Issue 14 | November 2022
Triannual | Bangalore

A publication of Azim Premji University together
with Community Mathematics Centre, Rishi Valley

RNI No.: KARENG/2018/76619

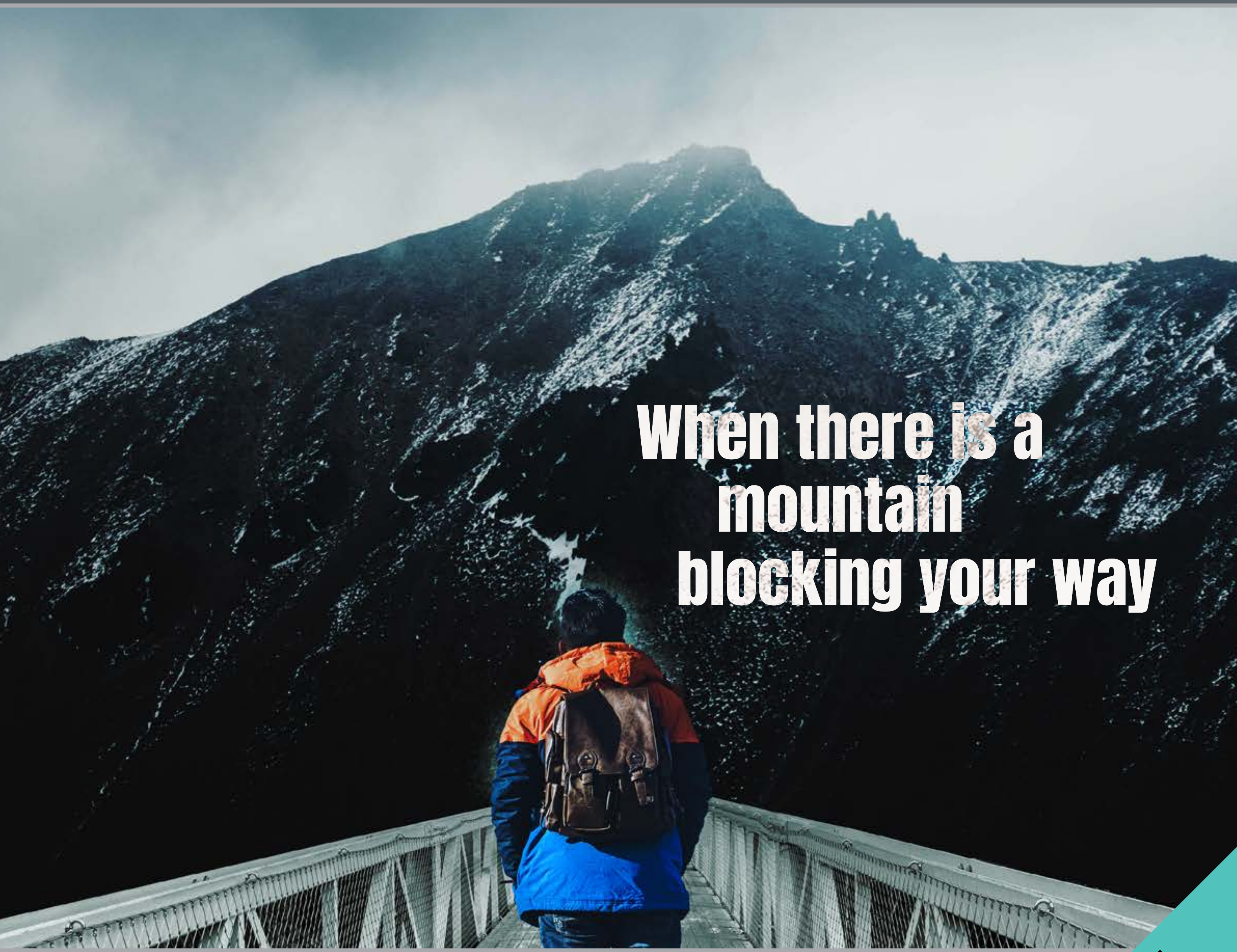
Community Math Centre Rishi Valley School (KFI)

Azim Premji University

Azim Premji University
At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

ISSN 2582-1873



PULLOUT
SCALE DRAWINGS

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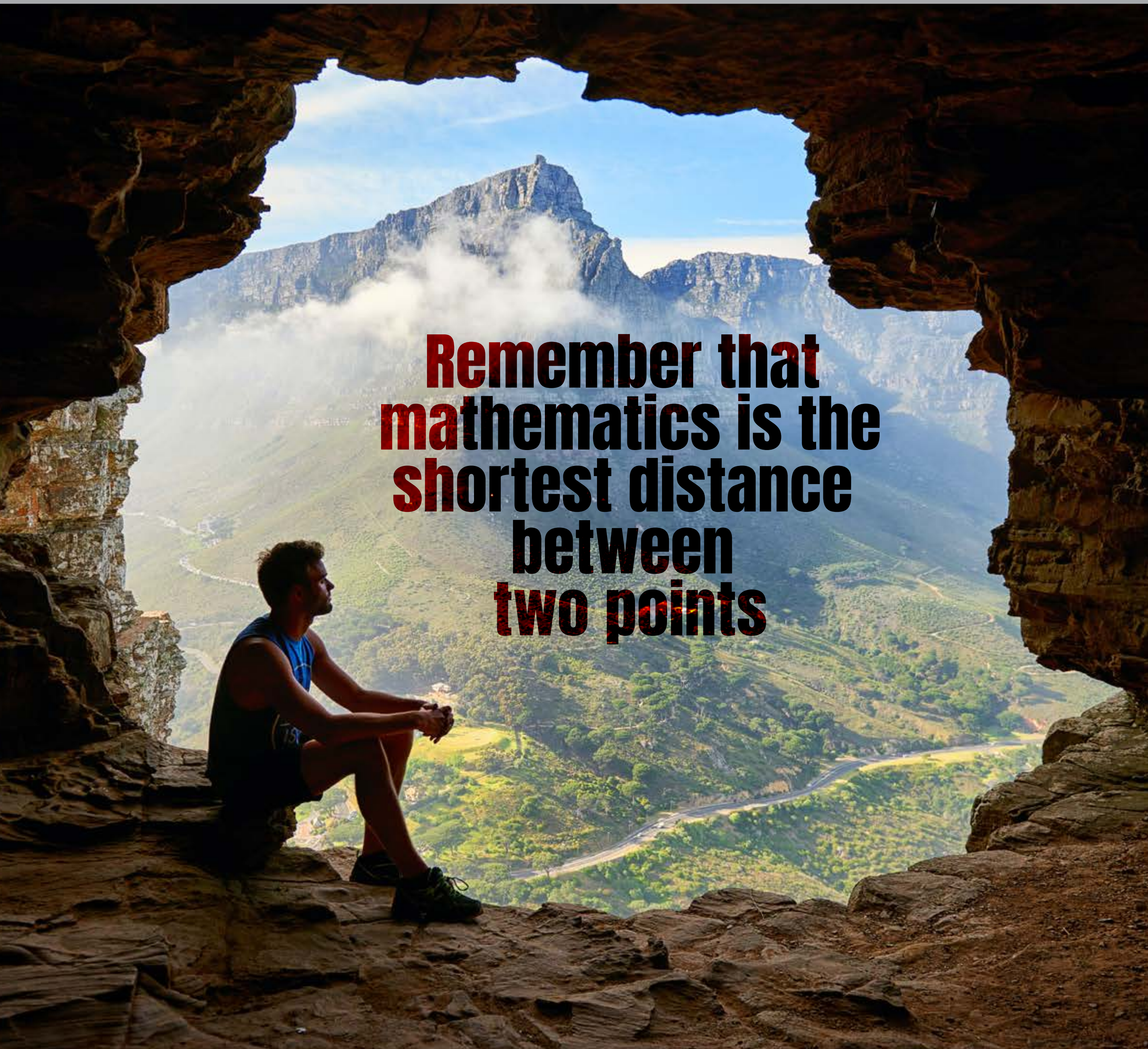

Problem Solving in mathematics is not just about finishing the exercises in the textbook.

It is about

- Framing the problem
- Identifying what needs to be solved
- Identifying the variables under our control
- Identifying the various constraints
- Connecting to concepts which have been learnt
- Applying these to arrive at a solution.

In between, we may look for analogies to solve the problem. Constructing a Tunnel through a Mountain is no big deal when it is approached with high school geometry and these problem-solving skills!

From the Editor's Desk . . .

Yet another year drawing to a close, with the usual festival flurry that tends to swamp our lives in this season. It's a good time to stop and reflect: in the March 2022 issue, we looked at Constructing Mathematical Understanding- the great Indian Rope Trick as we whimsically called it. In the July issue, we applied the Lens of Computational Thinking to the teaching of mathematics. It seems a good idea to close the year with a comprehensive Feature article that focuses on Mathematics and Life. What an intrinsic role mathematics plays in our life, beautiful in an unobtrusive yet pervasive way and what a pity that these attributes are so invisible in the mathematical experience of most school students. More power to the kind of teaching described in this article!

The ClassRoom section is full of the sort of articles which bring joy to the teaching of mathematics, from Sunil Bajaj and Jasneet Kaur's Harbans Puzzles which are based on the Japanese Ken-Ken puzzles to Math Space's TearOut on Symmetry. You can also find an article on Ganak, an innovative approach to teaching Place Value described by Pramod Maithil and the article featured on the cover of this issue, Radhakrishnamurty Padyala's Construction of a Tunnel through a Mountain.

In Problem Corner A Ramachandran poses some thought provoking problems on different number bases and Rajkumar Kanojiya proposes a method for constructing angles without the use of a protractor- the surprisingly good approximations to these angles are analysed by Shailesh Shirali in a companion article. Sasikumar defines and shares some properties of Repunit Numbers, and Anant Pratap Singh describes an attempt to make magic squares using square numbers. Explorations of Suma Numbers by Hara Gopal and of Mountain Numbers by young student Ragav Jayan Prabu give you a glimpse of what curious minds can stumble upon and the Intersecting Circles Investigation by Michael de Villiers, James Metz and Brad Uy brings in the advantages of using technology to fuel that curiosity.

The Professor and the Housekeeper- a book that is being much discussed recently is reviewed by student Nia Chari and the manipulative review by Math Space sheds light on an interesting and useful resource: ten-frames. We end as usual with the PullOut – this time it features Scale Drawing and brings the wheel full-circle from life to mathematics.

I hope you have fun reading this issue – remember to send in your feedback to AtRiA.editor@apu.edu.in

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

Azim Premji University
At Right Angles
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A Publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley and Sahyadri School, Pune

Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

Problem Corner

Student Corner

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Continue . . .

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Online Articles

Teaching Mathematics and Learning About Life…

A KUMARAN

I have been teaching high school Mathematics for a few years now. I always considered *teaching* as a sacred service, in engaging students, imparting valuable skills and creating enthusiasm about the subject; but over time I had come to realize that teaching students is as much or more of a *learning* opportunity for me. Here are my experiences as a teacher of mathematics and as a student of life. Invaluable in this journey, is the community of adults deeply involved in teaching and enquiry, in Shibumi and in KFI schools, and in particular, Kabir Jaithirtha, who helped me see and appreciate the richness of teaching, mathematics and life.

Teaching versus Learning…

I began volunteering a few years ago to teach mathematics in Shibumi. For the first few weeks, my primary activity was to attend the senior school math classes – actually to just sit and observe the discussions between the students and Kabir. In no time I realized that just sitting and observing was one of the hardest things to do in life. Quite often, I'd grab any opportunity to "teach" something to the students – lecturing on the topics or explaining away the steps, and often offering solutions even before the articulation of any questions! On such occasions, Kabir would generally veer the conversation away, leaving me feeling quite unfulfilled. Looking back now, it feels mildly comical, that I tried hard filling young minds with skills, instead of allowing them to flower naturally. But at that time, I was just too busy teaching mathematics.

In 2014-15, a school cultural program was arranged with the theme of mathematics (called *MathMela* or *Festival of Mathematics*) to let the children explore the beauty and versatility of mathematics.

Keywords: Mathematics and Life, Teaching Mathematics, Learning Mind, Philosophy of Mathematics, Philosophy of Mind, J Krishnamurti

It was a perfect opportunity for me, providing a structure to work with children on mathematical topics. Students picked, based on their interest or curiosity, various projects ranging from posters to programs, from poems to plays, from games to puzzles, and there were many opportunities to work with students on diverse topics. In one of the projects, I was trying hard to convince a student, that with some changes and extensions, the proposed project could become an excellent showcase of graph theory in the *Mela* and a great learning opportunity for him! But even after many sessions of cajoling, he would not budge! Kabir must have observed the struggle, and one afternoon we had a long conversation on what was happening. *Is the purpose of MathMela to showcase good mathematics projects, or helping the student explore mathematics?* he asked. *From what space is my enthusiasm to improve the project coming from? Shouldn't the natural curiosity and rhythm of the student be allowed to flower on its own?* It was a conversation that revealed to me deeply the very essence of the student-teacher relationship, and the purpose of that engagement – whether in *MathMela,* or in classroom, or even otherwise.

I came to the realization that the *primary* objective of the teacher in any student-teacher relationship is to foster a learning mind, and only *secondarily* to impart knowledge. The subject area is only a field in which a journey of exploration and discovery happens! And, from the teacher, it requires tremendous awareness of him/herself and the space he/she operates from, and watching the space that the student is operating from. After that interaction with Kabir, a significant shift happened in my interactions with the students; I started to try spending more time listening than lecturing; I try leaving the students with questions to explore, than with answers to digest. Importantly, I try watching myself to not let my own enthusiasm crowd out the student's own engagement with the subject. Of course, I fail often, only to remind myself of that very first (and the best) lesson on being an effective teacher: *To partake in a process of learning with the student, and in helping the student in a process of discovery.*

One of the outcomes of the *MathMela* experience was an informal and voluntary extra-curricular program that we started for the senior school students – called *MathLab* – to explore the beauty and versatility of Mathematics. The *MathLab* was literally a lab; started with the conviction that one must do something with one's hands to gain a deep understanding of a subject – even for mathematics. The very first explorations were mathematical artifacts, toys or games – the Abacus, Möbius strip, Slide-rule, interlocking metal puzzles, boardgames, etc. – interspersed with discussion on the mathematics behind the artifacts or the process to generate a solution. Over time, we moved on to reading some classical works in mathematics and exploring puzzles (such as Cissa's chessboard, Königsberg bridges, Zeno's paradox, Hilbert's hotel…). The puzzles and paradoxes provided the best means for mathematical explorations: A good puzzle usually has some history, a simple description, and one feels that the solution may be easy to find; yet, quite likely, either a solution doesn't exist or if it does exist it defies intuition. But either way, it generates good discussion, which is the critical ingredient for learning something new.

Does Provability dominate Truth?

Among the first explorations we took up in 2015 was the reading of the very first work in the development of modern Mathematics – *The Elements* by Euclid. We walked through the definitions, axioms and postulates, and worked through the proof procedure of the first few theorems, to understand the axiomatic systems framework on which mathematics developed over centuries.

In *The Elements,* Euclid developed the planar geometry based on just five simple and self-evident axioms and a rigorous rational process; this geometry – the *Euclidean Geometry* – thus developed is valid even today, more than two millennia later. It was a beautiful process to understand that most non-intuitive results (such as, "*The sum of all the angles of a triangle is two right angles")* emerge naturally out of the basic and simple assumptions. So far, good. Then many natural questions arose: Are all such known results thus proved from the axioms *("Yes")*; Are all possible results already known *("No")*; Are all possible results provable *("May be"),* Are all such possible results true *("Hmmm…")*. The last question of truth is one of the important questions that came up in the *MathLab* discussions… *How does one know if a given mathematical statement is true?* One thinks Mathematical theorems (such as, *"The sum of the angles of a triangle is two right angles"*) are truths. But, is it so?

Putting this question on hold, we moved on to the other flavours of geometries invented in the 19th century, by altering one or more of the original Euclidean axioms, and the resulting geometries – the Riemannian geometry in which *"The sum of the angles of a triangle is greater than two right angles"* and Bolyai-Lobachevskian geometry in which *"The sum of the angles of a triangle is less than two right angles"*. Much to the surprise of the inventors, these geometries turned out to be as consistent and as valid as the Euclidean geometry! The naturally expected question came up in the *MathLab: Which one of these geometries is true?* This turned out to be one of the very clarifying discussions we had had, as it teased out the two important, but different, concepts – *Provability* and *Truth* – but which are used synonymously in many situations. Each one of the flavours of geometries is consistent within itself – hence valid mathematics. Which one of the three statements about the sum of the three angles of a triangle is true, is a naïve question, as each is provable within the respective geometries. Perhaps a more-informed question is, which one of the three statements is relevant for our context. For example, simple high-school problems may need only Euclidean geometry, but modelling the universe accurately may require the Riemannian geometry. So, we choose the appropriate one for a phenomenon or a situation.

This discussion opened up a very important separation of the meanings (in scientific contexts) of *provability* and *truth*. *Provability* is a phenomenon in an axiomatic system – refers to the fact that a statement is derivable from the given axioms using a logical process – and only says that the *statement is proved in that system* – nothing more, but not trivial either! *Truth* is beyond *provability*, perhaps beyond mathematics, and certainly beyond the scope of this article! The mathematical theorems, though called truths, refer only to the fact that they are *provable* within a logical system. In the Sciences, logical axiomatic systems are used to formulate theories for modeling a given phenomenon (say, the planetary motions, the biological evolution, etc.), and the idea of *scientific truth* is tied to its *provability*. Of course, the more accurately a theory predicts a given phenomenon, the more confidence one has on the theory. Mathematics, of course, provides a rigorous rational framework for formulating and analyzing theoretical models, and thus plays an immensely important and successful role in developing a robust science program.

But, the immense success of the rational systems came to be regarded by many as the *best,* and perhaps, the *only* way to *any* truth in *all* areas of exploration. With rationality as the driving force, *provability* seems to have usurped the place of *truth* in human consciousness, even in situations where its role is not appropriate. The success of Mathematics in the realm of the Sciences is a much-debated topic… Wigner's 1960 paper titled *"The Unreasonable Effectiveness of Mathematics in Physical Sciences"* [1] and the slew of follow-up papers that followed, provide a delightful exploration of this very question from different perspectives. This series of papers laid bare in our minds that mathematics is a wonderful framework for modeling a physical phenomenon as a set of axioms and logical process, but the model's power and appropriateness depend on the fidelity of the axioms and the precision of the physical processes. There are vast areas of knowledge where this approach may not be applicable, such as psychological phenomena that give rise to the sense of individuals or culture. What axioms can we rely on for such analysis? Even if such axioms exist, do such psychological phenomena have the internal order that is demanded for such analysis? Are phenomena like *Beauty, Goodness* or *Truth* even model-able in such frameworks? Is rationality the right framework for such explorations? We just let the questions linger in our minds, so as to understand what mathematics or rationality themselves mean.

Hold ideas lightly; they distort actuality...

In 2016-17, groups of students took up for *MathLab* projects something to do with hands – one group explored Tessellation tiles and another group started the designing of a Geodesic structure... Both were geometric, artistic and required as much use of skillful hands as analyzing minds!

The tessellation project ran for a few weeks, with children learning how to identify basic tessellation shapes, and how to create complex tessellation tiles from simple geometric shapes. Enthused by the wonderful tessellation art by the Dutch artist M C Escher [2], the children started creating tiles of complex shapes: winged-horses, lizards, dancing clowns, etc. The mathematics behind the tessellations was fairly well understood, and they created templates meticulously, with the idea that the template can be copied repeatedly on a large canvas to create a large mural. The first few copies of the template on the canvas fitted very nicely with each other. But, as the canvas progressed, the copies would not fit properly… either there was insufficient space to place the template, or too much space! Reshaping the template tile slightly to make it fit only fixed the error temporarily, but the mis-alignment again happened a few tiles later! That was puzzling… there was precise mathematics behind tessellations and meticulous effort was put in making the templates!

In parallel, the geodesic dome team explored the concepts of geodesics, the mathematics and physics behind them, and even visited an existing large-scale dome. Finally, they decided to build a 20-foot half-dome that could be purposed as a play gym for the younger children, once built. Over the next many months, they spent an enormous amount of time and effort in cutting, pounding, shaping and painting the 100+ steel pipes. After the struts were readied, we started assembling the dome top-down, securing first the apex node with a bolt and a nut, and moving down to the nodes in the layers below. Curiously, a problem similar to that we encountered in the tessellation project started appearing in this project as well! The struts, if aligned perfectly in one side of the dome, would go out of alignment wildly in the opposite side of the dome (by several inches!). Dismantling and restarting from the opposite side or from the

bottom nodes did not help, as the mis-alignment only reappeared elsewhere in the dome! We were left wondering if we had made some serious mis-calculations in the strut design or hand-crafting – a dis-heartening proposition after several months of hard work! But our re-check of the design and re-measurement revealed that the struts were done right; yet, they would not assemble smoothly into a dome!

We let the projects simmer for a couple of weeks... The resolution happened quite unexpectedly; on precise remeasuring, we found that some of the struts were found to be off very slightly (couple of millimeters off over a metre), and bent perhaps a degree or two from the exact angle. When the node aligned perfectly on one end of the dome, the small errors in each strut just accumulated across the dome in the other end to large unsurmountable gaps! It occurred to us that if we keep nodes loosely bolted – *but not tightened fully* – then the resulting flexibility of the structure may allow all nodes to be aligned and bolted properly… Amazingly that was all that was needed to fix the problem! The dome stayed wobbly but flexible, allowing us to align every node, and once all bolts and nuts were in, we progressively tightened each of them to get a solid and rigid structure! Once the solution happened here, the same idea helped us to solve the tessellation project mis-alignment as well: The template tile that was created had some tiny unavoidable errors, but these errors kept accumulating as we progressed along the canvas, making either the space too narrow for the template, or too big, leaving wide gaps. All we had to do was to loosen our rigid holding on to the template!

In both the projects, we were holding rigidly on to the template or the strut, and the very rigidity was the source of our problems… and it can be solved only by maintaining a level of flexibility.

For a life of enquiry, it is absolutely necessary for one to have seriousness of intent and internal order, but also freshness in seeing and lightness of living, so as not to be deluded by fixed ideas.

All of life is relationship…

Around the same time (2016-17), a group of senior students were exploring the nature of the numbers (say, *Natural, Integer, Real, Imaginary,* etc.), that form the very basis for vast areas of mathematics[1]. A seemingly self-evident quest, but which started off some deep personal explorations. We discussed some philosophical perspectives on numbers – the *Platonic* view that the numbers exist only in the world of ideals, the *Fictionalisic* view that they are useful but merely fiction, and the *Nominalistic* view that they arise out of our phenomenological experiences before becoming a part of our language and mathematics. Mostly our group tended to be nominalists, that the numbers were defined as abstractions out of our experiences in the phenomenological world (say, the idea of *three* – written as *3* – abstracted from sensual experiences, such as, the three peaks, the three stars in *Orion belt*, etc.). After some initial readings on numbers, the discussions moved on to the axiomatic Number theory – the Peano axioms for Natural Numbers, where a simple definition of numbers (0, 1, 2, 3, …), and a set of simple operations (such as, +/- and */÷), give rise to fantastically complex results and intricate theorems (such as, "*There exist infinitely many prime numbers", "The prime factorization for a given composite number is unique", Fermat's Theorem,* etc.). Most of such theorems have been proven or disproven, many of them are conjectured and unproven, and there must be infinitely many such theorems that are unknown now, and perhaps never to be discovered!

[1] The idea for such discussions started with an earlier conversation during MathMela time with Kabir, when we wondered if any given number (say, 1, 42, √2, π or i) has any intrinsic meaning other than its relationship to all the other numbers?

