



Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

ISSN 2582-1873

The Great Indian Rope Trick Constructing Mathematical Understanding



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PULLOUT
NUMBER BASES

The Great Indian Rope Trick

This is the image that comes to mind on hearing of the 'Great Indian Rope Trick'. But, did you know of another more mathematical rope trick which was used in the Shulbasutras? Some simple ideas leading to powerful mathematics:

A rope stretched taut forms a straight line.

Multiples or fractions of these lengths can be marked by folding the rope.

Distance between two points can be marked by tying knots at the appropriate locations on the rope.

Keeping one end of the rope fixed and used as a pivot while we walk around with the rope stretched taut along the ground, will describe a circle on the ground.

All these and more by the use of a simple rope? Find more in the Features article on Mathematical Instruments.



From the Editor's Desk . . .

As we worked on the first issue of At Right Angles for 2022, what stood out so strongly was that human curiosity, study, reasoning and documenting thrive even in the middle of illness, loss and change. Submissions to AtRIA.editor@apu.edu.in continue to pour in and we are often hard pressed for choice - we request patience on the part of our growing tribe of contributors.

The March 2022 issue kicks off with a lovely study of approximations to Pi; if you thought that your students could only stand and marvel, we invite you to explore and investigate, as did the adult-student duo who wrote this article. The next article will 'rope you in' even further, as you study how a simple rope or thread can simulate and go beyond the 'compass box' which is so much a part of every student's school bag.

ClassRoom begins with the Power of Stories in Math Class – an account of a school's experiments with story writing, which brought rich dividends into their teaching and learning of mathematics. Math from Simple Grids describes the power of questions- read how the activity of filling a simple grid can build the art of thinking, reasoning and problem posing. In the TearOut, you move from joining unit squares to building families of polyominoes and from 2D to 3D as you visualise the nets of cubes and cuboids. Swati Sircar continues her series on formulas- this time she deconstructs the formula for the mode. Sense-making never made so much sense before!

Three articles on the pure joy of mathematics- in Arithmetic Escapade, Prithwijit De studies two, three and then n -digit numbers which continue to be in arithmetic progression even when written in reverse and A.S. Rajagopalan finds a cool way to reorder the digits of two-digit numbers and preserve the value of the product. A Ramachandran presents six problems on area and perimeter- though some of them are solved using trigonometry, other methods may be attempted and in fact, Shailesh Shirali has presented some beautiful visuals (almost proofs without words) for some of the problems.

In Student Corner, Anushka Tonapi presents multiple methods of factorising a quadratic expression and ends with a short comparison of them. Projectiles have always been connected with parabolas, but read Suketu Patni's fascinating account of Ellipses Hidden in Projectile Motion. Barry Kissane shows us how the table function in calculators can help students grasp functions, limits and much more! Shashidhar Jagadeeshan presents a well-rounded review of a Steven Strogatz book with a very provocative title, while the Math Space review of manipulatives talks of the variety of 'spinners' and the pedagogical opportunities available if you take a chance on them. The Pullout on Number Bases introduces you to the advantages of going beyond the Base Ten system – a hands free approach to numbers.

Do write in with your feedback to AtRiA.editor@apu.edu.in

Remember that you can find past issues on
http://publications.azimpremjifoundation.org/view/university_publications/fiel18=2E4/

Enjoy!

Sneha Titus

Associate Editor

The Opening Bracket . . .

Mathematics and Truth...

We live in a world of 'post-truths', where it is increasingly difficult to separate fact from spin. We assumed that being data driven would reduce biases and help us make better and more informed decisions. However, more data and more sophisticated analytical tools have not helped to the extent we would have liked, particularly in the humanities. Instead, selective representation of data, facile interpretations and convenient extrapolations have become the norm, all designed to fit a narrative. I have recently come across several instances where the same data was used to shore up two opposite arguments. Is this the best use of human intelligence?

How does Mathematics help us get to the truth? The elegance of Mathematics, at least the parts that I read and understood in school, is that it is built on a set of foundational truths that are universally true. That $2 + 2 = 4$, or that a circle can never be a square, is my limited understanding of Mathematics. Euclid's system, the basis of geometry taught in schools even today, is a set of principles (axioms) that are absolute truths. I do not claim an understanding of non-Euclidean Mathematics where this belief could come under question.

Why do we then hear that often numbers represent half-truths? A recent tweet which I thought was quite insightful said, 'Truth is almost always the combination of two opposite half-truths'. An HR friend of mine once told me in half jest that there are four kinds of lies in increasing order: lies, dark lies, statistics and résumés. That statistics comes second worst in this ignoble list is a sad truth. Why then does data (which is essentially numbers across time) not lead to the universal truths that we see in the real world?

First, data needs to be backed by rigour. It should lend itself to scrutiny and proof. Theorems in Mathematics are accepted only when they carry proof. In fact, when Sir Andrew Wiles, often considered to be the greatest mathematician alive, proved Fermat's Last Theorem, it was rejected the first time. He had to return with the correct proof two years later. How much of the data we see in today's surveys and research reports can we vouch for? Often, the statistical rigour is compromised or tailored to suit a narrative; for example, the sampling method, the sample size, the way questions are structured, and data collected across time. The controversy on the new GDP series a few years ago is a case in point. Perhaps, oversimplification is another aspect of lack of rigour. Most of us could be guilty

of this. For instance, while assessing performance of people, I often see managers making facile arguments of extraordinary or severely abysmally poor contributions based on shaky data, only to support a recommendation.

Second, we encounter several issues in the presentation of data. If you torture data long enough, it will squeal what you like to hear! For example, the deliberate mixing up of correlation with causality, slicing the time series to show convenient truths and presenting comparative figures only to suit a point. Often sophisticated data charts are no more than just optical illusions. During the COVID-19 crisis, when real time data was a problem and historical data did not exist, mathematical modelling such as 'the R factor' were used to project trends. While it undoubtedly helped in planning a response, the spread of an epidemic depends on far too many complex factors, and any mathematical model has limitations. Should the presenters be more mindful of this?

Third, there is spin. This takes many forms. For instance, quoting only from selective sources, exaggerating a trivial point because it fits the messaging and ignoring key findings that are inconvenient. The debate on 'jobs data' in the recent past was an illustration of how truth could be perceived differently, depending on which end of the political spectrum you were. In addition, demagoguery. A wise man once said, 'A lie told over and over is perceived as truth'. And we have leaders in business, society and geopolitics who have the capacity to convince anyone, anytime on anything, leading to catastrophic effects.

It is utopian to believe that we will have one version of the truth always. However, the mark of an educated mind is to put data to test and search for the truth. A sound knowledge of Mathematics helps, and our schools can prepare children to learn to question, become positive sceptics and not rest till they arrive at the truth. I heard Uday Kotak, the celebrated banker once say, 'If something is too good to be true, it is too good to be true'. We would like teachers to enable students to relentlessly pursue truth using Mathematics as an ally and *At Right Angles* is a humble endeavour in this direction.

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Please Note:

All views and opinions expressed in this issue are those of the authors and Azim Premji Foundation bears no responsibility for the same.

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.



Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

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How Archimedes showed that pi is approximately 22 by 7

ClassRoom

This section gives you a ‘fly on the wall’ classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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TechSpace

‘This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower

Continue . . .

the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

- 85 ▶ Barry Kissane
The Educational Use of Tables on Advanced Scientific Calculators

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea

is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

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PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Padmapriya Shirali

Number Bases

Online Articles

Ashwin, Shravan, Shivkumar & Gowri Satya
Addendum to Math from Simple Grids



Shailesh Shirali
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Review: Survey Responses to-Did Calculus Change the World



The Minimal Instruments of Geometry – I

MAHIT
WARHADPANDE

1. Introduction

Euclid's *Elements* (~300 BCE) built the edifice of (plane) Geometry using a toolkit comprising of two instruments: the 'straight edge' and the 'collapsible compass' [1]. Many centuries later (1941 CE), in *Basic Geometry*, George Birkhoff and Ralph Beatley provided an alternative construction of this edifice using a three-instrument toolkit which contemporary students continue to use: the 'ruler', the 'compass' and the 'protractor' [2]. In contrast, a few centuries before Euclid (~800 BCE), Indian vedic texts (*Shulbasutras*) recommended the 'rajju', i.e., a rope, as the lone instrument to be used for geometrical constructions [3].

In the first part of this two-part article, we introduce these three toolkits and explore some ideas for rope based geometrical constructions. In the second part of the article, we will compare the three toolkits in terms of their ability to do various geometric constructions and discuss some ideas to enable the construction of the tools themselves¹.

2. Euclid and Birkhoff-Beatley Toolkits

We begin by contrasting Birkhoff and Beatley's toolkit with that of Euclid.

2.1 Ruler vs. Straight Edge

A straight edge could be described as a ruler without markings (Figure 1). Thus, a straight edge and a ruler can both be used to draw a straight line but, unlike a ruler, a straight edge cannot quantify the length of that line.

¹ Historically, things may have been done differently from the ideas discussed here. Our focus however, is on the mathematical correctness of these ideas.

Keywords: Euclid, plane geometry, instrument box, ruler, compass, protractor, rope

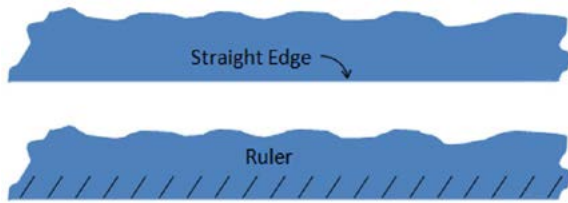


Figure 1: Straight edge (Euclid) vs. ruler (Birkhoff- Beatley)

Birkhoff and Beatley assumed that the ruler is marked with infinite resolution, so any length (rational or irrational) can be measured/constructed exactly using such a ruler.²

2.2 Normal vs. Collapsible Compass

The ‘normal’ compass is the one found in contemporary school geometry boxes. This compass ‘remembers’ the distance between its steady and adjustable legs when the ‘hinge’ is screwed tight (Figure 2).

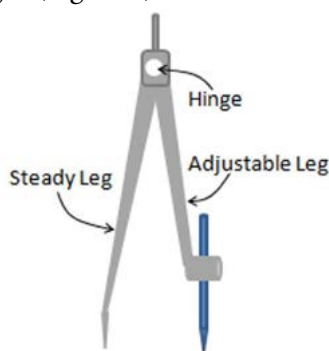


Figure 2: Basic parts of a ‘normal’ compass

If the hinge is loosened, the legs of the compass will ‘collapse’ together when they are lifted from the paper. In this case, the compass cannot ‘remember’ the distance between the two legs. This then becomes Euclid’s collapsible compass. Euclid showed that the straight edge and collapsible compass can perform the job of a ‘normal’ compass in terms of reproducing a given length at a different location [1 pp. 244-246]. We therefore, use the word ‘compass’ to refer to a ‘normal’ compass hereafter.

2.3 Protractor

The protractor is used to construct and measure angles (Figure 3). This instrument from the Birkhoff-Beatley toolkit has no direct equivalent in Euclid’s toolkit.

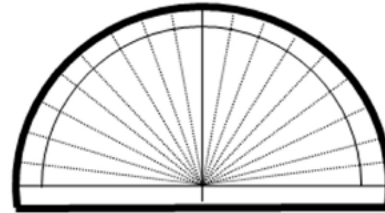


Figure 3: The protractor

Like the ruler, the protractor too is assumed to be marked with infinite resolution and can therefore be used to measure/construct any angle.

3. The Great Indian Rope Trick

Let us now see how the rope can be used as a geometrical instrument. In the *Shulbasutra* context, geometrical constructions were used to build life-size architectural structures with dimensions up to a few tens of metres. However, the geometrical principles involved in these constructions are also valid at smaller scales, just as those of Euclid or Birkhoff-Beatley are valid at scales larger than a sheet of paper.

3.1 Rope as a ‘Markable’ Straight Edge

A rope stretched taut forms a straight line. Thus, to construct a straight line between any two points on the ground, we simply stretch a rope taut between the two points and use it as a straight edge (Figure 4). Further, the distance between the two points can be marked by tying knots at the appropriate locations on the rope. In contrast, on Euclid’s straight edge we are not allowed to make any markings to record a length.

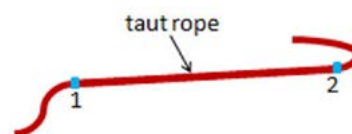


Figure 4: Rope as a straight edge between points 1 and 2

² If you have a ruler marked only at integer lengths, you can only measure lengths such as 1 unit, 2 units, 3 units... Or, if you have a ruler marked only at half-integer lengths, you can only measure lengths such as 0.5 units, 1 unit, 1.5 units... B & B take their ruler to have infinite resolution. You can measure any length with it, rational or irrational.

We assume that our rope based geometrical constructions are being performed over 'level' ground. This is equivalent to assuming that the Euclid or the Birkhoff-Beatley toolkit based constructions are being performed on a 'plane' sheet of paper.

3.2 Rope as a Compass

Let us say that we are given two points. One point marks the desired centre of the circle while the distance between the points is the desired radius. We can now stretch out a rope between them and mark it corresponding to the two points. If the point of the rope corresponding to the centre of the circle is now used as a pivot as we walk around with the rope stretched taut along the ground, the other mark on the rope will describe a circle on the ground as shown in Figure 5.



Figure 5: Using a rope to make a circle

Since the rope can be marked, it can 'remember' a distance and function as a 'normal' compass.

4. Unique Rope Constructions

The ability to use the rope as a straight edge and as a compass means that it is at least as good as Euclid's toolkit for geometrical constructions. Additionally, the flexibility of the rope, coupled with the ability to mark it, allows some alternatives to Euclidean constructions as well as some constructions that are impossible with the rigid instruments of the Euclid and Birkhoff-Beatley toolkits. We look at some such constructions here.

4.1 Multiples and Fractions of a Length

The *Shulbasutras* describe several constructions requiring the division of a finite straight line into a number of equal parts but do not explain how

this division is to be done [3 pp. 41–42]. That may be because with a rope, both multiplication and division of a length by any natural number is a fairly intuitive construction.

Figure 6 illustrates the multiplication of a given length. If L be the length marked off by points '1' and '2' on the rope, then the points '1' and '3' will be $2L$ apart while the points '1' and '4' will be $3L$ apart when the rope is unfolded and stretched taut again.

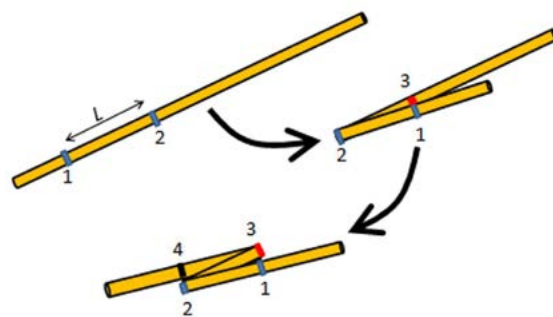


Figure 6: Multiplying a given length

Similarly, Figure 7 shows how a length can be divided. The division of L by 2 can be achieved in one shot by making the fold at 'a' such that the marks '1' and '2' align, with each of the folded sections stretched taut. When the rope is unfolded and stretched out again, the distances from '1' to 'a' and from 'a' to '2' will each be $L/2$.

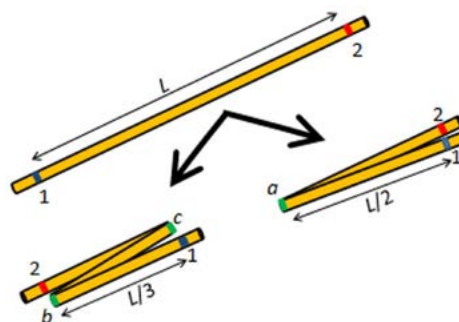


Figure 7: Dividing a given length

The division of L by 3 takes some 'tuning'. We need to make two folds between the '1' and '2' marks and adjust them such that one fold each aligns with these marks while keeping each folded section of the rope stretched taut. In Figure 7, these fold locations have been marked as 'b' and 'c'. Then, the distances '1' to 'b' or 'b' to 'c' or 'c' to '2' are each $L/3$.

In theory, this technique can be used to multiply or divide a length by any natural number N using $N-1$ folds. Consequently, any rational multiple of the length L can be obtained by this technique.

It is also possible to find the square root of a given length. This can be done by using the procedure set out in the *Shulbasutras* to construct a square that is equal in area to a given rectangle [3 pp. 83–85, 4]. Following this technique for a rectangle having the given length and unit breadth, we will get a square whose side is the square root of the given length.

4.2 Multiples and Fractions of an Angle

The rope can be used to measure curved lengths. In practice, the curves can be marked as a groove in the ground in which the rope can be fit and the length of the curve marked on it. This idea can be applied to a circle to convert a given angle θ° into an arc-length as shown in Figure 8 (using the property that angle (in radians) = arc/radius).

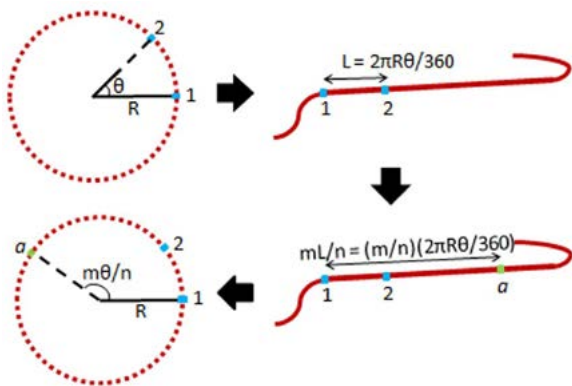


Figure 8: Constructing rational multiple of given angle

We use the vertex of the angle as the centre and the length of one of the arms forming the angle as radius to construct a circle around which we fit a rope. We mark the rope where the two arms of the angle (extended if required) cut the circle. This arc length can then be multiplied by any rational factor m/n to construct the angle $(m\theta^\circ)/n \pmod{360}$.

Apart from rational multiples, square root length constructions can also be used for corresponding irrational angle constructions.

4.3 Conic Sections

Though this has not been discussed in the *Shulbasutras* and is probably a later discovery, the rope enables the construction of ellipses, hyperbolas and parabolas [5]. This cannot be achieved by either the Euclid or the Birkhoff-Beatley toolkits.

Figure 9 illustrates the construction of an ellipse using a single rope F_1MF_2 of length R . We exploit the property of the ellipse that any point on it is such that the sum of its distance from two fixed points (the foci F_1 and F_2 of the ellipse) is some constant.

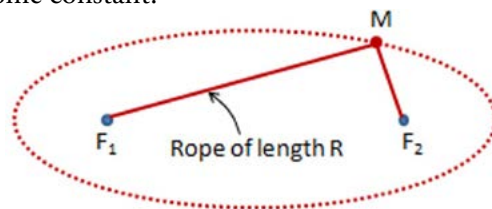


Figure 9: Construction of an ellipse

The marker M traces out the ellipse as it is moved around while keeping the two sections of the rope, F_1M and F_2M , stretched taut. Though the location of M along the length of the rope is not fixed, $F_1M + F_2M$ will always equal the total length R of the rope, which is a constant.

Figure 10 shows a hyperbola construction using ropes. A taut rope F_1P of length R is used as a straight edge pivoting at F_1 while another rope F_2MP of length L is used as a flexible rope which is folded into two sections of variable lengths PM and MF_2 by the marker M . The points F_1 and F_2 are fixed. The construction relies on the property of the hyperbola that all points on it are such that the difference in their distances from two fixed points (the foci F_1 and F_2) is some constant.

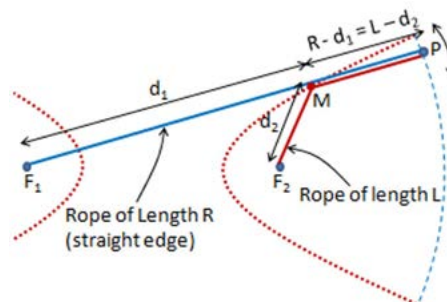


Figure 10: Construction of a hyperbola

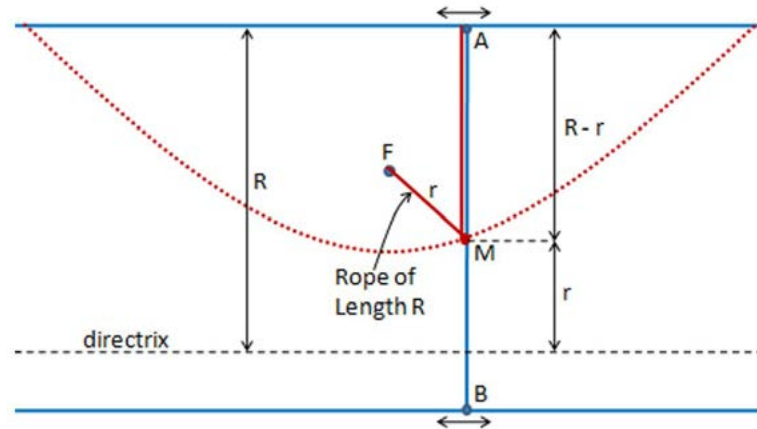


Figure 11: Construction of a parabola

As we can see in Figure 10, the length of the section PM of the flexible rope that lies along F_1P is $L - d_2$ where d_2 is the distance of the marker M from the focus F_2 . Further, PM is also equal to $R - d_1$ where d_1 is the distance of M from the focus F_1 . Thus, as F_1P rotates, the position of M varies, but we always have $PM = L - d_2 = R - d_1$, i.e., $d_1 - d_2 = R - L$, a constant. This is exactly the condition that defines a hyperbola.

The construction needs to be mirrored to get both halves of the hyperbola. In practice, it is far more convenient to use a rigid straight edge instead of a rope for F_1P .

Lastly, Figure 11 shows the construction of a parabola. There are two parallel (horizontal) fixed ropes on which a third (vertical) movable rope AB gets pushed (right or left) by the marker M placed on the flexible rope FMA . For A and B to always lie on the fixed horizontal ropes, AB must move parallel to its own previous position. During this movement, the section AM of the flexible rope lying along AB will be of length $R - r$ where R is the total length of the flexible rope, and r is the distance of the marker M from the fixed point F , the focus of the parabola. The length $FM = r$ makes up the remaining part of the flexible rope.

We now define the 'directrix' as a straight line parallel to the two fixed horizontal ropes and at a

distance R below the top horizontal rope. Then, the distance between M and the directrix must also be r . Thus, as the marker M pushes the rope AB to the right or left, it will describe the locus of points which are equidistant from F and the directrix, which is the definition of a parabola.

Again, in practice, the mechanical set-up is far more convenient if we use rigid materials for the fixed parallel ropes and movable vertical rope.

4.4 Practical considerations

The rope based geometrical constructions described here would ideally need ropes of zero thickness and infinite flexibility. In practice, therefore, these constructions will have inaccuracies. Also, as mentioned earlier, though the mathematical principles involved in these constructions are scalable, carrying out rope-based constructions for dimensions of a few centimetres or smaller would involve significant handling difficulties. For all such cases, a thread instead of a rope may work better. Another source of inaccuracies could be some inherent elasticity of the rope.

Having reviewed some uses of the rope as a geometrical instrument, we will analyze how it compares with the Euclid and Birkhoff-Beatley toolkits in the next part of the article.

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Some interesting patterns in palindromic numbers

Manoranjan Ghoshal

A palindromic number is a number that remains the same even when it is written in reverse.
(Example: 11, 101, 90909 etc.)

There are two types of such numbers:

1. **Mono** numbers have only one type of digit. (For example, 11, 222, 5555, etc.)
2. **Different** numbers have different digits. (For example, 101, 121, 12021, 12321, etc.)

Will the power of a mono palindrome also be a palindrome?

	1	1
×	1	1
	1	1
1	1	0
1	2	1

	1	2	1
×	1	1	1
	1	2	1
1	2	1	0
1	3	3	1

Similarly,

		1	1	1
	×	1	1	1
		1	1	1
	1	1	1	0
1	1	1	0	0
1	2	3	2	1

You can see similar patterns for higher powers of 111 and 1111

For the following examples, suggest answers and explain your reasoning. The first two have been done for you:

1. $(11111)^2 = 123454321$
[Since 11111 has 5 digits, write 12345 and thereafter 4321].
2. $(11.1)^2 = 123.21$

[There are 3 significant digits so first write 12321 and then account for the decimal place]

3. $\sqrt{12321}$
4. $\sqrt[3]{1.331}$
5. What can you say about $(333)^2$? Is the answer a palindrome?
6. What about $(125)^2$?

How Archimedes showed that pi is approximately 22 by 7

DAMINI D.B. and
ABHISHEK DHAR

Background

The mathematical constant π is defined as

$$\pi = \frac{C}{D}, \quad (1)$$

where C is the circumference and D is the diameter of a circle. The value of C/D is constant, regardless of the size of the circle. It is known that π is an irrational number, which means that it cannot be expressed as a ratio of two integers. Its value correct to the sixth decimal place is 3.141592 and fractions such as $22/7$ and $355/113$ are often used to approximate it. Since Archimedes was one of the first persons (around 200 BCE) to suggest a rational approximation of $22/7$ for π , the number is sometimes referred to as *Archimedes' constant*. In 2019, Emma Haruka Iwao from Japan numerically computed about 31 trillion digits of π , and the record for memorizing the maximum number of digits (71,000) is held by Rajveer Meena from India.

In the first part of this article, we briefly discuss the history of the computation of π and some formulas that mathematicians have discovered to represent this number. A detailed history of π can be found in [1] and [2]. Then, we describe the ideas and steps of Archimedes' derivation and close with some concluding remarks.

Keywords: pi, Archimedes, irrational number

A brief history of the number pi

The value of π was first calculated 4000 years ago. The ancient Babylonians and the Egyptians calculated an approximate value of π by actual physical measurements of the circumference or the area of a circle, and they estimated that π had a value close to 3.

About 1500 years later, the Greek mathematician Archimedes first used mathematics to estimate π and showed that its value lies between $22/7$ and $223/71$. The way he reasoned was as follows. He noted that a regular polygon circumscribed around a circle would have a perimeter *larger* than the circumference of the circle, while a regular polygon inscribed in the circle would have a *smaller* perimeter. He then observed that as one increased the number of sides of the polygon, the two perimeters close in on the circumference of the circle. Finally, he used Pythagoras's theorem to find the perimeters of the polygons and thus got upper and lower bounds for the value of π . Using a hexagon, a 12-sided polygon, a 24-sided polygon, a 48-sided polygon and then a 96-sided polygon, he proved that $223/71 < \pi < 22/7$. In this article, we discuss the basic ideas behind his derivation. (In fact, we provide an improved lower bound.) The details of his original method can be found in [1].

During the 5th century CE, the Indian mathematician Aryabhata calculated a value of π that was accurate up to 3 decimal digits. Zu Chongzhi, a Chinese mathematician and astronomer, calculated an approximate value of π using a 24576-gon, about 7 centuries after Archimedes. His estimated value, $355/113$, is approximately equal to 3.14159292. The Greek letter π was first introduced by William Jones in 1706. It was derived from the first letter of the Greek word 'perimetros,' meaning circumference.

Formulas for pi

Attempting to find the value of π by actual physical measurement of the circumference and diameter of a circle would involve a lot of errors and not be particularly reliable. There are many different formulas that mathematicians have discovered for representing π . Here are some examples [1]–[5].

Infinite series representations.

(A) The Madhava-Gregory-Leibniz series (1300-1700 CE):

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

(B) Nilakantha series (1500 CE):

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{2 \times 3 \times 4} - \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} - \frac{1}{8 \times 9 \times 10} + \dots$$

Using this series, we can get accurate estimates for π by including more terms in the series.

Continued fractions representation. The following form was obtained by Brouncker (1660 CE):

$$\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \dots}}}}$$

Infinite product representation. The following product form was found by Wallis (1656 CE):

$$\frac{\pi}{2} = \left(\frac{2}{1} \cdot \frac{2}{3}\right) \times \left(\frac{4}{3} \cdot \frac{4}{5}\right) \times \left(\frac{6}{5} \cdot \frac{6}{7}\right) \times \left(\frac{8}{7} \cdot \frac{8}{9}\right) \times \dots$$

Infinite nested radical representation. This involves taking an infinite number of nested square roots and was first found by Vieté (1593 CE):

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \times \dots$$

As we will see, Archimedes' method leads to a similar representation.

Archimedes's method for estimating pi

As discussed above, the basic idea is quite simple. We draw a regular polygon with n sides that circumscribes a circle with diameter 1. Let the perimeter of such a polygon be denoted by C_n . We also inscribe a regular polygon with n sides in the circle. Let its perimeter be denoted by c_n . Then, if C denotes the circumference of the circle, it is clear that we have

$$c_n < C < C_n. \quad (2)$$

As we increase n , both c_n and C_n should get closer to C . We illustrate this in Figure 1 for the cases with $n = 3, 6$ and 12 . It is clear that the polygon with 12 sides (dodecagon) has a circumference that is the closest to the circle.

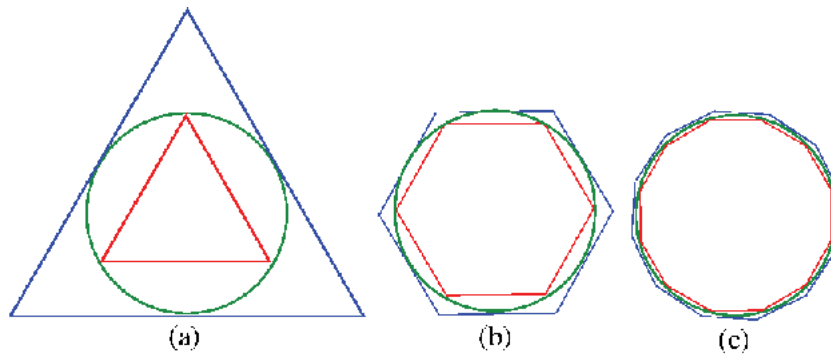


Figure 1. Here we show polygons of n sides inscribed (red) and circumscribed (blue) around a circle: (a) triangle ($n = 3$), (b) hexagon ($n = 6$), (c) dodecagon ($n = 12$)

Therefore, if we can find a way to compute the perimeter of an n -sided polygon, we can arrive at an accurate value of π simply by taking a large value for n . We now need a method to find the perimeter of an n -sided polygon. We can do this using trigonometry and the Pythagoras theorem though we must be careful not to use any result that already uses the value of π . From Figure 2a and Figure 2b, it is clear that the perimeters of the inscribed and circumscribed polygons for a circle of diameter 1 are respectively given by

$$c_n = n \cdot \sin \frac{180^\circ}{n}, \quad C_n = n \cdot \tan \frac{180^\circ}{n}. \quad (3)$$

It turns out that for some choices of n , it is easy to evaluate c_n and C_n .

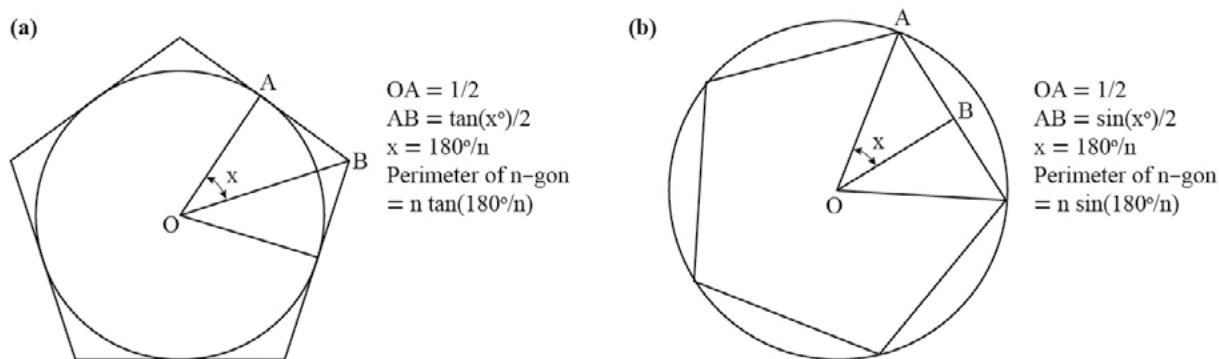


Figure 2. Here we show the calculation of (a) C_n and (b) c_n for $n = 5$

For example, take $n = 3$. Using Pythagoras theorem one can easily prove that $\sin 60^\circ = \sqrt{3}/2$, so we get $c_3 = 3\sqrt{3}/2$. Next, using the formulas

$$\cos x = \sqrt{1 - \sin^2 x}, \quad \tan x = \frac{\sin x}{\cos x}, \quad (4)$$

we get $\cos 60^\circ = 1/2$ and $\tan 60^\circ = \sqrt{3}$. Hence, we get $C_3 = 3\sqrt{3}$.

If we now double the number of sides, we get $n = 6$, and we need the values of $\sin 30^\circ$ and $\tan 30^\circ$ to find the perimeters of the inscribed and circumscribed hexagons. Now $\cos 30^\circ$ is easy to find from $\cos 60^\circ$ using the trigonometric identity

$$\cos x = \sqrt{\frac{1 + \cos 2x}{2}}. \quad (5)$$

This then gives us $\cos 30^\circ = \sqrt{3}/2$ and $\sin 30^\circ = 1/2$. And then we go to $n = 12$ for which we use the same rule to find $\cos 15^\circ = \sqrt{2 + \sqrt{3}}/2$ and $\sin 15^\circ = \sqrt{2 - \sqrt{3}}/2$. So the idea is to keep doubling n and each time we can find the cosine, sine and tangent of the relevant angle by using the formulas in (4) and (5). The results for $n = 3, 6, 12, 24, 48, 96$ are given in Table 1. We see a pattern in the expressions and after some time we can guess the next entry in the table. We also plot the numerical values (up to the 8th decimal place) and see that they become closer to π .

In Figure 3a we show how c_n and C_n for a circle with unit diameter change as we make n larger. Figure 3b shows a zoomed in version of Figure 3a, the values $22/7 = 220/70$ and $223/71$ that were given by Archimedes, and also the true value of π . We see that $22/7$ is just a bit larger than C_{96} while $223/71$ is just a bit smaller than c_{96} .

It is clear that to get an accurate value of π , we have to take a very large value of n . In Figure 3 we see that even with a 96-gon the difference is in the third decimal place. To 12 decimal places it is known that $\pi = 3.141592653589$, while from our formula we find that $C_{24576} = 3.141592670702$ and $c_{24576} = 3.141592645034$. The Chinese mathematician Zu Chongzhi in fact used these results for the 24576-gon to suggest the approximate rational value $355/113$ which is about 3.141592920354.

Rational approximations

We see from Table 1 that the perimeter of the polygon typically involves taking multiple square roots and is an irrational number. An interesting question is to find a rational number that is very close to this irrational number. Consider the value $c_{96} \approx 3.14103195$. A rational number approximating this (called a

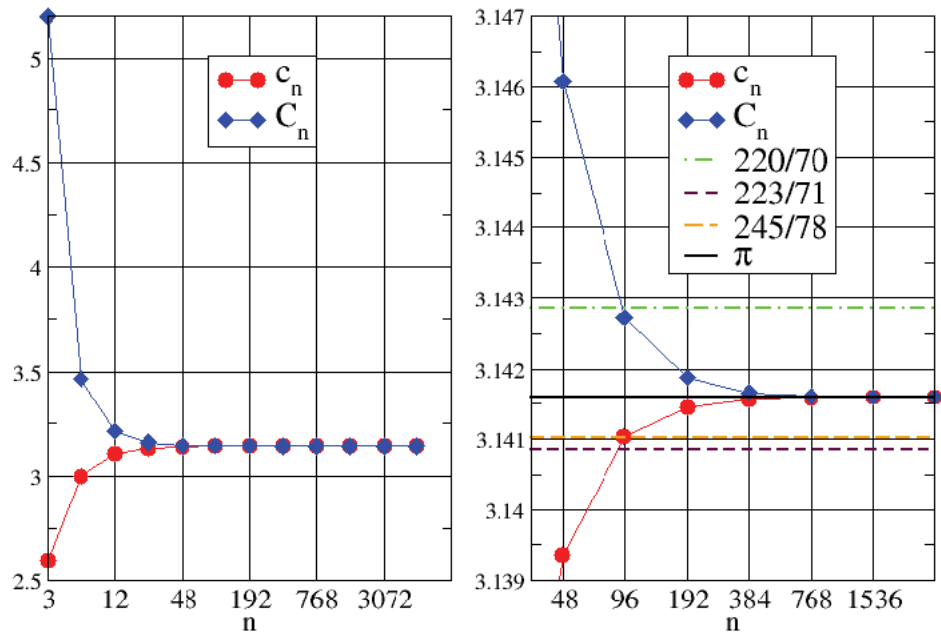


Figure 3. The values obtained for c_n and C_n plotted against n . The right panel shows a close-up so that we can see the values $22/7$ and $223/71$ obtained by Archimedes and also the true value of π . The improved lower bound $245/78$ is also plotted.

n	$c_n = n \cdot \sin(180^\circ/n)$	Approx value	$C_n = n \cdot \tan(180^\circ/n)$	Approx value
3	$3\sqrt{3}/2$	2.59807621	$3\sqrt{3}$	5.19615242
6	3	3	$2\sqrt{3}$	3.46410161
12	$12 \left(\frac{\sqrt{2-\sqrt{3}}}{2} \right)$	3.10582854	$12 \left(\frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}} \right)$	3.21539031
24	$24 \left(\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2} \right)$	3.13262861	$24 \left(\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{\sqrt{2+\sqrt{2+\sqrt{3}}}} \right)$	3.15965994
48	$48 \left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2} \right)$	3.13935020	$48 \left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}} \right)$	3.14608622
96	$96 \left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{2} \right)$	3.14103195	$96 \left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}} \right)$	3.14271460

Table 1.

rational approximant) is $314103/100000$. But can we get a rational approximant with a smaller denominator? Archimedes obtained the value $223/71 \approx 3.140845$ which has a smaller denominator and is still a good approximation. We are not sure how Archimedes made this guess.

However, one approach to find a good rational approximation for any number is to write it in the form of a *continued fraction expansion* that we then truncate after some term. We discuss this idea briefly and apply it to our problem.

A continued fraction expansion is given by the following form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

where $a_1, a_2, \dots$ are positive integers. For an irrational number, the series goes on forever, while for a rational number it ends after a finite number of terms. If we specify a number by its numerical value (to some decimal places), there is a simple procedure to write down its continued fraction expansion. For illustration let us consider the number $3.14 = 157/50$. We see that $a_0 = 3$. The remaining part of the expansion has the value $7/50$. To find a_1 , we have to take the integer part of its reciprocal, i.e., $50/7$. This gives us $a_1 = 7$. Repeating the procedure with the remainder, that is $1/7$, we find $a_2 = 7$. Hence we can write

$$3.14 = 3 + \frac{1}{7 + \frac{1}{7}}.$$

So we get the series of rational approximants for 3.14 as: $3/1, 22/7, 157/50$.

If we perform this procedure for $c_{96} \approx 3.14103195$, we get

$$\frac{314103195}{100000000} = 3 + \frac{1}{7 + \frac{1}{11 + \frac{1}{25 + \frac{1}{1 + \frac{1}{25 + \frac{1}{1 + \frac{1}{27 + \frac{1}{13}}}}}}}}.$$

Truncating the above series at various orders, we get the rational approximants $3/1, 22/7, 245/78, 6147/1957$, and so on. This does not include Archimedes' value of $223/71$. However, notice that $245/78 \approx 3.141025641$ is in fact a better approximation to c_{96} than $223/71$. Moreover we see that $245/78 < c_{96}$ and so we can use it as a lower bound. Applying the same procedure to C_{96} gives us the approximants $3/1, 22/7, 3149/1002$, and so on. In this case we see that $22/7 > C_{96}$ and so we can use this value as an upper bound. Hence we finally get the somewhat improved bound

$$\frac{245}{78} < \pi < \frac{22}{7}, \quad \text{i.e.,} \quad 3\frac{11}{78} < \pi < 3\frac{1}{7}.$$

The lower bound here is better than that given by Archimedes, in the sense that

$$\frac{223}{71} < \frac{245}{78} < \pi < \frac{22}{7}, \quad \text{i.e.,} \quad 3\frac{10}{71} < 3\frac{11}{78} < \pi < 3\frac{1}{7}.$$

If we find the continued fraction series for $C_{24576} = 3.14159267$, we get the approximants $3/1, 22/7, 333/106, 355/113, \dots$. The last number is the approximation discovered by Zu Chongzhi.

Conclusion

In this article, we have described Archimedes's method of determining the value of π by approximating the circumference of a circle of unit diameter by the perimeters of inscribed and circumscribed regular polygons. As we increase the number of sides, we get more accurate approximations for π . As we have shown, it turns out to be easy to find the required perimeters if we restrict ourselves to polygons with number of sides $n = 3 \times 2^k$, with $k = 0, 1, 2, 3, \dots$, i.e., $n = 3, 6, 12, 24, 48, \dots$. Archimedes went up to $n = 96$ and from this deduced the approximate value $22/7$ that is widely used today. Zu Chongzhi went up to $n = 24576$ and obtained the value $355/113$ which is a better approximation of π . Today we can use a calculator or computer to find the precise numerical value of the perimeter of a polygon, which typically involves finding a lot of square roots. It must have been very difficult in the times of Archimedes and Chongzhi to compute these numbers by hand. It would have required a lot of clever thinking to arrive at simple rational approximations such as $22/7$ and $355/113$, and it is an interesting question as to how exactly they arrived at these results [1].

Acknowledgements. We thank Ranjini Bandyopadhyay, Deepak Dhar, Rajesh Gopakumar, Anupam Kundu, Joseph Samuel, and Supurna Sinha for their comments and suggestions.

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The Power of Stories in Math Class

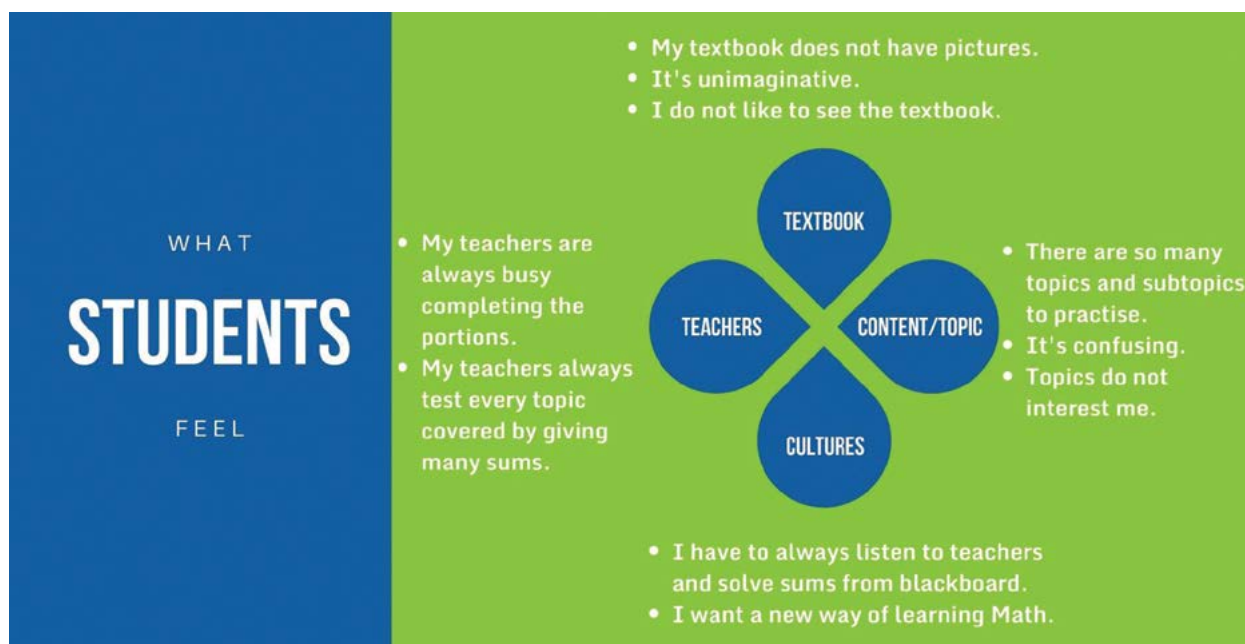
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Math is beautiful. But sometimes, this can be hard to see, and even harder to teach students who don't – *yet* – share the passion.

