives us a rough idea of the total amount. Estimation with numbers reinforces the understanding of place value. Students realise that in two-digit numbers, the tens' digit has greater bearing than the ones' digit in contributing to the answer.

219 + 345 + 564 would be summed as 200 + 300 + 500 and hence the sum is above 1000.

What is the rough value of 2453 × 312?

It is taken as 2000 × 300, so the product is 600000. The number of zeroes have to be placed right.

How does one attempt a question like the one below?

Estimate the approximate answer of 53687 + 8365 + 1638 + 28.

Example: The average amount of storage space needed for a photograph taken with a phone is 3940 KB. The available storage space on a laptop is 217 GB. By rounding these numbers estimate the number of photographs that could be stored on a laptop.

The space for the photograph can be rounded to 4000 KB. The storage space can be taken as 200 GB that is 200,000,000 KB. Hence the number of photographs that could be stored is 50,000!

6. CHOOSING SIMPLER FRACTIONS OR DECIMALS, ESTIMATING BY HALVING OR QUARTERING

A good understanding of fraction concepts is helpful in selecting appropriate fractions and adjusting the result.

Example. If we had to calculate five-eighth of 4,100 we would choose 1/2 as it is slightly less than 5/8 and compute 1/2 of 4000 which is 2000. Adjusting for the reduced fractional part of 1/8 we can increase it by 1/4 of 2000, i.e., by 500. Our estimate is 2500. The actual answer is 2562.5.

Estimating by halving and quartering

Heights of buildings, trees and electric poles are often difficult to estimate and in such cases, we contrast them with nearby objects and use the process of halving and quartering to estimate the heights of tall structures as explained in the example problem here.

Figure 13

Example: In Figure 13 the vehicle which is at a distance seems to be slightly less than one half of the height of the lamp post.

The vehicle's height can be estimated roughly as 9 feet. The lamp post is probably around 20 feet.

In Figure 14 a man is standing on the scaffolding reaching the top of the pole. If we imagine quartering the pole the man's height corresponds to a quarter.

The pole height must be roughly 25 feet.

Figure 14

7. MODELLING WITH SIMPLE SHAPES

Complex or irregular shapes are approximated to a simple shape for estimation purposes.

How many round cakes fit into a cuboid box?

Figure 15

For estimation purposes, we treat the round cakes as squares and calculate.

How many melons fit into a carton? In order to estimate the number of melons that can fit into the carton, we model the melon with an appropriate sized cube or cylinder and evaluate how many such cubes or cylinders will fit into the carton.

Figure 16

Using simple shapes like cubes, cuboids helps in the estimation process.

8. USAGE OF AVERAGES

(**Note:** Average does not necessarily equal arithmetic mean)

Here is the data of money collected by ten students at the school fair.

55, 75, 60, 45, 85, 60, 65, 55, 55, 70

What is the rough sum of the amount collected?

We notice that the figures hover around an average figure of 60. Hence the sum is roughly 60 × 10=600.

9. ADJUSTMENT OF THE ANSWER BASED ON ROUNDING DOWN OR UP

While doing estimations we round figures up or down. That affects the answer and in order to address any major discrepancy that might arise we adjust the answer after the computation.

Example: 44 × 41 = ?

We may round it as 40 × 40 obtaining 1600. But 44 has been rounded down as 40 and hence we adjust the answer and make it 1800.

10. ESTIMATION BY SAMPLING

One technique that biologists use to estimate the number of species in a particular area is random sampling. It can be used to estimate population size. In this procedure, the organisms in a few small areas are counted and projected to the entire area. It is the same process that we use to answer the question "How many hairs on a human head?"

A biologist collected 50 litres of pond water and counted 10 mosquito larvae. How many larvae would you estimate to be in that pond if the total volume of water in the pond was 80,000 litres?

What are some difficulties with this technique? What could affect its accuracy?

11. TIME ESTIMATION

Time estimation skills are very crucial for students and everyone else.

Students can start with simple estimation like how long would it take to complete 10 math problems?

At the airport, you are 30th in a line. How long will you need to wait?

How many hours do you spend on video games in a month?

Very often our time estimates go wrong as we do not factor in two crucial elements.

- How long has it taken us to do a similar task in the past?
- Anticipating unexpected delays.

Help students to make realistic time estimates.

As a second step, the students can learn to estimate the time they spend on doing homework in various subjects and learn to plan and create a schedule for themselves.

Some estimation problems from various topics

Numbers

Example: How will you estimate the solution to this subtraction problem? 217 – 96 – 46

From 200 you can remove a 100 and a 50. That leaves 50 as your estimate.

How many times would 71 divide 423 roughly?

What is 23% of 123?

Between which two whole numbers is $\sqrt{2021}$?

What would you approximate √(1250/10000) to?

