Issue 11 | Nov 2021
Triannual | Bangalore

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

RNI No.: KARENG/2018/76619

Community Math Centre, Rishi Valley School (KFI)

Azim Premji University

Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

ISSN 2582-1873

Networking: Mathematics and Embroidery

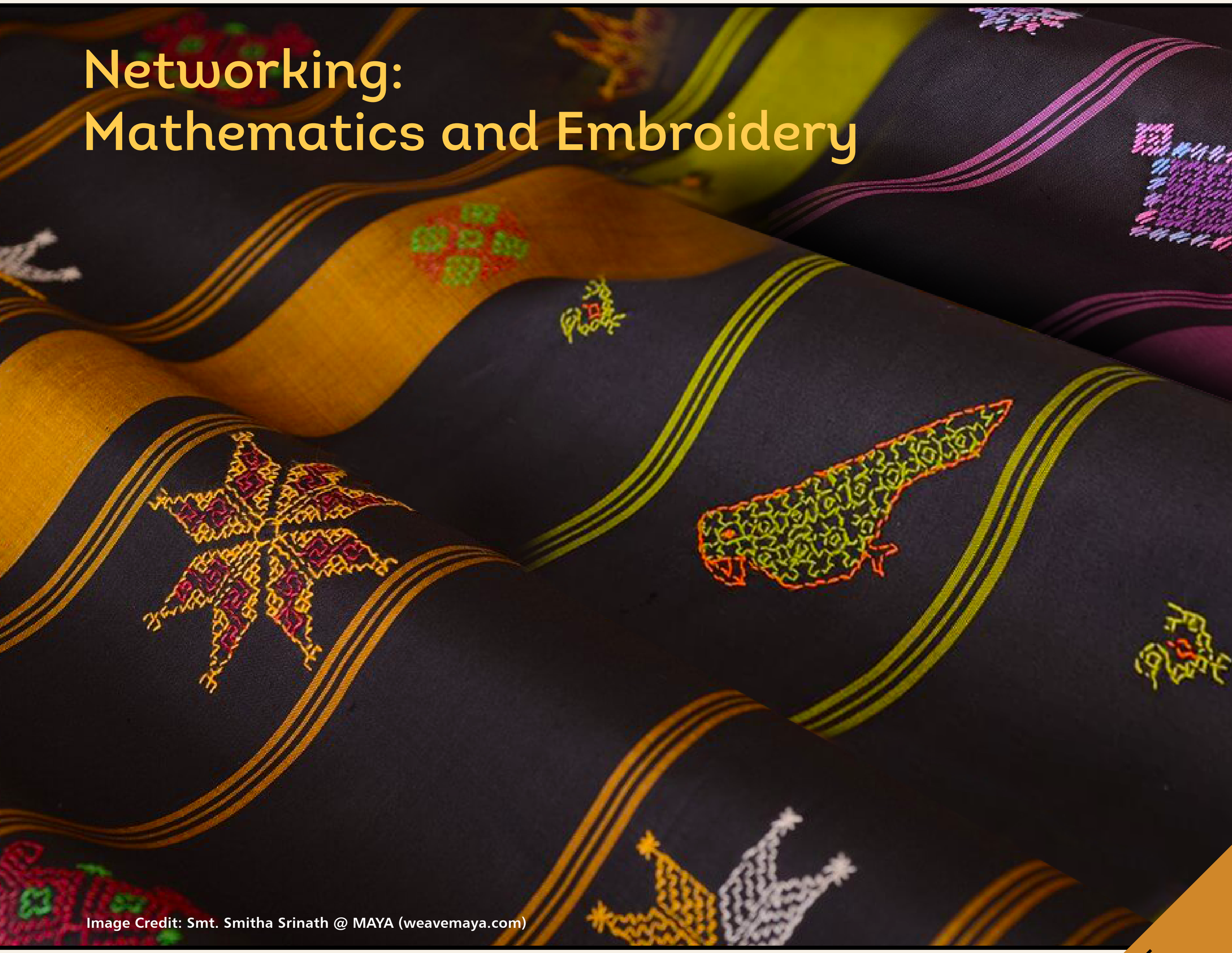


Image Credit: Smt. Smitha Srinath @ MAYA (weavemaya.com)

PULLOUT
COORDINATES

Bertrand Russel:
The pure mathematician like the musician is the free creator of the world of ordered beauty.

Paul Lockhart:
Beauty in mathematics should always mean finding patterns and crafting beautiful and meaningful explanations.

George Polya:
Beauty in maths is seeing the truth without effort.

Einstein:
How can it be that mathematics being after all, a product of human thought which is independent of experience is so admirably appropriate to the objects of beauty?

Archimedes:
Mathematics reveals its secrets, only to those who approach it with pure love, for its own beauty.

From the Editor's Desk . . .

Henri Poincare once said that *Mathematics is the art of giving the same name to different things.* What a beautiful way to express the amorphous task of 'mathematisation'. We feature on our cover, Kasuti work that is intrinsically Indian, typically done by women of a certain geographical region, seen and admired by many and having its own rules underpinned by beauty, symmetry, precision and more. Read the article by Nishtha Chabra to find amazing links to graph theory and ways of bringing needlework into math lessons. Preeti Dhasmana picks up the thread in ClassRoom with an article on symmetry and how she supplemented the textbook with worksheets on symmetry and tiling which enabled students to build their understanding of the concept of area. And with a hands-on activity on paper folding which establishes a deep understanding of the equality of areas of triangles on the same base and with the same height, we move seamlessly between doing and reasoning.

Features also carries V.G. Tikekar's article on Spoof Numbers, an engaging *What-If* which will help many teachers to convey the nuances of mathematical logic to students. Arddhendu Dash writes about how the pandemic taught him the importance of speaking with precision and care, the concise nature of mathematical language and the symbols used make perfect sense on the blackboard – do they mean the same thing to speaker and listener when conveyed in an audio lesson? In TechSpace, Mujahid shares with us the journey of discovery of Turtle Logo that he undertook with his daughter, her experiments with creating patterns, her growing understanding of mathematical language and the informal development of her programming skills convey the importance and naturalness of learning by doing. Our new Review series on the review of manipulatives used to teach mathematics, helps you do the same with Swati Sircar's guidance. The Review section also carries a review of a video from a math channel. I am sure that this will greatly add to the teaching and understanding of Bayes' Theorem.

In the PullOut, Padmapriya Shirali advocates setting the foundations of Coordinate Geometry from the middle school onwards and gives us several suggestions on activities with which to make the understanding and use of coordinates intuitive, natural and fun. Her views are echoed in Jayasree's article recounting a conversation with a class 7 student on a WhatsApp group, which elicited an understanding of the concept of slope and caused her to reflect deeply on guided teaching.

Rajesh Sadagopan uses the game of chess as a springboard to view Pascal's triangle from a new angle. A. Ramachandran takes a simple well-known riddle on weights to connect to sequences and we also explore the ever-engaging task of Trisecting an Angle both with an activity which will be followed (in the next issue) with a deeper analysis of the same. There is a lot more, including four gripping online articles – read, enjoy and get back to us on AtRiA.editor@apu.edu.in

The most satisfying editorial task is of receiving and reading articles from students, AtRiA is proud to be a platform to showcase their work and we take this opportunity to celebrate all those who write in with evidence that original thinking in mathematics is flourishing in our learning spaces. This is in no small measure due to the teachers who encourage them and mentor them, who find interesting problems for them to explore, who stimulate their thinking with the right questions… May their tribe increase!

Sneha Titus
Associate Editor

The Opening Bracket . . .

by Ravi Subramaniam

The pandemic has brought suffering into many homes and has disrupted our lives in a way that we could not have imagined two years ago. If, as a society, we are receptive, it can also teach us many lessons. Certainly, there are lessons to be learnt on how to manage healthcare better and how to deal with health disasters. But there are other important lessons too.

The pandemic brought home the deep inequality that exists in India. In the first wave, it was the poor mainly that suffered through loss of livelihood caused by the lockdown. The second wave hit the urban middle class too, when the fragile healthcare infrastructure in the country, weakened by decades of governmental neglect and privatisation, collapsed. The disaster was compounded by poor decision making and governance, which is unsurprising in an environment where manipulation of information and reluctance to admit facts is the norm among those in power. This is consistent with the progressive undermining of independent public institutions charged with the responsibility of gathering reliable data on the health of the economy and society. As educationists, we must partly bear responsibility for why our society does not care about facts and data. Education must aim beyond literacy, at developing among citizens a strong expectation and responsibility towards ensuring that information is publicly accessible and is free from distortion and manipulation. The pandemic may be a call for us as mathematics teachers and educators to move beyond numeracy, and to consider data literacy and data citizenship as an important part of the mathematics curriculum. For, mathematics education, like other subjects, must also contribute to the collective effort of building a better society.

It is the data that opens our eyes to the shocking fact that while millions lost their jobs and livelihood as a result of the lockdown and were pushed into poverty, a handful multiplied their wealth many times over. Should we not call into question an economic system that creates such deep inequalities even in times of crises? In a culture where there is receptivity and respect for data, and there is freedom to question, questions would necessarily be raised and change would follow. Education must lay the foundation for such a culture and bear responsibility for social transformation.

It is not only economic inequality that the pandemic has brought into focus. In the domain of education, the unequal impact of the pandemic is stark. Online teaching and learning during the pandemic has taken place regularly only for a miniscule fraction of students from well-endowed families. I am sceptical about the claims of success of online teaching, even with the best resources and practices. But let us grant that it works. Still, it is only the top rung of schools that may have successfully managed students' learning, possibly only in the higher grades. For the vast majority, online teaching is a mirage since they don't even have the minimum essential digital infrastructure as multiple studies have shown.

The economic stress of the lockdown has cascading effects on many sectors including the education sector. There is anecdotal evidence that a number of low fee private schools have folded up, unable perhaps to cope with the economic stress. Not only the students who studied in these schools, but also the teachers who taught there have been pushed out of school. One also hears of many parents being unable to pay private school fees, resulting in their students having to leave school and their scrambling to find other schools willing to admit them. Many are trying to get their children admitted back in government schools. There are indications that enrolment in government schools is going up. If these trends are true – studies are still awaited since this is largely a post-second wave phenomenon – we have a bigger crisis in education than the learning and emotional disruptions caused by a long gap in attending school. For many students, there may simply be no going back to their own school and their lives will likely be deeply disrupted. Government school capacities – classrooms, teachers – have to be expanded without delay to accommodate larger enrolments. We also have a lesson to learn – that an education infrastructure shaped by market forces has low resilience, and will fold up under stress, compounding an ongoing crisis.

All these crises add up to what could potentially be an educational disaster with socio-economic consequences that only play out years later. Remedial efforts need to be on a massive scale in order to make a dent, and these can only come through the initiative of the governments at all levels. They must prioritize, among other challenges, the task of bringing children and teachers who have been pushed out back to school. Educationists have urged that schools be reopened at once, following clear safety protocols, and teachers in all grades be given the freedom to begin from where students are in terms of their learning and emotional needs. They have urged that the practice of rigidly following the syllabus be suspended for at least the initial months. This is a minimalist demand, but it is proving hard to get even this minimal demand accepted by the educational bureaucracy. This is hardly reassuring given that only a massive intervention by the government can partly mitigate the larger, structural disruptions of schooling that has followed in the wake of the pandemic.

Editorial Office

The Editor, Azim Premji University
Survey No. 66, Burugunte Village,
Bikkanahalli Main Road, Sarjapura,
Bengaluru – 562 125
Phone: 080-66144900
Fax: 080-66144900
Email: publications@apu.edu.in
Website: www.azimpremjiuniversity.edu.in

Publication Team

Shantha K
Programme Manager
shantha.k@azimpremjifoundation.org

Shahanaz Begum
Associate
shahanaz.begum@azimpremjifoundation.org

Print

SCPL
Bengaluru 560 062
www.scpl.net

Design

Zinc & Broccoli
enquiry@zandb.in

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Azim Premji University

A Publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley and Sahyadri School, Pune

Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

Problem Corner

Student Corner

Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Azim Premji University

A Publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley and Sahyadri School, Pune

Continue . . .

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Online Articles

Features

Kasuti: The Mathematics of Embroidery

NISHTHA CHHABRA

This article is an attempt to understand *Kasuti* or Black-Work embroidery from a mathematical standpoint. Largely inspired by Joshua Holden's chapter in the book titled "Making Mathematics with Needlework", my article attempts to explore the subtleties of mathematics in *Kasuti* embroidery.

Kasuti - Tracing the origin

Kasuti is a thread-work embroidery that has historic ties to the Chalukyan dynasty of India. The word is derived from *Kai*, meaning 'hand' and *suti*, meaning 'cotton' in Kannada [1]. Traditionally, the artwork used silk yarns that were embroidered on pieces of cotton fabric. The embroidery derives inspiration from architecture, *rangoli*, palanquins, etc., and is often used in bedsheets, *kurtas*, sarees, table-runners, curtains and keychains [2].

Kasuti is distinct from other forms of embroidery. It is executed on a fabric which has significantly large holes that are carefully counted before the thread is passed through. Today, *Kasuti* embroidery is done on fabrics of matty or jute.

Keywords: art, craft, embroidery, mathematics, pattern, symmetry, networks

Figure 1 : Sample fabrics used for *Kasuti*

Stitches in *Kasuti*

Kasuti is a knot-less form of embroidery that entails four types of stitches -

1. *Gavanthi:* A double running stitch
2. *Murgi:* Used to form a zigzag design, akin to a ladder
3. *Negi:* A weaving stitch that uses a series of long and short lines
4. *Menthi:* A cross-stitch that literally means 'fenugreek' in Kannada

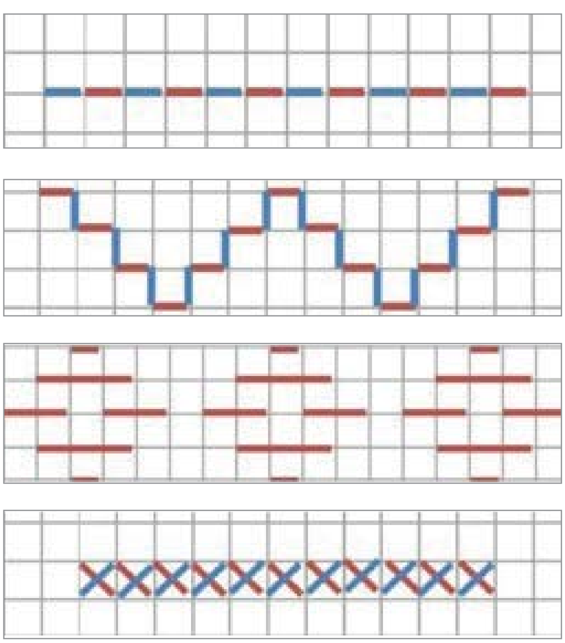


Figure 2: (Top to Bottom) *Gavanthi, Murgi, Negi, Menthi* [3]

In Figure 2, while the red lines denote the thread moving forward, the blue ones denote the stitch in the reverse direction.

Now let's take a look at a simple *Kasuti* motif.

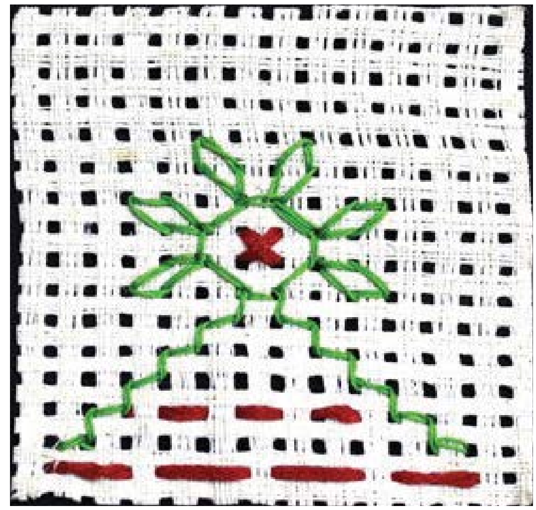


Figure 3: *Kasuti* Motif

Deconstructing the design, we identify the following stitches -

1. *Negi* - In this stitch, we keep moving forward in one direction, skipping the same number of holes every time. As a result, we get broken lines on the cloth.

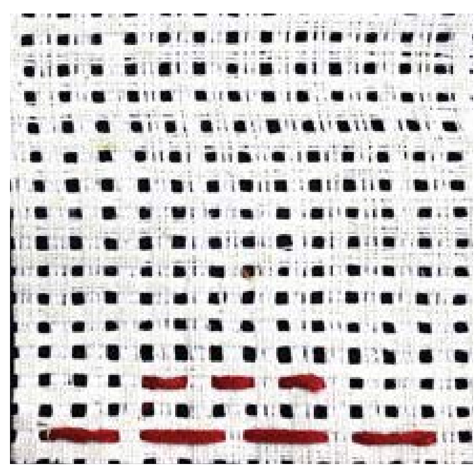


Figure 4: *Negi*

2. *Murgi* - Here, we first make the horizontal stitches (left), and then retrace the steps to create vertical lines (right). Doing this, we reach the starting point of the ladder, thereby completing the design.

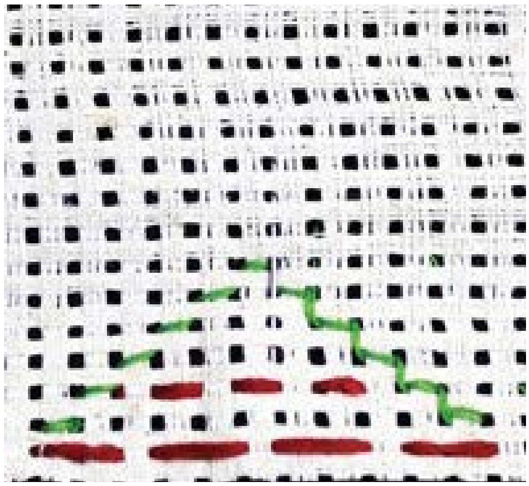
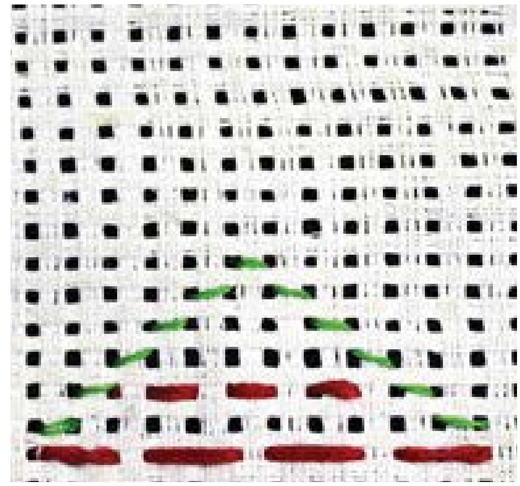
5a  5b 

Figure 5: *Murgi*

3. *Gavanthi* - Here, we move forward jumping alternate holes and retrace our steps. Retracing the stitch in the reverse direction produces stitches on the empty spaces that existed during the first stitch. This completes the stitch and we return to the starting point. This stitch can be performed vertically, diagonally or horizontally.

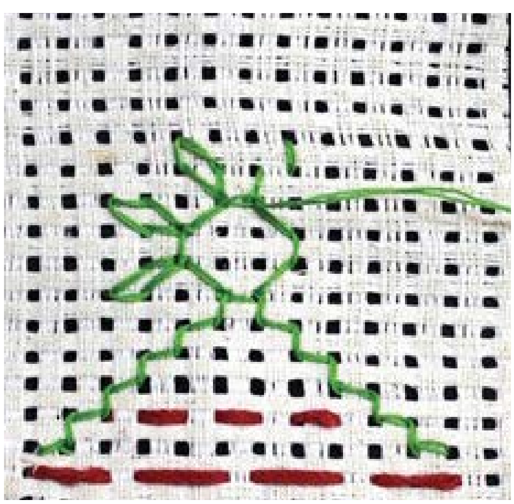


Figure 6: *Gavanthi*

4. *Menthi* - In *Menthi*, we start at one hole and move to the diagonally opposite hole. From there, we move to the hole adjacent to it and again to the hole in the diagonally opposite corner. Finally, we move to the adjacent hole, returning to the starting point of our stitch. Thus, the right side of the fabric has an 'x' appearing, whereas on the wrong side, we only have two vertical/horizontal lines.

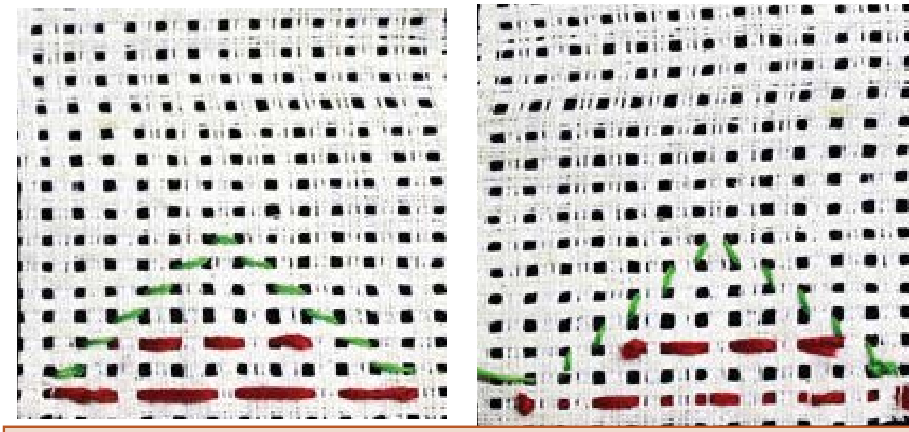
Figure 7: *Menthi*

Mathematics behind it

Symmetry

a. In *Murgi*, there is a rotational symmetry between the stitches on the right and the wrong sides of the cloth. The horizontal lines on the front side become the vertical lines on the wrong side and vice versa. In Figure 8, take a look at the green lines. The figure on the left has horizontal lines, whereas the same lines appear vertically on the wrong side of the fabric.

b. In *Negi*, there is a translational* symmetry between the stitches on the right and wrong side of the cloth. This means that the stitch moves by a few positions to the left/right on the wrong side (Figure 4). There is also a translational symmetry on the same side of the cloth.



A sneak peek at the 'Wrong' Side

In the textile industry, every fabric has a 'right' and a 'wrong' side. Often the 'right' side is smoother and is the side that is visible externally.

Figure 8: *Murgi* stitch on the right side and wrong side of the fabric respectively

c. In *Menthi*, the cross appears on the right side of the cloth and vertical lines appear on the wrong side of the cloth (Compare the red cross in Figures 3 and 9).

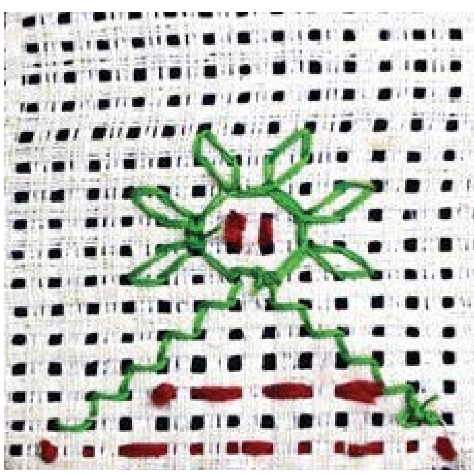


Figure 9: *Menthi on the wrong side of the fabric*

*Usually, translational symmetry is defined for infinite length. However, here we are restricted to a finite length.

Wrong is Right!

These symmetries can help to cross-check if the stitches are in the right position. One can always take a peek at the wrong side and verify if the symmetry holds.

Another way of knowing you are doing the stitch correctly, is to check if you are able to return to the starting point while using the same stitch in reverse direction.

2. Graph Theory - Graph theory is a study of vertices and edges of a drawing called 'graph'.

Figure 10 shows two kinds of graphs - undirected (wherein the edges are bidirectional) and directed (wherein the edges point in a particular direction).

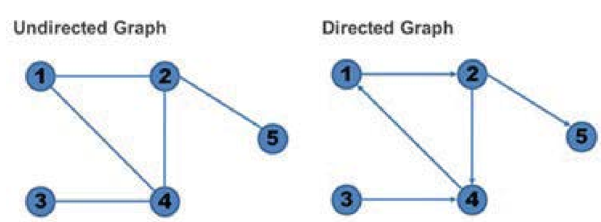


Figure 10: Types of Graphs [4]

Since *Murgi*, *Menthi* and *Gavanthi* involve returning to the same point, they form what is known in graph theory as a 'circuit'.

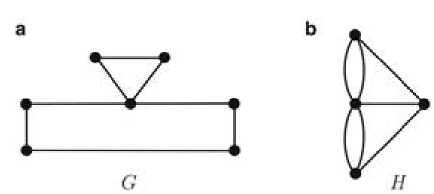


Figure 11: Eulerian and Non-Eulerian Graphs [4]

Eulerian graphs refer to graphs where it is possible to traverse across each edge exactly once and then return to the starting vertex. In Figure 11, while (a) is Eulerian, (b) is not. This means that in (b), it is impossible to traverse each edge exactly once, and return to the starting point. Note that here, there is no restriction on the number of times we can touch a vertex.

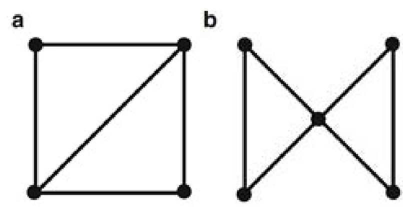


Figure 12: Hamiltonian and Non-Hamiltonian Graphs [4]

Hamiltonian graphs refer to graphs where it is possible to touch each vertex exactly once and then return to the starting vertex. In Figure 12, while (a) is Hamiltonian, (b) is not. This means that in (b), it is impossible to touch each vertex exactly once, and return to the starting point.

In *Kasuti*, all the stitches form an Eulerian circuit, i.e., we are able to touch each edge exactly once and return to the starting point. On the other hand, all the stitches in *Kasuti* are non-Hamiltonian since we touch each of the vertices (holes) multiple times.

Where do we use Eulerian and Hamiltonian graphs?

Google Maps - Imagine there is a bus driver who needs to find the shortest route possible to cover all stops. Here, the routes are edges and the stops are vertices. The driver will find an optimal route that begins at a stop, covers each stop exactly once, and finally return to the point where he started. Hence, we use Hamiltonian circuits to determine such routes.

Imagine you're taking a trip around the city to see its historical landmarks. You wish to travel along every road exactly once. Here, the places you visit are vertices and the route you take is an edge. In this trip, you are travelling across a particular route (edge) just once to come back to the place you began from. However, in this route it is possible that you touch a particular place more than once. Hence, this is an example of an Eulerian graph.

Pedagogical Suggestions for the Primary Level

Integrating mathematics with art and craft is a much-discussed topic now. *Kasuti* work is a good project for all levels right from primary, when students can follow instructions to complete (or observe) a *Kasuti* pattern to distinguish between types of lines (standing, sleeping, slanting). Understanding of Perimeter can be enhanced by counting the number of stitches (non-standard units) seen around a closed shape. Again, counting the number of squares inside a closed shape reinforces the idea of area. Meeting or interviewing a person who does *Kasuti* work after doing a project in class, will help students understand the monetary aspect of the craft, giving them real data about how much a piece of *Kasuti* embroidery is worth, how long it takes to complete, ideas of profit and loss, etc. And of course, as detailed in the article, at levels beyond primary, *Kasuti* embroidery helps students get a great introduction to symmetry, graphs and networks, topics not usually covered in school mathematics content.

References

[1] https://medium.com/direct-create/kasuti-embroidery-from-karnataka-9edd4a50da07 "*Kasuti* Embroidery from Karnataka"

[2] http://indianexpresss.in/kasuti-embroidery-of-karnataka-essay/ "*Kasuti* Embroidery of Karnataka"

[3] https://www.embroidery.rocksea.org/hand-embroidery/kasuti/kasuti-lesson-1/ "How to make Patterns in *Kasuti*"

[4] "Graphs in the Database : SQL meets Social Networks" HYPERLINK "http://alberton.info" alberton.info

[5] R. Balakrishnan, K. Ranganathan, "Eulerian and Hamiltonian Graphs" in *A Textbook of Graph Theory,* Universitext. Springer, New York: 2012, ch. 6, pp. 118-122.

NISHTHA CHHABRA is pursuing her M.Sc Mathematics at Christ (Deemed to be University) and is currently interning under Dr. Shantha Bhushan at Azim Premji University. She loves all things creative and enjoys working in the fields of Tactile Mathematics and Knot Theory. She has also worked as Programme Associate, Ranga Shankara Theatre and as a Guest Faculty member for the Civil Services certificate course, Christ (deemed to be University). Being someone who loves dipping her toes in various fields, she relishes reading, writing, theatre, documentaries, podcasts and current affairs. She may be contacted on nishwish28@gmail.com

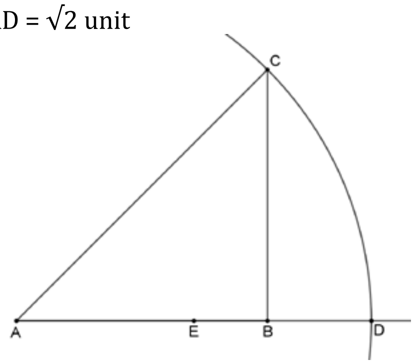
How to represent $1/\sqrt{2}$ on the Number Line.

In the figures

AB = BC = 1 unit,

AC = $\sqrt{2}$ units,

AD = $\sqrt{2}$ unit

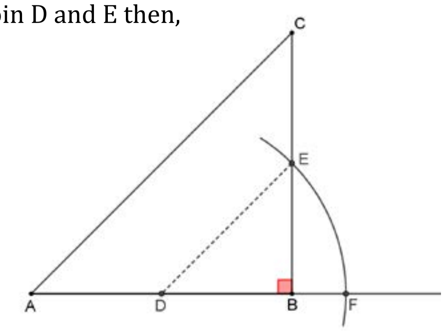


This is the standard construction for $\sqrt{2}$

How do we construct $\frac{1}{\sqrt{2}}$?

Method 1: Rationalising, we get $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2} \times \sqrt{2}$, hence if we construct the perpendicular bisector of AD, and mark its mid-point E, we get AE $= \frac{1}{2} \times \sqrt{2} = \frac{1}{\sqrt{2}}$

Method 2: Bisect the lengths AB and BC, so that D is the mid-point of AB and E is the mid-point of BC. Join D and E then,



$$DE^2 = DB^2 + BE^2$$

$$DE = \sqrt{DB^2 + BE^2}$$

$$DE = \sqrt{\left(\frac{AB}{2}\right)^2 + \left(\frac{BC}{2}\right)^2}$$

$$DE = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \text{ (as } AB = BC = 1\ unit)$$

$$DE = \frac{1}{\sqrt{2}}\ units$$

Contributed by **Toyesh Prakash Sharma**, Email: **toyeshprakash@gmail.com**

Features

Spoof Numbers and Spoof Solutions - Part I

V G TIKEKAR

In this two-part article, we consider the curious notion of spoof numbers and spoof solutions which we get when we partially relax the conditions needed to define particular number families.

In number theory, we study various families of numbers: prime numbers, composite numbers, triangular numbers, square numbers, Mersenne numbers, Ramanujan numbers, and so on. Each family has its own characteristic properties. In exhibiting an instance of a particular type of number, we have to be careful that none of the required properties is violated. This observation leads to the following definition.

Definition 1. While constructing a number belonging to a particular family, if we relax some of the required properties or rules of formation but ensure that all the other properties of that family are satisfied, then such a number is called a *spoof number* of that family. Sometimes, such a number is also called a *quasi number* of that family.

We can similarly define the notion of *spoof solutions* by considering the spoof numbers obtained in the context of solutions of equations.

We shall describe two important situations which give rise to spoof numbers and spoof solutions, related to *perfect numbers* and *Fermat's Last Theorem*, respectively.

The notion of spoof numbers can be applied to any type of number, but is most often used for those types of numbers that are not known to exist, at least at present.

Keywords: Spoof number, spoof solution, Fermat number, perfect number, Mersenne number, triangular number, Euler

Perfect numbers & non-perfect numbers

Definition 2 (Perfect number)**.** A positive integer is called a *perfect number* if all its divisors add up to twice that integer.

Example 1. 496 is a perfect number since all its divisors, namely, 1, 2, 4, 8, 16, 31, 62, 124, 248 and 496 add up to $992 = 2 \times 496$.

Readers can verify that 6, 28 and 8128 are other examples of perfect numbers. Euclid was aware of these four perfect numbers. (We remark here that the notion of perfectness of numbers goes back to the Greeks, who were fascinated by such properties.)

Leonard Euler, one of the greatest mathematicians the world has known (and certainly one of the most prolific) defined what he called the 'sigma (S) function.' It is very useful in handling perfect numbers.

Definition 3 (Euler's sigma function)**.** If n is a positive integer, then $S(n) =$ sum of all the divisors of n.

Example 2. $S(6) = 1 + 2 + 3 + 6 = 12$ and $S(8) = 1 + 2 + 4 + 8 = 15$.

So, using Definition 2, 6 is a perfect number, while 8 is not. Using Euler's S-function, the formal definition of a perfect number becomes:

Definition 4 (Perfect number)**.** A positive integer n is called a 'perfect number' if $S(n) = 2n$.

Example 3. $S(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56 = 2 \times 28$. So 28 is a perfect number.

Two important rules concerning the S function. Euler proved two important properties of the S function that come of great use when working with this function. We state them below and illustrate them with numerical examples, but leave the proofs to the reader.

Rule 1. $S(a \times b) = S(a) \times S(b)$ iff a and b are coprime. (Note that the rule extends in an obvious way to the product of more than two coprime numbers.)

Example 4. Since 3 and 8 are coprime, $S(3 \times 8) = S(3) \times S(8)$, so

$$S(3 \times 8) = (1 + 3)(1 + 2 + 4 + 8) = 4 \times 15 = 60.$$

Direct verification: $S(3 \times 8) = S(24) = 1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 60$.

Rule 2. If p is any prime number and a is any positive integer, then

$$S(p^a) = 1 + p + p^2 + p^3 + \dots\dots + p^a.$$

Example 5. As 2 is prime and 3 is a positive integer,

$$S(2^3) = 1 + 2 + 2^2 + 2^3 = 15.$$

Direct verification: $S(2^3) = S(8) = 1 + 2 + 4 + 8 = 15$.

We shall not give the proofs of these two rules here as they are available in all standard texts in number theory. The reader may refer to [3] for the proofs (the section 'Properties').

Combining the two rules. Now we present two examples in which we make use of both the rules given above.

Example 6. To compute $S(28)$:

$$\begin{aligned} S(28) &= S(4\times 7) = S(4)\times S(7) &&\text{(by Rule 1, as 4 and 7 are coprime)}\\ &= S(2^2)\times S(7^1)\\ &= (1+2+2^2)\times(1+7^1) &&\text{(by Rule 2)}\\ &= 7\times 8 = 56 = 2\times 28, \end{aligned}$$

implying that 28 is a perfect number.

Example 7. To compute $S(40)$:

$$\begin{aligned} S(40) &= S(5\times 8) = S(5)\times S(8) &&\text{(by Rule 1, as 5 and 8 are coprime)}\\ &= S(5^1)\times S(2^3)\\ &= (1+5)\times(1+2+2^2+2^3) &&\text{(by Rule 2)}\\ &= 6\times 15 = 90 \neq 2\times 40, \end{aligned}$$

implying that 40 is not a perfect number.

Remark. All the perfect numbers that we have met till now are even. Further, we note that:

(i) Every known perfect number is of the form $2^{p-1}\times(2^p-1)$, where both p and 2^p-1 are prime numbers;
(ii) The smallest known perfect number (namely, 6) corresponds to the case $p=2$ in the above formula, and the largest currently known perfect number corresponds to the case $p=82589933$.

Spoof perfect numbers

We shall now relax some of the rules and/or assume some incorrect properties to generate spoof perfect numbers.

Example 8. Let us assume (incorrectly, of course) that 4 is prime. Then we can get $60 = 3\times 4\times 5$ as the prime factorisation of 60. This would mean that

$$\begin{aligned} S(60) &= S(3)\times S(4)\times S(5) &&\text{(by applying Rule 1, as 3, 4, 5 are coprime)}\\ &= (1+3)(1+4)(1+5) &&\text{(by applying Rule 2, since 4 is taken to be prime)}\\ &= 120 = 2\times 60. \end{aligned}$$

Thus 60 satisfies the property of perfectness **if** we assume 4 to be a prime number.

But 4 is not really prime as we know. Further, the sum of all the divisors of 60 is

$$S(60) = 1+2+3+4+5+6+10+12+15+20+30+60 = 268 \neq 2\times 60.$$

So we cannot correctly call 60 a perfect number.

To accommodate such a situation, we create a new terminology and call 60 a 'quasi' or a 'spoof' perfect number under the assumption that 4 is prime. This indicates that 60 imitates the behaviour of a genuine perfect number under certain circumstances, namely, by treating 4 to be prime. Further, since 60 is even, we call it an "even spoof perfect number under the condition that 4 is prime."

Definition 5 (Spoof prime, quasi prime)**.** A number that we incorrectly assume to be prime in some situation is called a *spoof prime* or a *quasi prime* in the context of that situation.

For example, in Example 8 above, 4 is a quasi or a spoof prime (in the context of Example 8). In the following, we shall meet many such quasi primes (which are so designated only in the context of the corresponding situations).

Example 9. Here we shall again check 60 for perfectness, just as we did in Example 8, but instead of 4, we shall assume (incorrectly) that 10 is prime. Proceeding as earlier, we get:

$$\begin{aligned} S(60) &= S(2) \times S(3) \times S(10) \\ &= (1+2)(1+3)(1+10) = 132 \neq 2 \times 60. \end{aligned}$$

So 60 is *not* a spoof perfect number if we assume that 10 is prime.

From Examples 8 and 9, it should be absolutely clear that merely saying "The number n is a spoof perfect number" is not completely correct. We must explicitly state the rule that has been relaxed. Without this detail, calling a particular number a spoof number is an incomplete statement and hence not correct. To emphasize this point, let us state again: 60 *is a spoof perfect number if we assume that* 4 *is prime, but* 60 *is not a spoof perfect number if we assume that* 10 *is prime*.

We shall study more examples of this new category of numbers.

Example 10. We examine the number 90, assuming (again incorrectly) that 9 is prime. We have:

$$\begin{aligned} S(90) &= S(2 \times 5 \times 9) = S(2) \times S(5) \times S(9) && \text{(by Rule 1)} \\ &= (1+2) \times (1+5) \times (1+9) && \text{(by Rule 2)} \\ &= 180 = 2 \times 90. \end{aligned}$$

Hence 90 is a spoof perfect number if we assume 9 to be prime.

But if we assume (incorrectly) that 10 is prime (rather than 9), then:

$$\begin{aligned} S(90) &= S(3^2) \times S(10) && \text{(by Rule 1)} \\ &= (1+3+3^2) \times (1+10) && \text{(by Rule 2)} \\ &= 143 \neq 2 \times 90. \end{aligned}$$

Hence 90 is not a spoof perfect number if we assume that 10 is prime.

Example 11. We examine the number 84. We have:

$$\begin{aligned} S(84) &= S(2^2) \times S(3) \times S(7) \\ &= (1+2+2^2) \times (1+3) \times (1+7) = 224 \neq (2 \times 84) \end{aligned}$$

Hence 84 is not a perfect number.

But if we assume 6 to be a prime number, then we get

$$\begin{aligned} S(84) &= S(2) \times S(6) \times S(7) \\ &= (1+2)(1+6)(1+7) = 168 = 2 \times 84, \end{aligned}$$

so 84 is a spoof perfect number if 6 is considered to be prime.

Example 12. We examine the number 12. We have:

$$\begin{aligned} S(12) &= S(2^2) \times S(3) \\ &= (1 + 2 + 2^2) \times (1 + 3) = 28 \neq (2 \times 12). \end{aligned}$$

If we assume 4 to be a prime number, then:

$$\begin{aligned} S(12) &= S(3) \times S(4) \\ &= (1 + 3) \times (1 + 4) = 20 \neq (2 \times 12). \end{aligned}$$

We see that the number 12 is neither a perfect number nor a spoof perfect number under the assumption that 4 is prime.

Example 13. We know (from Example 6) that 28 is a perfect number. Now, if we assume that 4 is prime, then:

$$\begin{aligned} S(28) &= S(4) \times S(7) && \text{(by Rule 1)} \\ &= (1 + 4) \times (1 + 7) && \text{(by Rule 2)} \\ &= 40 \neq (2 \times 28) \end{aligned}$$

Hence 28 is a perfect number, but it is *not* a spoof perfect number if we take 4 to be a prime number.

Example 14. We examine the number 840. We are going to assume (incorrectly) that both 4 and 6 are prime numbers.

$$\begin{aligned} S(840) &= S(4) \times S(5) \times S(6) \times S(7) && \text{(by Rule 1)} \\ &= (1 + 4) \times (1 + 5) \times (1 + 6) \times (1 + 7) && \text{(by Rule 2)} \\ &= 1680 = 2 \times 840 \end{aligned}$$

Hence 840 is a spoof perfect number under the assumption that 4 and 6 are prime numbers.

Example 15. We leave it to the reader to verify that the integer

$$390405312000 = 4 \times 8 \times 9 \times 10 \times 15 \times 22 \times 46 \times 94 \times 95$$

is an even spoof perfect number if one wrongly assumes that all the factors stated on the right side are primes.

Remark. We close the article by noting that all the spoof perfect numbers that we have seen till now are even numbers, i.e., they are even spoof perfect numbers.

What about odd perfect numbers and odd spoof perfect numbers? We examine this question in Part II of this article.

References

1. Jason Chuang (Quanta Magazine), "Mathematicians Open a New Front on an Ancient Number Problem", https://www.quantamagazine.org/mathematicians-open-a-new-front-on-an-ancient-number-problem-20200910/
2. Wikipedia, "Descartes number" from https://en.wikipedia.org/wiki/Descartes_number
3. Wikipedia, "Divisor function" from https://en.wikipedia.org/wiki/Divisor_function

PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1995. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.

Verbal Communication in a Mathematics Classroom

ARDDHENDU SEKHAR DASH

Though we all know that proper use of mathematical language is an important aspect in teaching mathematics, this did not assume priority for me until face to face classroom interaction became impossible during the pandemic. Because of the lockdown, our school (Azim Premji School, Dhamtari, Chhattisgarh) reached out to students using different modes such as sharing hard copies of reading materials, telephonic conference calls with groups of students and even video conferences. However, since the majority of our students do not have facilities such as smart phones and internet connectivity, we engaged with them mainly through the first two methods. Using these, I continued mathematics classes with students of classes 6, 7 and 8.

It was then that I realised two things. The first was my reliance on old-fashioned 'chalk and talk' -written mathematical language accompanied by logical explanations. The second was my lack of attention to using proper verbal language in my classroom teaching. Both these hampered my 'distance-mode' lessons. I will write about my experience and my learning from it in this article.

Keywords: Mathematical language, precision, clarity, ambiguity, communication

We all know that visualization is an important aspect of teaching mathematics. Here the word visualization means both seeing the object and imagining the object. To visualize the concept, both verbal and written languages play important roles. During my telephonic conference calls, I faced problems in the step-by-step explanation of a procedure as described below.

Step by Step Explanation of a Procedure: We know that explaining a procedure requires a step-by-step process with the use of logic at each step. But, considering the limitation in the mode of interaction, we used examples from the textbook and asked students to open the specific examples or exercises in the textbook. However, what the students saw in the textbook was the final compilation of the procedure. Because they could not see the procedure being carried out as I would have done on the blackboard, it became extremely difficult to explain each step to them.

For example, if we wish to explain the division of one algebraic expression by another, we need to focus on different parts of the expressions in each step, do some procedures and then move to the next step. If the student only sees the image in Figure 1, it is too difficult for them to connect with the step-by-step process.

If the series of steps with appropriate highlighting of the relevant part of the expressions is shown, then students would definitely understand the procedure better.

$$
\begin{array}{r|l}
 & 2q^2 - 3q + 2 \\
4q + 2 & 8q^3 - 8q^2 + 2q - 1 \\
 & \pm 8q^3 \pm 4q^2 \\ \hline
 & -12q^2 + 2q - 1 \\
 & \mp 12q^2 \mp 6q \\ \hline
 & +8q - 1 \\
 & \pm 8q \pm 4 \\ \hline
 & -5
\end{array}
$$

Figure 1

Use of Verbal Language in Mathematics: A second difficulty that I faced was in communication when the only explanation that reached the student was the verbal language I used. I am also not clear whether there is any standard verbal language for communicating mathematical concepts because most of the textbooks have never mentioned the proper use of verbal language. The language I used was the language that my own mathematics teacher had used as it had been sufficient for me to grasp whatever he taught.

During one of the online classes, one of the students asked me to explain a question. He read the question as "two x plus 3 divided by 5 is equal to 3". Since the question was from the textbook, I searched in the book but could not find the expected question $2x + \frac{3}{5} = 3$. Then I asked him to share the question number. When I got it and searched by page and question number, I realised that the question was $\frac{2x + 3}{5} = 3$. It was clear that the student had used "divided by" instead of "whole divided by". Probably, I too would have read out $\frac{2x + 3}{5}$ as "two x plus 3 divided by 5" which is not correct. But in the physical classroom, there would be no confusion because I would have written it out and would also have been able to physically check what the student had taken down. I realised that this was not sufficient; I would always need to communicate in complete mathematical sentences and ensure that my students did too.

Think of reading $2x + \frac{3x - 4}{5} = 3$. It is not an easy task, but it should be necessary to read it with complete understanding (and no visual cues) on the part of the listener.

I am highlighting some more examples of similar challenges experienced in verbal communication -

- $x^m \times x^n = x^{m+n}$: Generally, we read it as "x to the power m times x to the power n which is equal to x to the power m plus n". A student could perceive this as $x^m \times x^n = x^m + n$.
- Communicating the presence of parentheses in a mathematics expression verbally: In most of our verbal communication, we generally fail to mention the parentheses or rather assume it is understood based on our board work. For example: we write $3x(2x + 5)$, but we generally say this out loud as $3x$ multiplied by $2x + 5$. Just imagine communicating $5a + \{3b - (2a - 4b)\}$ verbally! It definitely needs thoughtful practice.

- Inadequate use of parenthesis while writing: We read -3 – (-2), as "negative 3 minus negative 2". We must have observed that students write it as "-3 – -2" and we may have allowed that.

From the above examples, it is clear that we are careless about verbal communication in mathematics. Whether it is because we lack practice or awareness of the correct verbal language, this is a skill that teachers must practise and teach!

ARDDHENDU SHEKHAR DASH is the principal of Azim Premji School, Dhamtari. Earlier, he was a Resource Person at Azim Premji Foundation. He has an M.Sc in mathematics from Utkal University, Vani Vihar, Bhubaneswar. He has been closely working with teachers on issues related to Mathematics and conducts workshops focusing on conceptual understanding as well as pedagogical strategies used in teaching mathematics. He has been doing mathematics with children for more than 8 years and is deeply interested in exploring and designing tech resources. He is also engaged in the process of designing curriculum for Open Distance Learning and writing textbooks for Chhattisgarh. He may be contacted at arddhendu@azimpremjifoundation.org.