Though the pedagogy of mathematics has evolved over the years, it is still considered a tough subject. It is a known fact that when we ask students about their favourite subject at the elementary level most students choose math but when the same question is asked in middle and high school, the students' response is that math does not interest them anymore. There could be many reasons; it could be the unimaginative black and white textbook that only has numbers and symbols which do not help the students to visualize the abstract concepts that they are studying. Or it could be the pedagogy (students are not encouraged to think), or the relentless drill and practice with the sole end goal of getting better scores in assessment. Their learning is not connected with life and, in addition to this, the cultural inheritance of fear associated with the subject does not help the students.

To create interest in students towards the subject while catering to a heterogeneous group of learners is quite challenging.

Keywords: mathematics, math-phobia, pedagogy, stories



Mathematics is essential for everyone. Then how can we overcome student bias, improve pedagogy, and create interest in Mathematics?

One such strategy initiated 3 years ago by the Department of Mathematics, Delhi World Public School, Bangalore is *Math story writing*. *We have discovered that using stories and storytelling to teach mathematics can foster love for the subject.* Students enjoy listening to stories and reading stories. Stories are built upon human problems, conflicts and capabilities. Many times, the problems can be solved by mathematical thinking. Stories organically lead to discussions in the classroom. They can help to *humanize* mathematics, visualize Math concepts and create relatability. Then why not use stories and storytelling to develop critical thinking skills, problem solving and literacy skills amongst children.

Origin of stories

Stories are not new in Indian culture and history. For instance, in the Ramayana, Dasharatha distributes the divine payasam that he received from Lord Agni-this can be related to fractions. The story 'The Greedy Brahmin's Dream,' from Panchatantra in which the Brahmin gets a pot of flour and dreams of selling it for profit, and trading relates to money and estimation. Arjuna's archery can be used to illustrate Pythagoras theorem, area, perimeter, and coordinate geometry.

We used these as well as several books that are based on Math such as 'Mathematwist: Number Tales from Around The World,' by T.V. Padma, 'Tales from the history of Mathematics' written by Archana Sarat and stories published by Pratham.

Teacher's endeavor in using story-based pedagogy

One of the practices in school was to include relevant stories while introducing a math concept. The in-house publication 'Assignment Booklet' includes math stories for students to read. This story writing was not forced to fit into our lessons or was given to students as assignments but was planned according to the concepts taught in the classroom. The students were encouraged to weave a story using the concepts. The idea was to inculcate an intrinsic value of mathematical thinking that is required to solve the problem in the story.

After the first attempt of math story writing, we Math teachers could observe that our students were able to connect the concepts to their daily life and were using math concepts along with the required key vocabulary. But the stories lacked mathematical thinking. When we discussed this with our English department, the teachers suggested that the story must have a conflict and solution; this would lead to mathematical thinking. This discussion gave math teachers a clear understanding of how we could help students to further improve the math story writing process.

This strategy also helped the students to relate math stories with their English language writing and grammar. The children were able to identify the elements of story writing, such as the characters, setting, problems, situations, actions, and climax.

From listening to stories to writing stories - shift in pedagogy, advantages

We initially started narrating stories in the classroom related to the concept. We read the book *Sir Cumference and the Dragon of PI* written by Cindy Neuschwander and illustrated by Wayne Geehan. After reading the story in the class, we discussed the different story elements in the class through questionnaires. Later the students were given the *Math story analysis template* to fill in.

PHASE 1

Math story analysis

Name: Ananth B Grade: 9

Name of the story: _____ (books name and author)

Main Characters Sir Cumference Radius	Setting Castle Doctor's workroom Carpenters' workroom Ladyfingers' kitchen
Supporting Characters Guards Radius' Mother Geo Syn Lady Fingers	Problem Cumference and Radius were eating meal and Cumference felt fire in his stomach. Radius brought potion but cumference became dragon.
Title and Author Sir Cumference and the dragon of PI Cindy Neuschwander	
Solution Radius got another potion. He took it and read a poem which was written. It was to find out the circumference and radius and divide it. And that number will always be a constant. He got the value $3\frac{1}{7}$. He gave 3 spoon and $\frac{1}{7}$ spoon of potion and his father became a Human.	

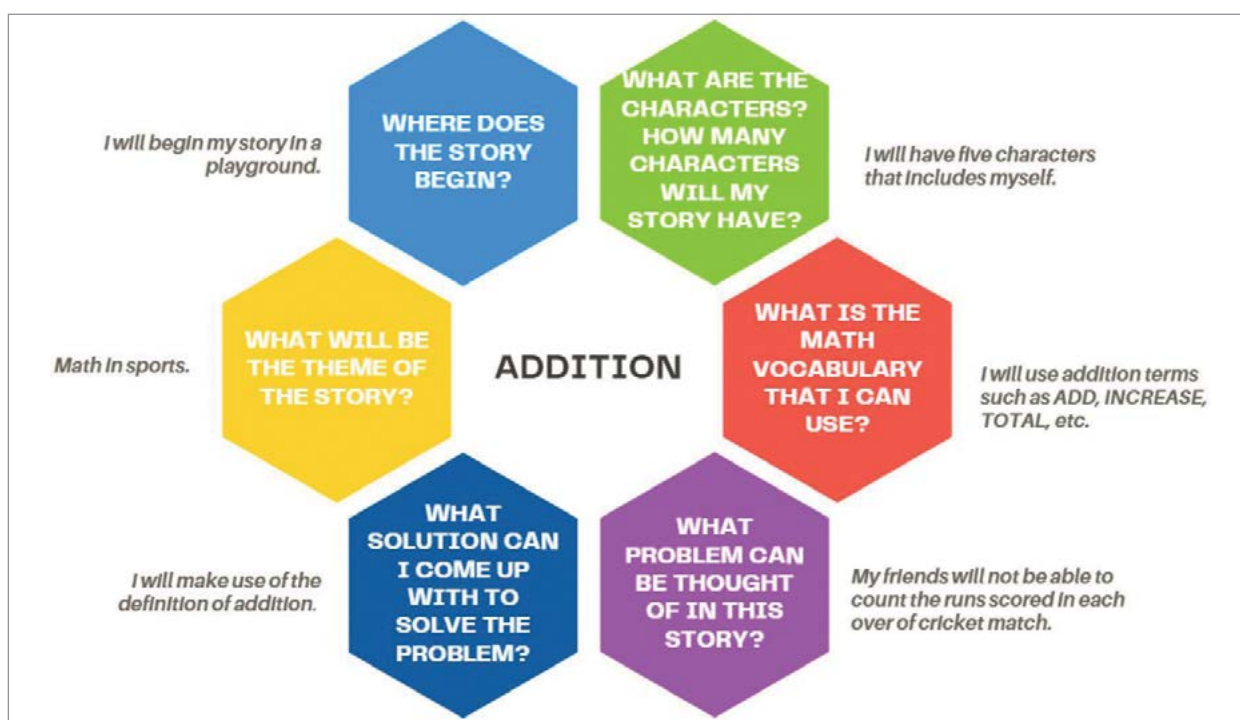
PHASE 2

Name: Pranav Bhattacharya Grade: IX

Where does my story begins? Giuseppe Piazzi demonstrated 61 Cygni had a large proper motion. Friedrich Bessel measured its distance using the stellar parallax method.	What would the characters be? How many characters should I have? Giuseppe Piazzi Friedrich Bessel Wilhelm Wein Ejnar Hertzsprung Henry Norris Russell Christian Doppler Edwin Hubble Henrietta Swan Leavitt Subrahmanyan Chandrasekhar
What are the maths vocabulary that I should use? Distance, light years, Temperature Luminosity, velocity, λ , Δ , Solar mass, wavelengths, related formulas	What problem should I think of? Stellar parallax can't be used to measure objects more than 500 light years away.
What solution will I come up with? Wein's Law, H-R Diagram, Hubble constant, Luminosity-period relationship of Cepheid variables and the Chandrasekhar limit are the solutions I will come up with.	What will be the theme? Astrophysics

This helped the students to analyse the story. After the teachers and students read the math stories together, they discuss and fill in the template. This helps the students to understand the various story elements that are present in a story. This is followed by another template, where the students, with the help of the math teachers, start planning about their individual story that they want to write. The process helps the students to visualize the story. After this, teachers introduced story writing activity to the whole class, later with small groups and finally, the students worked independently on writing stories as home assignments. The students were guided and mentored by the teachers to write math stories using some simple steps.

Create a mind-map – students were encouraged to create a mind map around a specific concept such as addition, fraction, multiplication, etc.



Come up with character, setting, problem & solution – students were encouraged to think about introduction, names of the characters, setting, problems and solutions.

Read the story - Once they had drafted the story, students were encouraged to read the story aloud in the class for the other students to listen to and provide feedback. The feedback activity was done from Grade VI to VIII in the age group between 9 to 12 years.

Feedback and mentoring by the teachers – The next step would be giving feedback to students. The students shared their write up of the story with the teachers who offered constructive feedback to improvise. Math teachers focused on the mathematical thinking and math vocabulary. We also kept a check on their grammar and sentence structure, as it was a multidisciplinary project. Teachers encouraged the students to observe and include logical, critical thinking and mathematical thinking in the stories that were being read and written.

In the month of November each class was given a list of topics based on the concepts that they had learned over the academic year. For example, grade 1 would write stories using shapes, grade 2 would write on addition, grade 6 would write on ratios and proportions, etc. Each child is encouraged to write one story.

Students who were slow learners or not interested in learning are slowly beginning to show progress. Most of the students are now able to visualize the problems by drawing pictures and coming up with solutions. They have started gaining confidence in solving math problems. They draw cartoon characters and write the formula. The students are happy to see their classwork not just filled with problems. The usage of visual representation, character and colours has made math a fun subject for the students.

We have been practising math through stories for the past 3 years and we have seen how it has led to the development of various skills such as (see mind map below):



Samir, a student of Grade VII, always had been mechanical in his presentation of math projects. He assumed Math is all about solving the sums from the textbook by following the steps and formulas. *Through this activity he was able to connect math with real life situations like connecting geometry to patterns in nature; he drew many pictures of flowers, shells, etc., connected with art during his art classes. I recollect an anecdote when a child came up to me and said that he was able to relate integers while using the elevator. There was another student who was fascinated about architecture and was able to connect architecture with math while learning practical geometry.* His way of looking at Math has changed and now he relates math with real-life problems.

It is more important to apply mathematical thinking to real life, than to apply real life to textbook mathematics. The focus is now gradually shifting to life, not a subject or examination.

Amith, a student of Grade III, was not interested in the subject. He avoided interacting in the classroom during the Math classes. When stories were introduced slowly the child started to participate in the classroom. *Now the child interacts in the class by asking questions like what exactly are addends? Why is there no end to numbers? Can I ever find an end to counting numbers? Why are shapes given specific names? He has started learning math by drawing simple drawings and completing the word problems with appropriate statements.*

Earlier we used to solve all the problems on the blackboard giving very little time for the students to think and come up with the key terms in the problem or formula used to solve the problems. But through the introduction of story writing, students are now developing confidence to read the questions and analyse the problems before solving them. They draw the shapes if they have to solve area-based problems or they draw small pictures considering them as characters and keeping them in real life situations and solving the problem.

When I was teaching addition word problems to Grade III, one of the children in the class drew a small picture and named himself as one of the characters in the picture in the rough column and wrote the statements required to solve the problem.

This is one of the student's exercises. The word problem actually began as Sam travelled...

③ Nikhil travelled from his home to school that covered a distance of 245 metres. From school he went to his Cricket coaching centre which was at a distance of 386 metres. How much distance did Nikhil cover totally?

Soln:- distance covered from home to school = 245m

distance covered from school to cricket coaching centre = 386m

So Nikhil covered 631m

245
+ 386

631

Let's take a sneak peek into some of the stories written by our students and see how they have used their imagination to relate mathematical concepts to their daily life.

Reema is an 8-year-old who lives in Bangalore with her parents. She is full of energy and enjoys playing with her friends. She is an inquisitive child & has always been full of questions about everything that she observes around her. She has always been interested in numbers & learnt to count from 1 to 20 at the age of five, from her father. Since then she counts everything she sees around her.... like the number of flowers that bloom in her garden each day, the number of butterflies that come to visit the flowers, the number of ants that crawl around fallen bread crumbs, etc. Her father, being an engineer himself, encouraged Reema's interest in numbers. Reema's mother, on the other hand, was trying to be cautious & always worried about Reema wandering off engrossed in her world of numbers.

Every year Reema & her parents travel to their native place in Kerala to celebrate Onam festival with her grandparents & cousins. Reema loved these trips and always waited eagerly for August so that she could enjoy the road trip, see new sights, eat different food, etc. This year also Reema is traveling along with her parents to Kerala. They left home by car at 5 a.m. Reema was very excited, but sleepy too. She dozed off! After some time, when she woke up, she heard her mother asking her father where they had reached.



Her father replied, "We have covered 50 Kilometres." Reema asked, "Daddy how many kilometres more do we have to travel to reach Kerala?" Dad replied, "Reema we have just covered 50 kilometres of the journey, we still have 6 times the distance to cover!" On hearing this, Reema started thinking.

Reema asked, "Daddy, so that means we have to travel 300 kilometres more! Am I right?"

Dad: "Absolutely correct!"

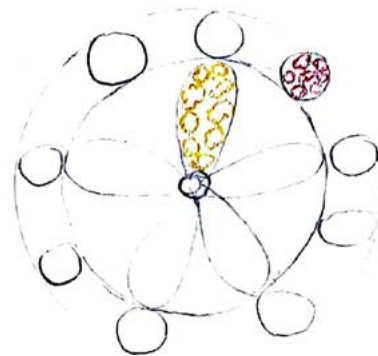
After some time, Reema saw mangoes being sold. They stopped the car to buy some mangoes for relatives and for themselves. Reema's mother said that they must buy mangoes for four families. Mom decided to buy 12 mangoes for each family. Reema again started calculating the number of mangoes they needed to buy in total: $4 \times 12 = 48$.



Soon it was lunchtime and Reema was very hungry. They stopped at a restaurant and ordered Thalies for all three of them. After lunch, Reema's dad went to the washroom. The waiter came with the bill and said that each thali costs ₹120/-. Reema quickly calculated the bill $120 \times 3 = 360$ before her father returned. When her father came, she asked for ₹360 and paid the bill. Her dad was proud to see Reema apply the math she learnt at school, in her daily life.

Finally, they reached home by 5 p.m. Reema was tired after the long journey. She met her grandparents and cousins. They had fun for some time. She had an early dinner and slept soundly.

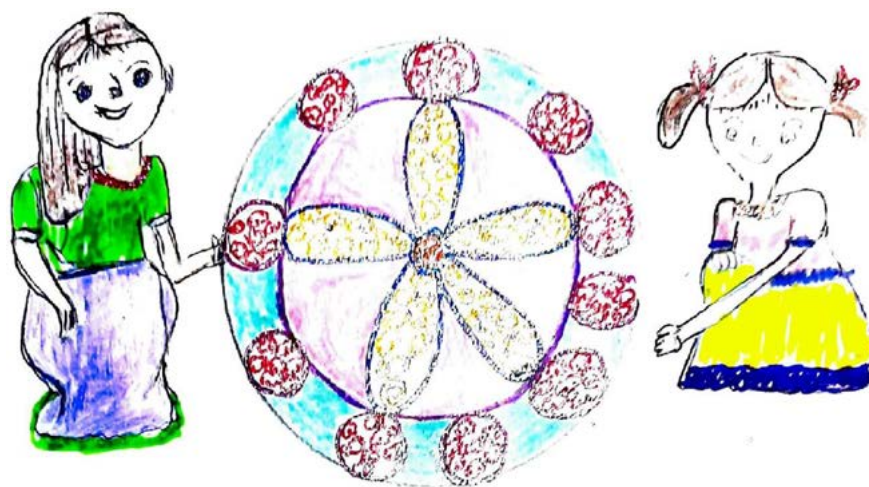
Next morning, Reema woke up to a flutter of activities in the house and saw her grandmother and mother worriedly rushing about in the kitchen. She asked her father and got to know that relatives called in the morning saying they would be visiting their house. So, grandmother and mom were busy making snacks and sweets for them. Reema felt bad to see her grandma so worried. She went to her cousins who were busy playing and told them, "Let's do something interesting this time. Shall we try and make the Pookalam? We have seen grandma doing it every year...let us try to do it this year. I will ask Amma for the design, then we can collect the flowers from the garden." All the kids were excited at the thought of making the pookalam all by themselves and immediately rushed to the garden. Reema went and told grandma and amma about their plan. Grandma was very pleased to know that Reema and her cousins were offering to help, understanding the difficult situation at home and happily gave her the design she had chosen. Reema took bath and wore her new "Pattu Pavada" for Onam celebrations and went down to see all her cousins in elegant traditional dresses, ready to make the pookalam.



They drew a design with 5 petal shapes and 10 small circular shapes inside a circle. Reema asked her cousins, "Simi, how many yellow flowers did you fill in one petal shape?"

"Nine yellow flowers," replied Simi. Reema calculated, "so for 5 petal shapes, we will need $9 \times 5 = 45$ yellow flowers!"

"Shall we fill the small circles with red flowers, Reema?" asked Simi. "OK Simi, how many flowers do we need to fill one circle?" asked Reema. Simi filled one circle with red flowers and counted. "We need 7 red flowers for 1 circle and there are 10 circles. So, in total we need $10 \times 7 = 70$ red flowers."



The Pookalam looked beautiful when they completed it. Everybody at home appreciated Reema and her cousins for the beautiful work they had done. Reema on the other hand spoke about shapes she had learnt in school and was fascinated to learn from her mother about rangolis and mandala art works.

All were hungry and tired and ready for the delicious feast prepared by Reema's mom, aunt and grandma. Lunch was served on a banana leaf. Mom asked Reema to serve three banana chips in each of the banana leaves. So Reema counted again! There were 8 members for lunch, so $8 \times 3 = 24$ chips. Reema took 24 chips in a bowl and served it equally on banana leaves.



After the delicious food, everyone took rest. In the evening, Reema, her cousins, and her parents went to the beach. Reema's father bought kites for them and they all flew their kites high in the sky. Reema enjoyed her Onam vacations!

My name is Maanasya Mirjith. I am studying in Grade III at Delhi World Public School, Bangalore. I love watching cartoons and playing with my friends. I have written this math story based on my Onam experiences. My parents helped me complete this story well. A big thanks to them and to my teacher Sowmya ma'am for encouraging me to think and apply the mathematical concepts in a fun way.

Maanasya has always found math interesting and ever since we started using stories, she also started to relate math with herself. And this is how she came up with this story, as she finds the mathematical operations quite easy.

She has beautifully described her family trip to Kerala using the 'Multiplication concept.' In the story, Maanasya has incorporated descriptive writing and applied the skills learned in the English language. She has explained the meaning and concept of multiplication in different situations such as calculating the kilometres travelled, while buying mangoes, while serving food and calculating the restaurant bill, and she has correlated multiplication with pookalam, a rangoli design made with flowers.

Every year, Delhi World Public School celebrates Math week and during this week we plan various activities such as making Math models, enacting math stories, integrating art with math etc. The students enacted maths plays referring to the book "Tales from the history of Mathematics" written by Archana Sarat. Our idea of writing stories was inspired from this activity. We wanted to do something different out of the box and that's where the idea flashed through to include story writing. This entire process happened over a period of about 3 months from November to January. Students from grade I to VIII wrote stories.

After the story is submitted, teachers review the stories in the class and individual feedback is given to the students. In the academic year 2019-20, stories were exhibited during the Math week and in the academic year 2020-21, the stories were published as an eBook. The link for the book is https://issuu.com/delhiworldpublicschool/docs/math_through_stories_37e547e733fb03

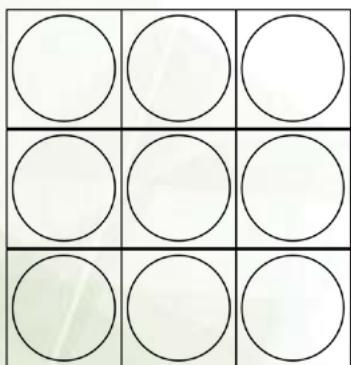
Written by **Ms. Sowmya H.S.** Mathematics Faculty at Delhi World Public School, Bangalore.

Maths from Simple Grids

GOWRI SATYA,
ASHWIN, SHRAVAN,
SHIVKUMAR

An exciting afternoon with friends and family, for me, often involves arguing endlessly over some trivial math or science question or solving a tricky puzzle or finding a pattern where you least expected it. When I started to teach math in an NPO (Non-Profit Organization) setting, I most of all wanted to bring this excitement and thrill of problem solving into our classroom. Easier said than done! I would pose what I thought was an interesting question and watch as students either struggled or simply did not even attempt it. Finding the right kind of problem that is challenging yet tantalizingly simple, that is familiar yet leads to new and exciting discoveries has proved to be difficult. Having to find something that is easy to implement, cost effective and that works for all the students in my mixed classroom even more so.

So, when Seed2Sapling Education conducted a teacher training session where they presented interesting, insightful math questions at various levels generated from common everyday things, I was completely hooked. Students' reactions to some of these ideas have been exactly what I had hoped for.



Here we present a few such interesting problems centered around the humble and modest **grid**. All the activities here can be worked out with paper and pencil. For younger or more kinesthetic learners, number tiles or even a life size grid on the classroom floor could be used. Answers to the puzzles are listed at the end of the article. Students from Grades 2 to 5 with a wide range of mathematical abilities can find something of appeal in these problems.

Keywords: critical thinking, problem solving, trial and error, generalisation

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

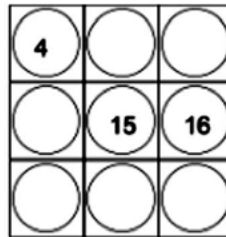


Figure 1

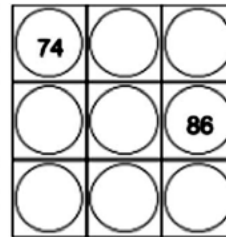


Figure 2

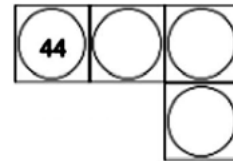


Figure 3

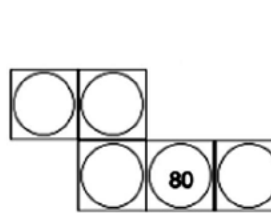


Figure 4

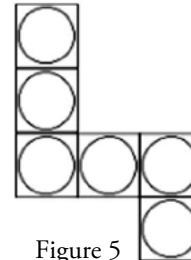


Figure 5

Preschool Corner - Number Grid Exploration

The most basic grid every child is bound to come across in school is the basic 1-100 number grid. While writing out the number chart is an oft repeated exercise, there are hours of fun to be had with finding patterns in it.

Given a 3 x 3 grid from the 10 x 10 number grid, fill in the missing numbers (Fig. 1, Fig. 2).

Or fill in the missing numbers in puzzle pieces of various shapes (Fig. 3, Fig. 4).

Or a puzzle piece can be left empty (Fig. 5). Then the challenge for the student is to find all the possible values that can work for it — first with a number grid for reference, then without.

Deeper understanding of the place value system can be fostered by asking students to explain in their own words their strategies for filling in the missing numbers. Mistakes made rather than just being corrected can be collected and students can observe for themselves the patterns of their

mistakes. Again, asking students to explain in their own words what mistakes they made and why, can lead to rich math conversations.

Students can also design their own puzzle pieces. What is the most difficult puzzle piece you can come up with?

This same familiar activity can be repeated in later classes with an addition grid or a multiplication grid to bring out the patterns in addition and multiplication.

Exploring Ordering - Inc/Dec Puzzle

Given a 3 x 3 grid and the numbers 1 to 9 arrange them in the grid such that each row and each column has numbers in increasing order (Fig. 6). Each number can and should be used only once.

What if each row and column had to be in decreasing order (Fig. 7)? Now what if all rows were to be in increasing order and columns in decreasing order (Fig. 8, solved for reference)? Or vice versa (Fig. 9)?

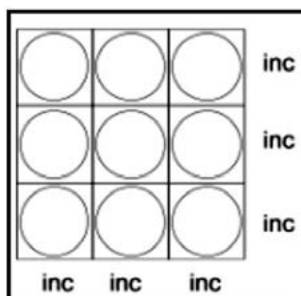


Figure 6

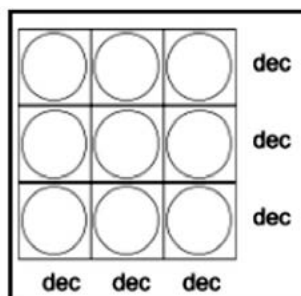


Figure 7

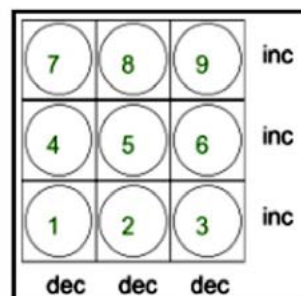


Figure 8

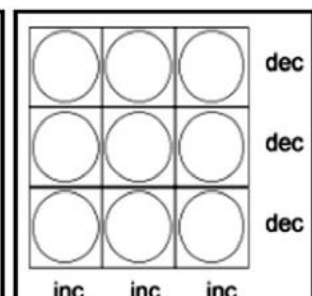


Figure 9

Now let us mix things up a bit. Have a go at the puzzles in Fig. 10 and Fig. 11. While the first four puzzles could be solved intuitively, the last two require a little more organized effort. A puzzle at this level would be ideal to introduce transitivity of 'greater than' and 'less than' to younger students.

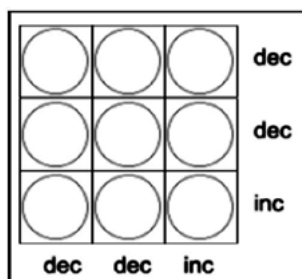


Figure 10

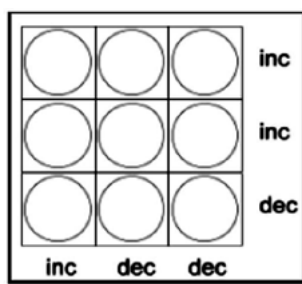


Figure 11

Now that we have six solutions under our belt, it is time to make some observations.

1. Are there any patterns in the solutions?

One thing that stands out is the placement of '1' and '9'. Being the smallest and largest numbers in the set, they naturally take their positions in one of the four corners of the grid. Can the same be said of '2' and '8'? Do they always take a position in the middle of a row or column?

2. Could a given grid configuration have more than one solution?

This is always an interesting question – can there be only one solution or are more solutions possible. Discussions around this can lead to insights that may be missed in just solving the problem.

3. Are there any patterns in our approach to solving them?

Based on our observations about 1 and 9, the first step could be to find possible positions for them.

Next, try to find possible positions for 2 and 8 and so on. Or perhaps take the opposite route and shortlist the possible numbers for each cell in the grid. How about using transitivity of the 'less than' and 'greater than' to solve the puzzles?

Once students have solved a few puzzles and understood the constraints, what works, what does not, it's time to up the ante. Let us flip the question. Create your own puzzle. Students can create a grid with the conditions of increasing and decreasing and give it to a friend or teacher to solve. (What student wouldn't like to school her teacher?) Some interesting grids that students came up with:

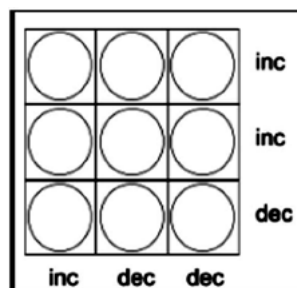


Figure 12

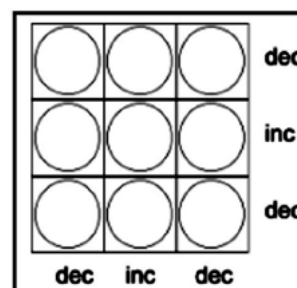


Figure 13

This exercise opens up some more questions:

4. How many ways are there to arrange Inc and Dec around a grid to set a puzzle like this?

There are 3 rows and 3 columns giving in all 6 positions to be filled with one of two values - 'Inc' or 'Dec'. What would be the total number of combinations?

If there were 2 positions to fill, we would have $2 \times 2 = 4$ possible combinations:

Inc Inc, Inc Dec, Dec Inc, Dec Dec

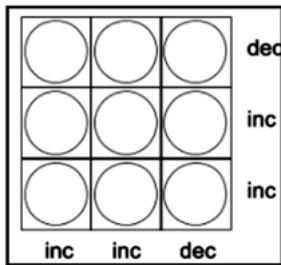


Figure 14

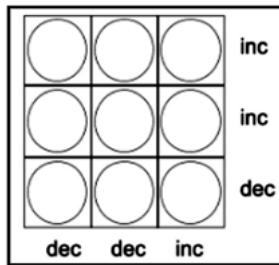


Figure 15

If there were 3 positions to fill, we would have an Inc- extension to all the above sequences and then a Dec- extension to all the above sequences like this:

Inc Inc Inc, Inc Inc Dec, Inc Dec Inc, Inc
Dec Dec
Dec Inc Inc, Dec Inc Dec, Dec Dec Inc, Dec
Dec Dec

That is, with each extra position that is added, the number of combinations doubles.

This gives us $2^6 = 64$ combinations for 6 positions.

5. Will all of those combinations have a solution or is it possible to create a puzzle with no solution?

Given that the two conditions of ‘increasing’ and ‘decreasing’ are contradictory, it seems likely that a particular configuration could lead to a contradiction that makes the puzzle unsolvable. Let’s try to create one then. After some attempts, we hit upon the configurations shown in Figures 14 and 15. The proofs that these configurations are unsolvable is given separately at the end (see

the Appendix, last page). The reader may enjoy looking for independent proofs.

6. How does adding or removing constraints affect the puzzle?

Along with the row and column constraints, we could add a constraint on the diagonals as well. Consider Fig. 16. Is this harder to solve? Can we create more unsolvable puzzles with the extra constraints?

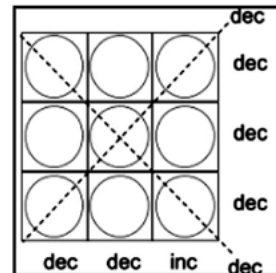


Figure 16

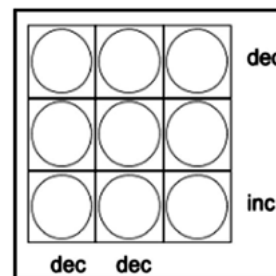


Figure 17

What if we drop a constraint? Instead of specifying Inc / Dec on every row and column, we could leave some with no constraints as in Fig. 17. Does this make the puzzle easier to solve? Is it harder to create an unsolvable puzzle if you could impose only 4 constraints instead of 6.

Challenge Question: How do we efficiently identify whether a given configuration can be solved or not?

We have one way to identify an unsolvable puzzle but is that the only condition resulting in no solutions? Consider the puzzle in Fig.18. Here, 9 and 1 have clearly identifiable cells that they could fill. Does that make this a solvable puzzle? Could there be a contradiction elsewhere?

How can we efficiently find out if a given configuration is solvable? Can this be extended to a generalized Inc/Dec puzzle on a $n \times n$ grid or even further to a $n \times m$ rectangular grid?

It turns out that this question has an unexpected twist!

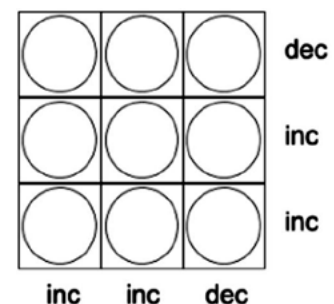


Figure 18

You can play Inc/Dec puzzles online at: <http://mathventure.in/games/incdec.html>



Solutions

Some of these puzzles have multiple solutions but only one possible solution has been listed here.

For more puzzles, please visit the addendum to this article in the online version available at <http://publications.azimpremjifoundation.org/3344/>

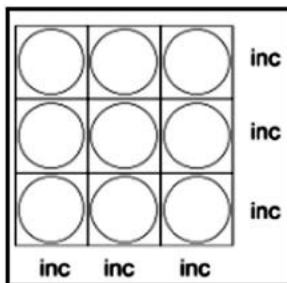


Figure 6

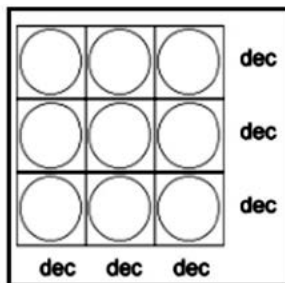


Figure 7

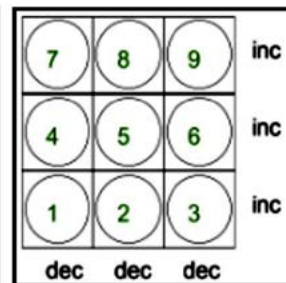


Figure 8

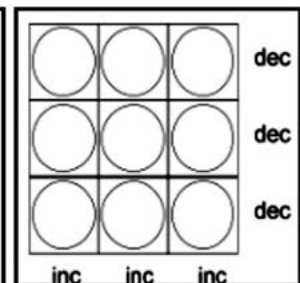


Figure 9

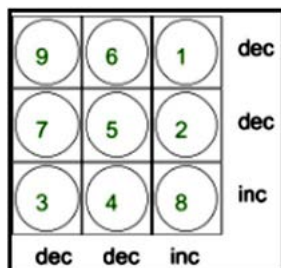


Figure 10

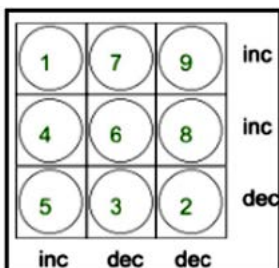


Figure 11

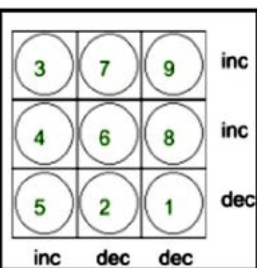


Figure 12

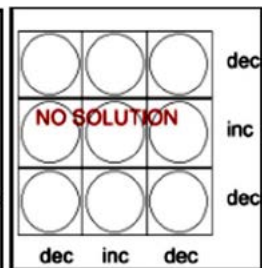


Figure 13

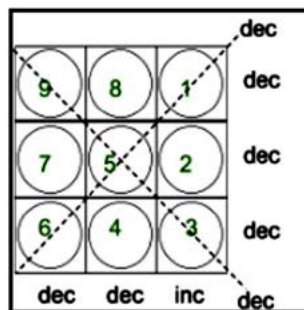


Figure 16

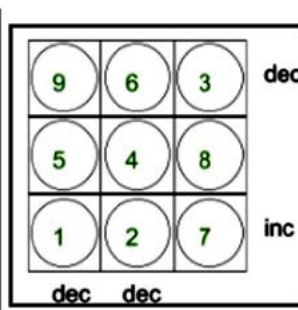


Figure 17

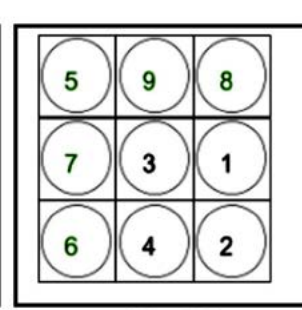


Figure 19

Appendix: Proof that the configurations shown in Figure 14 and Figure 15 are not solvable

Let the grid be as follows.

Let's look at Figure 14 first (shown alongside). Since column #1 is Inc and row #1 is Dec and column #3 is Dec, it must be that:

$$i < f < c < b < a < d < g.$$

This implies that $i < g$. However, row #3 is Inc, which implies that $g < i$. These two conclusions are contradictory. Therefore, this configuration is not solvable.

a	b	c	Dec
d	e	f	Inc
g	h	i	Inc
Inc	Inc	Dec	

Similarly, in Figure 15, since column #1 is Dec and row #1 is Inc and column #3 is Inc, it must be that:

$$i > f > c > b > a > d > g,$$

implying that $i > g$. However, row #3 is Dec, which implies that $g > i$. Once again, we reach a contradiction. Therefore, this configuration too is not solvable.

ASHWIN, SHRAVAN and **SHIVKUMAR** are all alumni of IISc, Bengaluru. They are now working full time on rejuvenating math education in schools through an education startup Seed2Sapling Education. They can be contacted respectively at Ashwin, guha.ashwin@gmail.com; Shravan, sra9449@gmail.com; and Shivkumar, shiv1729@gmail.com.



GOWRI SATYA teaches math at Swapaksh Learning Foundation, a learning center for underprivileged children in Bengaluru. An ex software engineer, she enjoys math and hopes to spread the same enthusiasm to her students. She can be contacted at s.gowri@gmail.com.

REASONING MADE SIMPLE

Contributor: Wallace Jacob

Question. In an x -metre race, A beats B by 11 metres, and A beats C by 90 metres. B beats C by 80 metres. What is the value of x ?

Is it possible to determine whether a race is a 100 m race or a 200 m race or a 300 m race simply by having data which projects who defeated whom by how many metres in the race? Well, Mathematics can help in unravelling hidden data!

Solution. It is easy to infer from the question that in the time in which A covered x metres, B covered $(x - 11)$ metres and C covered $(x - 90)$ metres.

Also, in the time in which B covered x metres, C covered $(x - 80)$ metres.

A	B	C
x	$(x - 11)$	$(x - 90)$
	x	$(x - 80)$

$$\frac{(x - 11)}{x} = \frac{(x - 90)}{(x - 80)}$$

$$x = 880\text{m.}$$

TearOut

Polyominoes and nets of a cube

In this 6th TearOut, we will explore polyominoes and all possible nets of cubes (and cuboids). As before, pages 1 and 2 are a worksheet for students while pages 3 and 4 give guidelines to the facilitator. Though dot sheets or square grid sheets are not a must, they may be useful to make the set of all polyominoes (up to hexominoes). Using these will help with the exploration.

Each polyomino is made of one or more congruent squares. Some of the sides of these squares are inside the polyomino while the rest are outside, forming the perimeter of the same. We are going to call each of the square-sides along the perimeter of the polyomino as an 'out-line'.



Figure 1

Now, observe that the monomino, which is just a square, has only one type of out-line because a monomino has rotational symmetry of order 4. So, if ABCD is the monomino, the out-line AB can coincide with BC, CD or AD if we chose a suitable rotation (or a reflection). Therefore, it doesn't matter where we add the next square to get a domino. So, there is only one type of domino, modulo rotation. However, there are two types of

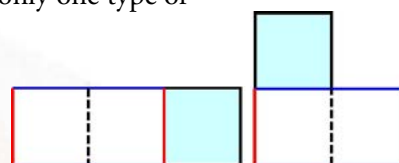


Figure 2

out-lines for the domino – (i) the red out-lines, which can be reflected on each other along the dashed line in the middle as shown in Figure 1, and (ii) the 4 blue out-lines which coincide with each other if we rotate by 180° or reflect along the perpendicular bisector of the red out-lines. So, we get two types of trominoes depending on where we attach the third square (Figure 2).

Also, if a polyomino is obtained by adding a square to an existing polyomino then the former is called a 'child' of the latter and the latter is a 'parent' of the former. So, monomino is the parent of domino and trominoes are children of the latter.

1. Types of 'out-lines' and number of children

- Find out how many types of out-lines there are for each tromino. So, how many possible tetrominoes are there? [Hint: consider the symmetries of each tromino]
- Is there any tetromino with two parents? If so, which one(s)?

2. Parents

- Now consider the 12 possible pentominoes (Figure 3). Find the parent(s) for each.
- Which pentominoes have only one parent? Which ones have two? Is there any with more than two parents? Create a suitable chart.
- Complete the tree diagram (Figure 4) to include all tetrominoes and all pentominoes.

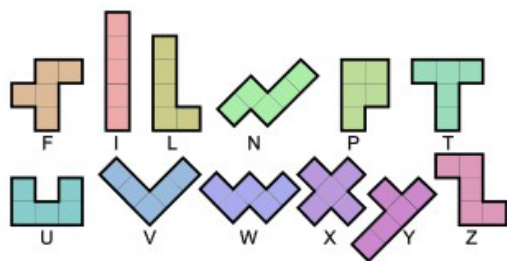


Figure 3: By R. A. Nonenmacher - Own work, CC BY-SA 4.0, <https://commons.wikimedia.org/w/index.php?curid=4412149>

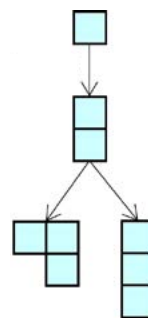


Figure 4

3. Wrapping around the cube

Monomino coincides with any face of a cube (of the same size). Similarly, domino can be folded and placed against a cube such that the two squares superimpose on two adjacent faces of the solid. So, when a polyomino is wrapped around a cube, each square must superimpose on a unique face of the solid. Therefore, the squares of a polyomino should not overlap with each other, neither should any square be left out, i.e., not overlapping a cube-face.

- Can both trominoes be wrapped around the cube?
- Is there any tetromino that can't be wrapped around the cube? Guess and reason.
- Based on the above, which pentomino(es) cannot be wrapped around the cube?
- Which other pentominoes cannot be wrapped around the cube? Guess and reason.
- When a pentomino is folded to form a cube, how should it look?
- List all pentominoes that can be wrapped around a cube.

4. Net of a cube

A cube has 6 square faces. So, if a cube is opened up to form its net, it is made of 6 squares, i.e., a hexomino. These are all the possible hexominoes (Figure 5).

- If a polyomino cannot be wrapped around a cube, can its children be? Why?
- Use above and 3f (cleverly) to strike out the hexominoes that cannot be the net of a cube. Give reasons for each.
- A cube's net can have 4 squares in a line (the walls) forming a 4×1 rectangle, with the remaining squares (floor and roof) on each of the longer sides of the rectangle. Identify the hexominoes fitting this description.
- Fold each of the remaining hexominoes and find out which ones are nets of a cube.
- How many possible nets are there?

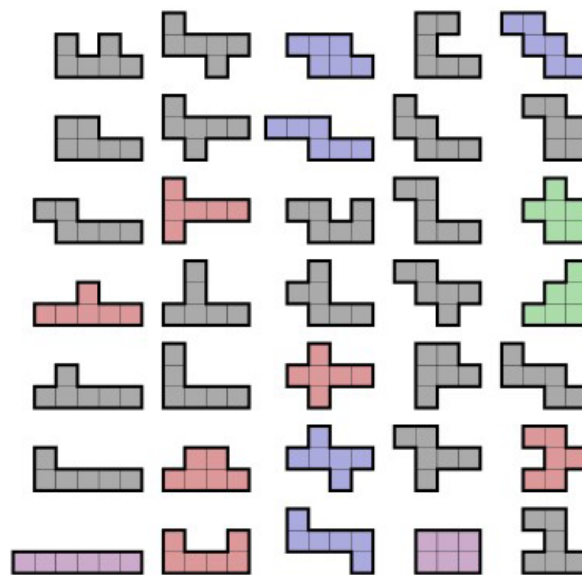


Figure 5: By R. A. Nonenmacher - Created by me, CC BY-SA 4.0, <https://commons.wikimedia.org/w/index.php?curid=4773113>

5. [Optional] Net of a cuboid

- Suppose the cube has been flattened to form a cuboid with dimensions $a \times a \times b$ with $0 < b < a$. How will each of the nets of the cube change?
- Repeat the above for an elongated cube, i.e., a cuboid with dimensions $a \times a \times b$ with $0 < a < b$.
- Repeat the above for a cuboid with dimensions $a \times b \times c$ with $0 < a < b < c$.