These observations led to a fundamental and tangential question: Are these theorems inventions or discoveries? There were enthusiastic arguments from students supporting each option; I can only recall with wonder when after a while we concluded that these theorems are all defined implicitly by the Peano axioms themselves, or perhaps when the idea of Natural numbers and their operations arose in human consciousness. How else can it be? As we discussed these ideas more, we could sense that while the phenomenological experiences only helped naming a few numbers and extended them indefinitely, it must *be the simple orderly arrangements and the arithmetic relationships defined by operations,* which imbibe them with rich meanings and enable the emergence of beautiful theorems! It was a sort of Copernican inversion that happened in our thinking about numbers and the relationships between them: The theorems are the expressions of the orderly patterns imposed by operations on numbers, just like the beautiful patterns in a kaleidoscope are nothing but simple pieces of coloured trinkets arranged in an orderly manner by their reflections in the mirrors.

Serendipitously, around the same time the geodesic dome project was getting completed…

The half-spherical dome stood about 6-foot tall, and about 20-foot across; and became literally a hang-out place for children – who sit on it to read books or eat snacks, or climb it or swing in it. The steel half bubble was quite a sight with its orderly spatial arrangement of the *struts* (the steel pipes) and the *nodes* (the bolt-and-nut holding all struts converging at a point). Standing in front of it, a natural question came up for discussion: Which one of the two – the *nodes* or the *struts* – is the basis for the shape of the dome? While visually the nodes of a dome are prominent, the physics and the mathematics of the geodesic dome reveal that the placement of the nodes itself is determined by the struts and their lengths. The nodes are just the points of convergence of the struts, and, depending on the numbers, lengths and arrangements of the struts, the nodes may be shifted at will, and the shape of the dome altered from being a simple sphere to a radically different shape! Aren't the nodes of a geodesic dome like the numbers, and the struts like the arithmetic relationships between them? Or, vice versa! And, the complex theorems are the patterns of relationships between struts and the shapes that are possible in a geodesic dome!

When an insight happens in one area, it brings clarity in many other situations as well. Our discussions moved to another fantastic phenomenon – the *Murmuration* of birds [6], in which a flock of birds fly together forming fantastic shapes that dynamically twist and turn like a real live amorphous animal with real sentience! Physically, that animated figure is nothing but a flock of birds flying as a group, with each and every bird flying not in perfect unison but relating to a small set of its neighbours in fairly simplistic ways. Murmuration of birds is beautifully patterned, but defined solely by the relationships between the simple birds of the flock.

Such ideas made me wonder whether the Self itself is an emergent phenomenon. Metaphorically, the Self is a vast geodesic dome, in which each node may be the individual symbols/ideas (about people, objects, events, phenomenon, etc.), relating to each of the other nodes in innumerable ways. Just as the shape of the dome is an expression of all the struts, the Self may be just an expression of all the relationships. Unlike a static geodesic dome, the Self is dynamic with ever-changing relationships, just as the murmuration of the birds, resulting in an illusion of Self that seems independent and alive!

This metaphor may even scale up: Isn't Humanity just a collection of individual selves and their relationships? An individual may be just a node in the geodesic dome of Humanity, with no independent existence other than the myriad relationships that he/she may have with others. The Humanity may be an expression of all relationships, but making prominent the individuals.

Such ideas made me wonder about *Humanity* itself. Metaphorically, if humanity is a vast geodesic dome, then each node is an individual *Self*, relating to each of the other nodes (people, objects, ideas, etc.) in innumerable ways. But, just as in the Geodesic dome, where the nodes are defined by the struts, perhaps an individual *Self* is nothing but the myriad relationships. Just like the flock of birds, Humanity appears to be composed of individual selves, but is only a collection of relationships giving rise even to the individual selves. Perhaps an individual is just like a node in the geodesic dome, a part and parcel of humanity, with no separate existence whatsoever, but appears prominently as a point in space and time. On the other hand, the sense of self in an individual is like the murmuration of the birds – with each idea and its interrelationships with other ideas interacting and reacting, giving rise to and sustain the complex sense of self that seems real and independent.

Mathematics shows how a well-defined order among numbers can bring about such great beauty as theorems. Geodesic domes are artifacts of beauty and utility, brought about by orderly relationships. Murmuration of birds shows how the fluidity of interrelationships can create immense beauty and dynamism to life. Perhaps order in relationships can bring harmony and beauty to the individual and humanity.

Rationality and the Other…

The great success of mathematics in many areas of knowledge had generally led to the conviction that mathematics *expresses only truths* or *is the truth.* Such conviction was expressed variously from the early Greeks (*"The highest form of pure thought is in mathematics"* by Plato), to medieval scholars (*"Mathematics is the language in which God has written the universe"* by Galileo), to modern mathematicians (*"Mathematics… possesses not only truth, but supreme beauty"* by Russell, *"God used beautiful mathematics in creating the world"* by Dirac, and *"An equation… expresses a thought of God"* by Ramanujan).

About a century ago, two great mathematicians – Alfred Whitehead and Bertrand Russell – were working on a logic framework to generate all the truths of a branch of mathematics (Number Theory), symbolically and systematically; their work, the *Principia Mathematica,* is comparable to the Biblical account of mankind's effort to build the *Tower of Babel* to reach the heavens. However, in 1931, an Austrian mathematician, Kurt Gödel, proved in his revolutionary work – the *Incompleteness Theorems* – that the goal to generate all mathematical truths systematically is bound to fail, as any such system will be incomplete. The great irony of Gödel's work is that he used the very same rigorous principles of mathematical logic to prove the incompleteness of any kind of logical system.

In my mind, Gödel's work is a jewel in the crown of mathematics. The logical framework and the Incompleteness Theorem taken together reveal mathematics *as it actually is:* a very powerful logical framework that helps in modeling and studying of many phenomena, but a far cry from being the revealer of all truths. If a rational system cannot assure discovery of *all* truths even with respect to its own self, then how can it be relied on as a means to discover *all* truths? In using rationality as a tool for understanding all phenomena, are we confusing *truth* with *provability?*

Even more importantly, the impeccable internal order of mathematics and how such an order can reveal its own limitation, holds an immense value for humanity. For me, it was a deep insight that revealed a parallel between Mathematics and the life of enquiry. Like mathematics, the internal order is an absolute necessary condition for any enquiry into the nature of truth; and, like mathematics, the rationality may not be sufficient for enquiry into the nature of truth. For a mind that is latched on to rationality – and hence provability – perhaps truth may be unreachable.

I wonder if the mind is in total internal order to observe *what is,* and is aware of its own structure and its inherent limitations, then perhaps such a mind may be free and pliable enough to move in a totally different dimension, for truth to be. And, the understanding of such a state of mind for a life of enquiry is perhaps the best learning I have gained by teaching mathematics.

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[1] Wigner, Eugene P. The unreasonable effectiveness of mathematics in the natural sciences. Richard Courant 1959 lecture in mathematical sciences at New York University. Published in Communications on Pure and Applied Mathematics, 1960.

[2] Wikipedia: M C Escher (https://en.wikipedia.org/wiki/M._C._Escher) & (https://mcescher.com/)

A KUMARAN is an engineer by training and a teacher by choice. After receiving his engineering and computer science degrees, he served for a decade and a half in the computing industry, from being a design engineer to a director. After that, he went back to academia for a doctoral degree and received his PhD in 2005 from the Indian Institute of Science. He spent the next decade as a researcher in the area of Computational Linguistics, focussed on Indian languages. Currently, he teaches Mathematics at Shibumi School. He may be contacted at a.kumaran@gmail.com

KABIR JAITHIRTHA (1949-2018) was an explorer of life throughout his life in the tradition of J Krishnamurti. He was deeply involved in the activities of Krishnamurti Foundation of India (KFI) for nearly four decades, and was a trustee of KFI till the end. Kabir started as a teacher in The Valley School, Bangalore, and later served as the Director of Rajghat Besant School in Varanasi. He was instrumental in starting two independent schools (Centre for Learning in 1990, and Shibumi in 2008), in Bangalore.

Kabir's zest for truth reflected in every aspect of his life – whether engaging a child in a conversation, exploring a subject with a student, enabling a learning environment at school or enquiring truth in dialogue. Our interactions – whether exploring our mutually favourite subject – Mathematics, discussing personal or school issues, or engaging in dialogue – had moved to a life of enquiry, for which I am forever grateful.

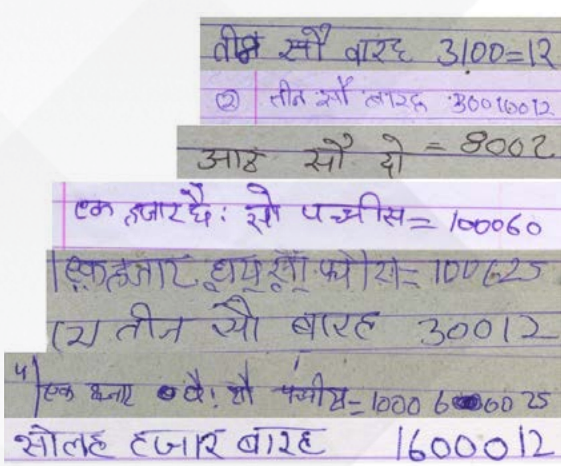
SHIBUMI (https://www.shibumi.org.in) is a learning centre for both adults and young people. For interested adults, it offers a space to understand oneself and one's relationship to life in the light of J Krishnamurti's teachings; for the child, it is an open and supportive environment for exploring itself and its relationship to nature and the society, and for developing academic skills. The centre's philosophy is to help the student to learn not only the subject, but to understand the whole activity of learning, and to become a complete human being. Shibumi is located in South India, on the outskirts of Bangalore.

The *'Ganak'* Exploration: Helping Children to Learn About Numbers

This article describes an investigation of a tool called Ganak. The author proposes that place value work should continue into middle school level with an emphasis on understanding the positional value of digits, and the additive, multiplicative properties of numbers. He demonstrates his work with students on sequentially exploring number patterns in various bases through a game based on abacus, pictorially representing them, deciphering their value in the decimal system, and allowing children to explore their own interpretive way of forming numbers.

PRAMOD MAITHIL

At times, children make sense of mathematical concepts differently and we term these interpretations as "mistakes". However, these mistakes highlight the children's understanding of concepts and gaps therein. During one of my interactions with children in 6th grade, I found that they weren't clear in the standard numerical representation of the numbers. I noticed confusion on using zeros while representing numbers. For e.g., three hundred and twelve was represented as 30012 (Pic 1). My premise was that their difficulty lies with a basic characteristic of the number system we use now and its written form. This classroom experience led me to go through some literature, documents & materials on mathematics education & particularly on teaching Place Value or Positional Notation System.



Pic 1

With a base 10 (often referred to as decimal) system we need only ten symbols (1, 2, 3, 4, 5, 6, 7, 8, 9 and 0). All these symbols & all possible combinations of these symbols are associated with unique numbers. Children often fail to differentiate between the face value of these symbols (or digits) and the actual value of the digit.

Keywords: Place value, number bases, decimal system, number operations

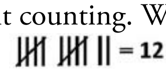
What is place value system?

There are a few terms used for this – The decimal number system & the place value system OR the positional notation system. I found a very clear definition written by Sharon R. Ross "Our numeration system is characterized by the following four mathematical properties:

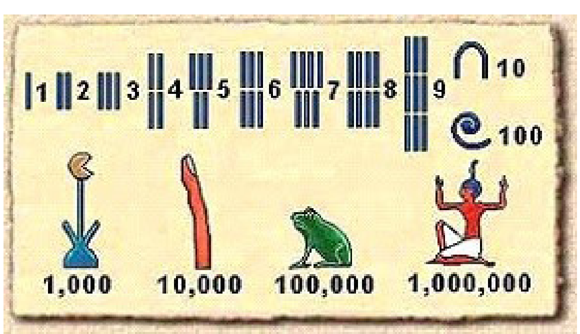
1. **Additive property** – The quantity represented by the whole numeral is the sum of the values represented by the individual digits.
2. **Positional property** – The quantities represented by the individual digits are determined by the positions that they hold in the whole numeral.
3. **Base-ten property** – The values of the positions increase in powers of ten from right to left.
4. **Multiplicative property** – The value of an individual digit is found by multiplying the face value of the digit by the value assigned to its position."

Example: $1324 = 10^3 \times 1 + 10^2 \times 3 + 10^1 \times 2 + 10^0 \times 4$

www.questia.com, "Place Value: Problem Solving and Written Assessment" by Sharon R. Ross (2002).

Historically, the tally system was often used to represent counting. We still use it sometimes. For e.g., 

Recognizing this as a lengthy process, people created more efficient benchmarks such as using new symbols at fixed intervals (5 or 10). Let's look at the Egyptian system:

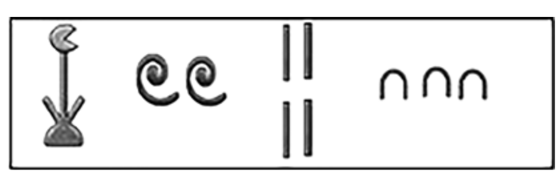


Using these symbols, if I need to write one thousand two hundred and thirty four, it will be as shown below.

If we follow the Egyptian system, we will need to keep generating infinitely many new symbols for ten, hundred, thousand, and so on. Moreover, it does not require a specific position for the symbols. In other words the symbols can be placed in any order and still represent the same number.

For e.g.,

OR



All these combinations stand for the number one thousand two hundred thirty four because there is no positional value associated with the individual symbols. Each symbol has the same value irrespective of where it is positioned. The idea of zero was also not needed in this system. It differs significantly from the decimal system in use today.

We traditionally teach the concept of place value as per the school textbooks at primary grades. Children are asked to write the numbers in columns marked out as units, tens and hundreds. In the alternative approaches prevalent in India, different concrete materials such as Dienes blocks and matchstick bundles are used (Usha Menon, Episteme-1)[1]. These manipulatives help children understand grouping and give them a concrete representation of a number. But this still falls short of "connecting" the positional, additive, and multiplicative properties of digits and instead focuses on the mechanics of writing digits in appropriate position. Curricula in India emphasize that the entire number system is to be taught in the elementary level but I strongly see the need for place value work to continue at middle school level as well.

In Bal Vaigyanik (Class 6 textbook of Eklavya's Hosangabad Science Teaching Programme), at the middle school level, the open abacus, Ganak is used as a tool to teach decimal & place value. The Ganak consists of thin rods vertically set to fit exactly 9 beads each. We begin by placing beads in the rightmost rod. After 9 beads are set, we take out all the beads and place 1 bead in the second rod from right. This 1 bead at second position represents 1 in the tens place. As the unit place is empty it represents the '0' in the numerical representation of 10.

From my own experience with Bal Vaigyanik, any activity with Ganak usually stops at highlighting the number indicated in each rod. I wondered if Ganak could be used to connect the positional nature of a number to its additive and multiplicative relation with other numbers. Further, I feel that the strength of positional notation system in base ten can be seen only when we expose numbers in different bases, in which we show the gradual increase & shift in the position and the acquisition of value accordingly. Zoltan P. Dienes[2] says:

> *"… different bases should be used at the start, and to facilitate understanding of what is going on, physical materials embodying the powers of various bases should be made available to children."*

To start with, I designed a game of base three mimicking the Ganak. The game was designed to give exposure to the positional notational system of different bases without explicitly mentioning the use of bases to the students. In this article, I call it a base-three game. I selected a 6th grade classroom in a village middle school with 22 boys & girls[3]. The school usually did not use any manipulatives to teach place value. Before introducing Ganak, I spent about one month with them using the Number Mala and the Pebble Card as materials for this study. The children built number sense using various counting, addition and subtraction games on the Number Mala and Pebble Card[4]. These games allowed them to develop number sense, grouping, and base-10 property. Taking this as the baseline for their work, we switched to the Ganak game to research sequencing of numbers, positional property of digits, and additive-multiplicative reasoning.

The rest of the article details the interventions with this game as a necessary step before reintroducing base 10 Ganak with proper understanding of place value system. Table 1 highlights the base 3 game. The students related to it as a game of two parts - "Do khand wala khel". The game involves placing a pebble at each column, starting from the rightmost. At the third, sixth and ninth moves, they have to consolidate the moves and replace it by a pebble to the next box or column.

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1 The Teaching of Place Value – Cognitive Considerations presented by Usha Menon: www.hbcse.tifr.res.in/episteme1/allabs/ushaabs.pdf

2 The Dienes blocks http://www.zoltandienes.com/what_is_a_base.pdf

3 Throughout the interactive study, there were consistently 10-12 students present in class.

4 Khushi-Khushi classes 1-5, Primary textbook and workbook for Prathamik Siksha Karyakram (Prashika), developed by Eklavya

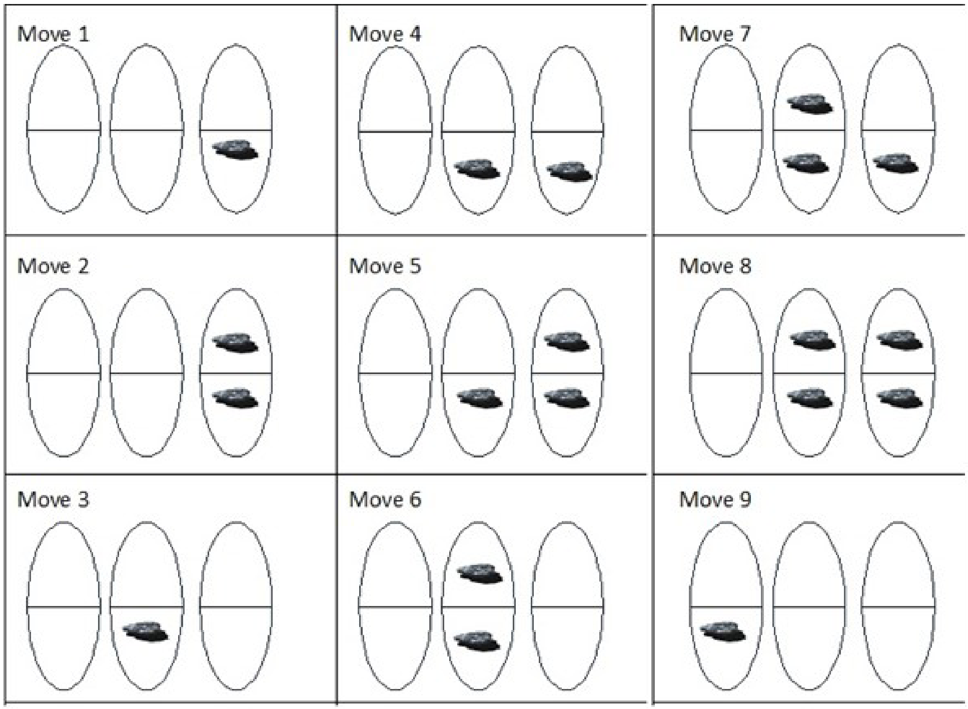


Table 1: *Do khand wala khel*

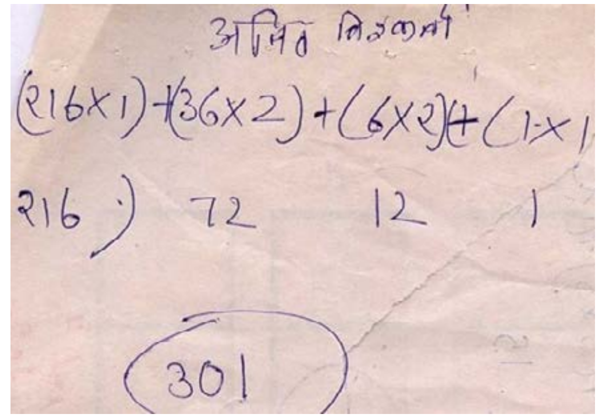
The children played this game several times in base 3 until they recognized the patterns of play/ position. In doing so, they were able to associate the representation with a unique number of moves. After several rounds, I concentrated on naming the column (position) as related with the number of moves. They agreed that each column could be numbered based on the least moves it represented. For e.g., the second column for a base 3 game was numbered 3, the third column was numbered 9 and so on (Pic 2). The students recognized that the next two columns could be numbered as 27 and 81 respectively.

Pic 2

I assessed the students by pictorially showing them the representation for 28 moves (in base 3) and asked them to decode the number of moves it represented. Then I gave them a set of blank templates (for base 3) and asked them to fill in the sequence of representations until 34 moves. They used worksheets, chalkboards or just paper to work out these using additive and multiplicative reasoning. Students numbered the sequence of moves using Indo-Arabic digits and recognized the pattern of digits for different bases. For e.g., they recognized that they only need to use the digits 0, 1, 2 for a base 3 system. This was a fun exploration for them. It was exciting to see children busy exploring new patterns and mathematizing.

I further challenged them by increasing the number of parts in each column to 3 (which signifies base 4) and higher bases. They worked in groups & designed their own games by extending the parts in the boxes, and progressed through different bases. They were able to represent moves pictorially, name each column, using the method for the earlier base 3 game and decode the number of moves for various representations using additive and multiplicative reasoning skills (Pic 3).

Pic 3



Pic 4

I had realized that children were effortlessly using additive-multiplicative reasoning to find connections between the pictorial representations and the number of moves (decimal value). I wondered if kids could understand equations using the extended notation to relate the different base numbers to the number of moves. I showed them a few simple examples using base 3, such as, $(102)_3 = (1 \times 9) + (0 \times 3) + (2 \times 1) = 11$ moves.

The students readily caught hold of such expressions and were able to convert other bases back to a number in the decimal system using this method. Thus far, the students were working in small groups and sharing their work. I asked students to work individually to be able to assess their understanding. I repeated the work with sequencing of moves, naming convention, picture representation & then the numerical representation with the 3-base game and recorded their work (Pic 4).

Towards the end, we returned to base 10 work. My intention was to see if students could recognize the connection between the positional nature of digits and the place value of each column. We extended the game to have 9 parts in each column and then used Ganak as a tool. Now the children found it very exciting to realize the direct correlation between the decimal system numbers and the digits in each position. There was no need for any extra work for figuring out the number of moves when interpreting the pictorial representation!

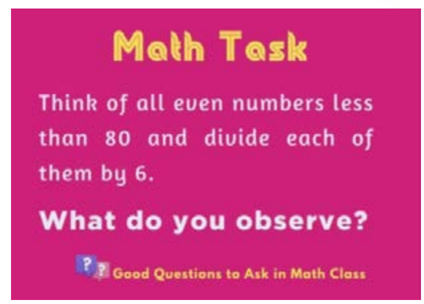
In this pilot study, it was interesting to see how Ganak can be utilized for the purpose of building positional value of digits, and additive, multiplicative reasoning skills. I am left with a few issues/questions:

a) Why and how should place value be taught?
b) Will this kind of intervention help students understand numbers better in the future?
c) What grade should this work be introduced in?
d) How do such detailed interventions gel with prescribed syllabus, teacher's time commitments and school schedules?

With further corroboration, this study[5] can be treated as an attempt to recommend Ganak to educational agencies as an efficient method for developing an understanding of place value.

5 I would like to thank ICICI Center for Elementary Education, Pune for facilitating this study, and Dr. Anju Saigal (ICEE) for guiding me in this research method. I am also thankful to Ajay Sharma (USA), Anjali Narhonha (Eklavya), Amitabh Mukharji (DU), CN Subramanian (Eklavya), H K Diwan (VBS), Jayasree Subramanian (Eklavya), Kamal Mahendru, K Subramaniam (HBCSE), Maheen (Bhopal), Rakhi (TISS), Reshma Madhusudan (Learning Network), Sushil Joshi (Hoshangabad), TulTul & Rajesh (Eklavya), Vijay Verma (Eklavya) and of course the children and teachers of middle school, Kulamari, who supported me at various stages till this article was written.