If 7 cones cost Rs 280 what is the best estimate for the cost of 100 cones?

Which of the following is the best approximation for √108?

8, 9, 10, 11, 12

Fractions and decimals

- $\frac{91^2}{9.9} - \left(35 - \frac{7.4}{0.12}\right) = ?$
- $13 \times 0.2 = ?$
- $\frac{36}{0.6} = ?$
- $468 \times 7.9849 + 71 + 38 = ?$
- $42 \times \frac{21}{1.77} = ?$
- $304 \times \frac{0.736}{0.099} = ?$
- $0.31^2 = ?$

Angles

Estimate the measures of angles between your fingers when you spread them out. Is it the same for everyone?

Too often we see students read protractor in the wrong manner and come up with absurd answers. Estimation can largely avoid this problem.

Ability to estimate angles grows by using clocks and circular fraction kits.

Ideally this should have happened in the primary school.

Figure 17

Mensuration

Estimations with regard to capacity or total area are based on estimations of length, width, circumference and height.

Take two identical rectangular papers. Roll one into a short cylinder and the other into a tall cylinder. Estimate which one will hold more popcorn (if it had a base!).

Can you estimate the mass of a tree?

Can you estimate the number of mangoes on a tree?

Figure 18

There is a pile of waste material from the building site which needs to be disposed of. The pile is about 6 feet high in the middle and about 9 feet across.

Can a small truck carry this rubbish?

Figure 19

How many bricks were used for your school building?

Figure 20

Graphs

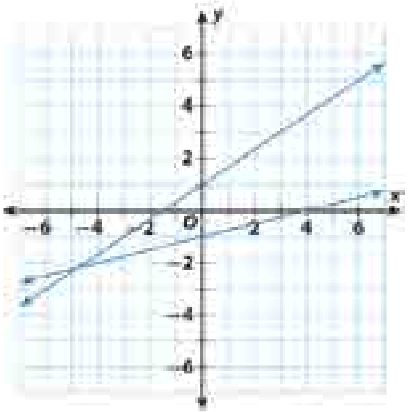

Figure 21

Example: Estimate the solution by sketching a graph of each linear function. Then solve the system algebraically. Use your estimate to judge the reasonableness of your solution.

$$x - 4y = 4$$

$$2x - 3y = -3$$

Example: Here is a pie chart of different types of books in a library. The blue part is fiction, the yellow is classics, the green portion is reference, and the red portion is encyclopaedias.

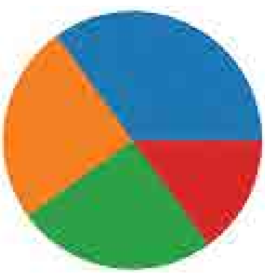

Figure 22

If the total number of books is 10,000 can you estimate the number of books of each type?

Example: How many soft drinks do you have in a month? Does it change over the year? Is it safe to consume such a quantity of soft drinks? Find out.

Example: How many sets of clothes do we buy in a month? What does your estimate show? Can you compute how much money we spend on clothes each year?

Example: Here is a child mortality graph of children in India over a period of time. What is your estimate for 2050?

Figure 23

Fermi question bank

1. What distance will a ballpoint pen write?
2. How much paper is used at our school each week?
3. How much money is spent in the school canteen each day? In a week? Over the year?
4. How big a block of chocolate could you make using all the chocolate eaten by the class in a year?
5. What is the weight of garbage thrown away by each family every year?
6. How many children are needed to have a mass the same as an elephant?

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PADMAPRIYA SHIRALI

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multi-grade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box.' Padmapriya may be contacted at padmapriya.shirali@gmail.com.

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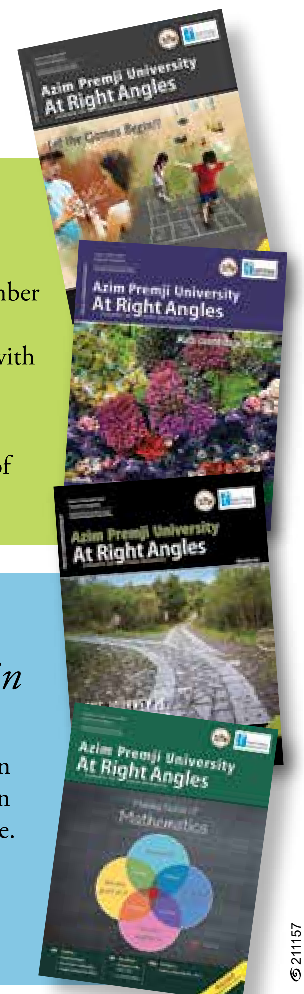

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Survey No. 66, Burugunte Village, Bikkanahalli Main Road, Sarjapura, Bengaluru – 562 125

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