This worksheet touches upon many ignored areas of mathematics. It starts with communication including creating terminology to convey ideas precisely and rigorously as per the need. Therefore, we introduce ‘out-lines’ as well as the children-parent relation among the polyominoes. This way the tree-diagram of polyominoes can be linked to the family tree – something that students may already be familiar with. It also fosters reasoning at every step.

1. This is a warm-up activity to get familiar with types of ‘out-lines’ for each polyomino and the notion of children and parent of the same. It is best not to show all possible tetrominoes to the students but let them discover the same. One may note that each child is a superset of a parent.

2. While the first question involved finding all children of the trominoes, the second one reverses the process and is about finding the parents. Since each parent is a subset of a child, one has to remove a square from the child to get a parent.

The task ends with extending the ‘family tree’ of the polyominoes till the 5th generation, i.e., the pentominoes. This tree is different since many branches often merge – something impossible in a usual family tree.

3. In this question, we go to the second part – nets of a cube. One can imagine wrapping trominoes around a cube. But it would help to actually have paper polyominoes and fold them to form partial cubes.

While both trominoes do wrap around a cube, there is one tetromino that doesn’t. Let us call this tetromino O. Only three squares meet at any vertex of a cube. So, if a polyomino has four squares sharing a vertex then it can’t be wrapped around a cube. Consequently, the pentomino P which is the only child (why only?) of tetromino O cannot be wrapped around a cube.

I is the other pentomino that can’t be wrapped around a cube. This is because there are only four squares as we go around a cube forming a loop. So, the 5th square of I becomes extra. The pentominoes V and U also fall in this category. However, it may be difficult to reason out why that is the case. But if one tries to fold a V or a U to a cube, it becomes clear.

Each pentomino has five squares. So, if one can be wrapped around a cube, then the folded one will look like a hollow cube with one face missing.

4. In this part we narrow down the list of all hexominoes to the possible nets of a cube. This is done in three ways – (i) elimination, (ii) justifying why it can be a net, and (iii) by actually folding to form a cube.

We get a child by adding a square to the parent. So, if the parent can’t be wrapped around a cube, adding one more square won’t change that. However, note that the converse is not true since the parent of tetromino O can be wrapped around a cube.

Using the above argument we can eliminate 21 out of the 35 hexominoes. These 21 are children of pentominoes P, I, V or U – the ‘or’ is very inclusive here!

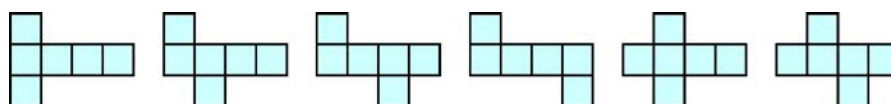


Figure 6

Six of the remaining hexominoes can be justified by the 4-walls-roof-floor argument (Figure 6). Four more can be considered as variations of some of these six (Figure 7).

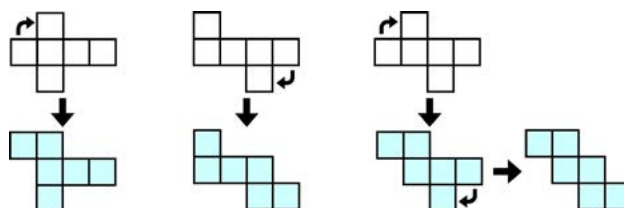


Figure 7

That leaves just four hexominoes which can be checked by folding. One of them does form a cube (Which one?). So, there are 11 possible nets of a cube.

5. This optional task stretches the exploration to cuboids by changing one dimension at a time. This can be given to groups of students, each exploring some of the 11 nets. Once all 11 nets are obtained for each set, there can be a further exploration on optimization using algebra.

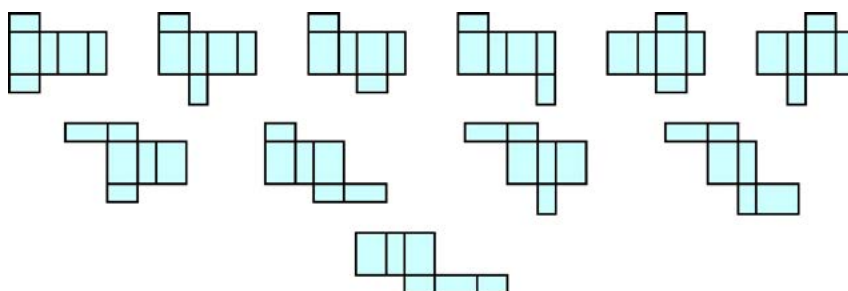


Figure 8: all possible nets for cuboid $a \times b \times c$ with $0 < a < b < c$

Suppose each of these nets has to be cut out of a rectangular piece of cardboard.

- a. What would be the dimensions of the smallest rectangle for each?
- b. How many pieces would have to be cut out of the rectangle to get the net? Note that fewer number of pieces to cut out means more efficiency with respect to effort.
- c. What is the total area of the pieces cut out (and thrown away)? Again, the smaller the area, the better optimization of the material, i.e., cardboard.
- d. Use the above to find the optimum net for each set, i.e., (i) $a \times a \times b$ with $0 < b < a$, (ii) $a \times a \times b$ with $0 < a < b$, and (iii) $a \times b \times c$ with $0 < a < b < c$.
- e. Can you arrange all 11 nets of each of the three sets from most to least economical? Are there some at the same optimum level? Are there any that are difficult to fit in the continuum? Why?

Questions d and e are best explored with some random numbers first and then algebraically. Set (i) generates 3 sizes of rectangles – $(2a + b)(4a)$, $(2a + b)(3a + b)$ and $(a + b)(4a + b)$, which can be easily ordered since $0 < b < a$. Similarly, set (ii) generates the same three sizes algebraically speaking. But because $0 < a < b$, the ordering can only be partial. Set (iii) however generates 5 rectangles: $(2a + c)(2a + 2b)$, $(a + b + c)(2a + 2b)$, $(2a + c)(a + 2b + c)$, $(a + b + c)(a + 2b + 2c)$ and $(a + c)(a + 2b + 2c)$. Some partial ordering can be argued among these, based on $0 < a < b < c$. These algebraic arguments would require playing with inequalities but that shouldn't be too hard since we are considering positive quantities only. It is worth pointing out that some of the most economical nets are usually seen while the less economical ones are rarely found anywhere.

Acknowledgement: This TearOut is based on the project of Shraddha Jain, MA Education student at Azim Premji University as part of her Curricular Material Development – Mathematics course.

Deciphering the Mode Formula

MATHEMATICS CO-DEVELOPMENT GROUP

In Parts 1 and 2 of this series on measures of central tendencies (published in Jul 2021 and Nov 2021 respectively) we discussed the median formula

$$M = l + \frac{\frac{N}{2} - m}{f} \times c$$

for grouped data from the corresponding histogram and the ogives. In Part 3, we will explore the mode formula $m = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times c \dots (1)$ from the corresponding histogram.

Mode can be obtained for both quantitative and qualitative data. For ungrouped data, it is simply the data value with the highest frequency. So, if data is ordered (according to some way for categorical data, for the rest there is an obvious order), then the longest run implying maximum frequency indicates the mode. For example, when we consider the choice of fruits for a class of 20 students and order it alphabetically, we get:

apple	apple	apple	apple	apple			
banana	banana	banana	banana	banana	banana	banana	banana
guava	guava	guava	guava				
orange	orange	orange					

Therefore, banana is the mode because it has the highest frequency. Note that neither mean, nor median work for categorical data.

Keywords: data, mode, modal class, frequency

Mode can be obtained from a frequency table also. For example, consider the academic qualifications of mothers of a class of 40 students (Table 1):

Academic qualification	Number of mothers
Primary school	7
Middle school	8
Secondary	10
Higher secondary	6
Graduate	5
Postgraduate	4

Table 1

Clearly, 'secondary' is the mode with the highest frequency of 10.

It is possible for a data set to have more than one mode.

Along the same lines, for grouped data, the class interval with the highest frequency is called the **modal class**. But how to figure out the exact location of the mode within this interval? That's what we are going to explore in this article.

Let us consider the marks obtained by two groups of students (Table 2):

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of students in Group 1	20	21	27	16	11	10	7	3	1
No. of students in Group 2	20	21	27	24	11	10	7	3	1

Table 2

The modal class for both groups is the same, i.e., 20-30 with frequency 27 each. However, the frequency distributions are different. So, the shapes of the histograms (Figure 1 and Figure 2) are different and therefore the modes should be different too. So, the midpoint of the modal class should not be the mode since it is independent of the frequencies. In fact, for Group 2, the class after the modal class has a much higher frequency (24) than the class before (21). So, the mode for Group 2 should be higher than that of Group 1, where the frequency of the class before (21) is higher than the class after (16).

It is easy to identify the modal class from the histogram. It corresponds to the tallest rectangle. Now, there are six points on that rectangle. Let us consider the histogram for Group 1 (Figure 1) for now. So, the three points on the left side of the rectangle are G (20, 27), F (20, 21) and T (20, 0) while those on the right side are H (30, 27), K (30, 16) and U (30, 0).

Since T and U have 0 as y -coordinates, they are independent of the frequencies and hence provide less information than the other four, i.e., G, F, H and K.

The y -coordinates of the top two points, i.e., G and H, are the highest frequency 27 and those of the middle two points, i.e., F and K, are the frequencies of the neighbouring classes, i.e., 21 and 16 respectively.

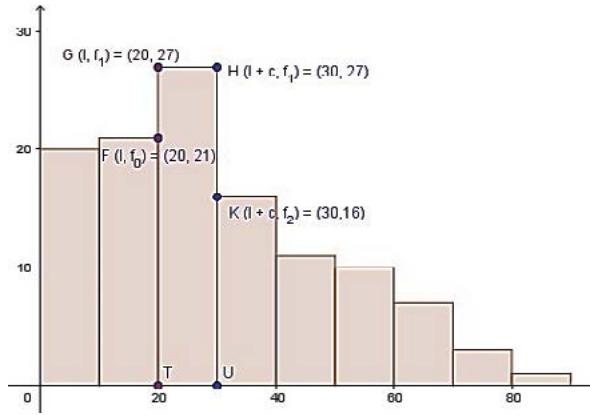


Figure 1. Marks from Group 1

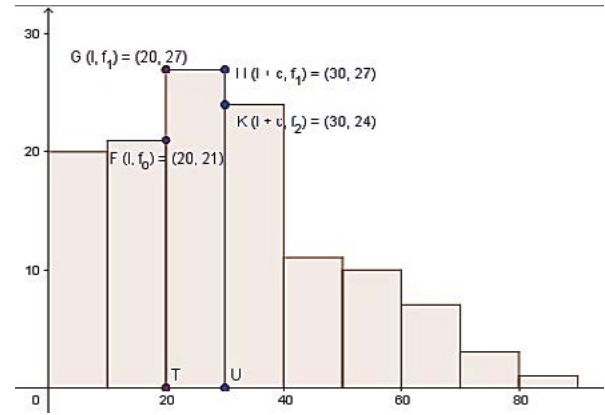


Figure 2. Marks from Group 2

It is understood that the mode lies somewhere within the modal class, i.e., mode is some x -value between 20 and 30 and is not necessarily the midpoint, i.e., $(20 + 30)/2 = 25$. So, the only way to get a x -value within the modal class or a point in the modal class rectangle is to draw two lines, each connecting points on either vertical side of the rectangle. Let's call these lines 'diagonals'. Since the choice of points must be uniform on both sides, there are three choices:

1. GU and HT: the diagonals intersect at the centre of the modal class rectangle making 25 the mode regardless of the frequencies, which is not desirable as mentioned already
2. FU and KT: modal frequency is not considered and therefore not desirable
3. GK and HF: only option left

So, mode is defined as the x -coordinate of the intersection point E of these two 'diagonals' GK and HF.

Let us look at the modal class in the histogram (Figure 3). Let the top border of the modal class be GH with G (20, 27) and H (30, 27) while F (20, 21) is the top-right corner of the rectangle for the class just before modal class and K (30, 16) is the top-left corner of the rectangle for the class just after the modal class.

Mode is the x -coordinate of the point of intersection E (m, f) of the line segments GK and HF. Let J be the point on GH directly above E, i.e., $EJ \perp GH$ and $J(m, f_1) = (m, 27)$.

$$\begin{aligned} \text{Now } \triangle GJE &\approx \triangle GHK & \text{and } \triangle HJE &\approx \triangle HGF \\ \Rightarrow JG : HG &= JE : HK & \text{and } HJ : HG &= JE : GF \\ \therefore JG \cdot HK &= JE \cdot HG = HJ \cdot GF \quad \text{i.e., } JG \cdot HK = HJ \cdot GF \dots (2) \end{aligned}$$

Now, $JG = m - 20$, $HK = 27 - 16$, $HJ = 30 - m$ and $GF = 27 - 21$

So,

$$\begin{aligned} (m - 20)(27 - 16) &= (30 - m)(27 - 21) \\ \Rightarrow 11(m - 20) &= 6 \times 10 - 6(m - 20) \quad \text{since } HJ = HG - JG = 10 - (m - 20) \\ \Rightarrow (11 + 6)(m - 20) &= 6 \times 10 & \Rightarrow m &= 20 + \frac{60}{17} = 23.53 \end{aligned}$$

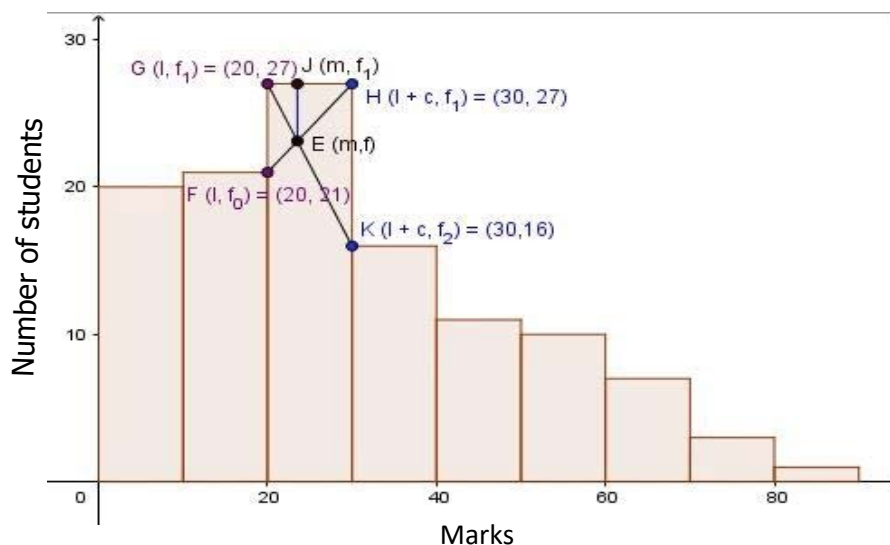


Figure 3. Group 1

Observe that for Group 2, the same process yields the mode to be:

$$m = 20 + \frac{27 - 21}{(27 - 21) + (27 - 24)} \times 10 = 20 + \frac{60}{9} = 26.67$$

which is higher than the mode for Group 1

Symbol	Meaning	In the examples
c	(Uniform) class-width	10
l	Lower limit of modal class	20
f_1	Frequency of modal class	27
f_0	Frequency of the class before the modal class	21
f_2	Frequency of the class after the modal class	16 (G1), 24 (G2)

Table 3

Note that this looks similar to the formula mentioned at the beginning. Now, let us generalize by algebraizing as follows:

So, $G = (l, f_1)$, $H = (l + c, f_1)$, $F = (l, f_0)$ and $K = (l + c, f_2)$. As before mode m is the x -coordinate of the point of intersection $E(m, f)$ of the line segments GK and HF . Also, $J(m, f_1)$ is the point on GH directly above E , i.e., $EJ \perp GH$.

So,

$$\begin{aligned}
 GJ \cdot HK &= HJ \cdot GF \Rightarrow (m - l) \cdot (f_1 - f_2) = (l + c - m) \cdot (f_1 - f_0) \\
 &\Rightarrow (m - l) \cdot (f_1 - f_2) = c \cdot (f_1 - f_0) - (m - l) \cdot (f_1 - f_0) \\
 &\Rightarrow (m - l) \cdot ((f_1 - f_2) + (f_1 - f_0)) = c \cdot (f_1 - f_0) \\
 &\Rightarrow (m - l) \cdot (2f_1 - f_0 - f_2) = c \cdot (f_1 - f_0) \\
 &\Rightarrow m - l = \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times c \\
 &\Rightarrow m = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times c
 \end{aligned}$$

It may make sense to let different groups of students work with different histograms, collate their work in a table like Table 3 and then crystalize the algebraic form of the formula from them.

But what happens when the modal class is at either end of the histogram? That is either the previous class with frequency f_0 is missing or the class after with frequency f_2 is not there. Let us consider the following case where modal class is the lowest class:

Age	Percentage of population
65-70	29.2
70-75	22.6
75-80	18.7
80-85	14.8
85-90	9.2
90-95	4.0
95+	1.5
Total	100.0

Table 4. Age distribution of US population of age 65yrs and over in 2008

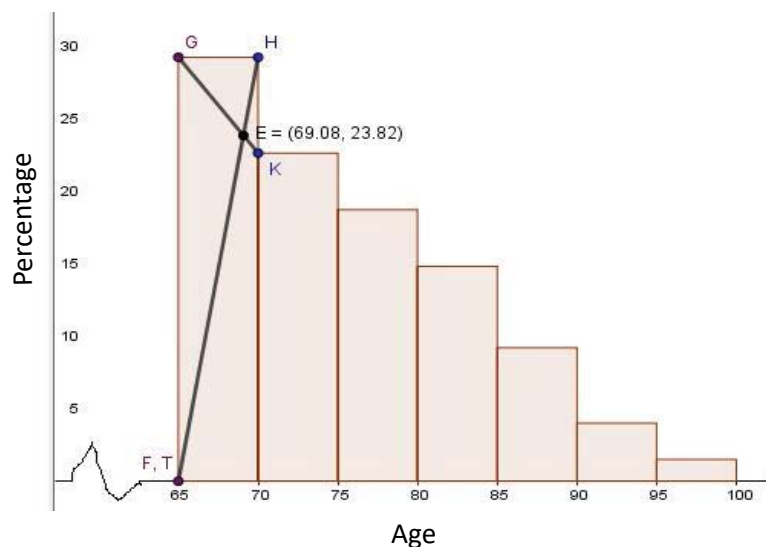


Figure 4. Age distribution

Note that in this case F and T have coincided and $f_0 = 0$ because there is no class before the modal class. So, the mode is

$$m = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times c = 65 + \frac{29.2 - 0}{2(29.2) - 0 - 22.6} \times 5 = 69.08.$$

So, if the modal class is the lowest class, then the formula holds with $f_0 = 0$. Similarly, if the modal class is the highest class, then the same applies to f_2 i.e., $f_2 = 0$.

Another related question: What if there are two (or more) consecutive modal classes (with the same frequency)? In that case, the modal classes should be combined to form a wider modal class with the class interval twice (or more times) that of other classes. Then this combined (and fatter) modal class would be

the rectangle GHUT in the histogram with the point F and K on either vertical side, E as the intersection of GK and HF, and J as the point on GH just above E. The same process holds.

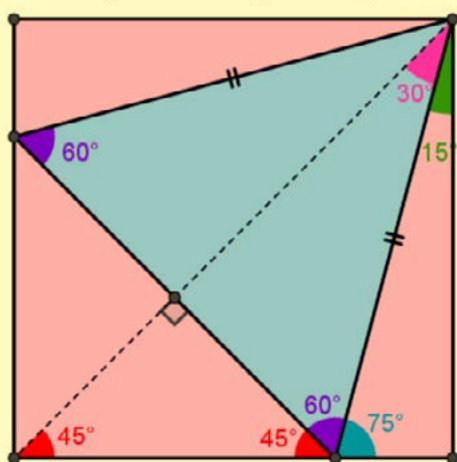
As we observed for the median formula, the derivation of the mode formula is also not difficult. It uses histogram and basic coordinate geometry along with a little bit of similar triangles — all very much part of secondary school syllabus. So, instead of prescribing a pretty complicated formula (without any clue to why or how it formed), it is better to teach the students the underlying reasoning involving the line segments GK and HF. The same principle can enable them to tackle cases like modal class at an end or multiple consecutive modal classes.

In the next and final article in this series we would discuss why the midpoints of the class intervals are used to compute the mean of a grouped data set. It would be particularly relevant since we argued that the midpoint (of the modal class) is not an ideal choice for the mode.

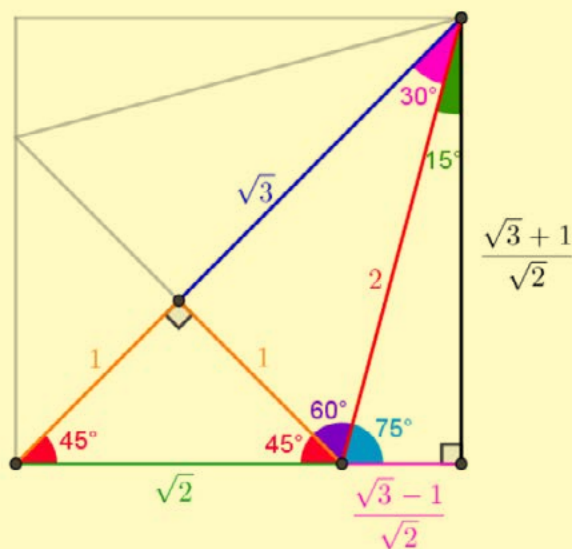
Math Co-dev Group or more elaborately **Mathematics Co-development Group** is an internal initiative of Azim Premji Foundation where math resource persons across states put their heads together to prepare simple materials for teachers to develop their understanding on different content areas and how to transact the same in their classrooms. It is a collaborative learning space where resources are collected from multiple sources, critiqued and explored in detail. Math Co-dev Group can be reached through yashvendra@azimpremjifoundation.org

A visual clue to trigonometrical ratios of some standard angles.

An equilateral triangle in a square.



@AmareshGS1



ಅಮರೇಶ್.ಜಿ. ಎಸ್., ಬಳ್ಳಾರಿ, ಭಾರತ.

An Arithmetic Escapade

PRITHWIJIT DE

Problem. Consider the numbers 75, 84 and 93. They are in arithmetic progression. Flip the digits to obtain 57, 48 and 39. These three numbers too are in arithmetic progression. Can we characterise all such three-term arithmetic progressions consisting of two-digit positive integers?

Solution. Let the three numbers be

$$10a_1 + b_1, \quad 10a_2 + b_2, \quad 10a_3 + b_3$$

where $a_i, b_i \in \{1, \dots, 9\}$ for $i = 1, 2, 3$. Note that we have assumed that the digits are non-zero so that the case of having a single digit number upon flipping is eliminated. Since

$$10a_1 + b_1, \quad 10a_2 + b_2, \quad 10a_3 + b_3$$

are in arithmetic progression we have

$$10(a_1 - 2a_2 + a_3) + (b_1 - 2b_2 + b_3) = 0.$$

Also, the numbers $10b_1 + a_1, 10b_2 + a_2, 10b_3 + a_3$ are in arithmetic progression, which implies

$$10(b_1 - 2b_2 + b_3) + (a_1 - 2a_2 + a_3) = 0.$$

Therefore

$$a_1 - 2a_2 + a_3 = b_1 - 2b_2 + b_3 = 0.$$

Thus the corresponding digits of the given positive integers are in arithmetic progression.

Keywords: Arithmetic progression, digits

The case of three-digit numbers. What if we want to characterise three-digit positive integers with the same property? Let the numbers

$$100a_i + 10b_i + c_i$$

with $a_i, b_i, c_i \in \{1, \dots, 9\}$, $i = 1, 2, 3$ be such that the numbers

$$100a_1 + 10b_1 + c_1, \quad 100a_2 + 10b_2 + c_2, \quad 100a_3 + 10b_3 + c_3$$

and

$$100c_1 + 10b_1 + a_1, \quad 100c_2 + 10b_2 + a_2, \quad 100c_3 + 10b_3 + a_3$$

form two arithmetic progressions. Then

$$100(a_1 - 2a_2 + a_3) + 10(b_1 - 2b_2 + b_3) + (c_1 - 2c_2 + c_3) = 0$$

and

$$100(c_1 - 2c_2 + c_3) + 10(b_1 - 2b_2 + b_3) + (a_1 - 2a_2 + a_3) = 0.$$

Whence, by subtraction,

$$a_1 - 2a_2 + a_3 = c_1 - 2c_2 + c_3 = t(\text{say}).$$

Then

$$10(b_1 - 2b_2 + b_3) = -101t.$$

Observe that $10|t$ and $|t| < 20$. Thus $t \in \{0, \pm 10\}$. If $t = \pm 10$ then $b_1 - 2b_2 + b_3 = \mp 101$, which is absurd. Hence $t = 0$ and $b_1 - 2b_2 + b_3 = 0$. We see that for three-digit positive integers to possess the desired property, the corresponding digits must be in arithmetic progression.

The general case. Now we proceed to tackle the case of positive integers with $n (> 3)$ digits. Let the numbers

$$x_i = \sum_{k=1}^n a_{i,k} 10^{k-1}$$

with $a_{i,k} \in \{1, \dots, 9\}$, $i = 1, 2, 3$ and $k = 1, \dots, n$ be such that x_1, x_2, x_3 is an arithmetic progression and so are the numbers y_1, y_2, y_3 where

$$y_i = \sum_{k=1}^n a_{i,n+1-k} 10^{k-1}.$$

By setting $\lambda_k = a_{1,k} - 2a_{2,k} + a_{3,k}$, it follows that

$$\sum_{k=1}^n \lambda_k 10^{k-1} = 0 \tag{1}$$

and

$$\sum_{k=1}^n \lambda_{n-k+1} 10^{k-1} = 0. \quad (2)$$

Observe that $10|\lambda_1|$ and $10|\lambda_n|$. Since $|\lambda_1| < 20$ and $|\lambda_n| < 20$, we must have $\lambda_1, \lambda_n \in \{-10, 0, 10\}$. Suppose $\lambda_1 \neq 0$. Then by (2) we have

$$\lambda_1 10^{n-1} = - \sum_{k=1}^{n-1} \lambda_{n-k+1} 10^{k-1}.$$

But since $|\lambda_{n-k+1}| < 20$ for $1 \leq k \leq n$ we have

$$\left| - \sum_{k=1}^{n-1} \lambda_{n-k+1} 10^{k-1} \right| < \frac{2(10^n - 10)}{9} < 10^n = |\lambda_1| 10^{n-1},$$

a contradiction. Hence $\lambda_1 = 0$.

Proceeding in a similar manner by using equation (1) we can show that $\lambda_n = 0$. Thus the terms independent of a power of 10 in both equations (1) and (2) are zero.

On dividing both sides of (1) and (2) by 10 we obtain two equations:

$$\sum_{k=1}^{n-2} \lambda_{k+1} 10^{k-1} = 0 \quad (3)$$

and

$$\sum_{k=1}^{n-2} \lambda_{n-k} 10^{k-1} = 0. \quad (4)$$

The same argument may be repeated with equations (3) and (4) to show that the constant terms are zero and the process continues till we show that $\lambda_k = 0$ for $1 \leq k \leq n$. This implies that the corresponding digits of the given numbers must form an arithmetic progression.



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Fun with Equivalent Fractions

A S RAJAGOPALAN

Start with any single digit pair of equivalent fractions such as $\frac{1}{2}$ and $\frac{2}{4}$ or $\frac{2}{3}$ and $\frac{4}{6}$. In general, let us write this as $\frac{a}{b} = \frac{c}{d}$. Note that we can also include special cases such as $\frac{a}{b} = \frac{b}{c}$.

Using the properties of proportions ('componendo' and 'dividendo'), we can write $\frac{a}{b} = \frac{c}{d}$ as

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Multiplying both sides by $\frac{11}{9}$, we get:

$$\frac{11a+11b}{9a-9b} = \frac{11c+11d}{9c-9d}.$$

Applying componendo-dividendo again we get,

$$\frac{20a+2b}{2a+20b} = \frac{20c+2d}{2c+20d}.$$

Removing the common factor 2, we get

$$\frac{10a+b}{a+10b} = \frac{10c+d}{c+10d}.$$

This may be written as a product in the following form:

$$(10a+b)(10d+c) = (10b+a)(10c+d).$$

Now observe that $10a+b$, $10b+a$, $10c+d$ and $10d+c$ are actually the 2-digit numbers $\overline{ab}$, $\overline{ba}$, $\overline{cd}$ and $\overline{dc}$.

Keywords: Equivalent fractions, componendo-dividendo

Therefore, by replacing a, b, c, d by suitable digits from different pairs of equivalent fractions $\frac{a}{b} = \frac{c}{d}$, we get many interesting equalities. For example, from the equivalent fractions $\frac{2}{3} = \frac{4}{6}$, we get

$$23 \times 64 = 32 \times 46.$$

Thus we have a case where the product of two 2-digit numbers is the same as the product of the numbers obtained by reversing the order of the digits.

We can thus generate the list of all such pairs, using nothing more than the properties of proportions. It turns out that there are 14 such pairs of numbers as listed below:

$$\begin{aligned}12 \times 42 &= 21 \times 24 = 504 \\12 \times 63 &= 21 \times 36 = 756 \\12 \times 84 &= 21 \times 48 = 1008 \\24 \times 63 &= 42 \times 36 = 1512 \\24 \times 84 &= 42 \times 48 = 2016 \\36 \times 84 &= 63 \times 48 = 3024 \\13 \times 62 &= 31 \times 26 = 806 \\13 \times 93 &= 31 \times 39 = 1209 \\26 \times 93 &= 62 \times 39 = 2418 \\14 \times 82 &= 41 \times 28 = 1148 \\23 \times 64 &= 32 \times 46 = 1472 \\23 \times 96 &= 32 \times 69 = 2208 \\46 \times 96 &= 64 \times 69 = 4416 \\34 \times 86 &= 43 \times 68 = 2924\end{aligned}$$



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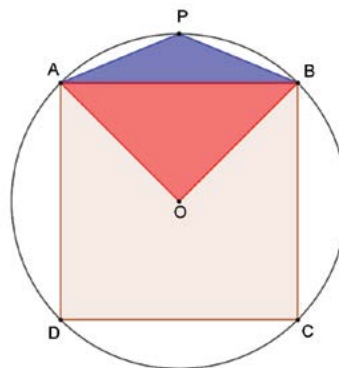
Six Problems on Area and Perimeter

A. RAMACHANDRAN

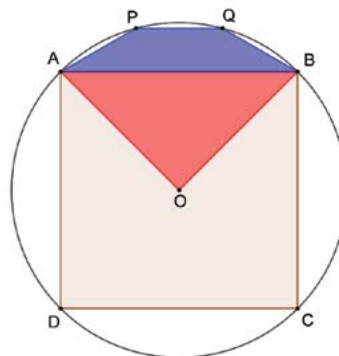
The following formula for the area of a triangle could be useful to you in solving the first two problems.

$$\text{Area of } \triangle ABC = \frac{1}{2}ab \sin C.$$

1. The accompanying figure shows a square ABCD inscribed in a circle with centre O. P is the midpoint of minor arc AB. Find the ratio of the areas of $\triangle AOB$ and $\triangle APB$.



2. The accompanying figure shows a square ABCD inscribed in a circle with centre O. Points P, Q are situated in the minor arc AB such that $AP = PQ = QB$. Find the ratio of the areas of $\triangle AOB$ and quadrilateral $\triangle APQB$.



Keywords: Area; Perimeter; Triangle; Trigonometry; Ratio; Composition; Decomposition; Dissection

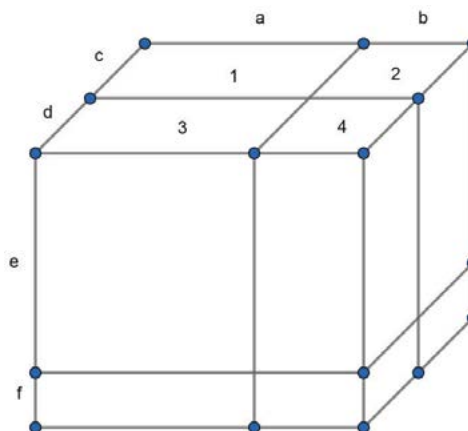
3. In the accompanying figure, a rectangle is divided into four smaller rectangles. If the perimeters of the top left, top right and bottom left rectangles are 74, 68 and 66 units respectively, find the perimeter of the rectangle at bottom right.



4. With reference to the figure of the previous question, if the areas of the top left, top right and bottom left rectangles are 135, 108 and 120 square units respectively, find the area of the rectangle at bottom right.



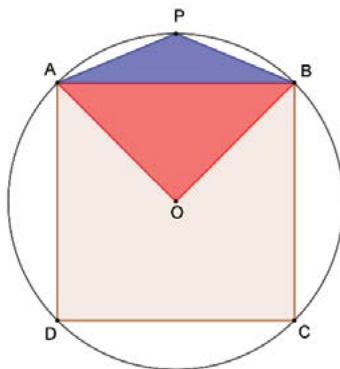
5. This problem is a 3-D analogue of the previous problem. We have a cuboid divided into eight smaller cuboids by three plane cuts one parallel to each pair of opposite faces. (See Figure.) The eight smaller cuboids have been numbered from 1 to 8. Cuboids 5,6,7,8 are directly below cuboids 1,2,3,4 respectively. The different edge lengths have been designated as a, b, c, d, e, f . If now the volumes of cuboids 1, 2 and 7 are given, how can you find the volume of cuboid 8?



6. This problem relates to the dissected cuboid of the previous problem. If the total surface areas of cuboids 1 to 7 are given, how can you calculate the total surface area of cuboid 8?

Solutions

- Let us take the radius of the circle to be 1 unit.



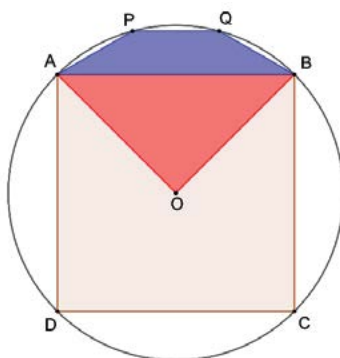
Then area of $\triangle AOB = \frac{1}{2}$ square units. If we join OP, $\angle AOP = 45^\circ$, by symmetry. Then area of $\triangle AOP = \frac{1}{2}(1)(1) \sin 45^\circ = \frac{1}{2\sqrt{2}}$.

$$\text{Area of quadrilateral APBO} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Area of $\triangle APB = \frac{\sqrt{2}}{2} - \frac{1}{2} = (\sqrt{2} - 1)/2$. [We could also find the length of the altitude on base AB in $\triangle APB$ and thereby the area of $\triangle APB$.]

Ratio of areas of $\triangle AOB$ and $\triangle APB = 1 : \sqrt{2} - 1$.

- We use an approach similar to that of the earlier solution. If we join OP, then $\angle AOP = 30^\circ$ (equal chords subtend equal angles at the centre).



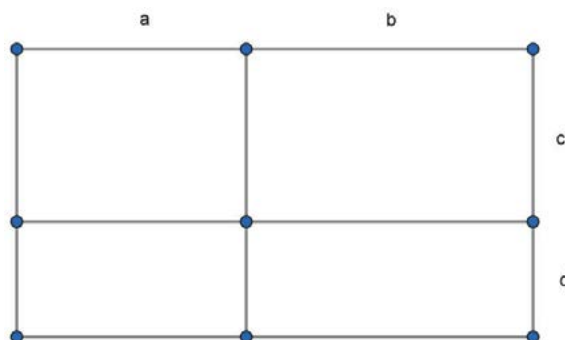
Area of $\triangle AOP = \frac{1}{2} (1)(1) \sin 30^\circ = \frac{1}{4}$. (Radius of circle taken as unity.)

Area of pentagon APQBO is then $\frac{3}{4}$.

Area of quadrilateral APQB = $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. Ratio of areas of $\triangle AOB$ and quadrilateral APQB = $2 : 1$.

3. Let us name the different lengths in the figure as a , b , c and d . (See accompanying figure.) Then we have $2(a+c) = 74$; $2(b+c) = 68$; $2(a+d) = 66$.

The required result is $2(b+d) = 2(b+c) + 2(a+d) - 2(a+c) = 68 + 66 - 74 = 60$.



4. Using the same designation of lengths as in the previous solution, we have $ac = 135$; $bc = 108$; $ad = 120$. The required result is $bd = bcad/ac = (108 \times 120)/135 = 96$ square units.
5. Volumes of cuboids 1, 2 and 7 are ace , bce and adf respectively. Volume of cuboid 8 is bdf . Notice that

$$\frac{bce \times adf}{ace} = bdf. \text{ That is, } (\text{volume}_2 \times \text{volume}_7) \div \text{volume}_1 = \text{volume}_8.$$

In other words the product of the volumes of two cuboids that are diagonally opposite to each other (do not share a face or edge but meet at a vertex) is the product of all six edge lengths, i.e., $abcdef$.

6. Let us write down the TSA (total surface area) of all eight cuboids, using the length designations of the previous problem:

$$\begin{aligned} \text{TSA}_1 &= 2(ac + ae + ce); & \text{TSA}_2 &= 2(bc + be + ce); \\ \text{TSA}_3 &= 2(ad + ae + de); & \text{TSA}_4 &= 2(bd + be + de); \\ \text{TSA}_5 &= 2(ac + cf + af); & \text{TSA}_6 &= 2(bc + bf + cf); \\ \text{TSA}_7 &= 2(ad + af + df); & \text{TSA}_8 &= 2(bd + bf + df). \end{aligned}$$

Note that $\text{TSA}_1 + \text{TSA}_4 + \text{TSA}_6 + \text{TSA}_7 = \text{TSA}_2 + \text{TSA}_3 + \text{TSA}_5 + \text{TSA}_8$.

So, knowing any seven TSA's, one can obtain the remaining TSA.



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Addendum to “Six Problems on Area and Perimeter”

SHAILESH SHIRALI

We present here geometric solutions to the first two questions in “Six problems on area and perimeter”. The solution for the second question involves a beautiful ‘proof without words’ (PWW).

Problem 1. Figure 1 (a) shows a square $ABCD$ inscribed in a circle with centre O . P is the midpoint of minor arc AB . Find the ratio of the areas of $\triangle APB$ and $\triangle AOB$.

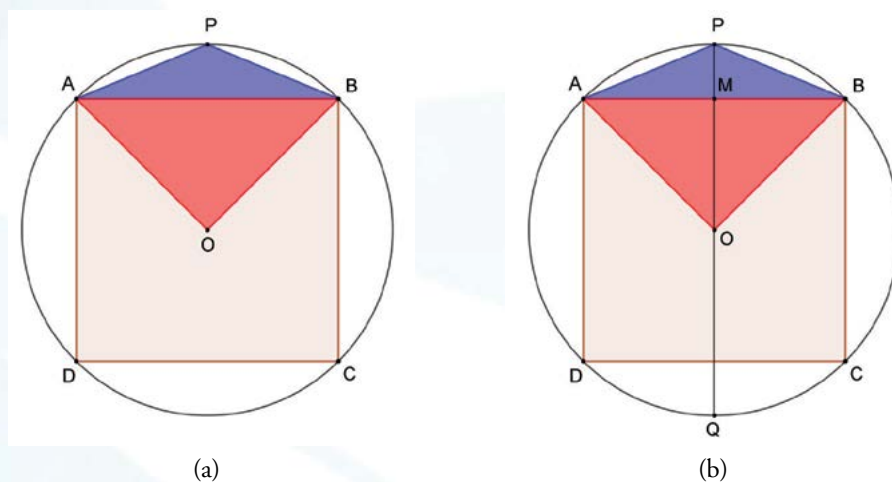


Figure 1.

Keywords: Intersecting chords theorem

Solutions. We take the circle to have radius 1. Let PQ be the diameter through P ; see Figure 1 (b). The midpoint M of AB lies on PO . Since $\triangle AMO$ is isosceles and right-angled, it follows (using Pythagoras's theorem) that $OM = 1/\sqrt{2}$ and therefore $PM = 1 - 1/\sqrt{2}$. It follows that

$$\frac{PM}{OM} = \frac{1 - 1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2} - 1.$$

Hence area of $\triangle APB$: area of $\triangle AOB = \sqrt{2} - 1 : 1$. (Areas of triangles on the same base are in the same ratio as their heights.)

Alternative solutions. Another approach is to use the intersecting chords theorem. This is the statement that if two chords PQ and RS in a circle intersect at point M , then $PM \cdot MQ = RM \cdot MS$. This may be shown using the fact that $\triangle PMR$ and $\triangle SMQ$ are similar to each other.

We refer again to Figure 1 (b). Let x be the length of PM . To find x , note that $AM = 1/\sqrt{2}$. Also, $MQ = 2 - x$. The intersecting chords theorem tells us that

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = x \cdot (2 - x), \quad \therefore 2x^2 - 4x + 1 = 0, \quad \therefore x \in \left\{ \frac{\sqrt{2}-1}{\sqrt{2}}, \frac{\sqrt{2}+1}{\sqrt{2}} \right\}.$$

The second solution does not fit here, as we must have $x < 1$. Hence

$$x = \frac{\sqrt{2}-1}{\sqrt{2}}, \quad \therefore 1 - x = \frac{1}{\sqrt{2}}.$$

Therefore

$$\frac{x}{1-x} = \frac{\sqrt{2}-1}{1}.$$

Hence area of $\triangle APB$: area of $\triangle AOB = \sqrt{2} - 1 : 1$, as earlier.

Problem 2. Figure 2 (a) shows a square $ABCD$ inscribed in a circle with centre O . Points P, Q are situated on the minor arc AB such that $AP = PQ = QB$. Find the ratio of the areas of $\triangle AOB$ and quadrilateral $APQB$.

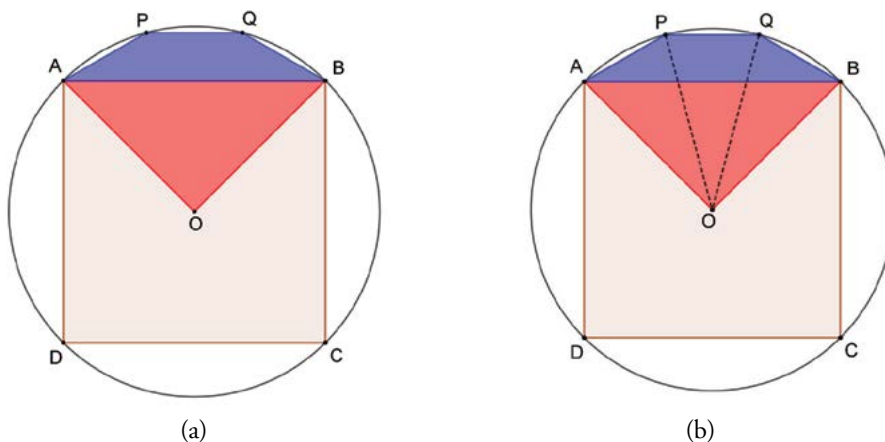


Figure 2.

Solutions. Join OP and OQ , and note that $\angle AOP = \angle POQ = \angle QOB = 30^\circ$. See Figure 2 (b). Four copies of pentagon $OAPQB$ can be used to make up a regular 12-sided polygon (i.e., a dodecagon) inscribed in the same circle, as shown in Figure 3. If we can find the area of this polygon, we will know the area of pentagon $OAPQB$, and from this we can find the desired ratio, as the area of $\triangle AOB$ is easily computed.