PRAMOD MAITHIL is a teacher, researcher and entrepreneur who has contributed to conceptualization and implementation work of well-known organizations over the past 20 years. He is involved in educational research with Eklavya (NGO), teaching at Sahyadri School (KFI), and has started an innovative school - Anand Niketan Democratic School (ANDS) in Bhopal. He documented his learning journey in his book *School for My Child* (Penguin). He is also a TEDx speaker. He has now founded a startup (Prakriti Initiatives) which creates Tinkering space [T--LAB] in schools for children to make them work on their ideas freely and innovate things. This article is about his study on a tool called Ganak, during his tenure in Eklavya. He can be reached at pramod.maithil@gmail.com



This task can be taken up in Grades 4, 5 and 6. It can be used to introduce Multiples and develop a deep understanding of factors and multiples. The pre-requisite for this activity is multiplication and division of numbers.

Students may also need some explanation on the remainders obtained when numbers smaller than 6 are divided by it.

Observations by students of Grade 4, Mind Tree School, Ambala

- All remainders are even.
- There is a pattern, we get only 0, 2 and 4 as the remainders and it repeats.

Further questions:

- Can you list down the numbers that give remainder 0 when divided by 6?
- Is there a pattern in these numbers?
- Can we extend these numbers beyond 80? Will they ever end?

Once they understood the concept of multiples of 6, we made the following table

Multiples of 6	(Multiples of 6) + 2	(Multiples of 6) + 4
6	8	10
12	14	16
18	20	22

Still more questions

- What is the difference between numbers in each row?
- Why aren't the numbers in the second and third columns multiples of 6?

Many series came up and we had a good time guessing their names.

Contributed by Reema Maggo Verma

Construction of a Tunnel Through a Mountain

RADHAKRISHNAMURTY PADYALA

This article describes a method for the construction of a straight tunnel through a mountain, given the two openings at the two ends of the tunnel.

Constructing a tunnel through a mountain sounds a herculean task and appears almost impossible. Yet we see that such tunnels are constructed for defence, for shortening the distance between two places for trade, etc. It involves great engineering design skills and civil construction skills. It also needs tools, implements and instruments such as the dioptra. This article describes a method to construct a tunnel through a mountain. I felt that this simple geometric method would be of interest to students at high school and college levels as well as to a general reader who would appreciate the simplicity of the solution to what looks to be an unsolvable problem. The problem is stated in simple terms thus: To make a straight tunnel through a mountain, given the locations of the openings at each end.

We have taken the material presented here from M. R. Cohen and I. E. Drabkin [1]. A point to note is that Drabkin's paper contains a serious conceptual problem with the diagram, which gives a wrong conceptual perception with respect to the magnitudes of the proportions dealt with.

The problem

To construct a straight tunnel through a mountain, given the openings at each end.

It may appear puzzling, 'What is the problem in drawing a straight line when two points are given?' The catch, however, is that the two points are not directly accessible from one another. We cannot see the line connecting the points or even see one point from the other – a mountain is there between the two points! Therefore, an indirect method is to be sought.

Keywords: geography, engineering, geometry, similarity

The method of solution described below is simple and easy. The strategy consists of two steps: **1.** Treating the line joining the two given points as the hypotenuse of a right triangle, we construct the remaining two sides. **2.** We construct two triangles similar to this right triangle. Our job is now almost done. We use the properties of similar triangles to get at the solution.

Geometrical Method of Solution

Let us imagine a mountain with base ABCD (Figure 1). Let B and D be the given openings at the two ends of the tunnel.

Draw a horizontal line BJ through B. From any point, say E, on BJ draw a perpendicular EF to BJ. Again, from any point, say Z, on EF draw a perpendicular ZG to EF. Now from any point, say H, on ZG draw a perpendicular HQ. From any point, say T, on HQ draw a perpendicular TK and similarly KL, perpendicular to TK. This process is repeated till we reach the level of point D.

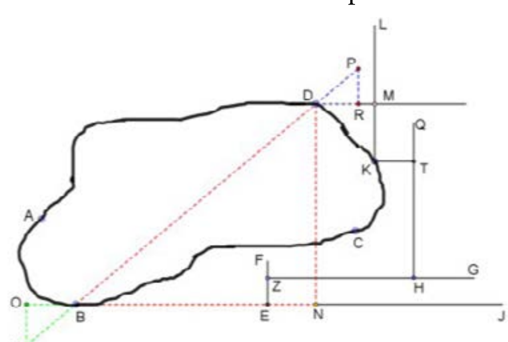


Figure 1. ABCD is the base of a mountain.
B and D are the given end points of the tunnel.

We now move the dioptra [2] along line KL (keeping the line of sight always perpendicular to KL) until point D comes into sight in a direction perpendicular to KL at point M. Draw a perpendicular from D to BE (or its extension) to meet it at N.

With the geometrical construction thus done, we can find length DN from lengths EZ, HT and KM (see Eq. (1)). Similarly, we can find length BN from lengths BE, ZH, TK and MD (see Eq. (2)).

$$DN = EZ + HT + KM \quad (1)$$

$$BN = BE + ZH - TK - MD \quad (2)$$

We now have two sides of the right-angled triangle whose hypotenuse is the line joining the two given points, the three sides together making the right triangle BDN (dotted line red triangle). Let $BN:DN = \nu:1$ (we shall use this ratio in constructing two more right triangles).

Our next step is to construct two more right triangles that are similar to the right triangle *BDN*.

Now imagine that DB is drawn and extended to a point X and XO drawn perpendicular to the horizontal BE. We get a right triangle BXO (dotted line green triangle). Similarly, suppose that BD is extended to a point P and PR drawn perpendicular to DM. We get a right triangle DPR (dotted line blue triangle). It is easy to see that triangles BND and BOX are similar:

$\angle BND = \angle BOX = 90°$,

$\angle NBD = \angle OBX$ ("vertically opposite angles"),

$\angle OXB = \angle NDB$.

Similarly, triangles BND and DRP are similar:

$\angle BND = \angle DRP = 90°$,

$\angle DBN = \angle PDR$ (since $BN \parallel DR$ and PB is a transversal),

$\angle BDN = \angle DPR$.

We now see that the big red right triangle BDN is similar to the two small right triangles BXO and DPR. Consequently, all the three right triangles BDN, BXO and DPR are similar. Therefore, the corresponding sides are proportional. Therefore, we get,

$$BD : DN : BN = BX : XO : BO = DP : PR : DR \quad (3)$$

$$\frac{BN}{DN} = \frac{BO}{XO} = \frac{DR}{PR} = \nu \quad (4)$$

Having thus completed the mathematical portion of the problem, we come to implementation. Produce the horizontal through B. Select an arbitrary point O, on it. Draw a perpendicular to the horizontal at O. With O as center and radius equal to (BO/ν) draw a circular arc to intersect the perpendicular, at X. It follows that line XB when extended passes through the point D which is on the other side of the mountain. Similar analysis on the side where D lies shows us that the line PD, if extended, goes through the point

B which is on the opposite side of the mountain. Let us do the analysis: On the horizontal through D select an arbitrary point R. Draw a perpendicular to the horizontal at R. With R as center and radius equal to (DR/ν) draw a circular arc to intersect the perpendicular at P. It follows that PD if extended passes through the point B which is on the other side of the mountain.

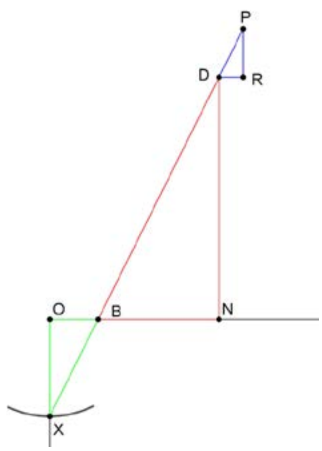


Figure 2.

We shall now commence our tunneling operations along the line XB from B and along line PD from D. For the rest of the tunnel, we proceed by setting our direction line along the determined lines XB and PD. If the tunnel is dug in this way, the workers (working from opposite ends) will meet at some point on the line BD [3].

In essence what was done to solve the problem of constructing a tunnel through a mountain is this: given two points B, D on the ground, on opposite sides of the mountain, we located two more points X and P so that all the four points lie on a straight line.

Another such problem, equally interesting, is the following:

> *Given two points on one side of a ditch or tank, determine the line on the other side of the ditch or tank that forms an extension of the line segment joining the two given points. In other words, given the points X, B (in the problem solved above) on one side locate points D, P on the other side such that all the four points lie on the same line. No dioptra to be used! You will be provided with marker pegs and assistants for completing the job.*

Interested readers could work on it. We shall present a solution in our next article.

Acknowledgement

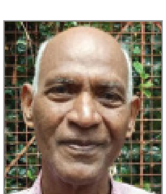
It is a pleasure to thank Mr. Arun Rajaram, Chennai, India, who supports and encourages my research pursuits in every possible way.

References and notes

[1] Morris R. Cohen and I. E. Drabkin, *'A Source Book in Greek Science'* Harvard Univ. Press, Cambridge, (1948) 341-342.

[2] The Dioptra in its simplest form is a sighting tube. In the form described by Hero of Alexandria, it is an instrument or pair of instruments for surveying and leveling, for the measurement of heights and distances of inaccessible places, and for the determination of angles in both terrestrial and astronomical problems. With it is combined a hydraulic level for the determination of the horizontal plane. The tilting of the line of sight to any plane between the horizontal and the vertical and to any position in the plane is made possible by sets of screws and geared wheels. The precise angles involved could, presumably, be read off when the position of the sighting piece was determined. In finding the relative height of two widely separated points a series of intermediate sightings was made with the help of leveling staffs.

[3] It appears that some time before 500 B. C. the engineer, Eupalinus of Megara tunneled through a hill in Samas for the purpose of carrying water, through pipes to the city. The tunnel, about 1000 yards long, is described by Herodotus; it was discovered in 1882. The tunneling operation was conducted from both ends with remarkable accuracy! Imagine the engineering skills the engineers had long before Christ.



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Harbans Puzzles

SUNIL BAJAJ &
JASNEET KAUR

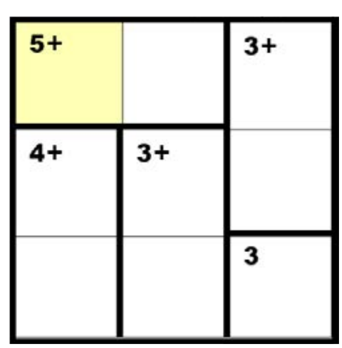
> Mathematical games, puzzles and stories involving numbers are useful to enable children to make connections between the logical functioning of their everyday lives to that of mathematical thinking and to build upon their everyday understandings. (NCERT, 2006)

According to Collins Dictionary, a Puzzle is- "a toy, problem, or other contrivance designed to amuse by presenting difficulties to be solved by ingenuity or patient effort". Most puzzles present a perplexing task that is not easily solved. At the same time, they are designed to be fun and engaging, so that the puzzle solver will invest time and energy in figuring out the solution. Mathematical puzzles can engage students in rich mathematical explorations and logical reasoning in the classroom (Evered, 2001). In this article, we will illustrate the extensions and variations of a Japanese Puzzle- 'KenKen' (which we have named Harbans Puzzle) and then discuss students' responses to these puzzles.

KenKen Puzzles were invented by a Japanese teacher-Tetsuya Miyamoto which he used as a learning tool for his students. KenKen means Cleverness and on these lines, Harbans Puzzle is an acronym for 'Har Banda Samajhdaar' in the local language, which means that everyone has the potential to do anything.

Getting started with KenKen Puzzle

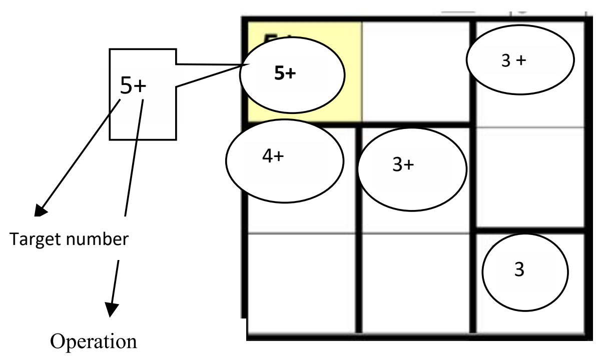
To play with these puzzles, take a grid of size 3 × 3 or 4 × 4 or 5 × 5 …….



Keywords: Math pedagogy, puzzles, reasoning, constraints, problem extension

The target will be to fill in the whole grid with numbers, making sure no number is repeated in any row or column. In this 3 × 3 grid, we will use three numbers 1, 2, 3 only. In a 4 × 4 puzzle, we use the numbers 1, 2, 3, 4; in a 5 × 5, we use the numbers 1, 2, 3, 4, 5, and so on. The heavily-outlined area is called a cage (Group Box). The top left corner of each group box has a 'Target number' and one math operation. The numbers you enter (in any order) into the squares of the cage must combine to produce the target number using the math operation indicated (+, –, × or ÷). A cage with one square is named as 'freebie'(single box). Just fill in the number that is given in the box. Remember! Numbers cannot be repeated within the same row or column, just as in 'Sudoko'.

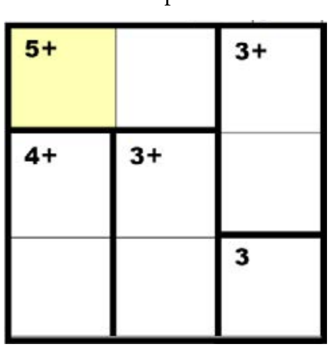
In the beginning, let us take the 3 x 3 grid shown above. In the first cage, the math operation to be used is **addition**, and the numbers must add up to 5. Since this group box has 2 squares, the possibilities are 2 and 3, in either order (3 + 2 or 2 + 3 = 5).



Let us Start

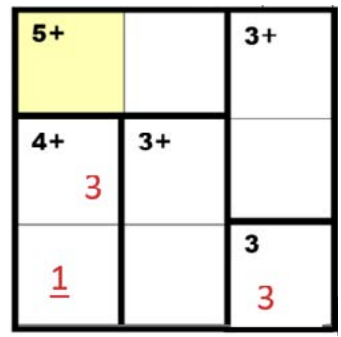
1. Enter 3 in the Freebie (Single Box). It's always best to begin with your singles.

Step 1



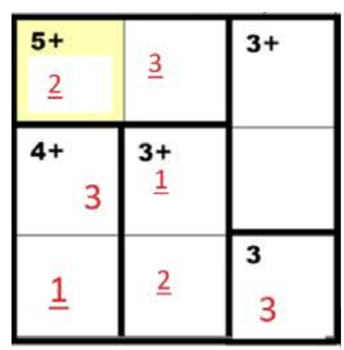
2. The lower-left group box must be filled in with a 1 down and 3 up in order to get 4 as 3 is already filled in the last row as shown in the figure.

Step 2



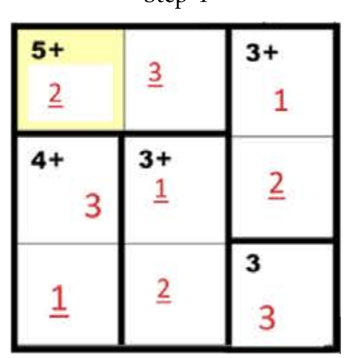
3. Each row and column must have a 1, 2 and 3 in it. The bottom row already has 1 and 3, so 2 is the only choice in the middle square of the lowest row and in the top left square. Then the middle square of the top row must be 3 and this completes the top cage.

Step 3



4. We can now enter 1 and 2 in the first and second squares of the right most column using the same logic.

Step 4



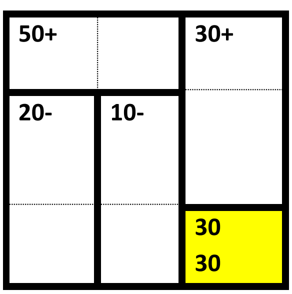
Let us Move to Harbans Puzzles

To extend a KenKen puzzle, we may use a random set of numbers (Consecutive or non- consecutive) and operations on them. In a 3 × 3 puzzle, you may use numbers such as 0, 1, 2 or 1, 2, 3 or 10, 20, 30 or three integers/prime numbers/ even numbers/ odd numbers - these need not be equally spaced. Similarly, we may take any random set of four numbers for a 4 × 4 puzzle, and so on. We would illustrate it using some examples.

Illustration 1

In this 3 × 3 puzzle, we will use the numbers 10, 20, 30 and operations - addition and subtraction.

Enter a 30 in the Single square. It's always best to begin with your singles.



Then using the same rules as of KenKen Puzzle, we can fill the grid using 10, 20 and 30 with the given operation to get the target numbers.

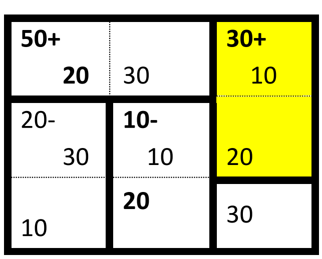


Illustration 2

Use three consecutive numbers from 1 to 5 (think and choose the numbers).

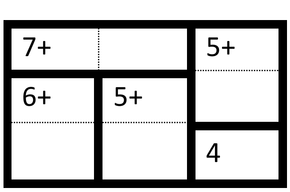


Illustration 3

Use three consecutive even numbers from 1 to 15.

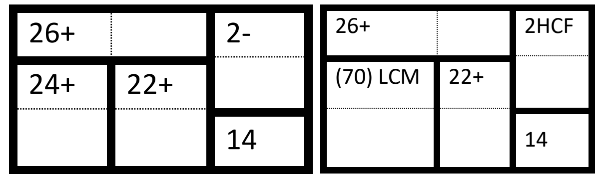


Illustration 4 Use 3 Prime Numbers from 1 to 10.

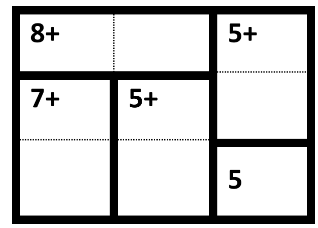


Illustration 5 Use 3 consecutive integers between -5 and 5

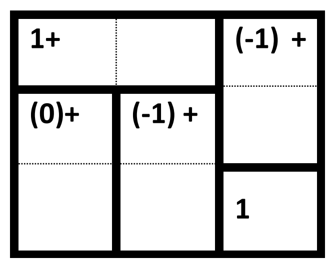
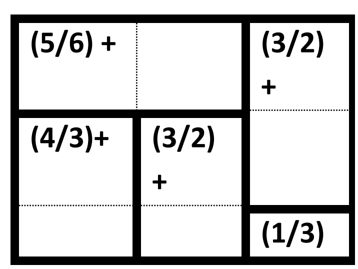


Illustration 6 Use 3 Fractions 1/1, 1/2, 1/3



Ilustration 7 Use 3 Fractions 1/1, 1/2, 1/3.

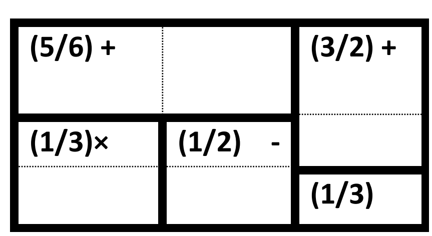


Illustration 8 (i) Choose the numbers to be used and consider them as x, y and z; here the operation is highlighted and the result is written before it. (ii) Use 3 consecutive integers -5 to 5

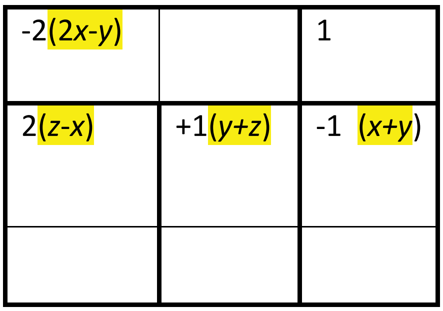


Illustration 9 Choose the numbers that can be used to solve this puzzle.

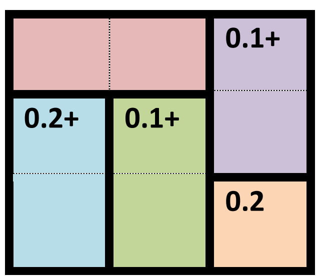
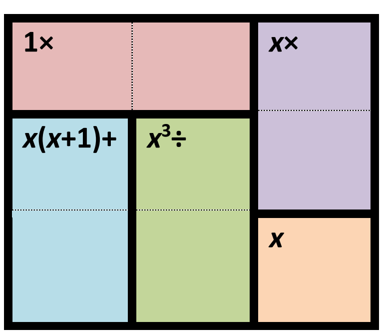
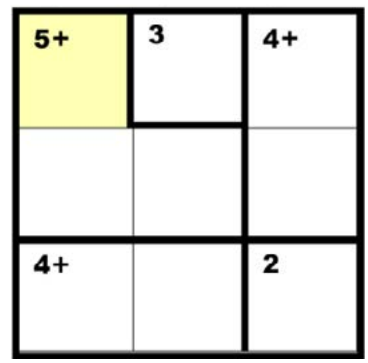
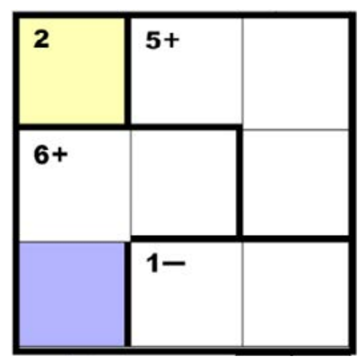


Illustration 10 Use $1/x$, x, x^2



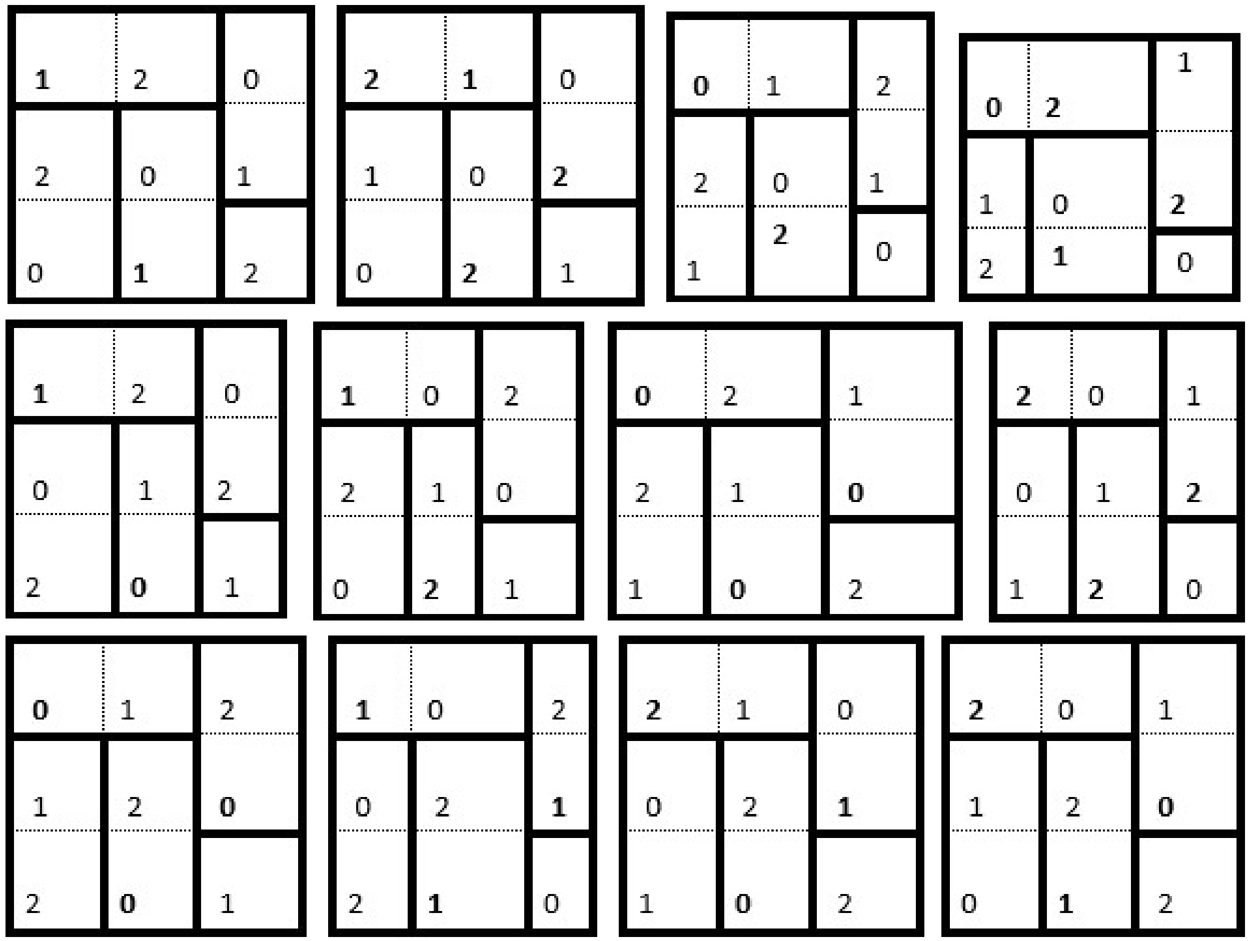
Here we can see how Harbans puzzles are different from KenKen puzzles as, in Harbans puzzles, a variety of number sets with different operations can be used, such as multiples of 10, even numbers, odd numbers, decimals, fractions, whole numbers, integers, variables, and so on. Further, the name 'cage' is replaced by 'Group box' and 'Freebie' is replaced by 'single box' to communicate in a simple language. The rule that no number will be repeated in any row or column is the same in both *Harbans* puzzles and *KenKen* puzzles and the symbolic representation is also the same, i.e., the target number is to be written on the top of a box with the operation. Further, we can use various types of caging having different number of cells. Shown below are examples of puzzles having cages with more than two cells.