PREMIER INSTITUTIONS IN INDIA FOR STUDYING MATHEMATICS AFTER COMPLETING 10+2

– Wallace Jacob, Pune

1. **The Institute of Mathematical Sciences, Chennai:** offers courses on *Mathematics, Theoretical Physics, Theoretical Computer Science and Computational Biology*. The institute offers doctoral programmes, post-doctoral programmes, summer research programmes, associateship programmes, visiting scientists programme and visiting students programme.
2. **Indian Statistical Institute:** was started in 1931 in Kolkata. It has campuses/ centres in Kolkata, New Delhi, Bengaluru, Chennai, Tezpur and Giridh. ISI offers study programmes at the Undergraduate level and Postgraduate level as well as Research Fellowships in several areas such as Mathematics, Statistics, Biological Science, Computer and Communication Science, Physics and Earth Science.
3. **Chennai Mathematical Institute:** was founded in 1989 and was recognized as a University by the Government of India in December 2006. The research groups in Computer Science as well as Mathematics at CMI are among the best in the country. The students can obtain admission into B. Sc., M. Sc., PhD programs of the institute after qualifying in the entrance exam conducted by CMI.
4. **Indian Institute of Science Bengaluru:** The Bachelor of Science (Research) Programme is one of the most sought-after courses after Class XII.
5. **IISER:** Indian Institute of Science Education and Research at Berhampur, Bhopal, Kolkata, Mohali, Pune, Thiruvanathapuram and Tirupati conduct BS, BS-Dual degree programmes in Mathematical and other sciences.
6. **National Institute of Science Education and Research Bhubaneswar:** offers higher education progammes in Mathematics and other basic sciences.

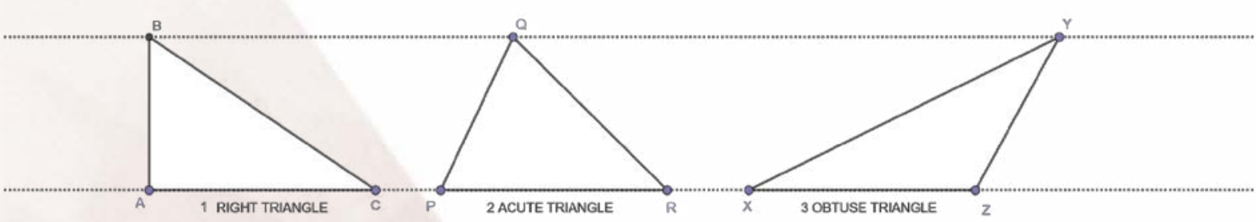
Areas of Triangles between Two Parallel Lines with Same Base are Equal

Proof(?) by Paper Folding

BABURAJAN K

Introduction

There are many methods to prove that the areas of triangles between two parallel lines and with the same base length are equal. Here, we are discussing the proof using a paper folding method. We give the geometric reasoning of the proof and follow it up with the implications of this in the paper folding activity. During the paper folding activity, some geometrical properties are used intuitively, and the questions alongside will serve to stimulate both observation as well as reasoning and deductive skills in the students. We hope that this approach illustrates that results which are observed during the paper folding are based on beautiful and rigorous mathematics.



In this article, we consider three triangles (right-angled, acute and obtuse), all of which are of the same height and the same base. We prove that all these triangles have the same area. Note that the following are **not** assumed:

1. The formula for the area of a triangle. This will be arrived at (for all three types of triangles).
2. The mid-point theorem.

Theorem to be proved

Triangles on the same base (or equal bases) and between the same parallel lines (i.e., of equal height) are equal in area.

Steps

Here we are considering three different types of triangles, i.e., the right triangle, acute triangle and obtuse triangle with the same base length and lying between the same parallel lines. For this we construct each of these triangles with base '*b*' and height '*h*' as shown in Figure 1.

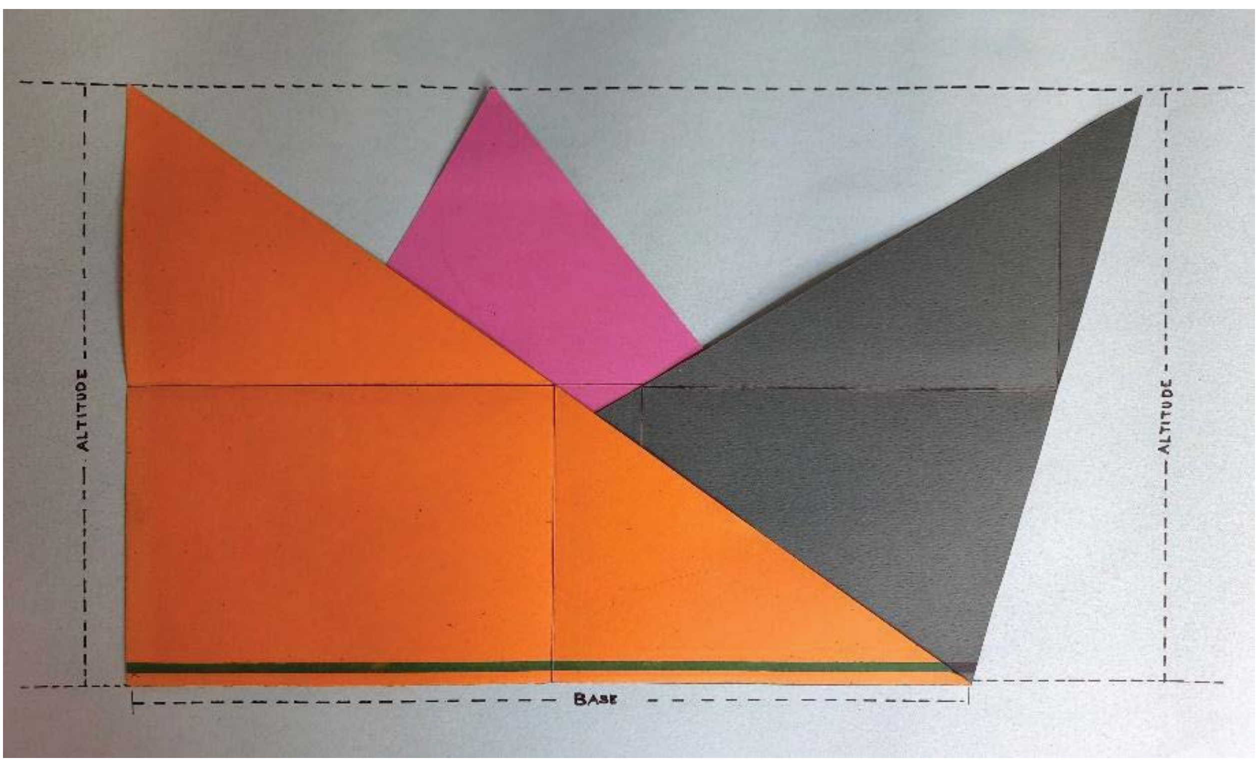


Figure 1.

Let us follow the steps for each triangle as given below.

(Note: The sequence of reasoning is to be read column-wise.)

1. Right Triangle

Geometric Reasoning	Paper Folding

Geometric Reasoning

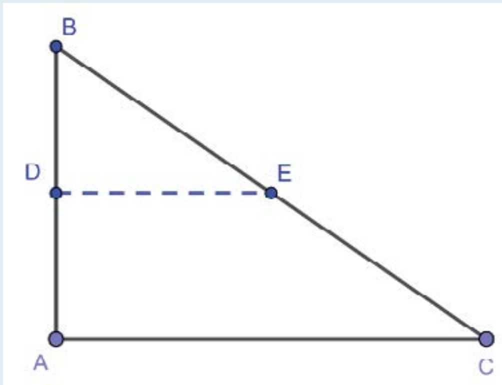


Figure 1.

Let ΔABC be a triangle (of base $AC = b$ and height $AB = h$) which is right-angled at A. Let DE be the perpendicular bisector of BA. We will prove that E is the mid-point of BC.

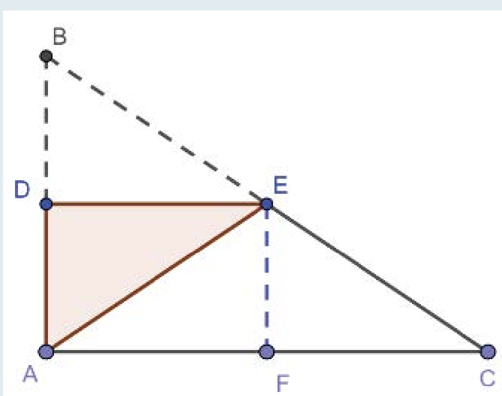


Figure 2.

Drop EF perpendicular to AC and join AE.

Clearly, $AFED$ is a rectangle since three of its angles are right angles.

$$\therefore DB = DA = EF.$$

And DE is parallel to AF (and hence to AC).

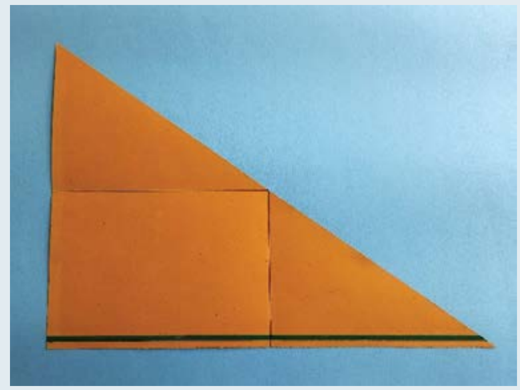
So, $\measuredangle DEB = \measuredangle FCE$ and this proves that $\Delta DEB \cong \Delta FCE$ (by AAS since both are right triangles).

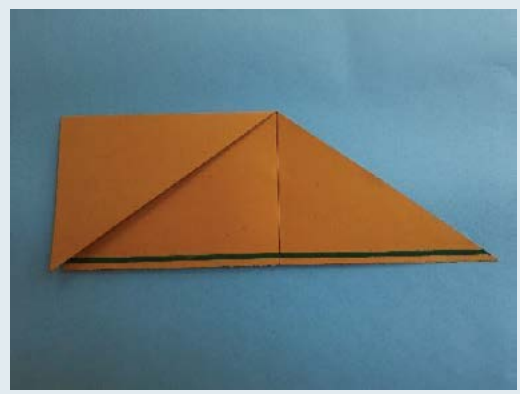
$$\therefore BE = EC$$

So E is the mid-point of BC.

Paper Folding

Make a paper cut-out of the right triangle ΔABC.

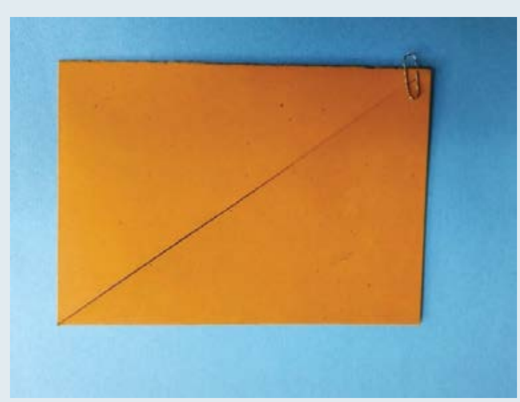




Fold the triangle so that the point B coincides with A. The fold line DE is the perpendicular bisector of BA.

Fold AC so that (i) the two parts of AC are aligned and (ii) the fold line passes through E

Let the point where the fold line meets AC be F.



<table>
<tr><th>Geometric Reasoning</th><th>Paper Folding</th></tr>
<tr><td>

We have just reasoned geometrically that E is the mid-point of BC. Can you also prove that the four triangles ADE, BDE, EFA and EFC are congruent?

$$BD = DA = EF = h/2$$

$$AF = FC = b/2$$

And, $AE = BE = CE$

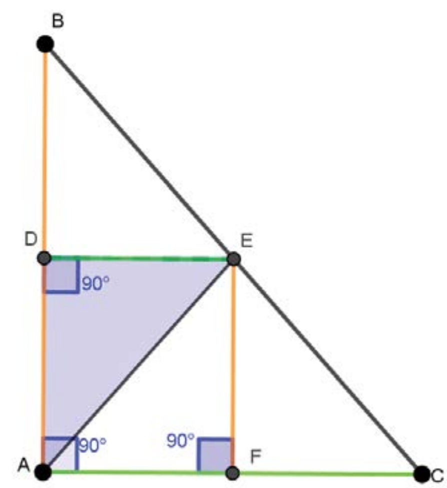


Figure 3

</td><td>

$\measuredangle AFE = \measuredangle CFE$ (since they superimpose on each other)

$$= \frac{1}{2}\measuredangle AFC = \frac{1}{2} \times 180^\circ$$

$$= 90^\circ \therefore EF \perp AC$$

Observe that C coincides with A superimposing ΔEFC on to ΔEFA, while ΔBDE superimposes on ΔADE thanks to the fold along DE. So, we get two rectangles with base $b/2$ and height $h/2$. The sum of the areas of these two rectangles is equal to the area of ΔABC.[1]

</td></tr>
<tr><td colspan="2">

The equalities we arrived at by geometric reasoning resonate in the paper folding. We see that the area of the rectangle

$DEFA = AF \times AD$,

$$AF = \frac{1}{2}AC = b/2 \text{ and } AD = \frac{1}{2}AB = h/2,$$

$$\Delta ABC = 2 \times DEFA = 2 \times AF \times AD = 2 \times b/2 \times h/2 = \frac{1}{2}b \times h$$

</td></tr>
<tr><td colspan="2">

We also observe both through geometry and through paper folding that $BE = AE = CE \Rightarrow E$ is the circumcentre $\Rightarrow$ We have also arrived at the result that the circumcentre of a right triangle is the midpoint of its hypotenuse!

</td></tr>
</table>

Now we can check the other triangles with similar steps.

1 Check Unfolding, At Right Angles, Jul 2014 issue for folding a perpendicular to a given line through a given point.

2. Acute[2] Triangle

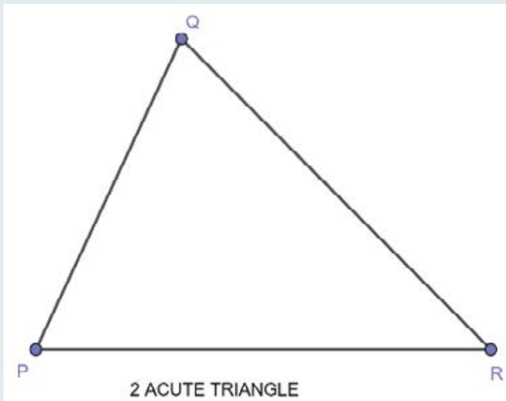
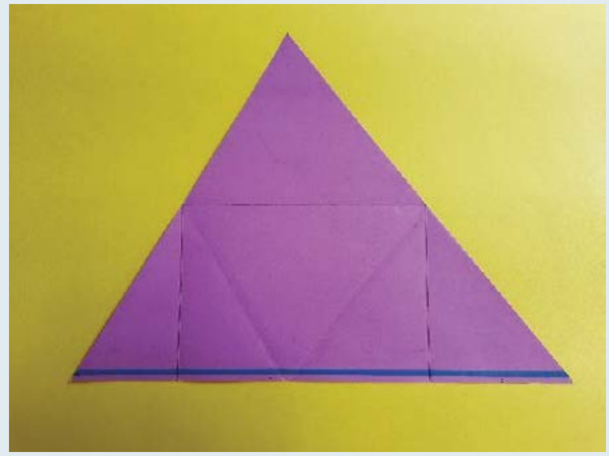
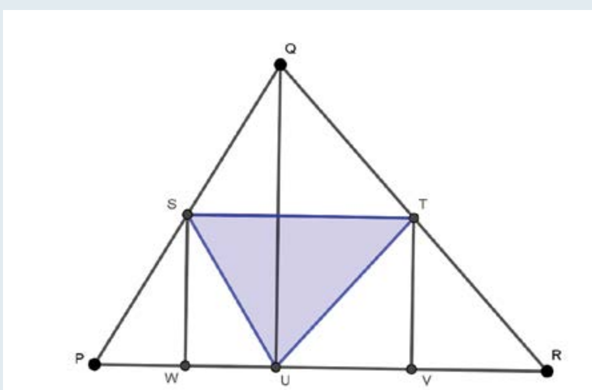
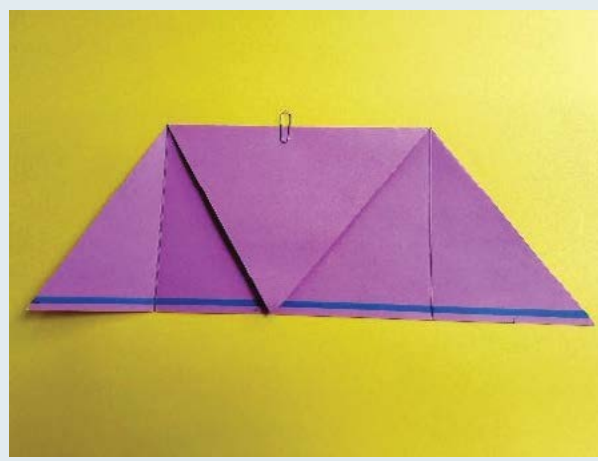
<table>
<tr><th>Geometric Reasoning</th><th>Paper Folding</th></tr>
<tr><td>
</td><td></td></tr>
<tr><td colspan="2">Let ΔPQR be an acute angled triangle with base $PR = b$ and height h.</td></tr>
<tr><td>

Figure 4

Here, we mark the mid-points S and T of sides QP and QR respectively and draw the line ST. Draw two perpendiculars SW and TV to the base PR. Let U be the foot of the perpendicular drawn from Q to PR. ($QU = h$)

In the right triangle ΔQUP, S is the mid-point of the hypotenuse QP.</td><td>

Mark the mid-points S and T of the sides PQ and QR by folding.

Mark the foot U of the perpendicular to PR through Q by folding.

Can you see why Q coincides with U on folding along ST?</td></tr>
</table>

2 This also works for right and obtuse triangles and for those cases we need to consider the longest side as the base. However, for an acute triangle the base can be any side.

Geometric Reasoning

We have proved above that therefore $SQ = SP = SU$.

Similarly, in the right triangle ΔQUR, T is the mid-point of the hypotenuse QR.

So, $TQ = TR = TU$.

$SQ = SU$, $TQ = TU \therefore SQTU$ is a kite. Its diagonals QU and ST will be perpendicular. If they intersect at Y, then $QY = YU = h/2$.

ST is the perpendicular bisector of QU.

We can prove that ΔSWP and ΔSWU are congruent. (How?)

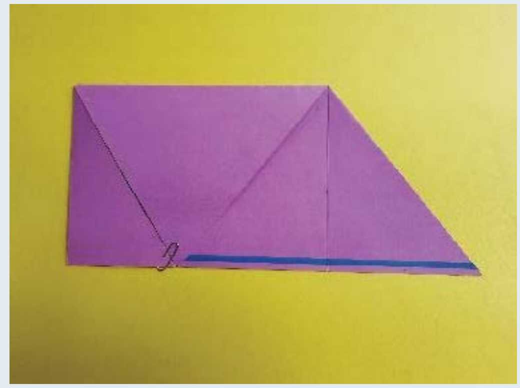
So, W is the mid-point of PU.

Similarly, $\Delta TVU \cong \Delta TVR$.

So, V is the mid-point of UR.

$$\therefore ST = WV = WU + UV = \frac{1}{2}PU + \frac{1}{2}UR$$
$$= \frac{1}{2}(PU + UR) = \frac{1}{2}PR = b/2$$

Paper Folding



Can you also see why P and R fold along SW and TV, the perpendiculars to PR, to meet at U?

[What kind of triangles are ΔPSU and ΔRTU?]

Again, the paper folding aligns perfectly with the geometric reasoning.

$$ST \perp QU \text{ and } QU \perp PR \Rightarrow ST || PR$$

$ST || PR$, and $SW, TV \perp PR \Rightarrow WSTV$ is a rectangle, and its area is $b/2 \times h/2$

Since $WSTV$ is ΔPQR folded into 2 layers, $\Delta PQR = 2 \times STVW = \frac{1}{2}bh$.

Find three angles which are equal to $\angle PQR$, $\angle QRP$ and $\angle RPQ$ respectively. Using these three angles can you prove that the sum of the angles in a triangle is 180° ?

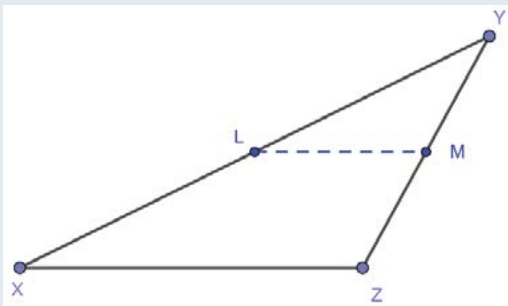
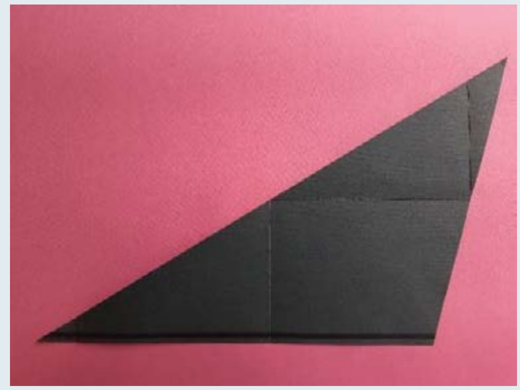
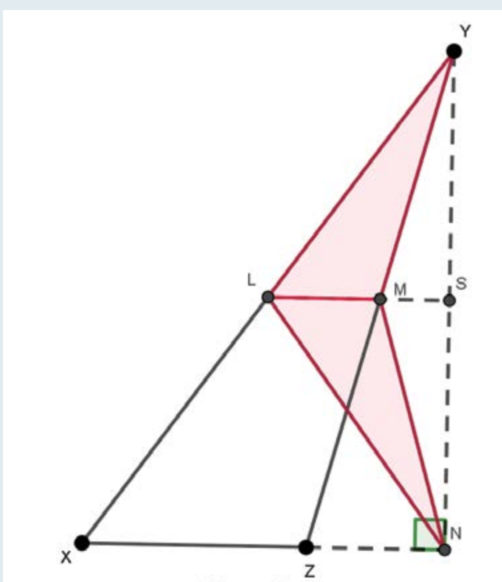
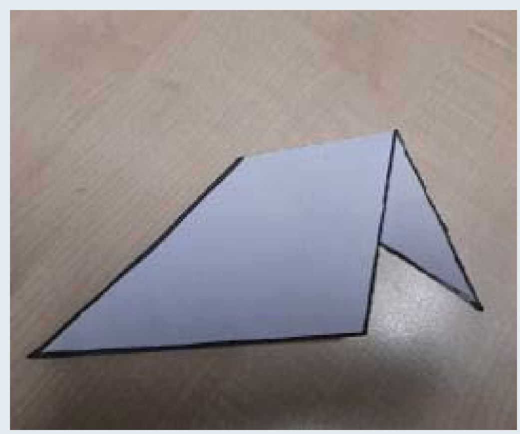
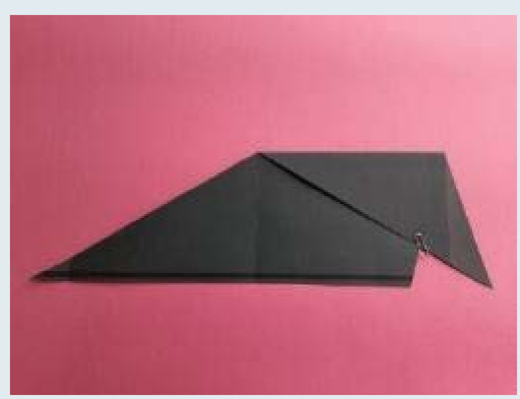
$$\angle PQR = \angle SUT \text{ since } USQT \text{ is a kite}$$
$$\angle QRP = \angle TUV \text{ since } \Delta TRV \cong \Delta TUV$$
$$\angle RPQ = \angle SUW \text{ since } \Delta PSW \cong \Delta USW$$

$$\therefore \angle PQR + \angle QRP + \angle RPQ = \angle SUT + \angle TUV + \angle SUW = \angle WUV = 180^\circ$$

We have arrived at the result that the sum of the angles of this acute angled triangle is 180° !

Note that the sum of the angles for any triangle can be explored this way provided that the longest side is considered the base.

3. Obtuse Triangle

Geometric Reasoning	Paper Folding
Let ΔXYZ be an obtuse angled triangle with base $XZ = b$ and height h. 	
Mark the mid-points L and M of the sides XY and YZ respectively of the obtuse angled triangle ΔXYZ. Figure 5	By folding, mark the mid-points L and M of the sides XY and YZ of the triangle XYZ. Fold[3] along LM. It helps to fold it back rather than front.  
Let N be the foot of the perpendicular from Y to XZ extended i.e. $YN \perp XN$. Let LM extended meet YN at S. We can prove that LS is the perpendicular bisector of YN in the following steps:	

3 Folding ΔLMY back may help with the subsequent folds (see blue triangle in Figure 8).

<table>
<tr><th>Geometric Reasoning</th><th>Paper Folding</th></tr>
<tr><td>

1. ΔXYN is a right triangle and L is the midpoint of its hypotenuse XY,

$$\therefore LX = LY = LN$$

ΔZYN is a right triangle and M is the midpoint of its hypotenuse ZY,

$$\therefore MZ = MY = MN$$

2. $\Delta LMY \cong \Delta LNM$ by SSS

$$\Rightarrow \measuredangle MLY = \measuredangle MLN$$

3. $\therefore \Delta LSY \cong \Delta LSN$ by SAS

$\Rightarrow YS = SN$ and $\measuredangle LSY = \measuredangle LSN = \frac{1}{2} \times 180° = 90°$ i.e. LS is the perpendicular bisector of YN

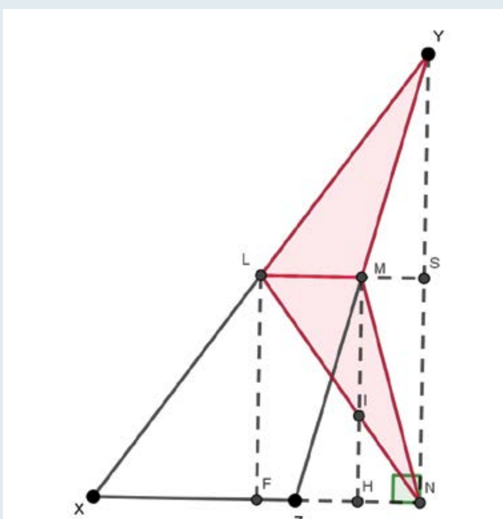


Figure 6

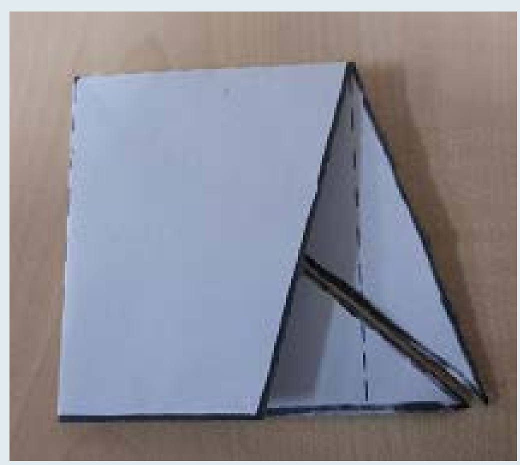
Drop perpendicular LF to side XZ.

ΔLXN is isosceles, $\therefore LF$ is the angle bisector of $\measuredangle XLN$, in fact it is the line of symmetry of ΔLXN, $\therefore LF$ is perpendicular bisector of XN

</td><td>

Our geometric reasoning proves that Y coincides with N when the triangle is folded along LM.

So, when you fold along $LF \perp XZ$, where does X end up? Note that the folded triangle becomes a trapezium with 2 adjacent right angles. Depending on the first fold along LM, these right angles may be on the right or on the left as we have shown above (and below).



Now if you fold along MH, can you see why the folded triangles ΔNIM and ΔNIH fit the remaining space beside the trapezium $LMZF$?

</td></tr>
</table>

Geometric Reasoning	**Paper Folding**
Drop perpendicular MH to side XZ (produced). Let LN intersect MH at I. Since ΔZMN is isosceles, MH is the line of symmetry of ΔZMN $\Rightarrow \Delta ZHM \cong \Delta NHM$ by $RHS \Rightarrow ZH = HN$ $\Rightarrow \Delta ZIM \cong \Delta NIM$ and $\Delta ZIH \cong \Delta NIH$ (Why?)	 

Again, the paper folding aligns perfectly with the geometric reasoning.

Now, LM $\|FH$, which is part of XZ (extended), $LF\|MH$ since both are perpendicular to XZ (produced), $\therefore$ $LMHF$ is a parallelogram.

Since $LF \perp XZ$, $LMHF$ is in fact a rectangle with area $b/2 \times h/2$ because,

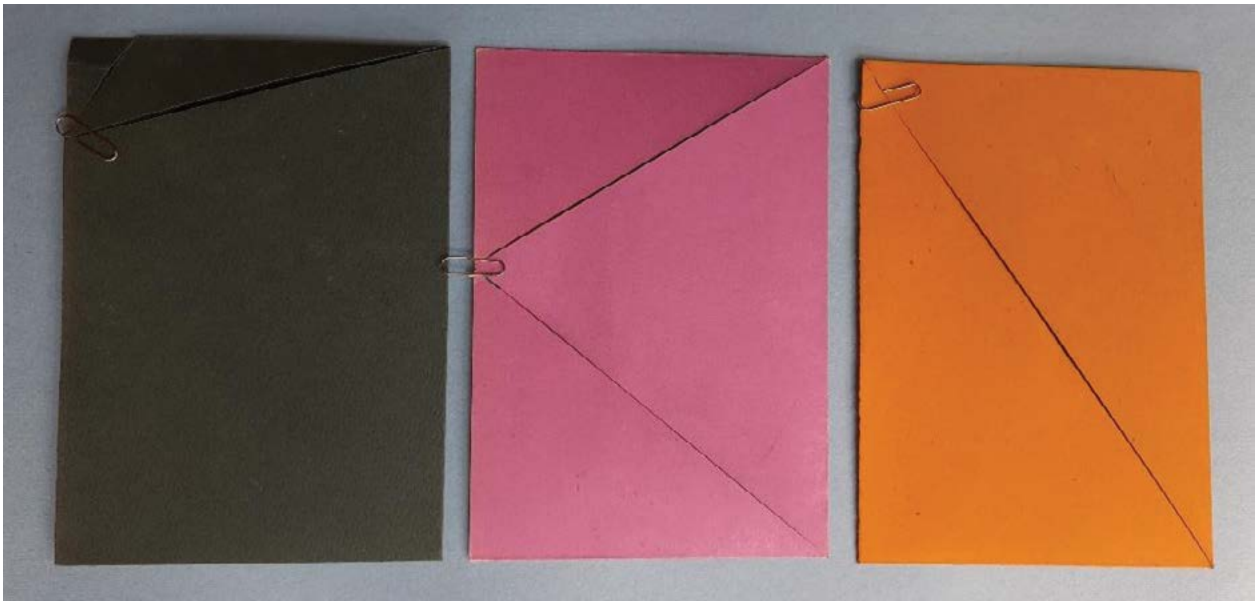
its base $FH = FN - HN = \frac{1}{2}XN - \frac{1}{2}ZN = \frac{1}{2}(XN - ZN) = \frac{1}{2}XZ = b/2$ and

its height $LF = SN = \frac{1}{2}YN = h/2$

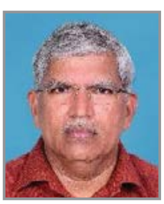
Since $LMHF$ is ΔXYZ folded into 2 layers, $\Delta XYZ = 2 \times LMHF = \frac{1}{2}bh$.

A critical question at this point is whether F, the foot of the perpendicular from L to XZ, would always be within XZ. We strongly encourage our readers to explore the position of F using GeoGebra (or otherwise) by varying the position of Y without changing the height of the triangle.

If F is outside XZ i.e. outside the paper triangle, can these steps be modified to get a double-layered rectangle? If so, how? We hope to discuss these in a future article.



Acknowledgment: The author wishes to thank Swati Sircar for the additional pictures and details added to the article.



BABURAJAN. K worked as a mathematics teacher for over 30 years in a secondary school in Kozhikode Dt. of Kerala. He retired as the headmaster of A K K R High School; during this period, he served also as a member of the district maths resource group for more than 10 years. Now he is interested in training students on recreational mathematics. He may be contacted at baburajkkavil@gmail.com

The Task

A Quadrilateral Inscribed in a Circle

– Moshe Stupel, Shannan College, Haifa, Israel

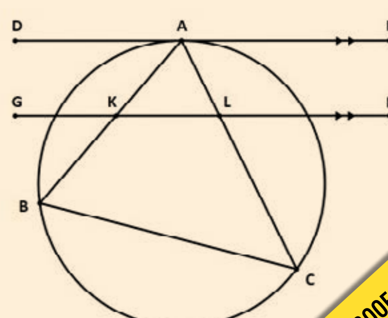
Given a triangle ΔABC which is inscribed in a circle.

The segment DE is a tangent to the circle at point A.

GF is parallel to the tangent and intersects the triangle at points K and L.

Prove that:

The quadrilateral BKLC can be inscribed in a circle.



Refer to page 37 for the Proof without Words

PROOF WITHOUT WORDS

Quantifying Steepness – A Reflection

JAYASREE S

Sometimes as teachers we tend to expect students to use a particular definition, notation, or representation, though what students come up with on their own may be a legitimate way of working as well. Some of the reasons why we do this could be that a) having seen ahead of the students in the subject, we anticipate that the definition or representation suggested by the student will lead to problems on the path ahead or b) we see specific advantages for our particular definition/representation, that the student suggested one may not have or c) simply because we are used to one particular representation, and the student suggested one happens to be different from what we are used to. In the following article I reflect on one such instance where I steered towards a 'preferred representation' and my reasons for doing so. I wonder whether the 'push' was justified.

In a nutshell

The instance I focus on in this article is a conversation with a 7th grader that happened through a chat medium, as part of an enrichment course on school science and mathematics. There were around 15 students and 2-3 course coordinators who were part of the discussion forum. In this exchange, my goal was to get the group to come up with a 'quantification' for the idea of steepness, so that the steepness of two straight

Keywords: alternative definitions, slope, student-centric pedagogy

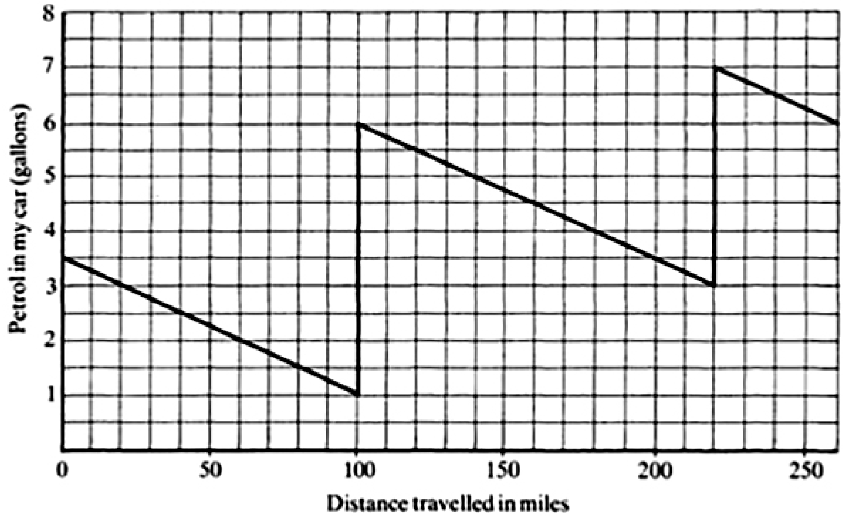


Figure 1. ***Image courtesy: Shell Centre Resource,***
The Language of Functions and Graphs (1985), ***p71 (193)***

lines could be compared, even when a 'visual comparison' is not possible, and they are not drawn alongside each other. The exchanges were spread across a time span of three weeks giving time for all involved to think about and respond to posts and served as a build up to a forthcoming class on graphs. In this article I focus on the interactions with one student – Siddharth. Siddharth had not been introduced to the formal notion of slope at school.

The conversation started with my inviting students to post whatever they could say about the journey represented in the graph in Figure 1.

One of the course coordinators set the ball rolling with the comment that the second 2.5 gallons consumed took him farther than the first 2.5 gallons. I decided to use the coordinator's oversight to open up a conversation and asked the group if the coordinator was missing something out. Siddharth promptly responded that in the 'second lap', 3 gallons of petrol were consumed (and not 2.5) and that explained the extra 20 miles. He also identified that the car travels 40 miles per gallon of petrol and that the sudden spikes in the graph could be explained by the refueling stops.

Now I asked them, if it were true that the car travelled a longer distance on the second 2.5 gallons (as the coordinator had said), how would the graph look? In what way would it be different from the one given? Siddharth responded that in this case, the graph would be *"less steeper"* than that before the first refueling. He went on to add that if a gallon had given 80 miles instead of 40, the line would have been *"half less steeper"*.

Now 'half' implies quantification of some kind and I wanted to elicit that. Siddharth's first explanation was that in the current graph, for every gallon of petrol the graph line goes down and right by four blocks. A line half as steep as this one would go 'two blocks right' instead of four. Now a line that goes right by two blocks for every block down would be twice as steep as one that goes right by four blocks and not half as steep as Siddharth said (see Figure 2). This could have been a slip of the pen (or the keyboard in this instance!) or it could be a struggle to quantify this intuitive notion of steepness, such that the relation '*half* as steep' holds between the two lines.

I asked the group to take some time to think over that response and soon enough, Siddharth came with the contrary answer that if a gallon had given 80 miles instead of 40, the fuel efficiency would have been *double* that now and in that case the line would be *half* as steep. Perhaps the fuel efficiency being double and that this line would go right by eight blocks instead of the

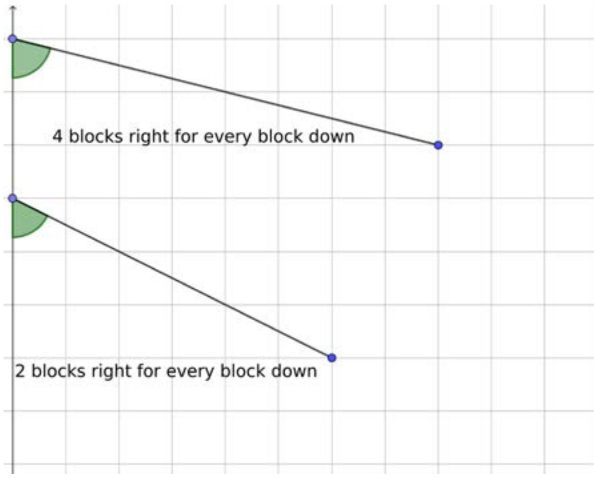


Figure 2: Angle made with y-axis as measure of steepness

current four as in Figure 2, connected well and made sense to him. *"As the fuel efficiency decreases, the graph line becomes more steeper"*, he said.

Calling attention to the "physical meaning" of steepness here, I asked for an explicit quantification of this concept of steepness. Siddharth came up with five ways to do this through the conversation. I kept pushing him on to something else with the first four ways, till he finally came to the conventional '"rise"÷"run"'.

The heart of the matter

I now describe his first four ways and the objections I raised to them.

1. His first idea was to look at the angle that the line makes with the y- axis. I granted that this is a possible way, but we usually look at the angle that the line makes with the x-axis. I also asked if there was some other way, especially one that can be read off from the graph, and not require a measurement, like the angle.

Now Siddharth came up with two more ideas.

2. The second idea was to look at the part above the graph in the first block to the right of the y-axis (which he abbreviated to 'patl' in the conversation, which I will also follow here!), as he has highlighted in Figure 3.

He was looking at how much the line 'dropped' in the first block immediately to the right of the y-axis. The smaller this drop is, the less steep the line, he said.

3. The third idea was *"See how many miles does the line cover for 1 unit on the y-axis."*

For reasons unknown to me then (and now!) my response to his third way was to tell him to look for a way that would quantify the steepness of *any line* and not just the petrol-distance graph here!

His method 2 is the reciprocal ratio of that described in method 3. I spelt it out for the benefit of the rest of the class as *"how much does the line fall when it moves right by one unit"*. Now this is essentially the same as the conventional '"rise"÷"run" ', but stated differently. I wanted him to clarify it more. Noticing that he had arrived at the measure 'patl' by considering only cases where the line cuts the y-axis at an integer value and there is a clearly visible drop in the first block to the right of the y-axis, I gave him examples of lines which do not cut the y-axis at an integer value. In such cases, the relevant drop spans more than a unit length (line AB in figure 4) or needs to be adjusted otherwise to get the slope (line CD in figure 4).

I asked him how he would modify his measure for these cases. Also, as in the case of the angle with the y-axis, 'patl' would need actual measurement with a ruler, and I suggested that he look for something that can be read off from the graph. He came up with a fourth way, saying

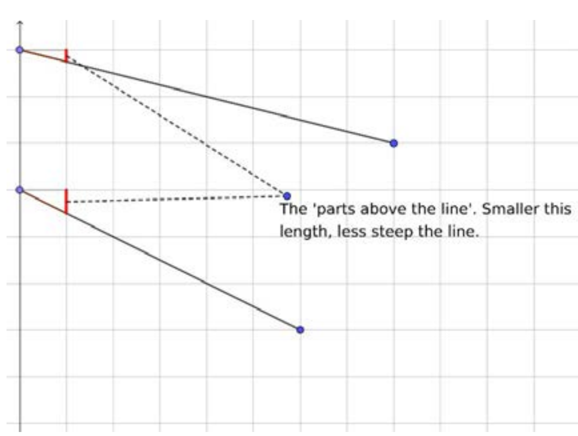


Figure 3: 'part above the line - (patl)' as measure of steepness

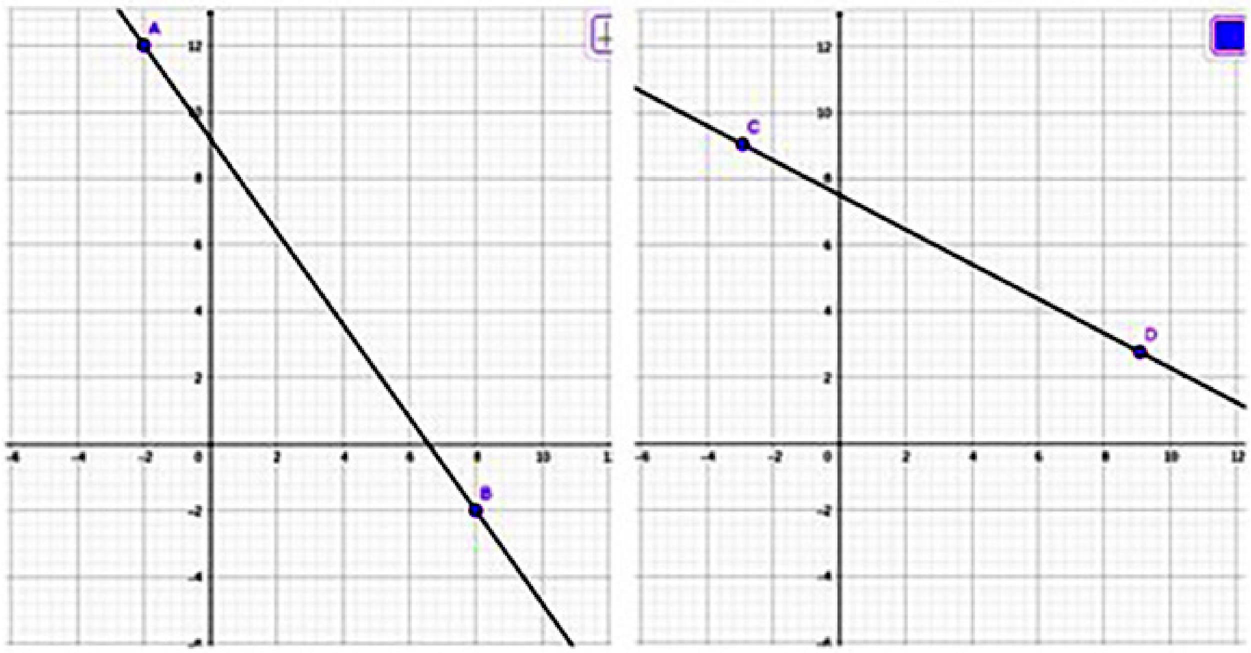


Figure 4: What is the 'patl' here? Does it measure steepness?

that he would come back later with a response to my comment.

4. *"One way to figure out the steepness in a numerical way is to use the formula*

$$\frac{\text{length of the graph line}}{\text{number of boxes it covers}}$$

From this formula you get the amount of the graph line in a box. "

He explained this with an example. In the figure that he drew (Figure 5), he measured the length of the red line to be 8.5 units and its horizontal span is 10 boxes, and so he said the steepness was $\frac{8.5}{10}$ which is 0.85.

Here he measured the length of the line with a ruler, whereas he counted the number of blocks on the graph sheet and took the ratio of these two quantities in two different units. I let the difference in units pass but chose to focus on the fact that this measure would not remain invariant if I extended, say, the blue line to the edge of the sheet. Extending the line straight on is a move which does not change what we intuitively understand as steepness and we wouldn't want its 'measure' to change as well after this move.

He said we could still preserve the measure, by allowing only specific extensions - extending by integer unit multiples of the initial line segment considered. *"If you increase the line length and the number of boxes it covers with the same number (that is by the same multiple), you will get a line with equivalent steepness."* He called this 'equivalent graph line' (EGL) drawing on the notion of equivalent fractions. The steepness of the extended line, if it is an EGL, would be

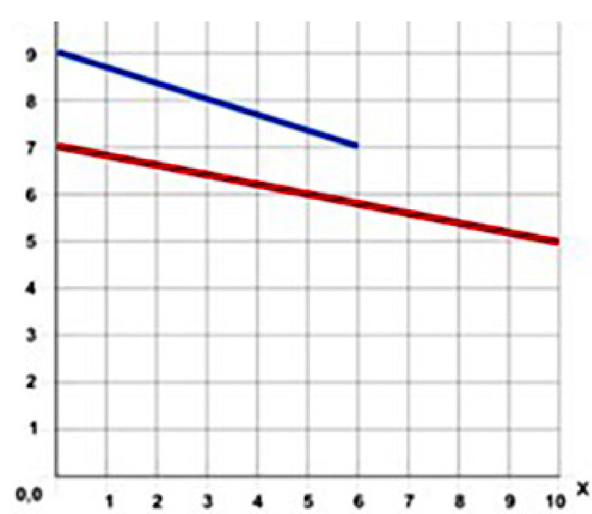


Figure 5: Secant as measure of steepness
(Image courtesy: Siddharth)

a fraction equivalent to the original ratio and therefore invariant.

At this point, driven by my concern of not dampening his spirit by my finding fault with four of his suggestions, I suggested that he think of comparing the steepness of staircases by placing ramps on them. Whether triggered by this, or on his own he came to the conclusion that *"The easiest way to figure out the steepness of the line is to see how many boxes does the graph-line covers in the x axis for every box in the y axis."* and that *"Steepness is a measure of how quickly/slowly something is happening."* This was as close to my preferred definition of '"rise"÷"run" ' as it could get and here the conversation was wrapped up in that state of elation which every teacher would have felt when a student comes up with such an insight.

In a quick glance at the conversation, we see Siddharth trying different ways to quantify steepness drawing on his intuitive understanding of the concept. As the teacher, I validated some of them, and pushed him on to find what in my view was a 'better' measure, which also happened to be the 'textbook-given' measure. There were reasons given to look for an alternative measure at each step – be it cases not covered by the definition, or the invariance of the measure under conditions where we would want it preserved, or readability from the graph.

In the process, Siddharth was functioning as a little mathematician, creating and naming concepts (patl, EGL) and noticing other related things as well. For example, there are lines which divide every square on a graph sheet diagonally into two halves and he termed these the 'Perfect Graph lines'. He also noticed that such lines make an angle of 45° with the y-axis. He then saw that lines steeper than this would lie above this and others below. Lines of slope 1 and -1 acquired a certain 'glow' for me after this conversation! It was an enjoyable exchange for both of us.