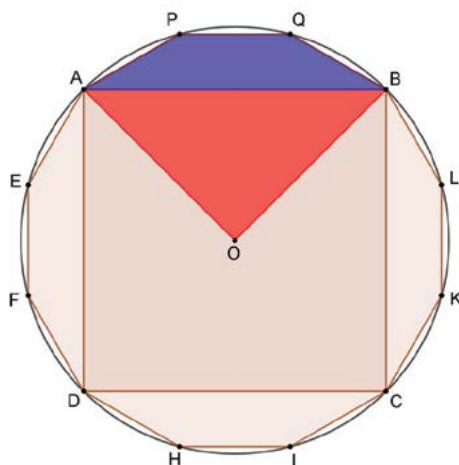


Figure 3. $AEFDHICKLBQP$ is a regular 12-sided polygon (a dodecagon)

Before proceeding, we make an important observation about triangle APQ : it is isosceles, with angles of $15^\circ, 15^\circ, 150^\circ$. Two copies of this triangle can be joined along their longest side to make a rhombus (a ‘diamond’) with angles of $30^\circ, 150^\circ$. See Figure 4.

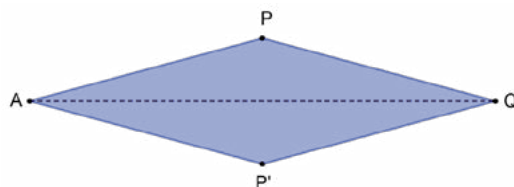


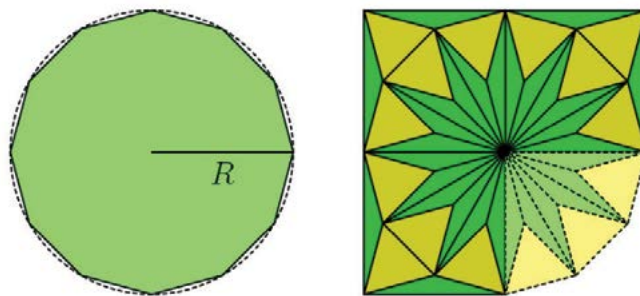
Figure 4.

Now there is a beautiful ‘proof without words’ which shows that the area of a regular 12-sided polygon inscribed in a circle of radius R is $3R^2$. See Figure 5. Observe how 24 congruent copies of $\triangle APQ$ and 12 congruent copies of the equilateral triangle with side AP fit together beautifully to make three-quarters of a square whose side is the diameter of the circumscribing circle. Note that a few extra copies of the two kinds of triangles are needed to make the full square, as shown in the right-hand corner (using a lighter colour; 6 congruent copies of $\triangle APQ$ and 3 congruent copies of the equilateral triangle with side AP). Since the area of the full square is $(2R)^2 = 4R^2$, it follows that the area of the 12-sided polygon is $3/4$ of this, i.e., $3R^2$.

Using this result, and referring once again to Figure 2 (b), we see that the area of $OAPQB$ is

$$\frac{1}{4} \cdot 3 \cdot 1^2 = \frac{3}{4}.$$

Since the area of $\triangle AOB$ is $1/2$, the area of quadrilateral $APQB$ is $3/4 - 1/2 = 1/4$. We deduce that area of quadrilateral $APQB$: area of $\triangle AOB = 1 : 2$.



The area of a dodecagon is $3R^2$, where R is the circumradius.

Figure 5. Source: https://artofproblemsolving.com/wiki/index.php?title=Proofs_without_words

Comment. Obviously, the solution for the second problem given in the article “Six problems on area and perimeter”, using trigonometry, is far more compact and simple than the solution presented here! (This only goes to underscore the great power of trigonometry—something which is not always appreciated by us.) However, we have shared these alternative solutions here only to show how different ideas in mathematics can fit together to make a beautiful and composite whole. We hope you have enjoyed reading through these solutions.



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On a method for Solving Cubic Equations

BIPLAB ROY

In this short note, we discuss a method to solve cubic equations. It is based on a method of factorisation developed by Abdul Halim Sk., a school teacher of West Bengal, so we call it ‘Halim’s method of factorisation.’

It involves a certain degree of hit-&-trial (or ‘guesswork’), and may be applied to cubic polynomials of the forms $x^3 + bx^2 + c$ and $x^3 + bx + c$. Here, b and c are integers.

Let us see how the approach works for the polynomial $x^3 + bx + c$. Suppose that

$$x^3 + bx + c = (x + p)(x^2 + qx + r).$$

The expression on the right side is equal to $x^3 + (q + p)x^2 + (r + pq)x + pr$. As this is identically equal to $x^3 + bx + c$, we may equate coefficients of like powers of x on both sides. We get:

$$\begin{aligned} q + p &= 0, \\ r + pq &= b, \\ pr &= c. \end{aligned}$$

These equalities yield $q = -p$ and $b = r - p^2$. Therefore we can rewrite the given polynomial as

$$x^3 + bx + c = x^3 + (r - p^2)x + pr.$$

Now we apply the above to solve a cubic equation.

Keywords: Cubic equation, polynomial, factorisation

The general form of a cubic equation is $ax^3 + 3bx^2 + 3cx + d = 0$.

We first reduce it to the standard form by the transformation $y = ax + b$. This removes the quadratic term, and we are left with the equation $y^3 + 3Hy + G = 0$ (for some H, G).

We factorize this using Halim's method:

$$y^3 + 3Hy + G = (y + p)(y^2 - py + r),$$

where $3H = r - p^2$ and $G = pr$.

For this, we must look for a pair of numbers p, r such that $3H = r - p^2$ and $G = pr$. This involves a certain amount of trial and error.

If we are easily able to find p and r , then by solving the linear equation $y + p = 0$ and the quadratic equation $y^2 - py + r = 0$, we find all three roots of $y^3 + 3Hy + G = 0$:

$$-p, \frac{p \pm \sqrt{p^2 - 4r}}{2}.$$

Finally, from the relation $y = ax + b$, we get all the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$.

We demonstrate this using two examples.

Example 1

Take the equation $x^3 - 6x - 9 = 0$. We must look for a pair of numbers p, r such that $-6 = r - p^2$ and $-9 = pr$. By inspection we find $r = 3$ and $p = -3$, because $-9 = (-3) \times 3$ and $-6 = 3 - 3^2$. So:

$$\begin{aligned} x^3 - 6x - 9 &= 0 \\ \implies x^3 - (3^2 - 3)x - 9 &= 0 \\ \implies (x - 3)(x^2 + 3x + 3) &= 0. \end{aligned}$$

From $x - 3 = 0$ we get $x = 3$, and from $x^2 + 3x + 3 = 0$ we get $x = \frac{1}{2}(-3 \pm i\sqrt{3})$.

So the roots of the given equation are

$$\left\{ 3, \frac{-3 \pm i\sqrt{3}}{2} \right\}.$$

Example 2

Take the equation $x^3 - 12x + 65 = 0$. We must look for a pair of numbers p, r such that $-12 = r - p^2$ and $65 = pr$. By inspection we find $r = 13$ and $p = 5$, because $65 = 5 \times 13$ and $12 = 5^2 - 13$. So:

$$\begin{aligned} x^3 - 12x + 65 &= 0 \\ \implies x^3 - (5^2 - 13)x + 65 &= 0 \\ \implies (x + 5)(x^2 - 5x + 13) &= 0. \end{aligned}$$

From $x + 5 = 0$ we get $x = -5$, and from $x^2 - 5x + 13 = 0$ we get $x = \frac{1}{2}(5 \pm 3i\sqrt{3})$.

So the roots of the given equation are

$$\left\{ -5, \frac{5 \pm 3i\sqrt{3}}{2} \right\}.$$

Closing remarks

Will this method always work? Given the cubic equation $y^3 + 3Hy + G$, it should be clear that the success of this approach depends on our easily finding a pair of numbers p and r such that $3H = r - p^2$ and $G = pr$.

As noted above, this requires trial and error. Unfortunately, there is no straightforward way to find such a pair of numbers. If we try to do it systematically, by setting it up as an equation, we end up with the very equation that we had started with.



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The Rascal Triangle Revisited

JAMES METZ

In the *At Right Angles* October 2021 webinar, Dr. Shashidhar Jagadeeshan and Dr. Shailesh Shirali discussed The Rascal Triangle, referencing a 2016 article of the same title by Ishaan Magon, Maya Reddy, Rishabh Suresh and Shashidhar Jagadeeshan.

The speakers elaborated on methods that students had discovered for finding a particular entry in the triangle and stressed the value of allowing students to explore and enjoy mathematics. While the webinar also explored many other interesting problems, I would like to share a few additional ideas about the fascinating Rascal Triangle. Please visit [1] to view the recording, and see the articles at [2] and [3].

Using the example from the webinar, suppose we want to know the 3rd term in row 5, as shown in Figure 1, with the diamond showing $a = 3$, $b = 5$, $c = 4$ and x .

Note: In this article, the rows and elements are numbered as in the original article, see [2]. Rows start at 0 and each element in the rows also starts at 0.

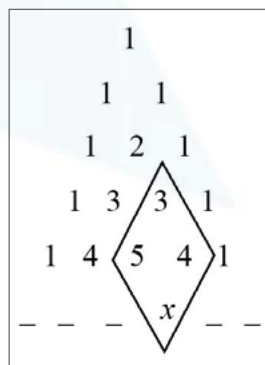


Figure 1

Keywords: Patterns, problem solving, exploration, Rascal Triangle

The value of x can be found using the expression $a + n - 1$, so we use a and n instead of b and c . With $a = 3$ and $n = 5$, $x = 3 + 5 - 1 = 7$. For another example, if we want to find the 2nd element in row 8, $a = 6$ by inspection and $n = 8$, so $x = 6 + 8 - 1 = 13$.

Figure 2 shows a modified diamond diagram from page 47 of [2], with $x = \text{Entry}(n, k) = k(n - k) + 1$ as established in the article.

$$\begin{aligned} a &= \text{Entry}(n - 2, k - 1) \\ &= k(n - k) - n + 2 \\ b &= \text{Entry}(n - 1, k - 1) \quad c = \text{Entry}(n - 1, k) \\ x &= \text{Entry}(n, k) = k(n - k) + 1 \\ &= [k(n - k) - n + 2] + n - 1 = a + n - 1 \end{aligned}$$

Figure 2

Thus, the value of x is $a + n - 1$.

We can also determine the 3rd number in the 5th row in a way similar to what is used with Pascal's Triangle (Figure 3).

We need to consider the triangle (not the diamond) as we do with Pascal, but we must also use n . We use the formula $x = (b + c + n)/2$, so $x = (5 + 4 + 5)/2 = 7$. It is easy to show this is valid since $x = a + n - 1$ and $b + c = a + x - 1$. Replace a in the second equation with $x - n + 1$ and the result follows easily. For another

example, the 5th number in the 8th row is $(13 + 11 + 8)/2 = 16$. When the row number is even, then the sum of b and c is even (both b and c are odd) and when the row number is odd, then the sum of b and c is odd (one is even and the other is odd), echoing the webinar exploration of the sequence of triangular numbers.

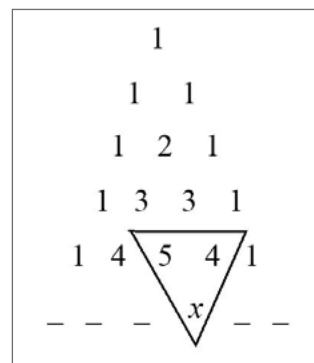


Figure 3

Another way to look at the two different formulas given in the webinar is to observe that $ax = bc + 1$ and also $a + x = b + c + 1$, and solving for x in each gives the two formulas. What a delightful set of 4 numbers! No doubt, both the American students and the CFL students might have discovered their respective formulas from this observation.

In Pascal's triangle, the sum of the numbers in a given row is a power of 2. In the Rascal Triangle, the sum of the numbers in row n is given by $(n^3 + 5n + 6)/6$, a nice result that students can get using successive differences.

References

- [1] *At Right Angles* Webinar: Rascal Triangle, <https://www.youtube.com/watch?v=fryhomqA0dI>
- [2] The Rascal Triangle, <https://bit.ly/3FF65en>
- [3] The Rascal Triangle, a Rascal full of Surprises, <https://bit.ly/3BxNuhM>



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Triangles with One Angle Twice Another

SHAILESH SHIRALI

Triangles with one angle equal to another are familiar objects; a lot is known about them. What can be said about triangles in which one angle is twice another? Can such triangles be characterised in any other way? We explore these questions in this article.

We have the following striking and compact result about triangles in which one angle is twice another (see Figure 1).

Theorem 1. In any $\triangle ABC$, the following is true:

$$\angle A = 2\angle B \iff a^2 = b(b + c). \quad (1)$$

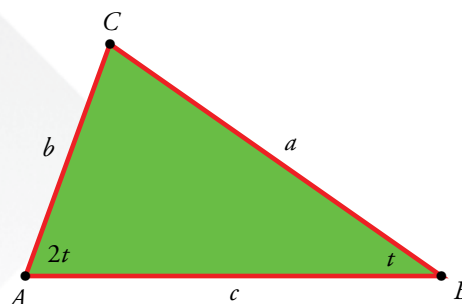


Figure 1. Triangle ABC with $\angle A = 2\angle B$

Observe that the result is an “if and only if” statement. We consider the forward and reverse implications separately.

We offer two different kinds of proofs of the result. It is interesting that the reverse implication presents a greater challenge using either approach.

Proof using trigonometry. We make use of the sine rule for triangles and the fact that supplementary angles have equal sine values.

Forward implication. We must prove that if $\angle A = 2\angle B$, then $a^2 = b(b + c)$. Let $\angle B = t$; then $\angle A = 2t$ and $\angle C = 180^\circ - 3t$. We therefore have:

$$\frac{a}{\sin 2t} = \frac{b}{\sin t} = \frac{c}{\sin 3t}. \quad (2)$$

Keywords: Triangles, similarity, trigonometry, proof, trigonometric identity

Multiplying through by $\sin t$ and remembering the double angle and triple angle identities, we get:

$$\frac{a}{2 \cos t} = b = \frac{c}{3 - 4 \sin^2 t}.$$

Since $3 - 4 \sin^2 t = 4 \cos^2 t - 1$, we have:

$$2 \cos t = \frac{a}{b}, \quad 4 \cos^2 t - 1 = \frac{c}{b}. \quad (3)$$

We may easily eliminate t from the above two equalities:

$$\frac{c}{b} = \frac{a^2}{b^2} - 1, \quad \therefore bc = a^2 - b^2,$$

and so

$$a^2 = b(b + c). \quad (4)$$

Reverse implication. We must prove that if $a^2 = b(b + c)$, then $\angle A = 2\angle B$. Invoking the sine rule again, the given equality leads to

$$\begin{aligned} \sin^2 A &= \sin B \cdot (\sin B + \sin C), \\ \therefore \sin^2 A - \sin^2 B &= \sin B \cdot \sin C. \end{aligned} \quad (5)$$

We now invoke the following striking and beautiful trigonometric identity (which looks extremely surprising at first glance, as it looks just like the “difference of two squares” identity):

$$\sin^2 A - \sin^2 B = \sin(A + B) \cdot \sin(A - B).$$

We also have $\sin(A + B) = \sin C$. Hence from (5) we get $\sin B \cdot \sin C = \sin C \cdot \sin(A - B)$, and so (since $\sin C \neq 0$),

$$\sin B = \sin(A - B). \quad (6)$$

From (6) the following two possibilities arise:

- the angles B and $A - B$ are equal; OR
- the angles B and $A - B$ are supplementary.

The second possibility leads to $A = 180^\circ$, which is absurd. Therefore we must have $B = A - B$, which leads to $A = 2B$, as required. $\square$

Proof using ‘pure’ geometry. Examining the form of $a^2 = b(b + c)$, we are led to expect that the proof will involve working with suitably constructed similar triangles. This is because the expression $a^2 = b(b + c)$ may be written as $a/b = (b + c)/a$, and this immediately suggests looking for a pair of similar triangles. (Actually, there is another possible line of inquiry, but we will say something about this at the end.)

Forward implication. We must prove that if $\angle A = 2\angle B$, then $a^2 = b(b + c)$. Since $A = 2B$, it makes sense to draw the angle bisector of $\angle BAC$, as this will give us an angle equal to $\angle CBA$. Drawing this angle bisector, we obtain Figure 2. The angle bisector meets side CB at D . Let $CD = x$; then $DB = a - x$.

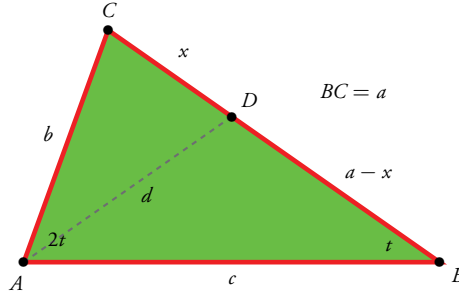


Figure 2. Triangle ABC with $\angle A = 2\angle B$

In Figure 2, $\triangle DAB$ is isosceles, so $DA = DB$, i.e., $d = a - x$. Also, $\angle CDA = 2t$ (note that it is an exterior angle to $\triangle DAB$). This means that $\triangle CAD$ has the same set of angles as $\triangle CBA$ (namely: $180^\circ - 3t, t, 2t$). The two triangles are therefore similar to each other, so their sides (namely: $\{x, b, d\}$ and $\{b, a, c\}$, respectively) must be in proportion. That is:

$$\frac{x}{b} = \frac{b}{a} = \frac{a-x}{c}. \quad (7)$$

These equalities give:

$$x = \frac{b^2}{a}, \quad \therefore \frac{b}{a} = \frac{a - b^2/a}{c} = \frac{a^2 - b^2}{ac},$$

and therefore

$$b = \frac{a^2 - b^2}{c}, \quad \therefore a^2 = b^2 + bc = b(b + c). \quad (8)$$

Reverse implication. This presents a greater challenge. Starting with the expression $a^2 = b(b + c)$, we must construct a pair of similar triangles. The challenge here is to make geometric sense of the expression $b + c$, which is a sum of two lengths which do not even lie in a straight line. We shall solve the problem using a suitable construction (Figure 3).

Extend side CA of $\triangle ABC$ to E such that $AE = AB$, i.e., $AE = c$. This results in a segment CE which has length $b + c$. Join BE . Now write the given relation $a^2 = b(b + c)$ as

$$\frac{a}{b} = \frac{b + c}{a}.$$

With reference to Figure 3, this states that

$$\frac{CB}{CA} = \frac{CE}{CB}. \quad (9)$$

Relation (9) immediately tells us that

$$\triangle CBA \sim \triangle CEB, \quad (10)$$

(note the order in which we have labelled the vertices—it indicates the vertex correspondence) and hence that we must have the following angle equalities:

$$\angle CBA = \angle CEB, \quad (11)$$

$$\angle CAB = \angle CBE. \quad (12)$$

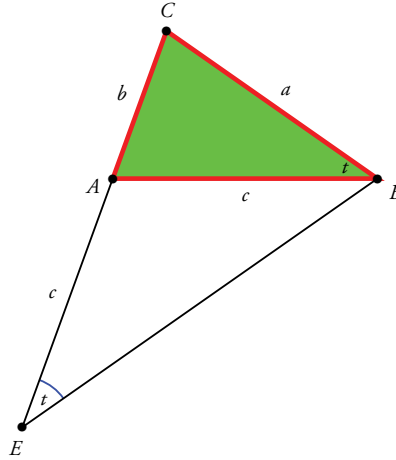


Figure 3. Triangle ABC with $a^2 = b(b + c)$

Relation (11) tells us that $\angle ABE = t$ (since $AE = c = AB$), and therefore that $\angle CBE = 2t$.

Relation (12) combined with the above finding tells us that $\angle CAB = 2t$.

Hence $\angle CAB = 2\angle CBA$, i.e., $\angle A = 2\angle B$. □

A pure geometry solution using a single figure. Examining the above two proofs (for the forward and reverse implications), the reader will notice that we have used different constructions for the two proofs. This may seem unsatisfactory. Is it possible to use a single figure to prove both the implications? We successfully answer this challenge. (It turns out to be simpler than expected!)

Figure 4 shows $\triangle ABC$ in which we extend side CA beyond A to E so that $AE = AB$. We then draw $\triangle ABE$. Since $AE = AB$, we have $\angle ABE = \angle AEB$. We now consider the claim. The reverse implication has already been dealt with above, so we consider only the forward implication.

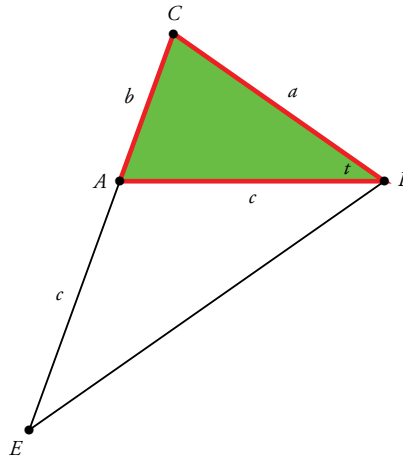


Figure 4. Triangle ABC with CA extended to E so that $AE = AB$

Suppose that $\angle A = 2\angle B$, i.e., $\angle A = 2t$. Since $\angle CAB = \angle ABE + \angle AEB = 2\angle AEB$, it follows that $\angle AEB = t = \angle ABE$.

Observing that the angles of $\triangle CEB$ are identical to those of $\triangle CBA$ (namely: $180^\circ - 3t, t, 2t$), we see that the two triangles are similar to each other. Therefore their sides are in proportion. From this it follows that

$$\frac{a}{b} = \frac{b+c}{a}. \quad (13)$$

Simplifying (13), we obtain $a^2 = b(b+c)$, as required. $\square$

Postscript: A trigonometric analysis that yields a strange conclusion. In the same spirit as the geometric analysis presented above, can we do a trigonometric analysis that yields both the forward and reverse implications in a single movement? This too is possible, but there is a slight twist. The ‘twist’ occurs with the isosceles case.

Consider first the case when the triangle has $\angle B = \angle C$ (i.e., $b = c$) and $\angle A = 2\angle B$. That is, we have $\angle A = 2\angle B = 2\angle C$. The triangle is now isosceles right-angled, with angles $90^\circ, 45^\circ, 45^\circ$. So we have $a/b = \sqrt{2} = a/c$, and the relation $a^2 = b(b+c)$ holds.

Conversely, if we have $b = c$ together with $a^2 = b(b+c)$, then $a^2 = 2b^2$, so $a/b = \sqrt{2} = a/c$, leading to the triangle having angles $90^\circ, 45^\circ, 45^\circ$, in which case the relation $\angle A = 2\angle B$ holds.

Therefore, the case with $b = c$ satisfies the conditions of the theorem and need not be considered further.

In the analysis below, we explicitly exclude the case when $\angle B = \angle C$. We have:

$$\begin{aligned} \angle A = 2\angle B &\iff \sin A = \sin 2B && \text{(because we have } \angle B \neq \angle C) \\ &\iff \sin A = 2 \sin B \cdot \cos B \\ &\iff \frac{a}{b} = 2 \cos B && \text{(by invoking the sine rule)} \\ &\iff \frac{a}{b} = \frac{a^2 + c^2 - b^2}{ac} && \text{(by invoking the cosine rule)} \\ &\iff a^2 c - b(a^2 + c^2 - b^2) = 0 \\ &\iff a^2(c - b) - b(c^2 - b^2) = 0 \\ &\iff (c - b) \cdot (a^2 - b(b + c)) = 0 \\ &\iff a^2 - b(b + c) = 0 && \text{(because we have } b \neq c). \end{aligned}$$

Therefore we have $\angle A = 2\angle B \iff a^2 = b(b+c)$, and the theorem is proved. $\square$



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Finding the Base Angles of a Triangle

MOSES MAKOBE

Consider a $\triangle ABC$ in which the following are specified: $\angle A$ (i.e., the apex angle BAC), the length a of the base BC , and the length h of the altitude from A to BC .

Is it possible to find expressions for the two base angles, $\angle B$ and $\angle C$, in terms of A , a , h ? We do so using trigonometry.

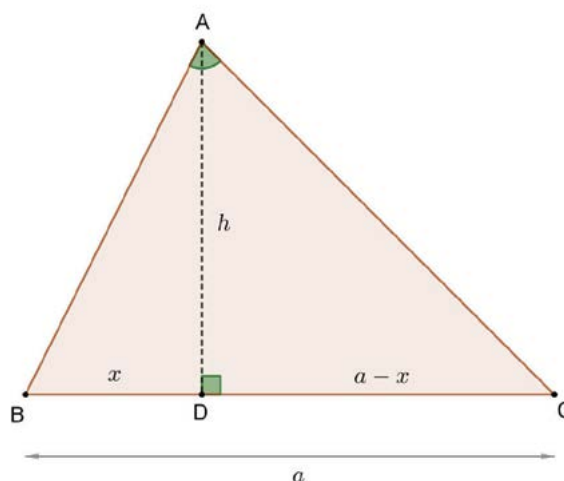


Figure 1.

Let AD be the perpendicular from vertex A to BC , and let $BD = x$, $DC = a - x$. (For convenience, we assume that $\angle B$ and $\angle C$ are acute, which means that D lies on the side and not on the extension of the side. We also assume that the triangle is not right-angled.)

Keywords: trigonometry, quadratics, problem solving

From right-angled triangles ABD and ACD , we have:

$$\tan B = \frac{h}{x}, \quad \tan C = \frac{h}{a-x}. \quad (1)$$

Hence:

$$x = \frac{h}{\tan B}, \quad a - x = \frac{h}{\tan C} = -\frac{h}{\tan(A+B)}, \quad (2)$$

where the last step comes from the fact that $C = 180^\circ - (A+B)$. Hence:

$$\begin{aligned} \frac{a}{h} &= \frac{\tan(A+B) - \tan B}{\tan B \cdot \tan(A+B)} \\ &= \left(\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} - \tan B \right) \div \tan B \cdot \left(\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} \right) \\ &= \frac{\tan A + \tan A \cdot \tan^2 B}{\tan B \cdot (\tan A + \tan B)}. \end{aligned}$$

From the last relation we obtain, by cross-multiplication:

$$(a - h \tan A) \tan^2 B + (a \tan A) \tan B - h \tan A = 0. \quad (3)$$

Here, (3) can be regarded as a quadratic equation in $\tan B$; the coefficients are known quantities, as they have been expressed in terms of a, h, A . Solving the equation, we get:

$$\tan B = \frac{-a \tan A \pm \sqrt{a^2 \tan^2 A + 4(a - h \tan A) \cdot h \tan A}}{2(a - h \tan A)}. \quad (4)$$

We have not attempted to simplify the expression in (4). An equivalent way of expressing the same result, in terms of sines and cosines, is the following:

$$\tan B = \frac{-a \sin A \pm \sqrt{a^2 \sin^2 A + 4(a \cos A - h \sin A) \cdot h \sin A}}{2(a \cos A - h \sin A)}. \quad (5)$$

Note the plus-minus sign. The two values given by the formula correspond to the values of $\tan B$ and $\tan C$ respectively. (There is an obvious symmetry in the problem between B and C .)

If the triangle is right-angled, then we may encounter fractions with zero denominator, so we need to be careful. We look at this possibility below.

The case when $A = 90^\circ$. In this case, $\angle B + \angle C = 90^\circ$, so $\tan B \cdot \tan C = 1$. Therefore (2) assumes the form

$$x = \frac{h}{\tan B}, \quad a - x = \frac{h}{\tan C} = h \tan B, \quad \therefore x(a - x) = h^2. \quad (6)$$

The quadratic equation $x(a - x) = h^2$ may be solved for x , and from this we get $\tan B$:

$$\begin{aligned} x(a - x) &= h^2, \quad \therefore x = \frac{a \pm \sqrt{a^2 - 4h^2}}{2}, \\ \therefore \tan B &= \frac{2h}{a \pm \sqrt{a^2 - 4h^2}}. \end{aligned} \quad (7)$$

Rationalising, we get:

$$\tan B = \frac{a \mp \sqrt{a^2 - 4b^2}}{2b}. \quad (8)$$

As earlier, the two values given by the formula correspond to the values of $\tan B$ and $\tan C$ respectively. (Note that the product of the two values is equal to 1, as it should be.)

The case when a denominator of 0 occurs in (4) and (5). This will happen when $a = b \tan A$ (equivalently, $a \cos A = b \sin A$). This means, clearly, that either $\angle B = 90^\circ$ or $\angle C = 90^\circ$.

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REASONING MADE SIMPLE

Contributor: Wallace Jacob

Some problems in math are interesting simply because of the reasoning used to solve them.

Question. How many three-digit numbers increase by 297 when written in the reverse order?

Solution. Let cba ($100 \times c + 10 \times b + a$) be a three digit number.

When cba is written in reverse order then it would be abc ($100 \times a + 10 \times b + c$).

$$100 \times a + 10 \times b + c - (100 \times c + 10 \times b + a) = 297$$

$$\Rightarrow 99a - 99c = 297 \Rightarrow a - c = 3$$

For $a - c = 3$ to hold true, the possible values of a and c are shown in the table.

a	c
4	1
5	2
6	3
7	4
8	5
9	6

Thus, there are 6 possible combinations for which $a - c = 3$.

b can be any integer in the range of 0 to 9, i.e., b can have 10 values.

Therefore, there are $6 \times 10 = 60$ three-digit numbers which increase by 297 when written in the reverse order.

[Note: Why can't the value of a be 3 and c be 0?

If $a = 3$ and $c = 0$, then one possible three-digit number can be 370.

When written in reverse order it would become $073 = 73$ (i.e., a two-digit number).]

Practice question: How many two-digit numbers decrease by 18 when their digits are reversed?

Factorising Non-monic Quadratic Equations

ANUSHKA TONAPI

In the November 2014 issue of At Right Angles, author Shashidhar Jagadeeshan, in the article "Completing the Square . . . A powerful technique, not a feared enemy!" talked about completing the square, quadratics having the shape of a parabola . . . In this article, student Anushka Tonapi explains a few methods of solving non-monic quadratic equations.

A **polynomial** in a variable x has the form $ax^n + bx^{n-1} + \dots + k$ (for some constants $a, b, \dots, k$ and some positive integer n). A quadratic expression has the form $ax^2 + bx + c$, where a, b, c are constants, with $a \neq 0$. If $a = 1$, the quadratic is **monic**, and if $a \neq 1$, it is **non-monic**. When we equate this to 0, it becomes a **quadratic equation**. Solving a quadratic equation means finding its **roots**. Some widely-used techniques to find the roots of quadratic equations are:

1. The Quadratic Formula
2. Completing the Square
3. Factorisation

There are other interesting methods, each with its advantages and disadvantages:

1. Lyszkowski's / Modified Trial and Error Method
2. Vedic Mathematics Method
3. Slide and Divide Method
4. Monic / Scaling Method
5. Po-Shen Loh's Method

We illustrate these methods for the quadratic expression $ax^2 + bx + c$ (where a, b, c are given integers).

Keywords: Quadratics, Factorisation, Multiple methods of solution

Lyszkowski's / Modified Trial and Error Method

We begin by finding constants y and z such that $yz = ac$ and $y + z = b$. Next, we write the form $\frac{(ax + y)(ax + z)}{a}$. Finally, we take a common factor p and q out from each bracket in the numerator, where $pq = a$, getting $\frac{p(qx + y)q(px + z)}{a}$. The result: $(qx + y)(px + z) = ax^2 + bx + c$.

Example: Factorise $6x^2 - 5x - 4$.

Solution: We find that $(y, z) = (-8, 3)$. We get $\frac{(6x - 8)(6x + 3)}{6}$, and we can factorise this further by taking 2 and 3 out of the numerator: $\frac{2(3x - 4)3(2x + 1)}{6}$, which simplifies to $(3x - 4)(2x + 1)$.

Vedic Mathematics Method

We start by finding numbers p, q that add to b and multiply to ac . The constants p, q must form the proportion $a : p = q : c$. Then, we simplify any one of the ratios by reducing it to lowest terms. This ratio provides the linear coefficient and constant for the first factor. The process is called *Anurupyena Sutra* (अनुरूप्येण सूत्र).

We obtain the linear coefficient of the second factor by dividing a by that of the first factor and get the constant of the second factor by dividing c by the constant in the first factor. The process is known as *Adyamadyenantyamantya* (आद्यमाद्ये नान्त्यमन्त्येन).

The Vedic Mathematics¹ method results in a clever way of indirectly factorising the quadratic by observing the ratios between its coefficients.

Example: Factorise $15x^2 + 26x + 8$.

Solution: We look for two constants that multiply to give $ac = 120$ and add to give $b = 26$; we find that they are 20 and 6. Setting up our ratio, we get $\frac{15}{20} = \frac{6}{8}$. Simplifying, we get $\frac{3}{4} = \frac{6}{8}$.

Anurupyena gives us the first factor: $3x + 4$. Next, we perform *Aadyamaadyenaantyamantya*. Dividing 15 by 3 gives 5, and dividing 8 by 4 gives 2; so we obtain the second factor: $5x + 2$. Therefore, $15x^2 + 26x + 8$ can be factorised as $(3x + 4)(5x + 2)$ using the Vedic Method.

Slide and Divide Method

We 'slide' a to the right and multiply by c to get $x^2 + bx + ac$, a monic quadratic. We factor this in the usual manner — factoring monics is easier than factoring non-monics. Suppose this can be factored as $(x + p)(x + q)$. Next, we divide each of the constants in the factorisation by a :

$$\left(x + \frac{p}{a}\right) \left(x + \frac{q}{a}\right).$$

Multiplying suitably, we get the required answer.

¹ Vedic Mathematics is a way of performing operations based on some aphorisms constructed by Swami Bharati Krishna Tirtha (a saint of Shankaracharya order) in the early 20th century CE. It must not be confused with the mathematics contained in the Vedic texts written thousands of years ago.

Example: Factorise $4x^2 + 17x + 15$.

Solution: We 'slide' 4 to the right and get $x^2 + 17x + 60$. Factoring, we get $(x + 5)(x + 12)$. Now, to compensate for our earlier step, we divide each constant by 4; we get:

$$\left(x + \frac{5}{4}\right) \left(x + \frac{12}{4}\right) = \left(x + \frac{5}{4}\right) (x + 3) = \frac{1}{4} (4x + 5) (x + 3).$$

From this we get the required answer $(4x + 5)(x + 3)$.

Monic/Scaling Method

We start by equating $ax^2 + bx + c$ to a dummy variable y to keep track of all our operations. Next, we multiply both sides by a to make the first term on the right-hand side a perfect square:

$$ay = a^2x^2 + abx + ac = (ax)^2 + b(ax) + ac.$$

Replacing ax by z , we get the monic quadratic

$$ay = z^2 + bz + ac.$$

Suppose that $z^2 + bz + ac$ can be factorised as $(z + p)(z + q)$. We have then:

$$ay = (z + p)(z + q) = (ax + p)(ax + q) = a(mx + n)(rx + s),$$

where $mr = a$, $an = p$, $as = q$. Dividing both sides by a , we get the factorisation

$$y = (mx + n)(rx + s).$$

Example: Factorise $9x^2 + 31x + 12$.

Solution: We set $9x^2 + 31x + 12$ equal to y and multiply both sides by 9:

$$9y = 81x^2 + 31(9x) + 12(9).$$

Replacing $9x$ by z , we rewrite this as $9y = z^2 + 31z + 108$. Factorising this monic quadratic the usual way, we get $(z + 4)(z + 27)$.

We have $9y = (z + 4)(z + 27) = (9x + 4)(9x + 27) = (9x + 4)9(x + 3)$. Dividing by 9, we get $y = (9x + 4)(x + 3)$. Therefore, $9x^2 + 31x + 12$ can be factorised as $(9x + 4)(x + 3)$.

Po-Shen Loh's Method

In 2019, Prof. Po-Shen Loh of Carnegie Mellon University came upon a method of finding the roots of a quadratic equation while thinking of an approach to introduce quadratic equations in his Expil Daily Challenge videos. To start with, he looks at the monic quadratic $x^2 + bx + c$. The method begins by reasoning that if one can find two constants r and s with sum $-b$ and product c , then we obtain the factorisation $x^2 + bx + c = (x - r)(x - s)$.

Now if two constants have sum $-b$, then their average is $-\frac{b}{2}$, so the two numbers must be of the form

$-\frac{b}{2} \pm u$ for some constant u . We must find this constant. Multiplying the two numbers, we get $\left(-\frac{b}{2} + u\right) \left(-\frac{b}{2} - u\right) = \frac{b^2}{4} - u^2$. As this must be equal to c , we must have $\frac{b^2}{4} - c = u^2$. Solving for u

(we obtain two values, each the negative of the other), we obtain the two roots. It doesn't matter whether we use the positive or negative value of u ; we get the same roots either way.

This results in a clean and straightforward process to factorise quadratics – with *no guessing*.

For the non-monic quadratic $ax^2 + bx + c$ where $a \neq 0$, we divide both sides by a , thus forming the monic quadratic $x^2 + \frac{b}{a}x + \frac{c}{a}$, and apply this method.

Example: Factorise $2x^2 - 4x - 5$.

Solution: Dividing by 2, we get $x^2 - 2x - \frac{5}{2}$.

We need two numbers with sum 2 and product $-\frac{5}{2}$. Let them be $1 + u$ and $1 - u$. Their product is $-\frac{5}{2}$.

Therefore, $(1 + u)(1 - u) = -\frac{5}{2}$, which gives $1 - u^2 = -\frac{5}{2}$, and $u^2 = \frac{7}{2}$, or $u = \pm\sqrt{\frac{7}{2}}$. So the two roots are $1 \pm \sqrt{\frac{7}{2}}$.

Conditions in which these methods work and don't work

Of the methods described above, the first four (Lyszkowski's Modified Trial and Error Method; Vedic Mathematics Method; Slide and Divide Method; Monic / Scaling Method) work under the same conditions: they give the factors when the quadratic expression has real roots and can be factored using rational numbers. The fifth method (Po-Shen Loh's Method) can be used under all conditions, without any limitations.

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Ellipses Hidden in Projectile Motion

SUKETU PATNI

Introduction and Problem Statement

We study here the following very interesting problem associated with projectile motion.

Problem. A man standing at the origin of the Cartesian plane throws a projectile with a fixed velocity v at varying angles of projection θ . What figure does the point at which the projectile achieves maximum height (above the x -axis) trace as θ moves from 0 to π ?

In other words, what is the shape formed when we mark the point where the projectile is highest from the ground, for all angles of projection from 0° to 180° ? (Keep in mind that the quantity v is fixed. For the sake of simplicity, we do not consider air resistance.)

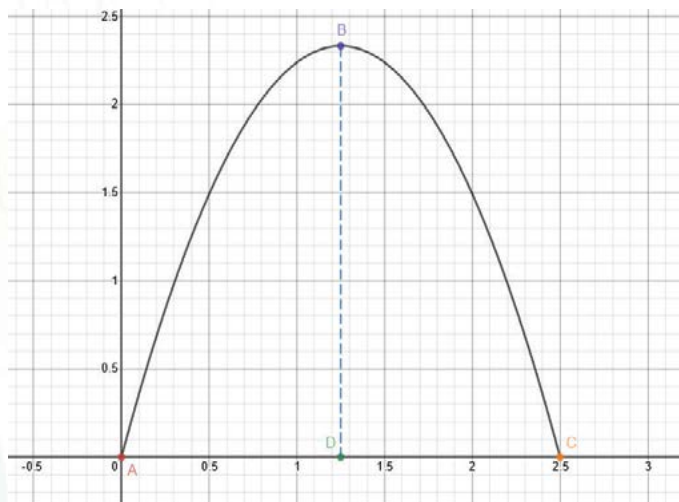


Figure 1. What figure does B trace as the trajectory of the projectile moves with v and g constant?

Keywords: Projectile motion, ellipse, Cartesian plane, parameter

I encountered this problem in an article by my favourite math blogger, Ben Orlin, who has a most humorous blog, *Math with Bad Drawings*. The link to his blog as well as the original tweet are given in the references.

(All graphs in the following article were made using the *Desmos* graphing calculator. The link to the graphs is given in the references.)

Parametrization and Required Formulae

So, how do we solve this problem? We first *parametrize* both the x -coordinate and y -coordinate of the moving point and then *eliminate* the changing variable θ to obtain an implicit equation in x and y which will describe this shape completely.

Parametrization means that we express how a point moves in space by expressing both its ordinate and abscissa in terms of some third variable (in this case θ). This means that the coordinates (x, y) of such a moving point satisfy

$$x = f(\theta), \quad y = g(\theta)$$

for functions f and g simultaneously. For simple examples, consider the following parametrizations.

Example 1.

$$x = 5k + 1, \quad y = 6k + 4.$$

This illustrates the simplest type of parametrization. Solving this is trivial. We get:

$$\frac{x-1}{5} = k, \quad \frac{y-4}{6} = k,$$

hence $6(x-1) - 5(y-4) = 0$, i.e.,

$$6x - 5y + 14 = 0, \tag{1}$$

which is the equation of a straight line, a consequence of the fact that both functions of k here are linear polynomials. More importantly, we note that (1) does not have any terms related to k .

Formally, we may write: $6x - 5y + 14 = 0$ is the locus of all points that move in space such that $x = 5k + 1$ and $y = 6k + 4$ for real values of k .

Example 2.

$$x = \tan \theta, \quad y = \cos \theta.$$

We recall the identity $\sec^2 \theta - \tan^2 \theta = 1$ and thus we get:

$$\frac{1}{y^2} - x^2 = 1.$$

This may be simplified further, but we have already got what we wanted: an equation in x and y , free of θ .

Also, in order to carry out the required parametrization, we require two key formulae, the proofs of which shall not be discussed here but which may be easily derived. For a projectile with velocity of projection v and angle of projection θ , the following hold (where g is the gravitational acceleration):

$$\text{Maximum height achieved by projectile} = \frac{v^2 \sin^2 \theta}{2g} \quad (2)$$

$$\text{Horizontal range of projectile} = \frac{v^2 \sin 2\theta}{g} \quad (3)$$

We are now ready to tackle the problem.

Solving the Problem

Consider the following projectile motion, where we have to trace the point MH (standing for 'Maximum Height') as it moves with a changing angle of projection.