Exploring the hidden patterns and potential of the puzzle to use it with students of all ages.

When we were exploring the puzzles, we came across a number of patterns while creating and solving these puzzles. We realized that such kind of puzzles work as low floor high ceiling tasks (i.e., they move in increasing order of difficulty) and also promote the computational thinking skills of students. To support our statement, we would illustrate the patterns observed in 3 × 3 grids and the possibilities to create 3 ×3 grid puzzles. Let us take an example of filling up 3 numbers 0, 1, 2 in a 3 × 3 grid; we can observe and can also compute using permutation that the possible number of arrangements is 12.



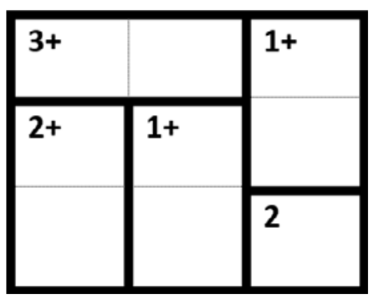
We can observe that the first square can be filled in 3 possible ways, second and fourth squares can be filled in 2 possible ways, the rest of the squares are left with a single choice only. Therefore, total number of possibilities is $3 \times 2 \times 2 = 12$.

I	II	III
IV	V	VI
VII	VIII	IX

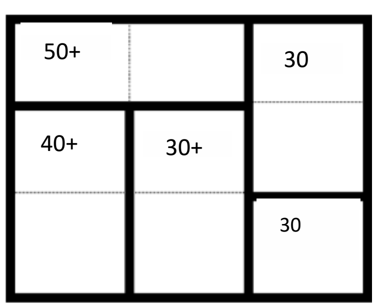
With just one grid, an ample number of puzzles can be created by varying the cages and operations. When these puzzles were given to students of different grades, we encountered different ways that students used to create and solve such puzzles. Many students of third and fourth grades used 'guess and error' strategy to write the target number to create the puzzle and rectify their errors while solving it. Another way to create the puzzle that two third graders used was 'solution to puzzle strategy,' i.e., they started with a solution (but did not write it in their puzzle sheet) to reach the target numbers and the operations used. For example, they decided to put 1, 2, 3 in the first row of a 3×3 grid and then subtracted 2 from 3 to get the target number as 1. This way they created a cage having 2 cells with a target number 1 and 'minus' as operation. Such opportunities made them think analytically and logically, which are important mathematical processes.

Further, while creating the puzzle, it was observed that just by taking one puzzle, we can create ample number of variations.

Suppose we have one puzzle where 0, 1, 2 numbers are to be filled, and we want to create a puzzle using 10, 20, 30 instead of 0, 1, 2. Observe the first target number, it is the sum of 1 and 2, i.e., second and third number. So, we would replace 3^+ by 50^+ as 50 is sum of second and third number. Similarly, 1 can be obtained by 1 + 0, i.e., first and second number, so 1 would be replaced by 30, i.e., sum of 10 and 20 and so on, as shown below.



So, our new Puzzle would be -



In this way we can take any three numbers (integers, fractions, etc.) that is to be filled in the grid, we can just replace the target numbers by using the same process as mentioned above. You can notice that all the illustrations (1 to 10) given above have been created using one puzzle only. Target Numbers have been replaced keeping the cages same.

Further, in the 3 × 3 grid, we can observe patterns in diagonal numbers and once those patterns are noticed, the puzzles can be solved in a minimum number of steps.

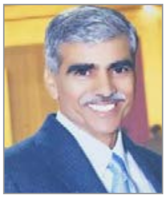
Explorations for Students

These puzzles can easily be accommodated in the school curriculum as such puzzles can be created for most of the school maths topics given from primary to secondary grades. These puzzles provide scope for problem solving and also problem posing in the classroom. Students can be given such puzzles as a practice material too which, though essential, is often boring and repetitive. Apart from this, there is a lot of scope to create and solve new puzzles, explore patterns and find more complex mathematical connections through these puzzles. Some of the questions for explorations would be-

- How many steps are used in solving the puzzle?
- What are the minimum number of steps to solve the puzzle?
- Is it always feasible to get the solution with the target numbers given?
- Do we always have a unique solution for a puzzle or more than one solution can exist?
- What are the possible number of arrangements in a 4 × 4 grid? Can we generalize to get the possible number of arrangements for an $n \times n$ grid?

References

1. Evered, L. (2001). Riddles, Puzzles, and Paradoxes: Having Fun with Serious Mathematics. Mathematics Teaching in the Middle School, 6 (8), 458-61.
2. NCERT (2006). National Position Paper on Teaching of Mathematics. NCERT: New Delhi. http://www.kenkenpuzzle.com/



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THE U TURN STAIRWAY

A stairway at the home of my brother features a stairway that makes a 180° turn. The architects were Thomas Eddy Tallmadge and Vernon S. Watson, 1912.

The photograph shows the stairway as viewed from the top. The three steps that effect a 90° turn use two 30-60-90 triangles with a right kite. The two such sets cause the U turn.

Submitted by James Metz

TearOut

Playing with Symmetry

In this 7th TearOut, we will be exploring the symmetry of various regular polygons. This time, pages 1-3 are a worksheet for students while page 4 provides guidelines for the facilitator.

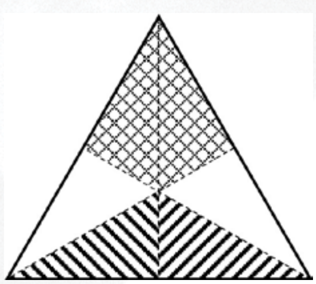
Let's warm up by filling this table. The column corresponding to the 'Square' has been done for you.

	Equilateral triangle	Square	Regular hexagon
No. of lines of symmetry		4	
Types of lines of symmetry		Diagonals, perpendicular bisectors of sides	
Order of rotational symmetry		4	

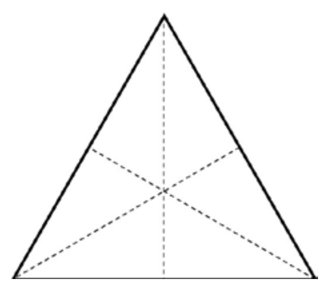
Colour or shade each polygon so that the resulting picture has the indicated symmetry. The first one is done for you.

Equilateral Triangle

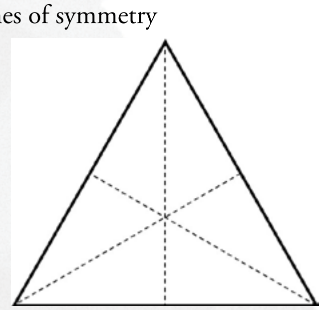
1. Exactly one line of symmetry and no rotational symmetry



2. Only rotational symmetry of order 3 and no line symmetry

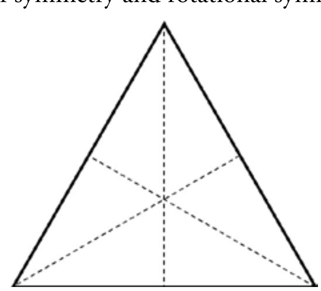


3. Two lines of symmetry



What is the angle between these lines? Does the coloured triangle have any rotational symmetry? What is the order?

4. A line of symmetry and rotational symmetry

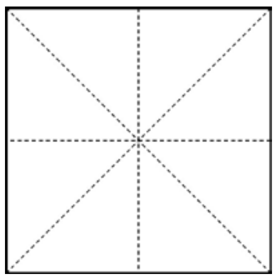


Does the coloured triangle have any other line of symmetry? What's the angle between them?

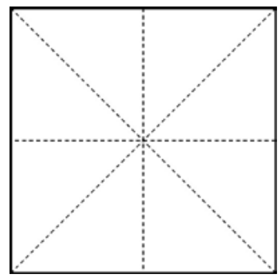
Square

1. Simple ones

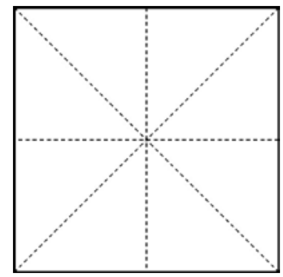
a. Exactly one line of symmetry and no rotational symmetry



b. Only rotational symmetry of order 2 and no line symmetry

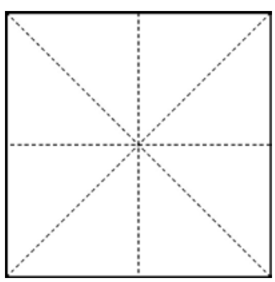


c. Only rotational symmetry of order 4 and no line symmetry

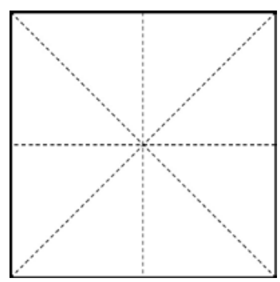


2. Multiple lines of symmetry

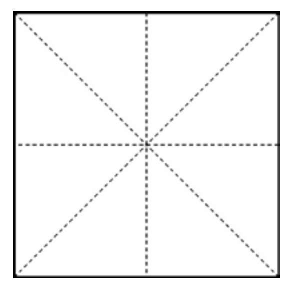
a. Two perpendicular lines of symmetry



Can the same be done with other lines of symmetry? Try.



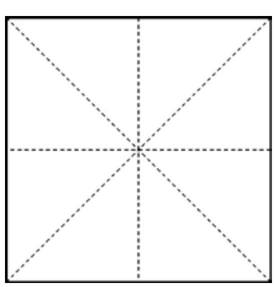
b. Two lines of symmetry at 45° angle



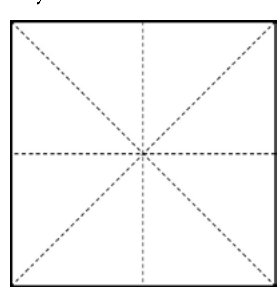
Does the coloured square have rotational symmetry? What is the order?

3. Mixing it up

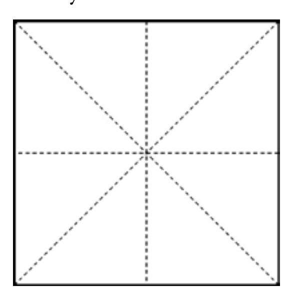
a. A line of symmetry and rotational symmetry of order 2



What if a different type of line of symmetry is selected?



b. A line of symmetry and rotational symmetry of order 4

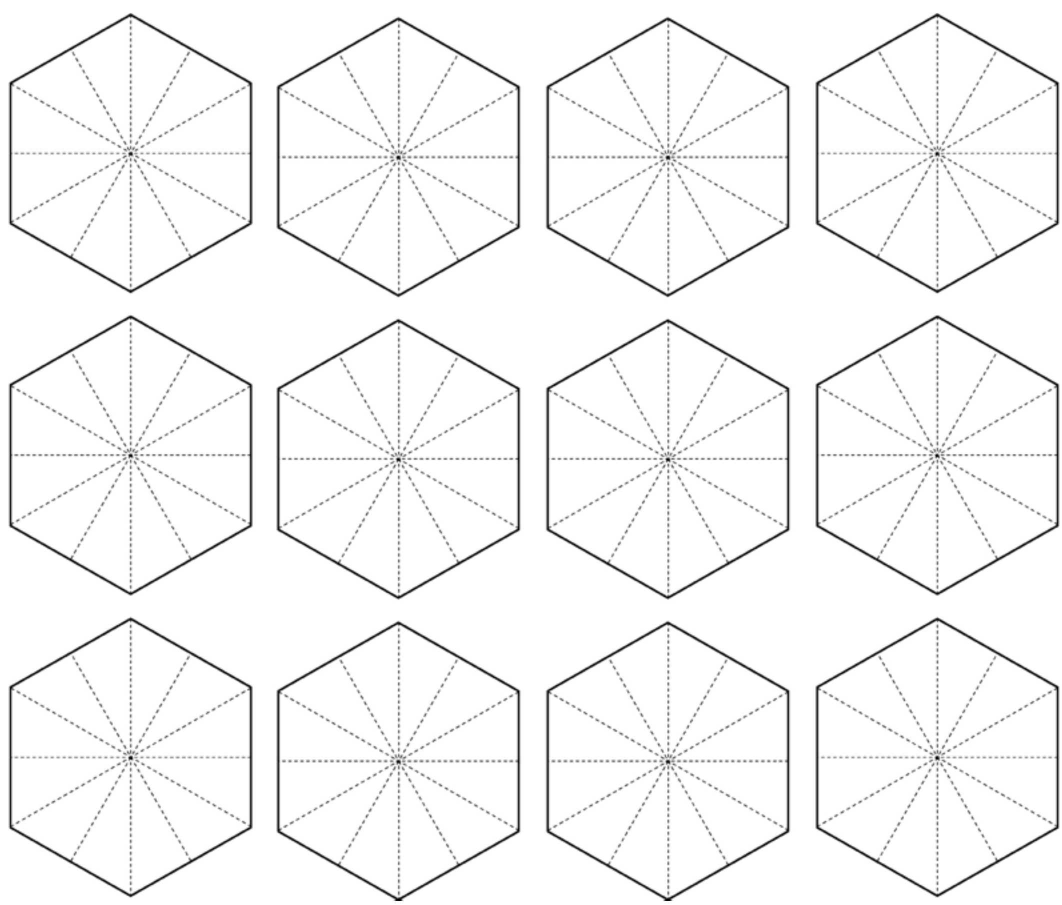


Does the coloured square have other line(s) of symmetry? What's the smallest angle between two such lines?

Regular Hexagon

1. Simple ones	2. Multiple lines of symmetry	3. Mixing it up
a. Exactly one line of symmetry and no rotational symmetry b. Only rotational symmetry of order 2 and no line symmetry c. Only rotational symmetry of order 3 and no line symmetry d. Only rotational symmetry of order 6 and no line symmetry	a. Two perpendicular lines of symmetry b. Two lines of symmetry with 60° angle between them Can the above be done with other lines of symmetry? Try. c. Two lines of symmetry with 30° angle between them Does the coloured hexagon have rotational symmetry? What is the order?	a. A line of symmetry and rotational symmetry of order 2 b. A line of symmetry and rotational symmetry of order 3 What if a different type of line of symmetry is selected? c. A line of symmetry and rotational symmetry of order 6 Does the coloured hexagon have other line(s) of symmetry? What's the smallest angle between two such lines?

[Optional] Now that you have explored three regular polygons, what do you think are the possibilities for a regular pentagon? What about a regular octagon? Try these on your own.



MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] – their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

This worksheet can be used to assess the understanding of symmetry. Note that most questions are open ended, i.e., have multiple correct options. All foster a sense of aesthetics and allow creativity. The first part for each regular polygon involves only one type of symmetry and avoids the rest. However, it may happen that other kinds of symmetry may creep in. The student should be guided to eliminate them. For example, Figure 1 has both line symmetry and rotational symmetry whereas Figure 2 has only line symmetry and no rotational one. So, Figure 1 is correct for Square 2.a, but not for Square 1.a.

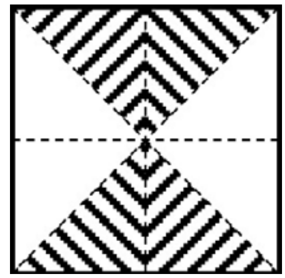


Figure 1

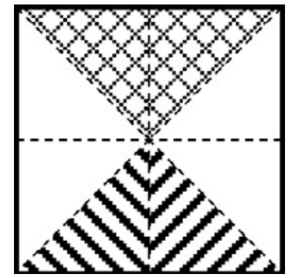


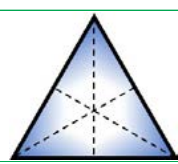
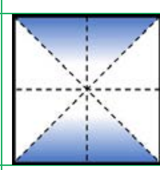
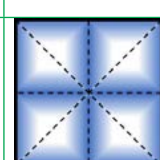
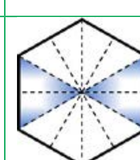
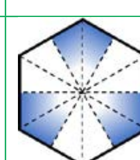
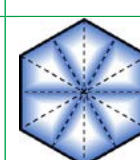
Figure 2

The order of rotational symmetry and its factors also play a crucial role. The order is 4 for square and it can be coloured to have order 2 or 4, which are factors of 4. Similarly, the hexagon can be coloured to have order 2, 3 or 6 – the factors of 6. Students can be encouraged to explore the possible orders for other coloured regular polygons.

The next set of questions with multiple lines of symmetry hints at the following:

1. Multiple lines of symmetry imply rotational symmetry
2. Order of the rotational symmetry = 360° ÷ (2 × the smallest angle between the lines of symmetry)
   Or, the angle of rotational symmetry = 2 × the smallest angle between the lines of symmetry

Students can be asked to tabulate their results of Triangle 3, Square 2 (a, b) and Hexagon 2 (a, b, c) to arrive at this relation.

	Equilateral triangle	**Square**		**Regular hexagon**		
Angle between lines of symmetry	60°	90°	45°	90°	60°	30°
Example of coloured polygons						
Angle of rotation	120°					
Order	3					

Note that, in the above example for Square 90°, the lines of symmetry pass through the midpoints of the sides. We could have chosen the diagonals to be the lines of symmetry to get another possibility. Similarly, in the above example for Hexagon 60°, the lines of symmetry are the diagonals. Instead, they could have passed through the midpoints of the sides. However, for Hexagon 90° such alternative choices are not available. One may wonder how this could be linked with the angle in question (second row of the above table).

The last set is the reverse of 1, illustrating

3. Line symmetry and rotational symmetry combined implies more than one lines of symmetry as well as the earlier relation between the order of rotational symmetry and the angle between the lines of symmetry. The above table can be utilized to cross reference the findings from this last set.

   In the optional part, students should realise that regular polygons with odd number of sides have only one kind of line of symmetry – each starting from a vertex and ending at the midpoint of the opposite side. On the other hand, regular polygons with even number of sides have two kinds of lines of symmetry – the diagonals and the lines connecting the midpoints of opposite sides.

This TearOut is based on the project of Keshvi, MA Education student at Azim Premji University as part of her Curricular Material Development – Mathematics course.

Stewart's Theorem for a Right Triangle – A Compact Proof

MOSHE STUPEL & VICTOR OXMAN

Stewart's theorem states the following (see Figure 1).

Let a, b, c be the sides of $\triangle ABC$.

Let D be any point on side AB, and let p be the length of cevian CD. Let the lengths of AD, BD be m, n.

Then $a^2 m + b^2 n = c\left(p^2 + mn\right)$.

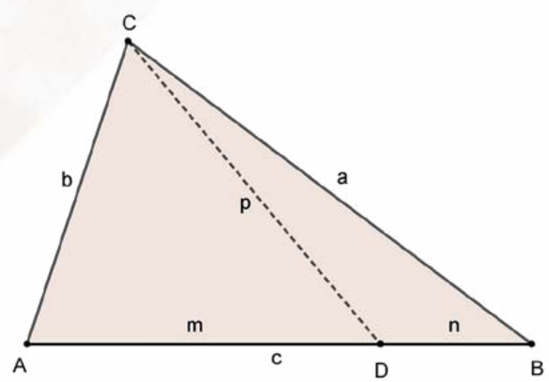


Figure 1

We give below a compact proof for the particular case when ABC is right-angled at C (see Figure 2).

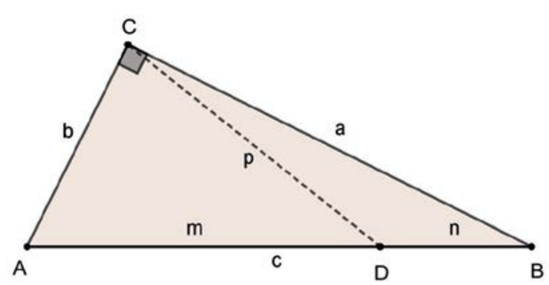


Figure 2

Keywords: Stewart's theorem, right-angled triangle, cevian

We shall prove that $a^2m^2 + b^2n^2 = c^2p^2$ and then show that this is equivalent to Stewart's theorem.

Proof. Construct $DE \perp AC$, $DF \perp CB$ (Figure 3). The proof follows on examining the resulting right-angled triangles.

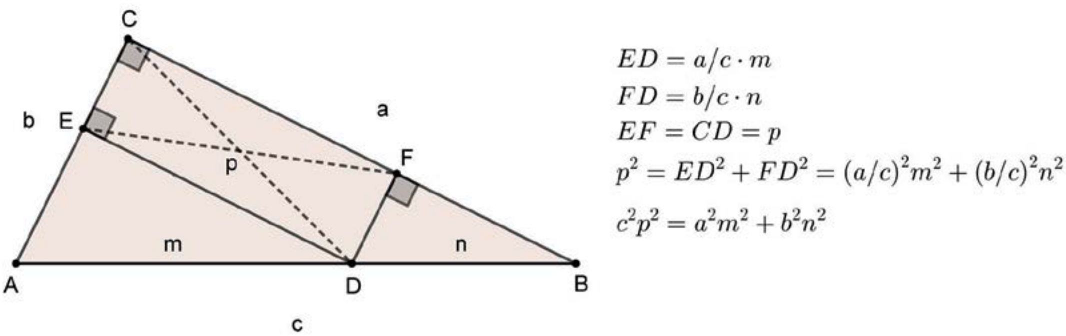


Figure 3

We now show that this result is a particular case of Stewart's theorem, $a^2m + b^2n = c(p^2 + mn)$.