Looking back

Two months past the conversation, having stepped down from cloud nine, and having taken a closer look at the conversation, I have other thoughts. Looking at the five different measures for steepness that I got, all of them were reasonable measures, of course in need of due modifications, but retaining the spirit of the definition. I accepted three of them (perhaps half-heartedly?), and rejected one, before Siddharth came up with a measure acceptable to me. I had some implicit criteria for a 'good enough' measure like invariance under extension, generality (applicability to all cases) and 'visual calculability' from the graph. I now look back on these criteria and wonder what should have been the criteria for a suitable measure.

For a 7th grader, the 'angle that a line makes with the y-axis' gives a much better intuitive sense of steepness or slope as we call it, than when it is defined as "the tangent of the angle that it makes with the positive direction of the x-axis". If the purpose were just to compare the steepness of two lines, the former may even be the better measure, whether or not it is possible to get the exact measure visually. Looking back, I now think the question "Would I want to consider a line that makes 60° with the y-axis to be twice/half as steep as one that makes 30°?" is a more pertinent question than "is it readable from the graph without actual measurement?", in deciding whether "the angle made with the y-axis" as a measure of slope is suitable enough.

The 'patl in the first block' needs to be redefined perhaps to cover all possible orientations of the line and modified as needed for lines with positive slope but boils down to the familiar '"rise"÷"run" ', though not articulated in so many words. My objection here was again *'not readable from the graph and needs measurement'* which I will come to in a while. The third method, which got scant attention is essentially 'run divided by rise' stated in the context of distance covered (run) and petrol consumed (rise).

The fourth method, if the mismatch in units is resolved, would be the secant of the angle that the line makes with the x-axis, which is a reasonable quantification too. But because Siddharth expressed the measure in terms of lengths of line segments, the issue of invariance when the same definition is extended for a line comes up and needs to be addressed.

Now why should ‘"rise"÷"run"’ be the preferred measure over the other four options put forward by this child? Just because the textbook says so? Should *"readability from the graph"* be the deciding factor? Though desirable, does ‘"rise"÷"run"’ meet this criterion? With the kind of problems we encounter in school textbooks, we somehow tend to think so! We are so used to having those integer points, from which we draw those perpendicular lines and come up with the ‘"rise"÷"run"’. But does it always work?

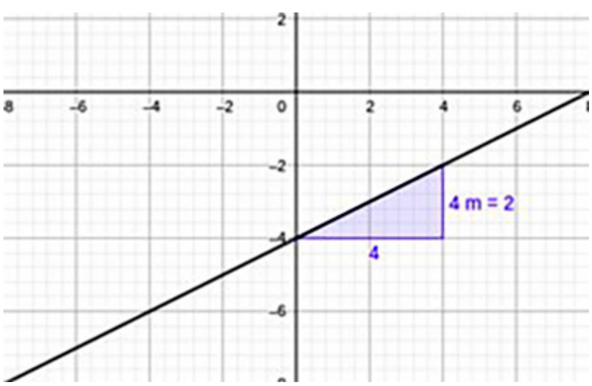


Figure 6: Slope as $\frac{rise}{run}$

Consider a line with irrational slope. It could have at most one integer point. (Note that if it had two integer points, its slope would be the ratio of two integers and hence rational.) In such a case, clearly the ‘rise’ and ‘run’ that we need cannot both be read off from the graph. Even if we consider a line with rational slope, it is very much possible that it will not pass through any integer points at all.

For example, consider the equation y = x + 1. Clearly it has infinitely many integer points. But if the intercept is changed to 1/3, instead of 1, the line y = x + 1/3 dodges all integer points. (Figure 7)

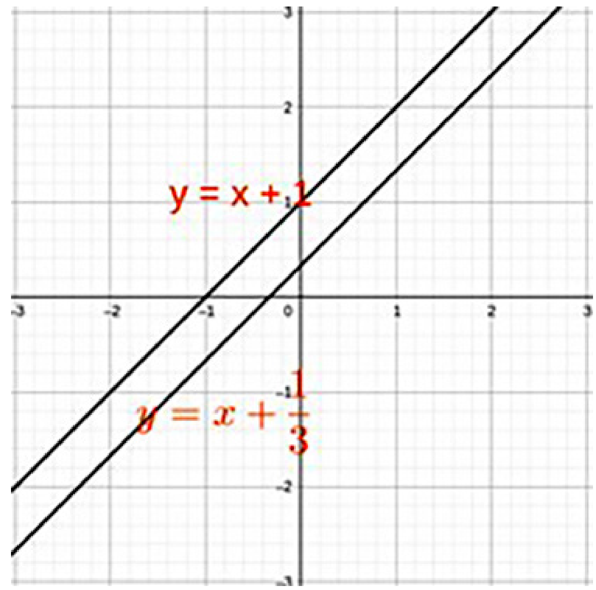


Figure 7: Line with rational slope and no integer points

Or consider the equation 3x + 6y = 19. If this line had integer points, at those points the left-hand side of the equation would a multiple of 3, but the right-hand side, 19, is not. So, this line cannot have integer points as well. Thus ‘"rise"÷"run"’ is not always ‘readable’ from the graph. So why then should it be the preferred measure for steepness? ‘"Rise"÷"run"’ has an advantage that the other four measures do not have.

As Siddharth rightly pointed out, *steepness is a measure of how quickly/slowly something is happening*, so the numerical measure that we define for steepness, should capture our intuitions about the varying ‘rates’ at which this happens. A straight line represents how two quantities vary with each other, and the steepness of the line captures how fast or slow one quantity varies with another. For example, if the variable on the y-axis grows much faster than the one on the x-axis, the graph would be a steep line. ‘"rise"÷"run" ’ tells us how fast the variable on the y-axis is growing compared to that on the x-axis. This meaning does not come through so clearly when slope is quantified as length of ‘patl in the first block’ or the $\frac{\text{length of the line}}{\text{horizontal span}}$ discussed above. Thus, the preference for ‘"rise"÷"run" ’ is because *it reflects the physical meaning underlying slope* better when compared to the other two.

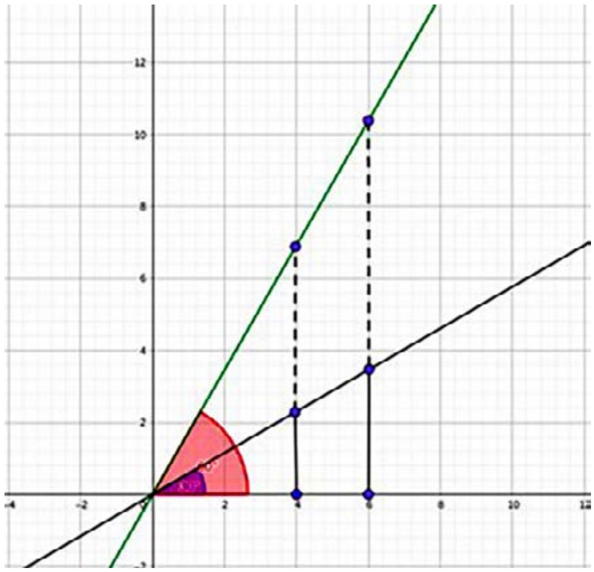


Figure 8: Is one line twice as steep as the other?

Similarly, the y-axis variable of a straight-line graph making 60° with the x-axis is not growing twice as fast as the y-axis variable of a line making 30° with the x-axis. In the context of rate change, *the tangent of the angle that the line makes with the x-axis captures our intuition better than the measure of the angle itself.*

So then the criteria should have been the ease with which the definition lends itself to meaning making in context, and not 'readability from the graph'. Perhaps the missing pieces 'flash upon that inward eye' only in 'vacant or pensive mood' and not amidst the chatter of the classroom! But there is always a next time…when I will be more accepting of student suggested and perhaps 'unconventional' measures. But at the same time, I realise that I may have to 'push' towards the measure accepted by the community. When I do 'push' the student along a preferred path, I will be more sensitive to the reason why I am doing so and clearly articulate it to the students as well.

Thanks to Siddharth Kothari for the enjoyable conversation and to Genwise Talentdev for the opportunity to interact with this group of students.

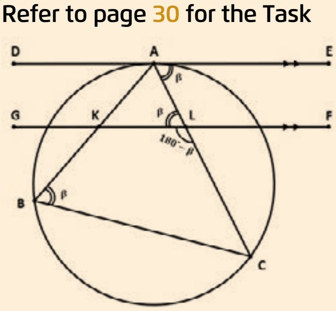
A passionate mathematics teacher, **JAYASREE** has taught math across levels, right from middle school to undergraduate. She loves to share her fascination for math with learners and initiate in them the joy of finding things out for themselves. She has also worked on creating diagnostic assessments, adaptive learning modules and teacher support materials for these. With a Masters in Mathematics from IIT Madras, she is currently enrolled for a PhD in math education from the Homi Bhabha Centre for Science Education (HBCSE), Mumbai. Jayasree may be contacted on jsree.t.s@gmail.com

The Proof

A Quadrilateral Inscribed in a Circle

– Moshe Stupel, Shannan College, Haifa, Israel

Refer to page 30 for the Task



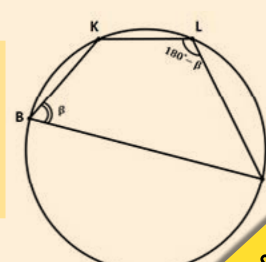
∢EAC = ABC = β

↓

∢EAL = β → ∢ALK = β → ∢KLC = 180° − β

↓

∢KBC + ∢KLC = 180°



PROOF WITHOUT WORDS

My Square is your Triangle! – The King said to Pascal

RAJESH SADAGOPAN

I. Introduction

Imagine a lonely King placed in the left bottom corner of a $n \times n$ chess board! We are aware that the King can move only one square in any possible direction on the chess board. The King should reach the right top corner of the chess board by choosing his own path. See Figure 1.

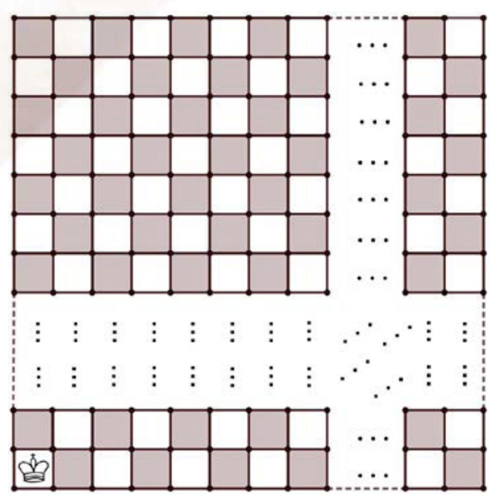


Figure 1

Let us define a path to be a combination of possible moves between the initial position of the King (left bottom corner) and the final position of the King (right top corner). Here's

Keywords: Problem-solving, Chessboard, Move, Path, Weights, Pascal's Triangle

an interesting question! *How many paths are there for the King to reach his destined position?* **Answer:** Infinite! Why? Because the King can keep moving back and forth and that leads to infinite choices!

We can keep the interest of the question alive by imposing certain conditions in the movement of the King as follows:

- The King is allowed to move either to the right or up only.
- No other movement such as left, down, diagonal is allowed.

II. Planner

First let us make a study with smaller values of n such as 2, 3 and 4. We can make observations and notes side by side while studying the paths of the King on 2×2, 3×3 and 4×4 chess boards. Later, we can establish the general theory.

2×2 Chess Board. The King is placed in the left bottom corner of a 2×2 chess board as shown.

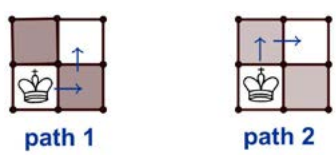


Figure 2

There are exactly two paths for the King to reach the right top corner. Note that each path is a two movement path for this to happen. See Figure 2.

Key observation. In a $n \times n$ chess board, the King can move from any square to his immediate diagonal (right and up) square in exactly two ways.

3×3 Chess Board. The King is placed in the left bottom corner of a 3×3 chess board as shown.

There are exactly six paths for the King to reach the right top corner. Note that each path is a four movement path for this to happen. See Figure 3.

Remember the conditions: the King can move either right or up only and no other move such as left, down or diagonal is allowed. Observe that the King enters the destined right top corner finally through the square just below it in paths 1;4; 6.

Also observe that the King enters his destination through the square to its immediate left in paths 2;3; 5.

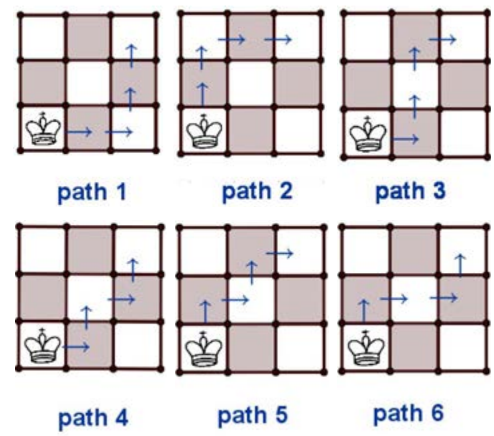


Figure 3

Key observations

- The King can enter the destined corner square either from the square to its immediate left or the square immediately below it; there are no other possibilities.
- The King can enter any intermediate square either from the square to its immediate left or the square immediately below it; there are no other possibilities.
- The King can enter any boundary square in the first column by an upward movement only (as there is no column to its left).
- The King can enter any boundary square in the bottom row by a movement to the right only (as there is no row below it).

4×4 Chess Board. The King is placed in the left bottom corner of a 4×4 chess board, as shown.

There are ten paths for the King to reach the right top corner through the square just below it (Figure 4), and there are ten paths for the King to reach the right top corner through the square to its immediate left (Figure 5). There are thus exactly twenty paths for the King to reach the right top corner. Note: Each path is a six movement path for this to happen.

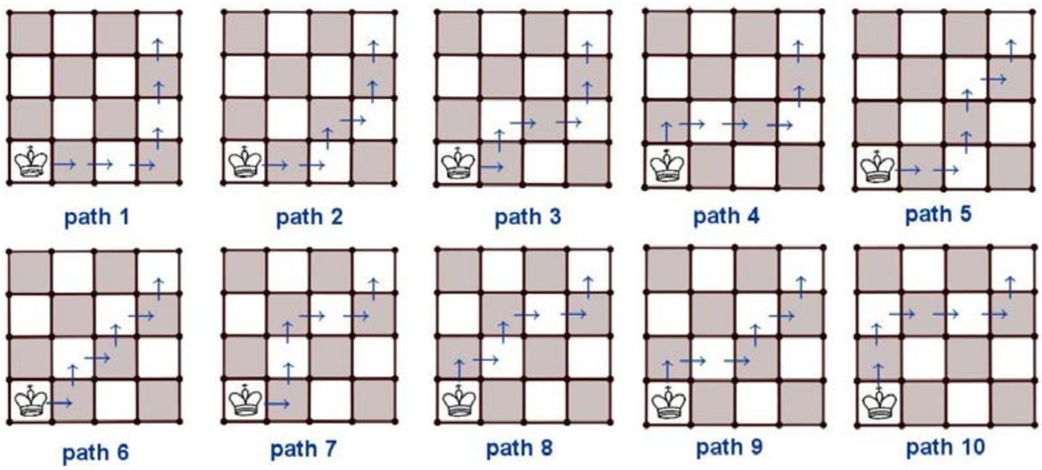


Figure 4

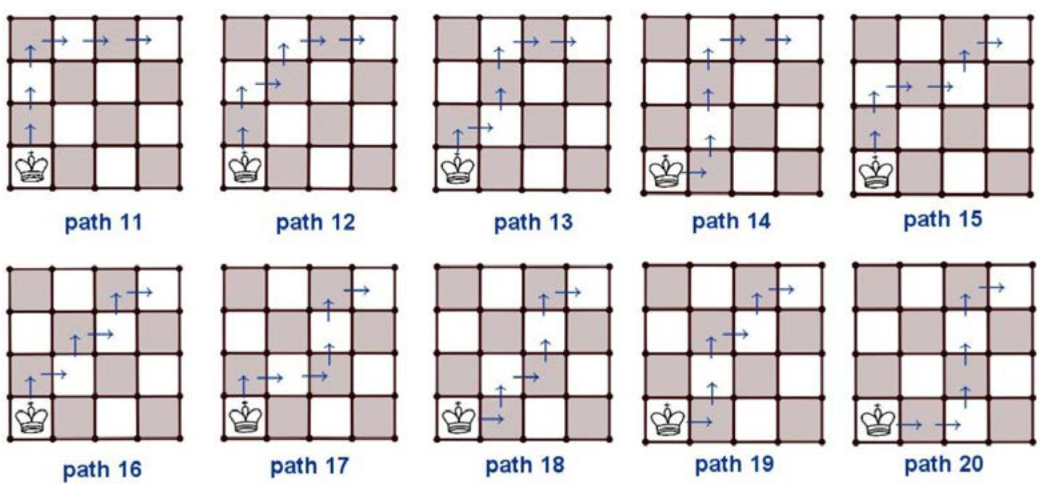


Figure 5

Key notes

- We observe that the first ten paths and the next ten paths are correspondingly symmetric. The up and right movements in the first ten paths are respectively replaced by right and up movements in the next ten paths.
- We also observe that the King's final move is either an up movement in exactly ten paths or a right movement in exactly ten paths.

III. Theory: Assigning Weights

Now, we have reached a stage for setting the theory. Let us not indulge in drawing paths any more! We assign weights to every square in the chess board other than the initial left bottom corner square, the starting position of the King.

Assigning weights

- Each square is assigned the number that equals the total number of paths by which the King can reach that square.
- Every square along the boundary of the first column or last row is assigned 1 as there is only one possible path for the King to reach the respective square. (See Figures 6 and 7.)

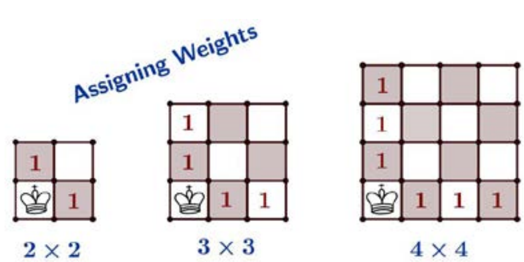


Figure 6

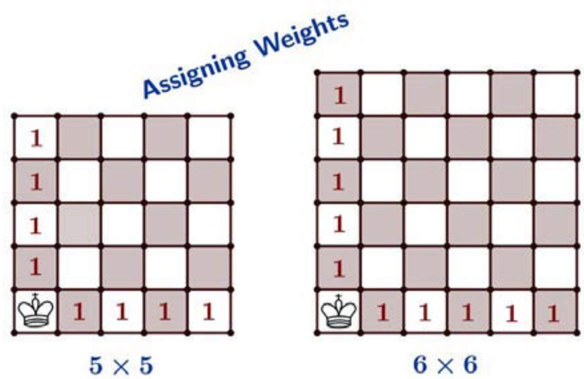


Figure 7

Remember that a path may have one or more moves! The King can reach any square in the first column or last row by only one path (way).

What about assigning weights to other squares? Let us try to develop a relation between assigned weights.

Connecting weights

- Let *A, C, B* be three squares in a $N \times N$ chess board assigned with weights *a, c, b* respectively, as shown (Figure 8).
- For the King to reach square *C*, its previous position is either square *A* or square *B* only.
- There are *a* ways (paths) for the King to reach square *A* from his left bottom corner; and only one way (right move) from square *A* to square *C*.

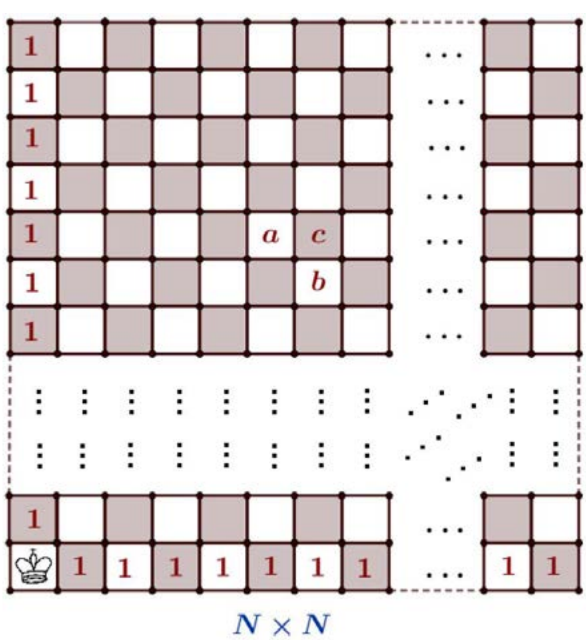


Figure 8

- Therefore there are $a \times 1 = a$ ways (paths) for the King to reach square *C* from his left bottom corner through square A.
- There are *b* ways (paths) for the King to reach square *B* from his left bottom corner; and only one way (up move) from square *B* to square *C*.
- Therefore there are $b \times 1 = b$ ways (paths) for the King to reach square C from his left bottom corner through square *B*.
- Therefore $c = a + b$.

It follows that the weight of a square (not in first column or last row) equals the sum of the weights of its immediate left square and immediate below square.

Inductively assigning weights. We have the weights (1) for each square in the first column and last row. Having laid the foundation theory, we can construct the entire $N \times N$ chess board with weights inductively, for any *N*. This is like constructing a *N*-storey building with a strong foundation.

We can now start constructing weights for 2×2, 3×3, 4×4, 5×5, 6×6 chessboards and observe how the entire construction looks! See Figures 9 and 10.

Likewise, we can construct weights for any $N \times N$ chess board! Do you observe something interesting here ? YES! (See Figures 11 to 14.)

Now, we know why the King says to Pascal, "My Square is your Triangle!"

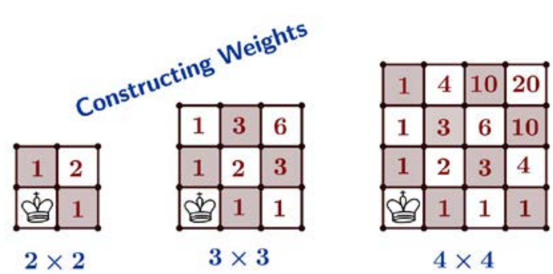


Figure 9

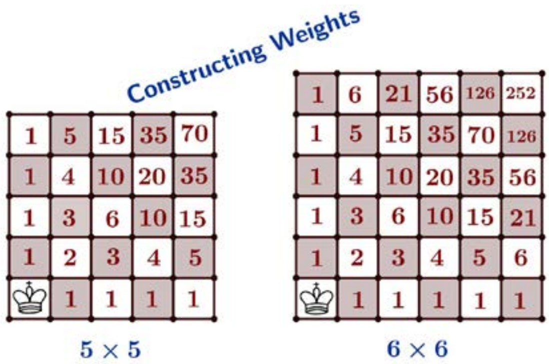


Figure 10

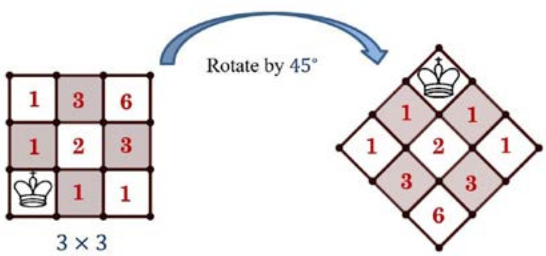


Figure 11

Exercises

- What is the total number of moves (not paths) in a 7 × 7 chess board for the King from left bottom corner to reach the right top corner?
- Among them, how many are up moves? how many are right moves?
- What is the weight of right top corner in a 7 × 7 chess board?
- Suppose the King makes *r, r, u, u, u, u* in a 6 movement path to reach an intermediate square in a 7 × 7 chess board. (Here, *r* means right move; *u* means up move.) Will the King reach a different square if the order of these 6 moves is changed?

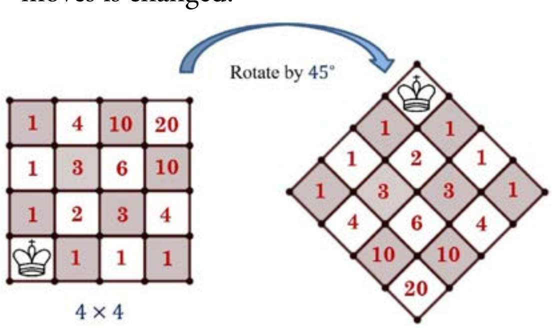


Figure 12

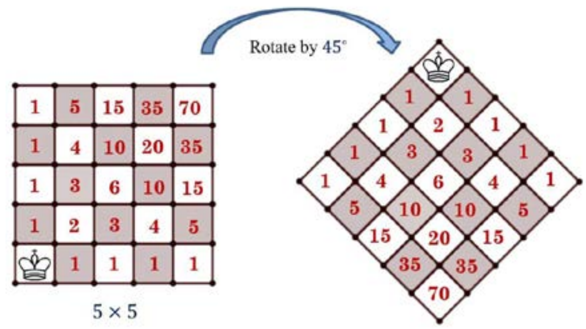


Figure 13

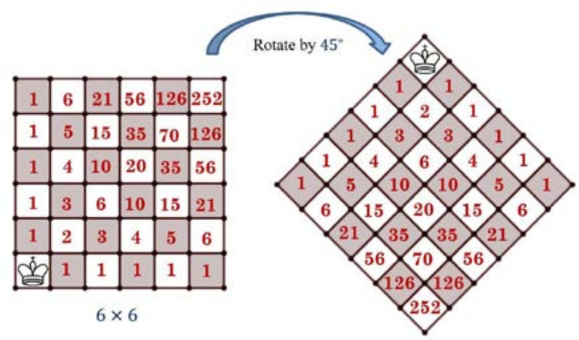


Figure 14

- Can we now say that the weight of a square attained by a 6 movement path with 2 right moves and 4 up moves is ${}^6C_2$ or ${}^6C_4$?
- What is the weight of a n movement path having r right moves and $n - r$ up moves in a $N \times N$ chess board?
- Can we make out combinatorial identities such as ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$, etc... from this setup?

Projects for further exploration

- Fix the number of moves between initial and final positions with no restrictions in the moves of the King. Find the total number of paths in such case.
- Make "No Entry" for certain squares between initial and final positions with no restrictions in the moves of the King and no repetition of positions (squares). Find the total number of paths in such case.

References

[1] Wikipedia, "King (chess)" from https://en.wikipedia.org/wiki/King_(chess)

[2] https://www.geeksforgeeks.org/

[3] AMTI/NMTC/2019/Stage-1/Junior/Problem 23

RAJESH SADAGOPAN is a Mathematics Educator and a member of AMTI. He has served as the editor of Junior Mathematician, a magazine published by AMTI. He is Director of Aryabhatta Institute and currently associate expert of Ahaguru. He has been teaching and motivating elementary, middle and high school students for the past three decades. He may be contacted at aryabhattainstitute97@gmail.com.

Math-e-Magical March

Let's find the magic of the numbers connected to two recently elapsed dates 10th March & 12th March, 2021! Each numeral is used once and only once

Date (dd-mm-yy)	Corresponding number	Numerical Relationship	Observation
10-03-21	10321 (Note that we have omitted the leading 0 in 03)	10321	It is prime and an emirp (its reverse, 12301 is also prime) It is also an additive prime (1 + 0 + 3 + 2 + 1 = 7 & 7 is a prime too)
		$10^2 - 3^1 = 97$ $10^2 + 3^1 = 103$ and $10^2 + 1^3 = 101$	The answers are all prime
		103 + 21 = 124	124 = 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29, which is the sum of 8 consecutive primes starting from 5
		103 – 21 = 82 = 2 × 41	82 is a semiprime (has only two distinct prime factors)
		$10^3 - 21 = 979$ 979 = 11 × 89	979 is also a semiprime, also a palindromic number
		$10^3 + 21 = 1021$	1021 is a prime and an emirp
		$10321 = 95^2 + 36^2$	10321 is expressed as a sum of two different perfect squares, so it is called a Pythagorean prime.

Can you find similar interesting connections for 12th March, 2021 i.e. the number 12321?

Write in to rushik.dharaiya@podar.org

Contributed by **Rushik M Dharaiya**

Finding out Area by using Symmetry

PREETI DHASMANA

"What do Bach's compositions, Rubik's cube, the way we choose our mates, and the physics of subatomic particles have in common? All are governed by the laws of symmetry which elegantly unify scientific and artistic principles"– Mario Livio.

Introduction

In the first drawings of a child, we often see flowers, butterflies, leaves and so on. Have you noticed that they have intuitively replicated the symmetry distributed everywhere in nature?

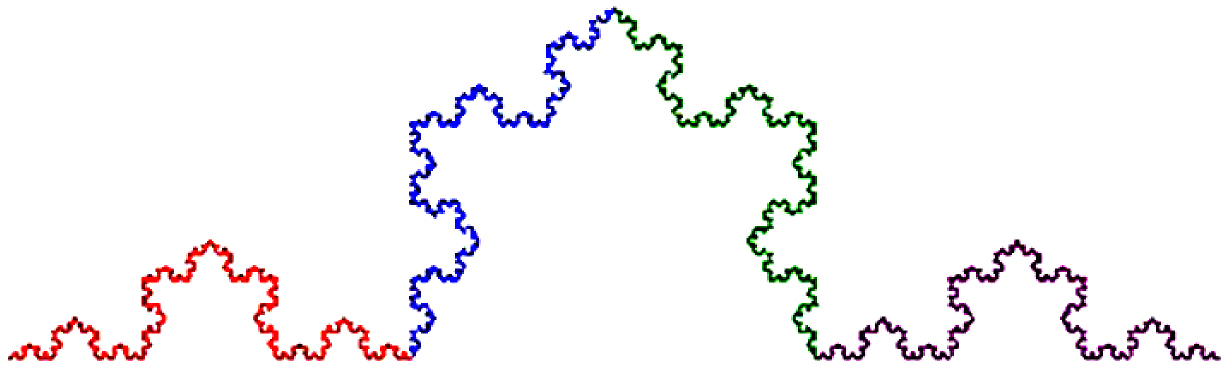
Let us understand the basics of symmetry first. What is symmetry? How does a young child perceive it? Here is a definition of symmetry from the Merriam Webster's dictionary – Symmetry is the property of correspondence in size, shape and relative position of parts on opposite side of a dividing line or median plane or about a center or axis.

Mathematically speaking we say an object is symmetric if it is invariant with respect to a transformation. More specifically, it means that after you transform the object it looks exactly the same as its original version.

Types of symmetry

1. **Reflective symmetry** – A 2D object has reflective symmetry if there is a line going through it which divides it into two pieces that are mirror images of each other
2. **Rotational symmetry** – A 2D object has rotational symmetry if the object can be rotated about a fixed point by a certain angle resulting in no apparent change

Keywords: symmetry, transformation, reflection, rotation, implication, area, pedagogy



When zoomed, the red portion (or the blue, or the green, or the purple) is identical to the whole

3. **Translational symmetry** – when a 2D or 3D object can be translated (moving every point of the object by the same distance and in the same direction) resulting in no apparent change
4. Other symmetries include glide reflection symmetry (a reflection followed by a translation) and retroflection symmetry (combination of a rotation and a reflection – a special type can be helical symmetry) and scale symmetry (where enlarging or shrinking does not change an object, for example, in the case of fractals).

Concept of symmetries in textbooks

Primary level textbooks include only the first three types of symmetries in the form of various activities and worksheets. In the Grade 2 NCERT textbook there are worksheets on visual recognition of patterns and symmetry (non- verbal reasoning). Though the patterns are nothing but the transformations or a movement of figures in a plane classified as translation (slide), a reflection (flip) and a rotation (turn), yet can we say that it is the introduction of symmetry in this textbook?

In the Grade 3 NCERT textbook, the very first chapter discusses concepts of symmetry using a story 'Tit for Tat' and follows it up with activities like completing mirror halves with the introduction of line of symmetry (or dotted line) by using figures and alphabets. Similarly, pattern activities given in Chapter 10 use ingrained concept of symmetry implicitly.

Symmetry is used in Grade 4 – specifically in Chapter 9 with activities on completing the other half and implicitly in Chapter 10 with the activity on floor design and tiling patterns.

Line of symmetry is discussed for more complex figures and complex mirror game activities in Grade 5, Chapter 5. These activities strengthen the idea of line symmetry as well as rotational symmetry (by rotating the objects using half, one-third, quarter or one-sixth turns). Chapter 7 directly links rotational symmetry and patterns with 'turns and pattern' activities.

Now – why, how and to what extent should we introduce symmetry at the primary level? Some of the answers to the why are:

- To develop the ability of identifying geometrical patterns in objects around them
- To develop problem solving skills
- To develop the reasoning power of children
- To develop imagination, creativity and a sense of aesthetics in children.

How and to what extent: An indirect approach is used in the primary textbooks. The concept of symmetry is utilized for different patterns in activities such as rangoli, brick tiling and weaving patterns, mirror halves, paper folding and in number patterns. At this stage, we also need children to explore the symmetry around them in

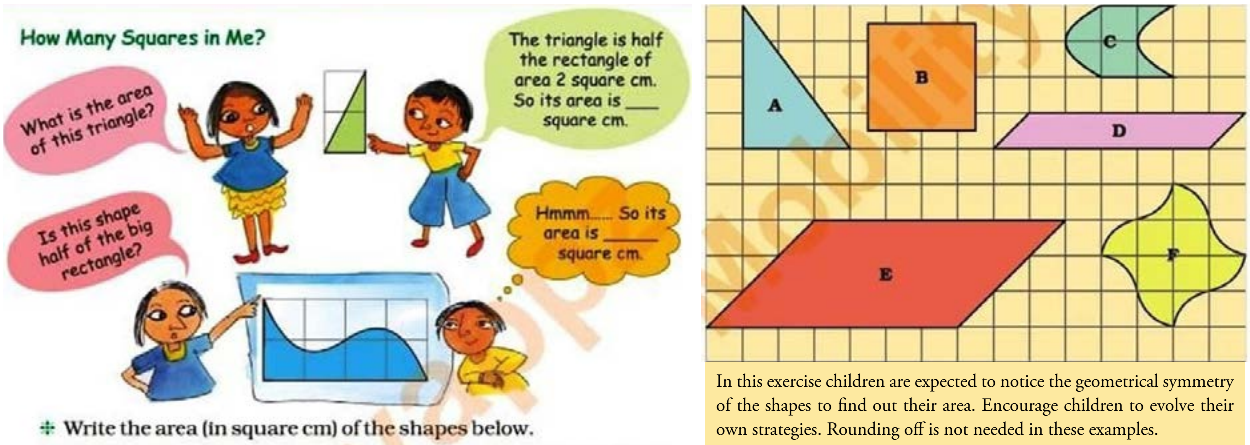


Figure 1

nature. This is consistent with the approach that mathematics is playing with numbers and shapes, their relations to each other, and the rules of the game. The activities used for this purpose are illustrated in the rest of the article.

Target Audience

These activities were conducted with four students of Class 5 of GPS Danda Malla, Block Ekeshwar. They came from underprivileged background with no academic support at home. They had the usual curiosity to learn new things.

Identification of the problem

While teaching Chapter 4 and Chapter 9 of the Grade 5 textbook, we came across a page (Figure 1) where an instruction has been given to the teachers to apply symmetry and to find out the area of different shapes in grid paper.

What the teacher wanted to do and why

We realised that although the students were introduced to 'symmetry' through various activities in the Class 3, 4 and 5 textbooks, they could not use it for these area related problems. So, I tried to help them link the concept of symmetry with the concept of area, specifically in the context of the given problem.

Work Plan: I made a series of worksheets for the following:

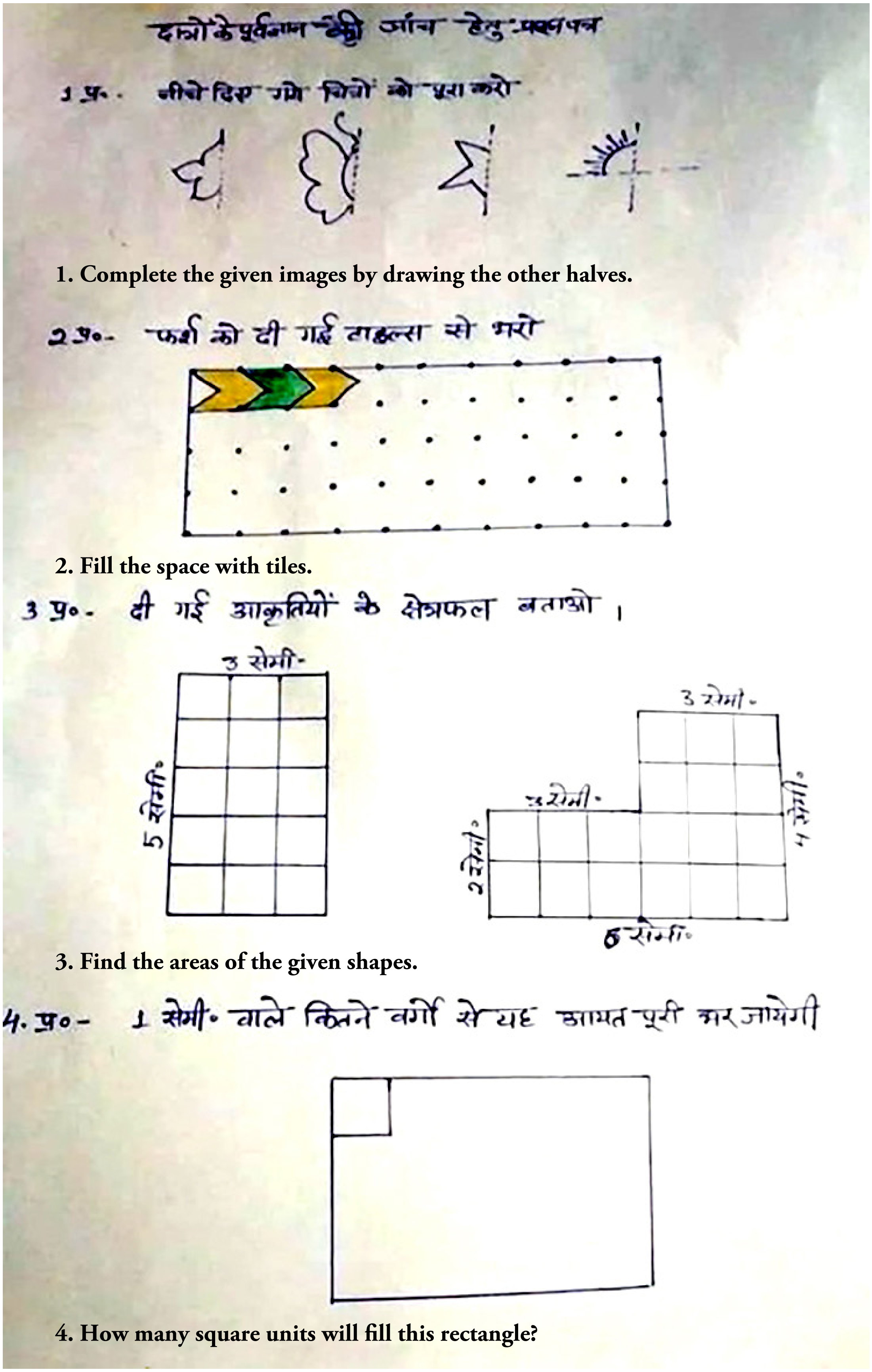
1. Pretest to identify the previous knowledge of students
2. Understanding the various symmetries such as rotational symmetry, translation symmetry, mirror reflection, line of symmetry and curve of symmetry[1]
3. Finding the areas of different shapes on grid paper using symmetry
4. Evaluating students' understanding of these concepts.

Day 1: Previous Knowledge of Students

Result: I came to the conclusion that:

1. Students had implicit understanding of symmetry in form of recognizing pattern, reflection and completing the other half.
2. They were able to get the area of a simple square and rectangle by counting the unit square.
3. They had no knowledge of line of symmetry.
4. They could not relate symmetry with the concept of area.

[1] A curved line that splits a shape in congruent parts (see Activity 3)



छात्रों के पूर्वज्ञान की जाँच हेतु प्रश्नपत्र

1 प्र.- नीचे दिए गये चित्रों को पूरा करो

1. Complete the given images by drawing the other halves.

2 प्र.- फर्श को दी गई टाइल्स से भरो

2. Fill the space with tiles.

3 प्र.- दी गई आकृतियों के क्षेत्रफल बताओ।

3 सेमी.
5 सेमी.
3 सेमी.
2 सेमी.
4 सेमी.
2 सेमी.
6 सेमी.

3. Find the areas of the given shapes.

4. प्र.- 1 सेमी. वाले कितने वर्गों से यह आयत पूरी भर जायेगी

4. How many square units will fill this rectangle?

Figure 2

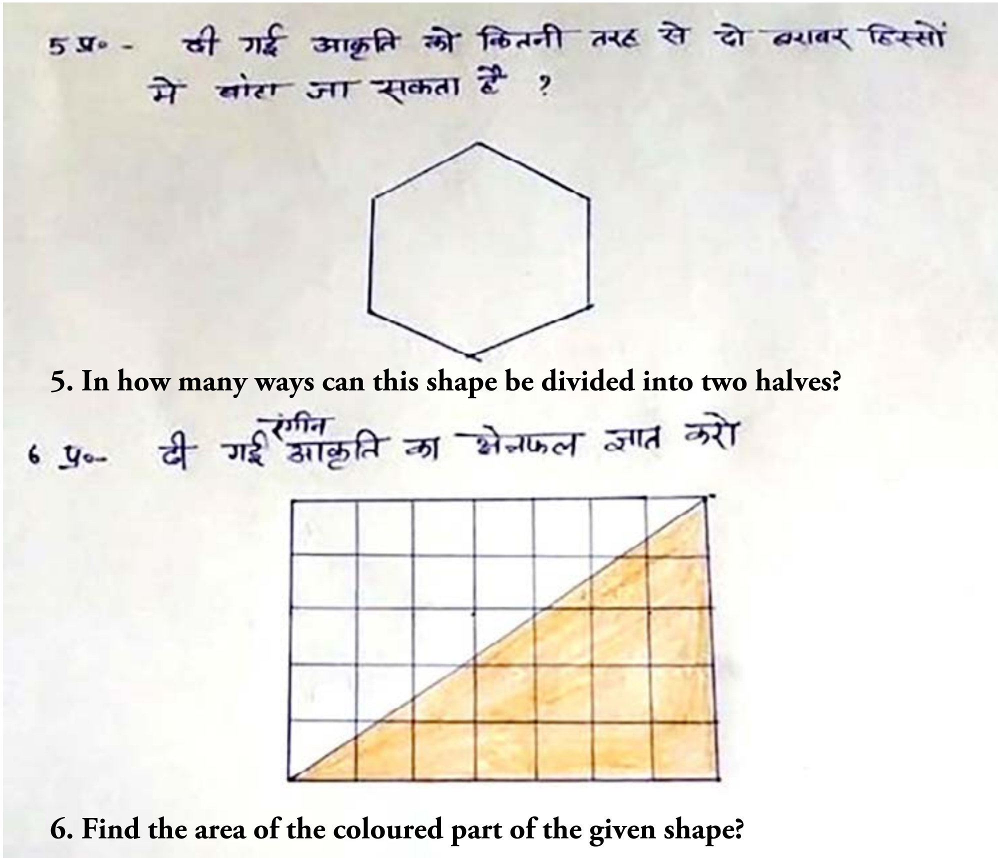


Figure 3: Pretest (Part 2)

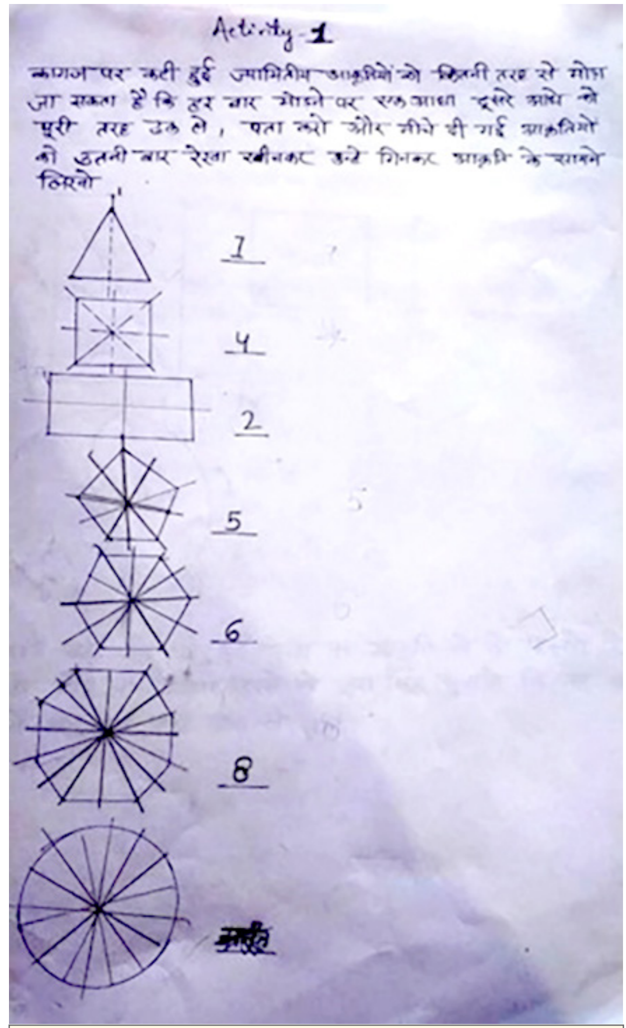
Day 2: Conducting the Work Plan

Activity 1: Lines of symmetry: With this worksheet, I provided students with paper cutouts of the relevant shapes. Children worked with the shapes, identified the lines of symmetry, and filled the worksheet easily. They could identify the line of symmetry in other shapes as well.

Figure 4: Activity 1

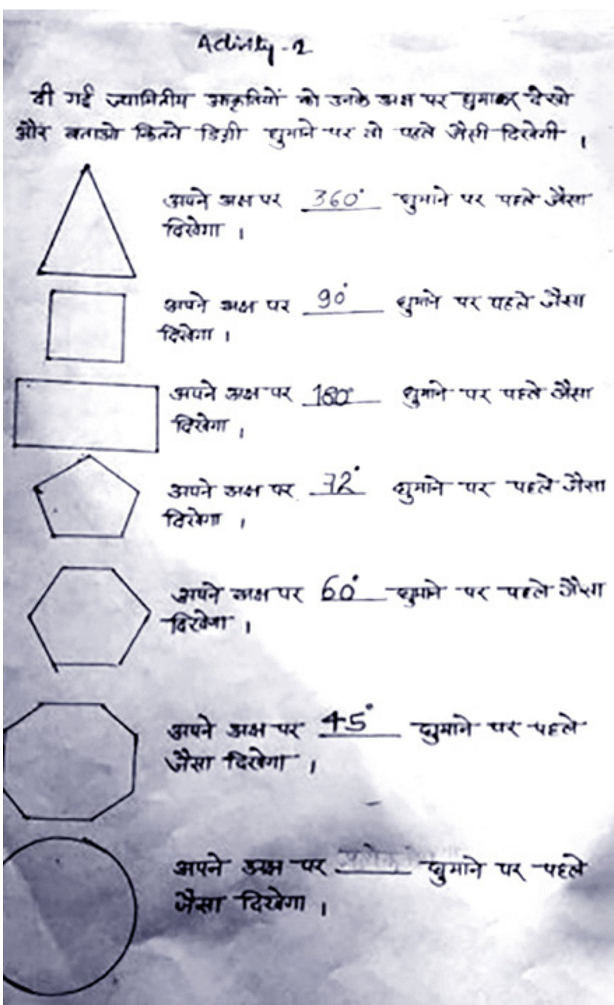
Activity 2: Rotational symmetry: I demonstrated with one shape and the children got the idea.