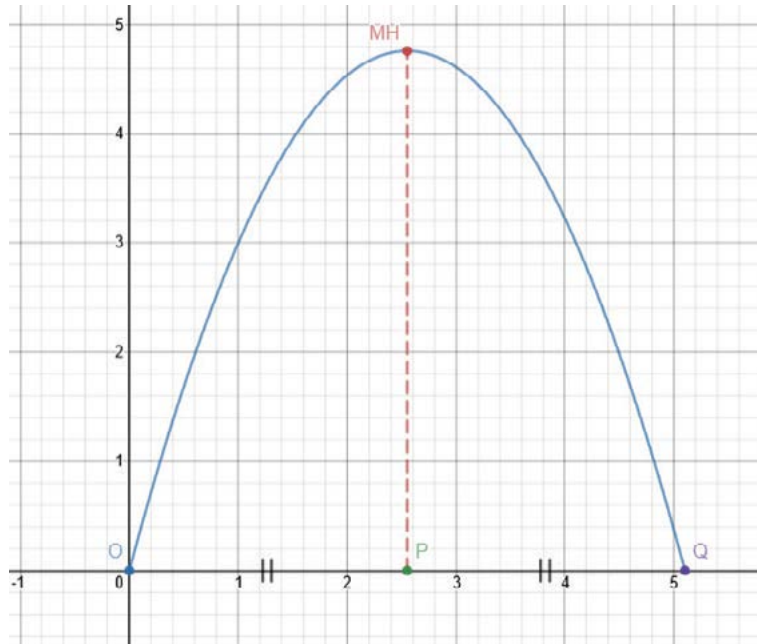


Figure 2. MH marks the Maximum Height

We have:

$$\text{Ordinate of } MH = \text{Maximum height achieved by projectile} = \frac{v^2 \sin^2 \theta}{2g}$$

$$\text{Abscissa of } MH = \frac{1}{2} \times \text{Horizontal range of projectile} = \frac{v^2 \sin 2\theta}{2g}.$$

Therefore, for a given v and θ ,

$$B = (x, y) = \left(\frac{v^2 \sin 2\theta}{2g}, \frac{v^2 \sin^2 \theta}{2g} \right). \quad (4)$$

We now need to eliminate θ from the given parametrization:

$$x = \frac{v^2 \sin 2\theta}{2g}, \quad y = \frac{v^2 \sin^2 \theta}{2g}. \quad (5)$$

Notice that all we must do is find a relation between the trigonometric terms, i.e., the terms involving θ . Then, we can transpose and substitute the terms containing x and y as per our requirements. In this case, the terms are $\sin^2 \theta$ and $\sin 2\theta$. Now we have the relations

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta, \quad \therefore \sin^2 2\theta = 4 \sin^2 \theta \cos^2 \theta, \\ \therefore \sin^2 2\theta &= 4 \sin^2 \theta (1 - \sin^2 \theta). \end{aligned} \quad (6)$$

From (5),

$$\sin^2 2\theta = \left(\frac{2gx}{v^2}\right)^2, \quad \sin^2 \theta = \frac{2gy}{v^2}. \quad (7)$$

Substituting, we get the following simplification. All cancellations are valid as all of the quantities, v , g , and θ , are non-zero.

$$\begin{aligned} \sin^2 2\theta &= 4 \sin^2 \theta (1 - \sin^2 \theta) \\ \implies \left(\frac{2gx}{v^2}\right)^2 &= 4 \left(\frac{2gy}{v^2}\right) \left(1 - \left(\frac{2gy}{v^2}\right)\right) \\ \implies \frac{4g^2 x^2}{v^4} &= \left(\frac{8gy}{v^2}\right) \left(\frac{v^2 - 2gy}{v^2}\right) \\ \implies gx^2 &= (2y)(v^2 - 2gy) \\ \implies gx^2 &= 2yv^2 - 4gy^2. \end{aligned} \quad (8)$$

We can stop at (8) and say that this describes the path that the moving point traces. But we want to find out the shape, so we must manipulate it further.

Notice that as (8) is a second-degree equation in x and y , it represents either a conic section or a pair of straight lines. Let us thus continue:

$$\begin{aligned} 4y^2 - \frac{2yv^2}{g} + x^2 &= 0 \\ \implies y^2 - \frac{yv^2}{2g} + \frac{x^2}{4} &= 0 \\ \implies \frac{x^2}{4} + \left(y - \frac{v^2}{4g}\right)^2 &= \left(\frac{v^2}{4g}\right)^2 \\ \implies \frac{x^2}{4(v^2/4g)^2} + \frac{(y - v^2/4g)^2}{(v^2/4g)^2} &= 1 \\ \implies \frac{x^2}{(v^2/2g)^2} + \frac{(y - v^2/4g)^2}{(v^2/4g)^2} &= 1. \end{aligned} \quad (9)$$

Et voilà, the equation of an ellipse! An ellipse, of all curves, has appeared unexpectedly.

The graph below clearly shows that the required figure is indeed an ellipse, because for all trajectories, MH_i for all $i = 1, 2, 3, 4, 5, 6$ always lies on the ellipse.

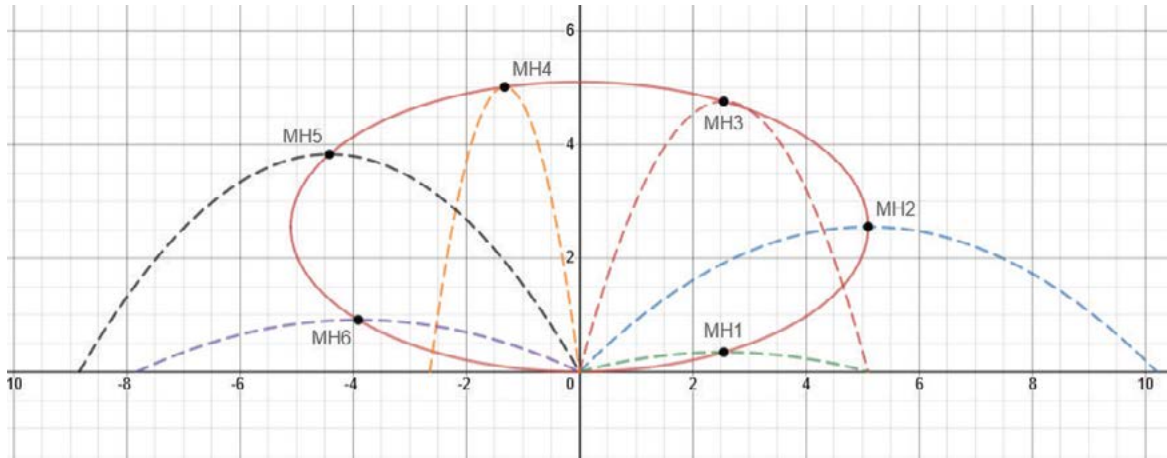


Figure 3. An ellipse!

The coordinates of the centre of the ellipse are

$$\left(0, \frac{v^2}{4g}\right),$$

and the lengths of the semi-major axis and the semi-minor axis are

$$\frac{v^2}{2g}, \quad \frac{v^2}{4g}.$$

Further questions to ponder

This problem isn't just an interesting exercise in itself; it is also a rich mine of further questions.

- (1) What happens if instead of keeping v and g constant and varying θ , we keep g and θ constant and vary v ? What is the figure so formed? What happens if we vary only the gravitational acceleration? (Although the last question quickly departs from reality, the problem is a thought experiment, and hence we need not be limited by reality!)
- (2) Note that we have only considered the 2D-Cartesian plane here. What would happen, if the man was standing at the origin of a 3 dimensional Cartesian space and could throw the projectile in any vertical plane while keeping v and g constant? What 3D-figure would be formed? What would be its implicit equation in x, y and z ?

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3. The original tweet on twitter which asked this question: <https://twitter.com/InertialObserver/status/1153356695474585600>
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5. Figure 3: <https://www.desmos.com/calculator/4jwskh8onn> The values of g, v and θ can be varied and its effects can be seen here.



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An Interesting Conservation Property

Two circles intersect at points D and G. Two parallel lines pass through the points D and G, and intersect the circles at points A, B, M and N (AB parallel to NM). Choose a point F on the arc NG and a point E on the arc GM and connect F with A, and E with B. The lines FA and EB intersect at point C, forming a triangle ABC with angles α , β , γ as can be seen in Figure 1.

Prove that quadrilateral FGEC can be inscribed in a circle.

If D and G coincide, will the same result be true for quadrilateral FDEC? (Figure 2)

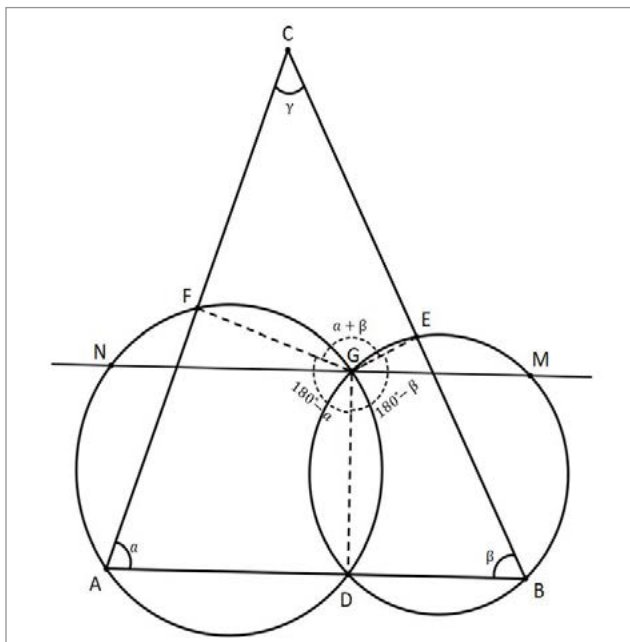


Figure 1

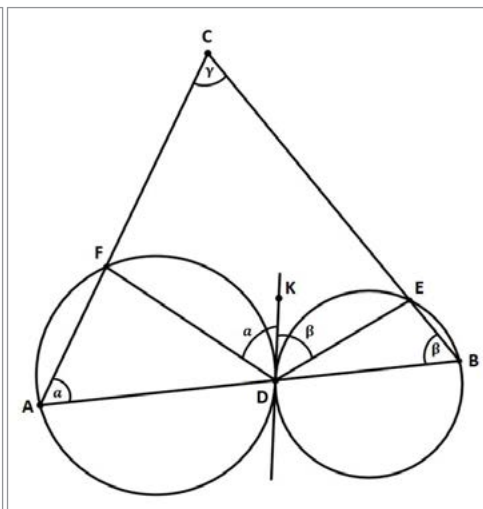


Figure 2

Explore with the GeoGebra sketch available at <https://www.geogebra.org/m/wz3vtfe7>

Prof. Moshe Stupel and Prof. David Ben-Chaim from the SHAANAN College of Teachers Education, Haifa, Israel

Proof is available on page 96

Solution to a 2018 Romanian Mathematics Olympiad Problem

PRANEETHA KALBHAVI

In this article, we solve an inequality problem involving logarithms.

Problem (Romania MO 2018). Let a, b, c be real numbers such that $1 < b \leq c^2 \leq a^{10}$ and

$$\log_a b + 2 \log_b c + 5 \log_c a = 12.$$

Show that

$$2 \log_a c + 5 \log_c b + 10 \log_b a \geq 21.$$

Solution. Let $\log_c b = k$, $\log_a c = l$, $\log_a b = m$. From $1 < b \leq c^2 \leq a^{10}$, we have

$$\log_c b \leq 2, \quad \log_a b \leq 10, \quad \log_a c \leq 5, \quad (1)$$

i.e., $0 < k \leq 2$, $0 < m \leq 10$, $0 < l \leq 5$. The hypothesis is now transformed to

$$m + \frac{2}{k} + \frac{5}{l} = 12. \quad (2)$$

Noting that $kl = m$, (2) gets simplified to

$$2l + 5k = m(12 - m). \quad (3)$$

The goal now becomes: show that

$$2l + 5k + \frac{10}{m} \geq 21. \quad (4)$$

Keywords: Logarithms, Romanian Mathematics Olympiad

In view of (3), the goal (4) further simplifies to:

$$m(12 - m) + \frac{10}{m} \geq 21,$$

which in turn is equivalent to

$$m^3 - 12m^2 + 21m - 10 \leq 0. \quad (5)$$

To see why (5) is true, we observe that $m - 1$ is a factor of the polynomial on the left side, for $1 - 12 + 21 - 10 = 0$. After performing the relevant division, we get

$$m^3 - 12m^2 + 21m - 10 = (m - 1)(m^2 - 11m + 10),$$

and we see that $m - 1$ is a factor of $m^2 - 11m + 10$ as well, for $1 - 11 + 10 = 0$. We have:

$$m^2 - 11m + 10 = (m - 1)(m - 10).$$

Hence

$$\begin{aligned} m^3 - 12m^2 + 21m - 10 &= (m - 10)(m - 1)^2 \\ &\leq 0, \quad \text{since } 0 < m \leq 10. \end{aligned}$$

The proof is now complete. □

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Counting some more Triangles

SUNDARRAMAN M

In the November 2016 issue of *At Right Angles*, I published an article, *Counting Triangles*, in which I discussed the following problem: *Count the number of triangles formed in a triangle if n segments are drawn from one vertex to its opposite side, and h segments are drawn from another vertex to its opposite side.*

This was a special case of a more general problem (stated below) which I left as an open question. Recently, I revisited this problem to see if I could show something interesting. In this article I provide a solution to the general problem and also show how the constraint can be done away with, i.e., show how to count the number of triangles even if there are concurrent segments inside the triangle.

The only prerequisite for reading this article is the *inclusion-exclusion principle* for three sets A, B, C :

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|,$$

where $|X|$ denotes the cardinality of a set X .

The problem may be formally stated as follows:

Problem. Count the number of triangles formed when n, h, k line segments are drawn from the vertices A, B, C of a given triangle ABC respectively to the opposite sides, assuming that no three of these $n + h + k$ line segments concur. (See Figure 1 for an example when $n = 1, h = 2, k = 3$.)

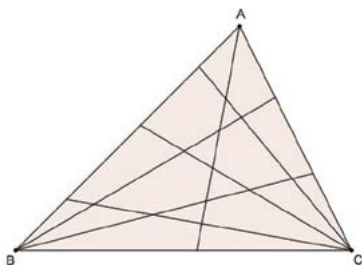


Figure 1.

Keywords: inclusion-exclusion principle

Now any triangle in Figure 1 must belong to one of the following two (disjoint) categories. 1) The triangle shares at least one vertex with $\triangle ABC$. 2) The triangle lies strictly in the interior of $\triangle ABC$.

Thus, the total number of triangles is equal to the number of triangles with at least one vertex in common with $\triangle ABC$, plus the number of triangles strictly in the interior of $\triangle ABC$.

It is relatively easy to count the number of triangles in the second category. For, each edge of such a triangle has to be one of the $n + k + h$ edges drawn from the vertices. Moreover, the triangle will have exactly one edge from among the n edges drawn from A , exactly one edge from among the h edges drawn from B and exactly one edge from among the k edges drawn from C . This is so because if there are two edges emanating from the same vertex, then that vertex will be a vertex of the triangle in question; and that is exactly what we wish to avoid. Thus the number of such triangles, if there are no concurrent edges, is nkh .

(Remark: In my earlier article, $h = 0$, which is why we didn't have this concept of internal triangles.)

So, the only thing left to calculate is the number of triangles with at least one vertex in common with $\triangle ABC$. This is where we use the inclusion-exclusion principle. Let the set of triangles containing vertex A be denoted by A , the set of triangles containing vertex B be denoted by B , and the set of triangles containing vertex C be denoted by C . Then the number we need is $|A \cup B \cup C|$. By the inclusion-exclusion principle, we know

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

(Remark: Note that any triangle in this whole case is not dependent at all on the number of concurrent edges and so will remain the same in the slightly general case as well. We'll throw some light on this when we discuss the concurrent edges case.)

Consider first $|A|$. If you look at Figure 1, you will see that the 2 edges originating from A will have to be among the $n + 2$ edges originating from A . Once you have chosen the 2 edges, the third edge can be any edge not originating at A . This is because any such edge will intersect these 2 lines at unique points other than A and thus will form a unique triangle. Thus, out of the $n + k + h + 3$ edges in the figure, $k + h + 1$ edges will each give a unique triangle (after removing the $n + 2$ edges originating in A). Hence,

$$|A| = \binom{n+2}{2} \cdot (h + k + 1).$$

The cardinalities $|B|$ and $|C|$ can be written using symmetry as follows:

$$|B| = \binom{h+2}{2} \cdot (n + k + 1),$$

$$|C| = \binom{k+2}{2} \cdot (h + n + 1).$$

Next, consider $|A \cap B|$. By definition, it is the number of triangles containing both A and B as vertices. This implies that edge AB will be part of the triangle. If you look at Figure 1, you will notice that every triangle with AB as an edge must have the remaining two edges originating from A and B respectively. Moreover, if you randomly select an edge originating at A (other than AB) and an edge originating at B (other than AB), you get a unique triangle. Therefore

$$|A \cap B| = (n + 1) \cdot (h + 1).$$

We have taken $n + 1$ and $h + 1$ and not $n + 2$ and $h + 2$ as we have excluded AB from the selection. The other 2 cardinalities $|B \cap C|$ and $|C \cap A|$ can be written using symmetry as follows:

$$|B \cap C| = (k + 1) \cdot (h + 1),$$

$$|C \cap A| = (n + 1) \cdot (k + 1).$$

Finally, $|A \cap B \cap C| = 1$. (Why?)

Therefore the number of triangles in the figure is

$$\begin{aligned} & \binom{n+2}{2} \cdot (h+k+1) + \binom{h+2}{2} \cdot (n+k+1) + \binom{k+2}{2} \cdot (h+n+1) \\ & - (n+1) \cdot (h+1) - (k+1) \cdot (h+1) - (n+1) \cdot (k+1) + 1 + nhk. \end{aligned}$$

So, this is a pretty clean formula. After simplification and some clever grouping, we get the following expression:

$$\begin{aligned} & \frac{n^2(h+k+1)}{2} + \frac{h^2(n+k+1)}{2} + \frac{k^2(h+n+1)}{2} \\ & + 2(nh + hk + kn) + \frac{3(n+h+k)}{2} + 1 + nhk. \end{aligned}$$

Solution to a more generalized problem. What happens when there are concurrent edges in the interior of the triangle? Note that it is not possible for more than three edges to concur at a point. Note also that each point of concurrence is like a triangle shrunk to a point, therefore every set of concurrent edges reduces the number of triangles by exactly 1.

From these observations, it follows that the number of triangles formed when n, h, k line segments are drawn from the vertices A, B, C respectively to the opposite sides and there are t sets of concurrent segments, is equal to

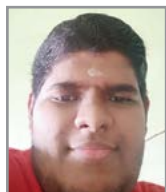
$$\begin{aligned} & \frac{n^2(h+k+1)}{2} + \frac{h^2(n+k+1)}{2} + \frac{k^2(h+n+1)}{2} \\ & + 2(nh + hk + kn) + \frac{3(n+h+k)}{2} + 1 + nhk - t. \end{aligned}$$

We close this article by asking the following question.

Problem. Count the number of triangles formed in a quadrilateral with $n_1, n_2, h_1, h_2, k_1, k_2, y_1, y_2$ edges from each of its vertices to points on the two non-adjacent sides respectively.

References

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The Educational Use of Tables on Advanced Scientific Calculators

BARRY KISSANE

Although advanced scientific calculators of recent years include many features intended for educational use, they are still frequently misunderstood as mostly devices for numerical computation. Following a model for the educational use of calculators, this article describes and illustrates several ways in which the use of a calculator facility to construct tables of values can have educational value in the secondary school. Examples include the study of linear and quadratic functions, algebraic equivalence, equations, sequences, series, limits, convergence, differentiation and integration. The article concludes by observing that the educational value of calculators derives from the experiences they offer students, not merely from their capacity to generate numerical answers.

Introduction

Hand-held calculators have been around for almost fifty years now. It was no surprise that they quickly replaced older technologies for computation when they first appeared in the 1970s, as they were much easier to use and much more accurate than alternatives. However, it is a surprise that the calculators continue to be described as *scientific* calculators, when their many significant developments and improvements since then have been concerned almost entirely with *education* and not with science. The most obvious improvements to scientific calculators over the last fifty years include increased functionality (what a calculator can do), increased usability (how easy it is to do it) and the use of natural displays (so that calculator screens increasingly show mathematics as it is represented elsewhere).

There have been significant expansions of scientific calculators to include a graphics screen and even to include computer algebra capabilities, but we restrict attention in this paper to devices that are still described as scientific calculators. In particular, we focus attention on one of these developments, the capacity to display tables of numerical values, and elaborate on some of the ways in which this capability is of value to school students of mathematics.

Keywords: calculators, pedagogy, functions, tables, variables

Calculators and education

Naïve thinking about calculators still regards them solely as devices for arithmetical calculation. In contrast, Kissane & Kemp (2014) elaborated a model for the educational role of calculators, highlighting three other aspects of calculators that are also important. Computation is helpful for students, especially when the numbers involved are awkward for mental or by-hand calculation, such as when real measurements are involved or students are dealing with genuine statistical data, rather than fabricated data of the kinds that often appear in textbooks. The other three aspects highlighted by the model are Representation, Exploration and Affirmation. Calculators allow mathematical concepts to be represented in various ways, helpful to learners; a good example is the use of a table of ordered pairs to represent a function, in addition to symbolic representations. Exploration of mathematical relationships and ideas has long been recognised as a powerful means of active learning and is at the heart of many student-centred approaches to learning. As this paper illustrates, tables offer many opportunities of this kind. Finally, calculators can be used thoughtfully by students engaged in affirmation of their own thinking, which also involves the possibility of contradiction of their own thinking; the responsiveness of calculators allows tables to be used to permit students to test out their own thinking and to engage actively in their own learning.

Advanced scientific calculators

Calculators are routinely improved by manufacturers, with advice from teachers and mathematics educators, so that modern versions are at least a fifth generation of the species. In recent years, some manufacturers have developed *advanced* scientific calculators with suites of mathematical capabilities focused on senior secondary school or the early undergraduate years. These have been mostly developed for environments in which graphics calculators are prohibited for use in formal examination systems. Student use of such calculators might

be preferred over more sophisticated forms of technology in some settings: they are much less expensive than technology requiring individual computer access; they are more acceptable to some examination boards than are computers; they are sufficiently portable to be easily taken by students between classroom, home and examination room.

Many advanced scientific calculator models (and some less advanced models) from different manufacturers now include a capacity for users to generate tables of values of functions, which is the main focus of this paper, and to which attention now turns.

Using tables

It is now almost twenty years since the first scientific calculators included a facility to construct tables of values, originally described by the manufacturer (CASIO) as ‘number tables’. Mechanisms for defining tables differ between calculator manufacturers, with some tables defined with a finite domain, while others are defined with a ‘scrolling’ infinite table. Regardless of this, typically, the small screen size of calculators constrains tables to showing only a few terms at any time, so that scrolling to see further rows of a table is generally necessary. The essential user input to generate tables is a definition of a function and a mechanism to determine which values will be tabulated.

Not all calculators are the same, as noted above, so for convenience in this paper we will use a particular calculator, the CASIO fx-991 EX *ClassWiz*, as a device to illustrate the key ideas. While relatively modern, this advanced scientific calculator has now been in regular educational use in many countries for at least seven years and is described in some detail in Kissane (2015). Although the first scientific calculators that included a capacity to generate tables were restricted to a single table, while graphics calculators generally permit three or more tables, *ClassWiz* allows users to generate two tables simultaneously. It should be noted that some scientific calculators intended for younger

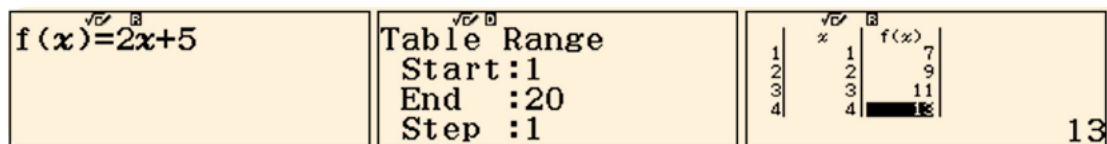


Figure 1: Defining a table on *ClassWiz*

students also include a capability to generate tables, so that some parts of this paper are also relevant to the earlier years of secondary school. In the remainder of this paper, brief glimpses of the potential educational benefits of a table capability are described and illustrated.

Understanding linear functions

Perhaps the most obvious use of tabulation is to explore the nature of functions. Essentially, a table of values for a function allows a user to generate a set of ordered pairs for the function, according to a defining rule. On *ClassWiz*, the process is quite direct, as shown in Figure 1, with a linear function (of x) defined on a finite domain generating a table of values instantly.

Early discussions about the use of computers and graphics calculators to represent and study functions frequently referred to a “Rule of Three”, referring to symbolic, numerical and graphical representations. The representation of a function in this way on the calculator helps users to regard it differently from the symbolic representation, and of course provides an efficient way for users to construct a suitable graphical representation, avoiding the need for extensive by-hand computation.

The educational value of the table extends beyond its efficiency for drawing a graph, however. In this case, the numerical values displayed allow users to see for themselves that the function values $f(x)$ increase by 2 when the value of the independent variable, x , increases by 1... which is the essential idea of the *slope* of the function, as well as the key idea that the increase is steady, and does not vary with changing x .

While the screen limitation restricts *ClassWiz* to displaying only four ordered pairs, this affords users an opportunity to test their understanding

of the function, by predicting the next term, in this case the value associated with $x = 5$ before using the cursor key to scroll down. In a similar way, *ClassWiz* allows users to change an x -value in a table directly (by entering a new value directly) in order to see its associated $f(x)$ value. Both of these operations are shown in Figure 2.

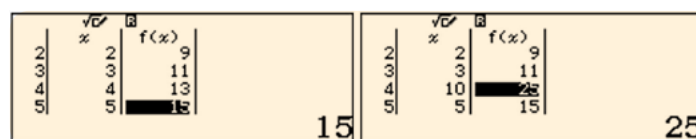


Figure 2: Exploring a table

It seems likely that most users would predict the next term of the table in the first screen, by continuing the process of increasing the previous value by 2 to predict that $f(5) = 13 + 2 = 15$. It also seems likely that users might predict $f(10) = 2 \times 10 + 5 = 25$. Both of these illustrate the idea of affirmation referred to in the model and make clear that the calculator allows students to manipulate what they see and to learn from it. Of course, the environment allows students to easily explore other linear functions, such as $f(x) = 2x + 6$ or $f(x) = 3x + 5$ in order to see for themselves how these differ from each other.

While such explorations are possible when a calculator has a tabulation facility, they are enhanced when a calculator, such as *ClassWiz*, allows for a pair of simultaneous tables to be generated, as illustrated in Figure 3, with $f(x) = 2x + 5$ and $g(x) = 2x + 6$.

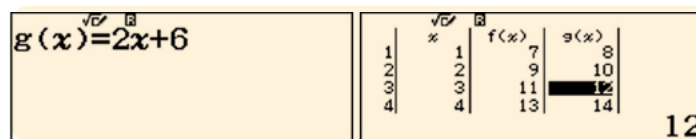


Figure 3: A pair of tables

Users can see for themselves in Figure 3 that the function values for g are always one more than

those for f , so that $g(x) = f(x) + 1$. They can also see that each function behaves in a similar way, which would be reflected in identical slopes of 2 when they are graphed. These observations can be readily tested – and affirmed – as before. Such an environment offers powerful and readily accessible learning insights for students on the nature of linear functions and the significance of their parameters.

Quadratic functions

Opportunities provided by a calculator for linear functions are of course also available for other species of functions typically studied in the secondary school years. Essentially all school curricula proceed from linear functions to quadratic functions, before examining other kinds of functions. The contrast between linear and quadratic functions is valuable for students, as these are used to model quite different kinds of phenomena and are represented by quite different kinds of graphs. To illustrate, the screens in Figure 4 show a table of values for $f(x) = x^2 + 3x + 2$ for $1 \leq x \leq 4$.

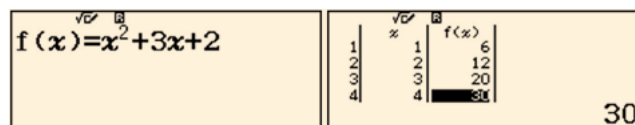


Figure 4: Tabulating a quadratic function

The table makes it clear that, unlike linear functions, the values of f do not change in the same way as x increases, helping users to understand why the associated graph would not be linear. The values do, however, show a consistent pattern, with the differences between successive function values changing in a systematic way. Users can see the pattern of increases, +6, +8 and +10 in the few table values shown in Figure 4. They might then predict that later increases will change in the same way, expecting +12, +14, +16, etc. To affirm that these predictions are correct involves merely scrolling down the next few terms, as shown in Figure 5. Collectively, these data give a strong suggestion that the function is increasing in an increasing way, helping to anticipate the shape of the graph.

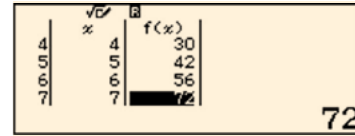


Figure 5: Extending a table to affirm predictions

Further evidence of the quadratic behaviour, and its differences from linear behaviour, can be found by students exploring different values of the independent variable, x , as shown below, with $-6 \leq x \leq -3$. Here the table shows that successive values are decreasing, instead of increasing, although still changing in what seems to be a predictable way, with changes of -8 , -6 and -4 shown in the first screen in Figure 6.



Figure 6: Further explorations of a quadratic function

When further tabulated values are revealed (by scrolling), as shown in the second screen shot in Figure 6, further phenomena are revealed, at first unlikely to be expected by naïve users. In this case, two function values are the same, $f(-2) = f(-1) = 0$. Furthermore, successive function values stop decreasing and start increasing and there is a clear symmetry of values shown.

These kinds of explorations provide students with a chance to understand some features of this particular quadratic function, and of its graphical representation, such as its general shape as well as whether and where it crosses the x -axis. Users can easily conduct similar explorations with different quadratic functions to gain even further insights into this family of functions. Explorations of these kinds are much more difficult if a tables facility is not available, requiring extensive by-hand computations.

Trigonometric functions

As for algebraic functions, the availability of tables allows a calculator to be used to get insight into the values of trigonometric functions. Although some insight is available via a conventional book of tables, a calculator allows

for a larger range of values to be explored. An example is shown in Figure 7, where the sine function has been tabulated for $0^\circ \leq x \leq 435^\circ$. The *ClassWiz* is constrained to a table of no more than 30 entries, so a suitable choice of increment is shown.

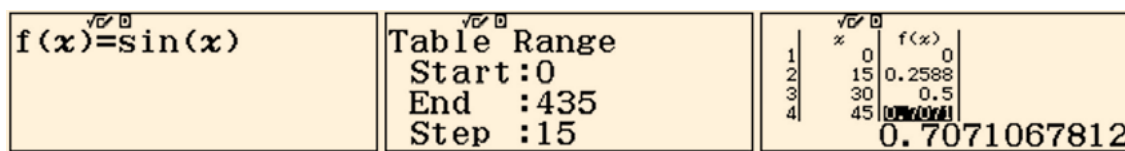


Figure 7: Tabulating the sine function

Scrolling a table like this allows a user to see that sine values increase from 0 at 0° to a maximum of 1 at 90° , and then decrease in the same way to 0 again at 180° . Sine values in the third and fourth quadrants are readily seen to be the opposite of those in the first two quadrants – and are always negative. The periodic nature of the function can be glimpsed by examining further values for $x > 360$. Of course, a table like this is not only helpful for visualising the sine function, but also for sketching it on paper. Finer-grained increments can be used if desired to obtain more precision.

Using a pair of tables of trigonometric functions offers users a chance to see some relationships between the functions, as the tables in Figure 8 of both the sine and cosine functions for $0^\circ \leq x \leq 180^\circ$ illustrate.

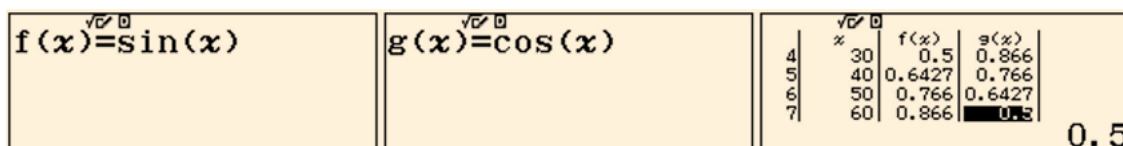


Figure 8: Tabulating the sine and cosine functions together

Exploring tables like these allow users to see that values for sines and cosines repeat; closer analysis will suggest further detailed relationships such as $\sin x = \cos(90^\circ - x)$, worthy of exploration in other ways.

Equivalence

As well as representing and exploring functions of various kinds, calculator tables offer an opportunity for the key notion of equivalence to be represented and understood. Researchers have long recognised the limitations of students regarding algebra as essentially a process of operating on symbols, or letters, but with little sense of the meaning of the operations used. However, a numerical meaning of equivalence can be provided by evaluating algebraic expressions at many points. An example is shown in Figure 9, in which two functions are tabulated.

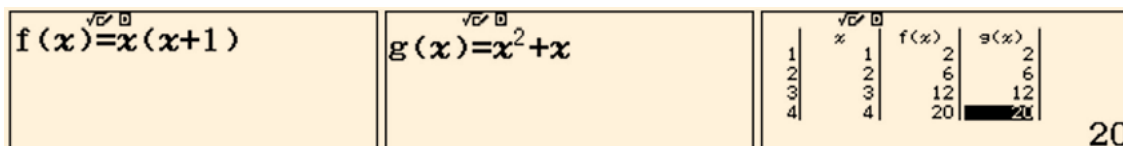


Figure 9: Tabulating a pair of equivalent functions

Inspecting the tabulated values, and scrolling to see values for other points, allows users to see that they are always the same for the two functions. This provides a powerful stimulus to understanding the meaning of an identity such as $x(x+1) = x^2 + x$; regardless of the value of x , the two functions have the same value, helping students to see that processes such as ‘expanding’ and ‘factorising’ algebraic expressions are merely ways to represent them differently. The screens in Figure 10 show a similar process for the quadratic function described in the previous section:

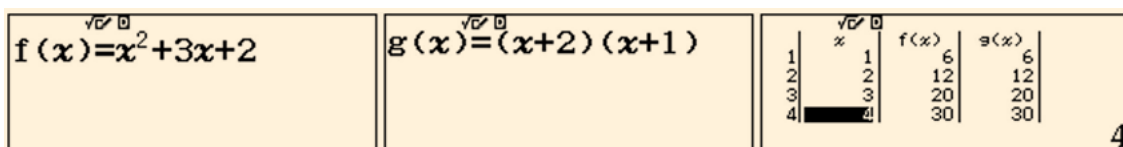


Figure 10: Exploring factorisation through tables

In a comparable way, users can explore for themselves some algebraic operations, to have their processes confirmed, or, as in the case shown in Figure 11, contradicted.

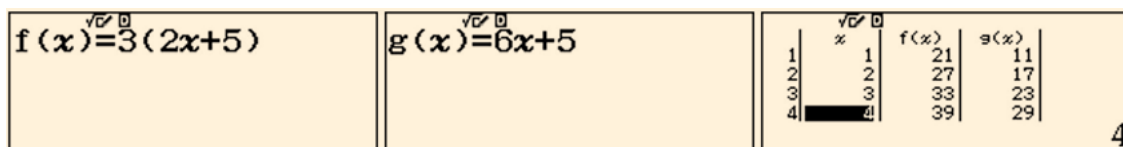


Figure 11: Checking equivalence through tables

As any teacher will attest, many students have difficulty with expressions like $3(2x + 5)$, because of conceptual errors, suggesting that an equivalent expression is $6x + 5$. Clearly, the calculator offers significant evidence that this is not the case. Using tables on calculators to explore these – and many other – examples offers another way of thinking about the matter. Of course, there is no suggestion here that students do not need to learn the symbolic manipulative processes eventually; rather, it is suggested that the calculator will help them to make sense of what the equivalence processes actually mean.

The same idea might be used to consider trigonometric identities too. The example in Figure 12 shows how students can easily see that, for almost all values of x , $\sin 2x \neq 2\sin x$.

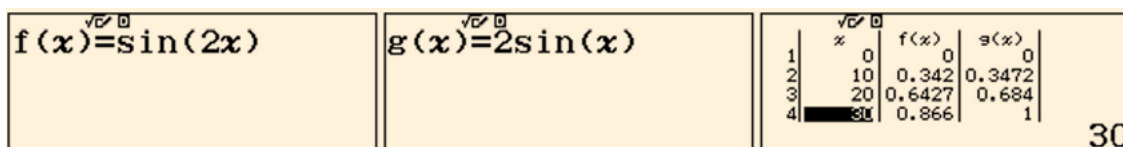


Figure 12: Exploring a trigonometric relationship

Identities can be affirmed as well, of course. The example in Figure 13 illustrates the Pythagorean relationship that $\sin^2 x + \cos^2 x = 1$. (Note that the conventional – albeit idiosyncratic – syntax of $\sin^2 x$ does not represent $\sin(\sin(x))$, but rather means $(\sin x)^2$. On *ClassWiz*, the corresponding Pythagorean expression is written in calculator syntax as $\sin(x)^2 + \cos(x)^2$, shown over the first two screens in Figure 13.)

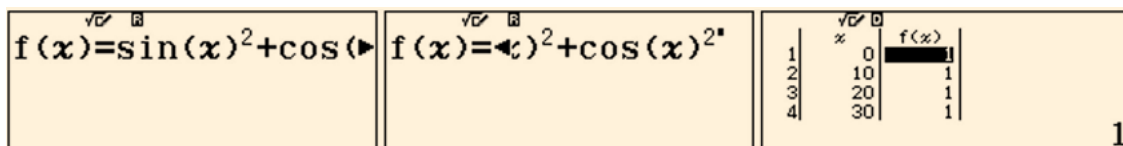


Figure 13: Using tables to represent the Pythagorean identity

Again, the calculator is supporting learning about these relationships; in other parts of student learning experience, alternative approaches will be used as well, including formal proofs. The fundamental role of the calculator is to help the user make sense of the mathematical ideas and relationships involved.

Solving equations

Solving equations is a key component of any secondary school mathematics curriculum, but attention is rarely focused on the very limited range of equations that is generally included. Table capabilities on calculators add a further dimension to equation solving, allowing numerical approximations to essentially any elementary equation that makes sense to students.

Consider for example the equation $x^3 = 3 - x^2$, which is generally beyond the capabilities of typical secondary school students to solve. Solutions, if any, to the equation will be the (irrational) roots of the function $f(x) = x^3 + x^2 - 3$. As shown in Figure 14, scrolling a table of values for this function on a wide interval, $-10 \leq x \leq 10$, suggests that there is a root between $x = 1$ and $x = 2$, as the function changes sign in this interval:

$f(x)=x^3+x^2-3$	Table Range Start:-10 End :10 Step :1	<table><tr><td>x</td><td>$f(x)$</td></tr><tr><td>0</td><td>-3</td></tr><tr><td>1</td><td>-1</td></tr><tr><td>2</td><td>9</td></tr><tr><td>3</td><td>33</td></tr></table>	x	$f(x)$	0	-3	1	-1	2	9	3	33
x	$f(x)$											
0	-3											
1	-1											
2	9											
3	33											

Figure 14: Using a table to find approximate roots of a function

To obtain a closer approximation, users can change the table interval to $1 \leq x \leq 2$, with a smaller increment of 0.1 and redraw the table, as shown in Figure 15.

Table Range Start: 1 End : 2 Step : 0.1	<table> <tr><td>x</td><td>$f(x)$</td></tr> <tr><td>1</td><td>-1</td></tr> <tr><td>1.1</td><td>-0.459</td></tr> <tr><td>1.2</td><td>0.168</td></tr> <tr><td>1.3</td><td>0.887</td></tr> </table>	x	$f(x)$	1	-1	1.1	-0.459	1.2	0.168	1.3	0.887
x	$f(x)$										
1	-1										
1.1	-0.459										
1.2	0.168										
1.3	0.887										

Figure 15: Refining a table to improve precision

Now it is clear that a solution (root) lies between $x = 1.1$ and $x = 1.2$. Redrawing the table again with the interval $1.1 \leq x \leq 1.2$ and a smaller increment of 0.01 allows a better approximation of $1.17 < x < 1.18$ to be made. A further refinement is shown in Figure 16, to demonstrate that there is a root in the interval $1.174 < x < 1.175$.

Table Range Start: 1.17 End : 1.18 Step : 0.001	<table> <tr><td>x</td><td>$f(x)$</td></tr> <tr><td>1.173</td><td>-0.01</td></tr> <tr><td>1.174</td><td>-0.003</td></tr> <tr><td>1.175</td><td>2.8118</td></tr> <tr><td>1.176</td><td>9.3118</td></tr> </table>	x	$f(x)$	1.173	-0.01	1.174	-0.003	1.175	2.8118	1.176	9.3118
x	$f(x)$										
1.173	-0.01										
1.174	-0.003										
1.175	2.8118										
1.176	9.3118										

Figure 16: Zooming in to show that a root lies in $1.174 < x < 1.175$

This process of ‘zooming in’ on the table can be repeated efficiently, and can be used to improve the approximation to a solution by one decimal place each time, to whatever accuracy is regarded as appropriate. Using this process, a good approximation to an irrational solution of the equation is found at $x \approx 1.1746$, which is correct to four decimal places.

Although some calculators (including *ClassWiz*) include numerical equation solving software, using a table of values in this way provides the user with control of the approximation process and a solid understanding of the nature of a solution, which are less likely to characterise other solution methods.

Sequences and series

Many curricula include the study of sequences and series, although typically they are restricted to arithmetic and geometric sequences. While students are expected to use formulas of various kinds to answer questions of interest, using tables on calculators provides an alternative – and frequently more insightful – approach. Essentially, if students know how to find a general term of a sequence, as a function of its place in the sequence, then the calculator can be used to generate terms explicitly and answer many questions of interest by inspecting the table.

To illustrate, consider a geometric progression with first term 3 and common ratio 2, so that each term is twice the previous term. Students are sometimes asked to find a particular term (such as the 12th term) or terms with certain properties (such as the first term larger than six lakhs). The n th term of the sequence is $T_n = 3 \times 2^{n-1}$, $n = 1, 2, 3, \dots$. To generate successive terms of this sequence on the calculator, a function of x is needed, as shown in Figure 17.

$f(x)=3 \times 2^{x-1}$	<table><tr><td>x</td><td>$f(x)$</td></tr><tr><td>10</td><td>1536</td></tr><tr><td>11</td><td>3072</td></tr><tr><td>12</td><td>6144</td></tr><tr><td>13</td><td>12288</td></tr></table>	x	$f(x)$	10	1536	11	3072	12	6144	13	12288	<table><tr><td>x</td><td>$f(x)$</td></tr><tr><td>16</td><td>98304</td></tr><tr><td>17</td><td>196608</td></tr><tr><td>18</td><td>393216</td></tr><tr><td>19</td><td>786432</td></tr></table>	x	$f(x)$	16	98304	17	196608	18	393216	19	786432
x	$f(x)$																					
10	1536																					
11	3072																					
12	6144																					
13	12288																					
x	$f(x)$																					
16	98304																					
17	196608																					
18	393216																					
19	786432																					

Figure 17: Tabulating a geometric sequence

As Figure 17 shows, users can scroll the table to see readily that the 12th term is 6144 and the first term larger than six lakhs is $T_{19} = 7,86,432$.

Series are a little more problematic on calculators, as the formulas involved tend to be quite complex. An easier approach on *ClassWiz* is to use a special summation function (Σ) that is available on the keyboard for evaluating series. For practical purposes, however, a table is a useful tool for users to see a series as a sequence of partial sums, as illustrated in Figure 18 for the same sequence with $T_n = 3 \times 2^{n-1}$, $n = 1, 2, 3, \dots$. In the second screen, note that the series is defined as $g(x)$ using the Σ function available on *ClassWiz*.

$f(x) = 3 \times 2^{x-1}$	$g(x) = \sum_{x=1}^x (3 \times 2^{x-1})$	<table> <tr><th>x</th><th>f(x)</th><th>g(x)</th></tr> <tr><td>1</td><td>3</td><td>3</td></tr> <tr><td>2</td><td>6</td><td>9</td></tr> <tr><td>3</td><td>12</td><td>21</td></tr> <tr><td>4</td><td>24</td><td>45</td></tr> </table>	x	f(x)	g(x)	1	3	3	2	6	9	3	12	21	4	24	45
x	f(x)	g(x)															
1	3	3															
2	6	9															
3	12	21															
4	24	45															

Figure 18: Tabulating a sequence and its associated series

As for the case of sequences, successive terms of the series can be read directly from the table as values of the function g . In the third screen shown in Figure 18, note that $g(4) = 3 + 6 + 12 + 24 = 45$, the fourth term of the series.

While algebraic formulas are helpful for resolving many problems, the calculator tables allow students to see several terms of the sequence or series at the same time and hence to potentially see them in better context.

Limits and convergence

The idea of a limit is often regarded as an important one in early studies in school calculus (although it is worth noting that neither Newton nor Leibniz used it explicitly in developing calculus). A key idea is that of getting 'closer and closer' to a value, without ever actually reaching it. On a calculator with table capabilities, successive values can sometimes be represented in a table, to help users see the limiting processes involved.