Since $\triangle ABC$ is right angled, $a^2 + b^2 = c^2$. Multiply both sides of the above by $c = m + n$:

$$\begin{aligned}
&a^2m + b^2n = c(p^2 + mn)\\
&\iff (a^2m + b^2n)(m + n) = c^2(p^2 + mn)\\
&\iff a^2m^2 + b^2n^2 + mn(a^2 + b^2) = c^2p^2 + c^2mn\\
&\iff c^2p^2 = a^2m^2 + b^2n^2, \quad \text{since } a^2 + b^2 = c^2.
\end{aligned}$$

Remark. If $m = n = c/2$ (i.e., if CD is a median), we get $c^2p^2 = a^2(c/2)^2 + b^2(c/2)^2 = c^2 \cdot (c/2)^2$ and so $p = c/2$.

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Problems on Number Bases

A. RAMACHANDRAN

We normally use the base 10 or decimal system of writing numbers. One could also use other bases to write down numbers. If n is the number base, then the place values are (from right to left) 1, n, n^2, n^3,…. For example, take the number 124, which is 'one hundred and twenty four' in the base 10 system.

That is, $1 \times 10^2 + 2 \times 10 + 4 \times 1 = 124$.

When 124 is read in base 5, its value is $1 \times 5^2 + 2 \times 5 + 4 \times 1 = 39$ in base ten.

To write a number in base n we need n numerals: 0, 1, 2,…, $(n - 1)$. In the problems that follow, you are required to move between different number bases and make connections. We denote the numerals by lower case letters and the base used is indicated by a subscript at the end of the number. For instance, abc_5 represents a 3-digit number in base 5. The letters a,b,c need not represent different digits – two different letters could stand for the same digit. The first digit in any string of numerals is never zero. There may be more than one solution. Please find all solutions.

Hint: In solving these problems, first form an equation reflecting the values in the given base systems. Find values for a,b,c,d that satisfy this. The numerals to be considered will depend upon the bases involved. Consider only those that are less than the lower of the bases used. For instance, in problem 3, the bases used are 6 and 9; consider only the numerals 0-5.

Problems

1. If $2(abc_2) = abc_3$, find (a,b,c).
2. If $abc_6 = 2(abc_4)$, find (a,b,c).
3. If $abc_9 = 2(abc_6)$, find (a,b,c).
4. If $abc_6 = cba_5$, find (a,b,c).
5. If $abc_{12} = cba_{11}$, find (a,b,c).
6. If $abcd_{10} = 6(abcd_5)$, find (a,b,c,d).

Solutions to the problems are on page 59

A Cousin of the Rascal Triangle

JAMES METZ

While working on a counting problem, I encountered the need to compute the sum

$$(5)(1) + (4)(2) + (3)(3) + (2)(4) + (1)(5)$$

and I was immediately reminded of Pascal's triangle and the Rascal Triangle.[a] I took a detour and looked at a relative of the Rascal Triangle, shown in Figure 1.

					(1)(1)					
				(2)(1)		(1)(2)				
			(3)(1)		(2)(2)		(1)(3)			
		(4)(1)		(3)(2)		(2)(3)		(1)(4)		
	(5)(1)		(4)(2)		(3)(3)		(2)(4)		(1)(5)	
(m)(1)										(1)(m)

1

2 2

3 4 3

4 6 6 4

5 8 9 8 5

6 10 12 12 10 6

Figure 1

---

a I needed to find the sum of $1+(1+2)+(1+2+3)+(1+2+3+4)+(1+2+3+4+5)$. If you count the number of 1's in this expression, the number of 2's, the number of 3's, …, you get the expression shown above.

Keywords: Number patterns, sums, Rascal Triangle

The first row is row 0. Along each diagonal is a sequence of numbers with a common difference, first of 1, then 2, then 3, etc. Also, the central vertical column consists of the squares of the integers. The central number in row n (for n even) is $\frac{(n+2)^2}{4} = \frac{n^2}{4} + n + 1$.

The sum of the numbers in row n is given by $\frac{(n+1)(n+2)(n+3)}{6}$, which is the product of three consecutive integers, divided by 6. For example, the sum of the numbers in row 5 is $\frac{(6)(7)(8)}{6} = 56$.

Suppose we want to know the 3rd term in row 5, as shown in Figure 2, with the diamond showing $a = 6$, $b = 8$, $c = 9$ and x.

The value of x can be found using the formula $x = b + c + 1 - a$, so $x = 8 + 9 + 1 - 6 = 12$. This is the same formula used with the Rascal Triangle. We cannot use the formula $x = (bc + 1)/a$, the second formula for a Rascal Triangle, but instead we can use $x = bc/a$, noting that the product of the North and South numbers equals the product of the East and West numbers. Note that the sum of the North and South numbers and the sum of the East and West numbers differ by 1.

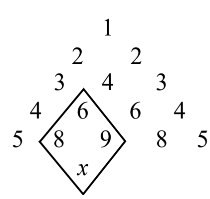

Figure 2

The value of x can also be found using the equation $x = a + n + 1$, which is different from the Rascal Triangle equation $x = a + n - 1$. For example, with $a = 6$ and $n = 5$, $x = 6 + 5 + 1 = 12$. For another example, if we want to find the 2nd element in row 8, $a = 7$ by inspection, and $n = 8$, so $x = 7 + 8 + 1 = 16$.

Substituting $a = b + c + 1 - x$ for a in $x = a + n + 1$ and solving for x, we have,

$$x = (b + c + n + 2)/2,$$

an equation that requires only b, c and the row number n, not unlike what was done with Pascal's Triangle but also using n. See Figure 3. To find the 3rd term in row 5, $x = (8 + 9 + 5 + 2)/2 = 12$.

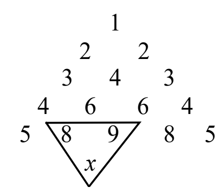

Figure 3

I never would have considered this investigation had I not known about the Rascal Triangle, so I thank the authors of "The Rascal Triangle."

References

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Constructions of Angles without Using a Protractor

RAJKUMAR KANOJIYA

Most students enjoy the topic of geometrical constructions using compass and ruler. When it comes to constructing specific angles, we consider only multiples of 15°. Students wonder: what about angles such as 10°, 20°, 40° and 50°? Can we construct these too?

In this article, we consider this problem and offer an approximate procedure which yields angles that are reasonably close to the desired measure. So, the solution that we give is not exact.

It is assumed that the student knows: (i) how to construct a 60° angle, (ii) how to divide a line segment in a given ratio, using compass and ruler only.

Construction procedure

- Step 1. Construct a 60° angle AOB (with OA = OB) using compass and ruler. Join AB.
- Step 2. Find the points C and D of trisection of segment AB. So, AC = CD = DB = AB/3.
- Then angle AOC is almost exactly 20° and angle DOA is almost exactly 40°. The reader may check this out.

Questions for the reader

1. Why does this work? We leave it to the reader to work this out.
2. How can this procedure be modified to yield angles measuring 10°, 50° and 70°?

Comment

Later in this issue, the editors have included a full analysis of this procedure.

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Approximate Angle Constructions

SHAILESH SHIRALI

In an article elsewhere in this issue, the author gives a construction procedure which yields angles close to 20° and 40°. In this article, we explain 'why' the procedure (depicted in Figure 1) gives good results and suggest an alternative procedure that gives better results.

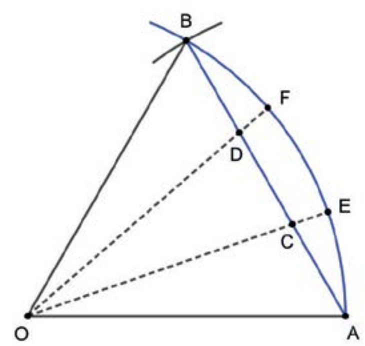

Figure 1. Construct $\measuredangle AOB = 60^\circ$ in the usual manner. Join AB. Trisect AB at points C and D (so $AC = CD = DB$). It will be found now that $\measuredangle COA \approx 20^\circ$ and $\measuredangle DOA \approx 40^\circ$. The approximation is reasonably close, with an error of 4.5%.

To check how close, we use coordinates. Assign coordinates so that $O = (0,0)$ and $A = (2,0)$. Then we have $B = 2 \cdot (\cos 60^\circ, \sin 60^\circ) = \left(1, \sqrt{3}\right)$. (Here we use the same symbol for the name of the point and the ordered pair giving the coordinates of the point.) Therefore the coordinates of C and D are given by:

$$C = \frac{2 \cdot A + 1 \cdot B}{3} = \frac{1}{3}\left(5, \sqrt{3}\right),$$
$$D = \frac{1 \cdot A + 2 \cdot B}{3} = \frac{1}{3}\left(4, 2\sqrt{3}\right).$$

Keywords: Construction, compass, ruler, approximation, section formula, trigonometry

This implies that

$$\tan \measuredangle COA = \frac{\sqrt{3}}{5}, \quad \therefore \measuredangle COA = \arctan\left(\frac{\sqrt{3}}{5}\right) \approx 19.11^\circ.$$

Similarly we find

$$\measuredangle DOA = \arctan\left(\frac{\sqrt{3}}{2}\right) \approx 40.89^\circ.$$

Another procedure that gives better results. Here is a small tweak on the above procedure.

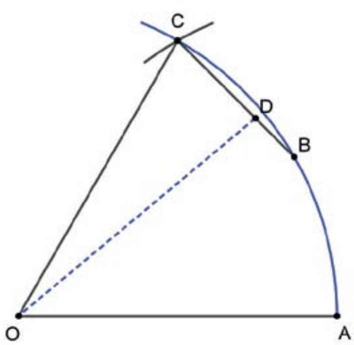

Figure 2. Construct $\measuredangle AOC = 60^\circ$ and $\measuredangle AOB = 30^\circ$. Join BC. Trisect segment BC at point D (so $BD = BC/3$). It is found that $\measuredangle DOA \approx 40^\circ$ and $\measuredangle COD \approx 20^\circ$.

We may analyse the procedure just as we did earlier. Let $O = (0,0)$ and $A = (2,0)$. Then we have $B = (\sqrt{3}, 1)$ and $C = (1, \sqrt{3})$. Hence

$$D = \frac{2B + C}{3} = \frac{1}{3}(2\sqrt{3} + 1, 2 + \sqrt{3}).$$

Therefore

$$\tan \measuredangle DOA = \frac{2 + \sqrt{3}}{2\sqrt{3} + 1} = \frac{1}{11}(4 + 3\sqrt{3}),$$

giving significantly better results than earlier:

$$\measuredangle DOA \approx 39.896^\circ, \qquad \measuredangle COD \approx 20.104^\circ.$$

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Problem Corner

Two Properties of Natural Numbers all of whose Digits are 1

SASIKUMAR K

In this short article we study two properties of natural numbers all of whose digits (when the numbers are expressed in base 10) are 1. (Such numbers, in which the digit 1 is repeated several times, are also known as *repunits*.)

Proposition 1. *The number* $111\ldots11$ *with '1' repeated n times cannot be a perfect square for any natural number* $n > 1$.

Proof of proposition 1. We need not consider the case $n = 2$ as we know that 11 is not a perfect square. Hence we assume that $n > 2$.

Let $x = 111\ldots11$ with '1' repeated $n > 2$ times. Suppose that x is a perfect square, say

$$x = (2m+1)^2 \qquad \text{(for some integer } m > 0). \qquad (1)$$

Then we have:

$$10^{n-1} + 10^{n-2} + \cdots + 10 + 1 = 4m^2 + 4m + 1,$$
$$\therefore\ 10^{n-1} + 10^{n-2} + \cdots + 10 = 4m^2 + 4m,$$
$$\therefore\ 5\left(10^{n-2} + \cdots + 10 + 1\right) = 2m(m+1). \qquad (2)$$

In (2), observe that the quantity on the left side is odd (since $n > 2$), whereas the quantity on the right side is even. Therefore, equality (2) cannot hold. It follows that our assumption that x is a perfect square is wrong.

Hence the number $111\ldots11$ with '1' repeated $n > 1$ times cannot be a perfect square for any natural number $n > 1$. □

Keywords: Repunit, natural number, divisible

Remark. Proposition 1 can also be established by looking at the remainder in the division

$$\underbrace{111\ldots11}_{n \text{ times}} \div 4.$$

If $n > 1$, the remainder will always be 3, since $111\ldots11 = 11 +$ a multiple of 100, and 11 leaves remainder 3 under division by 4.

On the other hand, no square leaves remainder 3 under division by 4. Hence the conclusion. $\square$

Proposition 2. *The number* $\underbrace{111\ldots11}_{n \text{ times}}$ *is divisible by n if n is a power of* 3.

Proof of proposition 2. We have:

$$\underbrace{111\ldots11}_{n \text{ times}} = 10^{n-1}+10^{n-2}+\cdots+10+1 = \frac{10^n - 1}{9}, \tag{3}$$

by summing the geometric progression.

Let $n = 3^k$ where $k \in \mathbb{N}$. We claim that

$$\frac{10^{3^k} - 1}{9 \cdot 3^k} \quad \text{is a positive integer for every } k \in \mathbb{N}. \tag{4}$$

In other words:

$$10^{3^k} - 1 \quad \text{is divisible by } 3^{k+2} \text{ for every } k \in \mathbb{N}. \tag{5}$$

We shall use the principle of mathematical induction to establish this.

The proposition is true for $k = 1$ because 999 is divisible by 27 (note that $999 = 27 \times 37$).

Let us assume that (5) is true for some positive integer $k = m$. That is,

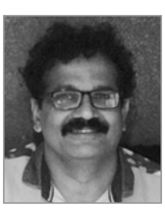
$$10^{3^m} - 1 \quad \text{is divisible by } 3^{m+2}. \tag{6}$$

Let $10^{3^m} - 1 = a \cdot 3^{m+2}$ for some positive integer a. Then we have:

$$\begin{aligned} 10^{3^{m+1}} - 1 &= \left(10^{3^m}\right)^3 - 1 \\ &= \left(10^{3^m} - 1\right) \cdot \left(10^{2\cdot 3^m} + 10^{3^m} + 1\right) \\ &= a \cdot 3^{m+2} \cdot \left(10^{2\cdot 3^m} + 10^{3^m} + 1\right). \end{aligned} \tag{7}$$

In (7), consider the quantity $10^{2\cdot 3^m} + 10^{3^m} + 1$. Since 10 is 1 more than a multiple of 3 (i.e., it is of the form $3t + 1$ for some $t \in \mathbb{N}$), every power of 10 will also be of this form. This implies that $10^{2\cdot 3^m} + 10^{3^m} + 1$ is a multiple of 3. It follows that $10^{3^{m+1}} - 1$ is a multiple of 3^{m+3}.

This establishes the inductive step and hence the stated proposition. $\square$



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Repunit Primes

SHAILESH SHIRALI

A *repunit* is a natural number all of whose digits are 1; see [2]. (Here, numbers are expressed in base ten.) Elsewhere in this issue of the magazine, Sasikumar proves two properties of such numbers. In this short note we consider the question of which repunits are prime numbers.

Definition. For any natural number n, we define R_n to be the repunit with n digits:

$$R_n = \underbrace{111\ldots11}_{n\ \text{1's}}. \tag{1}$$

A question which arises naturally when we define these numbers is the following:

Problem 1. For which positive integers n is R_n prime?

The first such prime is $R_2 = 11$, but the supply seems quite limited after that:

- $R_3 = 111 = 3 \times 37$ is not prime.
- $R_4 = 1111 = 11 \times 101$ is not prime.
- $R_5 = 11111 = 41 \times 271$ is not prime.
- $R_6 = 111111 = 11 \times 10101 = 3 \times 7 \times 11 \times 13 \times 37$ is not prime.
- $R_7 = 1111111 = 239 \times 4649$ is not prime.

The following should be clear:

Proposition 1. *If n is even, then $11 \mid R_n$.*

It follows that if $n > 2$ and is even, then R_n is composite.

More generally we have the following:

Proposition 2. *If n is composite, then R_n is composite.*

Keywords: Repunit, composite, prime, converse, contrapositive

Indeed, if $a \mid n$ where $a > 1$, then $R_a \mid R_n$. (Here the notation $u \mid v$ means that u is a divisor of v.) This can be seen by direct division. For example:

$$\begin{aligned} R_4 &= R_2 \times 101, \\ R_6 &= R_2 \times 10101, \\ R_8 &= R_2 \times 1010101, \\ R_9 &= R_3 \times 1001001, \\ R_{15} &= R_3 \times 1001001001001, \\ R_{15} &= R_5 \times 10000100001, \end{aligned}$$

and so on; the pattern should be clear. For a formal proof we note that for integers $m, k > 1$,

$$m - 1 \mid m^k - 1. \qquad (2)$$

Indeed, the quotient in the division $(m^k - 1) \div (m - 1)$ is

$$\frac{m^k - 1}{m - 1} = m^{k-1} + m^{k-2} + \cdots + m + 1.$$

In particular, we have:

$$10^a - 1 \mid 10^{ab} - 1. \qquad (3)$$

This implies that $9 \times R_a \mid 9 \times R_{ab}$ and therefore that $R_a \mid R_{ab}$.

This proves that if n is composite, then so is R_n.

The contrapositive of Proposition 2 is the following:

Proposition 3. *If R_n is prime, then n is prime.*

The converse of this proposition is not true; e.g., R_5 and R_7 are composite (as may be seen from the factorizations given above). Nor are R_{11}, R_{13} and R_{17}:

$$\begin{aligned} R_{11} &= 21649 \times 513239, \\ R_{13} &= 53 \times 79 \times 265371653, \\ R_{17} &= 2071723 \times 5363222357. \end{aligned}$$

However, R_{19} is prime.

So we ask again: for which primes n is R_n prime? After R_{19} the next prime is encountered quickly; we find that R_{23} is prime. But following that we have a very long stretch of composite numbers. The next prime number in the list is R_{319}, and after that it is R_{1031}. A strange pattern!

Clearly, there is some mystery here. It is conjectured that there are infinitely many repunit primes. However, the question remains open. See [1] for the list of known primes p for which R_p is prime.

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Searching for a 3rd Order Magic Square of Squares

ANANT PRATAP SINGH

Third order magic squares, i.e., of order 3×3, are the simplest kind of magic squares. Nevertheless, there are a few unsolved problems concerning such objects. For example we may ask:

Problem. Does there exist a 3×3 magic square composed only of square numbers?

This problem remains unsolved. As of today, no such magic square has been found. Nor has it been proven that no such magic square exists.

Recall that a magic square is a square array of distinct positive integers, arranged so that the sum of the numbers is the same for each row, each column, and the two main diagonals. This common sum is the *magic sum* of the square.

The structure of 3×3 magic squares

For a 3×3 magic square, this means that 8 sums need to be equal to one another. The most familiar such square is the following:

4	9	2
3	5	7
8	1	6

(1)

Keywords: Magic square, third order magic square, magic sum, arithmetic progression

Here the magic sum of the square is 15:

$$4+9+2=3+5+7=8+1+6=4+3+8=9+5+1=2+7+6=4+5+6=2+5+8=15.$$

It is known that for any 3×3 magic square, if the magic sum is S and the entry in the central square is a, then

$$S = 3a. \tag{2}$$

Readers who are unfamiliar with this result may enjoy trying to prove it.

An immediate implication of (2) is that with each 3×3 magic square, we can associate 4 distinct arithmetic progressions (APs), each of which has a as the central number.

In this article, we propose to find a 3×3 square of perfect squares in which 7 out of the 8 sums are equal to one another. This would be 'close' to being a magic square.

Let the central number of the square be a. From the observation made above about APs, it follows that the magic square must have the following form:

$a-b$	$a+b+c$	$a-c$
$a+b-c$	a	$a-b+c$
$a+c$	$a-b-c$	$a+b$

(3)

Observe the 4 APs:

$$\{a-b, a, a+b\}, \quad \{a-c, a, a+c\}, \quad \{a-b-c, a, a+b+c\}, \quad \{a+b-c, a, a-b+c\}. \tag{4}$$

We can view the 9 elements of the array as made up of 3 separate, disjoint APs of 3 terms each:

$$\left.\begin{aligned} &\{a-b-c,\ a-b,\ a-b+c\}, \\ &\{a-c,\ a,\ a+c\}, \\ &\{a+b-c,\ a+b,\ a+b+c\}. \end{aligned}\right\} \tag{5}$$

Observe that each AP has common difference c. This observation is crucial.

To find a magic square of the stated kind, we must find 3 different APs of squares sharing the same common difference. This seems a daunting task at the outset. Let us see where the algebra will lead us.

Looking for 3-term APs composed of perfect squares

If the perfect squares u^2, w^2, v^2 are in AP, then we have

$$u^2 + v^2 = 2w^2 \tag{6}$$

$$\iff \quad 2u^2 + 2v^2 = 4w^2$$

$$\iff \quad (u+v)^2 + (u-v)^2 = (2w)^2. \tag{7}$$

Hence the triple $(u+v, u-v, 2w)$ *is Pythagorean.* (Note, however, that it is not primitive; all the numbers in the triple are even.)

We now invoke a well-known way of generating Pythagorean triples: by using the formula

$$\left(m^2 - n^2, 2mn, m^2 + n^2\right), \tag{8}$$

with $m > n$. This formula generates infinitely many Pythagorean triples.

Equating the triple $(u + v, u - v, 2w)$ with the above triple, i.e., setting

$$\left.\begin{aligned} u + v &= m^2 - n^2, \\ u - v &= 2mn, \\ 2w &= m^2 + n^2, \end{aligned}\right\} \tag{9}$$

and solving for u, v, w, we get

$$u = \frac{m^2 - n^2 + 2mn}{2}, \qquad v = \frac{m^2 - n^2 - 2mn}{2}, \qquad w = \frac{m^2 + n^2}{2}. \tag{10}$$

Evidently, m, n must be both odd or both even for u, v, w to be integers.

Using (10), we can generate an unlimited supply of 3-term APs of perfect squares. Thus we have:

m, n	u, v, w	u^2, v^2, w^2	AP	Common difference
3, 1	7, 1, 5	49, 1, 25	1, 25, 49	24
5, 1	17, 7, 13	289, 49, 169	49, 169, 289	120
7, 1	31, 17, 25	961, 289, 625	289, 625, 961	336
9, 1	49, 31, 41	2401, 961, 1681	961, 1681, 2401	720
10, 4	82, 2, 58	6724, 4, 3364	4, 3364, 6724	3360
14, 4	146, 34, 106	21316, 1156, 11236	1156, 11236, 21316	10080

Looking for the required magic square

We now use the above data to generate a 3×3 square with the required property. Our strategy will be to generate a large number of 3-term APs of perfect squares and to look for common differences that are repeated 3 or more times. We find the following common differences repeated multiple times:

$$3360, \;\; 13440, \;\; 43680, \;\; 53760, \;\; 127680, \;\; 174720, \;\; 215040, \;\; \ldots. \tag{11}$$

Three APs for which the common difference is 3360 are the following:

$$\{2^2, \; 58^2, \; 82^2\}, \qquad \{46^2, \; 74^2, \; 94^2\}, \qquad \{97^2, \; 113^2, \; 127^2\}. \tag{12}$$

The 3×3 square that can be formed using these APs is:

46^2	127^2	58^2
82^2	74^2	97^2
113^2	2^2	94^2

Every sum is equal to 21609, except for one diagonal which adds to 16428. That's close!

Here are 3 other such squares:

103^2	446^2	218^2
302^2	233^2	334^2
394^2	62^2	313^2

2729^2	6271^2	1418^2
1838^2	2969^2	6049^2
6161^2	802^2	3191^2

4324^2	6382^2	2836^2
3676^2	4916^2	5458^2
5938^2	1604^2	5444^2

In all of them, the row sums, column sums and diagonal sums are all equal — except for one sum!

References

1. Wikipedia, "Magic squares" from https://en.wikipedia.org/wiki/Magic_square
2. Shailesh Shirali, *Adventures in Problem Solving*

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Are you an enigmathologist?

- Find the value of a and b if $a + b = a \cdot b = a / b$.
- Fill in the blank in the sentence below with a suitable number name. Is there more than one possibility?

The letter 'e' appears _____ times in this sentence.