Figure 5: Activity 2



In how many ways the given shapes can be folded so that the folded halves overlap each other exactly? Draw these fold-lines for each given shape and write the number of fold-lines.

Figure 6: Activity 1



Rotate each shape by an angle so that the rotated shape looks as if it hasn't been rotated. Write the angle in the blank next to each shape.

Figure 7: Activity 2

Activity 3: Finding halves using rotational symmetry: This dealt with shapes halved by a straight or curved line in such a way that there is rotational symmetry (half turn) but not line symmetry. Children had to understand how to use rotational symmetry to determine if two parts of a given shape were identical.

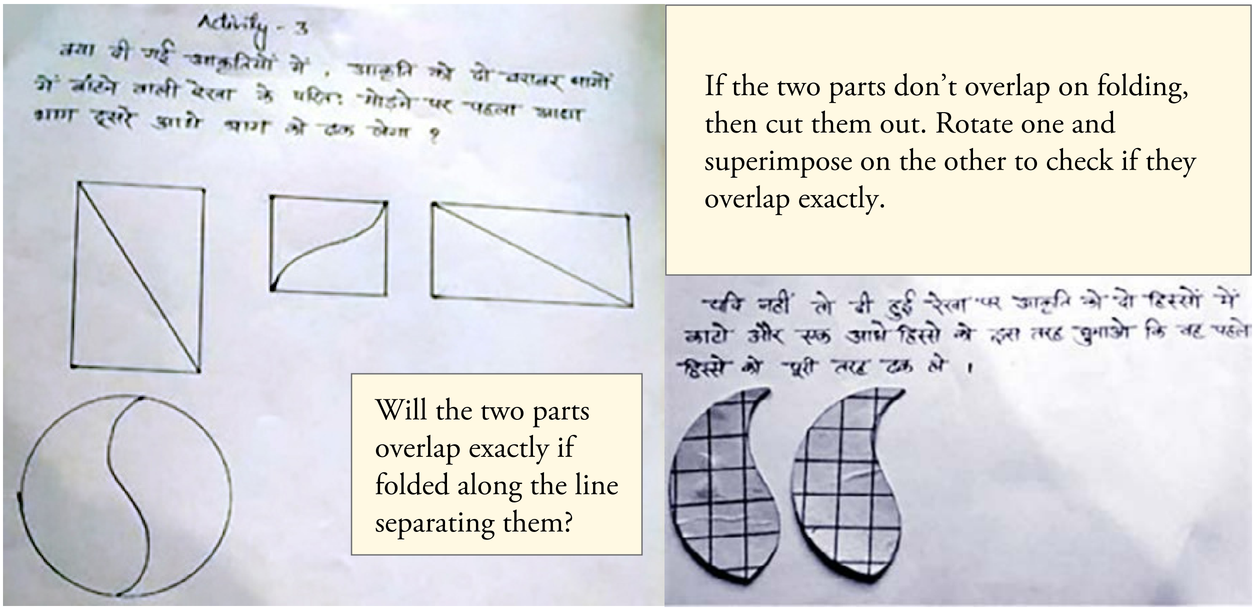


Figure 8: Activity 3

Activity 4: Folding to find equal parts: Children may not see symmetry directly in paper folding, when one part superimposes on the other. This activity was intended for them to understand which fraction of the whole had been shaded.

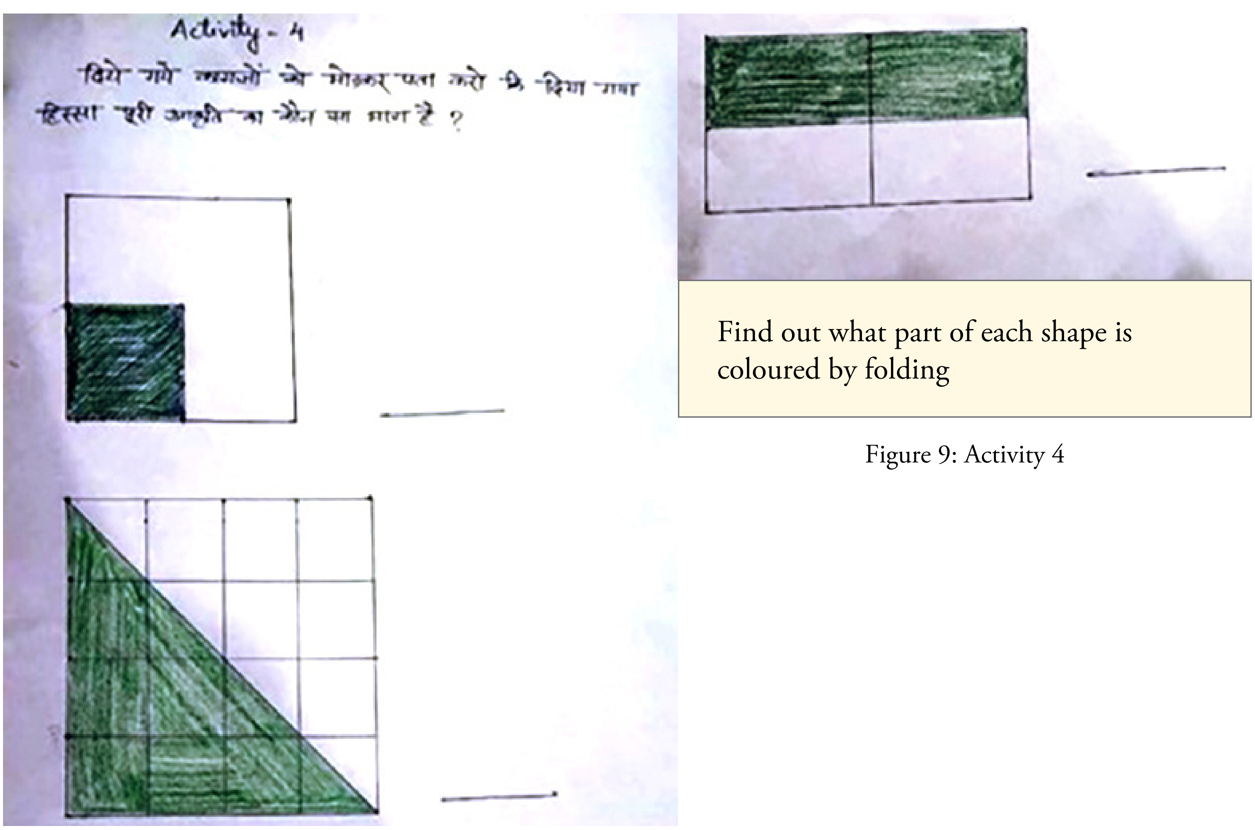
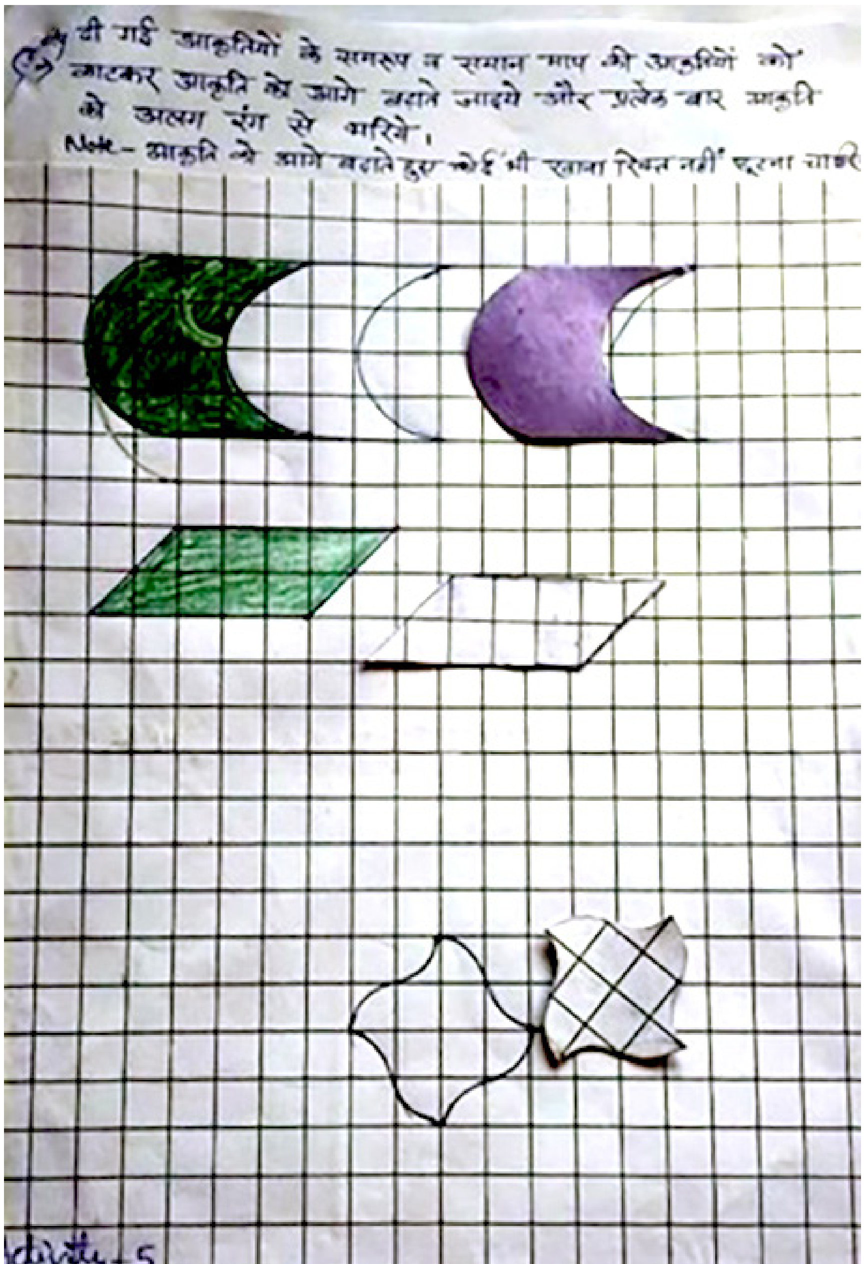


Figure 9: Activity 4

Activity 5: Tile with a given shape: This gave children an exposure to translational symmetry mostly used in tiling (and therefore easily seen on floors) and textile prints. A stencil was provided for the children to outline and tile the shapes on squared paper.

In the next part we demonstrate how this symmetry can be used to find areas of certain shapes as we move the given shapes forward, form a pattern and colour each part with a different colour.



Tile with the given shape. Colour adjacent shapes differently.

Figure 10: Activity 5

Using symmetry to find the areas:

After experiencing three types of symmetry, we went to the next level. It was time to make them understand how the knowledge and understanding they had gained through these activities could help in finding the areas of shapes other than squares or rectangles.

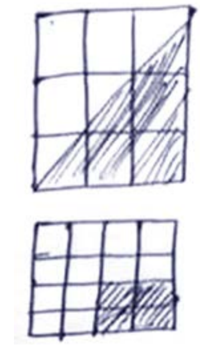


Figure 11

In **example 1**, we identified the line of symmetry and by imagining the paper folding through that line we got two halves or a complete square (or rectangle). Now it became easy for the student to count the number of unit squares in the given square (or rectangle), and then find half or quarter of that number to get the required area (Figure 11).

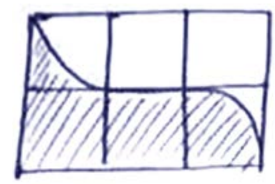


Figure 12

In **example 2**, we used the symmetry of the shapes which were halved by the curved line. Children understood by cutting, rotating and superimposing that the curved line divided the rectangle into two halves. So, they counted the number of unit squares in the rectangle and halved the number to get the required area (Figure 12).

In **example 3**, we use Activity 5 to show how by translating the given shape we can get a complete square or rectangle as shown in Figure 13.

Figure 13

Evaluation

After practising such examples for 3-4 days, it was time for evaluation. The worksheet simply asked them to find the areas of the given shapes.

The instructions that I gave them were that they could use the activities of completing the square or rectangle or folding the shapes or cutting and rotating as they had done in class. They were also given the option of simply visualizing these operations instead of

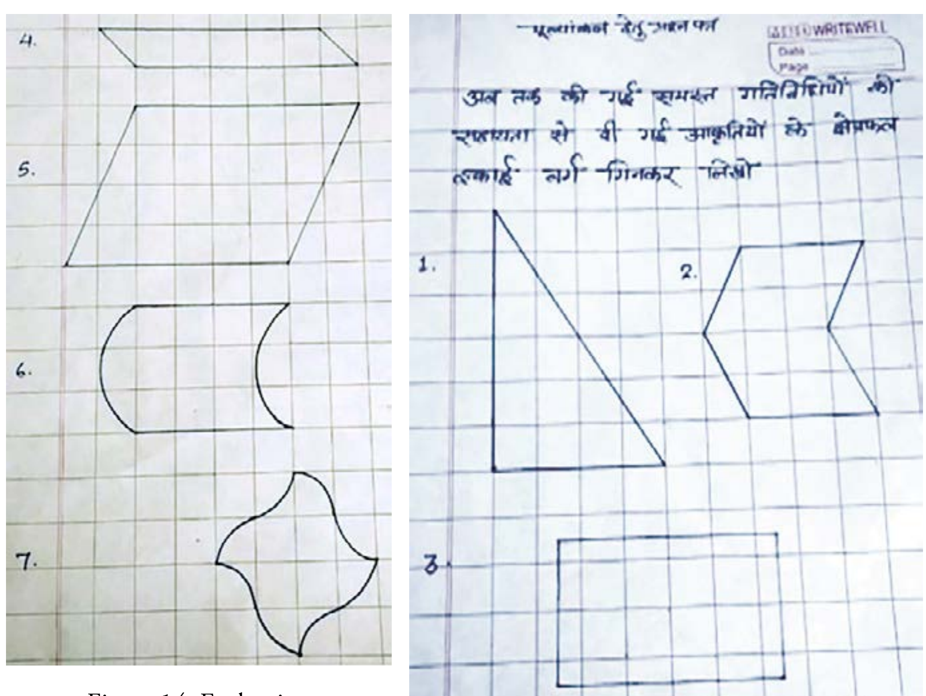


Figure 14: Evaluation

actually executing them. They had to however show their reasoning process (even with sketches), instead of just giving the answer. Each problem had 2 marks : 1 for obtaining a correct answer and 1 for the usefulness of the procedure they used.

Finding

Three children were able to get the area of different shapes using symmetry.

In question 1 of the evaluation, it became obvious that students were getting the idea of making the other half to get a full square or rectangle. But beyond this they also understood that this is not happening just by folding the shapes through the hypotenuse (let's call it AB); here the concept of rotation was used, and they knew that by rotating the shape through the line AB by 180°, they were getting a rectangle. Furthermore, they knew that a square or a rectangle is needed to be formed to get the area of the required shape.

In question 6, the only way to get the area of the required shape is by using the concept of translation and all but one of the students were able to get this question right.

In the rest of the figures, the children were able to use the activity of completing the square and forming a rectangle and square as well as using translational symmetry to get the area.

Conclusion

Since the whole concept and problem were completely new for the children, we cannot compare their pretest knowledge and post test results by standard means.

My observations:

1. Children were excited to play with the shapes and do the activities related to symmetry and could easily solve the given worksheets.
2. They still found it complicated when they had to use symmetry to find areas.
3. I myself had difficulties with these and had to practise several examples with them for a few days.

Finally, it was satisfactory to some extent as a completely new experience for both children and teacher. Children were able to get the idea and use it in solving the problem without using any terminology related to symmetry.

Limitations

We chose very specific and limited number of problems and we dealt with very few shapes. Further worksheets with more shapes are needed to fully evaluate the children's understanding in this matter.

PREETI DHASMANA is an assistant teacher at a basic school in Pauri Garhwal, Uttarakhand. She has a teaching experience of 8 years with children from rural backgrounds. She has done a Masters course in political science, with B.Ed. and D.El.Ed. She may be contacted at preetidhasmana884@gmail.com.

The Tachymeter and the Hyperbola

JAMES METZ

A tachymeter (ta kim' i ter) is a scale on certain chronograph watches that enables one to calculate speed, where "speed" is the amount of work done in one hour. The work can range from covering a certain distance to manufacturing a number of items. With x as the number of seconds/task and using the equation $y = 3600/x$, some convenient values of y are placed around the watch dial as shown in figure 1. The graph is shown in figure 2.

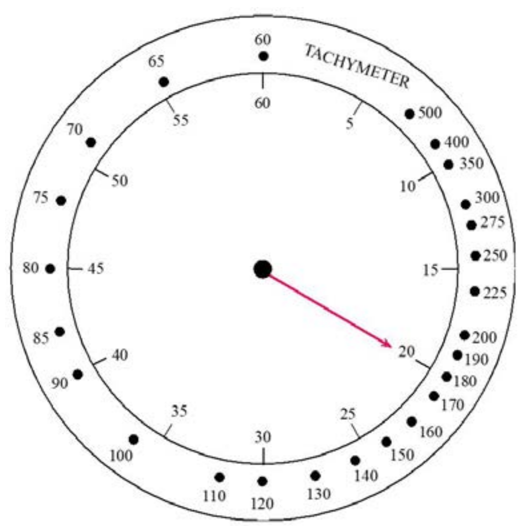


Figure 1. Tachymeter

Keywords: Rate, Speed, Work, Time

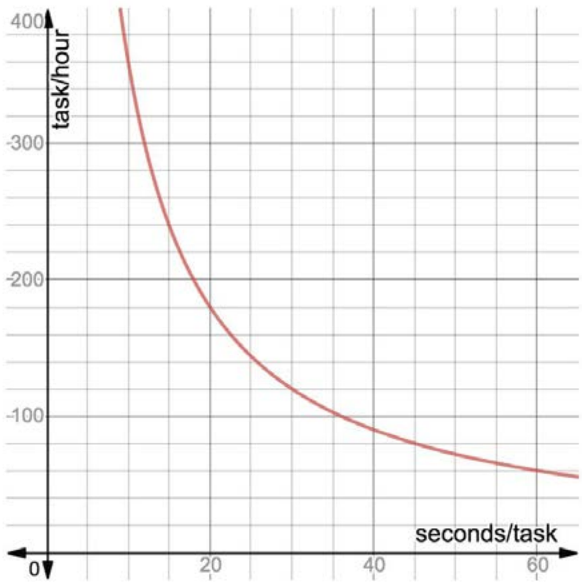


Figure 2. Graph of $y = 3600/x$

In the equation $y = 3600/x$, the units for the numerator are "seconds per hour," and the units for the denominator are "seconds per task." For example, the denominator may be "seconds per kilometre" or "seconds per light bulb produced." Consequently, the units for y, the numbers on the tachymeter, are "task per hour." Conveniently, the x value is given by the position of the second hand, assuming of course that the tip of the second hand started at the top of the watch. For example, when $x = 20$, $y = 180$. If it takes 20 seconds for a racecar to travel one kilometre, then the car is traveling at a speed of 180 kilometres per hour; (3600 seconds/hour)/(20 seconds/kilometre) = 180 kilometres/hour. If it takes 20 seconds to produce one light bulb, then 180 light bulbs are produced in an hour.

A nice feature of the hyperbola is that for every point (*a, b*) there is a point (*b, a*). So, in the example since the point (20, 180) is on the curve, so is (180, 20), which means that if it takes 180 seconds to cover 1 kilometre, the speed is 20 kilometres per hour.

Since the equation $y = 3600/x$ implies $x = 3600/y$, if we know the value of y, we can find the value of x. For example, if we know we are going 80 kilometres per hour, we can find 80 on the tachymeter and see 80 corresponds to 45 seconds/kilometre, so we know we travel one kilometre in 45 seconds.

The shape of the hyperbola tells us that the more seconds it takes to do something, the slower the speed, something that is intuitively obvious. The tachymeter basically converts from seconds per task to a speed of task per hour, and that is achieved with a simple, beautiful hyperbola.

JAMES METZ is a retired mathematics instructor. He volunteers for Teachers Across Borders, Southern Africa for one month each year and enjoys doing mathematics. He may be contacted at metz@hawaii.edu

Deciphering the Median Formula - Part 2: from the Ogives

MATHEMATICS CO-DEVELOPMENT GROUP

In Part 1 of this series, we discussed how to obtain the median formula $M = l + \frac{\frac{N}{2} - m}{f} \times c$ for grouped data from the corresponding histogram. In this article, we will discuss how the same formula can be derived from the corresponding ogive, the graph mapping the cumulative frequencies against the upper (or lower) class-limits. Since there are two kinds of ogives viz. (i) less than and (ii) more than, we will show how either can be used for this purpose. In addition, we will discuss why the ogives intersect at the median.

Ogives are a type of statistical graphs introduced at the secondary level. They plot the 'growth' (or 'decay') of the data and are linked with the cumulative frequencies. Let us consider a grouped data of ages of people living in an island (Table 1) and the (less than) cumulative frequencies (Table 2). Similarly, we can also calculate the 'more than' cumulative frequencies (Table 3).

Table 1

Age (in years)	No. of people
0-10	20
10-20	21
20-30	23
30-40	16
40-50	11
50-60	10
60-70	7
70-80	3
80-90	1

Table 2

Age (in years)	Less than Cumulative frequency
≤ 10	20
≤ 20	41
≤ 30	64
≤ 40	80
≤ 50	91
≤ 60	101
≤ 70	108
≤ 80	111
≤ 90	112

Table 3

Age (in years)	More than Cumulative frequency
≥ 0	112
≥ 10	92
≥ 20	71
≥ 30	48
≥ 40	32
≥ 50	21
≥ 60	11
≥ 70	4
≥ 80	1

Keywords: Statistics, median, ogive, reasoning, visualisation, inference

Figure 1 represents the 'Less than' ogive that plots the 'less than' cumulative frequencies against the data values. In other words, it is the graph that plots the points (10, 20), (20, 41), . . . (90, 112) from Table 2 and connects the consecutive points by line segments. Similarly, Figure 3 shows the 'More than' ogive based on the 'more than' cumulative frequencies in Table 3.

Let us recall that for ungrouped quantitative data, the median splits the entire data set in two parts – each with the same number of data points. Now, if we extend that to the grouped data, there should be $112 \div 2 = 56$ data points less than the median and 56 greater than it. Or in other words, the cumulative frequency for the median should be 56. Since data values are plotted along the x-axis, median is going to be an x-coordinate. Similarly, since cumulative frequencies are plotted along the y-axis, the y-coordinate, i.e., the cumulative frequency corresponding to the median should be 56. So, if M is the median, the point $(M, 56)$ should be on the ogive.

So, to find M, we draw the line $y = 56$, and find the x-coordinate of the point where this line intersects the ogive. This line intersects the ogive in the line segment connecting the points A (20, 41) and B (30, 64). Note that this is the line segment corresponding to the median class. Let the point of intersection between the ogive and the horizontal line be E. So, $E = (M, 56)$.

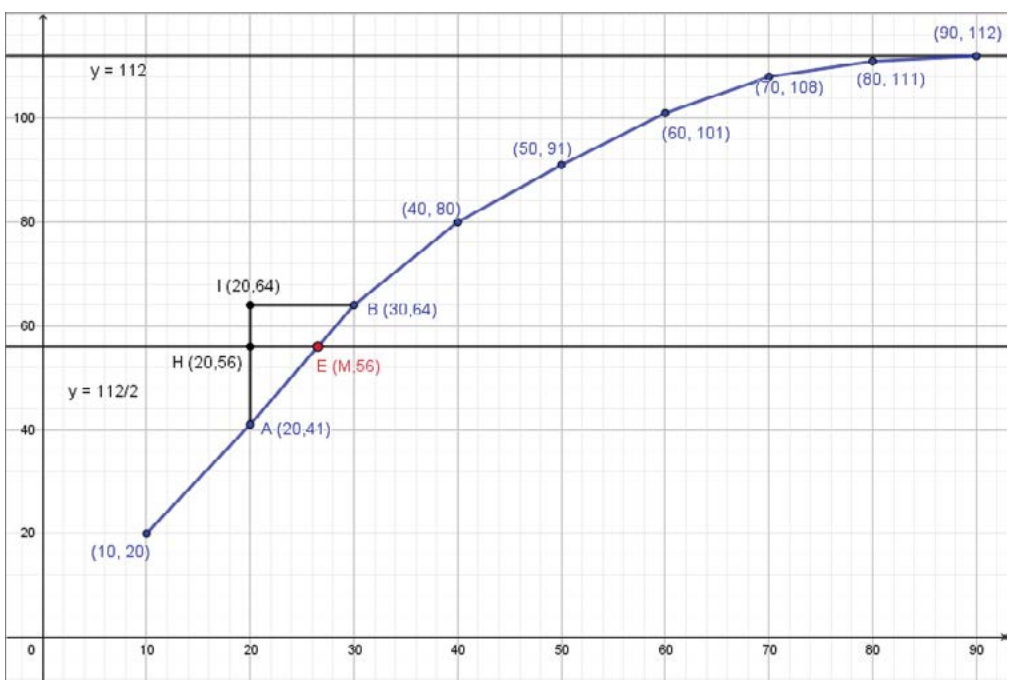


Figure 1. Less than ogive

Now, let us complete the similar right triangles ΔAEH and ΔABI with AE and AB as hypotenuses respectively by drawing their horizontal and vertical legs as shown in Figure 1 and Figure 2. Therefore, $I = (20, 64)$ and $H = (20, 56)$. So, $HE = M - 20$ and $IB = 30 - 20 = 10$, the class-width; while $HA = 56 - 41$ and $IA = 64 - 41 = 23$, the frequency of the median class.

Now, $HE : IB = HA : IA \ldots..$ (1)

$$\Rightarrow (M - 20) : 10 = (56 - 41) : 23 \Rightarrow (M - 20) \times 23 = (56 - 41) \times 10$$
$$\Rightarrow M - 20 = (56 - 41) \times 10/23 \Rightarrow M = 20 + (56 - 41) \times 10/23$$
$$\Rightarrow M = 20 + (112/2 - 41) \times 10/23 \ldots.. \quad (2)$$

which is very similar to the median formula!

Let us now generalize by using algebra (Figure 2):

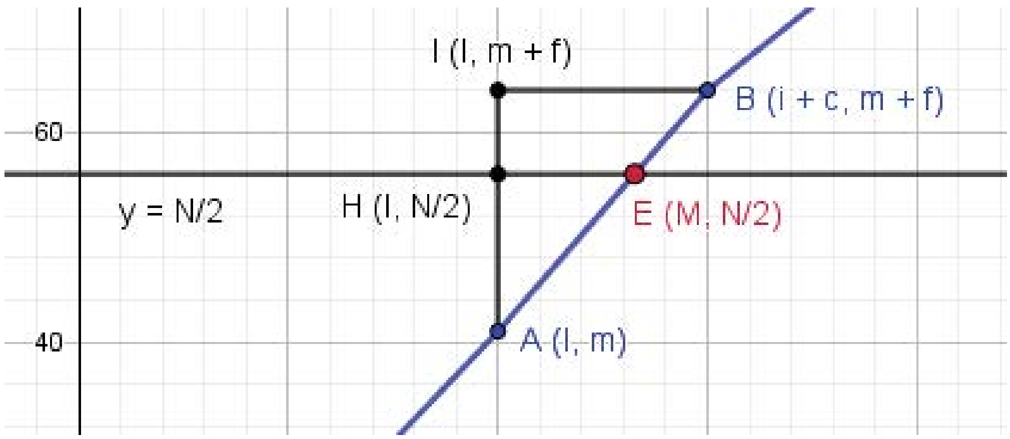


Figure 2.

$$\text{So (1)} \Rightarrow \frac{M - l}{c} = \frac{\frac{N}{2} - m}{f}$$
$$\Rightarrow M - l = \left(\frac{\frac{N}{2} - m}{f}\right) \times c$$
$$\Rightarrow M = l + \left(\frac{\frac{N}{2} - m}{f}\right) \times c \ldots. \quad (3)$$

Table 4		
Symbol	Meaning	In the example
N	Total frequency	112
c	(uniform) class-width	10
l	Lower limit of median class	20
f	Frequency of median class	23
m	(Less than) cumulative frequency for l	41
m'	(More than) cumulative frequency for l	71

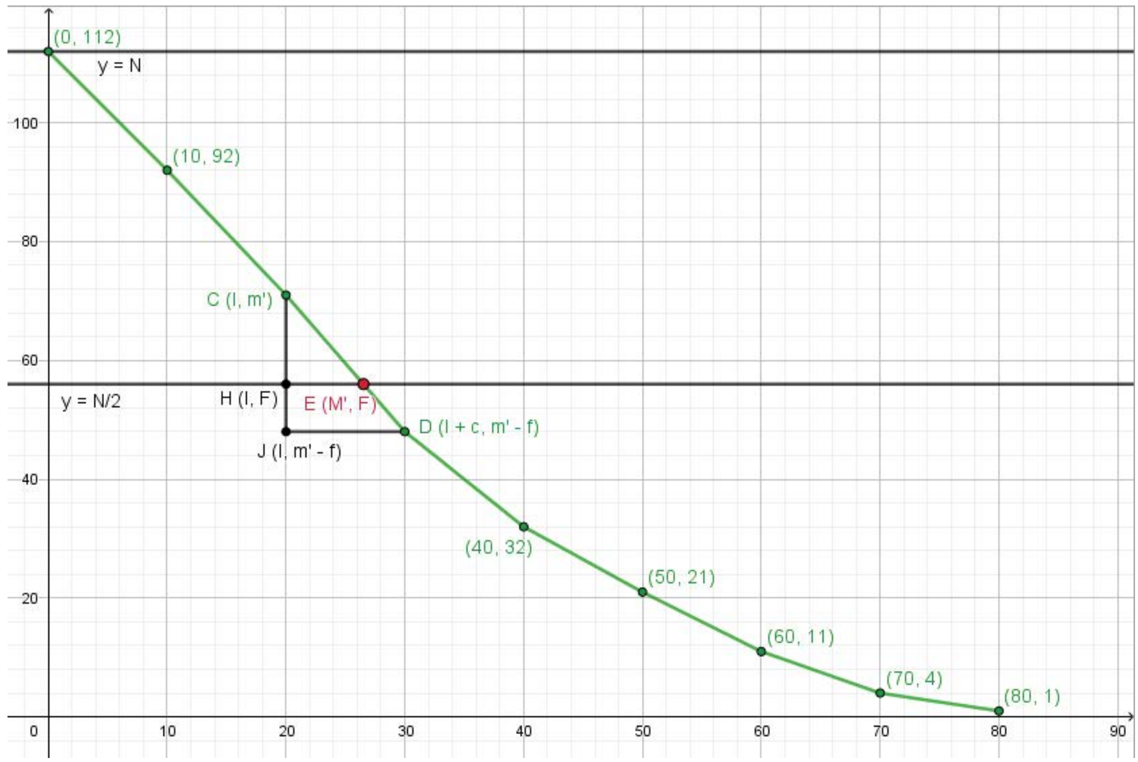


Figure 3. More than ogive

Similarly, we can get the same from 'more than' ogive. Note that this graph (Figure 3) is made by joining the points (0, 112), (10, 92), . . . (80, 1) from Table 3. The line $y = 56$ intersects the line segment CD corresponding to the median class at E (Figure 3 and Figure 4). So, $C = (20, 71)$, $D = (30, 48)$ and $E = (M, 56)$ as before. We complete the similar right triangles ΔCHE and ΔCJD on the hypotenuses CE and CD respectively. So, $H = (20, 56)$ as before and $J = (20, 48)$. Therefore, $HE = M - 20$ (as before), $CH = 71 - 56$, $JD = 30 - 20 = 10$ (class-width), and $CJ = 71 - 48 = 23$ (frequency of the median class).

$$\text{Since } HE : JD = CH : CJ, \text{ we get } (M - 20) : 10 = (71 - 56) : 23$$

$$\Rightarrow M = 20 + (71 - 56) \times 10/23$$

$$\Rightarrow M = 20 + (71 - 112/2) \times 10/23 \ldots \quad (4)$$

Note that (4) is identical to (2) except $112/2 - 41$ being replaced by $71 - 112/2$.

To generalize with algebra (Figure 4), we need to add only the 'more than' cumulative frequency m' for the lower limit of the median class l and the rest remain the same.

$$\text{So, (3) generalizes to } M = l + \frac{m' - \frac{N}{2}}{f} \times c \ldots \quad (5)$$

$\therefore$ we can combine (3) and (5) by $M = l + \frac{|\frac{N}{2} - m|}{f} \times c$ where m is the cumulative frequency corresponding to l

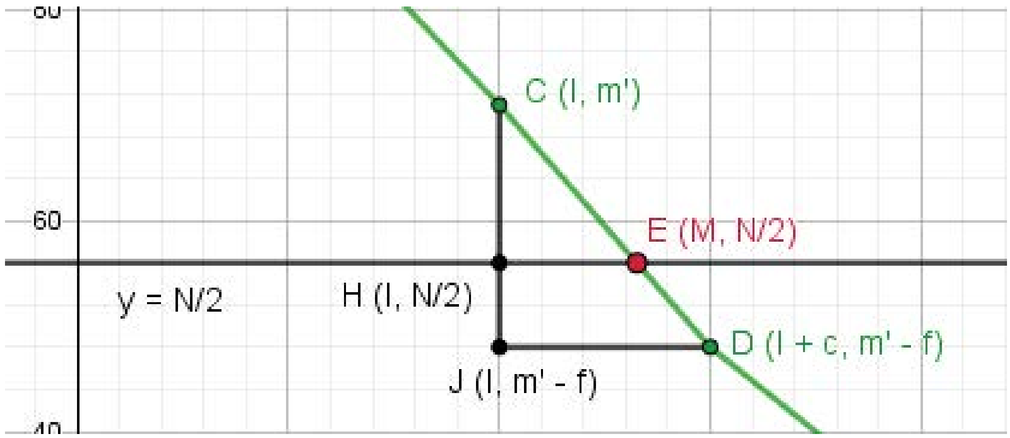


Figure 4.

The last part of the article explores where these two ogives intersect (Figure 5).

Consider the combined Table 5, which is basically Table 1-3 put together. Note that the shaded parts refer to the median class and the corresponding line segments (Figure 5).

Table 5					
Class-Interval	Frequency	Upper Class Limit	Less than CF	Lower Class Limit	More than CF
0-10	20	≤ 10	20	≥ 0	112
10-20	21	≤ 20	41	≥ 10	92
20-30	23	≤ 30	64	≥ 20	71
30-40	16	≤ 40	80	≥ 30	48
40-50	11	≤ 50	91	≥ 40	32
50-60	10	≤ 60	101	≥ 50	21
60-70	7	≤ 70	108	≥ 60	11
70-80	3	≤ 80	111	≥ 70	4
80-90	1	≤ 90	112	≥ 80	1

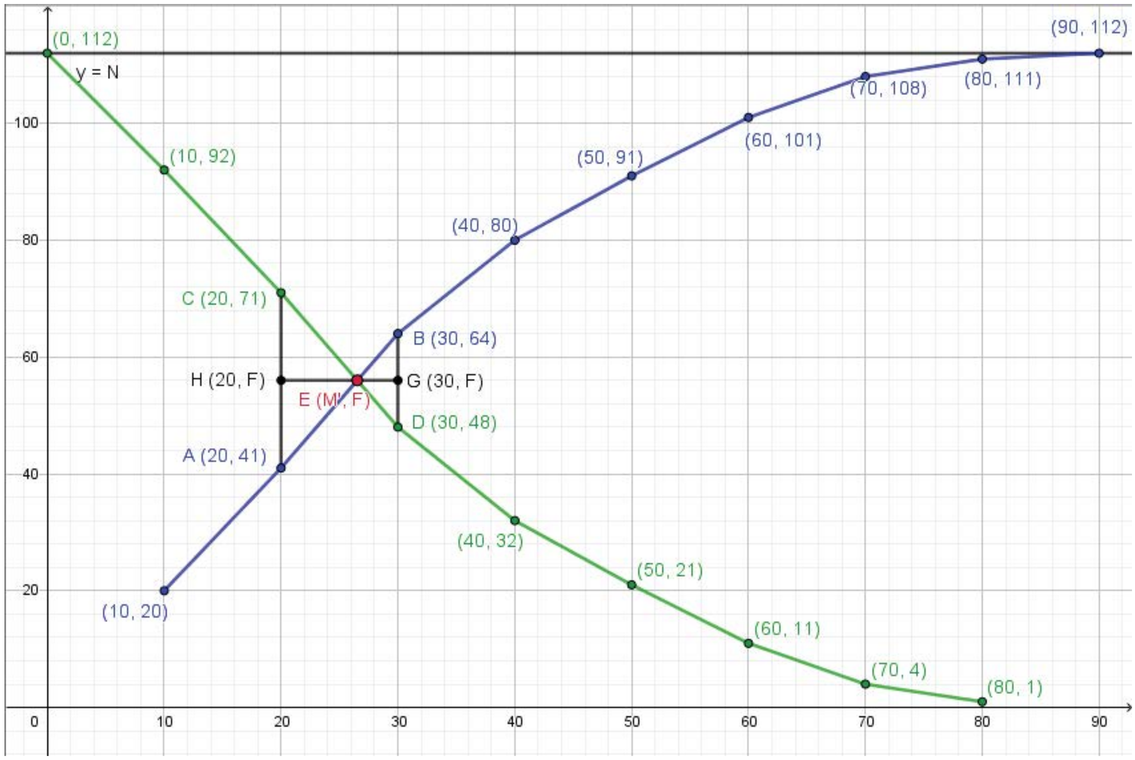


Figure 5.

A, *B*, *C* and *D* are defined as before. So, *AB* and *CD* are the line segments corresponding to the median class in the less than and the more than ogives respectively. *E* (*M'*, *F*) is the point of intersection of *AB* and *CD*. We have to show that E is on the line $y = 56$ i.e. $F = 56$. Then *M'* becomes the median. Join *AC* and *BD*, and let them intersect the line $y = F$ at *H* and *G* respectively.

Since $\Delta AHE \sim \Delta BGE$ and $\Delta CHE \sim \Delta DGE$, we get $HE : GE = AH : BG$ and $HE : GE = CH : DG$

$$\Rightarrow AH : BG = CH : DG \Rightarrow AH \times DG = CH \times BG \ldots. \quad (6)$$

$$\Rightarrow (F - 41) \times (F - 48) = (71 - F) \times (64 - F)$$

$$\Rightarrow F^2 - (41 + 48)F + 41 \times 48 = F^2 - (71 + 64)F + 71 \times 64$$

$$\Rightarrow (71 + 64 - 41 - 48)F = 71 \times 64 - 41 \times 48 \ldots. \quad (7)$$

To generalize with algebra, Table 5 becomes Table 6. Let $a_{k-1} - a_k$ be the median class

$$\text{So } a_{k-1} = l, a_k - a_{k-1} = c, f_k = f, \sum_{i=1}^{k-1} f_i = m, N - \sum_{i=1}^{k-1} f_i = m'$$

$$\text{So, } m' = N - \sum_{i=1}^{k-1} f_i = N - m \ldots \quad (8)$$

Table 6					
Class-Interval	Frequency	Upper Class Limit	Less than CF	Lower Class Limit	More than CF
$a_0 - a_1$	f_1	$< a_1$	f_1	$> a_0$	N
$a_1 - a_2$	f_2	$< a_2$	$f_1 + f_2$	$> a_1$	$N - f_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{k-2} - a_{k-1}$	f_{k-1}	$< a_{k-1}$	$\sum_{i=1}^{k-1} f_i$	$> a_{k-2}$	$N - \sum_{i=1}^{k-2} f_i$
$a_{k-1} - a_k$	f_k	$< a_k$	$\sum_{i=1}^{k} f_i$	$> a_{k-1}$	$N - \sum_{i=1}^{k-1} f_i$
$a_k - a_{k+1}$	f_{k+1}	$< a_{k+1}$	$\sum_{i=1}^{k+1} f_i$	$> a_k$	$N - \sum_{i=1}^{k} f_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{n-1} - a_n$	f_n	$< a_n$	$\sum_{i=1}^{n} f_i = N$	$> a_{n-1}$	$N - \sum_{i=1}^{n-1} f_i$

So, $A = \left(a_{k-1}, \sum_{i=1}^{k-1} f_i\right) = (l, m)$ and $B = \left(a_k, \sum_{i=1}^{k} f_i\right) = (l + c, m + f)$ in the less than ogive while $C = \left(a_{k-1}, N - \sum_{i=1}^{k-1} f_i\right) = (l, m')$ and $D = \left(a_k, N - \sum_{i=1}^{k} f_i\right) = \left(l + c, m' - f\right)$ in the more than ogive.

AB and CD intersect at $E\,(M', F)$. We need to show that $F = \frac{N}{2}$ since that will imply that $M' =$ median.

AC and BD intersect $y = F$ at $H\,(l, F)$ and $G\,(l + c, F)$ respectively (Figure 6).

So, (6): $AH \times DG = CH \times BG \Rightarrow (F - m) \times (F - m' + f) = (m' - F) \times (m + f - F)$

$$\Rightarrow F^2 - \left(m + m' - f\right) F + m\left(m' - f\right) = F^2 - \left(m' + m + f\right) F + m'\,(m + f)$$

$$\Rightarrow \left(m' + m + f - m - m' + f\right) F = m'\,(m + f) - m\left(m' - f\right)$$

$$\Rightarrow 2fF = \left(m' + m\right) f = Nf \ldots \text{by (8)} \Rightarrow F = \frac{N}{2}$$

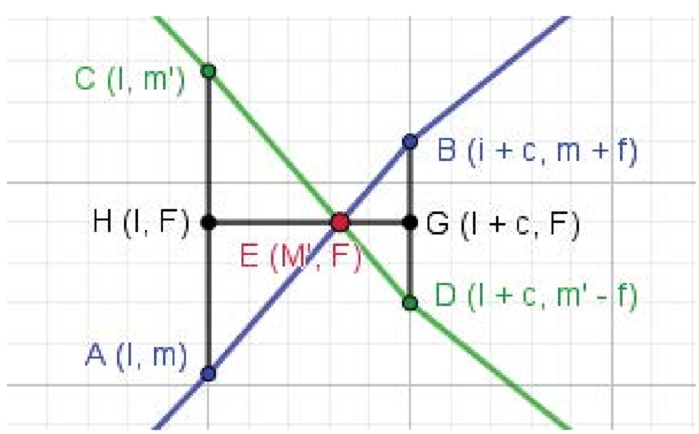


Figure 6.

We would like to observe how the ogives are symmetric about the line $y = \frac{N}{2}$ (Figure 5). This can be explained as follows:

- For any ogive, each line segment represents a particular class-interval
- So, except for the first and last line segments, the remaining ones in less than and more than ogives are paired
- Now, each such pair of line segments have the exact same run $= a_i - a_{i-1}$, upper class limit − lower class limit of the i^{th} class
- And each such pair has opposite rises, f_i i.e. the frequency of the i^{th} class for the less than ogive and $-f_i$ for the more than ogive

Even though the median formula is quite complicated and ogive is a new graph at the secondary level, note that none of the derivations require any sophisticated techniques. They were based on basic coordinate geometry viz. coordinates of points, lengths of horizontal and vertical line segments (which are essentially difference of x- or y-coordinates respectively), and similar triangles — all of which are part of secondary mathematics syllabus. If students can be taught to use the basic principle, then they can easily draw the relevant part of the less than ogive, as in Figure 2, for any given data and compute the median without having to remember the formula.

In the next article, we shall look into another such complicated formula — that of the mode.

Math Co-dev Group or more elaborately **MATHEMATICS CO-DEVELOPMENT GROUP** is an internal initiative of Azim Premji Foundation where math resource persons across states put their heads together to prepare simple materials for teachers to develop their understanding on different content areas and how to transact the same in their classrooms. It is a collaborative learning space where resources are collected from multiple sources, critiqued and explored in detail. Math Co-dev Group may be reached through yashvendra@azimpremjifoundation.org.

NEW RELATIONSHIPS, NEW FORMULAE

Contributed by
Vikram A Ghule

Area of a right-angled triangle = $\frac{1}{2}$ × base × height

Some interesting results – can you prove them?
If in a right-angled triangle hypotenuse = c and base = b and

i. $b = c - 2$, then area of triangle = $b\sqrt{c-1}$

ii. $b = c - 1$, then area of triangle = $\frac{b}{2}\sqrt{c+b}$

An application of formula (i)

Find the area of the right-angled triangle with hypotenuse c = 10,400 and base b= 10,336. Let the height be h, clearly (h, b, c) is a Pythagorean triplet.

Since computation with such big numbers is difficult, find the greatest common divisor of 10,400 and 10,336.

GCD (10400, 10336) is 32 and the triangle with sides $\left(\frac{h}{32}, \frac{b}{32}, \frac{c}{32}\right)$ is similar to the given triangle.

The right-angled triangle with hypotenuse $\frac{10400}{32}$ = 325 and base $\frac{10336}{32}$ = 323, satisfies the conditions of formula (i). So its area is $base\sqrt{hypotenuse - 1}$ = $323\sqrt{325-1}$ = $323\sqrt{324}$ = 323 × 18

And the area of the original triangle is $323 \times 18 \times (32)^2 = 5953536$

Acknowledgement: The author is grateful to Prof. B.N.Waphare and Prof. P.M.Avhad for encouraging him to send this finding to At Right Angles.

Problem Corner

Bachet's Problem

A.RAMACHANDRAN

There is a question, often posed as a puzzle or brainteaser, which has been popular for generations. It was probably in circulation earlier but came to be more widely known due to the Frenchman Claude Gaspar Bachet de Meziriac (1581-1638). The problem is not too easy to solve but the solution can be appreciated by anyone. It runs as follows:

A trader had a 40-pound standard weight which was dropped down accidentally and broke into four pieces, each weighing a different integral number of pounds. The trader then found that with the four weights thus obtained he could measure any integral number of pounds from 1 to 40, placing the weights on one, or both, of the pans of his balance. What are the weights of the four pieces?

Though I had come across the problem and its solution quite some time ago, I recently wondered how one could solve it by logical reasoning. I started by assuming the four weights to be a, b, c and d, with $a > b > c > d$.

Now, $b + c + d = 40 - a$. Let us assume that with the three weights b, c and d, one can measure all integer pound weights from 1 to $(b + c + d)$, both inclusive. Now, by taking weight a and some or all of b, c, d we can measure all integer weights from $(a + 1)$ to 40. Again, we can work downwards from a to obtain all integral weights till $a - (b + c + d) = a - (40 - a) = (2a - 40)$. But integers weights from $(b + c + d + 1)$ till $(2a - 41)$, both inclusive, cannot be obtained.

A numerical example to drive home this point: Let $a = 34$. Then $b + c + d = 6$. Let $b = 3$, $c = 2$ and $d = 1$. With these we can measure all the integer weights from 1 to 6. Taking these with $a = 34$, we can obtain all the integer weights from 35 to 40.

Keywords: Reasoning, equations, combinations, exponents

Similarly, we can obtain the integer weights from 33 down to 28. But the integer weights from 7 to 27, both inclusive, cannot be measured.

Returning to the general case, if we are to eliminate the above-mentioned gap, we should make $(2a - 40)$ the successor of $b + c + d$, i.e., the successor of $40 - a$. That is, we must have $40 - a + 1 = 2a - 40$ or, $3a = 81$ or, $a = 27$.

By similar arguments, we find that $b = 9$, $c = 3$, $d = 1$.

Now, $1 + 3 + 9 + 27 = 3^0 + 3^1 + 3^2 + 3^3 = \frac{(3^4-1)}{3-1} = 40$.

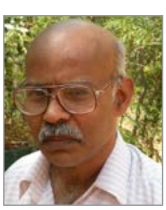
The situation can be extended by adding subsequent powers of 3. That is, by including a standard weight of 81 pounds, one could measure all integer pound weights from 1 to 121 $[=\frac{(3^5-1)}{3-1}]$, and so on.