Consider for example the fundamental trigonometric limit, $\lim_{x \rightarrow 0} \frac{\sin x}{x}$. To study the behaviour of this limit with a table, a suitable function first needs to be defined, after *ClassWiz* has been set to radian measure as shown in Figure 19:

$f(x) = \frac{\sin(x)}{x}$	<table><tr><th colspan="2">Table Range</th></tr><tr><td>Start</td><td>-0.1</td></tr><tr><td>End</td><td>0.1</td></tr><tr><td>Step</td><td>0.01</td></tr></table>	Table Range		Start	-0.1	End	0.1	Step	0.01	<table><tr><th>x</th><th>f(x)</th></tr><tr><td>9</td><td>-0.02</td></tr><tr><td>10</td><td>-0.01</td></tr><tr><td>11</td><td>0</td></tr><tr><td>12</td><td>0.01</td></tr></table>	x	f(x)	9	-0.02	10	-0.01	11	0	12	0.01
Table Range																				
Start	-0.1																			
End	0.1																			
Step	0.01																			
x	f(x)																			
9	-0.02																			
10	-0.01																			
11	0																			
12	0.01																			

Figure 19: Using a table to explore a trigonometric limit

A table of values generated on a small interval close to $x = 0$ gives some insight into the limiting process. The calculator recognises that the expression is undefined at $x = 0$, but the tabulated values make it clear that the values seem to get closer to the limit of 1 as the x -values approach zero from either side. Users scrolling the table of values can see for themselves the 'closer and closer' idea very well, as shown in Figure 20.

Table Range	
Start	-0.01
End	0.01
Step	0.001

x	f(x)
10	-1 × 10 ⁻³ 0.9999
11	0 ERROR
12	1 × 10 ⁻³ 0.9999
13	2 × 10 ⁻³ 0.9999

Figure 20: Examining values 'closer and closer' to 0

Users can readily define smaller and smaller intervals to do so, as shown in Figure 21. (Notice that columns are constrained in size, so that only the highlighted value is shown to full accuracy, so scrolling is necessary.)

Table Range	
Start	-1 × 10 ⁻³
End	1 × 10 ⁻³
Step	0.0001

x	f(x)
9	-2 × 10 ⁻⁴ 0.9999
10	-1 × 10 ⁻⁴ 0.9999
11	0 ERROR
12	1 × 10 ⁻⁴ 0.9999

x	f(x)
10	-1 × 10 ⁻⁴ 0.9999
11	0 ERROR
12	1 × 10 ⁻⁴ 0.9999
13	2 × 10 ⁻⁴ 0.9999

Figure 21: Further refinement to get 'closer and closer' to a limit

Together, these capabilities allow users to appreciate that the important result that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ at least seems to be reasonable.

A similar process can be used to explore limits to infinity, although of course the (finite) calculator can only approximate the concept of (unbounded) infinity. To illustrate this, consider the infinite limit, $\lim_{n \rightarrow \infty} \frac{(5n+7)}{2n-11}$. To represent the idea of a number increasing without bound, a calculator like *ClassWiz* is restricted to choosing larger numbers and inferring what will happen as they increase. While imperfect, this process still offers insights. In this case, the function needs to be defined using x instead of n , which is unlikely to be problematic for students. In the example shown in Figure 22, the range is from five to six lakhs – a very long way short of infinity, but sufficient to appreciate the limiting processes involved:

$f(x) = \frac{5x+7}{2x-11}$	Table Range	
	Start	500000
	End	600000
	Step	10000

x	f(x)
8	570000 2.5
9	580000 2.5
10	590000 2.5
11	600000 2.5

Figure 22: Using a table to explore an infinite limit

Scrolling the table shows that the limit seems to be approaching 2.5 as the values of the independent variable increase. (Notice that the value is actually shown as 2.5, unless it is highlighted to show more precision.) Of course, much larger values can be used – with due care that the interval does not result in a table with too many entries. With such a capability, users can appreciate the reasonableness of the

result that $\lim_{n \rightarrow \infty} \frac{(5n+7)}{2n-11} = \frac{5}{2}$.

A celebrated infinite limit is that used to define the exponential constant, $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$. This can be studied on the calculator in a similar way to other infinite limits, as shown in Figure 23, using a range for the variable of five to six lakhs, as previously.

$f(x) = \left(1 + \frac{1}{x}\right)^x$	Table Range	
	Start	500000
	End	600000
	Step	10000

x	f(x)
8	570000 2.7182
9	580000 2.7182
10	590000 2.7182
11	600000 2.7182

Figure 23: Exploring the exponential function

Although the result shown to four decimal places is close to the known result of e , notice that tabulated values are truncated rather than rounded, so only the first three decimal places are correct. This limit converges very slowly, so this is not a practical means of approximating e via a limit. A better means of doing so involves the use of the infinite series expansion for e :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, -\infty < x < \infty$$

The tables in Figure 24 show how this series can be used with $x = 1$ to explore the limit much more efficiently. The second screen shows the first four terms of the series, while the third screen shows that the first nine places of decimals for e are obtained with remarkably few (twelve) terms of the series:

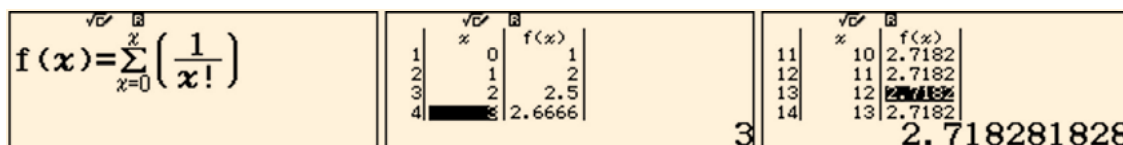


Figure 24: Exploring an infinite series for the exponential function

In these kinds of ways, tables on calculators can be used to provide experiences of limiting processes and of convergence, without the user being required to undertake excessive by-hand computation.

Calculus

Like other advanced scientific calculators, *ClassWiz* provides both a numerical derivative and a definite integral command, both illustrated in Figure 25. (In each case, the first line shows the command entered, while the second line shows the result generated by the command.)

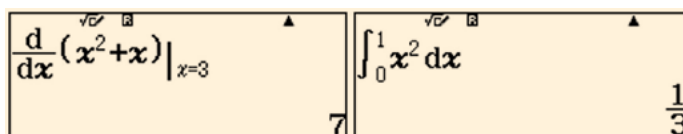


Figure 25: Calculus commands on *ClassWiz*

Once again, the availability of tables on a calculator allows users to understand important features of the mathematical concepts involved. The numerical derivative of a function $f(x)$ at a point $x = a$ can be understood through a limiting process via the First Principles definition, as suggested in the previous section:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

However, a powerful use of the table facility is to evaluate a succession of derivatives at a point and to see the emerging patterns involved. To illustrate with an unsophisticated example, often the first one studied by calculus students, consider the calculator screenshots shown in Figure 26. The first screen displays the expression for the derivative of the x^2 function at a given point (named as $f(x)$). The second screen displays the numerical values of this function.

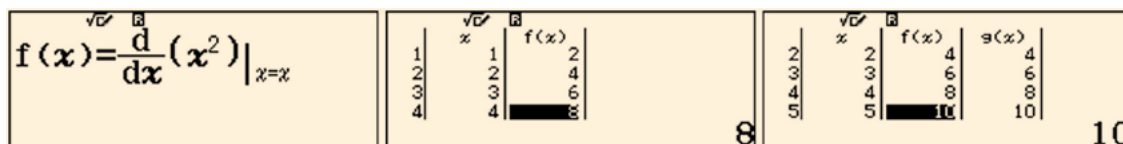


Figure 26: Using tables to represent a derivative function

Students studying the calculus, should have no difficulty in seeing the extraordinary pattern in the numerical derivatives that are shown in the table, with each value twice the value of the independent variable. Scrolling the table will allow them to predict the next values and affirm their understanding of the pattern, as suggested by the third screen above. Students are likely to find this insightful in making the conceptual leap from the idea of a derivative at a point to the more powerful idea of a derivative *function*, in this case $g(x) = 2x$, shown in the third screen in Figure 26.

Tables can be used in a comparable way, albeit limited, to see the general situation associated with integration, rather than the specific situation of an individual definite integral. This is illustrated in Figure 27, again in the case of an unsophisticated example, nonetheless of help to novice students.

x	$f(x)$
1	1
2	4
3	9
4	16

Figure 27: Exploring patterns in integration

In this case, as in the previous one, the pattern of results seems to be very clear, and likely to be recognised by novice students, helping them to get some insight into the general result $\int 2x \, dx = x^2$, a powerful instance of the relationship between differentiation and integration, ultimately reflecting the Fundamental Theorem of Calculus.

These results, of course, need further experiences for sound learning, and it is not suggested that a calculator like *ClassWiz* by itself is sufficient. However, it is suggested that the calculator – through its table facilities – offers opportunities for learning that are easily overlooked, especially if the calculator is regarded as merely a computational device to get numerical answers.

Conclusion

An advanced scientific calculator like *ClassWiz* is better understood as a device to provide students with experience than as a device to provide students with answers. The availability of tables on modern calculators opens up many opportunities for doing this, well reflected in the proposed model and showing how representation, exploration and affirmation are at least as important for learning as is computation. This paper has shown several examples of these relevant to typical secondary school curricula. Tables can be used to represent concepts such as functions, identities, equations, limits, convergence and derivative functions. They allow students to undertake personal explorations related to these concepts, to understand better the mathematics involved. Well used, the table features on calculators allow students to make and test their own conjectures, with opportunities to affirm these or to contradict them, supporting their learning in either case. The examples offered in the paper are illustrative, not exhaustive: students and teachers can find many more examples for themselves using advanced scientific calculators with a capability to generate and productively use tables of values.

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Kissane, B. (2015) *Learning mathematics with ClassWiz*. Tokyo, Japan: CASIO.



BARRY KISSANE is an Emeritus Associate Professor at Murdoch University in Perth, Western Australia. He has worked with teachers and students to make effective use of calculators for school mathematics education in many countries. In a career spanning more than forty years, he has worked as a mathematics teacher and mathematics teacher educator, publishing books and papers concerned with the use of calculators. He has held various offices, such as President of the Australian Association of Mathematics Teachers, editor of *The Australian Mathematics Teacher* and Dean of the School of Education at Murdoch University. He may be contacted at b.kissane@murdoch.edu.au.

Proof of An Interesting Conservation Property

Problem is available on page 79

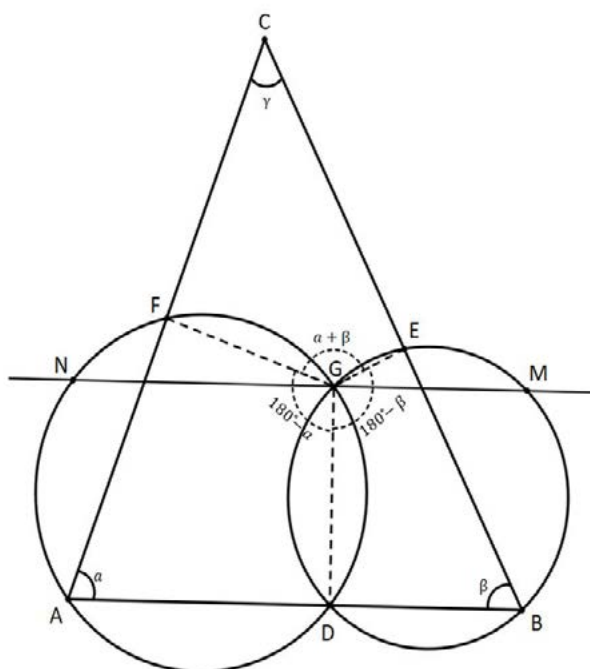


Figure 1

The proof

$$\angle FGD = 180^\circ - \alpha \rightarrow \angle DGE = 180^\circ - \beta$$

$$\angle FGE = \alpha + \beta$$

$$\angle GE + \angle FCE = \alpha + \beta + \gamma = 180^\circ$$

Notes:

The points on the arcs NG and GM always enable the lines AF and BE to intersect and hence to form a triangle.

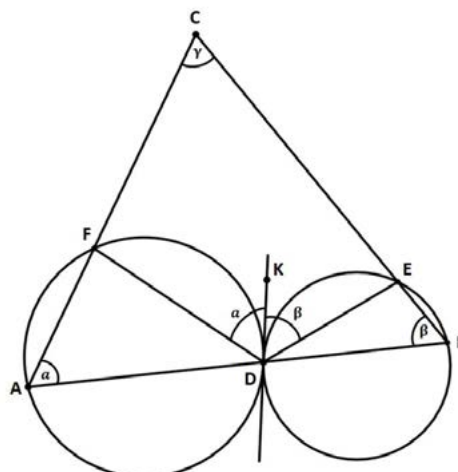


Figure 2

Notes:

If the two circles are tangent at point D (=G), then by applying twice the theorem regarding the angle between a tangent to a circle and the cord through the tangent point, we will receive $\angle FDE = \alpha + \beta$ and hence the quadrilateral FDEC can be inscribed in a circle as can be seen in figure 2.

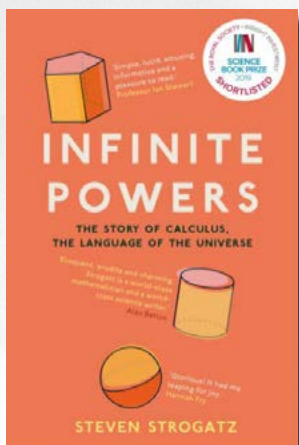
Prof. Moshe Stupel and Prof. David Ben-Chaim from the SHAANAN College of Teachers Education, Haifa, Israel

The Language God Talks?

A Review

Infinite Powers: The Story of Calculus - the Language of the Universe

Reviewed by Shashidhar Jagadeeshan



Author: Steven Strogatz

I. The infinity principle and the language God talks

The book by Steven Strogatz,¹ the highly accomplished applied mathematician, teacher and communicator of mathematics, begins with a bang:

“Without calculus, we wouldn’t have cell phones, computers, or microwave ovens. We wouldn’t have radio. Or television. Or ultrasound for expectant mothers, or GPS for lost travellers. We wouldn’t have split the atom, unravelled the human genome, or put astronauts on the moon. We might not even have had the Declaration of Independence.

It’s a curiosity of history that the world was changed forever by an arcane branch of mathematics.”

The unusual lens that Strogatz uses to view calculus is

“**The Infinity Principle:** To shed light on any continuous shape, object, motion, process or phenomenon -no matter how wild and complicated it may appear -reimagine it as an infinite series of simpler parts, analyse those, and then add the results back together to make sense of the original whole.”

A classic illustration of this principle is finding the volume of a sphere. The sphere is sandwiched between inscribed and circumscribed cylindrical slices. The radius of these discs vary, but their thickness is identical, and can be made smaller and smaller. One can easily find the volumes of these cylinders and by summing up an infinite series of volumes and taking limits, one derives the formula for the volume of a sphere.

¹ Regular readers of At Right Angles will be familiar with Prof Steven Strogatz who was introduced in the November 2018 issue (Volume 7 | Issue 3).

This allows him to take what he calls a ‘big tent’ view with which to question the conventional narrative of calculus. The narrative holds that calculus began by great insights from Newton and Leibniz, created a ‘gold-rush’ of results, but lacked proper rigour, which led to serious questions about the foundations of calculus. Mathematicians in the nineteenth century ‘expunged infinity and infinitesimals’ to clean up calculus.

This view is too ‘blinker’ for Strogatz, who prefers the lens of the Infinity Principle, according to which calculus has its origins way back in the history of mathematics. This allows him to include many descendants and spin-offs of calculus, and to feel that calculus is very much alive today with a vibrant future.

Along with re-examining the very idea of what calculus is, the book has two other aims, both illustrated in a story. In a conversation Richard Feynman asked the novelist Herman Wouk if he knew calculus, and when Wouk said he did not, Feynman told him *“You had better learn it. It’s the language God talks”*.

Strogatz wants to convince the reader of the truth of this statement (page viii of the introduction):

“By inadvertently discovering this strange language, first in a corner of geometry and later in the code of the universe, then by learning to speak it fluently and decipher its idioms and nuances, and finally by harnessing its forecasting powers, humans have used calculus to remake the world.”

That is the central argument of this book.

His other aim is slightly more modest! Apparently when Wouk finally got around to learning calculus, he found no suitable material. Given that calculus is considered one of the greatest intellectual achievements of mankind, Strogatz has written this book for a person from a non-mathematical background to learn all about it, without having to go through the grind of doing calculus.

II. The story of calculus and the calculus debate

There is no doubt that Strogatz is a master storyteller. Using the Infinity Principle as his point of view, he builds up a riveting history of calculus over eleven chapters.

Through the understanding of the work of an array of scientists from Archimedes to Zeno, we learn how the Infinity Principle has been working its way through mathematics, reaching its high point in The Fundamental Theorem of Calculus at the hands of Newton and Leibniz, after which it explodes and enters almost all areas of mathematics both pure and applied. Along the way we learn about Galileo’s work, Cartesian geometry, Descartes’s and Fermat’s idea of the slope of a curve, and the laws of Kepler and Newton.

Strogatz spends tremendous energy educating the reader in a variety of applications of calculus, ranging from how it is used in animation movies, GPS, PET scan and predicting the trajectory of a moving space craft. The chapter on HIV treatment is a personal favourite of mine. He also explains how calculus has played a role in modern physics - for example in the work of Maxwell, Einstein, Dirac and Feynman.

He has also taken great pains to include the work of many scientists, not all of whom are European men! Thus we learn about Al-Hassan Ibn al-Haytham, Katherine Johnson, Sophie Germain and Sophia Kovalevskaya, among many others.

This non-gendered, non-Eurocentric approach is refreshing. Yet I still have a bone to pick with Strogatz. It is to do with the calculus debate (see [2], pages 348-357) and will need some background material to explain.

As early as 1832, through the work of a Britisher, Charles Whish, and later on from the 1940’s, modern historians of mathematics have been aware of the work of the so-called Nila school of mathematics and astronomy from Kerala (see [1],[2],[3]). The founder of the school is believed to be Madhava (c. 1360-c.1430) of Sangamagrama, and the school was active from the

fourteenth to sixteenth century of the common era. The Nila school had discovered the power series expansion for trigonometric functions, the surface area and volume of the sphere, basic ideas of differentials and integrals, the discrete fundamental theorem of calculus, integration by parts, multiple integrals and much more!

Several scholars (such as Diwakaran and Mumford) strongly feel that the birth and the early steps of calculus took place on the banks of the river Nila and its inventor was Madhava, who too “stood on the shoulders of giants” like Aryabhata and Bhaskara.

On the other hand, historian Victor J Katz in a paper entitled *Ideas of Calculus in Islam and India* (see [4], page 173) says, “There is no danger, therefore, that we have to rewrite the history texts to remove the statement that Newton and Leibniz invented the calculus.”

Strogatz does not enter into this messy debate at all, and I can understand why. After all, issues of priority can get nasty, just look at what happened between Newton and Leibniz! Moreover, as we have seen earlier, Strogatz’s take is very different when it comes to the history of calculus. To give him due credit, he does acknowledge that the Nila school had discovered the infinite series expansion for trigonometric functions 250 years before Newton. My bone however, is that while he has taken great care to apply the Infinity Principle to study the work of many of the masters and explained how calculus emerged from their work, he has not done this for Madhava. To quote Mumford:([3], page 389) “It seems fair to me to compare him with Newton and Leibniz.”

It would also have been really fruitful to see an analysis of the so-called Indian method of mathematics. In Mumford’s words ([3], page 389) “Simply put, these are recursion, induction, and careful passage to the limit.”

On a separate note, while the birth of calculus might have happened in Kerala, it did not spread like wildfire, nor did it have a similar

impact on science and technology as in Europe. Traditionally, this has been attributed to the genius of Newton and Leibniz in their formulation of mathematics and its application to the natural world, the beginning of the scientific revolution and support from the ruling hierarchy. The author of *Sapiens*, Harari, alerts us that this view may be a bit naive, and one should not ignore the twin forces of ‘imperialism and capitalism.’ To quote him “The feedback loop between science, empire and capital has arguably been history’s chief engine for the past 500 years.” Kerala in the fourteenth to sixteenth centuries was hardly at the centre of the empire!

III. The language of the universe

As mentioned in the first section, Strogatz is convinced of and wants to sell the idea that calculus is the best language to understand the universe, whose mysteries are written in “sentences called differential equations”.

How much of the modern world has really been shaped by calculus? I thought it best to ask several scientists this question (see the box for the survey sent). I deliberately chose scientists who are not pure mathematicians. They work on a wide spectrum of problems: how bees smell flowers, properties of polymers (with applications from plastics to DNA), behaviour of nano-materials, the application of group theory in physics, cellular and tissue biophysics and supermassive black holes. Interestingly, all of them do use calculus to solve their problems. Madan says, “Newton’s math and its descendants pervade every aspect of my work and every physicist’s work . . . ” However, the jury is still out as far as Strogatz’s strong claims!

For example, Ravi feels: “It is an exaggeration to say that calculus is at the heart of all scientific and technological progress, and to modern civilisation as we know it. It is definitely an integral part, and it is central to many of the achievements of mankind, but how does one separate one element of mathematics and give it the most prominent role?”

Shobhana wonders: “I do not know what to make of the claim that the Declaration of Independence would not have been made were it not for calculus! I am not sure what Strogatz had in mind. Maybe if people hadn’t been able to calculate the trajectory of a cannonball using calculus, the outcome of some battle would have been different, thus changing the course of history?”

Prajval says: “Exceptionalizing calculus in technological developments unnecessarily undermines the importance of tinkering and trial-and-error in engineering...”

Knowledge creation is a process of building upon and constantly testing, interrogating and reaffirming (or not) past knowledge, and is perforce a collective, co-operative human endeavour. Therefore to create straw-persons that privilege some particular branch or toolkit that has contributed to the current positive aspects of the human condition becomes merely polemics.”

Mukunda points out: “However, calculus by itself cannot lead us to the laws of nature, like Newton’s equation of motion, or Maxwell’s equations, or wave mechanics, or relativity. These are independent inputs from physics, to express which calculus is the most convenient language. So, the basic physical laws are not determined by calculus, but when discovered are expressed in the language of calculus. This distinction is very important.”

Due to constraints of space we are not able to publish their complete, fascinating responses in this print issue. Please visit <http://publications.azimpremjifoundation.org/3350/> to find their responses.

There is of course the great wonder as to why the universe is comprehensible by a tiny species with a large brain; and why is mathematics the best language to express the laws of nature? This is a very deep philosophical debate and I will let the reader enjoy Strogatz’s point of view.

IV. Read the book!

I would strongly urge anyone interested in studying or teaching mathematics to read the

book, especially because Strogatz takes an unusual point of view and often makes dramatic and provocative statements! I am not sure if he has succeeded in conveying what calculus is to the lay person (one of his intentions as mentioned earlier), but he definitely tries very hard to lay out all the background in as non-technical a manner possible, and conveys the vast sweep of ideas using many metaphors, illustrations and examples.

If you are a teacher of calculus, then there is much to learn, both in terms of mathematical content and its exposition. I particularly enjoyed the way he explains the proof of The Fundamental Theorem of Calculus. You will also learn many anecdotes in the story of calculus which you can use in the classroom. And when your students ask you, as they inevitably do, “what is the use of learning all this stuff?”, you can regale them with mind boggling applications from their DNA to their smart phone!

V. To end on a sombre note

The book is an unbridled celebration of the human ability to comprehend the universe, control it and produce marvels of technology that have helped some humans enjoy longer life, better health and vast wealth.

The genus *Homo* evolved on earth 6 million years ago, and for the last 13000 years the only species in this genus to survive is *Homo sapiens*. For most of this time, we have had very little impact on the earth. The last five hundred years, however, has seen the mass extinction of several thousand species, serious biodiversity loss, intolerable inequality among humans, the possibility of all life on earth being wiped out by nuclear bombs, and as a direct result of human activity, a change in our climate that has started and will continue to unleash huge suffering.

If calculus helped us “reshape civilization”, should we also consider the role of human thought and knowledge, even specifically calculus, in creating our current crisis?

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- [4] Katz, V.J. *Ideas of Calculus in Islam and India*, Mathematics Magazine, Volume 68, Number 3, June 1995, p 163-174.



SHASHIDHAR JAGADEESHAN has been teaching mathematics for over 30 years. He is a firm believer that mathematics is a human endeavour, and his interest lies in conveying the beauty of mathematics to students and demonstrating that it is possible to create learning environments where children enjoy learning mathematics. He may be contacted at jshashidhar@gmail.com.

Did Calculus Change the World?

What do an astrophysicist, a biologist, a bio-physicist, a chemical engineer, a computational nano-scientist and a theoretical physicist have in common? They were all subject to the following survey:

- (i) Could you briefly describe for a layperson the problem(s) you work on?
- (ii) Can you explain how calculus plays a part in your work (again as non-technically as possible)?
- (iii) Steven Strogatz begins his book, *Infinite Powers*, with the following claim (page vi of the Introduction)

“Without calculus, we wouldn’t have cell phones, computers, or microwave ovens. We wouldn’t have radio. Or television. Or ultrasound for expectant mothers, or GPS for lost travellers. We wouldn’t have split the atom, unravelled the human genome, or put astronauts on the moon. We might not even have had the Declaration of Independence.

It’s a curiosity of history that the world was changed forever by an arcane branch of mathematics.”

Later on he tempers the above statements by saying (page x of the introduction):

“...This is a much broader view of calculus than usual. It encompasses the many cousins and spinoffs of calculus, both within mathematics and in adjacent disciplines. Since this big-tent view is unconventional, I want to make sure it doesn’t cause any confusion. For example, when I said earlier that without calculus we wouldn’t have computers and cell phones and so on, I certainly didn’t mean to suggest that calculus produced all these wonders by itself. Far from it. Science and Technology were essential partners-and arguably the stars of the show. My point is merely that calculus has also played a crucial role, albeit often a supporting one, in giving us the world we know today.”

(on page xi after describing Maxwell’s work on electromagnetism)

“Clearly, calculus could not have done this alone. But equally clearly, none of it would have happened without calculus. Or, perhaps more accurately, it might have happened, but only much later, if at all.”

What is your opinion about this?



PROFESSORS RAVI PRAKASH JAGADEESHAN, N. MUKUNDA, SHOBHANA NARASIMHAN, MADAN RAO and **PRAJVAL SHASTRI** were kind enough to respond to questions (i) to (iii). Avehi Singh, a graduate student, has responded to questions (i) and (ii). Please visit <http://publications.azimpremjifoundation.org/3350/> to read their responses to the survey.

Review: Spinner

Reviewed by Math Space

Data Handling, Statistics and Probability have gained prominence in school mathematics over the last 5-6 years and thanks to the pandemic we all understand how crucial it is to have a basic understanding of probability. This topic is included in Classes 7 and 8 according to the NCERT syllabus and is part of the respective data handling chapters. However, the probability portions of these two chapters seem very disjoint from the rest of the data handling parts. The integration happens later in Class 9 when chapter-wise statistics and probability are separated! However, one does not need to wait that long. Spinners provide an excellent way to integrate probability within the rest of data handling along with many other benefits.

Spinners are inspired by the spinning wheels used in gambling. They can be easily made using a corrugated cardboard or a cardboard box as a base of the manipulative. Ideally this base should be at least 8-10cm along each dimension. A spinner is usually circular, but other shapes can provide very interesting interlinkages among different topics – we will discuss this later. Figure 1 shows a circular spinner mounted on the rectangular base of the manipulative. The circle can be split into any number of sectors. Consecutive sectors should be coloured (or shaded) so that they can be easily distinguished from each other. It is better to draw the spinner on paper, cut out the circle and paste it on the cardboard base. A safety-pin can be used as the needle and push pin can be used to attach it to the base. The push pin should pass through the ‘eye’ of the safety-pin and the centre of the base (Figure 1).



Figure 1

Observe that there is no restriction on how many sectors the circle can be split into. Therefore, a circular spinner provides much more freedom than the standard manipulatives for probability, viz. (fair) coin – 2 options, (fair) dice – 6 options, playing cards – 52 (or 13 or 4) options.



Figure 2

Moreover, the rest, i.e., coin, dice and cards are good for equally likely outcomes only. And this often creates a misconception that if there are only two possibilities then the chances must be 50-50 or the probability of each must be $\frac{1}{2}$. This can again be easily rectified by suitably designed spinners. Would the probabilities of blue and red be equal for spinners A, B and C in Figure 2? Such spinners can help students relate probability with the areas of the respective sectors.

Not only that, but also observe that circular spinners look just like a pie chart! Therefore, any pie chart can be converted into a spinner!! Thus, it is possible to collect data in a class or from a group of students, create a pie chart, and convert that into a spinner. Now the relative frequencies of the distribution (which is represented by the pie chart) become the (theoretical) probabilities, thus connecting statistics (or the rest of data handling) with probability. Since such spinners are easy to make, students can make their own and have fun finding empirical probabilities

given theoretical ones (through the spinner). In fact, the Law of Large Numbers for probabilities, i.e., 'the empirical probability converges to the theoretical one as the number of trials grow larger and larger (or goes to infinity)', can be experienced with a spinner!

For this endeavour, online spinners like <http://www.shodor.org/interactivateJS/AdjustableSpinner/AdjustableSpinner.html> can be of great help! This online version allows one to generate a desirable circle very quickly by providing the number of sectors and the percentage for each sector (percentages must add up to 100). It also allows one to spin it multiple times much faster. In fact, one can specify number of spins and it automatically generates the empirical (or experimental) probabilities as percentages. One can try multiple sets of say 1 spin, 5 spins, 20 spins, 100 spins, 500 spins, 2000 spins, etc. [It is important to hit 'update' to avoid cumulative spins.] Table 1 includes the empirical probabilities from 4 sets of each (1 spin, 5 spins, 20 spins, etc.) for blue with theoretical probability 0.2 (Figure 3), i.e., the empirical probabilities for 100 spins are 0.23, 0.21, 0.21 and 0.26 from the 4 sets. Now, check the spread (we have used standard deviation) for the empirical probabilities across each row. Observe how the spread is reducing as the number of spins is increasing.

Note how this also utilizes spread or measure of dispersion to understand probability – another interlinkage with statistics.

Table 1					
No. of spins	Set 1	Set 2	Set 3	Set 4	Spread
1	0	0	0	0	0
5	0.2	0.2	0.4	0	0.163299
20	0.1	0.2	0.35	0.1	0.118145
100	0.23	0.21	0.21	0.26	0.023629
500	0.192	0.2	0.166	0.188	0.014549
2000	0.204	0.204	0.201	0.198	0.002872

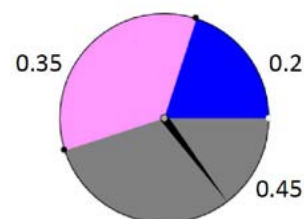


Figure 3

However, the NCTM version (<https://www.nctm.org/adjustablespinner/>) allows only equal sectors and therefore, only equally likely outcomes. So, it has very limited scope.

While online spinners can generate the data much faster and therefore is great for demonstrations or workshops where time may be limited, one does not know how they actually work. So, the actual spinner can provide a more grounded experience where each spin is observed and recorded. So, it would be more suitable for younger children giving them a more direct experience.

We would like to end this review by considering spinners of different shapes. In which of the following spinners in Figure 4 would the two colours have equal probability? Why? Note how this brings the focus from the area to the angle at the centre, where the diagonals intersect (which is where the push pin should be stuck) and thereby interlinking geometry with probability. This also opens the discussion on why area sufficed for circular spinners.

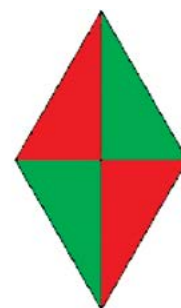
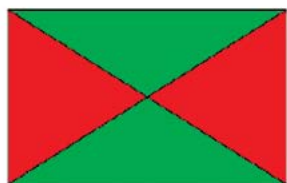
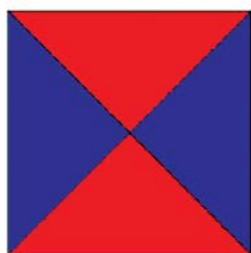


Figure 4

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] both in terms of uses and regarding possibility of low-cost versions that can be made from boxes etc. It tries to address both fear and dislike for mathematics as well as provide food for thought to those who like or love the subject. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space may be reached at mathspace@apu.edu.in

Acknowledgement: We would like to sincerely thank Angela Jain, MA Education student of Azim Premji University for exposing us to the world of spinners! She explored this as part of her project in the Curricular Material Development – Mathematics course. Most of the figures are thanks to her!

Addendum to Math from Simple Grids

Exploring Addition - RowColSum Puzzle

ASHWIN, SHRAVAN,
SHIVKUMAR &
GOWRI SATYA

5	1	8
4	3	6
7	9	2

Figure 19

This is a quick puzzle that could be used at the end of a session. Again with a 3 x 3 grid and the numbers 1 to 9, the teacher sets a focus number and students race to find an arrangement of 1 to 9 in the grid such that the sum of the numbers in the same row and column as the focus number is the highest.

For example, in the grid in Fig. 19, if the focus number is 5:

$$\text{RowColSum}(5) = 5 + 1 + 8 + 4 + 7 = 25$$

1. Is this the maximum possible RowColSum of 5?
2. Is there a strategy to arrange the numbers in the grid so that RowColSum of 5 is maximum?
3. Does the strategy change when you change the position of 5?
4. Is there a 'preferred' position of 5?
5. Does the strategy change when the focus number is changed?

I find that some students even in Grades 3 or 4 have not fully assimilated the nature of addition. While they can perfectly reproduce the addition algorithm, the insight that "adding a bigger number to a given number results in a bigger sum" is missing. This is a fun puzzle to drive that point home.

You can play RowColSum puzzles online at:
<http://mathventure.in/games/rowcolsum.html>

Keywords: puzzles, reasoning, mathematics

IV. Exploring Addition - Fubuki Addition Puzzle

The Japanese puzzle culture is rich in grid-based puzzles like Sudoku, Kakuro, etc. Another popular grid puzzle with Japanese origins is the Fubuki addition puzzle that is particularly suitable for the primary / middle school classroom.

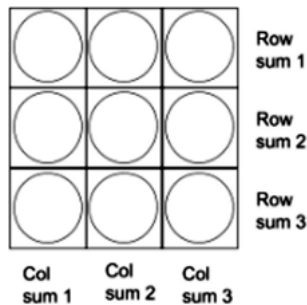


Figure 20

Given a 3 x 3 grid and target sums for each row and column, fill the grid with all the numbers from 1 to 9 such that the row and column sum targets are met. Each number can and should be used only once.

To start off, easy puzzles can be used where some of the numbers are filled into the grid (Fig. 21, Fig. 22) and slowly increased in complexity by decreasing the hint numbers in the grid (Fig. 23, Fig. 24).

Solving these puzzles requires students to make various connections and thrilling deductions.

1. What is the largest row or column sum possible? The smallest?

2. Can there be more than one solution?

3. Is there a pattern in the row and column sums?

One observation is that all four examples have two even numbers and one odd number in the row sums and similarly in the column sums. Does this always have to be so? Do the row sums always need to have the same pattern as the column sums in terms of parity? Of the 6 row / column sums, is it possible to have a configuration of say 3 odd numbers and 3 even numbers or 2 even numbers and 4 odd numbers?

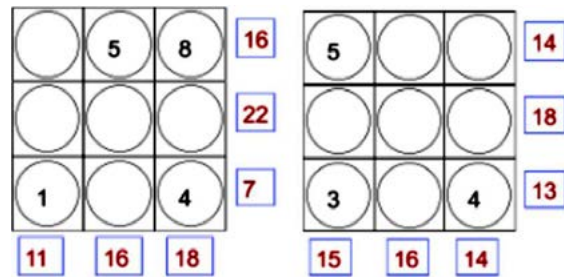


Figure 21

Figure 22

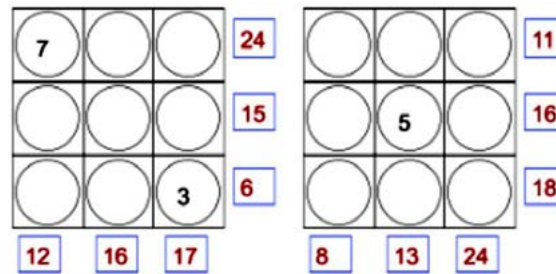


Figure 23

Figure 24

Again, we get to the most interesting part when we flip the question and ask students to create their own puzzles.

4. Can we have any randomly chosen six numbers as row and column sums?

Based on our previous observations, parity is one constraint. But what about the magnitude of the sums? Could there be a constraint on that? Observe the sum of the row sums and the sum of the column sums. Do they necessarily have to be that way? Why?

First, the sum of the row sums is equal to the sum of the column sums. This has to necessarily be so as it is the same nine numbers being added - just in a different order.

Second, the sum of the row sums and sum of the column sums have to necessarily be 45 as it is nothing but the sum of numbers 1 to 9.

5. Can we create a Fubuki addition puzzle that does not have any solution?

We have now identified two constraints on the row sums and column sums. Breaking either would lead to a puzzle without a solution.

6. How about if we remove the requirement of using numbers from 1 to 9?

If we could use any whole number inside the 3×3 grid, do the constraints on parity and magnitude of the row and column sums still hold? Can we still create an unsolvable puzzle under these revised conditions? How about if we expand the domain further - if we could have any integers inside the grid - how does this affect the solvability?

Challenge Question: *If any rational number could be used in the grid of a Fubuki addition puzzle, how does that affect the constraints on the row and column sums? What are the different ways to create a puzzle with no solutions?*

You can find Fubuki Addition puzzles online at: <https://www.mathinenglish.com/puzzlesfubuki.php>

V. Exploring Parity - Fubuki Addition Puzzle Variation

	5		even
			even
1			odd
odd	even	even	

Figure 25

A more basic variation to the Fubuki addition puzzle - instead of specifying the row and column sums, we can just specify that they are to be even or odd. Fill the grid using the numbers from 1 to 9 exactly once such that the row and column sums match the parity specified. Again, similar questions pop up:

1. What are the constraints on the conditions?
2. Can we create a puzzle without a solution?
3. Can there be more than one solution to a puzzle?

Challenge Question: *If there can be more than one solution, how many solutions are possible? Is there an efficient way to find these?*

VI. Hollow Grid Puzzle

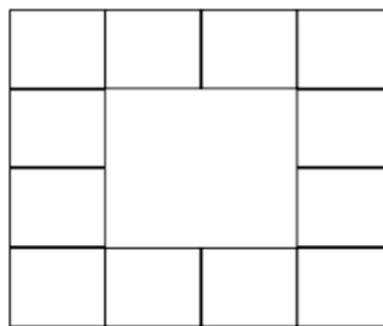


Figure 26

Given a 4×4 hollow grid (Fig. 26), fill each cell with pebbles such that each side of the grid has a total of 8 pebbles. Without actually solving the puzzle, can you predict how many pebbles are needed to do this?

Students are quick to use trial and error and come up with various solutions to this puzzle:

2	2	2	2	2	3	1	2	1	2	1	4
2			2	1			2	4			1
2			2	2			2	2			2
2	2	2	2	3	2	1	2	1	3	3	1

Figure 27

Figure 28

Figure 29

Now, if we add an extra pebble to an arbitrary cell in a given solution, is it possible to rearrange the pebbles so as to get the totals on each side back to 8? Amazingly, the extra pebble *can* (almost always) be accommodated. What if we add one more? This raises a host of questions:

1. **Can the required configuration be achieved for any and all numbers of pebbles?** Is there a minimum or maximum number of pebbles with which we can achieve this configuration? Can we achieve the 8 pebbles per side configuration for every number in between the min and max?
2. What's the key observation to help analyze and answer these questions?
3. For a given total number of pebbles, how do we come up with an arrangement that yields 8 pebbles per side? Is there more than one way to fill in the cells?

Let's take a look at the three solutions we have.
What is the relation between the sum of the pebbles along each side and the total number of pebbles?

Fig. 27 - total number of pebbles = 24

Fig. 28 - total number of pebbles = 23

Fig. 29 - total number of pebbles = 25

Sum of pebbles along each side in every case is $4 \times 8 = 32$

It can be seen that the pebbles in the corner cells are counted twice and so, we have:

Total number of pebbles = sum of pebbles along each side - sum of pebbles in corner cells.

Using this, how many pebbles should be put in corner cells to accommodate the maximum total number of pebbles? How many in the corners for the minimum total number of pebbles?

4. What if we change one of the variables - each side needs to have 9 instead of 8 pebbles?
5. How about a bigger grid - say a 6x6 hollow grid (Fig. 30)? Can we still achieve a total of 8 pebbles along each side of the grid?

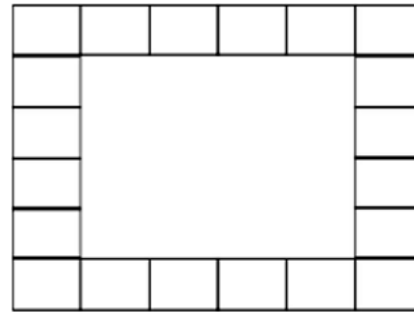


Figure 30

6. Is there a combination of grid size and side total for which there can be no solution? Can there be a grid with no solution if no empty cells are allowed?

I found that these puzzles could engage my students at many different levels. Each student learnt something new in the process and each student took home an unsolved question to mull on. The higher order questions about solvability have also kept me mulling for weeks. Now that makes for one happy math teacher.

Solutions

Some of these puzzles have multiple solutions but only one possible solution has been listed here.

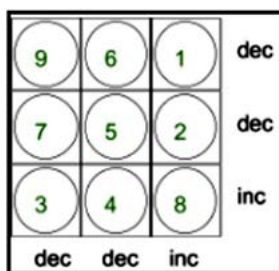


Figure 10

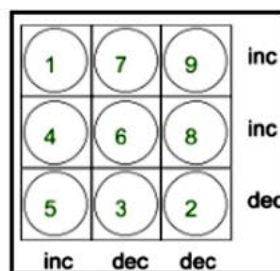


Figure 11

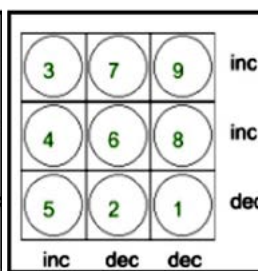


Figure 12

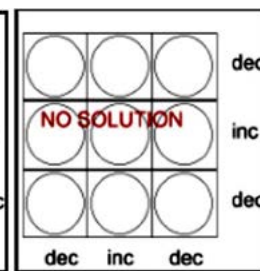


Figure 13

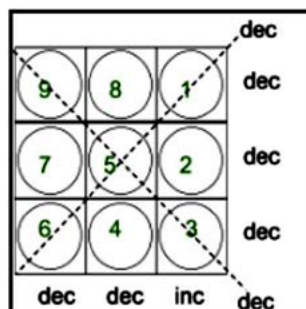


Figure 16

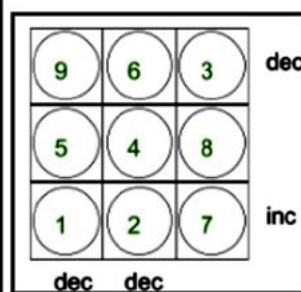


Figure 17

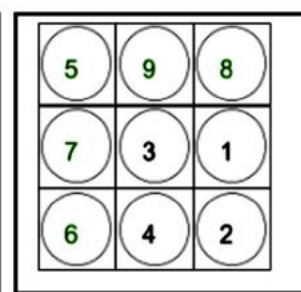


Figure 19

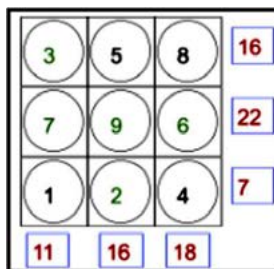


Figure 10

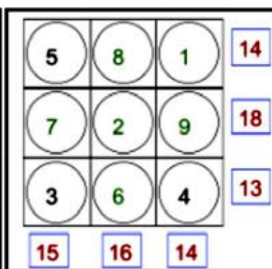


Figure 11

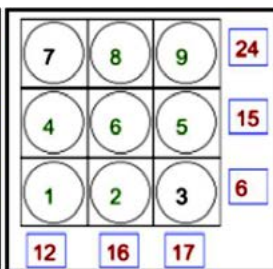


Figure 12

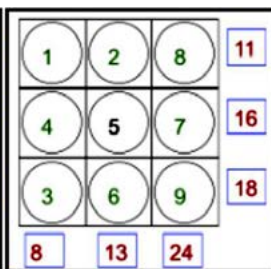


Figure 13

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Analysis of the Angle Trisection Procedure

SHAILESH SHIRALI

In the article by Shri Manoranjan Ghoshal which appeared in *At Right Angles*, November 2021, it is mentioned that approximate procedures for trisection of arbitrary angles exist, and one such procedure (elegant and simple to execute) was presented by the author. In this article, we analyse this procedure.