- Fill in the blanks with names of months:
  In any calendar year the first of __________ and the first of __________ fall on the same day of the week. Likewise, the first of __________ and the first of __________ fall on the same day of the week.
- What is the temperature at which the Celsius and Fahrenheit scales show the same numerical value but differ in sign?

Send in your students' answers to AtRiA.editor@apu.edu.in

Suma Numbers

HARA GOPAL R

Definition 1. *N is called a 'Suma' Number if it is the smallest positive integer for which the number of factors of N is equal to the units digit of N.*[a]

For Example:

(1) Consider **14**
Number of Factors of 14 = 4 [1, 2, 7 & 14]
Units digit of 14 is 4

(2) Consider **76**
Number of Factors of 76 = 6 [1, 2, 4, 19, 38, 76]
Units digit of 76 is 6

Before finding 'Suma' numbers, we must know how to calculate the number of factors of any given number.

To find the number of factors of N, we first need to express N as the product of prime numbers in exponential form, i.e.,

$$N = p_1^{\alpha_1} \times p_2^{\alpha_2} \times p_3^{\alpha_3} \times \ldots . \times p_n^{\alpha_n},$$

where $p_1, p_2, p_3, \ldots . p_n$ are distinct prime numbers and α_1, $\alpha_2, \alpha_3, \ldots \alpha_n$ are positive integers.

Then the number of factors of $N = (\alpha_1 + 1) \times (\alpha_2 + 1) \times (\alpha_3 + 1) \times \ldots . \times (\alpha_n + 1)$.

a The author usually names his findings with a specific name that reflects the property of that number or work. If no such name suits, then he uses the name of a person who motivates his mathematical observations. Mrs. Suma is one such person.

Keywords: Factors, number relations, reasoning, justification

For Example: To find the number of factors of 72.

$$72 = 2^3 \times 3^2$$
$$\therefore \text{Number of factors of } 72 = (3+1) \times (2+1)$$
$$= 4 \times 3 = 12.$$

Now let us find some 'Suma' Numbers, proceeding systematically as the units digit changes from 1 to 9. Remember the definition of a Suma Number: The smallest positive integer for which the units digits is equal to the number of factors.

Units Digit	Reasoning	Suma Number
1	There exists one and only one number with only one factor and that is '1'	The 'Suma' number with units digit '1' is 1
2	The only numbers which have exactly 2 factors are the prime numbers and of these, 2 is the only prime with units digit 2	The 'Suma' number with units digit '2' is 2
3	The only numbers which have exactly 3 factors are the numbers of the form p^2 where p is a prime number. No square number has units digit 3, so N (with units digit 3) $= p^2$ is impossible.	There is no 'Suma' number with units digit 3
4	Since $4 = 4 \times 1$ or 2×2, using the above statement regarding the number of factors of a number, the only numbers which have exactly 4 factors are numbers which are either in the form: p^3 where p is a prime number; OR	There exists no prime p such that the units digit of its cube is 4.
	$p_1{}^1 \times p_2{}^1$ where p_1, p_2 are distinct prime numbers. If $N = p_1{}^1 \times p_2{}^1$ Since $4 = 2 \times 2$ does not fit the constraints, consider the next positive integer with units digit 4, i.e., 14. $14 = 2 \times 7$ and has 4 factors namely 1, 2, 7, 14. [**Note:** for $p_1 = 2$ & $p_2 = 17$ also, we get units digit of N = 4. For any prime p_2 whose units digit is 7, $N = 2 \times p_2$ gets unit digit = 4; however, the Suma number, being the smallest positive integer satisfying this property, is 14]	The 'Suma' number with units digit '4' is 14.
5	To get the number of factors of 'N' equal to 5, N should be in the form of p^4 where p is a prime number and the units digit of p has to be 5 for the units digit of N to be 5. $5^4 = 625$	The 'Suma' number with units digit '5' is 625.

Units Digit	Reasoning	Suma Number
6	To get the number of factors of 'N' equal to 6, we consider that $6 = 1 \times 6$ or $2 \times 3 = (\alpha_1 + 1) \times (\alpha_2 + 1)$ So, $\alpha_1 = 0$ and $\alpha_2 = 5$ which gives $N = p^5$ where p is a prime number Or $\alpha_1 = 1$ and $\alpha_2 = 2$, which gives $N = {p_1}^1 \text{ x } {p_2}^2$, where p_1, p_2 are distinct prime numbers. If $N = p^5$ there exists no prime p such that the units digit of N is 6 If $N = {p_1}^1 \times {p_2}^2$ we proceed systematically through 6, 16, 26, 36, … until 76, when we get $p_1 = 19$ & $p_2 = 2$ $19 \times 2^2 = 19 \times 4 = 76$	The 'Suma' number with units digit '6' is 76
7	To get the number of factors of 'N' equal to 7, N should be in the form of p^6 where p is a prime number. But no prime exists whose sixth power ends with 7.	So there exists no 'Suma' number with the units digit '7'
8	To get the number of factors of 'N' equal to 8, N should be in the form of p^7 where p is a prime number OR	If $N = p^7$ there exists no prime p such that 'N' ends with 8
	${p_1}^1 \times {p_2}^3$ where p_1, p_2 are prime numbers. If $N = {p_1}^1 \times {p_2}^3$ for $p_1 = 11$ & $p_2 = 2$ we get units digit of $N = 8$ $[11 \times 2^3 = 11 \times 8 = 88]$ [**Note:** for $p_1 = 31$ & $p_2 = 2$ also we get units digit of $N = 8$ & for any prime p_1 whose units digit is 1, above 'N' gets units digit = 8]	'88' is the 'Suma' number with units digit '8'
9	To get the number of factors of 'N' equal to 9, N should be in the form of p^8 where p is a prime number or ${p_1}^2 \times {p_2}^2$ where p_1, p_2 are prime numbers.	If $N = p^8$ there exists no prime p such that 'N' ends with 9
	If $N = {p_1}^2 \times {p_2}^2$ for $p_1 = 3$ & $p_2 = 11$ we get units digit of $N = 9$ $[3^2 \times 11^2 = 9 \times 121 = 1089]$ [**Note:** for $p_1 = 31$ & $p_2 = 3$ also we get units digit of $N = 9$ & for any prime p_1 whose units digit is 3, p_2 whose units digit is 1, above 'N' gets units digit = 9]	So '1089' is the 'Suma' number with units digit '9'

Here is the list of 'Suma' Numbers with different units digits

For the Units Digit	'Suma' Number	Number of Factors
1	1	1
2	2	2
3	No such number exists	–
4	14	4
5	625	5
6	76	6
7	No such number exists	–
8	88	8
9	1089	9

There is a chance of extending this for the last two or more digits of given number:

Observe the number 2312.

To find the number of factors for 2312 …

Let us write 2312 as $2^3 \times 17^2$

$\therefore$ No of factors of 2312 $= (3 + 1) \times (2 + 1) = 12$

We get exactly 12 factors which means numbers of factors is equal to last two digits of 2312.

So 2312 is the 'Suma' number whose last two digits (12) is equal to number of factors of 2312, provided it is the smallest number which satisfies this property. Can you verify this? Which clearly indicates that there will be a chance for the existence of 'Suma' numbers with higher number of factors which are equal to last group of digits instead of units digit alone.

Closing Remarks: Such investigations improve students' observation skills. By understanding constraints, doing a systematic search and documenting their work, they find that even small connections between numbers are to be noted and they have beautiful patterns. This will encourage them to try and find such interesting relations and connections between the numbers, so that with the help of their logical & reasoning skills they may develop research abilities also.

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Dwelling on the Incircle

PRITHWIJIT DE

The purpose of this note is to encourage students and teachers alike to explore simple configurations in geometry and come up with interesting questions which can be answered by employing elementary knowledge of plane geometry.

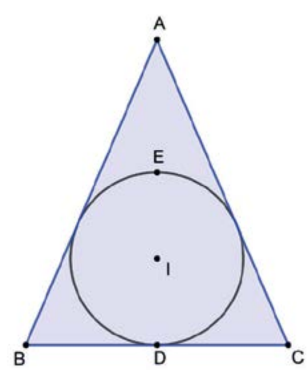

Figure 1

Given a triangle ABC with $AB = AC$ (see Figure 1). Let I be its incentre, and r its inradius. For such a triangle the circumcentre (O), centroid (G), orthocentre (H) and nine-point centre (N) lie on the line AI. It may so happen that any of these points lies on the incircle. Can we determine all such isosceles triangles ABC with $AB = AC$, up to similarity, for which any one of O, G, H and N lies on the incircle? Let us explore.

Keywords: Incircle, incentre, circumcentre, centroid, orthocentre, antipode

Let $AB = AC = x$ and $BC = y$. We consider each possibility in turn.

If the circumcentre lies on the incircle …. Suppose O lies on the incircle. Let D be the point of intersection of AI and BC and let E be the antipode of D on the incircle (i.e., the point diametrically opposite to D). Thus O can coincide with either D or E.

If O coincides with D, then $\measuredangle BAC = \frac{1}{2}\measuredangle BOC = 90^\circ$, so ABC is a right-angled isosceles triangle.

If O coincides with E, then $OD = R\cos A = 2r$, so

$$AD = AO + OD = R + 2r,$$

where R is the radius of the circumcircle, and

$$AD = AI + ID = r\csc(A/2) + r.$$

Combining these we obtain

$$\frac{r}{R} = \frac{\sin(A/2)}{1 - \sin(A/2)} = \frac{\cos A}{2}.$$

Using $\cos A = 1 - 2\sin^2(A/2)$ we reduce the last equation to

$$1 - 3\sin(A/2) - 2\sin^2(A/2) + 2\sin^3(A/2) = (1 + \sin(A/2))(1 - 4\sin(A/2) + 2\sin^2(A/2)) = 0.$$

Since $0 < \sin(A/2) < 1$, the only admissible root of this equation is $\sin(A/2) = 1 - 1/\sqrt{2}$ whence $\measuredangle BAC = 34^\circ$, approximately. See Figure 2.

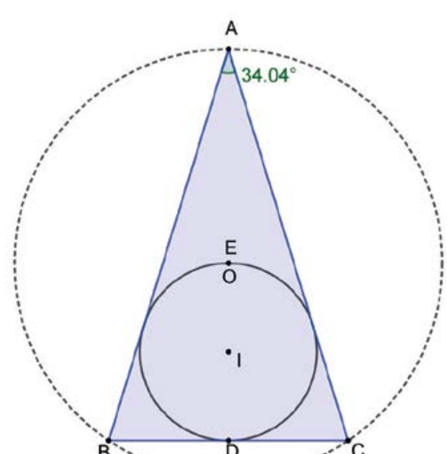

Figure 2

If the centroid lies on the incircle …. If G lies on the incircle, it must coincide with E. Hence $GD = 2r$. Since AD is the median we must have $GD = AD/3$. Therefore

$$2r = \frac{1}{3}\sqrt{x^2 - y^2/4}.$$

But

$$r = \frac{[ABC]}{s} = \frac{\frac{y}{2}\sqrt{x^2 - y^2/4}}{(2x+y)/2}.$$

Eliminating r from these relations, we get

$$6y = 2x + y, \quad \therefore \; \frac{y}{x} = \frac{2}{5},$$

and $\sin(A/2) = y/2x = 1/5$ and $\measuredangle BAC = 23°$, approximately. See Figure 3.

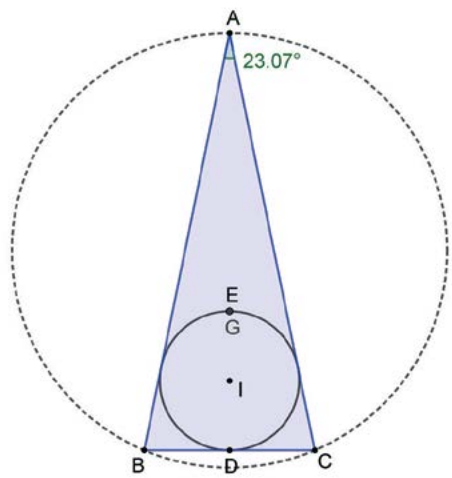

Figure 3

If the orthocentre lies on the incircle .... Similarly, if H lies on the incircle, then it has to coincide with E, so $HD = 2r$. Angle chasing gives $\measuredangle DAB = \measuredangle DBH$, therefore the triangles DAB and DBH are similar. Hence

$$\frac{BD}{AD} = \frac{HD}{BD}.$$

But $BD = y/2$, $AD = \sqrt{x^2 - y^2/4}$ and $HD = 2r = \dfrac{2y\sqrt{x^2 - y^2/4}}{2x+y}$. Therefore we obtain

$$y^2 = \frac{2y(4x^2 - y^2)}{2x+y}$$

whence $4x = 3y$ implying that $\sin(A/2) = y/2x = 2/3$ and $A = 84°$, approximately. See Figure 4.

If the nine-point centre lies on the incircle .... Lastly, we consider the case where N lies on the incircle. Recall that N is the centre of the circumcircle of the triangle (the *medial triangle*) whose vertices are the midpoints of the sides of ABC. Since the medial triangle of any triangle is similar to the original triangle with similarity ratio $1/2$, the radius of the nine-point circle is half of the radius of the circumcircle of the original triangle. Since the nine-point circle passes through D, the midpoint of BC, we assert that if N lies on the incircle of ABC then it has to coincide with E. Hence $ND = 2r$. But, if R is the circumradius of

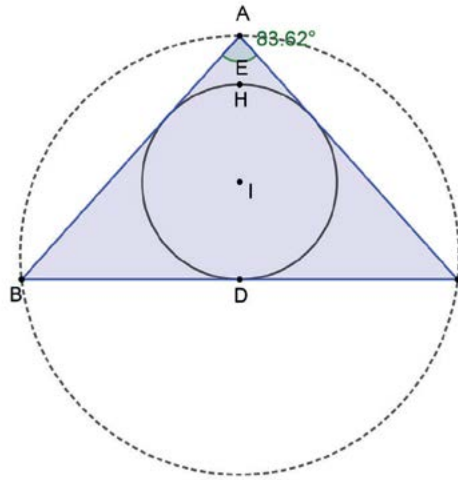

Figure 4

ABC then $ND = R/2$. Therefore $R = 4r$. Now

$$\frac{R}{r} = \frac{x^2y(2x+y)}{8[ABC]^2} = \frac{2x^2y(2x+y)}{y^2(4x^2-y^2)} = \frac{2x^2}{y(2x-y)},$$

whence

$$\frac{2x^2}{y(2x-y)} = 4$$

implying that

$$(x/y)^2 - 4(x/y) + 2 = 0.$$

Thus $x/y = 2 \pm \sqrt{2}$ and $\sin(A/2) = y/2x = (2 \pm \sqrt{2})/4$. The corresponding values of $\measuredangle BAC$ are 117° and 17°, approximately. See Figure 5. (The figure for the other possibility has not been drawn as there is a difficulty with the scale.)

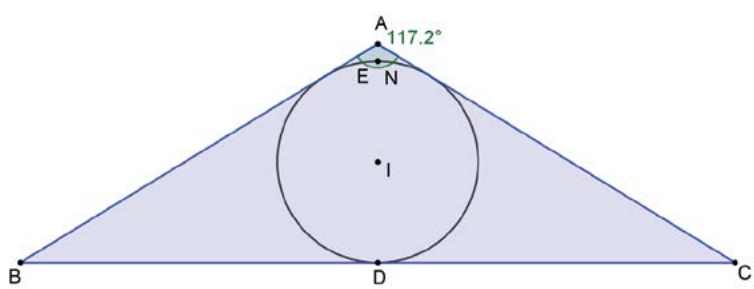

Figure 5

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Solutions to Problems on Number Bases

Problems may be found on page 37

1. Since one of the bases is 2, we need to consider only the numerals 0 and 1.

   We have $2(4a + 2b + c) = 9a + 3b + c$. On simplification this yields $b + c = a$. Since $a = 1$, one of b,c is 1 and the other is zero. If $b = 1$, $c = 0$, we have the solution $2(110_2) = 110_3 = 12_{10}$. If $c = 1$, $b = 0$, we have $2(101_2) = 101_3 = 10_{10}$.

2. We need to consider only the numerals 0-3. We have $36a + 6b + c = 2(16a + 4b + c)$. This leads to $4a = 2b + c$. The maximum value of RHS is 9 and so a can be only 1 or 2. The possible solutions are $112_6 = 2(112_4) = 44_{10}$, $120_6 = 2(120_4) = 48_{10}$ and $232_6 = 2(232_4) = 92_{10}$.

3. We need to consider only the numerals 0-5. We have $81a + 9b + c = 2(36a + 6b + c)$. This leads to $9a = 3b + c$. Maximum value of RHS is 20 and so a can be only 1 or 2. The possible solutions are (1,2,3), (1,3,0) and (2,5,3). That is, $123_9 = 2(123_6) = 102_{10}$, $130_9 = 2(130_6) = 108_{10}$ and $253_9 = 2(253_6) = 210_{10}$.

4. We need to consider only the numerals 0-4. We have $36a + 6b + c = 25c + 5b + a$ leading to $35a + b = 24c$. The only solution to this is (2,2,3). That is, $223_6 = 322_5 = 87_{10}$.

5. The relation given in the problem yields $144a + 12b + c = 121c + 11b + a$ leading to $143a + b = 120c$. A bit of trial and error yields the following as the only solution: (5,5,6). That is, $556_{12} = 655_{11} = 786_{10}$.

   (For bases higher than 10, we may sometimes need new numerals: for base 11, one numeral for ten, generally represented by 'A', and for base 12, one more for eleven, generally represented by 'B'. As an illustration consider the number $2A3B_{12}$. This stands for $2 \times 12^3 + 10 \times 12^2 + 3 \times 12 + 11 \times 1 = 4943$ in base 10.)

6. We need to consider only the numerals 0-4. We have $1000a + 100b + 10c + d = 6(125a + 25b + 5c + d)$ leading to $50a = 10b + 4c + d$. The maximum value of RHS is 60. So $a = 1$. Then we can have $b = 4$, $c = 2$, $d = 2$ and $b = 3$, $c = 4$, $d = 4$. That is, $1422_{10} = 6(1422_5)$ and $1344_{10} = 6(1344_5)$.

It is hoped that the reader sees that the common aspect of different number bases is that of place value, a key idea that enables one to write down large numbers in a compact form and using only a few symbols. Students develop greater flexibility in their problem solving skills with such activities.

Student Corner

A Property of a Class of Polygons

ANKUSH KUMAR PARCHA & TOYESH PRAKASH SHARMA

Consider a polygon $\mathcal{P}$ with the following three features.

(1) It is simple, i.e., it does not intersect itself, and it has no holes. $\mathcal{P}$ is permitted to be non-convex, i.e., with indentations.

(2) The number of sides of $\mathcal{P}$ is of the form $4k + 2$.

(3) If the polygon is $P_1P_2P_3 \ldots P_{4k+2}$, then the $2k + 1$ 'main' diagonals P_1P_{2k+2}, P_2P_{2k+3}, $\ldots$, $P_{2k+1}P_{4k+2}$ meet in a point.

Three examples of such polygons are shown in Figure 1. Note that a triangle with 3 concurrent cevians may be considered as a special case of such a polygon (see the figure in the middle; it may be regarded as a hexagon). Likewise for a quadrilateral with a line segment drawn through the point of intersection of the two diagonals (the third figure). All the polygons shown below have $4k + 2 = 6$ sides, i.e., with $k = 1$.

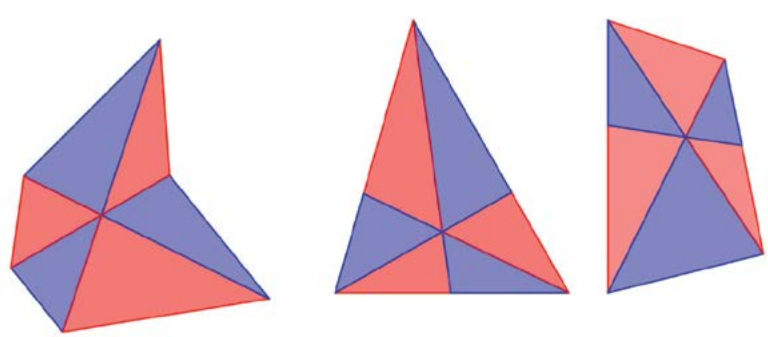

Figure 1

For such a polygon $\mathcal{P}$, let the $2k + 1$ main diagonals be drawn. These divide the interior of $\mathcal{P}$ into $4k + 2$ non-overlapping triangles. Let these triangles be coloured alternately blue and red, as shown.

Keywords: Simple polygon, diagonal, area

Theorem. *The product of the areas of the regions coloured blue is equal to the product of the areas of the regions coloured red.*

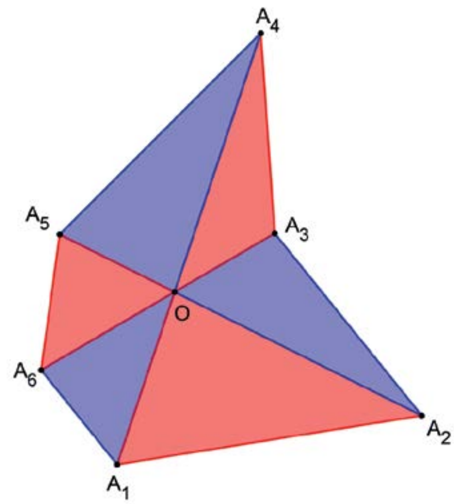

Figure 2

The case $k = 1$ is depicted in Figure 2. With reference to this figure, the claim is that

$$[OA_1A_2] \cdot [OA_3A_4] \cdot [OA_5A_6] = [OA_2A_3] \cdot [OA_4A_5] \cdot [OA_6A_1], \quad (1)$$

where the square brackets denote area.

Proof. Let the lengths of segments $\measuredangle OA_1, \measuredangle OA_2, \measuredangle OA_3, \ldots, \measuredangle OA_6$ be $a_1, a_2, a_3, \ldots, a_6$, respectively. Let the measures of $\measuredangle A_1OA_2, \measuredangle A_2OA_3, \measuredangle A_3OA_4, \ldots, \measuredangle A_6OA_1$ be $\theta_1, \theta_2, \theta_3, \ldots, \theta_6$, respectively. Then

$$[OA_1A_2] = \frac{1}{2} \cdot a_1 \cdot a_2 \cdot \sin\theta_1, \quad (2)$$

with similar expressions for the areas of the remaining triangles. It follows that

$$[OA_1A_2] \cdot [OA_3A_4] \cdot [OA_5A_6] = \frac{1}{2^3} \cdot a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6 \cdot \sin\theta_1 \cdot \sin\theta_3 \cdot \sin\theta_5, \quad (3)$$

$$[OA_2A_3] \cdot [OA_4A_5] \cdot [OA_6A_1] = \frac{1}{2^3} \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6 \cdot a_1 \cdot \sin\theta_2 \cdot \sin\theta_4 \cdot \sin\theta_6, \quad (4)$$

and equality (1) follows immediately because of the angle equalities $\theta_1 = \theta_4, \theta_2 = \theta_5, \theta_3 = \theta_6$.

The same reasoning works for any value of k. □

Now we see why n must be of the form $4k + 2$. The number of sides must clearly be even. Moreover, $n/2$ must be odd, else the regions corresponding to vertically opposite angles will receive the same colour and the stated result would not hold. Hence we write $n/2 = 2k + 1$, i.e., $n = 4k + 2$.

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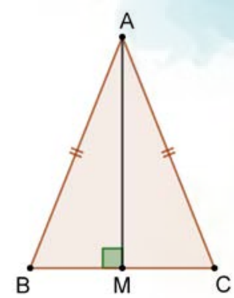
AREA OF AN ISOSCELES TRIANGLE

We know that area of triangle = $\frac{1}{2}$ × base × height. Here we must first find the height of the triangle. The following formula gives the area of an isosceles triangle in terms of its sides.