We now ask: In how many ways can four different weights be distributed in two pans of a balance if the two pans are indistinguishable (that is, they are not labelled A or B or left or right or in any other way)? We list the possibilities below. The possibility of not placing any weight on either pan is omitted.

Distribution of weights	Number of ways
All four in one pan	1
Three in one pan, one in the other	4
Three in one pan, none in the other	4
Two in each pan	3
Two in one pan, one in the other	12
Two in one pan, none in the other	6
One in each pan	6
One in one pan, none in the other	4

These give a total of 40 combinations. So, each integer pound weight from 1 to 40 can be obtained with the weights 1, 3, 9 and 27 pounds in a unique manner.

If the pans are designated in some way (say, left/right), then the number of possibilities doubles to 80. And if we include the possibility of not placing any weight on either pan, then we have 81 possibilities. This can be visualised in another way. Each of the weights can go into the left or right pan or be put aside: 3 possible outcomes. So, with four different weights we have $3^4 = 81$ possibilities.



A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

Problem Corner

An Easy Proof of Ptolemy's Theorem

RADHAKRISHNAMURTY PADYALA

Many proofs are available ([1], [2], [3]) for the famous and important theorem in geometry known as Ptolemy's theorem. For our discussion, we consider the proof presented by Shirali ([1]). In his article, he described a simple geometrical proof of the theorem and presented two elegant applications. He noted that the proof 'presents a challenge' because from the statement of the theorem we get no clue on how to tackle it. In the proof, there arises a crucial idea of locating a point E on a diagonal of the quadrilateral that enables the construction of two similar triangles. A recent demonstration by Tunsteno [2] demonstrates a simple and intuitively appealing method for locating the point E. We present it here for the benefit of school students and teachers.

Statement of Ptolemy's theorem

Theorem. If $ABCD$ is a cyclic quadrilateral, then we have the following equality:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD. \qquad (1)$$

In words: "The sum of the products of opposite pairs of sides of a cyclic quadrilateral is equal to the product of the diagonals" (see Figure 1).

The crucial idea in the proof described by Shirali is to locate a point E on diagonal AC such that $\triangle CDE \sim \triangle BDA$ (this amounts to $\measuredangle CDE = \measuredangle BDA$; see Figure 2).

Keywords: Ptolemy's theorem, cyclic quadrilateral, rotation, similar triangles, proof

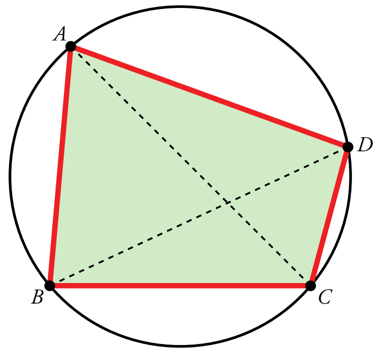
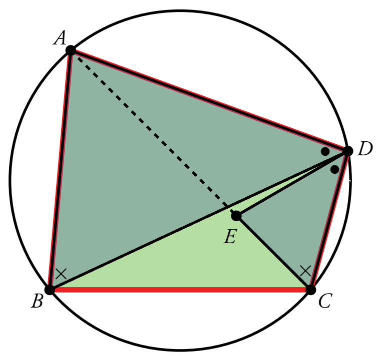


Figure 1. Cyclic quadrilateral $ABCD$ and Ptolemy's theorem



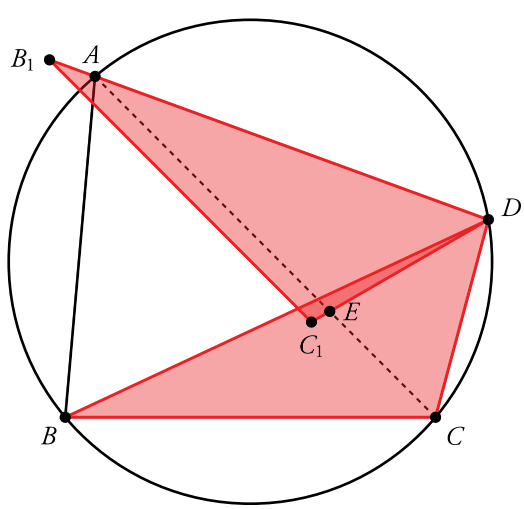
Locate point E on AC such that $\measuredangle CDE = \measuredangle BDA$. Then $\triangle CDE \sim \triangle BDA$.

Figure 2. Point E is located on diagonal AC such that $\measuredangle CDE = \measuredangle BDA$

Tungsteno's method of locating the point E

The idea proposed by Tungsteno [2] is ingenious. We rotate $\triangle BDC$ around vertex D as pivot (clockwise) through $\measuredangle ADB$, so that the image of side DB lies along line DA. Let the image of the triangle be $\triangle B_1DC_1$ (see Figure 3). Let C_1D intersect diagonal AC at E. Since $\measuredangle B_1DC_1 = \measuredangle BDC$, it follows that $\measuredangle C_1DC = \measuredangle ADB$. That is, $\measuredangle EDC = \measuredangle ADB$.

Similarly, we rotate $\triangle ADB$ around vertex D as pivot (counterclockwise this time) through $\measuredangle BDC$, so that the image of side DB lies along line



$$\measuredangle DAC = \measuredangle DBC$$

$$\measuredangle DBC = \measuredangle DB_1C_1$$

$$\therefore\ \measuredangle DAC = \measuredangle DB_1C_1$$

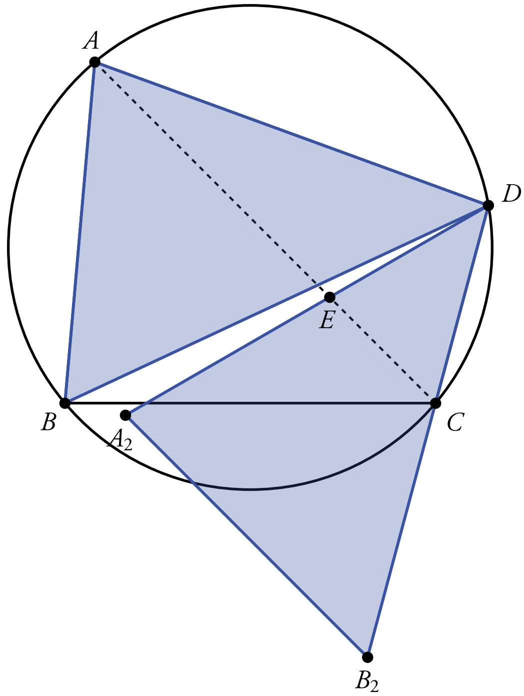
$$\therefore\ AE \parallel B_1C_1$$

$$\therefore\ \triangle DAE \sim \triangle DB_1C_1$$

$$\therefore\ AE : B_1C_1 = DA : DB_1$$

$$\therefore\ AE = \frac{BC \cdot AD}{BD}$$

Figure 3. $\triangle DB_1C_1$ is the image of $\triangle DBC$ under a clockwise rotation about D through $\measuredangle ADB$



$$\measuredangle DCA = \measuredangle DBA$$
$$\measuredangle DBA = \measuredangle DB_2A_2$$
$$\therefore\ \measuredangle DCE = \measuredangle DB_2A_2$$
$$\therefore\ EC \parallel A_2B_2$$
$$\therefore\ \triangle DEC \sim \triangle DA_2B_2$$
$$\therefore\ EC : A_2B_2 = DC : DB_2$$
$$\therefore\ EC = \frac{AB \cdot CD}{BD}$$

Figure 4. $\triangle DA_2B_2$ is the image of $\triangle DAB$ under a counter-clockwise rotation about D through $\measuredangle BDC$

DC. Let the image of the triangle be $\triangle A_2DB_2$ (see Figure 4). Since $\measuredangle A_2DB_2 = \measuredangle ADB = \measuredangle C_1DC = \measuredangle EDC$, it follows that side A_2D lies along side C_1D, so the intersection of A_2D with diagonal AC yields the very same point E as earlier.

From the derivations shown in Figure 3 and Figure 4, we get:

$$AE = \frac{BC \cdot AD}{BD}, \qquad EC = \frac{AB \cdot CD}{BD},$$

and therefore by addition,

$$AC = \frac{BC \cdot AD}{BD} + \frac{AB \cdot CD}{BD},$$
$$\therefore\ AC \cdot BD = BC \cdot AD + AB \cdot CD. \qquad (2)$$

Thus we prove Ptolemy's theorem easily by this method. □

References

1. Shailesh Shirali, "How to Prove it: Ptolemy's Theorem", *At Right Angles*, Vol 5, Nov 2016.
2. https://twitter.com/74WTungsteno/status/1373998345497219075
3. A Miraculous Proof (Ptolemy's Theorem) - Numberphile, https://www.youtube.com/watch?v=bJOuzqu3MUQ

DR. RADHAKRISHNAMURTY PADYALA is a retired scientist from CECRI (CSIR). He developed an interest in mathematics by reading Martin Gardner's 'Recreational Mathematics' column in *Scientific American*. At present he operates as a freelancer. He has a particular interest in analysing fallacies arising from the incorrect application of mathematics in natural phenomena. He is an ardent admirer of the works of Galileo and Ptolemy. His specialisations are Electrochemistry, Classical Thermodynamics and Kinematics. He may be contacted at padyala1941@yahoo.com.

Middle School Problems
Theme: Number of Digits

A.RAMACHANDRAN

In carrying out operations such as multiplication, division, squaring, cubing, finding square roots, finding cube roots, etc., with natural numbers, we may sometimes be interested in knowing the number of digits in the solution before obtaining the precise result. Often this can help in quickly spotting errors. This exercise aims to raise awareness in this area.

1. A two-digit, a three-digit and a five-digit number are multiplied together. How many digits can the product have?

2. A three-digit number is raised to the power of two. How many digits can the resulting number have?

3. A four-digit number is raised to the power of three. How many digits can the resulting number have?

4. This question takes up the reverse of the situations of questions 2 and 3. If the fourth power of a natural number N has 25 digits, how many digits does N have?

5. Referring to Problem 2, we now ask: What is the dividing line between three-digit numbers with five-digit squares and those with six-digit squares? Or, what is the largest three-digit number whose square has five digits? (Equivalently, what is the least three-digit number with a six-digit square?)

6. A similar question about the cube of a four-digit number: What is the largest four-digit number whose cube has (a) ten digits (b) eleven digits?

Solutions

1. The smallest two-digit number is 10 ; the least three-digit number is 100 ; the least five-digit number is 10000. The product of these numbers is 10000000, which has eight digits. So this is the least possible number of digits in the product. The greatest corresponding numbers are (respectively) one less than 100, 1000 and 100000, whose product (10000000000) has eleven digits. So the product of the greatest two-digit, three-digit and five-digit numbers must have ten digits. So the product could have eight, nine or ten digits.

2. The smallest three-digit number is 100, whose square (10000) is a five-digit number. The greatest three-digit number is one less than 1000, whose square (1000000) has seven digits. So the maximum number of digits that the square of a three-digit number can have is six. That is, the square of a three-digit number can have five or six digits.

3. The smallest four-digit number is 1000, whose cube has ten digits. The greatest four-digit number is one less than 10000, whose cube has thirteen digits. Thus we can see that the cube of the greatest four-digit number would have twelve digits. So the cube of a four-digit number can have ten, eleven or twelve digits.

4. By now you may be able to see that the fourth power of a natural number of d digits can have $4d$, $4d-1$, $4d-2$ or $4d-3$ digits. So one of these expressions must be equal to 25. Obviously, $4d-3=25$ is the only permissible relation (as the others lead to values of d which are not integral); it leads to $d=7$.

   (In the case of fourth powers we could take up another line of argument. The square root of a 25-digit perfect square must have 13 digits, and the square root of a 13-digit perfect square must have 7 digits.)

5. If n is a three-digit number whose square has five digits and $(n+1)$ is a three-digit number with a six-digit square, then n^2 is at most 99999, so n^2 is less than 100000 ; and $(n+1)^2$ is at least 100000.

   Taking square roots of all these quantities, we see that n is less than $\sqrt{10}\times 100$, while $(n+1)$ is at least equal to $\sqrt{10}\times 100$. So $n<316.2$ and $316.2\leq n+1$. Since n and $(n+1)$ differ by 1, we get $n=316$ and $(n+1)=317$. That is, 10^5 lies between 316^2 and 317^2.

6. Let's look at the two cases separately.

   a. If n is a four-digit number with a ten-digit cube, and $(n+1)$ is a four-digit number with a eleven-digit cube, then n^3 is less than 10000000000, and $(n+1)^3$ is at least 10000000000.

      Taking cube roots of all the quantities, we see that n is less than $\sqrt[3]{10}\times 1000$, while $(n+1)$ is at least equal to $\sqrt[3]{10}\times 1000$. This works out as $n<2154.4$ and $2154.4\leq n+1$. This means that $n=2154$ and $(n+1)=2155$. That is, 10^{10} lies between 2154^3 and 2155^3.

   b. If n is a four-digit number with a eleven-digit cube and $(n+1)$ is a four-digit number with a twelve-digit cube, then $n^3<100000000000\leq(n+1)^3$ giving $n<\left(\sqrt[3]{10}\right)^2\times 1000\leq(n+1)$. This works out as $n<4641.6\leq n+1$. Thus, $n=4641$ and $(n+1)=4642$. That is, $4641^3<10^{11}<4642^3$.

   You should be able to obtain the square and cube roots of 10 from the internet.

Angle Trisection – An Approximate Procedure

MANORANJAN GHOSHAL

It is well-known that there is no exact procedure for trisection of arbitrary angles if one uses only the available Euclidean instruments (the straightedge and the compass) and stays within the constraints specified in Euclidean constructions. However, approximate procedures do exist, and the challenge can be to find simple procedures for which the error is very small.

In this article, we present one such procedure.

Motivation for the procedure

The idea behind the procedure emerges from the following observation.

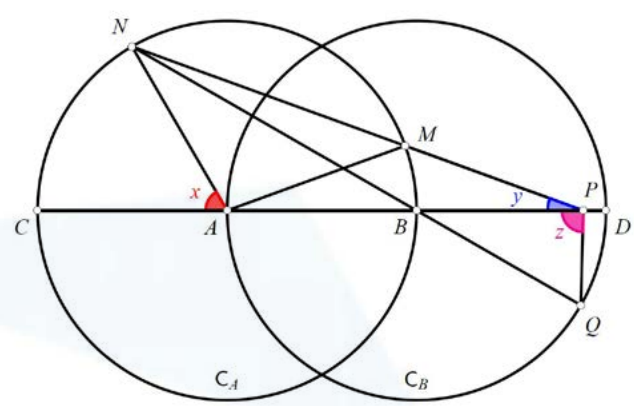


Figure 1

- Given any two points A and B. Let the distance AB be referred to as 1 unit.
- Draw circles C_A and C_B centred at A and B respectively, with radius 1 unit each (Figure 1).

Keywords: Angle trisection, Euclidean instrument, Euclidean construction

- Join AB and extend it to intersect C_A and C_B again at points C and D, as shown.
- Let P be any point on line AB, between B and D.
- Draw a circle C_P centred at P, with radius 1 unit. Let C_P intersect C_B at M (and one other point which we do not name as it is not needed).
- Join PM and extend it till it meets C_A again at point N. Join NA. Let $x = \measuredangle NAC$, $y = \measuredangle NPA$.
- Join NB and extend it till it meets C_B at point Q. Join PQ.
- It is now easy to verify that $\measuredangle NAC = 3\measuredangle NPA$, i.e., $y = x/3$. This relation is exact.
- If we try out this construction and measure the size of $\measuredangle BPQ$, we notice that *the angle is almost a right angle. Moreover, this remains true regardless of where P is located between B and D.*

It is the observation made at the end that gives rise to an approximate trisection procedure.

Proposed approximate procedure for trisection of angle

As noted above, the fact that $\measuredangle BPQ$ is almost a right angle plays a crucial role in the procedure. Here are the actual steps (see Figure 2).

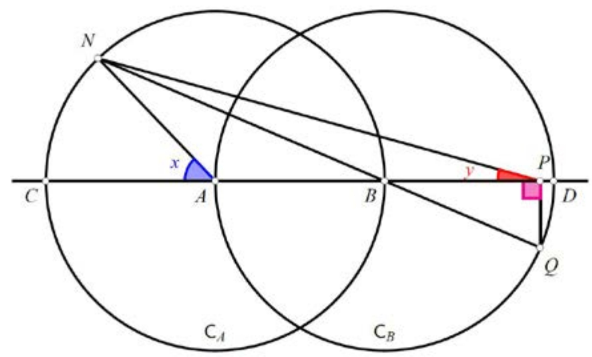


Figure 2

- Let the angle to be trisected be denoted by x.
- Mark any two points A and B, 1 unit apart. Draw circles C_A and C_B centred at A and B respectively, with radius 1 unit each. Join AB and extend it to intersect C_A and C_B again at points C and D.
- Locate a point N on C_A such that $\measuredangle NAC$ is equal to x (the angle to be trisected).
- Join NB and extend it beyond B till it meets C_B again at point Q.
- Draw QP perpendicular to line CD (with P on CD). Join NP.
- Then it will be found that $\measuredangle NPC \approx 1/3$ $\measuredangle NAC$, i.e., $y \approx x/3$.

The reader could try out the procedure and check the closeness of the approximation.

MANORANJAN GHOSHAL has a B Sc degree and operates his own LIC agency. He has a keen interest in popular science and in his free time engages in private tuitions and the study of science and mathematics. He may be contacted at ghoshal.manoranjan@gmail.com.

Exploring and Connecting: Inradius and Factorisation

SWASTI PATIL, SIDDHARTH LAXSHMISHA, DAVIN CHRISTINO, with comments by VINAY NAIR

During a few online sessions with students an exploration was made on inradius and its relationship with integer-sided right triangles. This article compiled by 9-year-old Swasti, explains the thought process that evolved during the sessions. In a regular classroom setting, a problem on calculating the inradius for a particular right triangle may be tackled and considered solved when the required value is obtained. But in these sessions, the goal was not only to arrive at answers but also to pose new problems that came to mind during the process of solution. Thus, you will see how the problem did not end with finding the inradius and how the students probed (and were sometimes nudged or even pushed) into questions involving how many possible solutions exist. In this process, they wandered into a factorization problem through a problem in geometry.

This is an account of a discussion with students of different age groups (9 to 14 years). All of them had the understanding of congruence, tests of congruence in triangles, inradius, generalisation of Pythagorean triplets, some properties of circles – such as the radius and tangent are perpendicular at the point of contact, solving equations and manipulating algebraic expressions.

Keywords: Inradius, Right angled triangle, Pythagorean Triplets, Exploration

Question: Can we find how many right triangles with integer side lengths exist for a given integer inradius?

This question was an outcome of an exploration in Pythagorean triplets that further led to finding the inradius of a right-triangle whose sides are integers. While observing the list of such Pythagorean triplets along with their inradii, it was seen that for a given inradius, there could be more than one integer-sided right triangle. It was here that the teacher pushed the class into exploring the question stated above (without the teacher himself knowing that it would lead to a factorisation problem).

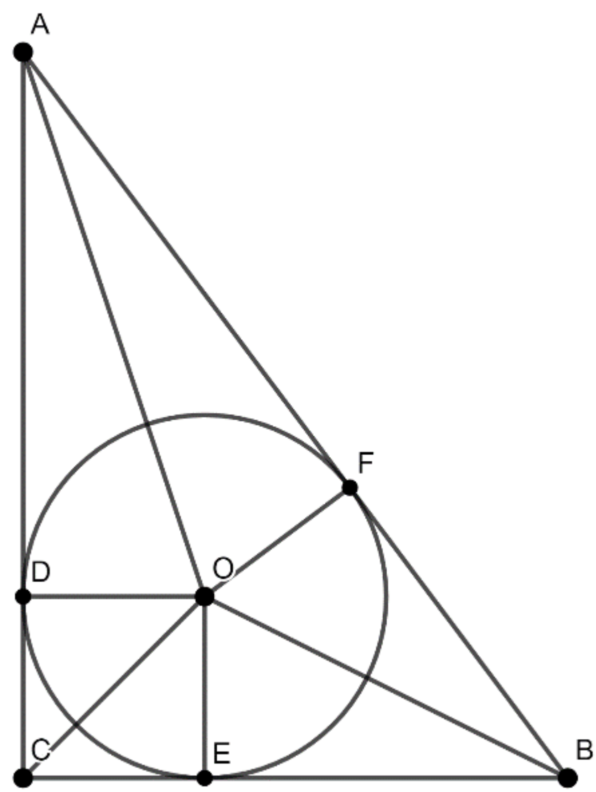


Figure 1.

In Figure 1, ΔABC is a triangle right angled at C. Point O is the incentre of ΔABC.

We started with the above question which led us to an exploration and we arrived at something interesting with respect to the inradius of a right triangle with integer side lengths.

The well-known formula $[\Delta ABC] = \frac{1}{2} \cdot r \cdot (AB + BC + CA)$ can be further simplified to get the value of r as $\frac{BC \cdot AC}{AB + BC + CA}$. Here $[\Delta ABC]$ refers to the area of the triangle ABC.

We use this formula and explore further.

Let D, E and F be the points where the incircle meets the sides AC, CB and BA, respectively.

Let $\overline{BC} = a$, $\overline{AC} = b$, and $\overline{AB} = c$. [(a, b, c) is a Pythagorean Triplet]

We know that Pythagorean triplets can be found using *Brahmagupta's formula*, i.e., $\left(m^2 + n^2, m^2 - n^2, 2mn\right)$, which generates *Pythagorean Triplets* (here m and n are natural numbers such that $m > n$). [*Comment.* By imposing additional conditions (that m and n are coprime and have opposite parity), we can ensure that the Pythagorean triplet is primitive, i.e., that the numbers in the triplet are coprime. However, we do not give the proof of this statement in this article, for brevity.]

The table below lists Pythagorean Triplets generated using m and n such that $m = n + 1$ and the corresponding inradii computed using the above formula. Here (a, b, c) is a Pythagorean Triplet such that $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$.

m	n	a	b	c	inradius
2	1	3	4	5	1
3	2	5	12	13	2
4	3	7	24	25	3
5	4	9	40	41	4
6	5	11	60	61	5

Observation 1. When $m - n = 1$, the inradius is n.

Let us observe the inradius when $m - n > 1$.

m	n	a	b	c	inradius
3	1	8	6	10	2
4	2	12	16	20	4
5	2	21	20	29	6
5	3	16	30	34	6
6	3	27	36	45	9

Observation 2. When $m - n = 2$, the inradius is $2n$.

Claim (based on the observations; this claim was made by a student). The inradius is $n\,(m - n)$.

Proof:

In polygon $CDOE$,

$OD = OE$ (radii of the same circle)

$\measuredangle D = \measuredangle O = \measuredangle C = \measuredangle E = 90°$.

$\therefore$ Polygon $CDOE$ is a square.

$\therefore CD = CE = OE = OD = r$.

In $\Delta AOD \& \Delta AOF$,

$$\overline{AO} = \overline{AO}$$
$$\measuredangle ADO = \measuredangle AFO = 90^\circ$$
$$OD = OF$$

$\therefore$ By RHS,

$$\Delta AOD \cong \Delta AOF$$
$$\implies \overline{AD} = \overline{AF}$$

Let side $\overline{AD} = \overline{AF} = q$.

Similarly, we can also show that $\Delta FOB \cong \Delta EOB \implies FB = EB$.

$$\therefore \text{Let } FB = EB = p.$$

We obtain the following equations:

$$r + q = \overline{AC} = b$$
$$p + r = \overline{BC} = a$$
$$q + p = \overline{AB} = c$$

We need to find r.

Adding the 3 equations, we get

$$2p + 2q + 2r = a + b + c$$
$$\implies p + q + r = \frac{a+b+c}{2}$$
$$\implies r = \frac{a+b+c}{2} - (p+q) = \frac{a+b+c}{2} - c = \frac{a+b-c}{2}$$

$\therefore$ The inradius of a right-angled triangle with hypotenuse c and sides a and b is $\frac{a+b-c}{2}$.

Substituting $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$:

$$r = \frac{(m^2 - n^2) + (2mn) - (m^2 + n^2)}{2} = \frac{2mn - 2n^2}{2} = mn - n^2 = n(m-n).$$

Hence proved. Using this result, we can easily prove Observation 1 & Observation 2.

We can get a more general result by multiplying $m^2 + n^2$, 2mn, $m^2 - n^2$ by a constant k (it should be a positive integer). This ensures that all Pythagorean Triplets have been covered. The inradius of the same will now be $kn(m-n)$.

Finally, we get to the problem of how many right triangles exist for a given inradius where the side lengths and inradius are integers.

We know that $r = kn(m-n)$, hence the problem reduces to factorisation. [*Comment.* The students did not use the word 'factorisation' explicitly, but they said – we need to find the factors of ' r ' to find all possible side lengths.]

Let us look at an example. When the inradius is 4, possible values for $(m - n)$ and n are:

k	**n**	**m − n**	**m**	**a**	**b**	**c**
1	1	4	5	24	10	26
1	2	2	4	12	16	20
1	4	1	5	9	40	41
2	2	1	3	10	24	26
2	1	2	3	16	12	20
4	1	1	2	12	16	20

If we discard repetitions, the possible triangles will be

1. (24, 10, 26)
2. (12, 16, 20)
3. (9, 40, 41).

Here's another way to proceed:

We know that the inradius $r = \dfrac{a+b-c}{2}$.

$$\therefore 2r = a + b - c$$

$$\Rightarrow 2r - a - b = -c = -\sqrt{a^2 + b^2}$$

Squaring both the sides, we get

$$a^2 + b^2 = (2r - a - b)^2 = 4r^2 + a^2 + b^2 + 2ab - 4ar - 4br$$

$$\Longrightarrow 2r^2 = 4r^2 - 2ar - 2br + ab = (a - 2r)(b - 2r)$$

This again reduces to a factorisation problem.

In this entire exploration, we were intrigued by the way exploration with Pythagorean triplets and inradius led us to a factorisation problem. We also saw that there are multiple ways to compute the inradius of an integer-sided right triangle.

Takeaways from a teacher's perspective

1. Usually, teachers are well prepared for a session and the aim is to enable the students to learn concepts and apply them in solving problems. In my experience of working with students who are passionate about mathematics and willing to go that extra mile in exploration, going to a class with a good question that I myself hadn't explored, also helped in multiple ways. The first advantage is that I don't end up steering them to a desired goal and that helps in bringing out their creativity. As we don't have a chalked-out path to walk, we ask ourselves and each other – what is the next good question to explore? This way, new questions come up from the students' side. That is the second advantage. Slowly, this becomes a thinking habit. Thus, whenever they see a problem, they start to think of more problems that can be constructed from the given problem, whether a generalisation would be possible, and so on.

2. While working on the problem in this article, when they tried to find all possible side lengths of a right triangle with integer side lengths, they found out that some solutions weren't getting generated by the factorisation of $r = n(m - n)$ and that made them realise that not all Pythagorean Triplets can be generated using Brahmagupta's formula. The first example that they hit was 9, 12, 15 where the inradius was 3, but for inradius 3, while considering $n(m - n)$, the triangle they obtained was 7, 24, 25. This reminded them of the fact that Brahmagupta's expressions generate only Primitive Pythagorean Triplets. After that, multiplying Brahmagupta's expressions by a new variable 'k' was a natural process because Brahmagupta's expressions when multiplied by another natural number k, give all possible Pythagorean Triplets.
3. Normally, factorisation is taught as a separate topic and many students don't find any purpose in learning or applying the ways to factorise an algebraic expression. However, in a problem like this, they started with a geometry problem, did some smart algebraic manipulation and ended up in factorisation. Such explorations result in getting into a decompartmentalised way of looking at mathematics.

SWASTI PATIL is a 10 year old unschooled girl from Pune. Apart from Mathematics and Coding, she also has a love for theatre, for plants, for reading, and singing. She also plays the instrument *jaltarang*.

SIDDHARTH LAXMISHA is a 5th grader from The Study L'école Internationale, Puducherry. He loves Mathematics, Physics, Coding, playing the Keyboard and most of all, eating french fries.

DAVIN CHRISTINO is a 10th grader from Good Earth School, Chennai. He loves Physics, Math, Coding, Chemistry and playing Football.

VINAY NAIR works with all three of them (and others like them who are precocious in mathematics) in a course, *Ganit Manthan*, where they together explore problems, create and prove/disprove conjectures. The focus of the sessions is not just to solve problems but also to pose interesting problems that the students come up with. He can be reached at nairvinayr@gmail.com

Solution to a Problem from the CMI Entrance Test 2020

PRANEETHA KALBHAVI

Problem (CMI entrance paper 2020)**.** Suppose A, B, C, D are points on a circle such that AC and BD are diameters of the circle. Suppose $AB = 12$ and $BC = 5$. Let P be a point on the arc of the circle from A to B (the arc that does not contain the points C and D). Let the distances of P from A, B, C, D be a, b, c, d respectively. Find the values of $\frac{a+b}{c+d}$ and $\frac{a-b}{d-c}$. You may assume $d \neq c$ so the second ratio makes sense.

Solution using Ptolemy's theorem. As AC and BD are diameters, $\measuredangle ABC = 90°$. As $AB = 12$, $BC = 5$, we get $AC = BD = 13$. Also, $\measuredangle BAD = \measuredangle ADC = \measuredangle BCD = 90°$, and $ABCD$ is a rectangle (see Figure 1).

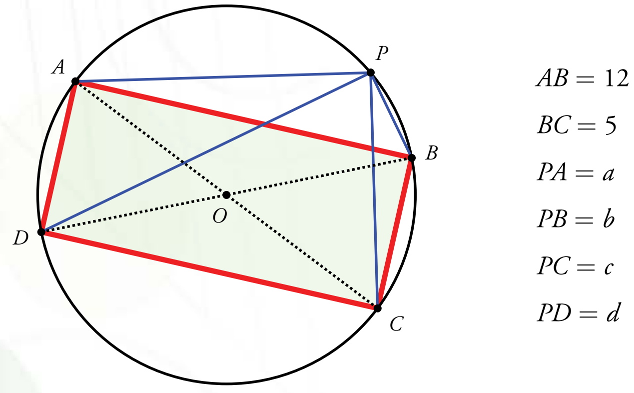


Figure 1.

Keywords: Ptolemy's Theorem, cyclic quadrilateral, properties of a circle, trigonometry

Since $\measuredangle APC = 90^\circ = \measuredangle BPD$, we get:

$$a^2 + c^2 = 13^2, \tag{1}$$
$$b^2 + d^2 = 13^2. \tag{2}$$

From (1) and (2), we get:

$$a^2 - b^2 + c^2 - d^2 = 0,$$
$$\therefore a^2 - b^2 = d^2 - c^2,$$
$$\therefore (a+b)(a-b) = (c+d)(d-c),$$
$$\therefore \frac{a+b}{c+d} = \frac{d-c}{a-b}. \tag{3}$$

Now apply Ptolemy's theorem (for the statement of the theorem, please refer to the appendix at the end of the article) in turn to the cyclic quadrilaterals *APBC*, *APBD*, *APCD* and *PBCD*:

$$12c = 13b + 5a, \qquad \therefore\ 13b = 12c - 5a,$$
$$12d = 13a + 5b, \qquad \therefore\ 13a = 12d - 5b,$$
$$13d = 12a + 5c,$$
$$13c = 12b + 5d.$$

By adding the above we get:

$$13(a+b+c+d) = 12(a+b+c+d) + 5(c+d-a-b),$$
$$\therefore\ 6(a+b) = 4(c+d),$$
$$\therefore\ \frac{a+b}{c+d} = \frac{2}{3}. \tag{4}$$

Recalling (3), we get:

$$\frac{a-b}{d-c} = \frac{3}{2}. \tag{5}$$

Solution using trigonometry. Let the given circle be regarded as the unit circle, with its centre at the origin. Rotating the figure as needed (see Figure 2), we may suppose that $B = (1, 0)$ and $D = (-1, 0)$.

Let $\measuredangle POB = x$ and $\measuredangle AOB = y$. (Here $0 < x < y$.) The coordinates of P, A, C are, respectively, $P = (\cos x, \sin x)$, $A = (\cos y, \sin y)$, $C = (-\cos y, -\sin y)$.

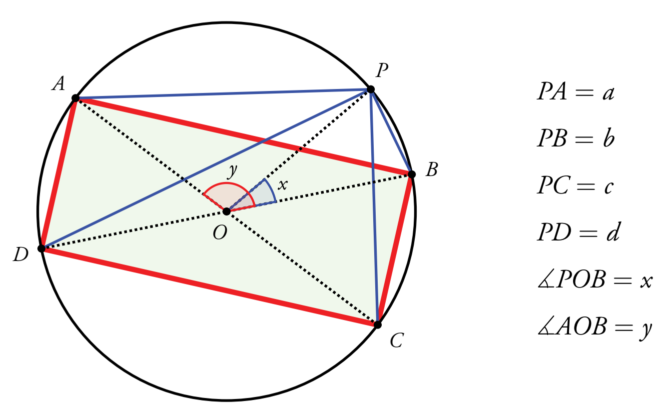


Figure 2.

Using the cosine rule, we get:

$$a = PA = \sqrt{2 - 2\cos(y-x)} = 2\sin\frac{y-x}{2}, \tag{6}$$

$$b = PB = \sqrt{2 - 2\cos(x)} = 2\sin\frac{x}{2}, \tag{7}$$

$$c = PC = \sqrt{2 + 2\cos(y-x)} = 2\cos\frac{y-x}{2}, \tag{8}$$

$$d = PD = \sqrt{2 + 2\cos(x)} = 2\cos\frac{x}{2}. \tag{9}$$

Therefore:

$$\frac{a+b}{c+d} = \frac{\sin\frac{y-x}{2} + \sin\frac{x}{2}}{\cos\frac{y-x}{2} + \cos\frac{x}{2}}$$

$$= \frac{2\sin\frac{y}{4}}{2\cos\frac{y}{4}} = \tan\frac{y}{4}. \tag{10}$$

From (10) we see that the required ratio is independent of the position of the point P.

So we opt for the most convenient choice and let $P = A$. For this choice of P, we have $a = 0$, $b = AB = 12$, $c = AC = 13$, $d = 5$. Hence:

$$\frac{a+b}{c+d} = \frac{12}{18} = \frac{2}{3}, \tag{11}$$

and a similar calculation yields (or we may use (3) and get the same result):

$$\frac{a-b}{d-c} = \cot\frac{y}{4} = \frac{3}{2}. \tag{12}$$

Appendix: Ptolemy's theorem. The theorem states that if $ABCD$ is a cyclic quadrilateral, then the following equality holds:

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

That is, *the sum of the products of opposite pairs of sides is equal to the product of the diagonals.*

References

1. C. R. Pranesachar, V Krishnamurthy, K. N. Ranganathan and B. J. Venkatachala, *Challenge and Thrill of Pre-College Mathematics*, New Age International Publication, Delhi, 2016

2. www.cmi.ac.in

PRANEETHA KALBHAVI is currently a student studying in Class 12 at The Learning Center PU College, Mangalore, Karnataka. She has deep love for Mathematics and Science. She participated in INMO 2020. Her hobbies include reading and badminton. She wishes to pursue a career in mathematics or computer science. She may be contacted at praneethakalbhavi@gmail.com.

A Functional Equation from APMO 2019

SHREYAS S ADIGA

In this article, we give a solution to Problem 1 from the Asia Pacific Mathematics Olympiad 2019:

Problem 1. Let $\mathbb{Z}^+$ be the set of positive integers. Determine all functions $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a and b.

Notation. If m and n are integers ($m \neq 0$), the notation $m \mid n$ means that m is a divisor of n. For example, $4 \mid 12$.

Solution. We use mathematical induction and show that $f(x) = x$ is the only function satisfying the given functional equation.

Let $P(a, b)$ denote the statement that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$. We argue as follows.

- $P(1, 1)$ tells us:
$$\begin{aligned} &f(1)+1 \mid f(1)^2+1,\\ \therefore\ &f(1)+1 \mid f(1)^2+1-(f(1)+1),\\ \therefore\ &f(1)+1 \mid f(1)^2-f(1),\\ \therefore\ &f(1)+1 \mid f(1)\cdot(f(1)-1). \end{aligned}$$
- Since $f(1)$ and $f(1)+1$ are coprime, $f(1)+1 \mid f(1)-1$. This implies that $f(1)+1 \mid 2$, hence $f(1) = 1$.

Keywords: Functional equation, integers, function, Asia Pacific Mathematics Olympiad

- Let the induction hypothesis be: For some positive integer $k > 1$, the relation $f(x) = x$ is true for $x = 1, 2, 3, \ldots, k-1$. We shall now prove that $f(k) = k$.
- $P(k-1, k)$ tells us (since $f(x) = x$ for $x < k$, by assumption):
$$2k - 1 \mid (k-1)^2 + (k-1)f(k),$$
$$\therefore\ 2k - 1 \mid (k-1) \cdot (k - 1 + f(k)).$$
Since $\gcd(2k-1, k-1) = \gcd(2k-1, 2k-2) = 1$, it follows that
$$2k - 1 \mid k - 1 + f(k). \qquad (1)$$
- Next, $P(k, k-1)$ tells us:
$$f(k) + k - 1 \mid k^2 + (k-1)f(k),$$
$$\therefore\ f(k) + k - 1 \mid k^2 + (k-1)f(k) - (k-1) \cdot (f(k) + k - 1),$$
$$\therefore\ f(k) + k - 1 \mid k^2 - (k-1)^2,$$
$$\therefore\ f(k) + k - 1 \mid 2k - 1. \qquad (2)$$
- From (1) and (2) we get $f(k) + (k-1) = 2k - 1$, so $f(k) = k$. We have thus completed the induction step.

It follows that the only function satisfying the given functional equation is $f(x) = x$. $\square$

SHREYAS S ADIGA is a student in Class 12 at The Learning Center PU College, Mangalore, Karnataka. He has deep love for Problem Solving in Mathematics, especially Geometry and Number Theory. He received a Winner Diploma in the 16th I F Sharygin Geometry Olympiad, and he cleared the Indian National Mathematical Olympiad (INMO) in 2020-21. He also cleared the American Mathematics Contest (AMC 10) in 2020-21. He has a keen interest in chess. He may be contacted at ssa135228@gmail.com.

A Relation between Polynomial and Exponential Functions

SOHAM PURKAIT

In this note, we hit upon an interesting property of polynomial functions that mimic the behavior of exponential functions.

Statement of the problem. Consider the following series of questions about a polynomial $P(x)$ which mimics the behaviour of an exponential function.

(1) Suppose that P is quadratic, and $P(x) = 2^x$ for $x = 0, 1, 2$. What is the value of $P(3)$?

(2) Suppose that P is cubic, and $P(x) = 2^x$ for $x = 0, 1, 2, 3$. What is the value of $P(4)$?

(3) Suppose that P is quadratic, and $P(x) = 3^x$ for $x = 0, 1, 2$. What is the value of $P(3)$?

(4) Suppose that P is cubic, and $P(x) = 3^x$ for $x = 0, 1, 2, 3$. What is the value of $P(4)$?

(5) Suppose that P is a polynomial in x of degree n, and $P(x) = 2^x$ for $x = 0, 1, 2, \ldots, n$. What is the value of $P(n + 1)$?

(6) Suppose that P is a polynomial in x of degree n, and $P(x) = 3^x$ for $x = 0, 1, 2, \ldots, n$. What is the value of $P(n + 1)$?

We shall prove an elegant result here which answers all such questions at once.

Theorem. Let $P(x)$ be a polynomial in x of degree n. Suppose that $P(x) = a^x$ for $x = 0, 1, 2, \ldots, n$, for some number a. Then $P(n + 1) = a^{n+1} - (a - 1)^{n+1}$.

So if P is quadratic, and $P(x) = 3^x$ for $x = 0, 1, 2$, then (going by the theorem) the value of $P(3)$ is $3^3 - 2^3 = 19$.

Keywords: Polynomial function, exponential function

Let us check this claim the 'long way' – by actually computing the expression for $P(x)$. Let $P(x) = a + bx + cx^2$. Then we have:

$$\begin{aligned} a &= 3^0 = 1, \\ a + b + c &= 3^1 = 3, \\ a + 2b + 4c &= 3^2 = 9. \end{aligned}$$

The first two equations yield $b + c = 2$, and the first and third equations yield $2b + 4c = 8$. From these we get $c = 2$ and $b = 0$. Hence $P(x) = 1 + 2x^2$, so $P(3) = 19$, as claimed.

Proof. We shall prove the claim using the principle of induction, by induction on the degree of the polynomial.

Let us first establish the claim for polynomials of degree 1 (i.e., linear polynomials). Let $P(x)$ be a polynomial in x of degree 1, and suppose that $P(0) = 1$, $P(1) = a$. Then the claim is that $P(2) = a^2 - (a-1)^2$, i.e., $P(2) = 2a - 1$. To prove this, note that since $P(x)$ is of degree 1, we have

$$P(2) - P(1) = P(1) - P(0), \qquad \therefore\ P(2) = 2P(1) - P(0) = 2a - 1,$$

as claimed.

The key result used is the following.

Lemma. *Let $f(x)$ be a polynomial in x of degree n, where n is a positive integer. Define $g(x)$ by:*

$$g(x) = f(x+1) - f(x). \tag{1}$$

Then $g(x)$ is a polynomial in x of degree $n - 1$.

For example, if $f(x) = x^3$, a polynomial of degree 3, then $g(x) = (x+1)^3 - x^3 = 3x^2 + 3x + 1$, a polynomial of degree 2. (Editor's note. It should be clear why this is true: the highest degree term in $f(x)$ gets cancelled as a result of the subtraction, so the degree of g is lower than that of f. In fact, the degree of g is $n - 1$, which is 1 lower than the degree of f. We shall say more about this lemma in the appendix.)

The induction hypothesis is the following.

> Suppose that $f(x)$ is a polynomial in x of degree k (a positive integer), such that $f(x) = a^x$ for $x = 0, 1, 2, \ldots, k$, for some a. Then $f(k+1) = a^{k+1} - (a-1)^{k+1}$.

To prove the induction step, we must assume the above and prove the following.

> Suppose that $g(x)$ is a polynomial in x of degree $k + 1$, such that $g(x) = a^x$ for $x = 0, 1, 2, \ldots, k, k+1$, for some a. Then $g(k+2) = a^{k+2} - (a-1)^{k+2}$.

Proof of the induction step. Let $g(x)$ be a polynomial in x with the stated properties: it has degree $k + 1$, and $g(x) = a^x$ for $x = 0, 1, 2, \ldots, k, k+1$, for some a. We must compute the value of $g(k+2)$. Now consider the function $h(x) = g(x+1) - g(x)$. We have:

$$\begin{aligned} h(0) &= g(1) - g(0) = a - 1, \\ h(1) &= g(2) - g(1) = a^2 - a = a(a-1), \\ h(2) &= g(3) - g(2) = a^3 - a^2 = a^2(a-1), \\ h(3) &= g(4) - g(3) = a^4 - a^3 = a^3(a-1), \end{aligned}$$

and in general:

$$h(x) = a^x(a-1) \qquad \text{for } x = 0, 1, 2, \ldots, k. \tag{2}$$

Define

$$f(x) = \frac{h(x)}{a-1}. \tag{3}$$

Then the function f satisfies the conditions of the induction hypothesis: it is a polynomial in x of degree k (by the lemma), and $f(x) = a^x$ for $x = 0, 1, 2, \ldots, k$. Hence $f(k+1) = a^{k+1} - (a-1)^{k+1}$.

Therefore we have, using the definitions of h and g:

$$\begin{aligned} h(k+1) &= (a-1)\cdot f(k+1) \\ &= (a-1)\cdot\left(a^{k+1} - (a-1)^{k+1}\right), \\ g(k+2) &= g(k+1) + h(k+1) \\ &= a^{k+1} + (a-1)\cdot\left(a^{k+1} - (a-1)^{k+1}\right) \\ &= a^{k+2} - (a-1)^{k+2}, \end{aligned}$$

as required. This proves the induction hypothesis. Hence the theorem is proved. □

Proof of the lemma

We must prove that if $f(x)$ is a polynomial of degree n, where n is a positive integer, and $g(x)$ is defined by $g(x) = f(x+1) - f(x)$, then $g(x)$ is a polynomial of degree $n-1$.

Proof

Let

$$f(x) = ax^n + bx^{n-1} + \cdots,$$

where $a \neq 0$. Then:

$$\begin{aligned} g(x) &= f(x+1) - f(x) \\ &= \left[a(x+1)^n + b(x+1)^{n-1} + \cdots\right] - \left[ax^n + bx^{n-1} + \cdots\right] \\ &= \left[a(x+1)^n - ax^n\right] + \left[b(x+1)^{n-1} - bx^{n-1}\right] + \cdots. \end{aligned}$$

In the expression $b(x+1)^{n-1} - bx^{n-1}$, the terms involving x^{n-1} cancel out, so the degree of that portion is less than $n-1$.

In the expression $a(x+1)^n - ax^n$, the terms involving x^n cancel out. The term involving x^{n-1} is ax^{n-1}, and this term survives, since $a \neq 0$ by assumption.

Hence the degree of g is $n-1$. □

Box 1

SOHAM PURKAIT is currently a student in Class 11 of Sarada Vidyapith, West Bengal. He has a passion for problem solving in mathematics and physics. He also has a keen interest in Rubik's cube, coding, and sketching. He has qualified for Indian National Mathematical Olympiad (INMO) 2020. He secured a gold medal in the Kangaroo Mathematical Olympiad, and a silver medal in the Southeast Asian Mathematical Olympiad. He may be contacted at mathematicsandfun1729@gmail.com.

Playing with Turtle Blocks

MUJAHIDUL ISLAM

Turtle Blocks is an Open Digital Canvas, especially designed for children. Here they can explore mathematical concepts through play. It functions as a digital playground where children can connect naturally with mathematics and perhaps even begin to appreciate what they have studied in their mathematics textbooks. Turtle Blocks has been developed on the lines of Logo Programming which was created at MIT Media Lab way back in the 1960s. It can be accessed on different devices such as desktop computers, laptops, tablets and smartphones.

One cannot talk about Logo programming without mentioning the name of Seymour Papert, to whom we owe the term *computational thinking*[1] and who had truly propounded the constructionist theory of learning. Papert had worked closely with Jean Piaget and had developed what we know today as *constructionism*, a learning theory based on Piaget's constructivism. According to Ackerman [1], the thrust of the constructionist theory is on learning by doing with one's own hands and helps us understand "how ideas get formed and transformed when expressed through different media, when actualized in particular contexts, when worked out by individual minds."

At MIT Media Lab, Papert had created Logo Programming with a physical turtle robot whose movement could be controlled through a given set of instructions. He had envisioned an open space where children could experience mathematics and he decided to call it Mathland [3]. Mathland can, in fact, be any space, not necessarily virtual, where learners can play with objects and experience mathematics in a natural setting. However, the digital space affords many possibilities which may be difficult to experience in the finite analogue world.

Keywords: learning, play, reasoning, logic, computational thinking

In this manner, Turtle and Logo programming was simulated digitally and now we can see it in the form of Turtle Programming Software [5]. In this software, there are virtual programming blocks (similar to Lego blocks in the tangible mode) which can be arranged as a set of instructions to guide the movement of the turtle on the computer screen. Thus, if the user wishes the turtle to move in a straight line or in a circle, the programming blocks can be arranged accordingly. When the turtle moves, it traces out its path in a color (chosen by the user) thus creating a shape. In this manner, one can draw many shapes and patterns by giving suitable instructions. The very process of articulating the instructions to create a specific shape with certain properties elicits mathematical thinking. Figure 1 illustrates the four main programming blocks which make the turtle move in a specific direction. Each block has a number associated with it which regulates the movement of the block in a specific direction and by a certain distance.