Proposed approximate procedure for trisection of angle

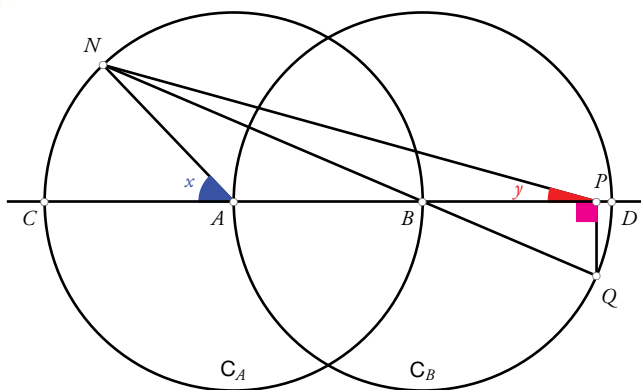


Figure 1.

- Let the angle to be trisected be denoted by x .
- Mark any two points A and B , 1 unit apart. Draw circles C_A and C_B centred at A and B respectively, with radius 1 unit each. Join AB and extend it to intersect C_A and C_B again at points C and D .

Keywords: Angle trisection, Euclidean instrument, Euclidean construction, error analysis

- Locate a point N on C_A such that $\angle NAC$ is equal to x (the angle to be trisected).
- Join NB and extend it beyond B till it meets C_B again at point Q .
- Draw QP perpendicular to line CD (with P on CD). Join NP .
- Then it will be found that $\angle NPC \approx \frac{1}{3}\angle NAC$, i.e., $y \approx \frac{1}{3}x$.

Analysis of the procedure

The task before us is to explain why this method gives such good results. We start by finding an expression for y in terms of x . For convenience, we have redrawn Figure 1.

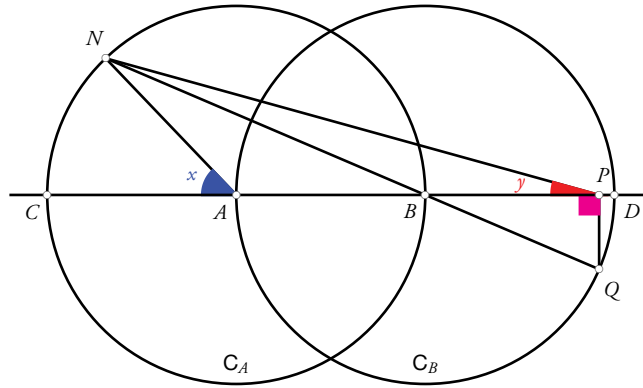


Figure 2.

We now make use of straightforward trigonometry. We have:

$$\angle NBA = \frac{x}{2} = \angle PBQ, \quad \therefore PQ = \sin \frac{x}{2}.$$

Next,

$$NB = 2 \cos \frac{x}{2}, \quad \therefore NQ = 1 + 2 \cos \frac{x}{2}.$$

Also, $\angle NQP = 90^\circ - x/2$, so by the sine rule,

$$\begin{aligned} \frac{NQ}{\sin(90^\circ + y)} &= \frac{PQ}{\sin(x/2 - y)}, \\ \therefore \frac{\sin(x/2 - y)}{\cos y} &= \frac{\sin x/2}{1 + 2 \cdot \cos x/2}. \end{aligned}$$

Expanding the sine term on the left side, we get:

$$\frac{\sin x/2 \cos y - \cos x/2 \sin y}{\cos y} = \frac{\sin x/2}{1 + 2 \cdot \cos x/2}.$$

This yields:

$$\begin{aligned}\sin \frac{x}{2} - \cos \frac{x}{2} \tan y &= \frac{\sin x/2}{1 + 2 \cdot \cos x/2}, \\ \therefore \cos \frac{x}{2} \tan y &= \sin \frac{x}{2} - \frac{\sin x/2}{1 + 2 \cdot \cos x/2} = \frac{2 \cdot \sin x/2 \cos x/2}{1 + 2 \cdot \cos x/2}, \\ \therefore \tan y &= \frac{2 \sin x/2}{1 + 2 \cdot \cos x/2},\end{aligned}$$

so:

$$y = \tan^{-1} \left(\frac{2 \cdot \sin x/2}{1 + 2 \cdot \cos x/2} \right).$$

We have thus been able to express y in terms of x , explicitly. (It does not seem possible to simplify the expression further.)

The above expression allows us to compute y for any given value of x . For example, if $x = 60^\circ$, we get:

$$y = \tan^{-1} \frac{1}{1 + \sqrt{3}} = \tan^{-1} \frac{\sqrt{3} - 1}{2} \approx 20.104^\circ,$$

which is not too bad. The following table shows the closeness of the approximation for select angles:

x	15°	30°	45°	60°	75°	90°
y	5.002°	10.013°	15.043°	20.104°	25.206°	30.361°

We see that the relative error is significantly smaller when x is small; also that the trisected angle in each case is slightly *larger* than it should be.

A more precise error analysis

To obtain a precise estimate of the error and also explain the two observations made above, we shall express y as a power series in x . We make use of the following known power series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad (\text{valid for all real } x),$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad (\text{valid for all real } x),$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad (\text{valid for } -1 < x < 1).$$

Using these results, we obtain:

$$\begin{aligned}2 \cdot \sin \frac{x}{2} &= 2 \left(\frac{x}{2} - \frac{1}{3!} \cdot \frac{x^3}{2^3} + \frac{1}{5!} \cdot \frac{x^5}{2^5} - \cdots \right) \\ &= x - \frac{x^3}{24} + \frac{x^5}{1920} - \cdots.\end{aligned}$$

Next,

$$\begin{aligned} 1 + 2 \cdot \cos \frac{x}{2} &= 1 + 2 \left(1 - \frac{1}{2!} \cdot \frac{x^2}{2^2} + \frac{1}{4!} \cdot \frac{x^4}{2^4} - \cdots \right) \\ &= 3 \left(1 - \frac{x^2}{12} + \frac{x^4}{576} - \cdots \right), \end{aligned}$$

so

$$\begin{aligned} \frac{1}{1 + 2 \cdot \cos x/2} &= \frac{1}{3} \left(1 - \frac{x^2}{12} + \frac{x^4}{576} - \cdots \right)^{-1} \\ &= \frac{1}{3} \left(1 + \frac{x^2}{12} + \frac{x^4}{192} + \cdots \right). \end{aligned}$$

The last step shown above may not look entirely clear, but we need to use the well-known expansion $(1 - t)^{-1} = 1 + t + t^2 + t^3 + \cdots$, which is valid for $-1 < t < 1$. By substituting the appropriate expression for t and ploughing through a lot of algebra, we obtain the stated result. Note that at each stage of the simplification, we make sure that we never carry with us terms beyond x^5 .

From the above we obtain:

$$\begin{aligned} \frac{2 \cdot \sin x/2}{1 + 2 \cdot \cos x/2} &= \frac{1}{3} \left(x - \frac{x^3}{24} + \frac{x^5}{1920} - \cdots \right) \cdot \left(1 + \frac{x^2}{12} + \frac{x^4}{192} + \cdots \right) \\ &= \frac{x}{3} + \frac{x^3}{72} + \frac{13x^5}{17280} + \cdots. \end{aligned}$$

Hence we obtain:

$$\begin{aligned} \tan^{-1} \left(\frac{2 \cdot \sin x/2}{1 + 2 \cdot \cos x/2} \right) &= \left(\frac{x}{3} + \frac{x^3}{72} + \frac{13x^5}{17280} + \cdots \right) - \frac{1}{3} \left(\frac{x}{3} + \frac{x^3}{72} + \frac{13x^5}{17280} + \cdots \right)^3 + \cdots \\ &= \frac{x}{3} + \frac{x^3}{648} + \frac{x^5}{31104} + \cdots, \end{aligned}$$

after a substantial amount of simplification. This means that

$$y = \frac{x}{3} + \frac{x^3}{648} + \frac{x^5}{31104} + \cdots.$$

We deduce the following from the above relation:

- If x is very small, then the terms involving x^3 and higher powers may be neglected, and we have

$$y \approx \frac{x}{3},$$

as it should be. Figure 3 shows a graph of y against x . Observe the (extremely surprising) fact that for the portion $-1 < x < 1$, the graph almost exactly coincides with the graph of $y = x/3$. A graph of the same function is shown over a wider domain (Figure 4); we see some signs of non-linearity towards the edges.

- From the expression obtained for y , we see that y always *exceeds* $x/3$. This agrees with our earlier computations.

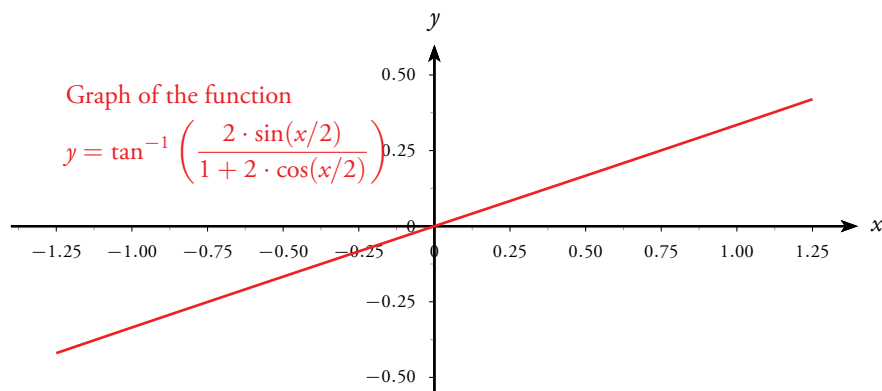


Figure 3.

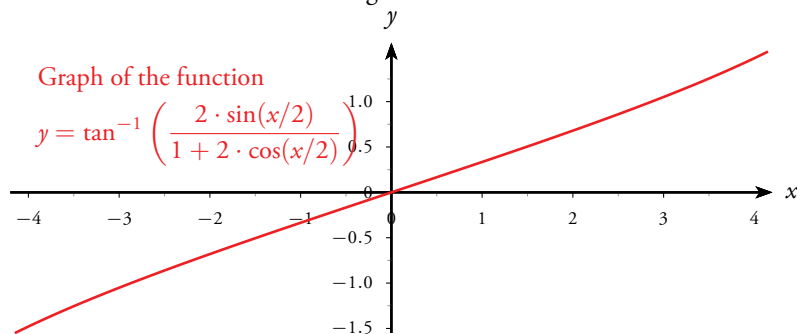


Figure 4.

- The relative error in taking y to be equal to $x/3$ is

$$\frac{x^3/648 + x^5/31104 + \dots}{x/3} = \frac{x^2}{216} + \frac{x^4}{10368} \approx \frac{x^2}{216}.$$

So the percentage error in taking y to be equal to $x/3$ is about $x^2/2.16$, i.e., roughly $x^2/2$.

- The above finding explains why the accuracy of the procedure is so much better when the angle is small, and why the accuracy goes down as the angle increases in size.
- If the angle to be trisected is close to 1 radian, then the percentage error according to the above analysis should be roughly 0.5. This agrees with the computation when the angle to be trisected is 60° . We had earlier found the error to be 0.104. So the percentage error is $0.104/20 \times 100$, which is close to 0.5, just as expected.

Concluding remark. To come across an approximate angle trisection procedure of such a simple nature (i.e., where the number of steps in the construction is relatively small) and which at the same time gives such accurate results (at least for small angles), is quite surprising. It just shows that there are unexpected but most pleasing surprises to be found in mathematics at every level!



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Prime Generating Polynomials

ANAND PRAKASH

Is there some polynomial $f(z)$ in a variable z , with integer coefficients, that always generates prime numbers, for all possible non-negative integers z ? The answer is **No**, and this has been known for long; no such polynomial exists. (See the addendum for a proof.)

However, mathematics enthusiasts from time to time have sought polynomials which generate prime numbers for many consecutive positive integers.

For example, **Leonard Euler** found the quadratic polynomial $f(z) = z^2 + z + 41$ which generates primes for 40 consecutive numbers, namely $z = 0, 1, 2, \dots, 38, 39$. (It also generates primes for $z = -40, -39, \dots, -2, -1$, but these primes are identical to the primes generated earlier, because $f(-z - 1) = f(z)$, identically.) Other mathematicians have found more examples of such low degree polynomials which display a similar behaviour.

In this article, we play with this theme and offer a few variations of such polynomials.

- Euler's polynomial is of the kind $f(z) = z^2 + z + p$, where p is a prime number. We noted above that it assumes prime values for all z in the set $\{0, 1, 2, \dots, p - 2\}$. There are other polynomials of the same 'shape' that share the same property, for example:

$$f(z) = z^2 + z + 3: \text{ prime for } z \in \{0, 1\};$$

$$f(z) = z^2 + z + 5: \text{ prime for } z \in \{0, 1, 2, 3\};$$

Keywords: Prime number, polynomial, integer coefficient

$f(z) = z^2 + z + 11$: prime for $z \in \{0, 1, 2, \dots, 8, 9\}$;

$f(z) = z^2 + z + 17$: prime for $z \in \{0, 1, 2, \dots, 14, 15\}$. (This example was given by the French mathematician **Adrien-Marie Legendre**.)

There do not appear to be any more polynomials of this kind. In other words, it appears that the only values of p for which the polynomial $f(z) = z^2 + z + p$ takes prime values for all $z \in \{0, 1, 2, \dots, p-2\}$ are the following: $\{2, 3, 5, 11, 17, 41\}$.

Note that for each of these polynomials, the list of consecutive values of z for which it takes prime values cannot be extended beyond $z = p-2$, for $z^2 + z + p$ is composite for $z = p-1$ (and for $z = p$ as well).

- As noted above, Legendre gave the polynomial $f(z) = z^2 + z + 17$ which is prime for all $z \in \{0, 1, 2, \dots, 14, 15\}$. Closely resembling this are the following:

$f(z) = z^2 + 3z + 19$: prime for $z \in \{0, 1, 2, \dots, 13, 14\}$;

$f(z) = z^2 + 5z + 23$: prime for $z \in \{0, 1, 2, \dots, 12, 13\}$;

$f(z) = z^2 + 7z + 29$: prime for $z \in \{0, 1, 2, \dots, 11, 12\}$.

- Legendre also gave the polynomial $f(z) = 2z^2 + 29$ which is prime for $z \in \{0, 1, 2, \dots, 27, 28\}$. Closely resembling this are the following:

$f(z) = 8z^2 + 29$: prime for $z \in \{0, 1, 2, \dots, 13, 14\}$;

$f(z) = 10z^2 + 13$: prime for $z \in \{0, 1, 2, \dots, 11, 12\}$;

$f(z) = 10z^2 + 7$: prime for $z \in \{0, 1, 2, \dots, 5, 6\}$;

$f(z) = 10z^2 + 19$: prime for $z \in \{0, 1, 2, \dots, 17, 18\}$;

$f(z) = 12z^2 + 59$: prime for $z \in \{0, 1, 2, \dots, 13, 14\}$.

- Here are some polynomials that closely resemble Euler's polynomial $z^2 + z + 41$ but differ in the coefficient of z :

$z^2 + 3z + 43$: prime for $z \in \{0, 1, 2, \dots, 37, 38\}$;

$z^2 + 5z + 47$: prime for $z \in \{0, 1, 2, \dots, 36, 37\}$;

$z^2 + 7z + 53$: prime for $z \in \{0, 1, 2, \dots, 35, 36\}$.

Each of these generates prime values for a relatively long stretch of consecutive values of z .

- Here is an example given by Fung and Ruby: $f(z) = 36z^2 - 810z + 2753$. It takes prime values for $z \in \{0, 1, 2, \dots, 43, 44\}$. Closely resembling this are the following:

$f(z) = 36z^2 - 828z + 4363$: prime for $z \in \{0, 1, 2, \dots, 24, 25\}$;

$f(z) = 36z^2 - 960z + 7993$: prime for $z \in \{0, 1, 2, \dots, 14, 15\}$.

- Similar to the above is another example by Fung and Ruby: $f(z) = 47z^2 - 1701z + 10181$. It takes prime values for $z \in \{0, 1, 2, \dots, 41, 42\}$. Closely resembling this are the following:

$$f(z) = 47z^2 - 1591z + 9631: \text{prime for } z \in \{0, 1, 2, \dots, 10, 11\};$$

$$f(z) = 47z^2 - 371z + 8761: \text{prime for } z \in \{0, 1, 2, \dots, 9, 10\};$$

$$f(z) = 47z^2 - 901z + 9151: \text{prime for } z \in \{0, 1, 2, \dots, 7, 8\};$$

$$f(z) = 67z^2 - 1261z + 9491: \text{prime for } z \in \{0, 1, 2, \dots, 7, 8\};$$

$$f(z) = 67z^2 - 561z + 9241: \text{prime for } z \in \{0, 1, 2, \dots, 11, 12\}.$$

- Lastly, we have this cubic polynomial given by SM Ruiz: $3z^3 - 183z^2 + 3381z - 18757$. It takes prime values for $z \in \{0, 1, 2, \dots, 41, 42\}$. Closely resembling this are the following:

$$f(z) = 3z^3 - 183z^2 + 3138z - 13487: \text{prime for } z \in \{0, 1, 2, \dots, 6, 7\};$$

$$f(z) = 3z^3 - 183z^2 + 2148z - 15277: \text{prime for } z \in \{0, 1, 2, \dots, 8, 9\}.$$

In this way, we can explore variations in prime generating polynomials which generate prime values for long stretches of consecutive values of the argument. Readers may find more such examples.

References

1. Ed Pegg Jr., "Prime Generating Polynomials", from https://www.mathpuzzle.com/MAA/48-Prime%20Generating%20Polynomials/mathgames_07_17_06.html
2. Weisstein, Eric W. "Prime-Generating Polynomial." From MathWorld—A Wolfram Web Resource. <https://mathworld.wolfram.com/Prime-GeneratingPolynomial.html>

Addendum: A note from the editor

Theorem. It is not possible for a non-constant single-variable polynomial with integer coefficients to take only prime values.

The word 'non-constant' is needed to avoid trivial cases. For example, suppose $f(x) = 2$ for all positive integers x (so it is a constant function); this clearly takes only prime values!

To prove the theorem stated above, we need the two lemmas given below.

Lemma 1. If $f(x)$ is a polynomial with integer coefficients, then $a - b$ is a divisor of $f(a) - f(b)$ for any two unequal integers a, b .

To see why this is true, observe first that $a - b$ is a divisor of $a^k - b^k$ for any positive integer k . (The reader should be able to verify this using the factor theorem.) Next, observe that if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where $a_0, a_1, \dots, a_{n-1}, a_n$ are integers, then

$$f(a) - f(b) = a_n (a^n - b^n) + a_{n-1} (a^{n-1} - b^{n-1}) + \dots + a_1 (a - b).$$

Since $a - b$ is a divisor of each bracketed term on the right, it follows that $a - b$ is a divisor of $f(a) - f(b)$.

Lemma 2. *If $f(x)$ is a polynomial with positive leading coefficient, then if x is sufficiently large, $f(x)$ is positive and strictly increasing. (In short, any polynomial function with positive leading coefficient ultimately becomes “positive and strictly increasing.”)*

Outline of proof:

- Since $f(x)$ is a polynomial with positive leading coefficient, this is also true for $f'(x)$. That is, the derivative is a polynomial with positive leading coefficient.
- It follows that the derivative will be positive for large enough x , because the leading term in $f'(x)$, which will dominate all terms with lower degree for large enough x , will be positive.
- Since $f'(x)$ is positive for large enough x , $f(x)$ is strictly increasing for large enough x .
- Since $f(x)$ has integer coefficients, it is integer-valued. Therefore, when x increases from any integer to the next higher integer, $f(x)$ increases by ≥ 1 . Hence $f(x)$ assumes positive values for large enough x . (More can be said: $f(x)$ grows without bound.)

Proof of the main claim. Let $f(x)$ be a non-constant polynomial with integer coefficients, with positive leading coefficient. To show that $f(x)$ cannot take only prime values, we only need to exhibit a single composite value taken by f . We shall exhibit such a value.

Suppose that for $x \geq N$, $f(x)$ is increasing, and $f(x) > 1$. Let $f(N) = q$. Then $q > 1$. Now consider the number

$$f(N + q) - f(N).$$

Invoking Lemma 1, we infer that $f(N + q) - f(N) = f(N + q) - q$ is divisible by q . Since $f(N) = q$, this implies that $f(N + q)$ too is divisible by q . Also, $f(N + q) > q$, since $f(N + q) > f(N)$. This implies that $f(N + q)$ is composite. We have thus succeeded in exhibiting a composite value taken by $f(x)$. $\square$



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The Painter's Paradox

Comparative Analysis of Gabriel's Horn and Triangular Pipe with Koch's Fractal Shaped Cross Section

RIDA SURAIYA KHAN

Abstract

This paper presents the Painter's Paradox—a highly counterintuitive situation where a painter is able to fill a certain 3-dimensional object with paint but is unable to fully paint the surface of that object.

Mathematically, this paradox illustrates that a 3-dimensional object can have a finite volume while having an infinite surface area.

A well-known object like Gabriel's Horn is a classic example used to illustrate this paradox. To study it, we require a basic understanding of integral calculus and the concepts of surface area and volume. However, one can construct other objects that illustrate the same paradox, using only high school geometry and geometric series.

At the heart of this paradox lies the counterintuitive nature of infinite series.

The Study

The “**Painter's Paradox**” is presented in two sections.

Section 1 uses the ‘Gabriel Horn’ to illustrate the paradox; it assumes the reader's acquaintance with integral calculus and concept of a solid of revolution—in this case, the solid obtained by revolving the region bounded by the function $f(x) = 1/x$ and the x -axis around the x -axis.

Keywords: Paradox, Geometric progression, infinite series

Section 2 illustrates similar results using a new 3-D object based on fractals and requires only the reader's familiarity with high school mathematics.

Section 1. Gabriel's Horn

Figure 1 shows the plot of function $y = 1/x$ for $x \geq 1$. The 3D region is generated by rotating the curve by 2π radians about the x -axis to generate the object known as Gabriel's Horn with volume V and surface area S .

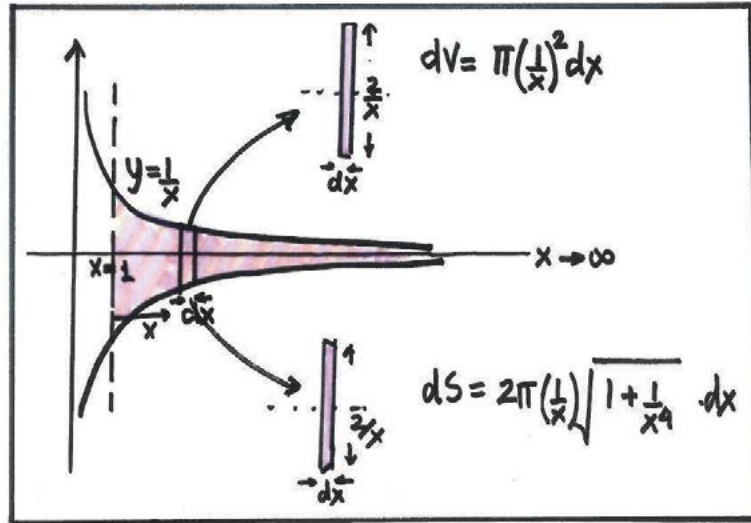


Figure 1.

Volume (dV) of a thin circular disk of thickness dx at distance x from the origin is given by:

$$dV = \pi y^2 dx \text{ where } y = \frac{1}{x} \text{ which gives } dV = \frac{\pi}{x^2} dx.$$

Volume (V) of the Gabriel Horn is obtained by integration as shown below:

$$V = \int_0^V dV = \int_1^\infty \frac{\pi}{x^2} dx = \pi \left[-\frac{1}{x} \right]_1^\infty = \pi \left[-\frac{1}{\infty} + \frac{1}{1} \right] = \pi.$$

Surface area (dS) of a thin circular disk at distance x from origin of thickness dx is given by:

$$dS = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ where } y = \frac{1}{x} \text{ which gives } \frac{dy}{dx} = -\frac{1}{x^2}$$

Hence:

$$S = \int_0^S dS = \int_1^\infty 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_1^\infty \frac{2\pi}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} dx$$

So:

$$S = \int_1^\infty \frac{2\pi}{x^3} \sqrt{x^4 + 1} dx$$

Therefore $S \int_1^\infty \frac{2\pi}{x^3} \sqrt{x^4 + 1} dx = \int_1^\infty \frac{2\pi}{x} dx = 2\pi [\ln x]_1^\infty$, which is not finite, since $\ln x \rightarrow \infty$ when $x \rightarrow \infty$.

Thus the surface area of Gabriel Horn is infinite while its volume is finite, i.e., π .

Section 2. Triangular Pipe with Koch's fractal shaped cross section

Figure 2 shows the cross section of a triangular pipe of unit height (i.e. $h = 1$) which undergoes repeated iterations following Koch's curve rule.

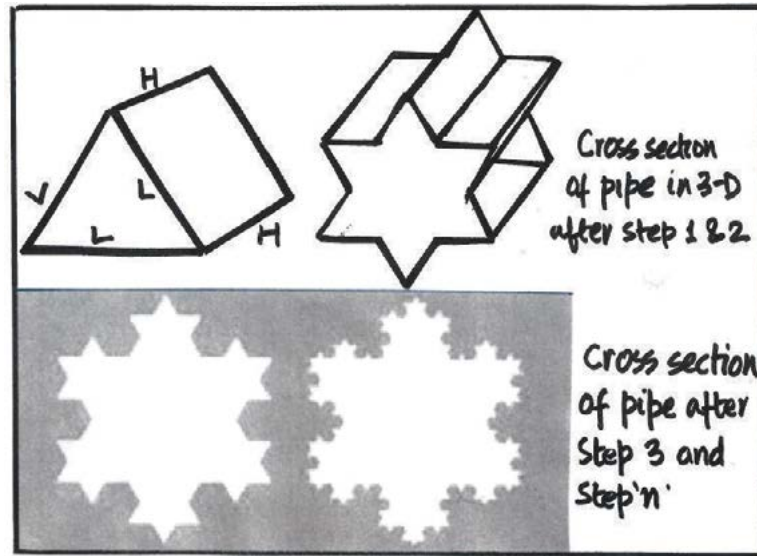


Figure 2.

Table 1 and Table 2 give the explicit formulae for volume and surface area of the object shown in Figure 2 after n iterations.

Hence as $n \rightarrow \infty$, we also have $S \rightarrow \infty$.

Thus the surface of a triangular pipe whose cross-section is shaped like Koch's fractal has infinite area while its volume is bounded to $9/5^{\text{th}}$ of its original volume.

Iteration	Number of new added $\triangle$	Area of each new $\triangle$	Total Volume (fig 2) ($V = A \times h = A \times 1$)
0	0	0	0
1	1	A	V
2	4	$\frac{A}{9}$	$V = V + \frac{4V}{9} = \frac{13V}{9}$
3	16	$\frac{A}{81}$	$V = V + \frac{4V}{9} + \frac{16V}{81} = \frac{133V}{81}$
4	64	$\frac{A}{729}$	$V = V + \frac{4V}{9} + \frac{16V}{81} + \frac{64V}{729} = \frac{1261V}{729}$
n	4^{n-1}	$\frac{A}{9^{n-1}}$	$V = V + \frac{4V}{9} + \frac{16V}{81} + \dots + \frac{4^{n-1}V}{9^{n-1}}$
∞	∞	0	$V = \sum_{n=1}^{\infty} \frac{4^{n-1}V}{9^{n-1}} = \frac{9V}{5}$

Table 1.

Iteration	Number of Segments	Length (L) of each segment	Perimeter (fig 2) $P = 3 \times L$	Total Surface area (fig 2) ($S = P \times h = P \times 1$)
0	3	1	3	3
1	12	$\frac{1}{3}$	4	4
2	48	$\frac{1}{9}$	$\frac{16}{3}$	$S = \frac{16}{3}$
3	192	$\frac{1}{27}$	$\frac{64}{9}$	$S = \frac{64}{9}$
4	768	$\frac{1}{81}$	$\frac{256}{27}$	$S = \frac{256}{27}$
n	34^n	$\frac{1}{3^n}$	$\frac{4^n}{3^{n-1}}$	$S = \frac{4^n}{3^{n-1}}$

Table 2.

Section 3. Conclusion

The investigation of volume and surface area of two different objects using calculus and high school mathematics provides an interesting glimpse of the Painter's Paradox.

Further study can be carried out to investigate the logical fallacy of the statement that if we can fill the Gabriel's Horn with a finite amount of paint, then the inner surface area is automatically painted completely.



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Triangles and Pell's Equation

HARAN MOULI

In a WhatsApp group of Math enthusiasts, a question was posed some time ago:

13,14,15 are the sides of a triangle with rational area and side lengths that are consecutive integers. Can we find more such triangles? This question led me to ask, how many such triangles exist? Can we come up with a general formula to generate such triangles? In this article, I explore these questions further.

Problem. Find all triangles with sides as consecutive integers and rational area.

Solution. Let the side lengths of the triangle be $x-1, x, x+1$, where x is a positive integer, $x > 1$.

Such a triangle exists if and only if the triangle inequality is obeyed, which implies here that $(x-1) + x > x+1$. This reduces to $x > 2$.

Next, we find the area of this triangle using Heron's formula:

$$A = \frac{1}{4} \sqrt{(3x)(x+2)(x)(x-2)} = \frac{x}{4} \sqrt{3(x^2-4)}.$$

This shows that the area of the triangle is rational if and only if $3(x^2-4)$ is the square of a rational number. Since $3(x^2-4)$ is an integer, this means that it is a perfect square, say y^2 . So we must look for positive integer pairs (x, y) such that $y^2 = 3(x^2-4)$.

We first show that x cannot be odd. For, if x were odd, then we would have $x^2 \equiv 1 \pmod{4}$, and this would lead to

$$y^2 \equiv 3 \pmod{4}$$

which is not possible, as no perfect square is of the form $3 \pmod{4}$. Therefore x is even. But this implies that y too is even.

Let $x = 2a$ and $y = 2b$, where a and b are positive integers (with $a > 1$, since $x > 2$). Substituting, we get $4b^2 = 12a^2 - 12$, and therefore

$$b^2 = 3a^2 - 3.$$

Keywords: Triangle, area, rational number, perfect square, Pell's equation

From this we deduce that b is a multiple of 3, say $b = 3c$, where c is a positive integer. Substituting, we get $9c^2 = 3a^2 - 3$, and therefore $3c^2 = a^2 - 1$, i.e.,

$$a^2 = 3c^2 + 1.$$

So the task reduces to finding positive integer pairs (a, c) such that $a^2 = 3c^2 + 1$. We have arrived at a well-studied kind of equation: the *Pell equation*.

The first solution in positive integers is $(a, c) = (2, 1)$. It is possible to show that the positive integer solutions are all given by $(a, c) = (a_k, c_k)$, where

$$a_k + c_k\sqrt{3} = (2 + \sqrt{3})^k.$$

This means that the solutions are $(a, c) = (2, 1), (7, 4), (26, 15), \dots$. To solve for a_k we have:

$$\begin{aligned} a_k + c_k\sqrt{3} &= (2 + \sqrt{3})^k, \\ a_k - c_k\sqrt{3} &= (2 - \sqrt{3})^k, \end{aligned}$$

which yield

$$a_k = \frac{(2 + \sqrt{3})^k + (2 - \sqrt{3})^k}{2}.$$

Substituting $x = 2a_k$ gives:

$$x = (2 + \sqrt{3})^k + (2 - \sqrt{3})^k.$$

We thus get the solutions $x = 4, 14, 52, 194, 724, 2702, 10084, \dots$

These lead to the following triangles:

x	4	14	52	194	724	...
Sides	3, 4, 5	13, 14, 15	51, 52, 53	193, 194, 195	723, 724, 725	...
Area	6	84	1170	16296	226974	...

Can you see why all the areas are integers?

Closing comment. An interesting question to explore further would be to characterize all integer-sided triangles with sides in an arithmetic progression and rational area. (Hint: The idea is similar to the original question: the arithmetic progression constraint ensures that the expression that needs to be a perfect square is a quadratic in x .)



HARAN MOULI is a 12th grade student of the PSBB group of schools. A fervent math enthusiast, he has a keen interest in problem solving and the learning and discussion of mathematical concepts, with a particular fondness for number theory. He loves teaching and volunteers at *Raising a Mathematician Foundation* to guide high school students passionate about Mathematics. He may be contacted at mouliharan@gmail.com.

Connecting φ and π

MAITREYO
BHATTACHARJEE

In this short article, we bring out an interesting connection between two fundamental constants of mathematics: π (which is sometimes known as “Archimedes’s constant”) and ϕ (the golden ratio). The connection comes about through consideration of an improper integral.

In integral calculus, integrals may be broadly classified into two categories: proper and improper. The term ‘improper’ means that the limits of the integral are not finite, i.e., one or both of the limits may be infinity (positive or negative). An integral having any one limit infinite can be expressed similarly as the following limit:

$$\int_0^{\infty} f(x) dx = \lim_{n \rightarrow \infty} \int_0^n f(x) dx. \quad (1)$$

Here, we study a particular improper integral and discuss its solution, which has a nice - looking closed form expression in terms of known universal constants. In the end, we comment on the general form of the integral. The first crucial claim we make is as follows:

Lemma.

$$\lim_{x \rightarrow \infty} x \tan^{-1} \left(\frac{2}{x^2 + 1} \right) = 0.$$

Proof. We use the method of series expansion. We have,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots,$$

so:

$$x \tan^{-1} \left(\frac{2}{x^2 + 1} \right) = x \left(\frac{2}{x^2 + 1} \right) - \frac{x}{3} \left(\frac{2}{x^2 + 1} \right)^3 + \frac{x}{5} \left(\frac{2}{x^2 + 1} \right)^5 - \dots$$

Keywords: Integration, improper integral, golden ratio, Archimedes’s constant

Now $\frac{x}{x^2+1} \rightarrow 0$ as $x \rightarrow \infty$. Thus, defining $T(x, n) := \frac{x}{(x^2+1)^n}$, we see that

$$\lim_{x, n \rightarrow \infty} T(x, n) = 0, \quad \text{for } n = 1, 3, 5, \dots$$

Therefore $\lim_{x \rightarrow \infty} c T(x, n) = 0$, for any $c \in \mathbb{R}$.

Using the above, we see that the required limit, which is the sum of above similar terms, is 0. Thus

$$\lim_{x \rightarrow \infty} x \tan^{-1} \left(\frac{2}{x^2+1} \right) = 0.$$

Now, we come to our main problem.

Problem

Evaluate the integral

$$\int_0^\infty \tan^{-1} \left(\frac{2}{x^2+1} \right) dx.$$

Solution

We use integration by parts, taking $\tan^{-1} \left(\frac{2}{x^2+1} \right)$ as the first function. Hence, denoting the given integral by I , we get :

$$\begin{aligned} I &= \int_0^\infty \tan^{-1} \left(\frac{2}{x^2+1} \right) \cdot 1 \, dx \\ &= \left[x \tan^{-1} \left(\frac{2}{x^2+1} \right) \right]_0^\infty - \int_0^\infty x \left[\frac{d}{dx} \tan^{-1} \left(\frac{2}{x^2+1} \right) \right] dx \\ &= \left[x \tan^{-1} \left(\frac{2}{x^2+1} \right) \right]_0^\infty - \int_0^\infty x \left(\frac{(x^2+1)^2}{x^4+2x^2+5} \cdot \frac{-4x}{(x^2+1)^2} \right) dx \\ &= \lim_{x \rightarrow \infty} x \tan^{-1} \left(\frac{2}{x^2+1} \right) - 0 + \int_0^\infty x \left(\frac{(x^2+1)^2}{x^4+2x^2+5} \cdot \frac{4x}{(x^2+1)^2} \right) dx \\ &= 0 + \int_0^\infty \frac{4x^2}{4+(x^2+1)^2} dx \\ &= \int_0^\infty \frac{4x^2}{x^2(x^2+2+5/x^2)} dx = \int_0^\infty \frac{4}{x^2(x^2+2+5/x^2)} dx \\ &= \int_0^\infty \frac{4}{(x^2-2\sqrt{5}+5/x^2)+22\sqrt{5}} dx \\ &= \int_0^\infty \frac{4}{(x-\sqrt{5}/x)^2+2+2\sqrt{5}} dx. \end{aligned}$$

A crucial step here is the introduction of the factor $2\sqrt{5}$.

Now we carry out this transformation: $x \mapsto x/\sqrt{5}$. The integral becomes

$$I = \int_{-\infty}^0 \frac{4}{(\sqrt{5}/x - x)^2 + 2 + 2\sqrt{5}} \cdot \left(\frac{-\sqrt{5}}{x^2} \right) dx.$$

Taking into account the negative sign, the expression becomes

$$I = \int_0^{\infty} \frac{4}{(\sqrt{5}/x - x)^2 + 2 + 2\sqrt{5}} \cdot \left(\frac{\sqrt{5}}{x^2} \right) dx.$$

So, we have:

$$\begin{aligned} I + I &= \int_0^{\infty} \frac{4}{(x - \sqrt{5}/x)^2 + 2 + 2\sqrt{5}} dx + \int_0^{\infty} \frac{4}{(\sqrt{5}/x - x)^2 + 2 + 2\sqrt{5}} \cdot \left(\frac{\sqrt{5}}{x^2} \right) dx \\ \Rightarrow 2I &= \int_0^{\infty} \frac{4}{(x - \sqrt{5}/x)^2 + 2(1 + \sqrt{5})} \cdot \left(1 + \frac{\sqrt{5}}{x^2} \right) dx \\ \Rightarrow I &= \int_0^{\infty} \frac{2}{(x - \sqrt{5}/x)^2 + 2(1 + \sqrt{5})} \cdot \left(1 + \frac{\sqrt{5}}{x^2} \right) dx. \end{aligned}$$

We need one more substitution. Putting $t = x - \sqrt{5}/x$, we get

$$I = \int_{-\infty}^{\infty} \frac{2}{t^2 + 2(1 + \sqrt{5})} dt.$$

It is well-known that $\phi = \frac{1 + \sqrt{5}}{2}$ is the Golden Ratio (for more details, one can refer to [4]), a fundamental constant appearing in theory as well as the practical world, and having numerous beautiful properties. Observe that $2(1 + \sqrt{5}) = 4\phi$. So,

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{2}{t^2 + 2(1 + \sqrt{5})} dt = \int_{-\infty}^{\infty} \frac{2}{t^2 + 4\phi} dt \\ \Rightarrow I &= \int_{-\infty}^{\infty} \frac{2}{t^2 + 4\phi} dt = \frac{1}{\sqrt{\phi}} \tan^{-1} \left(\frac{t}{2\sqrt{\phi}} \right) \Big|_{-\infty}^{\infty} = \frac{1}{\sqrt{\phi}} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = \frac{\pi}{\sqrt{\phi}}. \end{aligned}$$

Therefore, we have our answer:

$$\boxed{\int_0^{\infty} \tan^{-1} \left(\frac{2}{x^2 + 1} \right) dx = \frac{\pi}{\sqrt{\phi}}}.$$

Closing remarks.

- We have used only “high-school techniques” in finding out the answer.
- I first saw this problem in a post by Prof. Brian Sittinger, in Quora.
- An aspect of this problem which needs to be mentioned is that the appearance of the golden ratio in the answer is due to the 2 in the numerator of the original integral. Thus, we would not be able to get this compact form had there been any other natural number in place of 2, i.e., if we had the ‘general’ form of the integral. Thus, the 2 plays a crucial role.

Readers interested in exploring more such beautiful integrals involving special functions and mathematical constants could refer to [1], [3], [5] and [6].

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Responses to: Did Calculus Change the World?

I. The Survey

- (i) Could you briefly describe for a layperson the problem(s) you work on?
- (ii) Can you explain how calculus plays a part in your work (again as non-technically as possible)?
- (iii) Steven Strogatz begins his book, *Infinite Powers*, with the following claim (page vi of the Introduction)

"Without calculus, we wouldn't have cell phones, computers, or microwave ovens. We wouldn't have radio. Or television. Or ultrasound for expectant mothers, or GPS for lost travellers. We wouldn't have split the atom, unravelled the human genome, or put astronauts on the moon. We might not even have had the Declaration of Independence.

It's a curiosity of history that the world was changed forever by an arcane branch of mathematics."

Later on he tempers the above statements by saying (page x of the introduction):

"...This is a much broader view of calculus than usual. It encompasses the many cousins and spinoffs of calculus, both within mathematics and in adjacent disciplines. Since this big-tent view is unconventional, I want to make sure it doesn't cause any confusion. For example, when I said earlier that without calculus we wouldn't have computers and cell phones and so on, I certainly didn't mean to suggest that calculus produced all these wonders by itself. Far from it. Science and Technology were essential partners-and arguably the stars of the show. My point is merely that calculus has also played a crucial role, albeit often a supporting one, in giving us the world we know today."

(on page xi after describing Maxwell's work on electromagnetism)

"Clearly, calculus could not have done this alone. But equally clearly, none of it would have happened without calculus. Or, perhaps more accurately, it might have happened, but only much later, if at all."

What is your opinion about this?

II. Professor Ravi Prakash Jagadeeshan

The problems I work on: I am interested in a field known as "Polymer dynamics and rheology."

Polymers are very large molecules, composed of many smaller molecules called monomers. They are usually very long, like a piece of string or spaghetti, and in this case they are called linear polymers. However, they can also have complex architectures such as stars, or rings with no ends, or have branched topologies like combs and even have hierarchically branched structures like dendrimers, all made up of many monomers attached together.

I am interested in the properties of solutions of polymers, and the way they behave when they flow.

Knowing this is important for various reasons. All the plastics around us are processed with polymers either in solution or in the melt state. We may want to know how they behave when they go through various types of machinery, such as pumps or extruders. What is the rate at which they will flow, how much force will they exert on the walls of the geometry they are flowing in. How much force should we apply to push the solutions through such devices, etc. Importantly, most molecules in biology are polymers as well, such as DNA, proteins, etc. Many physical properties of biological systems can be understood by studying them from the perspective of polymer physics. For instance how does DNA fold itself into the very small dimensions of a cell's nucleus. How does a protein fold into its final state after it is made.

The 3D structure of a protein is extremely important for its function.

The aim of my research is to develop molecular theories using simple models for polymers, such as beads connected to each other with springs, and to use these theories to predict the behaviour of solutions. Such simple models account for the fact that a polymer is a long chain molecule made up of many monomers (or beads), that a polymer experiences the drag when it moves through a solution, and that it can be stretched and oriented.

Many properties of polymer solutions only depend on these aspects of polymers; the details at the level of chemistry don't affect the qualitative behaviour when the properties involve large length and time scales of the order of the size of the polymer and not of the monomers of which they are made. For instance, the viscosity of a polymer solution, which describes how easily the solution will flow if one deforms it, is such a macroscopic property.