Theorem.

In $\Delta\, ABC$, if $AB = AC = b$ and $BC = a$, perimeter $p = 2s = 2b + a$, then

$$\text{Area of } \Delta\, ABC = \frac{a}{4}\sqrt{p(2b-a)}.$$

Proof: Let $AM \perp BC$; then $\Delta\, ABM \cong \Delta\, ACM$ and $BM = CM = a/2$. Hence

$$AM = \text{height} = \sqrt{b^2 - \frac{a^2}{4}} = \sqrt{(4b^2 - a^2)/4} = \frac{1}{2}\sqrt{(2b+a)(2b-a)} = \frac{1}{2}\sqrt{p(2b-a)}.$$

Therefore:

$$\text{Area of triangle } = \frac{1}{2} a \times \frac{1}{2}\sqrt{p(2b-a)} = \frac{a}{4}\sqrt{p(2b-a)}.$$

Illustration. To find the area of isosceles triangle by new method where sides are 5, 5, 8.

Here $b = 5$, $a = 8$, $p = 18$, $2b - a = 2$; so

$$\text{Area } = \frac{8}{4}\sqrt{18 \times 2} = 2 \times 6 = 12.$$

Acknowledgement: The author is grateful to Prof. B. N. Waphare and Prof. P.M. Avhad for encouraging him to send this finding to *At Right Angles*.

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On Mountain Numbers

RAGHAV JEYAN PRABU

Definition. A *mountain number* is a (base ten) number with the following features (the references throughout are to its decimal digits):

(i) Its first digit and last digit are both 1.

(ii) Its digits increase up to a point and then start decreasing.

(iii) Two consecutive digits in the number cannot be the same.

Examples of mountain numbers:

$$1, \ 1431, \ 125631, \ 147851, \ 12345678987654321.$$

Examples of non-mountain numbers:

$$11, \ 1331, \ 14351, \ 23561, \ 12389, \ 235643, \ 1235461.$$

Clearly, the smallest possible mountain number is 1, while the largest possible mountain number is 12345678987654321. The largest possible number of digits in a mountain number is 17. Also, there is no mountain number with exactly 2 digits.

In this article, we pose and answer the following two questions.

(1) What is the total number of mountain numbers with n digits, for $3 \leq n \leq 17$?

(2) What is the total number of mountain numbers?

Keywords: Mountain number, peak value, combinatorial identity

Let the **peak value** of a mountain number refer to its largest digit. Denote the peak value by z. We shall first try to find an expression for the total number of mountain numbers with n digits (where $3 \le n \le 17$) and with a peak value of z.

A mountain number with a peak value of z has the following appearance:

$$1 \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots\ldots}_{A = \text{ digits between 1 and } z} z \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots\ldots}_{B = \text{ digits between } z \text{ and 1}} 1. \tag{1}$$

Let A denote the string of digits between 1 and z; these occur in strictly increasing order. Let B denote the string of digits between z and 1; these occur in strictly decreasing order. The digits in A and B are all drawn from the set $\{2, 3, \ldots, z-2, z-1\}$. Note that this set has $(z-2)$ elements.

Let A have k elements; then B has $n-3-k$ elements. The number of choices for the string A is

$$^{z-2}C_k,$$

since after selection of the digits, there is just one way of arranging them. Similarly, the number of choices for the string B is

$$^{z-2}C_{n-3-k}.$$

It follows that the total number of mountain numbers with n digits and with a peak value of z is

$$\sum_k {}^{z-2}C_k \cdot {}^{z-2}C_{n-3-k}, \tag{2}$$

the summation being over an appropriate interval of values for k. (We do not need to worry too much about the precise interval because of the useful convention that ${}^aC_b = 0$ if $a < b$.)

Now we recall the following well-known combinatorial identity: for fixed values of m, n, p,

$$\sum_k {}^mC_k \cdot {}^nC_{p-k} = {}^{m+n}C_p. \tag{3}$$

(The identity is obtained by equating the coefficients of x^p on the two sides of the equality

$$(1+x)^m \cdot (1+x)^n = (1+x)^{m+n}.$$

We leave the details to the reader.) It follows that the total number of mountain numbers with n digits (for $n \ge 3$) and a peak value of z is

$$^{2z-4}C_{n-3}. \tag{4}$$

Hence the total number of mountain numbers with n digits (for $n \ge 3$) is

$$\sum_z {}^{2z-4}C_{n-3}. \tag{5}$$

For the range of z in the summation, we must have $2z-4 \ge n-3$, i.e.,

$$z \ge \frac{n+1}{2}. \tag{6}$$

Therefore, the summation is over $\lceil (n+1)/2 \rceil \le z \le 9$.

Sample calculations.

- To count the mountain numbers with 3 digits, we put $n = 3$; the required count is:

$$\sum_{z=2}^{9} {}^{2z-4}C_0 = 1+1+1+1+1+1+1+1 = 8.$$

The 8 mountain numbers are obviously 121, 131, 141, ..., 191.

- To count the mountain numbers with 4 digits, we put $n = 4$; the required count is:

$$\sum_{z=3}^{9} {}^{2z-4}C_1 = 2+4+6+8+10+12+14 = 56.$$

- To count the mountain numbers with 6 digits, we put $n = 6$; the required count is:

$$\sum_{z=4}^{9} {}^{2z-4}C_3 = {}^{4}C_3 + {}^{6}C_3 + {}^{8}C_3 + {}^{10}C_3 + {}^{12}C_3 + {}^{14}C_3 = 784.$$

The individual counts are given in the following table:

n	Total number of n-digit mountain numbers
1	1
3	8
4	56
5	252
6	784
7	1792
8	3108
9	4166

n	Total number of n-digit mountain numbers
10	4352
11	3544
12	2232
13	1068
14	376
15	92
16	14
17	1

The total number of mountain numbers can be computed by adding all these individual counts:

$$1+8+56+252+784+1792+3108+4166+4352+3544+2232+1068+376+92+14+1 = 21846.$$

Thus, there are 21846 mountain numbers altogether.

But there is a nicer way of getting the total number. Consider the mountain numbers with a peak value of z, having 3 or more digits. Between the leftmost 1 and z is an ascending string of any length, drawn from the set $\{2, 3, 4, \ldots, z-1\}$; there are 2^{z-2} such strings (because set with k elements has 2^k subsets). Similarly, between z and the rightmost 1 is a descending string of any length, drawn from the same set;

there are 2^{z-2} such strings. Therefore the total number of mountain numbers with the peak value of z is $2^{z-2} \cdot 2^{z-2} = 4^{z-2}$. Hence the total count of mountain numbers having 3 or more digits is

$$\sum_{z=2}^{9} 4^{z-2} = 1 + 4 + 4^2 + \cdots + 4^7$$
$$= \frac{4^8 - 1}{4 - 1} = 21845.$$

Hence the total count of mountain numbers with all possible lengths is $21845 + 1 = 21846$.

References

1. MathEd (blog), "On Mountain Numbers" from https://mathsedideas.blogspot.com/search/label/Mountain%20Numbers

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Simple proof for the formula Area of Circle = πr^2

First divide a circle of radius r into n identical sectors and suppose this n tends to infinity, then these sectors approach triangles.

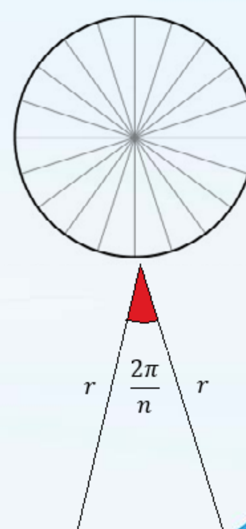
$$\textit{Area of each triangle} = \frac{1}{2} r^2 \sin\left(\frac{2\pi}{n}\right)$$

If we add up all the triangles then we obtain

$$\sum \textit{Area of triangle} = \frac{n}{2} r^2 \sin\left(\frac{2\pi}{n}\right)$$

Then,

$$\textit{area of circle} = \pi r^2 \lim_{n\to\infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} = \pi r^2$$



Contributed by Ankush Kumar Parcha, Email id: ankushparcha11@gmail.com

Intersecting Circles Investigation

MICHAEL DE VILLIERS,
JAMES METZ & BRAD UY

The reader is first invited to explore a dynamic geometry sketch of the investigation at: http://dynamicmathematicslearning.com/intersecting-circles.html where the stage is set for conjecturing and proving interesting properties of intersecting circles.

For those readers who would like to create their own sketches in GeoGebra, the following instructions may be helpful.

Construction Steps for GeoGebra	
Step 1:	Construct a unit circle with its centre at the origin and with a radius of 1.
Step 2:	Construct another circle with its centre OB on the x-axis and with radius 1 unit.
Step 3:	Construct (and name) points B and B' where the circle with centre OB intersects the x-axis and measure the x-coordinates of these points.
Step 4:	Construct a perpendicular to the x-axis at OB, and then label its intersection with the circle in the 1st quadrant as H.
Step 5:	Measure the y-coordinate of the point H.
Step 6:	Trace the path of point H when OB is dragged.

Figure 1 shows four intersecting circles that share a common vertical chord that is the diameter of a unit circle whose center is at the origin. Their intersections on the negative x-axis are separated by equal intervals of $\frac{1}{4}$. The four arcs to the left of 0 may remind you of longitude lines on a two-dimensional rendering of a globe, which was the inspiration for this investigation. These few examples satisfy the conjecture that the x-coordinates of the endpoints of the diameters lying on the x-axis of each circle are negative reciprocals of each other. A proof for the general situation follows.

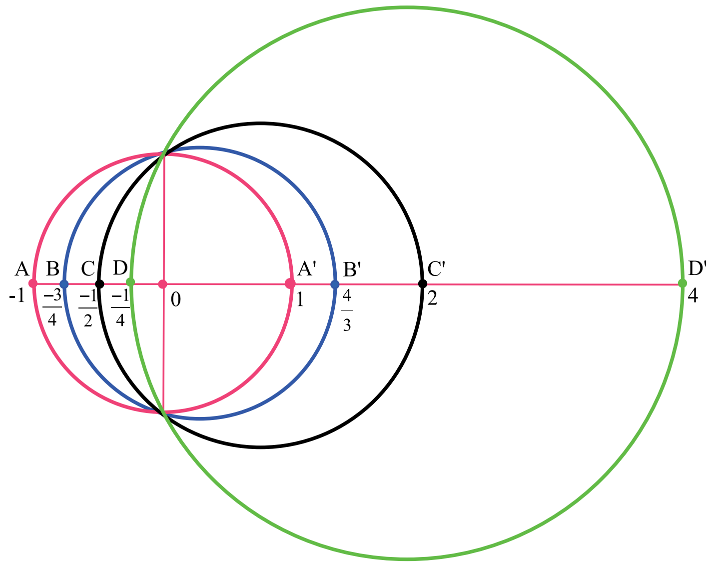


Figure 1. Four intersecting circles, with x-coordinates on the x-axis noted

General Situation

Figure 2 shows unit circle A with center at (0,0). Circle B with center D(x,0) is drawn to pass through point C(0,1) and the point B with coordinates ($-p$,0), with p a non-zero real number. Prove that the x-coordinates of the endpoints of the diameter on the x-axis of circle B are negative reciprocals of each other.

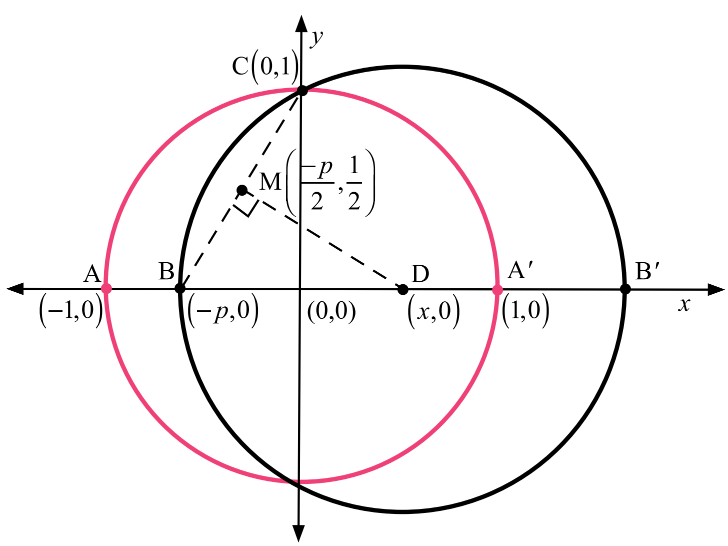


Figure 2. General proof

Proof. Point M is the midpoint of $\overline{CB}$ and has coordinates M $\left(\frac{-p}{2}, \frac{1}{2}\right)$.

The slope of $\overline{CB}$ is $\frac{1}{p}$, so the slope of $\overline{MD}$ is $-p$. Then $\dfrac{\frac{1}{2} - 0}{\frac{-p}{2} - x} = -p$, so $x = \dfrac{1 - p^2}{2p}$.

The radius of circle B is $x + p$ so the radius of circle B is $\frac{1+p^2}{2p}$.

Then $x + \text{radius} = \frac{1-p^2}{2p} + \frac{1+p^2}{2p} = \frac{1}{p}$, so B′ has coordinates $\left(\frac{1}{p}, 0\right)$ as required.

The Converse

We now prove the converse:

> Given a circle with center on the x-axis and x-intercepts that are negative reciprocals of each other, the circle intersects (0,1) and (0, −1).

Proof. The center of the circle is at $\left(\frac{-p+\frac{1}{p}}{2}, 0\right)$ or $\left(\frac{1-p^2}{2p}, 0\right)$. The radius of the circle is $\frac{\frac{1}{p}-(-p)}{2} = \frac{1+p^2}{2p}$. Thus, the equation of the circle is $\left(x - \frac{1-p^2}{2p}\right)^2 + y^2 = \left(\frac{1+p^2}{2p}\right)^2$. To determine the y-intercepts, replace x with 0 and solve for y. Hence, $\left(-\frac{1-p^2}{2p}\right)^2 + y^2 = \left(\frac{1+p^2}{2p}\right)^2$, so $y^2 = \left(\frac{1+p^2}{2p}\right)^2 - \left(-\frac{1-p^2}{2p}\right)^2 = \frac{4p^2}{4p^2} = 1$, so $y = \pm 1$, proving that the circle passes through (0,1) and (0, −1).

Connection with Inversive Geometry

A closely related geometric idea to finding the reciprocal of a number is that of "inverting" a point. In the plane, the inverse of a point P (lying inside, on or outside a reference circle with center O and radius r) is defined as a point P′, lying on the ray from O through P such that $OP \cdot OP' = r^2$ (see Figure 3).

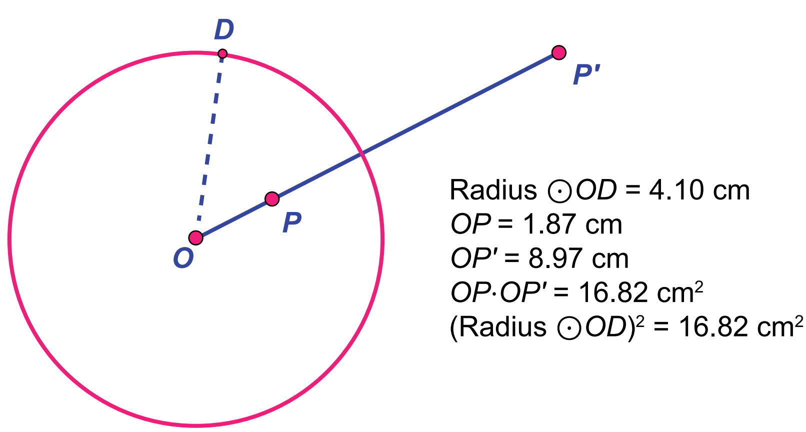


Figure 3. Definition of Inversion in Geometry

Now, with reference to Figure 2, if we reflect point B in the y-axis and label the image B″, then it follows from the already proven result at the beginning that $OB' \cdot OB'' = 1 = r^2$, where $r = 1$ is the radius of the unit circle. Hence, by definition B″ is simply the inversion of B′ in the unit circle, and vice versa.

The interested reader can read more about Inversive Geometry at:
https://en.wikipedia.org/wiki/Inversive_geometry

High Points

Consider Figure 4, which is a screen capture of the dynamic sketch at the URL given at the start. If the center of the circle, O_B, is $(x,0)$, the radius from O_B to $(0,1)$ is the hypotenuse of a right triangle, so the height, y, of the circle above the x-axis is given by $y = \sqrt{x^2+1}$, the upper branch of a hyperbola. The lower branch, with equation $y = -\sqrt{x^2+1}$, passes through the lowest point of the circle. The asymptotes are

$$y = \pm x.$$

Pythagoras and Triples

We next discover a proof of the Pythagorean theorem and relate p to Pythagorean triples. Suppose p is a non-zero rational number $\frac{n}{m}$, with $m > n$. The center of the circle is $\left(\frac{m^2-n^2}{2mn}, 0\right)$ and the radius is $\frac{m^2+n^2}{2mn}$, so the highest point on the circle is $\left(\frac{m^2-n^2}{2mn}, \frac{m^2+n^2}{2mn}\right)$. Define positive integers $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$. Then the highest point on a circle has coordinates $\left(\frac{a}{b}, \frac{c}{b}\right)$. Figure 5 shows that $\frac{a}{b}$, $\frac{b}{b}$ and $\frac{c}{b}$ are the three sides of a right triangle, the same right triangle we used to establish the equation of the curve passing through the highest points of the circles.

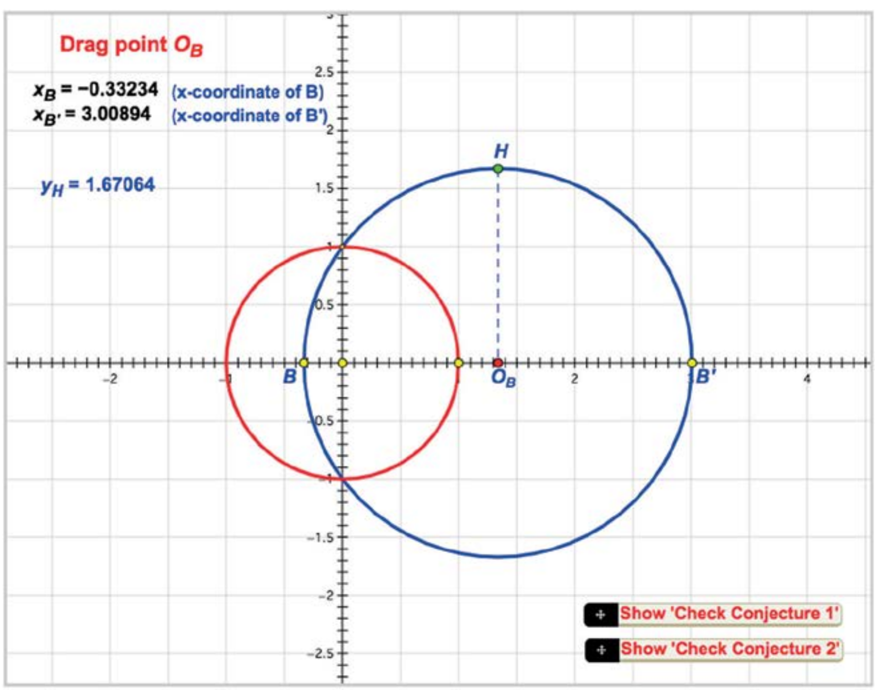


Figure 4. Dynamic Sketch

Thus, $2m^2 = c + a$, so $m = \sqrt{\frac{c+a}{2}}$ and $2n^2 = c - a$ so $n = \sqrt{\frac{c-a}{2}}$. Then $\frac{n}{m} = \sqrt{\frac{c-a}{c+a}}$. Furthermore, the difference between the radius and the x-coordinate of the center, $\frac{c-a}{b}$, is also equal to $\frac{n}{m}$, hence $\sqrt{\frac{c-a}{c+a}} = \frac{c-a}{b}$. The last equation becomes $\frac{c-a}{c+a} = \frac{(c-a)^2}{b^2}$, so $\frac{1}{c+a} = \frac{c-a}{b^2}$ and thus $a^2 + b^2 = c^2$, or equivalently $\left(\frac{a}{b}\right)^2 + \left(\frac{b}{b}\right)^2 = \left(\frac{c}{b}\right)^2$, which relates the three sides of the right triangle in Figure 3, thereby proving the Pythagorean theorem, since the argument holds for all positive real values of a, b and c. Looking at $\left(\frac{m^2-n^2}{2mn}, \frac{m^2+n^2}{2mn}\right)$ we can clearly see what Euclid taught us so many years ago, that for positive integers m and n, $m > n$, with $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, a, b and c form a Pythagorean triple.

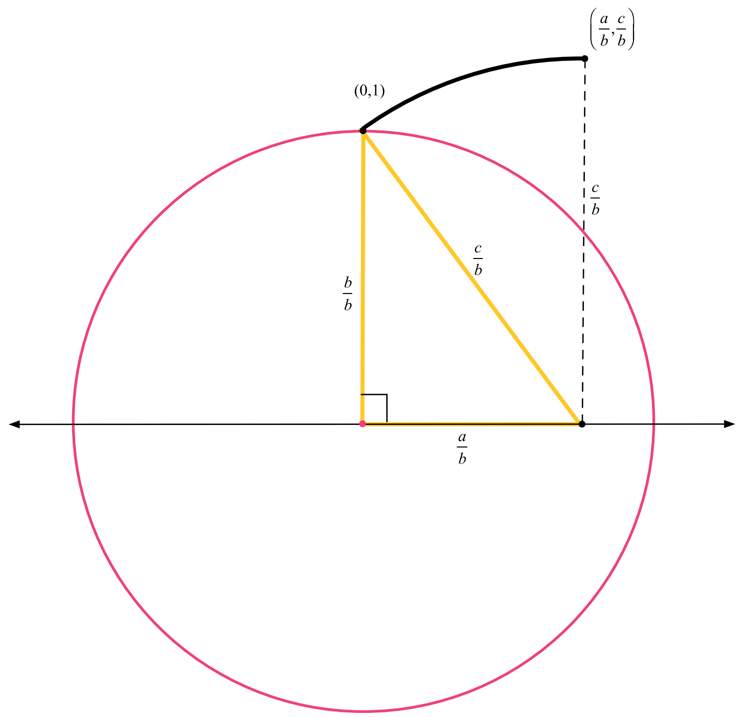


Figure 5. Stumbling upon a proof of Pythagoras theorem

If we consider only positive integer values of n and m, then for each of our circles we can use the known values of n and m to determine the coordinates of the highest point on the circle, which in turn can tell us a corresponding Pythagorean triple.

For example, the circle passing through $\left(\frac{-3}{4}, 0\right)$ will have its highest point at $\left(\frac{7}{24}, \frac{25}{24}\right)$ revealing the Pythagorean triple 7–24–25.

Furthermore, we can start with a Pythagorean triple and determine the circle. Suppose we select the Pythagorean triple 9–40–41. Since $n = \sqrt{\frac{c-a}{2}}$ and $m = \sqrt{\frac{c+a}{2}}$, $n = 4$ and $m = 5$. This means that a circle with center on the x-axis through $\left(\frac{-4}{5}, 0\right)$ will have the highest point of the circle at $\left(\frac{9}{40}, \frac{41}{40}\right)$. The other circles in Figure 1 have high points at $\left(\frac{15}{8}, \frac{17}{8}\right)$, generating the 15–8–17 triangle and $\left(\frac{12}{16}, \frac{20}{16}\right) = \left(\frac{3}{4}, \frac{5}{4}\right)$ generating the 12–16–20 triangle, a multiple of the 3-4-5 primitive Pythagorean triple, which comes from $n = 1$ and $m = 2$.