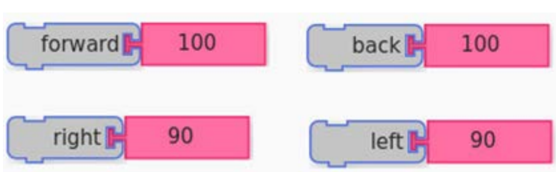
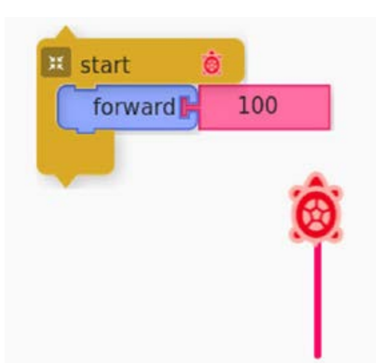


Figure 1. The Turtle Programming Blocks

The forward and backward commands are used to move the turtle in a straight line. Thus forward 100 means that the turtle will move 100 steps forward and backward 100 means it will move 100 steps backward. Similarly the right and left commands will rotate the turtle right or left on its axis. Hence, right 90 means the turtle will turn right through 90 degrees and left 90 will mean the turtle will turn left through 90 degrees.



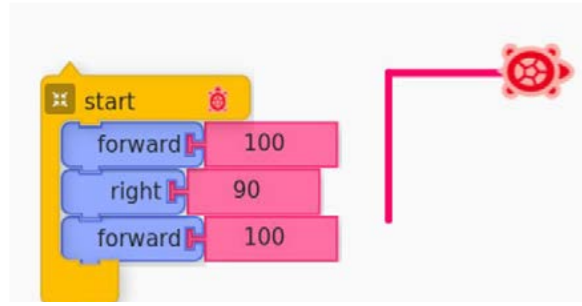


Figure 2. The movement of the turtle specified by the turtle programming blocks.

Why mathematical thinking?

Let us see how a rectangle can be drawn using the programming blocks. First, we may instruct the turtle to move in a straight line for a specified distance (say 100 units), then turn 90 degrees to the right. This will be followed by the turtle moving again in a straight line for a specified distance (say 50), followed by making a right turn. This is repeated until the turtle reaches the starting point. In this manner, one can create many shapes and experience their mathematical properties as well.

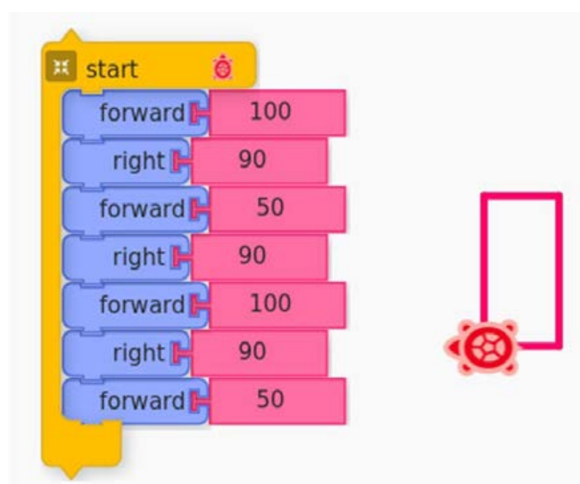


Figure 3. Instructions to the turtle block to trace out a rectangle.

Turtle Blocks may be explored from a pedagogical perspective. I had an opportunity of exploring the potential of this tool when I introduced it to my eight year old daughter. In fact, Turtle Blocks can be introduced to very young children. My daughter thoroughly enjoyed seeing the turtle 'follow' her instructions.

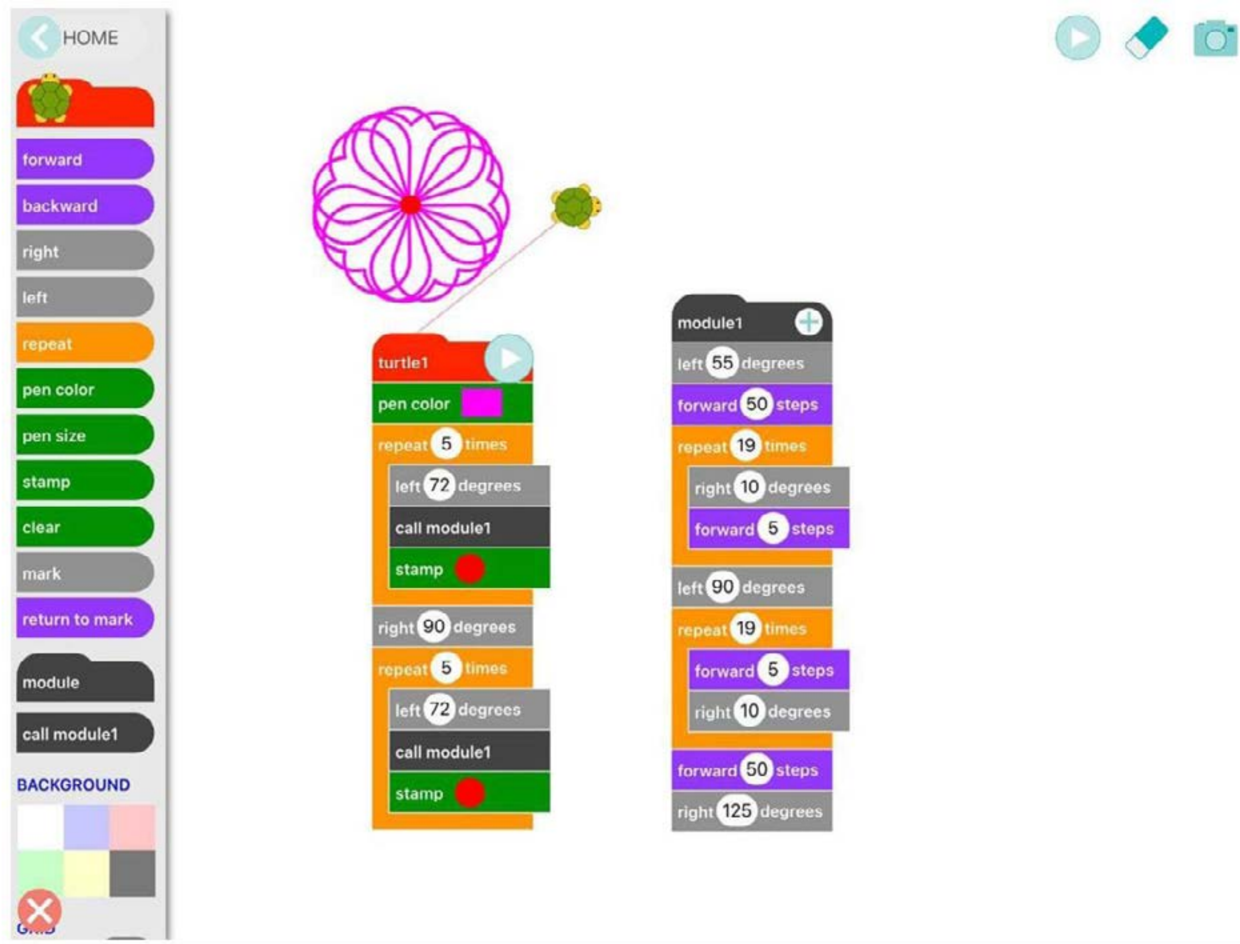


Figure 4. Complex patterns created using the Turtle Blocks.

Programming the turtle to make it draw a pattern of her choice gave her a sense of control and also fueled her imagination to create more shapes and patterns. It also motivated her to think about the shapes and patterns mathematically.

Open Digital Canvas or Playground

Unlike most digital games and platforms, Turtle Blocks is like a blank notebook where the child can put her imagination to work. She can decide on the shape or pattern she wishes to create and then use her mathematical thinking to program the blocks to trace out the required shape. The tool is open ended in the sense that it functions as a playground with certain objects which the child can use to play with. This leads her to engage creatively with the tool rather than play with ready made objects.

After the initial stage of playing with the tool, the child creates her own challenges and finds new problems to explore. The experience is not only fun filled but also an enriching one. However, initiation into the tool many need some scaffolding.

Problem Solving while playing

While playing with Turtle Blocks a child may begin by creating simple shapes but soon she will be encouraged to delve into more complex shapes and patterns. Figure 6 illustrates some complex designs which may be created using Turtle Blocks.

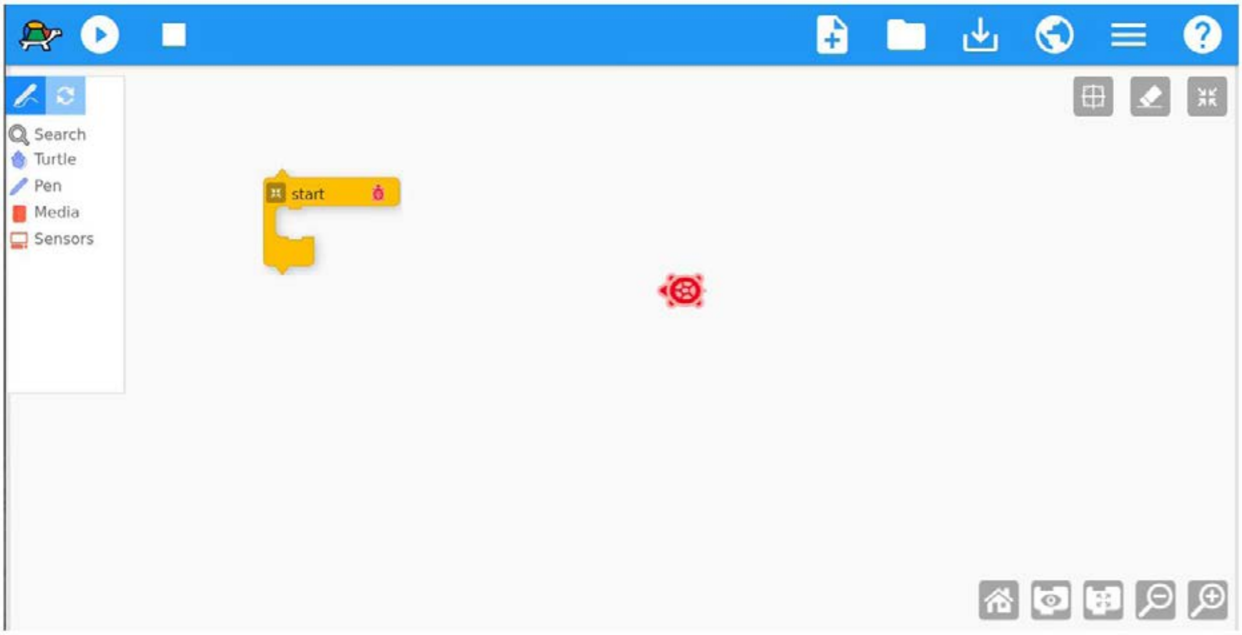


Figure 5. Turtle Blocks when launched will display an open canvas.

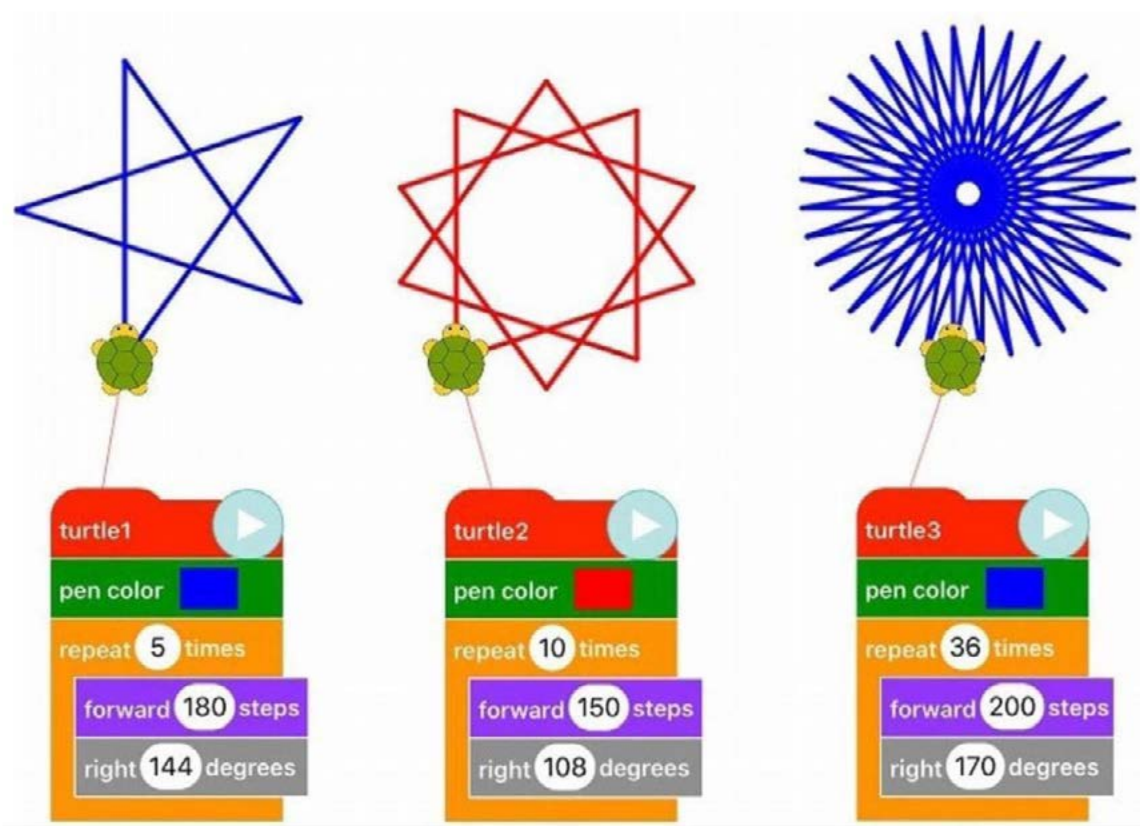


Figure 6. Examples of patterns generated using Turtle Blocks

The software doesn't provide readymade problems to be solved. However, it encourages the user to think about new patterns once they have learnt to create simple shapes. For creating more complex patterns, such as the ones illustrated in Figure 6, the user needs to create a new set of instructions. A method of trial and error which entails refining a set of instuctions to obtain the intended pattern is employed. Being able to accomplish the task of creating a specific shape leads to a sense of achievement. I have seen this in my own daughter as she gradually moved from simple shapes to more interesting and compex shapes. Figure 7 illustrates an example where she first drew a circle and then used the code to create a pattern of intersecting circles.

Figure 7 illustrates how circles have been drawn repeatedly to form a pattern. Each circle comprises four colours as each quarter of the circumference has a different colour. As there are five circles in the above pattern, we can see the module (or subroutine) being called five times in the main program. The module contains the instructions for drawing a circle using four colours. It instructs the turtle to trace one-fourth of the circle in red, another one-fourth in blue and so on. After performing all the steps in the module, one can see a complete circle emerge in four different colors. Once the first circle is drawn, the position of the turtle is shifted to the starting point of the new circle and the subroutine for the circle is repeated. After five repetitions a beautiful pattern of overlapping circles emerges.

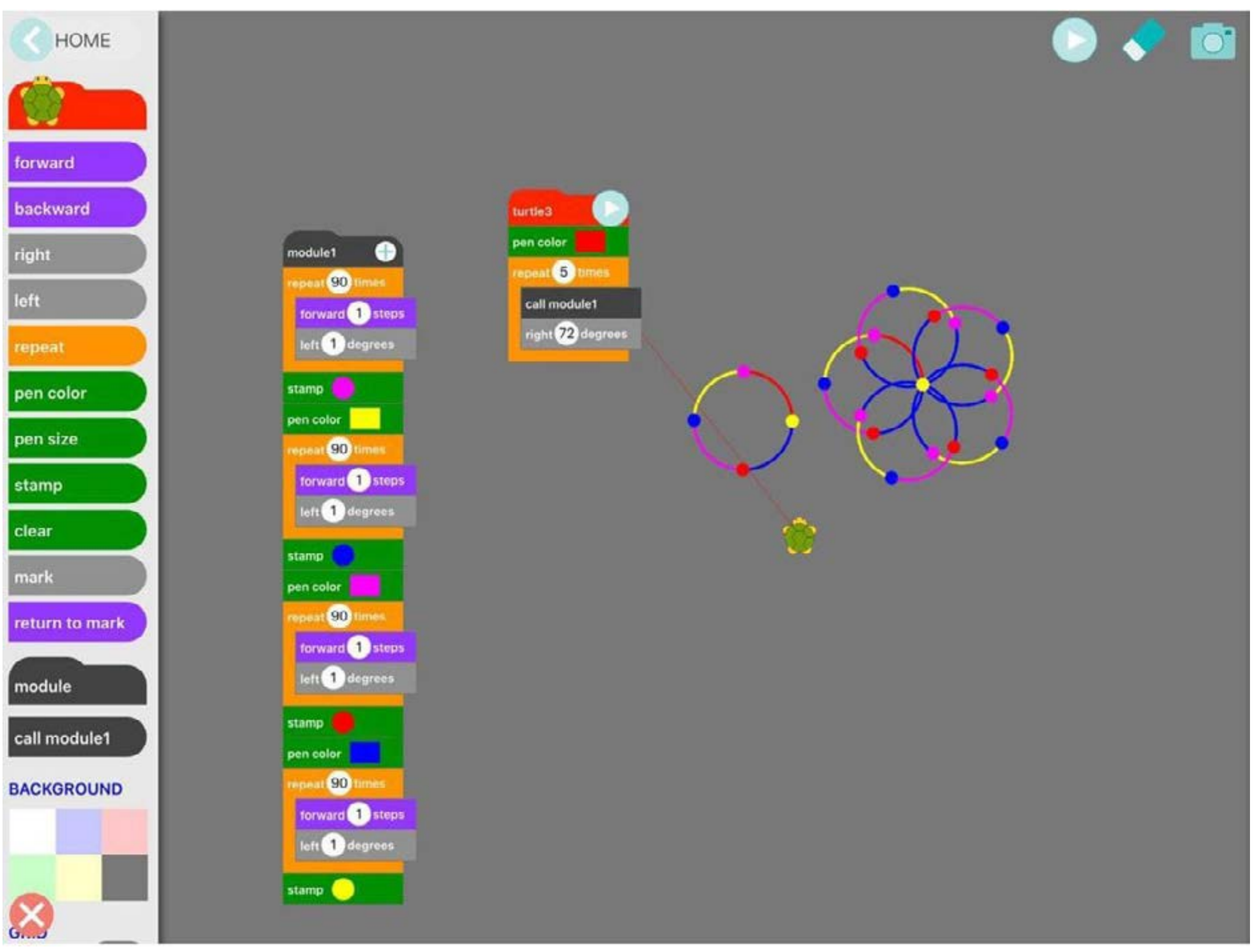


Figure 7. A pattern of intersecting circles using Turtle Blocks.

Abstracting concepts in a natural setting

Drawing a circle using Turtle Blocks may lead to the intuitive understanding of the degree as an angle measure. This happened with my daughter as well. She abstracted the idea that the the amount of rotation acquired by the turtle is determined by the degree of rotation. This helped her to develop an intuitive understanding of the concept of angle and its measure even before encountering the concept in class.

Creating your own algorithm

The Turtle Block encourages children to pose challenges to themselves. Further, the solution to a problem may be achieved in multiple ways and this leads to a learning experience widely different from the drill and practice routine of a typical mathematics classroom.

Every time the child encounters a new challenge, she is required to devise her own algorithm to address the problem. Very often strategies such as ‘trial and error’, ‘building from examples’ and ‘breaking a problem into simpler problems’ may be adopted. Figure 8 illustrates how an unsual and intricate pattern was created using repetitions of a subroutine.

Breaking a problem into simpler problems

Breaking down a problem into simpler problems is an important and well known problem solving strategy. I have observed my daughter using this approach too. In the examples given in Figures 7 and 9 we observed how the larger program consists of smaller subroutines. A complex pattern may actually be made up of several simple patterns which may be coded as subroutines. One

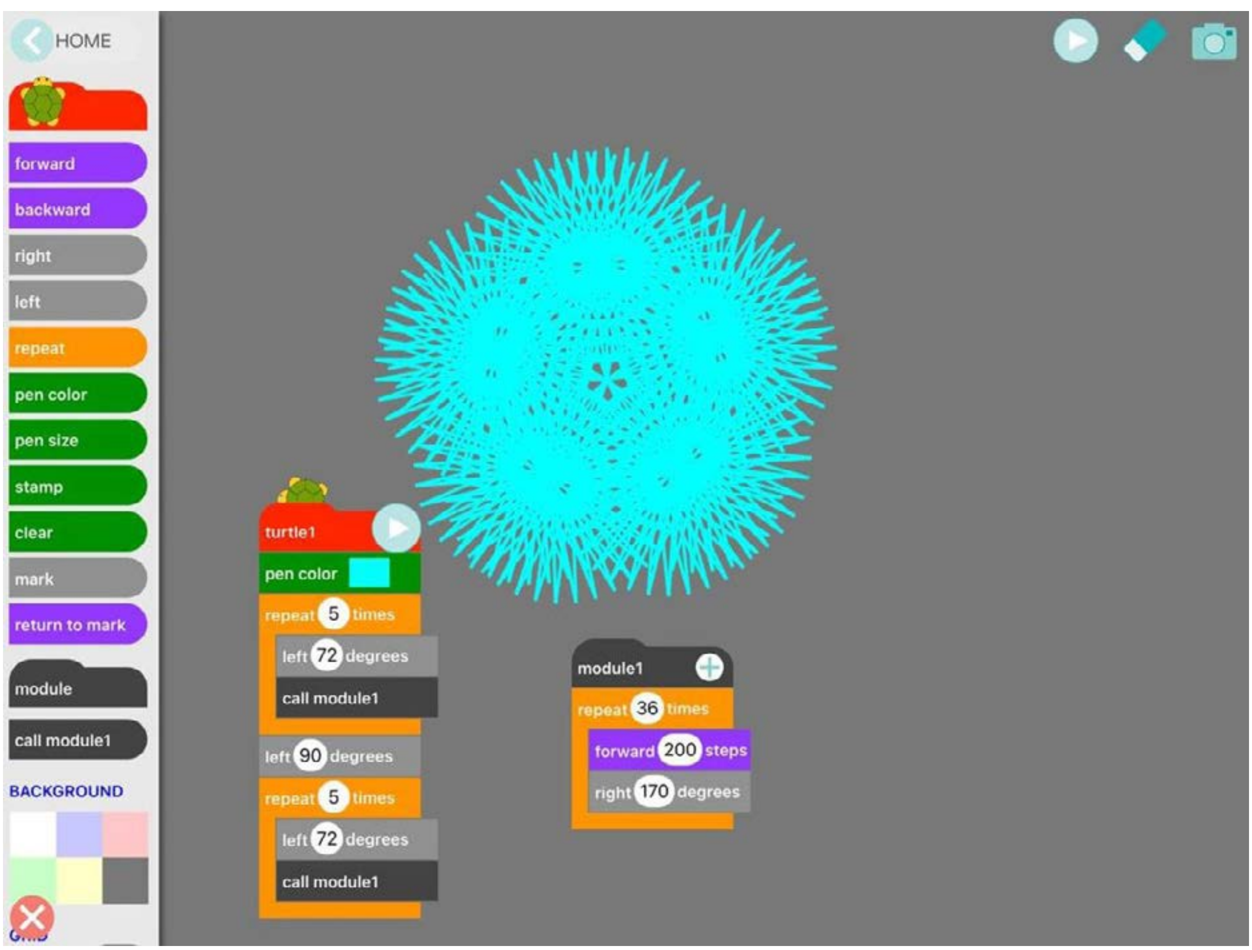


Figure 8: A pattern created by incorporating a subroutine in the main program.

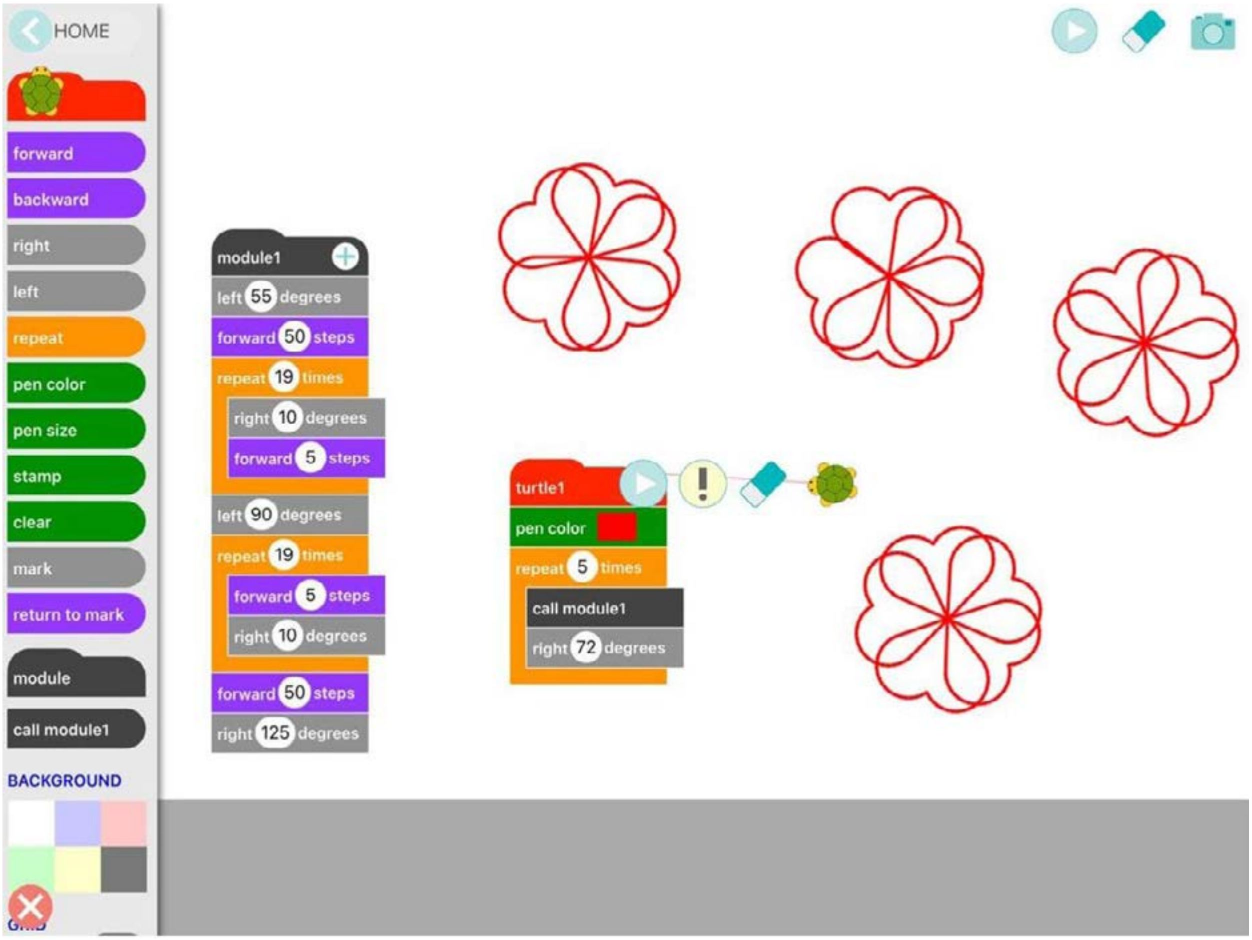


Figure 9. An example of a pattern drawn by trial and error

approach is to set up subroutines for the simpler tasks and then combine them to form the larger (main) program.

Learning from mistakes

When children play with Turtle Blocks, making mistakes may actually lead them to explore further and think about the process involved in obtaining the desired pattern. Every time a child fails to make the turtle draw the desired figure she has to revisit and rectify her algorithm. Thus rectifying errors can lead to interesting challenges in themselves.

The pattern in Figure 9 illustrates how the method of trial and error can be used to arrive at a symmetrical pattern. Module1 in the figure comprises instructions for drawing a heart. My daughter tried to create a pattern of overlapping hearts using this module. In the first attempt, she called Module1 five times, selecting the rotation angle as 63 degrees. However, this resulted in an asymmetrical pattern. Note that the main program calls the module for drawing a heart multiple times and each time the heart rotates by some angle.

In the second attempt, she entered the rotation angle as 69 degrees but this choice also didn't yield the desired result. Finally selecting the angle as 72 degrees led to the beautiful symmetrical pattern shown in the extreme right of Figure 9. This eureka moment also led to a discussion as to why the choice of 72 degrees led to the required pattern. How can one arrive at 72 degrees? Some scaffolding led her to conjecture that 360 degrees divided by 5 is 72. Hence the choice of rotation by 72 degrees leads to hearts being arranged in a symmetrical manner.

Appreciating the mathematics of shapes and patterns

An appreciation of mathematical properties of patterns comes naturally when young children use Turtle Blocks. They tend to extend these ideas in daily life as well. I have observed my daughter looking at patterns around her and trying to create codes using Turtle Blocks to replicate them. We also discuss different ways of creating patterns thus analysing them from a mathematical point of view. One of the important processes of learning mathematics is about observing patterns, not only in geometry, but also in numbers and their representations, algebra, trigonometry and various other topics of the school curriculum. Turtle Blocks help to initiate an exploration of geometrical patterns in young children.

Multiple ways of approaching a problem

Using Turtle Blocks, one can challenge oneself to draw a shape or pattern using different approaches. Sometimes a particular approach may reduce or increase the number of steps involved in a set of instructions. For example, while drawing a rectangle one can observe a sequence of steps being repeated. These steps may be incorporated in the form of a loop within the code of the main program. Loops help to modify programs thus enabling the user to try different approaches to arrive at the solution. The process of modifying codes to make them more concise is also a great learning opportunity for young children. I have often observed with great satisfaction how my daughter explains the codes created by her and then modifies them herself to obtain a different set of instructions. This opportunity provided

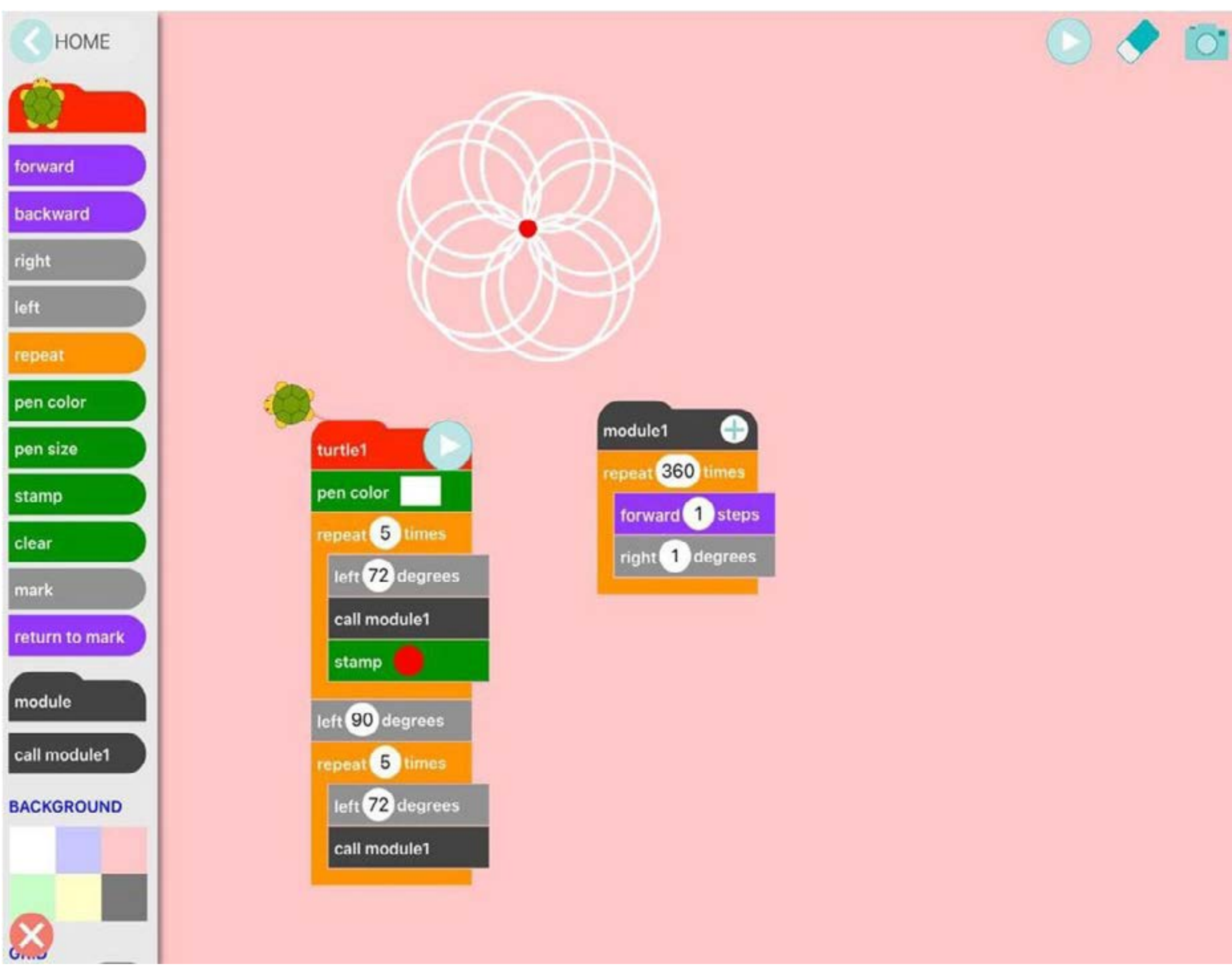


Figure 10: How patterns emerge from circles

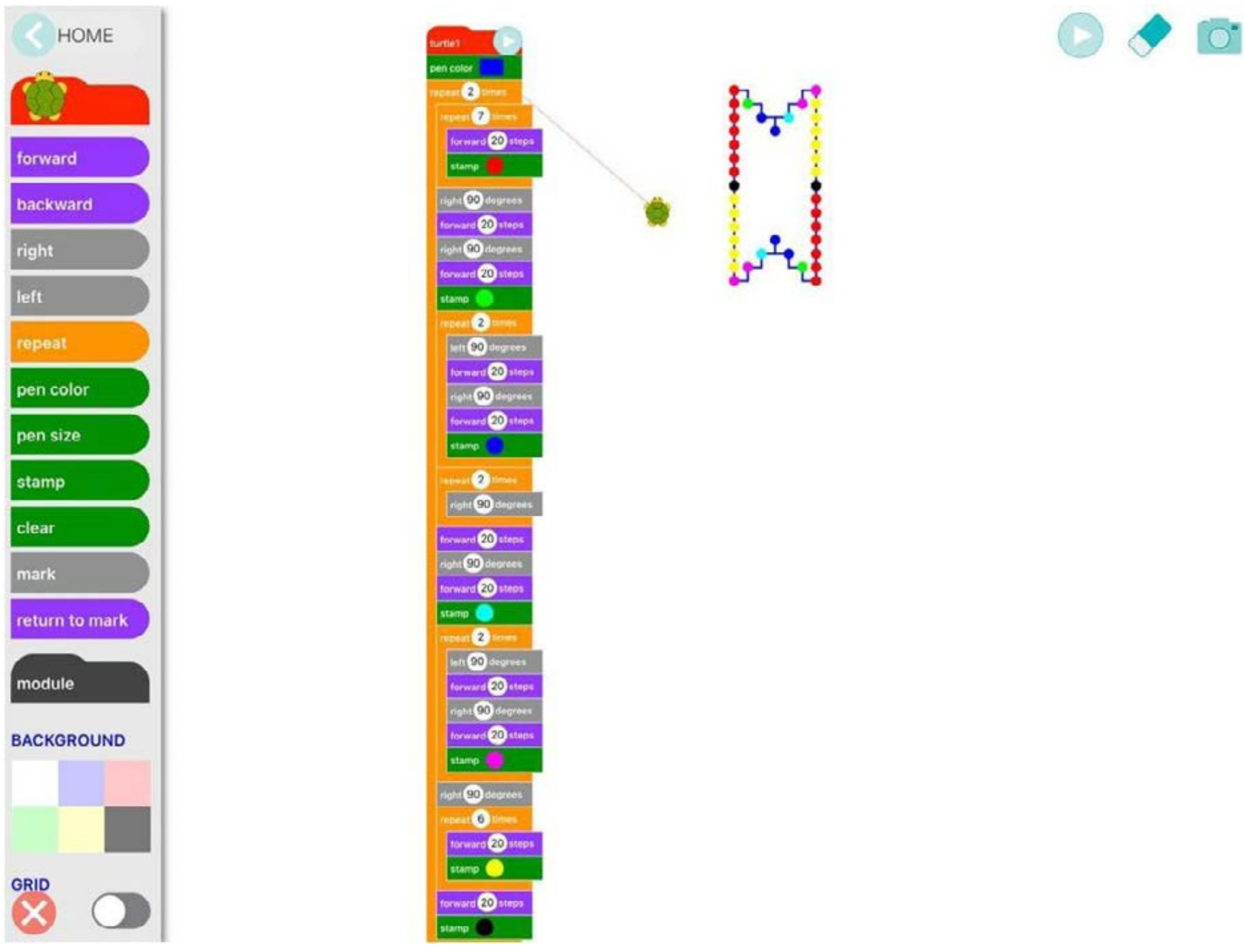


Figure 11: A symmetrical pattern obtained using Turtle Blocks.

by the tool also enables the learner to look at problem solving in different ways.

Exploring symmetry

Using Turtle Blocks can lead to drawing shapes and patterns which are symmetric. I have observed that whenever my daughter creates patterns, they have symmetry in some form. If the symmetry is broken, she modifies her instruction set until she is able to visualise some symmetry. Figure 11 illustrates an example of a symmetric pattern created through free exploration.

Is Turtle Blocks about coding?

Coding is the buzzword these days and it might appear that Turtle Blocks is yet another Software designed for learning programming. Although it does involve programming it is quite different from the plethora of commercial tools available for introducing children to programming techniques. It encourages the user to develop an intuitive understanding about patterns as well as engage in problem solving. It inculcates the skills required for writing codes to create new patterns or solve problems. In fact one might say that such a tool can help to develop computational thinking which entails the processes of decomposition, abstraction, developing algorithms and debugging codes. According to Jenette Wing [4]

Computational thinking is a fundamental skill for everyone, not just for computer scientists. To reading, writing, and arithmetic, we should add computational thinking to every child's analytical ability.

Wing [2,4] further articulates that computational thinking entails thinking like a computer scientist which "means more than being able to program a computer. It requires thinking at multiple levels of abstraction." Thus computational thinking is not merely about writing codes but more about understanding how machines work and about how the computer can be used to solve problems efficiently.

Turtle Blocks may be a very effective tool in enabling young children to engage in computational thinking. The tool is open ended in nature and offers a canvas of possibilities. This is almost similar to handing a canvas and a paintbrush to a child. Knowing how to program a computing machine has been recognised as an important 21st century skill. A tool such as Turtle Blocks can indeed provide a natural way to engage in Computational Thinking.

Sites and Apps to play with Turtle Blocks

1. https://turtle.sugarlabs.org/
2. For iOS - https://apps.apple.com/us/app/turtles-learn-to-code-for-fun/id1454902715
3. Javascript/HTML5 port of Turtle Blocks: https://github.com/sugarlabs/turtleblocksjs
4. https://blockly.games/turtle?lang=en

References

[1] Ackermann, Edith K. (2004): *Constructing Knowledge And Transforming The world, A learning zone of one's own: Sharing representations and flow in collaborative learning environments* [M. Tokoro and L.Steels (Eds.). Amsterdam, Berlin, Oxford, Tokyo, Washington, DC. IOS Press, 2004. Part 1. Chapt 2. pp. 15-37.

[2] Cuny, J., Snyder, L., & Wing, J.M. (2010): Demystifying computational thinking for noncomputer scientists. Unpublished manuscript in progress, referenced in http://www.cs.cmu.edu/~CompThink/resources/TheLinkWing.pdf

[3] Papert, S. (1980): *Mindstorms. Children, Computers and Powerful Ideas*. New York: Basic books.

[4] Wing J.M. (2006): *Computational thinking. Communications of the ACM*, 49(3), 33-35.

[5] https://el.media.mit.edu/logo-foundation/what_is_logo/history.html

MUJAHIDUL ISLAM is an Edtech Specialist who works at the crossroads of technology, media and education with a special emphasis on digital ways of learning. He researches, collaborates and executes technological interventions that promote equitable access in the Edtech space. He actively constructs learning experiences through making and tinkering, with the belief that mathematics is a language of finding, creating and expressing patterns. He may be contacted at mujahiddbg@gmail.com.

Review: Counters

Reviewed by Math Space

Simply put, anything that can be counted may be considered to be counters. So, pebbles, chalk pieces, bottle caps, seeds, buttons – all these are familiar objects which can be utilised as counters. Square counters can be made easily from (corrugated) cardboard boxes by drawing a square grid with squares of side at least 2cm and then cutting it up. Cutters (with steel scales) work much faster for corrugated cardboard than scissors. Making round counters from cardboard is more tedious but not impossible (with difficulty decreasing as size increases). Carrom coins, bottle caps (ideally of the same size), tooth paste caps are great as round counters. It is advisable to paint the counters in bright colours especially for younger children.

Counters can help a child establish a one-to-one correspondence between the quantity (count) and the numeral or number name that represents that quantity (Figure 1). They are the first manipulatives that any child should encounter. Counters can help one focus on the quantity and not how spread apart they are (Figure 2). As manipulatives, counters can be picked up (and set apart in a bowl or basket) – thereby making it easier to separate out what has been counted from what is yet to be counted. In fact, one may say that without adequate practice of counting, a child will not be able to have a strong foundation of numbers. Therefore, counters are very crucial at the beginning.

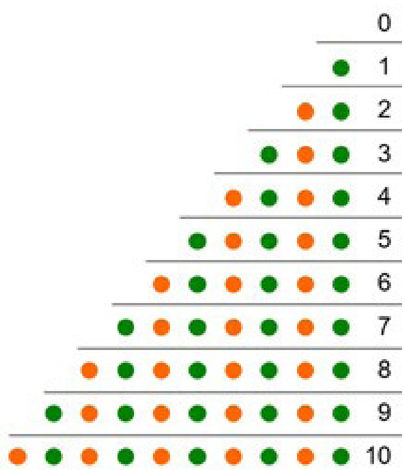


Figure 1

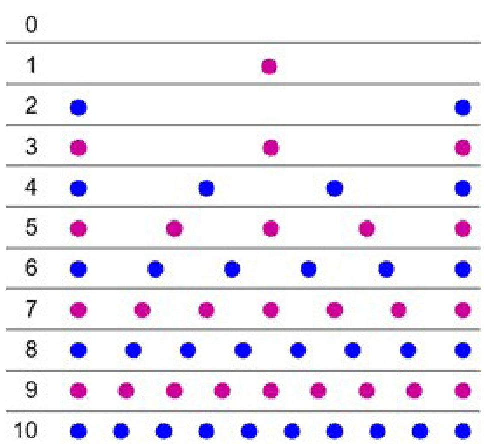


Figure 2

However, the advantages of counters are not limited to just counting. They help equally in terms of identifying any pattern. Therefore, they also provide a great starting point for algebraic thinking and algebra in general. Growing (Figure 3) and repeating (Figure 4) patterns can initiate a discussion around algebraic expressions and equations.

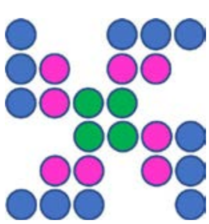


Figure 3

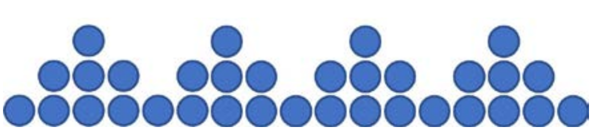


Figure 4

Moreover, they illustrate properties of addition and multiplication beautifully for discrete number sets, viz., whole numbers and integers. Here are some examples:

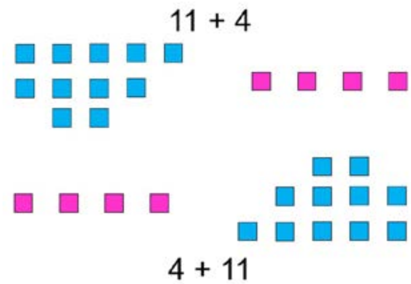


Figure 5

Figure 5 depicts the same sum from two perspectives while Figure 6 illustrates that no matter which two groups are combined first, the sum remains the same (see reference [2] for further details). For integers, since there are two types of numbers – positive and negative, we need two types of counters. Basically, the number of counters indicates the distance of the integer from zero, i.e., the absolute value and the colour (or type) indicates which side of zero it is on, i.e., the sign. Figure 7 and Figure 8 illustrates this with 3 and -2, respectively. Figure 5 and Figure 6 can be extended in a similar manner for integers with such coloured counters (see reference [3] for further details).

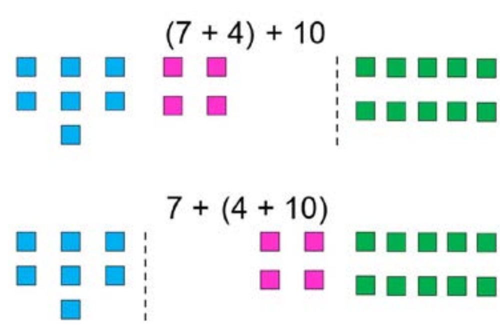


Figure 6

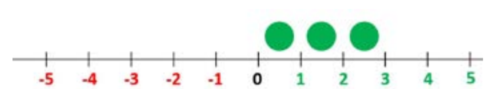


Figure 7

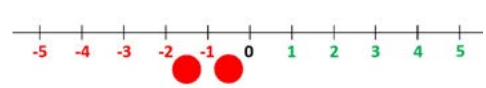


Figure 8

Figure 9 illustrates commutativity of multiplication for whole numbers. The same array depicts two products on changing its orientation. Figure 10 and Figure 11 on the other hand show the distributive property for whole numbers and for integers, respectively (see references [2] and [3] for further details).

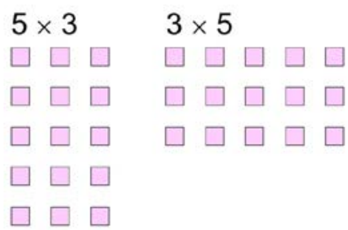


Figure 9

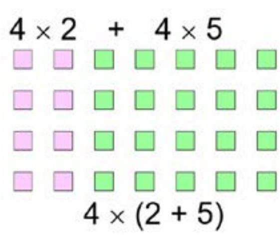


Figure 10

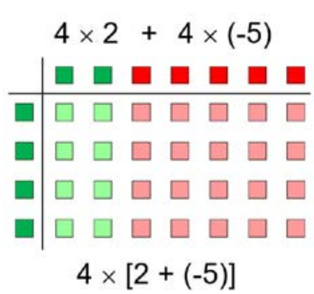


Figure 11

They also help in finding factors, characterising the primes including why 1 is not a prime, and understanding square numbers! Figure 12 depicts all possible arrays with 12 counters. Note that the number of rows (or columns) in each array maps to a factor of 12 and thus the number of possible arrays equals the number of factors.

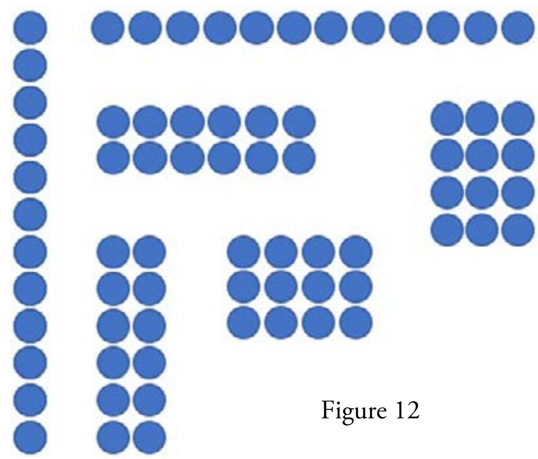


Figure 12

Note how the arrays are portrait-landscape pairs. So, what happens for squares?