Once we have a successful molecular theory we can run computer simulations to predict how a system composed of polymers will behave, and we can study many of the problems I gave earlier as examples.

On the role of calculus: Once we have developed a model for a polymer, we want to know how an ensemble of polymers behaves as a function of time. To do this we essentially use Newton's second law of motion: The mass times acceleration of each polymer is equal to the sum of forces acting on it. These forces include the force from all the other polymers around it, and from the molecules of the solvent. Acceleration is the rate of change of velocity, and velocity is the rate of change of position. So in order to apply Newton's second law we have to solve differential equations in order for us to find the position and velocity of each polymer as a function of time. This is where calculus enters the picture! The numerical solution of these equations with computers is usually called Molecular Dynamics Simulations.

Once we know how a system evolves in time, that is, once we know the position and velocity of every part of the system as a function of space and time, we can then calculate all the properties we are interested in. This methodology of finding the macroscopic properties of a system, as one would measure in a laboratory, from the behaviour of many molecules on a microscopic scale, is called Statistical Mechanics, which is a very beautiful subject, and a great achievement of mankind. Basically, one has to carry out averages of the properties of many many molecules, which involves numerical or analytical integration. Calculus again!

Often one is not interested in the motion of all the solvent molecules, but only in that of the polymers. The solvent molecules are then replaced with an effective force. Since they keep bombarding the polymer in random directions and at random times, the force is assumed to be a random noise. In this case, the differential equations for the polymer motion are no longer deterministic, but rather they become stochastic differential equations. The numerical integration of these equations needs entirely different methods from conventional calculus, and the field is called Stochastic Calculus. As you can imagine, the trajectory of each polymer is a random process in space, everywhere continuous but nowhere differentiable. Once we have integrated the stochastic differential equations, and we know the ensemble of trajectories for a system of polymers in time and space, we can again use Statistical Mechanics to solve for the properties of the solution as a

whole. Numerically integrating the stochastic differential equations using computers is known as "Brownian dynamics simulations".

Calculus is undoubtedly an integral part of everything we do in our research.

Regarding Strogatz's claims regarding

calculus: It is an exaggeration to say that calculus is at the heart of all scientific and technological progress, and to modern civilisation as we know it. It is definitely an integral part, and it is central to many of the achievements of mankind, but how does one separate one element of mathematics and give it the most prominent role? It would be more accurate to say that mathematics is central to modern society and all the technology and science we take for granted. Mathematics is the language with which we translate our mental construct of the world around us into a symbolic form that reveals the structure of the world and its order through the recognition of patterns. The representation of ideas in a symbolic form enables one to manipulate the symbols and make predictions, and this is the key to exploiting our understanding towards achieving practical outcomes. Mathematics has many branches, all of which seem necessary to create a symbolic representation of the world. Arithmetic, algebra, geometry, topology, logic . . . one could go on . . . surely there are many aspects of these subjects that do not have to do with the rates of change of variables, which is my understanding of calculus, and yet they are essential for the development of a mathematical framework.

Ravi Prakash Jagadeeshan is currently a Professor in the Department of Chemical Engineering at Monash University, Australia, where he heads the Molecular Rheology group. He has been at Monash since January 2001. Before joining Monash, Ravi was an Associate Professor at the Indian Institute of Technology, Madras, and did postdoctoral work on Sandpile dynamics with Prof. S. F. Edwards at Cavendish Laboratory in Cambridge, and on Polymer solution rheology with Prof. H. C. Öttinger at ETH Zürich. He was a Humboldt Fellow in the Techno-Mathematik Department at the University of Kaiserslautern in 1999/2000. His research is focussed on developing a theoretical and computational description of the flow behaviour of polymer solutions using a multiscale approach that combines molecular simulations at the mesoscopic scale with continuum simulations on a macroscopic scale. He is also interested in applying methods of soft matter physics to studying problems in biology.

III. Professor N. Mukunda

The problems I work on: My main interest has been in problems which are interesting from the physics point of view, in different areas which involve some amount of the mathematics of groups. This is a very important part of mathematics and it has overlap with various areas within mathematics and is also very important for problems in theoretical physics: classical mechanics, quantum mechanics, theoretical optics, statistical optics and quantum optics. All these areas have a group theoretical flavour. So I have been attracted to such problems. In addition I have looked at some examples of group representations which are suggested by physical ideas because they are in the same overall scheme.

On the use of calculus: Calculus comes into the picture among the groups that I have been interested in. That is the so-called continuous groups. There are other groups which are discrete, so the mathematics which goes with discrete groups is very different from that which we use when dealing with continuous groups. Continuous groups are also called Lie groups. For physical purposes the study of Lie groups involves calculus in a very basic way. This is one part of the answer. The other part is that in the second half of the 19th century there was a lot of work done in mathematics which led to what are called special functions of mathematical physics. These are solutions of important differential equations which arise in physical problems especially in the quantum mechanical framework. These functions are also closely related to group representations, I mean continuous group representation. All these ideas gel together: the use of differential equations

based on calculus, the use of group theory for continuous groups, and applications of this in interesting physical problems.

Regarding Strogatz's claims regarding calculus: Calculus played a crucial role in understanding the mechanics of moving bodies starting with Newton. Years later it was crucial in understanding the laws of electromagnetism in Maxwell's classical theory. These were the basis of a lot of engineering and technology in the early part of the 20th century. Then when quantum mechanics came one of its versions was the wave mechanics of Schrödinger, and it turned out that that too was based on differential equations. So the mathematics based on calculus has been crucial for quantum physics, for classical electromagnetism, for the classical description of motion in all these fields. It is also important in the context of general relativity based on Riemannian geometry and other kinds of geometry. In fact, the calculus in that context is called the absolute differential calculus. You can see even by the choice of the name the importance of calculus in relativity. So there is little doubt that calculus is most important for practical understanding and uses in all these fields. I don't know much about how it is related to the human genome or the Declaration of Independence.

However, calculus by itself cannot lead us to the laws of nature, like Newton's equation of motion, or Maxwell's equations, or wave mechanics, or relativity. These are independent inputs from physics, to express which calculus is the most convenient language. So, the basic physical laws are not determined by calculus, but when discovered are expressed in the language of calculus. This distinction is very important.

Professor N. Mukunda got his PhD in physics from the University of Rochester, NY, USA in 1964. He worked at the Tata Institute of Fundamental Research, Bombay from 1959 to 1972 and the Indian Institute of Science from 1972 to 2001. His research interests are in quantum and classical mechanics, optics and theoretical physics.

IV. Professor Shobhana Narasimhan

The problems I work on: I work in the area of computational materials design. I use the techniques of a field called 'density functional theory' to understand why materials have the properties that they do. I then use this understanding to design novel materials that

possess desired properties for specific applications. I focus on nanomaterials, that are either structured at the nanometer scale or are composed of just a few atoms.

On the role of calculus: The basic equations in my field are known as the Kohn-Sham equations. They are differential equations, similar to the Schrodinger equation, and hence based on calculus. This is used for static properties. If one wants to solve for dynamic properties, such as the trajectory of an atom over time, we usually integrate Newton's equation of motion. " $F = ma$ " may not immediately look like something that uses calculus, but since acceleration is the second derivative of position with respect to time, if one wants to find the position as a function of time, then if one knows the force at each time, one can find the position by integrating. I think the principle of calculus that I use most in my work is the idea that the minimum (or, more generally, extremum) of a function occurs where the first derivative of the function is equal to zero. I use this all the time in different ways. We use it to find the equilibrium geometry by searching for where the first derivative of the energy with respect to positions (which is equal to the force) is equal to zero. We also solve the Kohn-Sham equations by matrix diagonalization, and to diagonalize large matrices very fast on the computer, one can recast them as a minimization

problem. We also solve for the density (modulus squared of wave function) of the system we are working on in a self-consistent way, and that too is recast as a minimization problem. To find the minimum using a computer algorithm, we use various numerical techniques, many of them are modifications of techniques meant to search for the zeros of a function (since finding the minimum of a function is the same as finding the zeroes of its first derivative). All this makes use heavily of ideas from calculus, even though something like matrix diagonalization may seem to be more of a problem from linear algebra.

Regarding Strogatz's claims regarding calculus: Well, I am not an expert in the history of science. But certainly very many of the fundamental equations of physics (Newton's equations, Maxwell's equations, the Schrodinger equations, etc.) are differential equations and hence make use of calculus. I do not know if there are alternative ways of solving these problems. I remember reading that though Newton solved many problems in classical mechanics using calculus, he kept his invention of calculus secret and, once he found the answer, re-derived the formulae using older techniques. I suppose it might be possible to do something analogous for some of the problems mentioned, but it would be incredibly tedious, and I am not sure how accurate it would be. The one I do not know what to make of is the claim that the Declaration of Independence would not have been made were it not for calculus! I am not sure what Strogatz had in mind. Maybe if people hadn't been able to calculate the trajectory of a cannonball using calculus, the outcome of some battle would have been different, thus changing the course of history?

Shobhana Narasimhan has a MSc in Physics from IIT Bombay, PhD in Physics from Harvard University, Postdocs at Brookhaven National Laboratory and Fritz Haber Institut, Berlin. Since 1996 on the faculty of the Theoretical Sciences Unit, Jawaharlal Nehru Centre for Advanced Scientific Research. Area of research is computational nanoscience, using density functional theory.

V. Professor Madan Rao

The problems I work on: I am a theoretical physicist working in the area of Statistical physics, both equilibrium and non equilibrium, and Soft Matter physics. Of late I have been engaged with understanding the physical principles underlying living systems across scales. We study the interplay between active mechanics, molecular organisation, geometry, and information processing in a variety of cellular contexts. We are interested in how living systems, composed of physical entities such as molecules and molecular aggregates, driven far from equilibrium, have self-organised (evolved) to perform engineering tasks, such as the efficient processing of information, computation, and control. This potentially brings together many fields of research, including nonequilibrium statistical physics, soft active mechanics, information theory, and control theory, to the study of biology.

On the role of calculus: Newton's math and its descendants pervade every aspect of my work and every physicist's work – Calculus, Real and Complex Analysis, Statistical physics and Probability/Information theory, Approximation and Numerical analysis, Dynamical systems, Differential geometry of curves and surfaces, and Partial and Ordinary Differential equations.

Regarding Strogatz's claims regarding calculus: There is a consensus amongst mathematicians that Newton is amongst the top 10 mathematicians of all times, and I do remember a poll that declared that Newton was at the top of this list (I am afraid I can't recall who conducted the poll, possibly AMS?). (If I recall correctly, the list has Archimedes, Aryabhata, Newton, Euler, Gauss, . . .) Why is this so? In my opinion there are three parts to the answer – Natural Philosophy, Mathematics and as a Language of Science (Physics). Natural Philosophy: Briefly, Aristotelian natural philosophy posited that nature could be comprehended by rational thought, by the application of hard logic. Newtonian philosophy

went beyond this and asserted that nature could not only be comprehended, but that nature was, in principle, calculable and hence predictable, using the language of mathematics. Mathematics: While Newton's contribution to mathematics is many fold, Calculus is indeed one his priceless contributions. This includes the notion of infinitesimals, the notion of limits – through which one may get at hidden features of functions. Newton then systematically developed the ideas of Differentiation, Integration and the fundamental theorem of calculus. This leads directly to Analysis (...Jacobi, Bernoulli), Non Euclidean geometries, (Differential) geometry of curves and surfaces, higher dimensional spaces (...Gauss, Riemann) that brings together geometry and analysis, Series expansions (...Euler, Jacobi), Approximations and numerical analysis (...Gauss), Differential equations – local and global analysis leading to dynamical systems, Vectors / Tensors and their analysis, Continuous distributions and probability theory, Measure theory, ...and many more. Almost the whole edifice of mathematics.

Physics: Newton heralded a period which saw an intimate relation between physics and math. His deep philosophical insight, standing on the shoulders of Galileo, was to see and make precise Kinematics – the nature of space and time – as the underlying foundation on which a Mechanics of the Movement of Bodies could be built. With the relativity principle and inertial frames, Galileo and Newton potently refuted Aristotelian natural philosophy. Galileo however could make progress in rectilinear movement and simple curvilinear movement. The general motion of bodies needs the concept of instantaneous velocity, which in turn can only be defined by a limiting procedure and differential calculus – the hidden mathematical kernel that was gleaned by Newton. Newton's proofs combine the calculus he discovered with geometry, and are considered a thing of beauty (see book by S. Chandrasekhar). Newton's creative mind shed light on every aspect of the natural (non-living) world that he could perceive at that time.

Madan Rao is a theoretical physicist at the Simons Centre for the Study of Living Machines, National Centre for Biological Sciences (TIFR), Bangalore and affiliated with the International Centre for Theoretical Sciences (TIFR). He received his Ph.D from Indian Institute of Science, Bangalore in 1989 and post-doctoral training at Simon Fraser University, Vancouver. Subsequently, he held positions at the Institute of Mathematical Sciences, Chennai and Raman Research Institute, Bangalore. Madan works in the areas of nonequilibrium statistical physics, and soft and biological physics. At the Simons Centre, Bangalore, he has helped build a vibrant centre for Theory in Biology.

VI. Professor Prajval Shastri

The problems I work on: I investigate how giant black holes that are in the centres of galaxies like our Milky Way behave, by imaging their environs using telescopes. A black hole that is over a million to several billion times the mass of our Sun typically inhabits the centres of galaxies, even to very early cosmic times. They draw in any matter that approaches very close to them, accelerating it enormously and swirling it in, and consequently this matter is heated to enormous temperatures to become a beacon in the sky. These beacons are picked up by telescopes on earth even from billions of light years away. We can therefore image them, how they change, how they influence the evolution of their host galaxies, and also how they are formed.

On the role of calculus: Calculus is so integral to a physicist's thinking that most often we are not even consciously thinking "I am using calculus here". Mathematics enables us to describe the phenomena that we observe and measure, in a self-consistent language, that in turn enables us to both unearth something deeper than what an individual phenomenon suggests, and anticipate phenomena that we can observe and measure (or not, and thereby refute the earlier descriptions). In my own work, calculus plays a role in the tools used as well. For example astrophysicists use a technique of measuring the Fourier components of the Fourier transform of a piece of the sky, in order to reconstruct the original image at a level of sharpness that would otherwise be impossible, which is akin to constructing the sound of a flute from a whole bunch of "pure" or "tuning-fork-type" notes. Calculus plays a key role in this technique.

Regarding Strogatz's claims regarding calculus:

Exceptionalizing calculus in technological developments unnecessarily undermines the importance of tinkering and trial-and-error in engineering. For example, while GPS technology would not work without taking into account predictions of General Theory of Relativity, in a hypothetical situation where we had no General Relativity but only satellite-borne transmitters and receivers, it is quite possible that the need to offset transmitted and received frequencies (i.e., the General Relativistic offset) might have been arrived at by mere tinkering.

Knowledge creation is a process of building upon and constantly testing, interrogating and reaffirming (or not) past knowledge, and is perforce a collective, co-operative human endeavour. Therefore to create straw-persons that privilege some particular branch or toolkit that has contributed to the current positive aspects of the human condition becomes merely polemics. Furthermore, the extent to which any particular insight from any discipline realises its potential by impacting global knowledge creation is heavily dependent on the socio-political context in which the insight emerged. Therefore, in evaluating the role of mathematics to our current understanding of the world, it is far more useful to (a) evaluate the paths that specific insights took in their socio-political-historical context, and (b) foreground the pitfalls of how privileging mathematics can lend false credibility to predictive mathematical modelling, especially of environmental, public health, economic and other complex systems critical to human well-being, when the modelling could be fundamentally flawed because of flawed assumptions.

Prajval Shastri is an astrophysicist, formerly at the Indian Institute of Astrophysics, Bengaluru. Her core research interest lies in empirical investigations of supermassive black holes. Her core angst is discontent with early science education. Acutely aware of her privilege of being paid to be fascinated by the universe, Prajval is continually amazed that amateur astronomers often stand out as the impassioned ones. She can be contacted at prajval.shastri@gmail.com.

VII. Avehi Singh

The problems I work on: I am studying the ways in which bees find flowers using their sense of smell. Flowers emit blends of multiple volatile chemicals that have been shown to attract insects to them. For my work, I am testing whether individual chemicals from flower scents are detected by the brains of bees. Information is passed through the insect brain in the form of small electrical pulses (not exceeding 70 millivolts or 0.07 volts). By measuring the voltage change across the antenna (this is the primary organ of smell in insects) in response to a particular chemical from a flower, I identify chemicals that may be helping bees find these flowers. For example, if a chemical induces a particular voltage change in the antenna, I conclude that the insect brain is detecting it and can subsequently test whether this chemical produces any behavioural response in the bees.

On the role of calculus: A main challenge of using this technique (called electrophysiology) is differentiating between signal and noise. The voltage fluctuations caused by brain activity are very small (for reference, the voltage across an ordinary light bulb is 120 volts) and transient. A sensitive electrode can detect changes in voltage caused by humidity, static and any number of other random factors, which make it hard to identify brain activity. In order to determine whether a voltage change is caused by brain activity and not noise, we use mathematical models that can describe these electrical pulses. A main class of these models are the Hodgkin-Huxley models, which are a series of ordinary differential equations that describe the flow of current into and out of brain cells during brain activity. By using these equations, we can parse the output from the antenna into signal and noise.

Avehi Singh is a first-year PhD student in the Ecology program at Penn State University. She has a bachelor's degree in Biology from Reed College, Oregon. Her PhD research is on the evolution of sensory systems in pollinating insects and she is interested in interdisciplinary research questions that span biology, chemistry and mathematics. She attributes her interest in science to her schooling in Bangalore, where she spent many hours exploring the beautiful deciduous forests around the city. She enjoys hiking, cooking and spending time with her dogs.

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

Foundational topics include, but are not limited to, the following:

- Number systems, patterns and operations
- Fractions, ratios and decimals
- Proportional reasoning
- Integers
- Bridging Arithmetic-Algebra
- Geometry
- Measurement and Mensuration
- Data Handling
- Probability

We invite articles from teachers, teacher educators and others that are helpful in designing and implementing effective instruction. We strongly encourage submissions that draw directly on experiences of teaching. This is an opportunity to share your successful teaching episodes with AtRiA readers, and to reflect on what might have made them successful. We are also looking for articles that strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

Articles in this section may address key questions such as -

- What challenges did your students face while learning these fundamental mathematical topics?
- What approaches that you used were successful?
- What preparations, in terms of knowing mathematics, enacting the tasks and analysing students work were needed for effective instruction?
- What contexts, representations, models did you use that facilitated meaning making by your students?

Send in your articles to
AtRiA.editor@apu.edu.in

Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University

holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to 'At Right Angles'.

While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

The Closing Bracket . . .

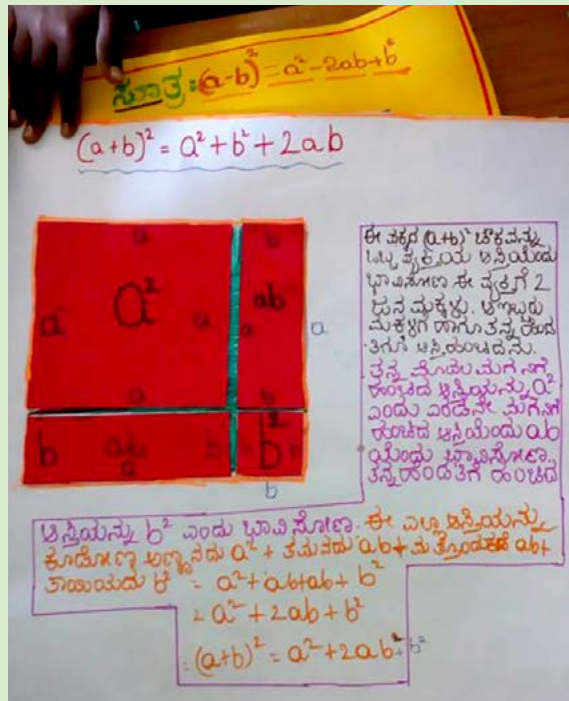
This column attempts to highlight individual teachers who make a difference to their pedagogical practice with reflective practice, innovative tinkering and a constant desire to improve their students' understanding and appreciation of the subject they teach.

Geetha R is one such, she teaches at Sri Sarada Devi Vidya Kendra, her alma mater. This is her first year of teaching and she currently teaches grade 3 EVS and grade 5 Mathematics. Speaking of her own student life, Geetha speaks frankly about her pursuit of a Master's course in Mathematics and how she found that with each year of study, she found the level of abstraction increase in leaps and bounds. The math that she did in school was so different from the math she did in university. While in college, she kept asking for representations which would enable her to visualise what she was learning. At the same time, she realised that there were many concepts in school mathematics which she had engaged with even without really visualising them.

While doing her M.Sc. Geetha, along with a few of her classmates, visited government schools with puzzles and problems to share the joy of learning math. As she engaged with school mathematics, she began to see different ways of visualising abstract concepts at that level. This made her want to become a teacher and to teach differently, so that students could visualise, could represent, could communicate and engage with mathematics in ways that made them realise the power and beauty of the subject. On the next page, are a few pictures of the manipulatives and tools that she, along with her friends, designed and used over this year. We salute this young teacher's spirit and wish her all success in her journey of discovery.



A geoboard created with a wooden plank by a local carpenter, she uses it not just to teach shapes-but also multiplication, factorisation, area, perimeter, angles and more.



The story of a farmer dividing his field, it helped to reinforce a much-used algebraic identity. (Text in Kannada.)



Painted sponge cubes used to make different tiling patterns, leading to better understanding of the types and properties of quadrilaterals and triangles.

A game played with marbles can be used to practise the four operations; the problems vary depending on the numbers touched by the marble as it is dropped along the white board. (Notice that the numbers can be changed depending on the age of the students.)



NUMBER BASES

PADMAPRIYA SHIRALI



**Azim Premji
University**

A publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley

NUMBER BASES

Students learn to count, read and write numbers in the decimal system for the first seven years of their schooling. They also learn how to read analogue clocks and use Roman numerals, and they may come across words such as feet, dozen and pound.

If they are to gain a good understanding and appreciation of the decimal base system, it is desirable that they are exposed to other number base systems like binary and hexadecimal, and do some operations in non-positional systems like the Roman system.

Through this exercise, students begin to understand the underlying structure of the positional system and its usage in other number bases. It helps them realise that the decimal base system is only one amongst other possible number systems. My experience of teaching number bases to students of class 6 has always been very rewarding, and I recommend it to all maths teachers of classes 6 and 7.

The Mohenjo-Daro culture of the Indus Valley was using a form of decimal numbering some 5,000 years ago as weights: $1/20$, $1/10$, $1/5$, $1/2$. The poet-mathematician Pingala (3rd/2nd century BCE) developed the binary number system for Sanskrit prosody, with a clear mapping to the base ten decimal system.

It is, of course, well known that the invention of zero as a number happened later in India. (The notion of zero did arise in some earlier cultures, but it served only as a placeholder, and it never entered into any arithmetical operations.)

The Roman system, which is not a positional system with its many symbols (I, V, X, L, C, D and M), is cumbersome and difficult to use as the numbers begin to grow in size. To write a million, one would have to use a thousand M's! Creating more and more symbols poses its own difficulties. Another major difficulty with the Roman system is the complexity of doing number operations. Try adding CLMDV to LDVXC, and you will see this for yourself!

Historically, vocations like carpentry and masonry needed facility with fractions. Wrenches are made in inches with half, quarter, eighth and sixteenths as their measures. Working with these measures and performing mathematical operations can be difficult, e.g., when fractions have to be multiplied. Yet, in many areas we continue to use fractions. Usage of fractions is evident in pizzas: a pizza is generally divided into 8 pieces. It would be difficult to divide a pizza into ten equal pieces!

In measurement of weight and volume, base 16 has often been used. An ounce equals 16 drams, a pound is 16 ounces, a cup equals 16 tablespoons, and a gallon is 16 cups.

Base 12 is familiar to us in the measurement of time, in cooking, etc. We often count items such as fruits and eggs in dozens. A dozen dozens is called a *gross*. And there are 12 months in a year. <http://en.wikipedia.org/wiki/Duodecimal>

In today's world, computers use the binary system as binary systems can be easily represented as on/off in electrical circuits. The input is either a zero or a one. This simplifies the information as there are only two states of representation. Computers also use groups of four digits, eight digits, and sixteen digits. Easy conversions between binary systems and octal or hexadecimal systems aid in the computations. However, we will not discuss this connection in this article.

While we have used varied systems through history, for everyday purposes we generally use the decimal system. One must see, however, that each system has its advantages and disadvantages. In some systems, the number representations may be very long (as there are too few symbols), while in some other systems, there may be too many symbols.

HOW DOES THE DECIMAL SYSTEM COMPARE WITH OTHER NUMBER BASED SYSTEMS?

Finger counting: The fact that human beings have ten fingers makes counting in tens easier. Counting in other bases using the fingers can prove to be difficult.

Length of representation: A binary number system leads to long representations of numbers. For instance, the base 10 number 365 is written in base two as 101101101. The same number in the hexadecimal system is written as 16D. These systems are explained further down in the article.

Number of symbols: If the number of symbols is large as in the case of the hexadecimal system (0 to 9, A, B, C, D, E, F), then one has to commit to memory several symbols and learn to handle many more operations. For example, we must know the values of $C + F$, $D \times E$, and so on.

In daily life, one needs to choose a base for which representations are not too long, and at the same time the number of symbols is manageable. If we

had to make a choice today, perhaps the decimal system will continue to be an obvious choice.

However, the binary, hexadecimal, and duodecimal systems have their rightful place in many areas.

In this article, we will explore the binary, hexadecimal and duodecimal (base twelve) systems.

Comment

I prefer to help students discover the underlying common structure by playing with and manipulating the numbers. As a second step I connect it with their understanding of writing numbers in exponential form and discovering the common structure. Others might prefer to explain the underlying common structure that holds for all number bases at the beginning and then develop the topic. I will leave this choice to the reader.

BASE 2: BINARY SYSTEM

The binary system has been used in different forms in the distant past, in ancient civilisations like Egypt, China and India. In the recent past, it was Leibniz and George Boole who studied these systems and worked on them.

https://en.wikipedia.org/wiki/Binary_number

The word 'Bi' means *two*. It is a system with two digits. Each digit is referred to as a bit (from binary and digit).

A single binary digit is called a 'bit'. 1001 is read as 'one, zero, zero, one'. It is four bits long.

Binary system or base two system implies the usage of two digits 1 and 0 in the framework of a positional system.

How do we represent different numbers in this system?

- Obviously, 1 (base ten) gets represented by 1, and a zero by 0.
- To represent 2 (base ten), a new place is needed; 2 is represented by 10.

The teacher can bring in the connection with the bundling idea that students have learnt in their early years. Ten units are bundled to make one ten which is represented in a new place as 10. Ten tens are bundled to make one hundred which is represented as 100.

Similarly, in the binary system, a set of two is bundled to make 10. Since 3 (base ten) is one more than 2 (base ten), 3 (base ten) is represented by 11.

4 (base ten) which is two sets of two is bundled to make a 100.

5 (base ten) is one more than 4 (base ten), hence is represented in base two by 101.

Decimal number	0	1	2	3	4	5	6	7	8	9	10	11	12
Binary number	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100

The sequence would begin to look like this:

Students should be encouraged to build a table like this, in sequence, till 25 (base ten).

Also, they need to maintain the sequence; that is, after reaching (say) 1000, the whole process begins again, starting from the rightmost place: 1001, 1010, 1011, etc. Do the students see the similarity between this and the way decimal numbers increase: 101, 102, 103, 104...?

After writing numbers till 25 in binary form, the students should answer these questions:

At which points did we have to create a new place? What would be the next number that requires a new place? Check and see. It turns out to be 32.

Observe the pattern and determine what would be the next number after 32 that requires a new place. It is 64.

What can you say about these numbers?

2, 4, 8, 16, 32, 64, ...

They are powers of 2. Each power of 2 becomes a transition point to a new place. Notice that in the decimal system too, each power of 10 becomes a transition point.

After 9, to write ten (10^1), we require the tens place (which is a bundle of 10 units). To write a hundred (10^2), we again require a new place

(i.e. 10 bundles of ten units). Similarly, to write a thousand (10^3), we require a new place yet again (i.e. 10 bundles of hundred units).

Students can contrast the positional system of the Binary and Decimal numbers.

Students need to observe the similarities to understand the underlying structure of different based systems by studying the figure below. In a decimal system what would replace the numbers from 2^7 to 2^0 ?

What digits may appear instead of just 0 and 1?

Here is a decimal number: 59012.

Note: The word *coefficient* will need to be explained.

$$59012 = 5 \times 10^4 + 9 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 2 \times 10^0$$

Look at the coefficients in this expansion. What are they? They are the digits 0, 1, 2, 5, 9. In a decimal number the coefficient can be any digit from 0 to 9.

Here is a binary number: 10110011.

$$10110011 = 1 \times 2^7 + 0 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$

Look at the coefficients in this expansion. What are they? They are the digits 0, 1. In a binary number the coefficients can be either 0 or 1.

What is common to the two expansions?

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	0	1	1	0	0	1	1	
1×2^7		1×2^5	1×2^4			1×2^1	1×2^0	
128		32	16			2	1	

$$128 + 32 + 16 + 2 + 1 = 179$$

DRAWING PATTERNS WITH BINARY NUMBERS

Build up a table of binary numbers. Shade all the 1's. Does it give rise to a nice pattern?

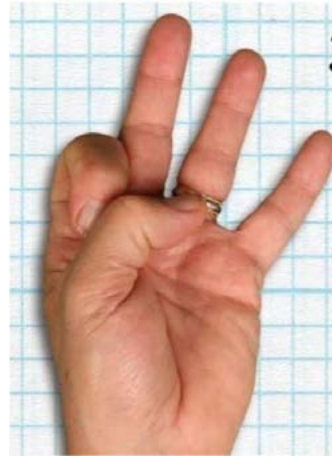
0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

Fun activity

How many numbers can you count using your fingers? Most students will answer 'ten' as we have ten fingers.

Show the students how we can use our fingers to count till 31 using just one hand, and till 1023 using both hands!

Use the fingers as binary digits starting from the little finger. If the finger is down, it is a 0. If it is up, it is a 1. Each finger represents a power of 2 if it is up.



Thumb down, pointer finger down, middle up, ring finger up, little finger up. What number is that? 7.



Thumb down, pointer finger up, middle finger up, ring finger up, little finger down. What number is that? 14.

With ten fingers, it is possible to show all the numbers from 0 to 1023.

BINARY CONVERSIONS

Can the students now convert a binary number to a decimal number? Their facility in doing this conversion will reveal their understanding of the structure. In the process of converting binary to decimal, they will use powers of 2 and expanded notation.

Would they be able to convert a decimal number to a binary number?

Figuring out the process of converting decimal to binary is an interesting investigation to try in the class. Will the students use the highest power of 2 less than the given number and work out the answer, step-by-step? Example: Take 1050. The power of 2 closest to and less than 1050 is 1024. When that is removed, the remainder is $1050 - 1024 = 26$. The power of 2 closest to and less than 26 is 16. When that is removed, the remainder is $26 - 16 = 10$. Then we have 8, leaving remainder 2. So the number 1050 in binary form is

$$1 \times 2^{10} + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^1.$$

Therefore, 1050 (base ten) is 10000011010 in base two.

By writing a number as sum of powers of 2, students can study the polynomial structure of base two:

$$25 = 16 + 8 + 1 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0.$$

This can be connected to the division approach and the usage of the remainder at each stage.

$$\begin{aligned} 25 &= (2 \times 12) + 1 \\ &= 2 \times (2 \times 6) + 1 \\ &= 2 \times (2 \times (2 \times 3)) + 1 \\ &= 2^3 \times (2 + 1) + 1 \\ &= 2^4 + 2^3 + 2^0 \end{aligned}$$

Ex. To convert 25 (base ten) to binary, divide repeatedly by 2 and note the remainders.

Write the result bottom up. We get 11001:

2		25	
2		12	
2		6	
2		3	
2		1	
		0	
			1
			0
			1
			0
			1

OPERATIONS IN THE BINARY SYSTEM

Students can explore addition and subtraction in the binary system to understand the usage of carry over, exchange with fours, twos, etc.

What is of interest is to see the same mechanism at work as in the familiar decimal system.

This can be demonstrated through a few problems requiring addition and subtraction operations.

$$\begin{array}{r} \\ 01010 \\ + 11101 \\ \hline 100111 \end{array}$$

What is $0 + 1$? What does $1 + 1$ become? What digit goes as a carry over?

$$\begin{array}{r} 10 \\ - 101 \\ \hline 1 \end{array}$$

In the subtraction problem, why has a 10 replaced the zero in the right most place?

Are these addition and subtraction methods similar to the methods of the decimal system?

Do commutativity, associativity, transitivity and distributivity laws hold for binary numbers?

MULTIPLICATION

How does multiplication work out in the binary system? Is there an identity element in binary numbers?

Here is a sample multiplication problem in the binary system.

$$\begin{array}{r}
 110 \\
 \times 11 \\
 \hline
 110 \\
 1100 \\
 \hline
 10010
 \end{array}$$

What are the possible multiplications that can arise in a binary system?

$$0 \times 0 = 0$$

$$0 \times 1 = 0$$

$$1 \times 1 = 1$$

So much simpler to memorise these tables!

Here is a sample binary division problem. In decimal terms, it would be the same as 45 divided by 5 which equals 9.

Does the usual division method work in the binary system? Explore and see.

Are there fractional numbers in the binary system? What would .1 mean in the binary system?

$$\begin{array}{r}
 101 \overline{) 101101} \quad (1001 \\
 \underline{(-) 101} \\
 101 \\
 \underline{(-) 101} \\
 0
 \end{array}$$

Just as the numbers to the left of a decimal point in the decimal system are whole numbers, the numbers to the left of a decimal point in the binary system too represent whole numbers.

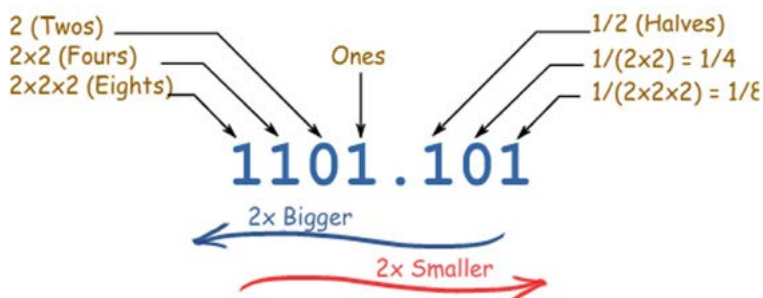
As we move to the left, with each step, place values get doubled (i.e., they are twice as big). As we move to the right, with each step, place values get halved (i.e., they are half as big). Hence the first digit on the right (.1) means 'half', and .01 means 'quarter'.

How does one identify a number as binary? In order to indicate the base, the practice followed is to show the base as a subscript. For example, 1011_2

Investigation

Do the binary representations of all rational numbers terminate? Or do some recur?

Binary positional system



BASE 16: HEXADECIMAL NUMBERS

The word 'Hexadecimal' means 'based on 16.' The value of the base in the hexadecimal system is 16.

What are the basic digits in the hexadecimal system?

Apart from 0 to 9, 10 is represented by A, 11 by B, 12 by C, 13 by D, 14 by E, and 15 by F.

Decimal number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Hexadecimal number	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10
Decimal number	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
Hexadecimal number	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F	20	21

Let students experiment with hexadecimal numbers by converting hexadecimal numbers to decimal numbers, and decimal numbers to hexadecimal numbers.

Let them build the positional table for a hexadecimal system and use it for conversions.

16^3	16^2	16^1	16^0
1	B	7	E
1×16^3	11×16^2	7×16^1	14×16^0
4096	2816	112	14

$4096 + 2816 + 112 + 14 = 7038$

What would the value of .1 be in the hexadecimal system?

In the hexadecimal system, as we move left, each place is 16 times bigger. The first number to the right of the point is one-sixteenth. As we move right, each place is 16 times smaller.

How would fractional numbers like 1/2 or 1/4 get represented in this system?

How would negative numbers get represented?

Students can check whether the procedures used for the binary system conversions work in this setting. How will they adapt the procedures to suit this system?

Here is a sample division procedure to generate the hexadecimal number for 447. The result is read bottom up as 1BF. (Remember that the symbols in this system are the digits 0 to 9 and A, B, C, D, E, F.)

Division	Quotient	Remainder
$447 \div 16$	27	$15 = F$
$27 \div 16$	1	$11 = B$
$1 \div 16$	0	$1 = 1$

Here are sample addition and subtraction problems for hexadecimal numbers. (The digits shown in red are the 'carry-overs'.)

Sample addition problem	Sample subtraction problem
$ \begin{array}{r} 1 \quad 1 \\ 7 \quad F \quad B \quad 3 \\ + 1 \quad B \quad 6 \quad 2 \quad 1 \\ \hline 2 \quad 3 \quad 5 \quad D \quad 4 \end{array} $	$ \begin{array}{r} E \quad 21 \quad 6 \quad 26 \\ 3 \quad F \quad 5 \quad 7 \quad A \\ - C \quad 8 \quad 5 \quad E \\ \hline 3 \quad 2 \quad D \quad 1 \quad C \end{array} $

What does a carry-over of 1 represent in this system?

In the subtraction problem, why is there a 26 over A?

Are these addition and subtraction methods similar to the methods of the decimal system? In what way are they similar? In what way are they different?

BASE 12: DUODECIMAL SYSTEM

Humankind has had a strong connection since ancient times with base 12, and it would be a pity if students do not get to study it.

The analogue clock has twelve hours displayed on its face. The number of hours in a day (24) is a multiple of 12. An hour has sixty minutes which is a multiple of 12. The number of degrees in a circle is 360 which again is a multiple of 12.

The segments on the four fingers are 12 in number, and can be used to count in base 12. It is said that the Babylonians used to count the three segments of their four fingers to get 12. They marked that 12 by raising a finger on the other hand. This way they could count up to 60 (twelve times five fingers being 60). Now 60 has excellent properties, being divisible by 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60 itself. This means that fractions will not pose too many difficulties!



<https://www.earthdate.org/how-10-fingers-became-12-hours>

Decimal number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Duodecimal number	1	2	3	4	5	6	7	8	9	A	B	10	11	12	13	14	15	16	17	18

Students can create a table of addition facts for base 12 as an extension in their study of operations in base 12.

	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	11
3	3	4	5	6	7	8	9	A	B	10	11	12
4	4	5	6	7	8	9	A	B	10	11	12	13
5	5	6	7	8	9	A	B	10	11	12	13	14
6	6	7	8	9	A	B	10	11	12	13	14	15
7	7	8	9	A	B	10	11	12	13	14	15	16
8	8	9	A	B	10	11	12	13	14	15	16	17
9	9	A	B	10	11	12	13	14	15	16	17	18
A	A	B	10	11	12	13	14	15	16	17	18	19
B	B	10	11	12	13	14	15	16	17	18	19	1A

Multiplication in base 12 produces nice patterns and makes for an interesting project.

	1	2	3	4	5	6	7	8	9	A	B	10
1												
2		4	6	8	A	10	12	14	16	18	1A	20
3		6	9	10	13	16	19	20	23	26	29	30
4		8	10	14	18	20	24	28	30	34	38	40
5		A	13	18	21	26	2E	34	39	42	47	50
6		10	16	20	26	30	B	36	40	46	50	60
7		12	19	24	2E	B	41	48	53	5A	65	70
8		14	20	28	34	40	48	54	60	68	74	80
9		16	23	30	39	46	53	60	69	76	83	90
A		18	26	34	42	46	5A	68	76	84	92	A0
B		1A	29	38	47	50	65	74	83	92	A1	B0
10		20	30	40	50	60	70	80	90	A0	B0	100

To study and contrast different number systems is enjoyable and stimulating for the students of an upper primary school.

CLOSING REMARK

In comparing base ten with other bases, there are two other factors which we did not consider earlier.

- **Tests of divisibility:** In base ten, divisions by 2 and 5 are very easy to carry out. This is because both 2 and 5 are divisors of 10. For the same reason, the tests for divisibility by 2 and by 5 are easy to understand and carry out in base ten. In general, for a given base b , the tests of divisibility by different numbers are easy to carry out when the numbers are divisors of b (or divisors of powers of b). Base 12 (duodecimal system) scores well in this sense, as 12 has many divisors (proper divisors: 2, 3, 4 and 6).
- **Terminating decimals:** In base ten, the decimal form of a fraction terminates when the denominator of the fraction is a product of powers of 2 and 5. If not, the decimal form is non-terminating (it recurs). One disadvantage of the binary system is that comparatively few fractions (those whose denominators are powers of 2) terminate. E.g., $1/10$ does not have a terminating binary representation.



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FORM IV

Statement about ownership and other particulars about newspaper (Azim Premji University At Right Angles) to be published in the first issue every year after the last day of February.

1. **Place of publication:** Azim Premji University Survey No. 66, Burugunte Village, Bikkanahalli, Main Road, Sarjapura, Bengaluru – 562125
2. **Periodicity of its publication:** Triannual
3. **Printer's Name:** Manoj P
Nationality: Indian
Address: Azim Premji University Survey No. 66, Burugunte Village, Bikkanahalli, Main Road, Sarjapura, Bengaluru – 562125
4. **Publisher's Name:** Manoj P
Nationality: Indian
Address: Azim Premji University Survey No. 66, Burugunte Village, Bikkanahalli, Main Road, Sarjapura, Bengaluru – 562125
5. **Editor's Name:** Shailesh Shirali
Nationality: Indian
Address: Community Mathematics Centre, Rishi Valley School, Rishi Valley – 517352, Madanapalle, A.P
6. **Name of the owner:** Azim Premji Foundation for Development
Address: #134 Doddakannelli, Next to Wipro Corporate Office, Sarjapur Road, Bengaluru - 560035

I, Manoj P, hereby declare that the particulars given above are true to the best of my knowledge and belief.

Date: 24th February, 2022


Signature of Publisher

Azim Premji University At Right Angles

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Azim Premji University together with Community Mathematics Centre, Rishi Valley.

Editors

Currently drawn from Rishi Valley School, Azim Premji Foundation, Homi Bhabha Centre for Science Education, Lady Shri Ram College, Association of Math Teachers of India, Vidya Bhavan Society, Centre for Learning.

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linguists and specialists in pedagogy being part of this community, posts are varied and discussions are in-depth.

Hard Copy

At Right Angles magazine is published in March, July and November each year. If you wish to receive a printed copy, please send an e-mail with your complete postal address to AtRightAngles@apu.edu.in

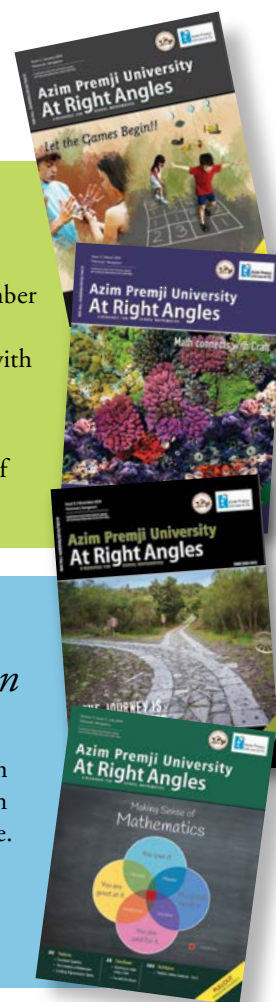
The magazine will be delivered free of cost to your address.

On e-mail:

AtRiA.editor@apu.edu.in

We welcome submissions and opinions at AtRiA.editor@apu.edu.in. The policy for articles is published on the inside back cover of the magazine.

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