Number	No. of arrays	Number	No. of arrays	Observations
1	1	9	3	• Which numbers have exactly 2 possible arrays? • Which numbers have more than 2 arrays? • So, is 1 a prime? Is it composite? • Which numbers have odd number of arrays?
2	2	10	4	
3	2	11	2	
4	3	12	6	
5	2	13	2	
6	4	14	4	
7	2	15	4	
8	4	16	5	

The other proof of 'why square numbers have odd number of factors' involves the Fundamental Theorem of Arithmetic (or the unique prime factorization theorem) and uses algebra (and is therefore much more demanding)!

The observation column scaffolds the path to abstraction of concepts as students begin to extend their thinking to pictorial representations and then mental images.

In addition, counters are great for generating Proofs Without Words (PWW) for results involving natural numbers. A resource group has in fact made a model of Gauss's famous childhood discovery of sum of natural numbers! Figure 13 shows the series of odd numbers adding up to a square. In fact, it should be possible to create PWW for every result involving natural numbers using counters!

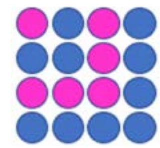
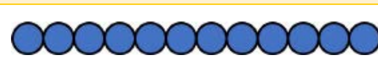


Figure 13

When we go into such higher and exciting levels of math, it helps if the counters are identical. There are two popular shapes – circle and square – each with their own pros and cons.

	Circle	**Square**
Pros	Most symmetric shape ∴ independent of orientation ⇒ best for very young children ⇒ best for patterns since it gives maximum freedom Also, when lined up, they are easier to count than square counters because of the gap between counters	A regular shape that tiles ∴ great to generate anything based on the square grid, e.g., polyominoes Unit of area ∴ great to explore area and perimeter ⇒ when lined up forms a rectangle whose area is proportionate to its length ∴ can be generalized to length
Cons	Does not tile	When arranged in a line forming a rectangle, it may be difficult to count them unless the borders are prominent Unless arranged properly, they may not look nice or neat Less symmetry compared to circles ∴ certain symmetry may not be clear if square tiles are used instead of circular ones
Picture		

References

[1] Multiplication Pullout, *At Right Angles* (Mar 2014)

[2] Exploring Properties of Addition with Whole Numbers and Fractions, *At Right Angles* (Jul 2018)

[3] Exploring Properties of Addition and Multiplication with Integers, *At Right Angles* (Jul 2019)

[4] Nelson, Roger B: *Proof Without Words* (Volume 1)

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] both in terms of uses and regarding possibility of low-cost versions that can be made from boxes, etc. It tries to address both fear and dislike for mathematics as well as provide food for thought to those who like or love the subject. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space may be reached at mathspace@apu.edu.in

Mini Review:
Bayes' Theorem and Independence by 3Blue1Brown

Reviewed by Swati Sircar

Bayes' theorem (15:45min):
https://www.youtube.com/watch?v=HZGCoVF3YvM

The quick proof of Bayes' theorem (3:47min):
https://www.youtube.com/watch?v=U_85TaXbeIo

3Blue1Brown is a YouTube channel that discusses various topics and problems from undergraduate level mathematics. The key features are very good visuals – animated pictures and diagrams that illustrate various concepts – and balanced, logical arguments.

Bayes' Theorem is one of the most counter intuitive results in probability. Many struggle to make sense of the complicated formula of this theorem,

$$P(H|E) = \frac{(P(H) \times P(E|H))}{P(H) \times P(E|H) + P(H') \times P(E|H')}$$

where H and E are two events of interest, P(H) is the probability of H happening, P$(E|H)$ is the conditional probability of E happening given H has happened, H' is the complement of H (which is referred to as "¬H" in the video), etc. This theorem is widely applied in various fields and is useful for anyone especially during a pandemic!

Let us say H is the event that you have Covid and E is the event that you have been tested positive. Now, P(H) or probability of you having Covid can be computed based on empirical data and P(H') = 1 – P(H).

P$(E|H)$ i.e. the probability of testing positive if you actually have Covid should be very high and P$(E|H')$ i.e. probability of false positive should be low. (Otherwise the test would not be recommended!) Now, if you do get tested positive, then you would be interested in the probability that you actually have Covid or P$(H|E)$. It may seem that this probability should also be quite high. But if P(H) i.e. the proportion of people actually infected is low, say 5% then P$(H|E)$ can actually be closer to half.

Many of the formulas and results in probability can be understood through Venn diagrams. That's exactly what the above-mentioned videos do. They make the connection between probability and proportion and then take the help of geometry to represent the proportions as areas and lengths. Thus, they provide a strong intuitive understanding of the heart of the matter from which the seemingly complicated formula can be easily derived.

More importantly, the first video starts by mentioning uses of this result, followed by outlining the levels of understanding as:

i. What is it saying?

ii. Why is it true?

iii. When is it useful?

Then it exemplifies with a case and numbers (not percentages) along with aptly drawn visuals. This is followed by the transition from numbers or counts to area and how that helps generalise the result – all within 5 minutes! It then systematically derives the formula with the example, counts, and visuals.

The second video, which is like a footnote for the first video, deals with P$(H$ and $E)$ or the probability of both events happening. As a result, it also touches upon the notion of independence. This is also illustrated with a suitable example.

How does this help a teacher?

It helps mainly in three ways:

First, by providing a visual than can be used to model any situation involving conditional probability and can help one understand what such probabilities really mean (Figure 1).

1. It uses the unit square as a sample space which can be partitioned in vertical strips given any partition E_1, E_2 … E_k such that the widths of the strips are proportional to their probabilities. Note that both the probabilities of the partition and the widths of the strips add up to 1.
2. Now each strip can be split into two strips (top and bottom) according to the conditional probabilities P$(A|E_1)$, P$(A|E_2)$ … P$(A|E_k)$. That is, the vertical dimensions of the bottom strips should be proportional to these conditional probabilities.
3. So, the bottom strips represent the events A and E_1, A and E_2 … A and E_k whose areas are P(E_1) × P$(A|E_1)$, P(E_2) × P$(A|E_2)$ … P(E_k) × P$(A|E_k)$, respectively. Also, event A is represented by the collection of the bottom strips.
4. Now, Bayes' theorem can be deduced by considering the sum of the areas of the bottom strips i.e. P(A) = P(E_1) × P$(A|E_1)$ + P(E_2) × P$(A|E_2)$ + … + P(E_k) × P$(A|E_k)$ and the area of any of the bottom strips i.e. P(E_i) × P$(A|E_i)$ for any i = 1, 2 … k.
5. So, if the heights (vertical dimension) of the bottom strips vary, then the partition has some effect on the event A. And therefore, A is not independent of the partition. But if the heights are same for all the bottom strips, then the partition has no effect on A. In that case, A is independent of the partition and the common height can be factored out as P(A). So, in that case, P$(A$ and $E_i)$ = P(E_i) × P(A).

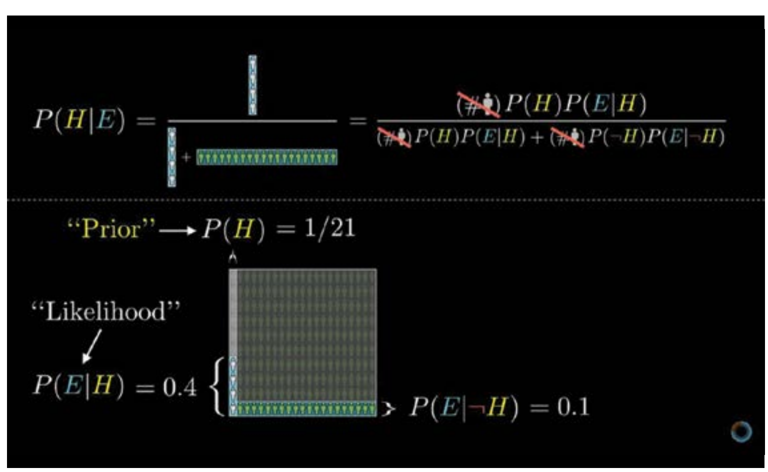


Figure 1

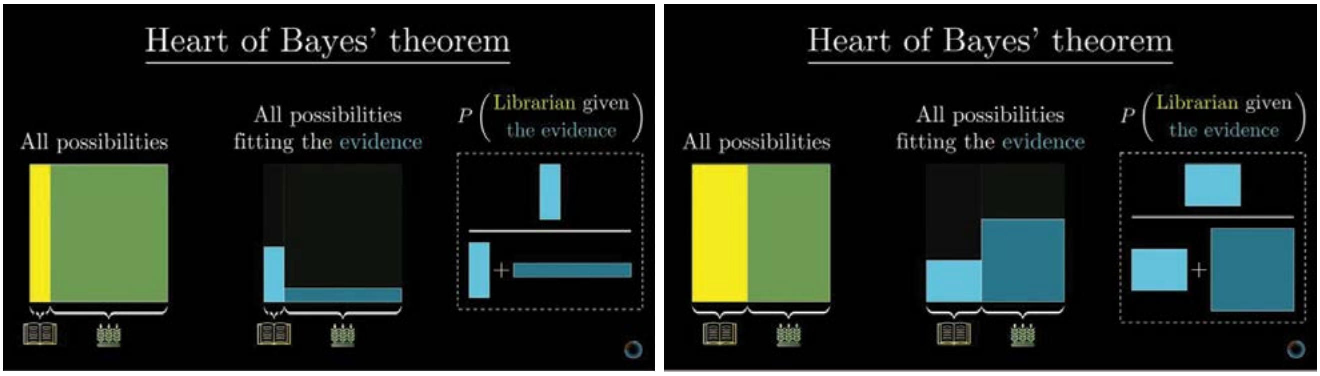


Figure 2

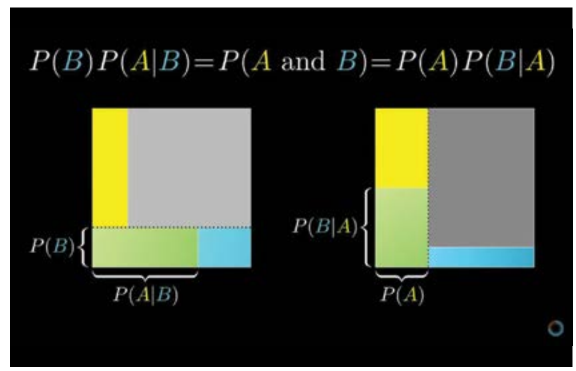


Figure 3

Second, it also draws attention to the fact that people understand proportions better in terms of odds (example: 7 out of 25) than as percentages (e.g. 28%), which is a very useful pedagogic tip.

Finally, it wraps up by linking this theorem to beliefs *(H)* and evidence *(E)* and how the latter should influence the former.

The second video discusses why there is a wide misconception that P*(H* and *E)* = P*(H)* × P*(E)* (regardless of independence).

These videos will undoubtedly help the teacher explain concepts in probability with visual representation and logical reasoning, both of which exemplify mathematical processes.

SWATI SIRCAR is Assistant Professor at the School of Continuing Education and University Resource Centre of Azim Premji University. Math is the 2nd love of her life (1st being drawing). She is a B.Stat-M.Stat from Indian Statistical Institute and an MS in math from University of Washington, Seattle. She has been doing mathematics with children and teachers for more than a decade and is deeply interested in anything hands on - origami in particular.

Problem Corner

Adventure with Quadratic Equations

PRITHWIJIT DE

Quadratic equations form an important topic in most school curricula and students are exposed to problems of the following types:

(a) Finding the roots;

(b) Determining the nature of roots without explicitly finding them;

(c) Determining the range of values of a quadratic expression or a rational function having a quadratic numerator or denominator;

(d) Word problems where the quadratic equation has to be formulated and then solved.

More often than not the problems on quadratic equations discussed in the school curricula can be solved by a routine application of a handful of formulae and laborious algebraic manipulations. As a result the topic may appear to be charmless to the students and teachers alike. The goal of this article is to debunk this notion by way of illustrative examples that will bring the adventurous side of quadratic equations to the fore.

The first example is a word problem formulated as a simple game being played by two friends, Akbar and Birbal.

Keywords: Quadratic equations, roots

Example 1. Akbar writes down the quadratic equation

$$ax^2 + bx + c = 0$$

with positive integer coefficients a, b, c. Then Birbal changes one, two or all three '$+$' signs to '$-$'. Akbar wins if both roots of the (modified) equation are integers. Otherwise (if there are no real roots or at least one of them is not an integer), Birbal wins. Can Akbar choose the coefficients in such a way that he will always win?

Solution. How do we start? There are eight possible equations (including the one written by Akbar) which can be divided into two groups, P and N, according to whether the sign of a is "$+$" or "$-$"; each group has four equations. Note that every equation in N may be obtained by multiplying by -1 an equation in P, and distinct equations in N are obtained from distinct equations in P. Thus it suffices to deal with the equations in any one of the two groups. We choose that group to be P. So, Akbar has to choose the coefficients a, b, c in such a way that all the four equations

(i) $ax^2 + bx + c = 0$

(ii) $ax^2 + bx - c = 0$

(iii) $ax^2 - bx + c = 0$

(iv) $ax^2 - bx - c = 0$

have integer roots. Since the sum of the roots and their product for the equations listed above are $\pm b/a$ and $\pm c/a$, to ensure that the roots are integers, it is wise for Akbar to choose $a = 1$. Observe that the roots of equations (iii) and (iv) differ from those of (i) and (ii), respectively, by a factor of -1. Thus, ensuring that the following equations have integer roots,

$$x^2 + bx + c = 0,$$
$$x^2 + bx - c = 0$$

is enough for Akbar to win the game.

Playing with small integers, Akbar will sooner or later obtain the winning quadratics

$$x^2 + 5x + 6 = 0 \quad \text{and} \quad x^2 + 5x - 6 = 0.$$

If Akbar decides to continue his search with vigour and enthusiasm, he will obtain several possible choices for the pair (b, c) that will guarantee his win against Birbal. To list a few:

$$(b, c) = (13, 30),\ (17, 60),\ (25, 84),\ \ldots.$$

But are there finitely many such pairs or infinitely many? This question is answered below. □

Proposition. *Let the positive integers b and c be such that both the quadratic equations*

$$x^2 + bx + c = 0, \quad x^2 + bx - c = 0$$

have integer roots. Then there exists a right-angled triangle with hypotenuse b and area c.

Proof of proposition. As both quadratics have integer roots, both $b^2 - 4c$ and $b^2 + 4c$ are perfect squares. Let

$$x^2 = b^2 - 4c, \quad y^2 = b^2 + 4c,$$

where x, y are positive integers. Then

$$b^2 = \frac{y^2 + x^2}{2} = \left(\frac{y + x}{2}\right)^2 + \left(\frac{y - x}{2}\right)^2$$

and

$$c = \frac{1}{2}\left(\frac{y+x}{2}\right)\left(\frac{y-x}{2}\right).$$

Observe that both x^2 and y^2 have the same parity because their difference is even, and $y > x$. Thus x and y have the same parity, hence $(y+x)/2$ and $(y-x)/2$ are positive integers, and indeed b is the hypotenuse of the right-angled triangle with sides

$$\frac{y-x}{2}, \quad \frac{y+x}{2}, \quad \sqrt{\frac{y^2+x^2}{2}}$$

whose area is c. (The reader may verify that the three numbers are the sides of a non-degenerate triangle.)

So Akbar can choose any Pythagorean triple (r, s, t) and set $b = t$, $c = \frac{1}{2}rs$ to ensure his triumph over Birbal. Since there are infinitely many Pythagorean triples, Akbar has infinitely many pairs (b, c) at his disposal. □

The next example involves permutation of the coefficients of a quadratic trinomial.

Example 2. Three nonzero real numbers a, b and c are given. We are told that if they are written in any order as the coefficients of a quadratic trinomial, then each of these trinomials has a real root. Does it follow that each of these trinomials has a positive root?

Solution. Since each quadratic trinomial has real coefficients, if each has a real root, then each must have two real roots. Suppose there is one trinomial which does not have a positive root. Without loss of generality, let it be $ax^2 + bx + c$. Since $c \neq 0$, we know that 0 is not a root of this trinomial. Let $-u$ and $-v$ be its roots where $u > 0$ and $v > 0$. Then

$$ax^2 + bx + c = a(x+u)(x+v),$$

and we notice that the signs of $b = a(u+v)$ and $c = auv$ are the same as that of a. Therefore, without loss of generality, we may assume that a, b, c are positive.

But according to the problem, each of $ax^2 + bx + c$, $bx^2 + cx + a$ and $cx^2 + ax + b$ has two real roots. Therefore

$$b^2 \geq 4ac, \quad c^2 \geq 4ab, \quad a^2 \geq 4bc.$$

These inequalities lead to

$$(abc)^2 \geq 64(abc)^2,$$

an absurd result unless $abc = 0$, which is impossible. Thus each of the six quadratic trinomials has a positive root. □

Here is a teaser for the reader.

Problem. Let a, b, c be three integers in arithmetic progression. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are integers, find the ratio $a : b : c$.

Using the Intermediate Value Theorem. Sometimes it is easier to establish the existence of a real root of a quadratic trinomial by exhibiting two real numbers at which it has opposite sign, rather than by explicitly computing the discriminant and showing that it is non-negative. The existence of a real root then follows from the Intermediate Value Theorem for a continuous function.The following example highlights this fact.

Example 3. Suppose P_1, P_2, P_3 are quadratic trinomials with positive leading coefficients and real zeros. Show that if each pair of them has a common zero, then the trinomial $P_1 + P_2 + P_3$ also has real zeros.

Solution. Let the common zero of P_1 and P_2 be β, and that of P_2 and P_3 be γ. Then the zeros of P_2 are β and γ, and if α is the other zero of P_1, then the zeros of P_3 are γ and α. Without loss of generality we may assume that $\alpha \leq \beta \leq \gamma$. So $P = P_1 + P_2 + P_3$ can be written as

$$P(x) = a_1(x-\alpha)(x-\beta) + a_2(x-\beta)(x-\gamma) + a_3(x-\gamma)(x-\alpha), \tag{1}$$

where $a_i > 0$ for $i = 1, 2, 3$. Then

$$P(\alpha) = a_2(\alpha-\beta)(\alpha-\gamma), \qquad P(\beta) = a_3(\beta-\gamma)(\beta-\alpha), \qquad P(\gamma) = a_1(\gamma-\alpha)(\gamma-\beta).$$

Observe that $P(\alpha) \geq 0 \geq P(\beta)$ and $P(\beta) \leq 0 \leq P(\gamma)$. This shows that P has a real zero between α and β, and another real zero between β and γ. □

The following example is from the Belarusian Mathematical Olympiad and it illustrates the level of challenge that a problem on quadratic equation can have.

Example 4. We call two quadratic trinomials $P(x) = x^2 + ax + b$ and $Q(x) = x^2 + cx + d$ **friendly** if each of them has distinct real roots, and if $x_1 < x_2$ are the roots of $P(x)$ and $x_3 < x_4$ are the roots of $Q(x)$, then $x_1 + x_3$ and $x_2 + x_4$ are the roots of the quadratic trinomial $x^2 + (a+c)x + (b+d)$. Let M be the set of pairwise friendly trinomials consisting of at least three trinomials. Prove that 0 is a root of every trinomial from the set M.

Solution. Let $P(x) = x^2 + ax + b$, $Q(x) = x^2 + cx + d$ and $R(x) = x^2 + ex + f$ be three pairwise friendly trinomials in M. Suppose $x_1 < x_2$ are the roots of $P(x)$, $x_3 < x_4$ are the roots of $Q(x)$, and $x_5 < x_6$ are the roots of $R(x)$. Then $x_1 + x_3$, $x_2 + x_4$ are the roots of

$$P(x) + Q(x) - x^2 = x^2 + (a+c)x + (b+d),$$

$x_3 + x_5$, $x_4 + x_6$ are the roots of

$$Q(x) + R(x) - x^2 = x^2 + (c+e)x + (d+f),$$

and $x_5 + x_1$, $x_6 + x_2$ are the roots of

$$R(x) + P(x) - x^2 = x^2 + (e+a)x + (f+b).$$

Observe that

$$b + d = x_1x_2 + x_3x_4 = (x_1 + x_3)(x_2 + x_4),$$

whence

$$x_1x_4 + x_2x_3 = 0.$$

Similarly from

$$d + f = x_3x_4 + x_5x_6 = (x_3 + x_5)(x_4 + x_6)$$

and

$$f + b = x_5x_6 + x_1x_2 = (x_5 + x_1)(x_6 + x_2)$$

we obtain

$$x_3x_6 + x_4x_5 = 0$$

and

$$x_1x_6 + x_2x_5 = 0.$$

From the above we obtain:

$$x_1x_4 = -x_2x_3, \qquad x_3x_6 = -x_4x_5, \qquad x_2x_5 = -x_1x_6,$$

which lead to

$$(x_1x_4)(x_3x_6)(x_2x_5) = (-x_2x_3)(-x_4x_5)(-x_1x_6),$$

or:

$$x_1x_2x_3x_4x_5x_6 = 0.$$

Therefore at least one of $x_1, x_2, x_3, x_4, x_5, x_6$ is zero. Moreover, since the roots of each of $P(x)$, $Q(x)$ and $R(x)$ are distinct, at most one root of each can be zero. Suppose $x_1 = 0$. Then $x_2 \neq 0$ and it follows that $x_3 = 0$. Therefore $x_4 \neq 0$. From the above we get $x_5 = 0$ and hence $x_6 \neq 0$. This shows that 0 is a root of each of $P(x)$, $Q(x)$, and $R(x)$. The reader may verify that we would have reached the same conclusion if we had assumed that $x_k = 0$ for $k = 2, 3, 4, 5, 6$ instead of $x_1 = 0$.

The next example is from the Russian Mathematical Olympiad.

Example 5. A quadratic polynomial $f(x) = ax^2 + bx + c$ has no real roots. It is given that b is a rational number, and exactly one of c and $f(c)$ is a rational number. Is it possible for the discriminant of $f(x)$ to be a rational number?

Solution. Suppose c is a rational number. Then, by hypothesis, $f(c) = c\,(ac + b + 1)$ is irrational. Since b and c are rational, a must be irrational. Therefore the discriminant $D = b^2 - 4ac$ is irrational.

Suppose $f(c)$ is rational but c is irrational. Note $f(c) \neq 0$, since f does not have any real root. Then $(ac + b + 1) \neq 0$ and is irrational. But b is rational. Therefore ac is irrational and hence $D = b^2 - 4ac$ is irrational.

Our last example is a problem with a simple statement which can be generalised to a much deeper result.

Example 6. Let a and b be positive integers such that $n^2 + 2an + b$ is a perfect square for all integers n. Then the quadratic trinomial $x^2 + 2ax + b$ is the square of a linear polynomial.

Solution. Let $f(x) = x^2 + 2ax + b$. Then $f(-a) = b - a^2$ is a perfect square. Let c be such that $b - a^2 = c^2$. If $c = 0$ we are done. We will prove that under the given hypothesis c cannot be nonzero. Suppose $c \neq 0$. Then

$$f(c - a) = c^2 + b - a^2 = 2c^2.$$

But $2c^2$ cannot be a perfect square unless $c = 0$, as the exponent of 2 in $2c^2$ is odd if $c \neq 0$. Therefore $c = 0$, implying $b = a^2$ and

$$f(x) = (x + a)^2.$$

The general statement illustrated by the above example is the following:

Proposition. *If $P(x)$ is a polynomial with integer coefficients such that for every integer n, $P(n)$ is a k^{th} power for some positive integer k, then there exists a polynomial $Q(x)$ with integer coefficients such that $P(x) = (Q(x))^k$.*

We hope that the reader will find this escapade involving quadratics stimulating enough to plunge into another on his or her own.

PRITHWIJIT DE is the National Coordinator of the Mathematical Olympiad Programme of the Government of India. He is an Associate Professor at the Homi Bhabha Centre for Science Education (HBCSE), TIFR, Mumbai. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. He may be contacted at de.prithwijit@gmail.com.

Inequalities - Part 2

SOURANGSHU GHOSH

In Part 1 of this series of articles, we had considered the arithmetic mean-geometric mean inequality along with some variants and some applications. Now in Part 2, we look at the Cauchy-Schwartz inequality and some applications.

The Cauchy Schwartz Inequality

An important result often used in mathematical Olympiads is the Cauchy-Schwartz inequality. It is especially useful in proving inequalities that have cyclic and symmetric terms. It can also be used to prove many established inequalities. Here is the statement of the result.

Theorem 1 (Cauchy-Schwartz inequality)**.** If $a_1, a_2, \ldots, a_n > 0$ and $b_1, b_2, \ldots, b_n > 0$, all numbers being real, the following inequality holds

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \cdot \left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq \left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)^2, \tag{1}$$

with equality precisely when the sequences are proportional to each other, i.e.,

$$a_1 = tb_1, \quad a_2 = tb_2, \quad \ldots, \quad a_n = tb_n \tag{2}$$

for some real number t.

Proof. The proof of the theorem is from [1]. Recall the AM-GM inequality for two positive numbers x, y: it states that $x + y \geq 2\sqrt{xy}$, with equality if and only if $x = y$. We apply this as shown below.

Let $A = a_1^2 + a_2^2 + \cdots + a_n^2$, $B = b_1^2 + b_2^2 + \cdots + b_n^2$. For $i = 1, 2, \ldots, n$, we have:

$$\frac{a_i^2}{A} + \frac{b_i^2}{B} \geq \frac{2a_i b_i}{\sqrt{AB}}.$$

Summing the n inequalities, we get:

$$\frac{A}{A} + \frac{B}{B} \geq \frac{2\left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)}{\sqrt{AB}},$$

$$\therefore \quad \sqrt{AB} \geq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

Keywords: Inequality, AM-GM inequality, Cauchy-Schwartz inequality, Holder's inequality.

i.e.,

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \cdot \left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq \left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)^2.$$

Equality holds if and only if for each subscript i we have:

$$\frac{a_i^2}{A} = \frac{b_i^2}{B},$$

i.e., $a_i = tb_i$ where $t = \sqrt{A}/\sqrt{B}$. □

Theorem 2 (Alternative form of Cauchy-Schwartz)**.** If $a_1, a_2, \ldots, a_n > 0$ and $b_1, b_2, \ldots, b_n > 0$, all numbers being real, the following inequality holds

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}. \tag{3}$$

Moreover, equality holds if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

Proof of the alternative form. The proof is from [1]. We proceed inductively, starting with the case $n = 2$. This specific case may be stated as follows: If $a, b > 0$ and $x, y > 0$, all numbers being real, then

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y},$$

with equality if and only if $a/x = b/y$. To prove this, we bring all the terms to one side and multiply through by $xy(x+y)$, which is a positive quantity. We obtain:

$$\begin{aligned}(x+y)\left(a^2 y + b^2 x\right) - xy(a+b)^2 &= a^2y^2 - 2abxy + b^2x^2 \\ &= (ay - bx)^2 \geq 0,\end{aligned}$$

with equality if and only if $ay - bx = 0$, i.e., $a/x = b/y$.

Now consider the case $n = 3$. The result may be stated as follows: If $a, b, c > 0$ and $x, y, z > 0$, all numbers being real, then

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z},$$

with equality if and only if $a/x = b/y = c/z$. To prove this, we use the result just proved (for the case $n = 2$). We have:

$$\begin{aligned}\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} &\geq \frac{(a+b)^2}{x+y} + \frac{c^2}{z} \\ &\geq \frac{(a+b+c)^2}{x+y+z} \qquad \text{(by applying the result once again).}\end{aligned}$$

The condition for equality should be clear: $a/x = b/y = c/z$.

The extension to the general case is now a matter of detail; we do not give the steps here. □

Some applications of the Cauchy-Schwartz inequality

We now look at some inequalities that make use of the Cauchy-Schwartz inequality.

Example 1: Nesbitt's inequality. For positive real numbers a, b, c, prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}. \tag{4}$$

We had proved this inequality in the first part of this series, but we give a different approach now.

Proof. The solution is from [1]. Adding 1 to each term, the inequality is transformed to:

$$(a+b+c) \cdot \left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \geq \frac{9}{2}.$$

Let $x = \sqrt{b+c}$, $y = \sqrt{c+a}$, $z = \sqrt{a+b}$, so that $x^2 + y^2 + z^2 = 2(a+b+c)$. Then the inequality may be rewritten as:

$$\left(x^2 + y^2 + z^2\right) \cdot \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) \geq 9.$$

This now follows immediately from the Cauchy-Schwartz inequality, for we have:

$$\left(x^2 + y^2 + z^2\right) \cdot \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) \geq \left(x \cdot \frac{1}{x} + y \cdot \frac{1}{y} + z \cdot \frac{1}{z} \right)^2 = 9.$$

Example 2: A problem from IMO 1995. Prove that for any three positive real numbers a, b, c such that the product $abc = 1$,

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}. \tag{5}$$

Proof. The solution is from [1]. We use the alternative form of the Cauchy-Schwartz inequality. Let $x = 1/a$, $y = 1/b$, $z = 1/c$. Then by the given condition we obtain $xyz = 1$. Note that

$$\begin{aligned}
&\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \\
&= \frac{1}{(1/x^3)(1/y + 1/z)} + \frac{1}{(1/y^3)(1/z + 1/x)} + \frac{1}{(1/z^3)(1/x + 1/y)} \\
&= \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}.
\end{aligned}$$

Now by the Cauchy-Schwartz inequality (in its 'alternative form'),

$$\begin{aligned}
\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} &\geq \frac{(x+y+z)^2}{2(x+y+z)} = \frac{x+y+z}{2} \\
&\geq 3 \cdot \frac{(xyz)^{1/3}}{2} = \frac{3}{2}.
\end{aligned}$$

Example 3. Prove that for all $a, b, c \geq 1$,

$$\sqrt{a^2 - 1} + \sqrt{b^2 - 1} + \sqrt{c^2 - 1} \leq \frac{ab + bc + ca}{2}. \tag{6}$$

The problem was posted on Leo Giugiuc's *Cut-The-Knot* Math Facebook page by Professor Dorin Marghidanu. The proof was given by C. Nanuti, D.Trailescu, D. Sitaru and L. Giugiuc.

Proof. Let $x = \sqrt{a^2 - 1}$, $y = \sqrt{b^2 - 1}$, $z = \sqrt{c^2 - 1}$; then $x, y, z \geq 0$. By the Cauchy-Schwartz inequality,

$$\sqrt{x^2 + 1}\sqrt{y^2 + 1} = \sqrt{x^2 + 1}\sqrt{1 + y^2} \geq x + y.$$

So we have, by addition,

$$\sqrt{x^2 + 1}\sqrt{y^2 + 1} + \sqrt{y^2 + 1}\sqrt{z^2 + 1} + \sqrt{z^2 + 1}\sqrt{x^2 + 1} \geq 2(x + y + z).$$

In other words we have:

$$ab + bc + ca \geq 2\left(\sqrt{a^2 - 1} + \sqrt{b^2 - 1} + \sqrt{c^2 - 1}\right),$$

which is the required inequality.

Equality holds if and only if $x = y = z = 1$, i.e., if and only if $a = b = c = \sqrt{2}$.

References

1. Riasat, Samin. (2008). Basics of Olympiad Inequalities, https://www.researchgate.net/publication/264892067_Basics_of_Olympiad_Inequalities
2. Mildorf, T J, (2005),"Olympiad Inequalities". Preprint, https://artofproblemsolving.com/articles/files/MildorfInequalities.pdf
3. AoPs Online, https://artofproblemsolving.com/wiki/index.php/1964_IMO_Problems/Problem_2
4. AoPs Online, https://artofproblemsolving.com/wiki/index.php/Proofs_of_AM-GM
5. Hardy, G.H.; Littlewood, J.E.; Pólya, G.; "Inequalities"; Cambridge University Press: Cambridge, UK, 1934; ISBN 978-0-521-35880-4.

SOURANGSHU GHOSH is a fourth-year undergraduate student of the Dept of Civil Engg at IIT Kharagpur. He is also pursuing Mathematics as a minor degree. He is interested in Discrete Mathematics and its applications. He has written several articles in mathematics, many in international journals. He has worked on the Reliability of Weld Joints of Nuclear Power Plants in collaboration with IGCAR Kalpakkam, and on the structural reliability of coherent systems, as part of his undergraduate thesis. He enjoys playing the violin in the Indian classical style. He may be contacted at sourangshug123@gmail.com.

Trigonometric Proof of Ptolemy's Theorem

K SASIKUMAR

In this short note, we provide a trigonometric proof of Ptolemy's theorem.

Theorem (Ptolemy)**.** If quadrilateral $ABCD$ is cyclic then $AB \cdot CD + BC \cdot AD = AC \cdot BD$.

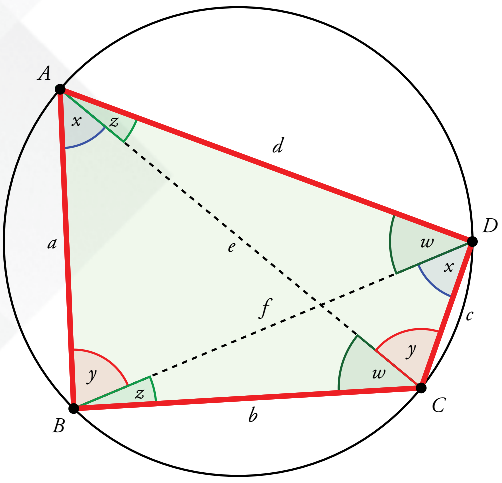


Figure 1.

Proof. Let

$$\measuredangle CAB = \measuredangle CDB = x, \qquad \measuredangle ABD = \measuredangle ACD = y,$$

$$\measuredangle DAC = \measuredangle DBC = z, \qquad \measuredangle ACB = \measuredangle ADB = w.$$

Also let $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, $BD = f$.

Keywords: Ptolemy's theorem, quadrilateral, cyclic, sine rule

By the sine rule, since quadrilateral *ABCD* is cyclic, we have

$$\begin{aligned} a/\sin w &= d/\sin y = f/\sin(y+w), \\ b/\sin x &= a/\sin w = e/\sin(x+w), \\ c/\sin z &= b/\sin x = f/\sin(x+z), \\ d/\sin y &= c/\sin z = e/\sin(y+z). \end{aligned}$$

From the second and third equalities we get:

$$\frac{ac}{\sin w \cdot \sin z} = \frac{ef}{\sin(x+w)\cdot\sin(x+z)},$$

and from the third and fourth equalities we get:

$$\begin{aligned} \frac{bd}{\sin x \cdot \sin y} &= \frac{ef}{\sin(x+z)\cdot\sin(y+z)} \\ &= \frac{ef}{\sin(x+z)\cdot\sin(x+w)}, \end{aligned}$$

since $x+y+z+w=\pi$ and $\sin(\pi-u)=\sin u$ for any angle u. Therefore:

$$\begin{aligned} \frac{ac}{ef} &= \frac{\sin z\cdot\sin w}{\sin(x+w)\cdot\sin(x+z)}, \\ \frac{bd}{ef} &= \frac{\sin x\cdot\sin y}{\sin(x+z)\cdot\sin(x+w)}. \end{aligned}$$

Therefore:

$$\begin{aligned} \frac{ac+bd}{ef} &= \frac{\sin z\cdot\sin w+\sin x\cdot\sin y}{\sin(x+w)\cdot\sin(x+z)} \\ &= \frac{\cos(z-w)-\cos(z+w)+\cos(x-y)-\cos(x+y)}{\cos(z-w)-\cos(2x+z+w)} \\ &= \frac{\cos(z-w)+\cos(x+y)+\cos(x-y)-\cos(x+y)}{\cos(z-w)+\cos(x-y)} \\ &= 1 \qquad \text{(repeatedly using } x+y+z+w=\pi). \end{aligned}$$

Therefore $ac+bd=ef$, or $AB\cdot CD+BC\cdot AD=AC\cdot BD$. □

SASIKUMAR K is presently working as a PG Teacher in Mathematics at Jawahar Navodaya Vidyalaya, North Goa. He completed his M Phil under the guidance of Dr.K.S.S. Nambooripad. Earlier, he worked as a maths Olympiad trainer for students of Navodaya Vidyalaya Samiti in Hyderabad. He has research interests in Real Analysis and Commutative Algebra. He may be contacted at 112358.ganitham@gmail.com.

Student Corner

Rational 'Twin' Isosceles Triangles

HARAN MOULI,
MAHAVIR GANDHI,
SARTH CHAVAN

Problem. Can there exist two non-congruent isosceles triangles with same perimeter and same area? If yes, how can you find them? How many solutions exist? Can you find a complete parametrisation for such triangles?

This problem is from the *Ganitasarasangraha*, an ancient mathematical text written by the Indian mathematician Mahavira in 850 CE. It is one of the 'Paishachika' problems, i.e., Devilishly Hard Problems!

We address a modified version of the same problem. Specifically, we provide a general formula to generate all pairs of rational-sided isosceles triangles that share the same perimeter and the same rational area. By appropriate scaling, we can also make all these quantities integers.

Consider two isosceles triangles $\triangle ABC$ and $\triangle XYZ$ with rational sides:

$$(AB, AC, BC) = (a, a, b), \qquad (XY, XZ, YZ) = (u, u, v)$$

The perimeter of $\triangle ABC = 2a + b$, and the perimeter of $\triangle XYZ = 2u + v$. As the triangles have equal perimeter, we have $2a + b = 2u + v$.

We now compute the areas of the triangles using Heron's formula:

$$\text{Area of } \triangle ABC = \sqrt{\left(\frac{2a+b}{2}\right)\left(\frac{2a+b}{2} - a\right)\left(\frac{2a+b}{2} - a\right)\left(\frac{2a+b}{2} - b\right)}$$

$$= \frac{b}{2}\sqrt{\left(\frac{2a+b}{2}\right)\left(\frac{2a-b}{2}\right)}.$$

Keywords: Area, perimeter, isosceles triangle, rational, Mahavira

Similarly, we can conclude that

$$\text{Area of } \triangle XYZ = \frac{v}{2}\sqrt{\left(\frac{2u+v}{2}\right)\left(\frac{2u-v}{2}\right)}.$$

Now, since $\triangle ABC$ and $\triangle XYZ$ have equal area, we have:

$$\frac{b}{2}\sqrt{\left(\frac{2a+b}{2}\right)\left(\frac{2a-b}{2}\right)} = \frac{v}{2}\sqrt{\left(\frac{2u+v}{2}\right)\left(\frac{2u-v}{2}\right)},$$

$$\therefore\ b^2(2a+b)(2a-b) = v^2(2u+v)(2u-v). \tag{1}$$

In equation (1), we can note that $(2a+b)$ and $(2u+v)$ are the same because they are the perimeters of $\triangle ABC$ and $\triangle XYZ$ respectively. Therefore we have:

$$b^2(2a-b) = v^2(2u-v). \tag{2}$$

Let P denote the (equal) perimeters of $\triangle ABC$ and $\triangle XYZ$, i.e., $2a+b = P = 2u+v$. Write $2a-b$ as $P-2b$ and $2u-v$ as $P-2v$.

Substituting in equation 2 we have,

$$b^2(P-2b) = v^2(P-2v). \tag{3}$$

Evaluating and rearranging equation (3), we have:

$$P = \frac{2(b^3-v^3)}{b^2-v^2} = \frac{2(b-v)(b^2+bv+v^2)}{(b+v)(b-v)} = \frac{2(b^2+bv+v^2)}{b+v} \tag{4}$$

Now, we can find the equal sides of $\triangle ABC$ and $\triangle XYZ$ as follows:

$$a = \frac{P-b}{2} = \frac{1}{2}\left(\frac{2(b^2+bv+v^2)}{b+v} - b\right)$$

$$\therefore\ a = \frac{b^2+bv+2v^2}{2(b+v)}.$$

Similarly,

$$u = \frac{2b^2+bv+v^2}{2(b+v)}.$$

Let A denote the (equal) area of the two triangles. From Heron's Formula, we know that:

$$16A^2 = b^2(2a+b)(2a-b) = v^2(2u+v)(2u-v).$$

Let

$$g^2 = \left(\frac{2a+b}{2}\right)\left(\frac{2a-b}{2}\right), \qquad h^2 = \left(\frac{2u+v}{2}\right)\left(\frac{2u-v}{2}\right),$$

where g and h are rational numbers. Substituting in terms of P yields:

$$g^2 = \left(\frac{2a+b}{2}\right)\left(\frac{2a-b}{2}\right) = \left(\frac{P}{2}\right)\left(\frac{P}{2}-b\right),$$
$$g^2 = \frac{1}{4}\left(\frac{2(b^2+bv+v^2)}{b+v}\right)\left(\frac{2(b^2+bv+v^2)}{b+v}-2b\right) = \frac{(b^2+bv+v^2)v^2}{(b+v)^2}.$$

Now, since

$$g^2 = \frac{(b^2+bv+v^2)v^2}{(b+v)^2},$$

it must be that b^2+bv+v^2 is a rational square.

We reach the same conclusion if we work with h.

We proceed to find the rational parametrization. For now, we take $b^2+bv+v^2=1$. Later we scale the sides of both triangles to allow this value to be any rational square.

For $b^2+bv+v^2=1$, an obvious solution is $(b,v)=(1,0)$. Now we simply draw a secant via this trivial solution, to the coordinates (b,v), the non-trivial solution. Let this line be $v=-k(b-1)$.

By substitution we conclude that $b^2-k(b-1)b+k^2(b-1)^2=1$. One solution to this quadratic is $b=1$ (from our trivial solution). We can use Vieta's formulas to find our other non-trivial root.

$$(k^2-k+1)b^2-(2k^2-k)b+(k^2-1)=0 \implies b=\frac{k^2-1}{k^2-k+1}$$

Now, we can evaluate v using the value of b:

$$v=-k(b-1)=-k\left(\frac{k^2-1}{k^2-k+1}-1\right)=\frac{2k-k^2}{k^2-k+1},$$
$$(b,v)=\left(\frac{k^2-1}{k^2-k+1},\frac{2k-k^2}{k^2-k+1}\right).$$

Thus we have found (b,v).

Note that (b,v) must be positive as they are the lengths of the sides of the triangle. The denominator k^2-k+1 is always positive, as it has negative discriminant. Hence the numerators must be positive.

$$k^2-1>0 \implies k\in(-\infty,-1)\cup(1,\infty), \tag{5}$$
$$2k^2-k>0 \implies k\in(0,2). \tag{6}$$

From equations (5) and (6) we can conclude that $1<k<2$, that is, k is a positive rational number between 1 and 2. Now we again evaluate the perimeter P.

$$\begin{aligned}P &= \frac{2(b^2+bv+v^2)}{b+v}\\ &= \frac{2}{\left(\frac{k^2-1}{k^2-k+1}\right)+\left(\frac{2k-k^2}{k^2-k+1}\right)}\\ &= \frac{2(k^2-k+1)}{2k-1}.\end{aligned} \tag{7}$$

Now, from equation (7), we know what the perimeter is. We can evaluate a, u again using the perimeter to finally obtain the quadruple:

$$a = \frac{P-b}{2} = \frac{1}{2}\left(\frac{2(k^2-k+1)}{2k-1} - \frac{k^2-1}{k^2-k+1}\right) = \frac{2k^4-6k^3+7k^2-2k+1}{2(2k-1)(k^2-k+1)} \tag{8}$$

Similarly, we have:

$$u = \frac{P-v}{2} = \frac{1}{2}\left(\frac{2(k^2-k+1)}{2k-1} - \frac{2k-k^2}{k^2-k+1}\right) = \frac{2k^4-2k^3+k^2-2k+2}{2(2k-1)(k^2-k+1)} \tag{9}$$

Thus we can conclude that:

$$(a,u) = \left(a\left(\frac{2k^4-6k^3+7k^2-2k+1}{2(2k-1)(k^2-k+1)}\right),\ a\left(\frac{2k^4-2k^3+k^2-2k+2}{2(2k-1)(k^2-k+1)}\right)\right),$$
$$(b,v) = \left(a\left(\frac{k^2-1}{k^2-k+1}\right),\ a\left(\frac{2k-k^2}{k^2-k+1}\right)\right),$$

where $k \in (1,2)$ is rational and a is a positive rational (scaling factor).

Corollary 1. *There exist infinitely many pairs of non-congruent isosceles triangles having rational (or integer) sides, and both equal and rational (or integer) area and perimeter.*

We also make another claim regarding the number of triangles simultaneously having the same area and perimeter:

Corollary 2. *Three pairwise non-congruent isosceles triangles with rational sides cannot have equal and rational perimeter and area.*

Proof. Assume the contrary. Let the sides of the three isosceles triangles be (a,a,b), (u,u,v) and (c,c,d). From our previous arguments:

$$P = \frac{2(b^2+bv+v^2)}{b+v} = \frac{2(b^2+bd+d^2)}{b+d} \implies \frac{v^2}{b+v} = \frac{d^2}{b+d},$$
$$v^2(b+d) = d^2(b+v) \implies b(v^2-d^2) + vd(v-d) = 0,$$
$$b(v+d) + vd = 0 \implies b = -\frac{vd}{v+d}.$$

This is clearly impossible as b is the side of a triangle and hence cannot be negative. This proves the corollary by contradiction.

Since the area and the sides of the triangles are all rational, all of the altitudes of the triangles are rational as well. The median AD in $\triangle ABC$ as well as the median XU in $\triangle XYZ$ are also rational, since they are altitudes.

An interesting follow-up problem would be to ask whether it is also possible to have the medians $BE = CF = m_x$ and $YV = ZW = m_y$ to be rational as well. We leave this for the reader to explore!

HARAN MOULI is a grade 12 student in the PSBB Group of Schools. A fervent math enthusiast, he has a keen interest in problem solving and learning mathematical concepts, with a particular fondness for Number Theory. He loves teaching and volunteers at Raising a Mathematician Foundation to guide high school maths enthusiasts. He may be contacted at mouliharan@gmail.com

MAHAVIR GANDHI is a grade 12 student from Pune. An avid lover of Number Theory and Geometry, he has won prizes in several math contests. He is particularly interested in Additive and Analytic Number Theory and wishes to settle conjectures in these fields. He may be contacted at gandhimahavir822@gmail.com

SARTH CHAVAN is a grade 11 student currently studying at M. Prakash Institute, Pune. He is keenly interested in analytic number theory, Fourier analysis and combinatorial game theory. His hobbies include swimming, chess and cycling. He may be contacted at sarth5002@outlook.com

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

Printed and Published by Manoj P on behalf of Azim Premji Foundation for Development;
Printed at Suprabha ColorGrafix (P) Ltd., No. 10, 11, 11-A, J.C. Industrial Area, Yelachenahalli, Kanakapura Road, Bengaluru 560062.
Published at Azim Premji University, Survey No. 66, Burugunte Village, Bikkanahalli Main Road, Sarjapura, Bengaluru – 562 125;
Editor: Shailesh Shirali

A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

Foundational topics include, but are not limited to, the following:

- Number systems, patterns and operations
- Fractions, ratios and decimals
- Proportional reasoning
- Integers
- Bridging Arithmetic-Algebra
- Geometry
- Measurement and Mensuration
- Data Handling
- Probability

We invite articles from teachers, teacher educators and others that are helpful in designing and implementing effective instruction. We strongly encourage submissions that draw directly on experiences of teaching. This is an opportunity to share your successful teaching episodes with AtRiA readers, and to reflect on what might have made them successful. We are also looking for articles that strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

Articles in this section may address key questions such as -

- What challenges did your students face while learning these fundamental mathematical topics?
- What approaches that you used were successful?
- What preparations, in terms of knowing mathematics, enacting the tasks and analysing students work were needed for effective instruction?
- What contexts, representations, models did you use that facilitated meaning making by your students?

Send in your articles to
AtRiA.editor@apu.edu.in

Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to 'At Right Angles'.

While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

The Closing Bracket . . .

Teaching Mathematics during the Pandemic – Some Experiences

Devendra Kumar, Sandeep Diwakar and Swati Sircar

Since March 2020, school education has been severely impacted by the pandemic and the ensuing lockdowns. As of date, children have missed more than one whole school year – the studies, the socialization, the discipline of a regular routine and so much more. While older children can (and ideally, should) take responsibility for their own learning and be more self-reliant, the same cannot be expected from the younger ones, especially those in primary school (Classes 1-5).

The Azim Premji Foundation undertook a study 'Loss of Learning during the Pandemic' (https://archive.azimpremjiuniversity.edu.in/SitePages/pdf/Field_Studies_Loss_of_Learning_during_the_Pandemic.pdf) in January 2021. The study reveals the extent and nature of the 'forgetting/regression' kind of learning loss (i.e. what was learnt earlier but has now been lost) among children in public schools across primary classes because of school closure during the COVID-19 pandemic. It covered 16067 children in 1137 public schools in 44 districts across 5 states and focused on the assessment of four specific abilities each in language and mathematics, across classes 2-6. These four specific abilities for each grade were chosen because these are among the abilities for all subsequent learning – across subjects – and so the loss of any one of these would have very serious consequences on all further learning.

The study highlights the following aspects of the learning loss in mathematics

- 82% of children on an average (67% of children in class 2, 76% in class 3, 85% in class 4, 89% in class 5, and 89% in class 6) have lost at least one specific mathematical ability from the previous year across all classes.
- Specific abilities include: Identifying single- and two-digit numbers; performing arithmetic operations; using basic arithmetic operations for solving problems; describing 2D or 3D shapes; reading and drawing inferences from data.

The alarming part is that not only do children miss out one year of school but also, they forget a lot of things they had learnt before the pandemic. Since mathematics builds up on previous knowledge (because of its axiomatic and hierarchical nature), it becomes a more serious problem.

So, what did the teachers and the school education system do? What was the impact of these efforts? What were the difficulties? Did anything work? Here, we consolidate the experiences of teaching mathematics during the pandemic, especially at the primary level - from three sources –Government school teachers of Madhya Pradesh and Uttarakhand, and teachers from one of the Azim Premji Schools.

When the chance of schools reopening faded, teachers turned to mobiles and especially WhatsApp to connect with their students. They reached out to check on how the students were doing, to assign them some work and to share resources. Online teaching was attempted. However, most children didn't have access to smartphones. Also, the limitations of online teaching were quickly felt. So, the next attempt was through worksheets, which wasn't very successful either. Then community classes were held as the best alternative given the circumstances. Each of these methods (i) online teaching, (ii) worksheet- based communication and (iii) community classes – came with its own challenges.

Government efforts: Both Madhya Pradesh and Uttarakhand governments organized online courses to enable the teachers to become more tech-friendly. WhatsApp groups were set up to connect and reach out to teachers from the cluster level and above in Madhya Pradesh. The Madhya Pradesh government provided some videos and used Door Darshan to broadcast them to the students. However, that didn't help much. The same videos shared over WhatsApp was more effective as the teachers showed them to the students from their own mobiles. The government used the WhatsApp groups to send resources and guidelines to the teachers. The Uttarakhand government developed worksheets and then a workbook (with the help of several organizations) and allowed the teachers to use them in both online and offline (i.e., face-to-face) mode. The workbook is supposed to be a key resource to bridge the learning loss and help plan for a 12-week calendar (as opposed to a full year). They have now come up with the weekly schedule for Class 1-5 on how to sequence the content for 8 weeks. They have also allowed the provision that if a child cannot cope with, say, Class 3 content, then the child will be allowed to work on Class 2 topics i.e., return to the previous class to brush up on the content in that class.

Effort from the teachers: In Madhya Pradesh, some teachers made class-wise WhatsApp groups while some made their own videos through which they guided the children (and their parents) in their own words and in a familiar voice. They organized community classes which focused more on the children who didn't have smartphones. In these classes they engaged with children in small groups of about 5 students and used Teaching Learning Materials. Parents and elder siblings or cousins were roped in to help and supervise the children. Some of the teachers utilized the experience of children who had some experience of working in local shops. This helped particularly in the content domain of numbers and number operations– the main focus of mathematics at the primary level during the pandemic. Meanwhile, in Uttarakhand, teachers reached out to children by telephoning them. Video calls were used over and above text messages to teach the children. Some spent their own money to print and distribute worksheets to children when such budget allocations were not available. Some paid for the data packs for the children's mobiles. Since the teachers could not be with the children as frequently as needed, some not only engaged local youth to fill the void but also paid them some remuneration from their own pockets! Some also mobilized the community to get the children together and arrange the space to hold the community level classes under CoVid protocol.

Difficulties: There were 3-fold difficulties in terms of teaching during the pandemic when schools were not operational –

1. Most children didn't have adequate access to devices needed to connect 'online'
   a. Most didn't have smartphones, but more ordinary mobiles which can't be used for WhatsApp or videos
   b. Since many parents didn't have work and were not earning, they couldn't recharge the phones w.r.t. data packs
   c. Power cuts hampered charging the phones
   d. Network continued to be inadequate especially in the hilly regions of Uttarakhand

   Creating and distributing worksheets haven't been easy either. Even when it was done, children were not necessarily motivated in doing the same. The supervision they needed was largely missing. Though some children did help each other in terms of showing videos and helping each other with worksheet, it was far from what they needed.

Community classes involved greater exposure to Covid-19 but while the adults were apprehensive about this risk, the children enjoyed using and learning with the TLMs. It meant meeting smaller groups of children from multiple classes together as well as fewer engagements per week compared to a regular school routine. Since practice is crucial for mastering anything, including mathematics, this reduced frequency was a big handicap.

2. Teachers had never been trained to teach in an online mode. Becoming more tech-conversant, learning to create effective worksheets and dealing with children spread across several classes are skills that take time to pick up.

3. Children were not prepared to take ownership of their learning. Both children and their parents were dependent on the teachers to a large extent for continuous guidance. In fact, in some cases, parents started ignoring the calls from the teachers.

The school experience: Things were not significantly different for the Azim Premji School. However, the teachers could get the parents more involved. The latter became more aware of their children's education thanks to weekly calls from the teachers.

Worksheets were not very effective since children 'helped' each other in the wrong way. Also, while they helped with practising what they knew, they were not helpful in introducing new concepts. Children who were not fluent readers or had lost that ability had difficulties especially with word problems and terminology.

Attendance at the community classes were good and teachers could provide more individual attention to each child and got to know them even more. The focus was more on the children who required critical help and they did improve significantly.

Conclusion: We have a long way to go. No one has much clarity on what is the best way forward especially how much to be decided centrally vs how much freedom to be given to teachers, which decisions to be taken at what level (state, district, block etc.), how to involve parents (and the community at large) who are struggling themselves who loss of livelihood at a large scale. Efforts are being made at various levels and they are far from perfect.

Hopefully, all these will help us figure out (i) what is most important with respect to school education in general and mathematics in particular, and (ii) how to help our teachers be more prepared. And hopefully we have all understood how crucial it is that the learner takes responsibility for his/her own learning.

At Right Angles
A Resource for School Mathematics

COORDINATES

PADMAPRIYA SHIRALI

Azim Premji University

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

COORDINATES

I currently teach coordinate geometry in Class 9 (ICSE Curriculum) and it is not clear to me why the basics of coordinate geometry are not included in the school syllabus (in India) in primary and upper primary levels. My experience of using it with young students has been enjoyable. Coordinate geometry can be introduced at an early stage, in both the primary and upper primary levels.

The basic concept can be appreciated by young children and the topic has plenty of scope for generating geometric exploration. It is a topic that lends itself to active participation and the usage of pegboards and geometric software. Mathematical games incorporating coordinates are great fun and develop strategic thinking.

Coordinate geometry at the middle school level provides a connection between algebra and geometry through its usage of lines. It helps students visualise simultaneous linear equations in a graphic form.

These coordinates are Cartesian coordinates and are also referred to as rectangular coordinates. The positions in the Cartesian system are defined by the distances of the points from the two axes. The scales used are linear.

There are also other types of coordinates used in maps, polar coordinates and 3-dimensional coordinates. In this article, we confine ourselves to Cartesian coordinates.

Coordinates can be introduced at the primary level with positive numbers and a transition to the usage of negative integers can be made at the upper primary level.

At the primary level, coordinates are used to describe the locations of objects using the language of position, i.e., across from the left (column) and then up or down (row). Rows and columns – words that are familiar to students – are the starting point. In an ordered pair, e.g., (3, 2), the convention is to state the column (horizontal reference) first and the row (vertical reference) second.

In an array arrangement, the coordinates refer to a discrete object. On a square grid, the location is a specific point at the intersection of the lines.

Students are generally made to stand in rows and columns either during assembly or games time. Indoors, in classrooms, they are generally seated in rows and columns. This array arrangement lends itself very well to locate each student using row and column numbers. While introducing the topic teachers must ensure that the students are familiar with the words row and column.

Note: There are some differences in the way we begin to count rows. In a physical arrangement, as in a theatre the first row is in the front. In a table format it begins from the top row, i.e., we start at the top and work our way down (top row is row 1). In the coordinate system we start at the x-axis (row 0) and work our way up. Numbering for the columns remains the same in all the situations. This need not pose a problem to understand naming of locations as the class arrangement corresponds to the graphing system.

ACTIVITY 1: POSITION, PLEASE!

Objective: Give and follow instructions using the language of position (coordinates)

Figure 1

Figure 2

The teacher points to the student in the first row and the first column and gives him/her the location number (1,1) and explains the ordered pair as "He/she is in the first column and the first row, so the location number is (1,1)." The teacher continues to give location numbers for others in row 1 by going in order: (2,1), (3,1), … (6,1), explaining each time the reason for the numbering process. After finishing with the first row, the teacher draws the attention of the students to the second row and starts with (1,2) by emphasising the words first column, second row, (2,2), etc.

The teacher must emphasise that the first number in the ordered pair refers to the column number, and the second number in the pair refers to the row number. Use the words row and column repeatedly till the students can comfortably associate the first number of the ordered pair with the column and the second number of the ordered pair with the row.

Practice

Once the students are clear about their location numbers (coordinates) the teacher can call out the name of a student, he/she has to give their location.

The teacher can call out a coordinate pair and the student in that position has to respond with an appropriate action like standing up.

Further interesting questions that can be asked are:

"Can all those whose row number equals column number stand up?" What will be the result?

"Can all those whose row number is less than the column number stand up?" What will be the result?

"Can all those whose numbers add up to 7 stand up?" Do students notice any symmetry here?

ACTIVITY 2: DOT PUZZLE

Objective: Use ordered coordinates to plot and connect

A coordinate dot puzzle consists of a set of given coordinates that students plot on a square dot/ grid paper. They plot and join the points in the order that they are given. The figure can be a cartoon character or any other picture that children can recognise.

They could also create problems of this kind with their designs, note the coordinates in order and share it with others.

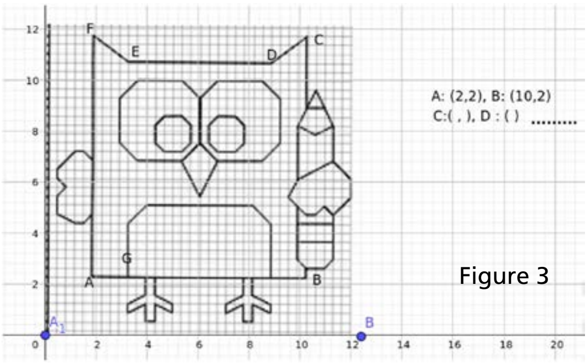


Figure 3

ACTIVITY 3: TREASURE MAP

Objective: Use coordinates to locate treasure on a map superimposed by a grid.
(GeoGebra can be used to prepare such pictures)

Pose questions: 'Where is the casket?' 'Where is the boat?' And so on.

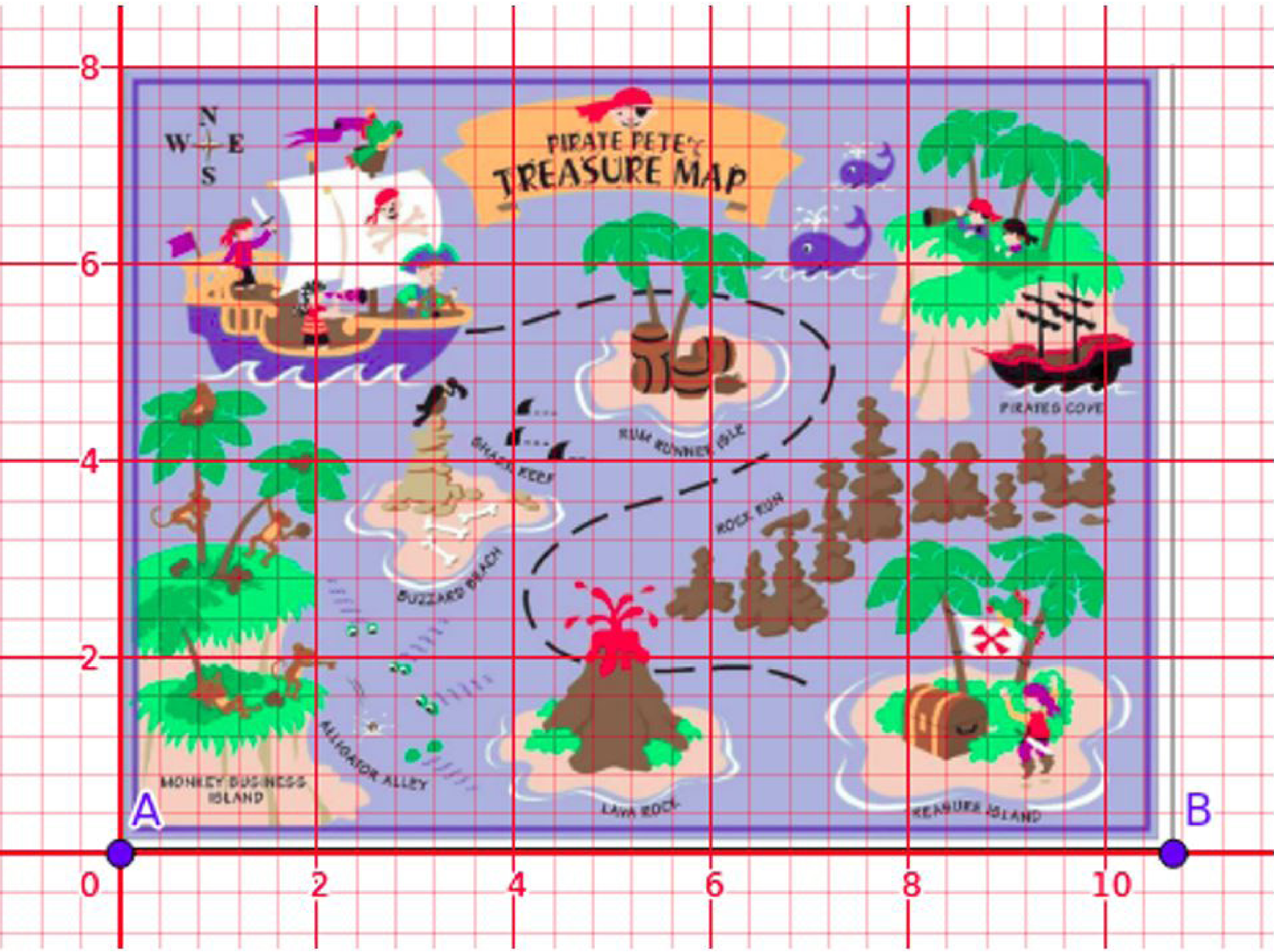


Figure 4

ACTIVITY 4: OUTDOOR TREASURE HUNT

Objective: Maps and usage of coordinates

Treasure hunts produce a lot of excitement and are generally organised using clues involving puns. It is interesting to modify the game and use coordinates. Teachers can use the school playground map with a coordinate system drawn over it. Each team will need copies of this map with the superimposed coordinate system.

Students can be separated into two teams. One team can bury some treasure at four different places on the school playground and plot the location of these places on the school map. They can then provide the coordinates of these locations to the other team. The second team has to locate the treasure based on the set of coordinates that have been given to them.

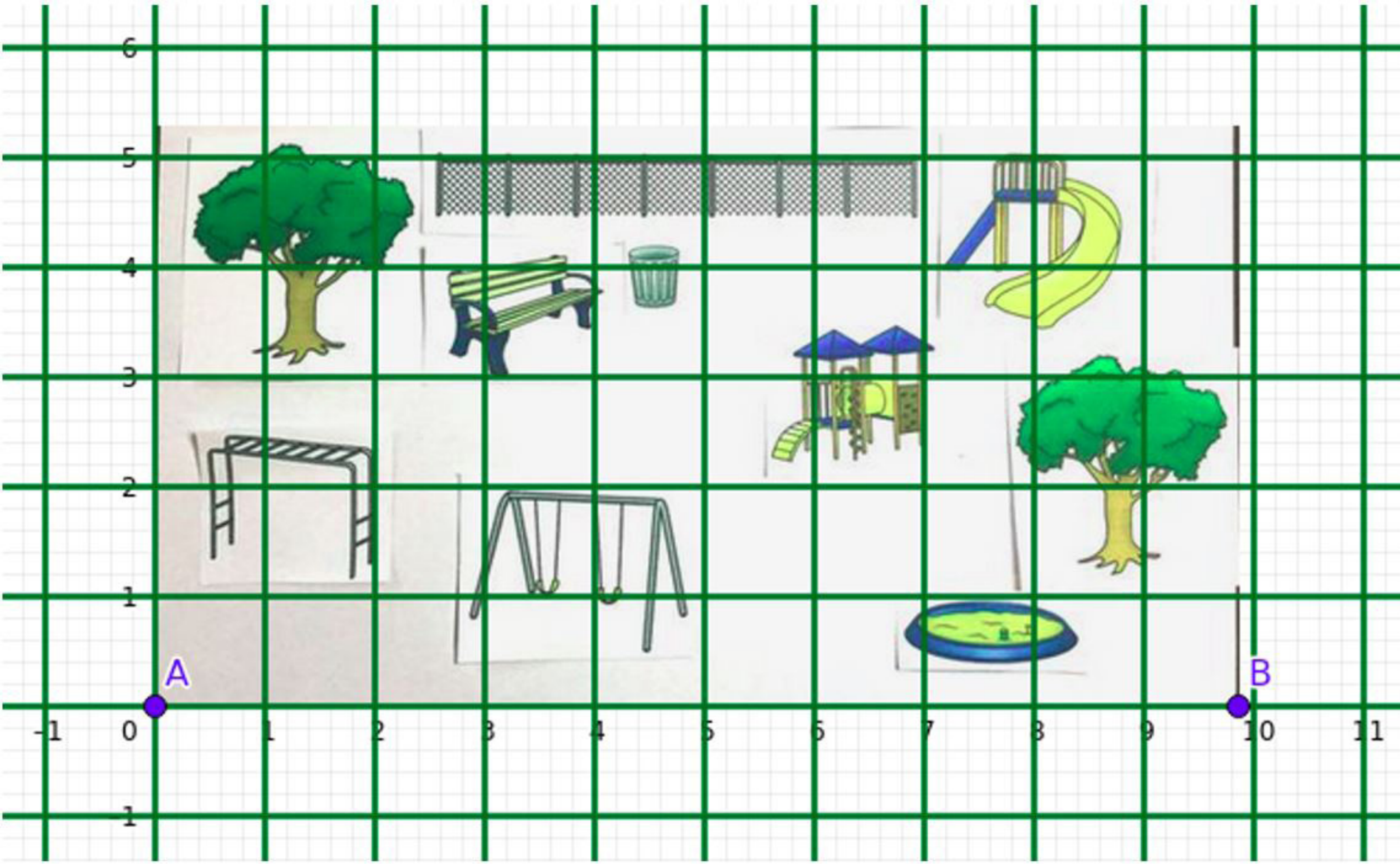


Figure 5

ACTIVITY 5: DOTS AND LINES

Objective: Plot points on a square dot paper and notice the line shape that emerges.

Students can use different colours for each question or create multiple grids on a single square sheet of paper and use one grid for each question.

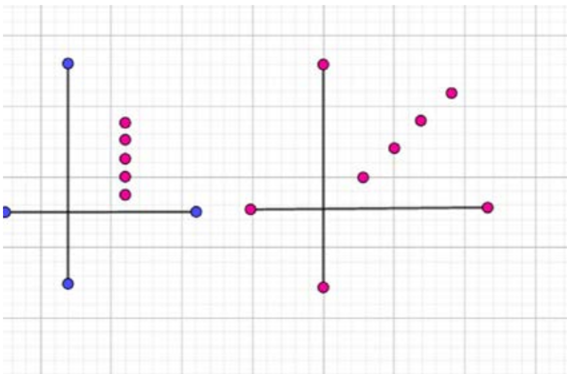


Figure 6

Questions:

Plot all the locations which have 4 as their column number. What do the points look like?

Plot all the locations which have 3 as their row number. What do the points in the figure form?

Plot all points where the row number equals the column number. What kind of a line is this? (Students may not know the word *diagonal*, so they may use their own words to describe it.)

Plot all points whose sum adds up to 8. What do you notice?

ACTIVITY 6: FIND MY ALPHABET

Objective: Visualisation, practice with coordinates and deduction of the alphabet

This activity can be done between two students. One student makes a letter of the alphabet (the digital form) on the grid and shares one coordinate pair at a time with the other student. Based on the information the other student has to deduce the letter. The second student must try to figure it out before all the coordinate pairs are given.

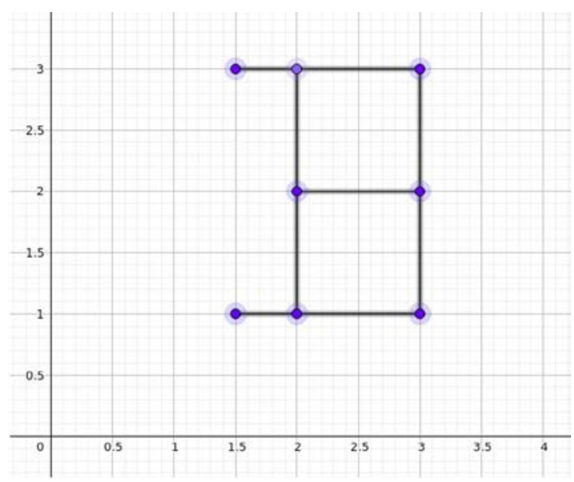


Figure 7

ACTIVITY 7: DOTS AND PATTERNS

Objective: Predicting the coordinates of a patterned figure

Students note down the coordinates of a given diagram in a table format and predict the next two coordinates.

	row	Column
A	0	3
B	1	4
C	2	5
D	3	6
E	4	7

Figure 8

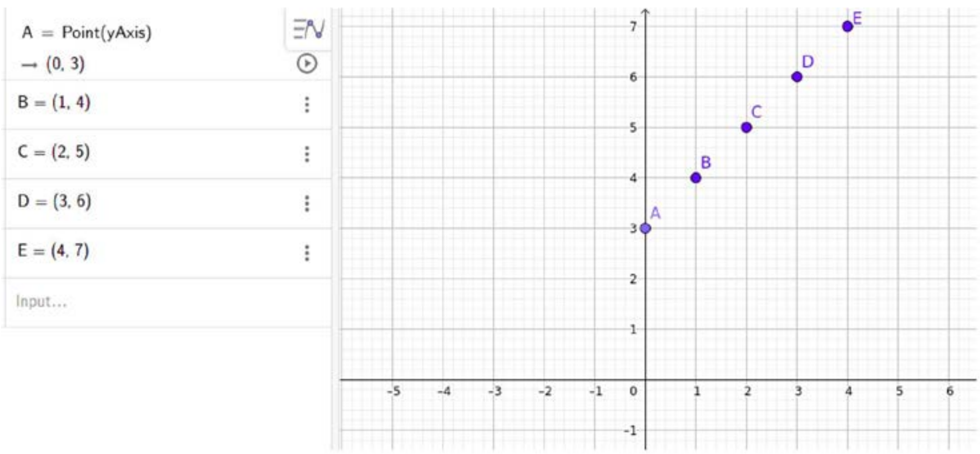


Figure 9

Here is one more example of a figure for prediction of the next two points.

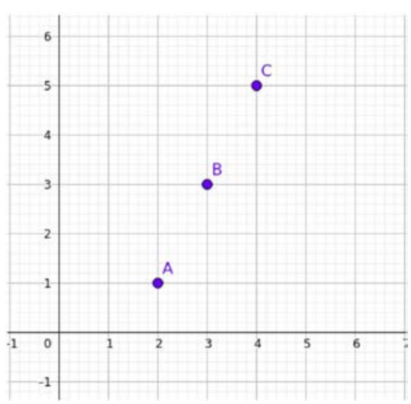


Figure 10

Note: At the upper primary level, teachers can begin to refer to column and row numbers as *X* coordinate and *Y* coordinate. Also, introduce quadrants and let students note the signs of numbers in each quadrant.

ACTIVITY 8: DOTS, LINES (FOR UPPER PRIMARY)

Objective: Plot points on a square dot paper and notice the lines or regions that emerge.

Plot all locations which have -4 as their row number. Plot all the locations which have -1 as their row number. In what way is this the same as the previous figure? In what way is it different?

Connect two points A (4,0) and B (-2,0). What can you say about the column number of all other points on this line? What is staying the same (constant)? What is changing (variable)?

Can you answer this without plotting the points?

If the two points (2, 1) and (6, 1) are joined, will they make a vertical line? Or will they make a horizontal line? How did you make out?

Where do you think the midpoint of this line segment will lie? How did you find out?

Now check your answers by plotting them.

ACTIVITY 9: GRIDLOCK

Objective: Strategic thinking

Materials: 6 x 6 Grid, two dice, 10 counters of two colours

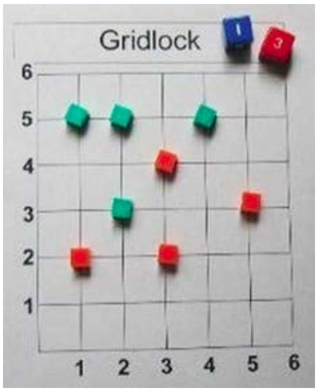


Figure 11

The game can be played by two players at a time. Each player casts the two dice and places his/ her counter on the grid. Ex. If the dice shows 2 and 3, the player can choose to place the counter either on (2,3) or (3,2) positions. Both the players take turns to place their counters on the grid. If the position is already occupied, the player skips his/her turn. Each one tries to get a series of 4 counters in a line to win the game.

ACTIVITY 10: REGIONS

Objective: To show inequalities

Use a 4 x 4 grid. Plot the points where the row number is less than or equal to 2 and shade the region. How will you describe the picture?

Plot the points where the row number is less than the column number in the first quadrant. Now plot points following the same rule in the second quadrant. What do you notice? Shade these regions.

Now plot the points observing the same rule in the third and fourth quadrants. What happens?

What would happen if you did the reverse, i.e., the column number is less than or equal to the row number?

In Figure 12 observe each quadrant separately to state what is happening out there.

How are the coordinates of the points lying on the separating line related to each other?

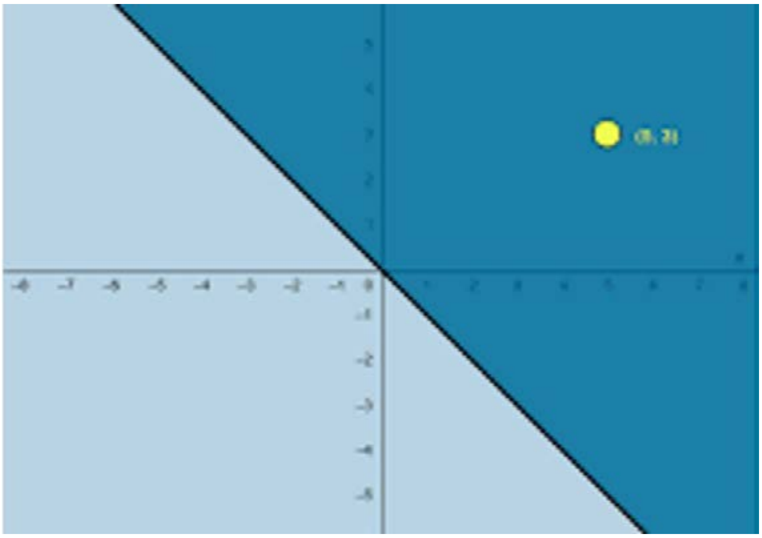


Figure 12

ACTIVITY 11: HIT OR MISS GAME!

Objective: Give and interpret instructions using coordinates

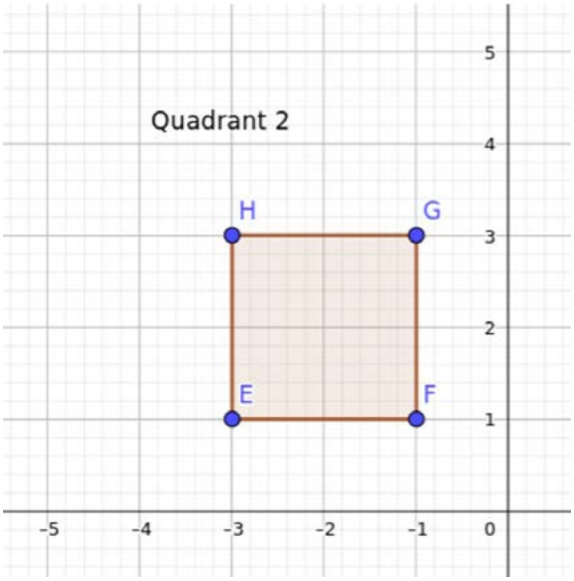


Figure 13

This can be played by two students. Players use a grid paper of 5 x 5 size. One player shades a 2 x 2 square box in his/her grid paper. It is hidden from the second player. They need to share where the shape lies, in which quadrant. The second player calls out a coordinate pair and the first player responds by saying hit or miss. If the coordinates called out are part of the shaded box it is a hit, if not it is a miss. Based on the response the second player will call out another set of coordinates.

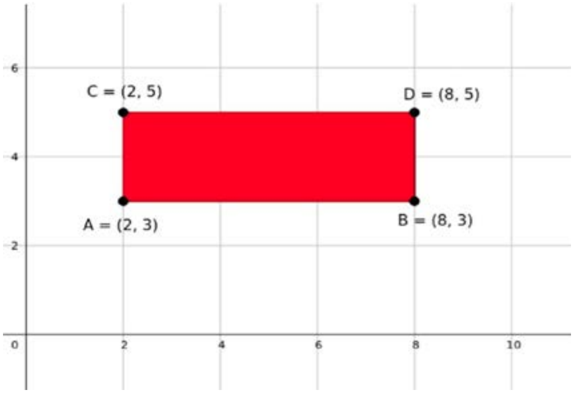


Figure 14

Players are given a maximum number of 12 chances to call out. If all the coordinates of the corners of the box are found then the player has won.

The game can be modified to find a rectangle but the level of challenge rises.

ACTIVITY 12: HIDDEN TREASURE

Objective: Visualisation
Materials: 6 x 6 square grid

The students can be made into two teams. The first team decides on a set of coordinates as the location for a treasure on a 6 x 6 grid.

The second team makes an initial guess of the location and gives a coordinate pair and plots the pair on their grid. Based on the guess the first team gives the number of steps required to get to the treasure. The number of steps is obtained by using the shortest route along the horizontal and the vertical lines (not diagonal) from the guessed location to the treasure. The second team notes down the information on their grid as shown in Figure 15. (Ex. A is 5 steps away from the treasure.) They use this information to make their second guess. At each step, the first team gives them the number of steps from the given coordinates to the treasure. This process continues till the second team manages to locate the treasure. The goal is to find the treasure in as few guesses as possible.

Where is the treasure in the example given below?

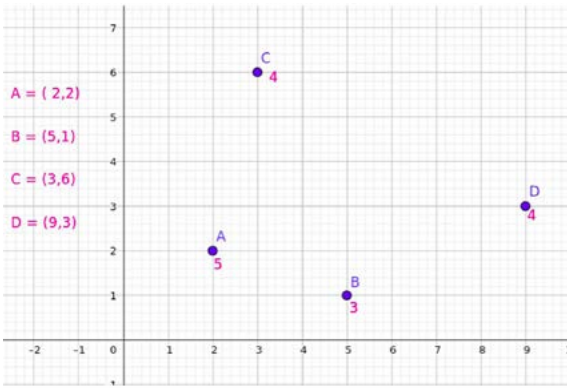


Figure 15

ACTIVITY 13: LINES

Objective: Figuring lines
Materials: 10 x 10 square grid

The line AC continues downwards to a point B. If AC is one-third of the line segment AB, what are the coordinates of B?

If AC is one-fourth of the line segment AB, what are the coordinates of B?

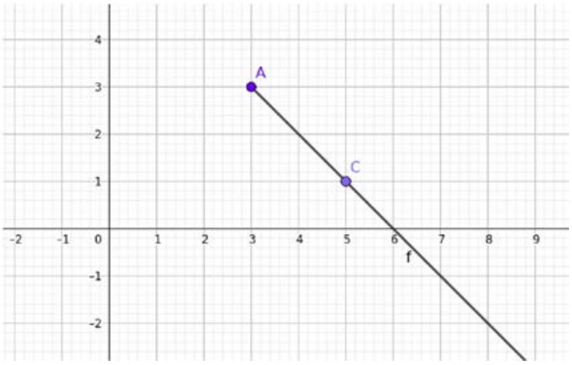


Figure 16

The line segment AB continues in both directions ending at points O and C. If AB is the middle one of three equal segments (one third of OC) what are the coordinates of O and C?

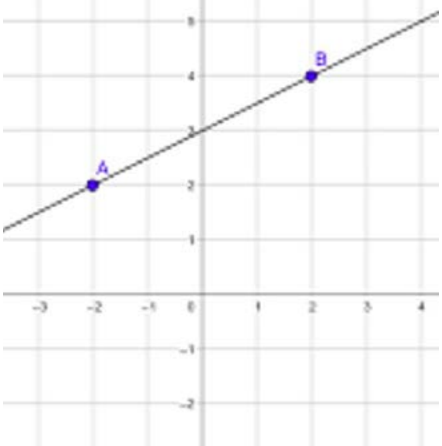


Figure 17

ACTIVITY 14: REFLECTING POINTS, LINES AND SHAPES

Objective: To reflect lines in both the axes
Materials: 10 x 10 square grid

Students can be initially asked to draw the reflections of given points along X-axis and along Y-axis.

As a second step they can reflect lines along x-axis and along y-axis.

They could also do shape reflection to generate symmetric designs.

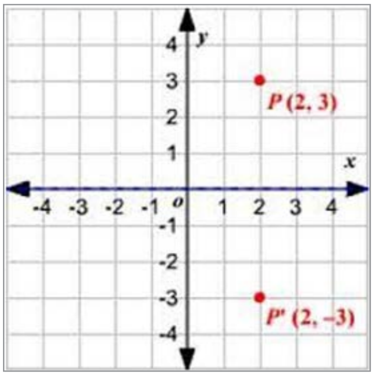


Figure 18

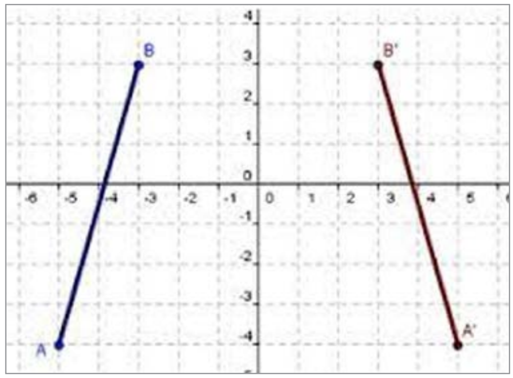


Figure 19

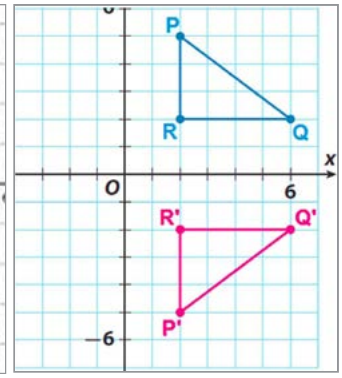


Figure 20

ACTIVITY 15

Objective: Deduce shape
Materials: 10 x 10 square grid

What would be the coordinates of the fourth vertex of this square?

Where can the coordinates lie if it were a kite? What are the possible answers?

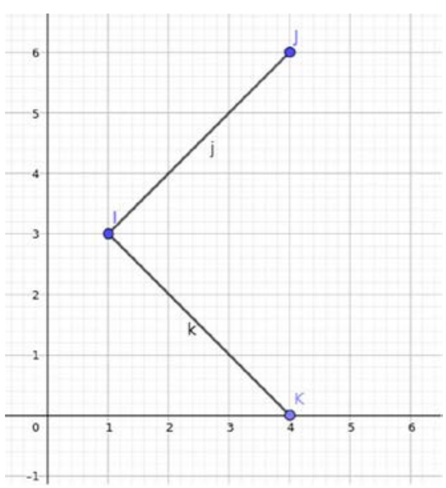


Figure 21

The centre of this rectangle lies at (-1,-0.5). Where do the vertices C and D lie?

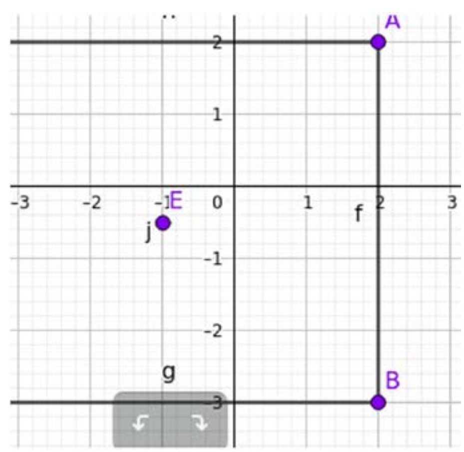


Figure 22

What shape would you get if you were to join the three sets of points (2,0), (8,0) and (5, 2)?

ACTIVITY 16

Objective: Finding patterns in coordinates to predict the next pair.
Materials: 10 x 10 square grid

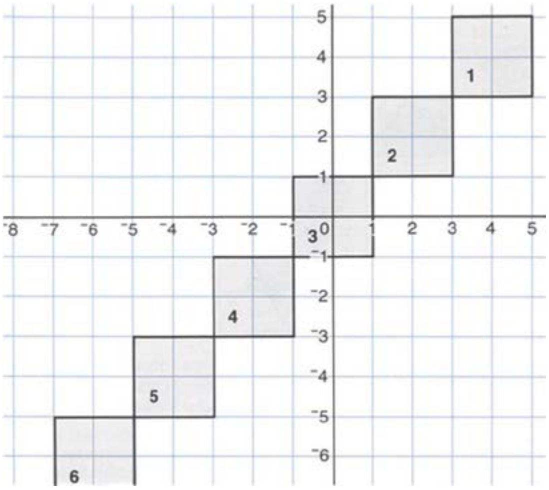


Figure 23

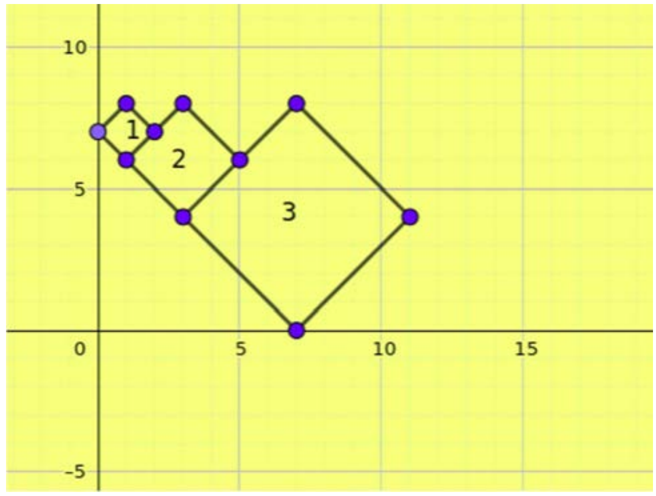


Figure 24

Here is a set of numbered squares in Figure 23 going downwards and to the left.

What will be the coordinates of the centre of the tenth square in Figure 23?

What are the coordinates of the centre of each square in Figure 24?

(The shapes, angled at 45 degrees, are growing downwards in increasing size.)

Here is a problem taken from the Nrich website. See Figure 25.

Where will the vertices of triangle 23 be?

Suggest a quick way of working out the coordinates of any triangle in this figure.

Will your method work if the triangles extended to the left?

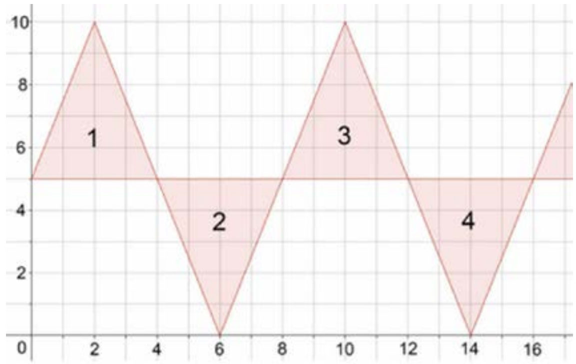


Figure 25

Acknowledgements

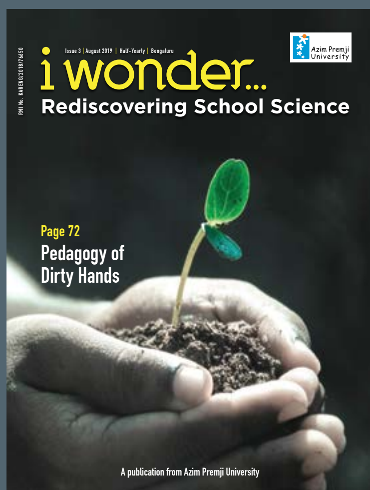
SMP
Nrich

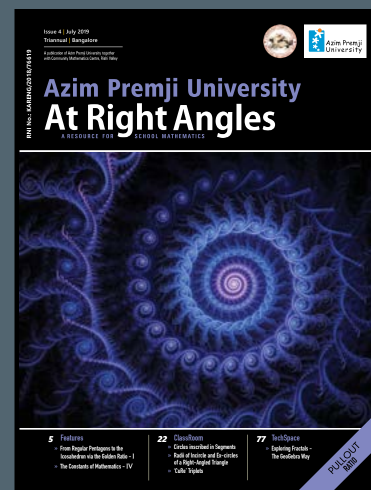
PADMAPRIYA SHIRALI

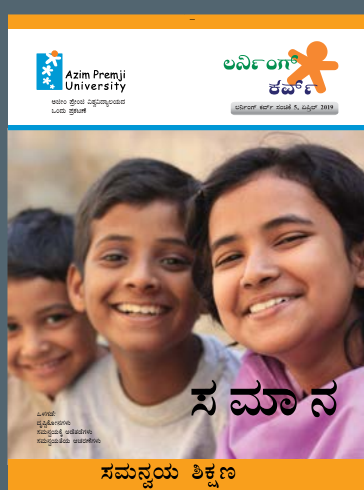
Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multi-grade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box.' Padmapriya may be contacted at padmapriya.shirali@gmail.com.

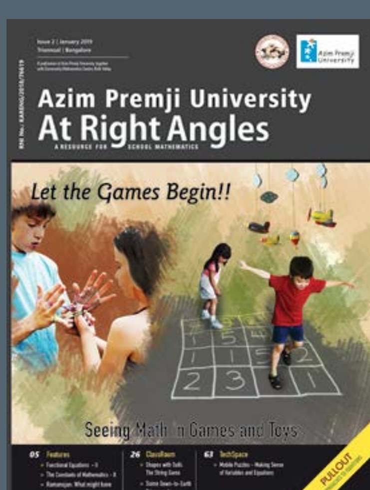
Azim Premji University

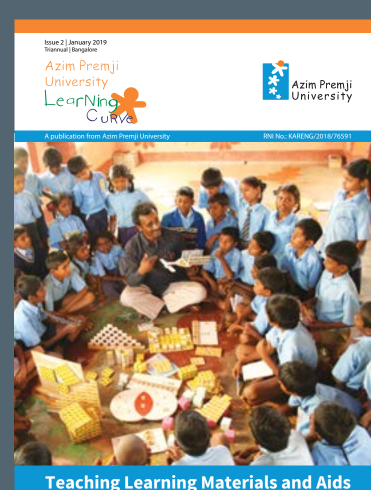
Other Magazines of Azim Premji University











Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

An in-depth, serious magazine on mathematics and mathematics education.

For teachers, teacher educators and students connected with the subject.

In this magazine, teachers and students can:

- Access resources for use in the classroom or elsewhere
- Read about mathematical matters, possibly not in the regular school curriculum
- Contribute their own writing
- Interact with one another, and solve non-routine problems
- Share their original observations and discoveries
- Write about and discuss results in school level mathematics.

Publisher
Azim Premji University together with Community Mathematics Centre, Rishi Valley.

Editors
Currently drawn from Rishi Valley School, Azim Premji Foundation, Homi Bhabha Centre for Science Education, Lady Shri Ram College, Association of Math Teachers of India, Vidya Bhavan Society, Centre for Learning.

You can find At Right Angles here:

Free download from

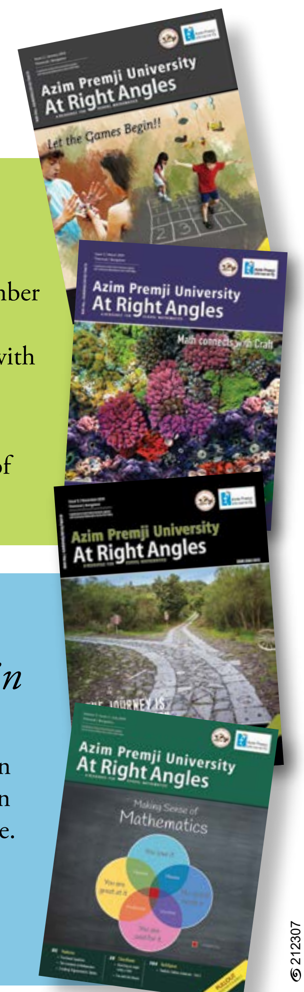
http://azimpremjiuniversity.edu.in/SitePages/resources-at-right-angles.aspx

At Right Angles is available as a free download in both hi-res as well as lowres versions at these links. Individual articles may be downloaded too.

Hard Copy

At Right Angles magazine is published in March, July and November each year. If you wish to receive a printed copy, please send an e-mail with your complete postal address to AtRightAngles@apu.edu.in

The magazine will be delivered free of cost to your address.



On FaceBook

https://www.facebook.com/groups/829467740417717/

AtRiUM (At Right Angles, You and Math) is the Face - Book page of the magazine which serves as a platform to connect our readers in e-space. With teachers, students, teacher educators, linguists and specialists in pedagogy being part of this community, posts are varied and discussions are in-depth.

On e-mail:
AtRiA.editor@apu.edu.in

We welcome submissions and opinions at AtRiA.editor@apu.edu.in The policy for articles is published on the inside back cover of the magazine.

Your feedback is important to us. Do write in.

Azim Premji University
Survey No. 66, Burugunte Village, Bikkanahalli
Main Road, Sarjapura, Bengaluru – 562 125

080-6614 4900
www.azimpremjiuniversity.edu.in

Facebook: /azimpremjiuniversity **Instagram:** @azimpremjiuniv **Twitter:** @azimpremjiuniv