Figure 6 shows the circles from Figure 1 and the coordinates of their highest points.

Our investigation of lines of longitude has taken us from negative reciprocals to inversive geometry, to a hyperbola whose branches contain the highest and lowest points of a series of circles, to a proof of the Pythagorean theorem and finally a connection to Pythagorean triples.

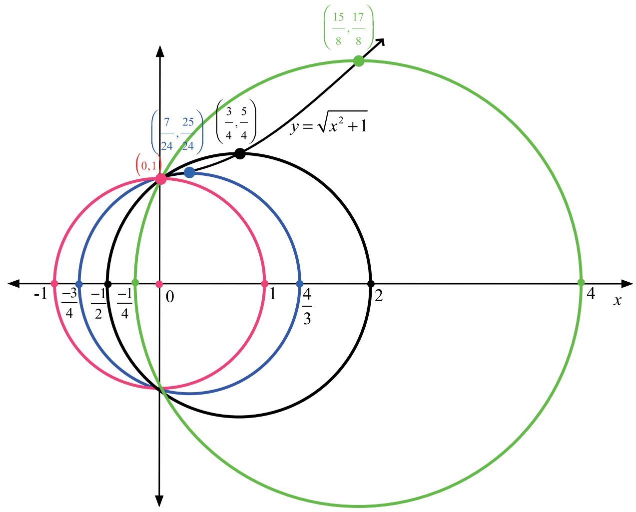


Figure 6. Graph of $y = \sqrt{x^2 + 1}$ with 4 circles

The pedagogic benefits of such an exploration are manifold. The dragging facility of dynamic geometry software is valuable because it quickly generates numerous examples that comply with the constraints of a particular construction. Not only can dragging produce numerical data, but it also provides visual stimuli for conjecturing.

While dynamic geometry provides valuable experimental confirmation, a deductive proof is needed to explain and understand why one's observations are true, which refers to the explanatory function of proof (De Villiers, 1999). More-over, since the software cannot check all possible cases, one ultimately needs a deductive proof to verify that the conjecture is generally true.

In this case, after observing the result about the product of the x-coordinates as well as the path traced out by H, we needed a proof to generally verify, as well as to explain our observations. Further reflection on the proof of the results led us to Pythagorean triples and a proof of the theorem of Pythagoras, which also nicely illustrates the discovery function of proof (De Villiers, 1999).

MICHAEL DE VILLIERS has worked as researcher, mathematics and science teacher at institutions across the world. From 1983-1990, he was at the University of Stellenbosch, and from 1991-2016 he was part of the University of KwaZulu-Natal. After retirement in 2016, he was appointed Honorary Professor in Mathematics Education at the University of Stellenbosch. He was editor of Pythagoras, the research journal of the Association of Mathematics Education of South Africa, and is currently chair of the Senior South African Mathematics Olympiad problems committee. His main research interests are Geometry, Proof, Applications and Modeling, Problem Solving, and the History of Mathematics. His home page is http://dynamicmathematics-learning.com/homepage4.html. He maintains a web page for dynamic geometry sketches at http://dynamic-mathematicslearning.com/JavaGSPLinks.htm. He may be contacted at profmd1@mweb.co.za.

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NUMBER FLEXIBILITY

Breaking down numbers to your advantage for problem solving is called number flexibility, which is a prerequisite for doing good math. Let's look at this with an example. In how many ways can you compute 18 × 5?

a. Repeated addition: 18 + 18 + 18 + 18 + 18 =90

b. Doubling to shorten the above: 18 × 2 × 2 + 18 → 36 → 72 → 90 (Double 18 twice and then add 18)

c. Using distributive property:

i. (20 – 2) × 5 = 100 – 10 = 90

ii. (12 + 6) × 5 = 60 + 30 = 90

iii. (9 + 9) × 5 = 45 + 45 = 90

d. The last method is just halving and doubling: (18 ÷ 2) × 5 × 2 = 45 × 2 = 90

e. Or it can be doubling and halving: 18 × (5 × 2) ÷ 2 = 180 ÷ 2 = 90

Why are d. and e. equivalent?

Halving is the same as multiplication by ½ and multiplication is both commutative and associative.

Let's look at some more examples of doubling and halving. Which number is doubled and which is halved? How does this make the calculation simpler?

a. 12 × 50 = (12 ÷ 2) × (50 × 2) = 6 × 100

b. 16 × 25 = 8 × 50 = 4 × 100

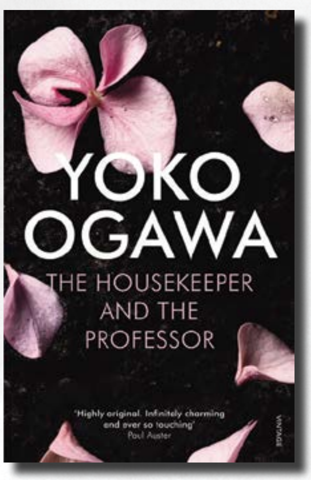
When do we use the distributive property: 24 × 11 = 24 × (10 + 1) = 240 + 24

Now, in how many ways can you compute 12 × 15? Which do you find the easiest?

Contributed by Vikas Sharma

The Housekeeper and The Professor

Reviewed by Nia Chari



The *Housekeeper* and the Professor by Yoko Ogawa, translated by Stephen Snyder, is a story of maths, memory, and the beautiful human connections that change and shape us. It centres around a Professor of maths, the Housekeeper who works for him, and her son, Root. The crux of this story is that the Professor's memory only lasts 80 minutes. Because of an accident almost 20 years prior to the beginning of the story, he cannot retain new memories for longer than 80 minutes. To quote a character from the book, he can remember a theorem he developed thirty years ago, but he has no idea what he ate for dinner last night. As a result of this, the Housekeeper needs to introduce herself to him every morning, because he forgets about her. Ogawa describes how he sticks little post-its onto his coat-sleeves, to remind him of the most important pieces of information.

One interesting thing about this book is that none of the characters are given names beyond who they are—everyone is referred to as "the Professor", "the sister-in-law", and so on. The almost-exception to this rule is Root, the Housekeeper's son. Upon meeting him, the Professor declares that his head is flat, like a square root sign. After that, we only know him as Root.

In addition to memory loss, the book has two other important themes. The first is the Professor's love for baseball, which he shares with Root. The second is, of course, his love for maths. The Professor manages to earn a little money by solving problems posed by magazines, but much more important is the way he teaches maths to the Housekeeper. (But more on that later.) It was fascinating to me how these two themes interacted.

Keywords: memory loss, love for mathematics, beauty, value

My absolute favourite thing about this book is how Ogawa manages to communicate such a strong love of maths through the writing. Because the Professor's whole life once revolved around the subject, it would have been easy to have him now dismiss it as a painful reminder of the life he used to live. But he kept to it, and it was almost presented to the reader as a lone lifeline in a sea of confusion. To the Professor, maths makes sense in a way his life doesn't. He remembers the concepts he knew before the accident, and whatever new proofs he is working on can be understood by simply re-reading them; he isn't thwarted by his short memory.

In connection to that is the way he teaches maths to the Housekeeper. In the book, they cover some things, but it's implied that he teaches her more. A particularly poignant scene occurs in the beginning, where the Professor shows her how her birthday, February 20^{th} (220) is an amicable number with the number imprinted on a watch he got as an award (284). Amicable numbers are those pairs where the sum of the proper divisors of one number equals the other, and vice versa. In this case, these divisors are: 1, 2, 4, 71 and 142, which add up to 220, and 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110, which add up to 284. The former set of numbers are the proper divisors of 284, and the latter are the proper divisors of 220. Note that this only works with the proper divisors of numbers; if we count the number itself as a divisor, 220 and 284 are no longer amicable numbers. (This definition also holds true for perfect numbers, see below.)

The book also introduces readers to:

Imaginary numbers—the Professor christens Root thus because of his apparent resemblance to the square root sign ($\sqrt{}$). When explaining this name, he teaches the Housekeeper and Root about imaginary numbers, and how they're special because they can't be seen; 'they live in your heart instead'.

Perfect numbers—this scene comes about when the Housekeeper herself discovers that 28 is a perfect number (the sum of its proper divisors is itself; $1 + 2 + 4 + 7 + 14 = 28$). The Professor explains that 28 is the second-smallest perfect number, and that the smallest is 6 (again, $1 + 2 + 3 = 6$). When the Housekeeper exclaims that they aren't that rare after all, the Professor tells her that the next smallest one is 496, and then 8,128.

Sum of consecutive numbers: the Professor assigns Root to figure out the sum of all the numbers from 1-10. Root just adds them up manually, but then the Professor ups the ante—what's the sum of all numbers from 1-100, or 1-1000? The two then have a contest to see if Root can figure out the answer before the Professor does a challenge of his own.

Ogawa also covers Euler's formula, Fermat's last theorem, triangular numbers, twin primes, and several other things.

In many ways, these lessons remind me of classes I've had at school, where maths is shown as a wondrous language, filled with secrets I can discover, rather than a fearful dragon to be bested and slayed.

As someone who is interested in reading, the stylistic choices Ogawa made were fascinating to me. Firstly, the book is in first person—which is difficult to pull off without sounding tacky. Ogawa passes the test with flying colours, crafting a narrative that sounds (in the best possible way) a fusion of a friend telling you a story in a coffee shop and a journal entry with poignancy the rest of us can only dream of.

Second, because none of the characters were mentioned by their 'real' names (Root being the almost-exception, it's a nickname), the book had a timeless, spaceless quality to it. Despite me knowing this book is set in Japan, it felt like it could have been set anytime, anywhere. It helped make this (all things considered, fairly niche book) feel much more accessible.

My reasons for why teachers should read this book and why students will like it are the same—the way maths is handled. As I said before, it's treated as a wonderful adventure, a multifaceted language. Underscoring every equation is a love and wonder for it, that the Professor effectively teaches the Housekeeper, and us, the readers. To teachers, it might serve as a reminder that this love for the subject is essential to any progress. To students, especially those (like me!) struggling with exams, it might serve as a reminder for what makes maths special to them, or why they chose the subject in the first place.

I would say that the content is understandable to anyone who has a 10th grade education in Maths. But even if you last interacted with the subject years ago (and since you're reading this magazine, I'm assuming the answer to that is no) concepts are explained with enough simplicity for you to pick it up anyway.

To reiterate, specific things you may enjoy in this book are: its simple, yet masterful handling of the topics at hand, how it focuses so beautifully on human relationships and their poignancy, the writing style, and the way that maths is handled. The book is also less than 200 pages, making it a quick read.

The Housekeeper and the Professor, ((博士の愛した数式, hakase no ai shita suushiki: literally "The Professor's Beloved Equation") by Yoko Ogawa, translated to English by Stephen Snyder (2003, 2009)

NIA CHARI is a committed bookworm and (slightly less committed) writer. She also loves working with wool, and can so be found attempting to read, eat, and crochet at the same time, in a cosy nook. Nia is a 12th grade student at Centre for Learning, Bangalore.

GOING FORWARD WITH MATH

A person sets out on a journey by car, driving at a uniform speed. As he starts the journey, he notices that the odometer shows a two-digit number. After an hour, he notices that the odometer shows the two-digit number that is obtained by switching the digits of the first number. After another hour, the odometer shows the three-digit number that is obtained by inserting a zero between the digits of the first number. What is the speed of the car? What were the odometer readings?

(An odometer shows the distance travelled by a vehicle since the last time it was set to zero.)

Send in your answers to AtRiA.editor@apu.edu.in

Manipulative Review: Ten-frames

Reviewed by Math Space

Ten-frames are not well known in India. But they do have certain advantages and can be made very easily. There are also virtual ten-frames in several websites.

A ten-frame is a 2 × 5 frame with slots on which counters have to be put, to represent various numbers up to ten.

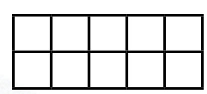


Figure 1

This is very useful at the beginning when children are learning to count. The act of putting counters on the frame requires eye-hand coordination as well as the ability to count. There are two possible ways of representing numbers on ten-frames:

1. Orient the frame as two rows and five columns. Fill the cells in the top row – left to right – and then the bottom one.

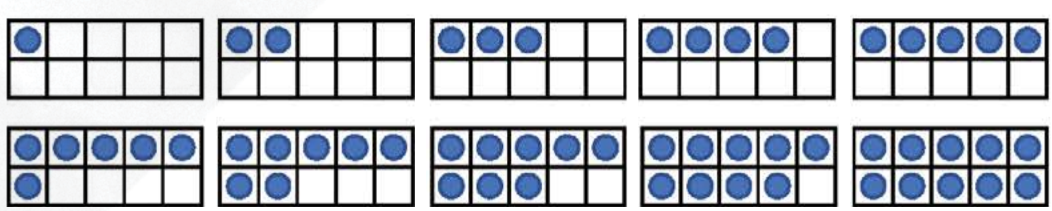


Figure 2

2. Orient the frame as five rows and two columns. Fill the cells from the bottom most row – left, then right – and move up (Figure 3).

Keywords: Base ten system, addition facts, patterns, parity

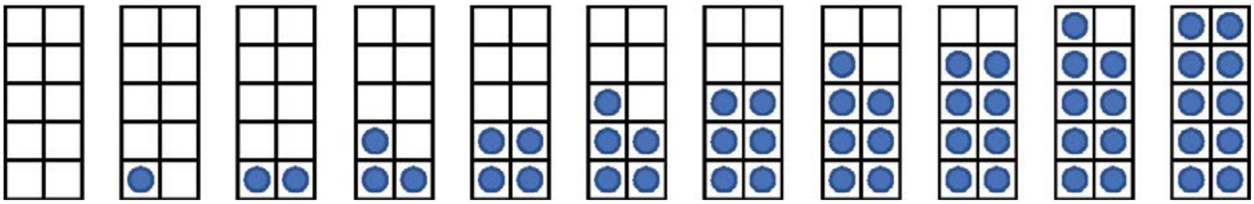


Figure 3

Both arrangements have certain advantages. Both help automatize[1] the number pairs that add up to ten. This can be done in two ways – (i) consider the number of filled cells and that of empty ones, and (ii) use counters of two colours to fill the frame (Figure 4) – for either arrangement.

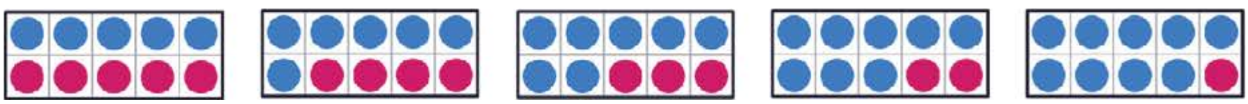


Figure 4

The first arrangement helps in automatizing single-digit addition facts, especially for sums greater than ten. Consider 6 + 7. If the first frame is rotated and put against the second one, then the two lines in the middle form a full frame, i.e., a ten. This and the remaining one and two add up to 13 (Figure 5). Here, 6 and 7 splitting into 5 + something is crucial. Others of this category are 5 + 6, 5 + 7, 5 + 8, 5 + 9, 6 + 6, 6 + 8, 6 + 9, 7 + 7, 7 + 8, 7 + 9, 8 + 8, 8 + 9 and 9 + 9. Similarly, one can also explore 7 + 4 (Figure 6), 8 + 3, 8 + 4, 9 + 2, 9 + 3 and 9 + 4.

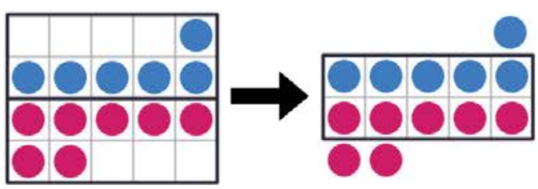


Figure 5

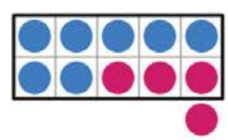


Figure 6

So, when children play with the ten-frames at the beginning of learning numbers, they can generate these visuals themselves, start observing many things and make various connections.

The second arrangement essentially represents odd and even numbers. The names can be connected to the 'shapes' as follows:

- Odd: numbers which have a single at the top
- Even: numbers which have a pair at the top

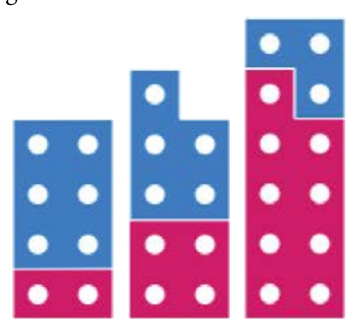


Figure 7

This representation can then be connected to the algebraic forms $2n + 1$ or $2n - 1$ for odd numbers and $2n$ for even ones. With this image, it becomes very clear what n stands for in each of these three cases: even $2n$, odd $2n + 1$ and odd $2n - 1$.

It also becomes clear what happens when these numbers are added (Figure 7). Clearly, adding an even number doesn't change the parity since that (even) number aligns smoothly below the other number. Therefore, even + even remains even and odd + even remains odd. Also, one can see how the sum of two odd numbers is even since the 'odd ones' pair up! One can also discuss whether zero is odd or even and why.

[1] Ability to respond immediately almost without thinking, along with the ability to justify if asked (because the fact has been understood and internalized, this makes it different from rote learning).

More importantly, this can be extended to bigger numbers. Any number bigger than 9 is made of tens and ones. And if the number is hundred or more, at least some of the tens are in hundred or bigger groups. Nonetheless, all bigger groups, hundred, thousand, etc., are made of tens. So, all groups – ten, hundred, thousand, etc., – are even. Therefore, the number of ones determine the parity of any number. This can be clearly seen with ten frames – see how 25 or 34 look (Figure 8).

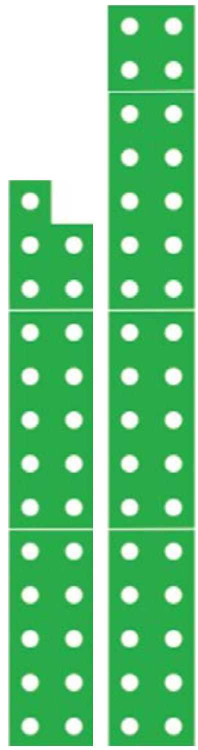


Figure 8

As mentioned earlier, several websites include virtual ten-frames:

Toy Theater (https://toytheater.com/two-color-counter-ten-frame/) provides simple frame and counters in two colours. The user will have to put the counters on the frame similar to a real frame. However, the orientation of the frame is fixed, and it is not possible to use multiple frames. So, this is good for a beginner but has limited scope.

Math Learning Center (https://apps.mathlearningcenter.org/number-frames/) provides a wide range of frames including the ten-frame as well as counters in two colours. Counters can be brought onto the screen as singles, in groups of fives and in groups of tens. The orientation of the frame can be changed, and multiple frames can be used simultaneously. Colours of the counters can be changed as well. The other pre-determined frames are 1 × 5, 2 × 10 and 10 × 10. The 2 × 10 frame may be useful for automatizing the addition facts for sums ≤ 20. They can provide alternative visuals corresponding to Figures 5-6 (Figure 9). This site definitely allows plenty of explorations going much beyond ten-frames.

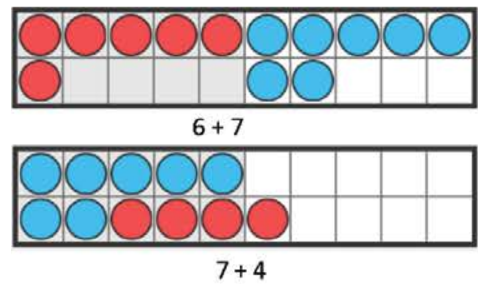


Figure 9

Mathigon polypad (https://mathigon.org/polypad#number-frames) includes ten-frames and counters in two colours. The orientation of the frame and the colours of the counters can be changed. Figures 4-8 have been generated using this site. So, multiple frames can be used as well. In addition, it also has the numbers pre-grouped in the odd-even format which have been used in Figures 7-8. These can provide a lot of exploration possibilities by allowing quick manipulation.

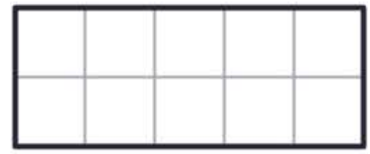
It is interesting that none of the sites provide the '5 + something' pre-grouped version. However, those can be made easily. The Math Space site (https://sites.google.com/apu.edu.in/mathspace/matherials#h.a21m13dektic) includes:

- How to make ten-frames – 3 versions including (i) empty frames for children, (ii) pre-filled frames for teachers, and (iii) pre-grouped (filled and cut) version – both odd-even and '5 + something' – for older children and more
- Posters showcasing number patterns for addition – (i) using '5 + something' and (ii) the odd-even layout

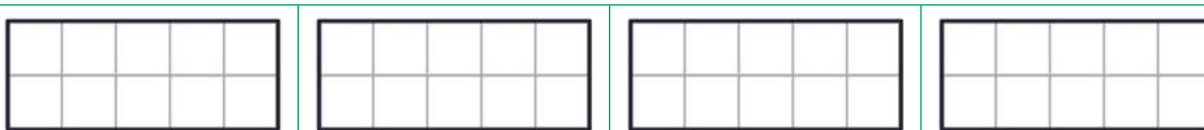
MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] – their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

Worksheet based on Ten-Frames (Class 1)

1. Draw 6 counters on the frame.
   a. How many cells are empty?
   b. Number of counters + number of empty cells = ___ + ___ = ____



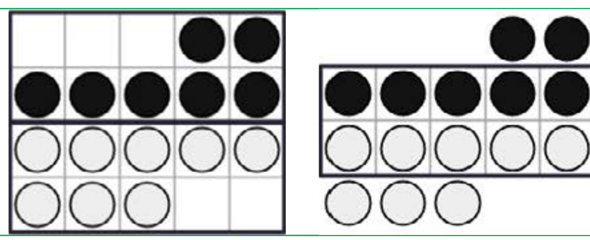
2. Make more addition facts as above:



____ + ____ = ____ ____ + ____ = ____ ____ + ____ = ____ ____ + ____ = ____

3. Sum in the picture

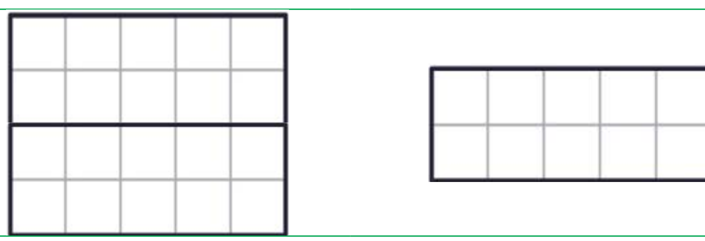
This picture shows
____ + ____



The sum is
= ____

4. Make similar pictures for any of these: 5 + 6, 5 + 7, 5 + 8, 5 + 9 and find the sum.

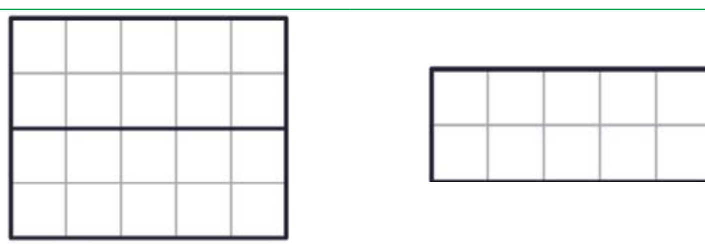
____ + ____



= ____

5. Make similar pictures for any of these: 6 + 7, 6 + 8, 6 + 9, 7 + 9, 8 + 9 and find the sum.

____ + ____



= ____

Problems Based on the AM-GM Inequality – I

TOYESH PRAKASH SHARMA

In this two-part article, we consider problems from various sources which are solved using the AM-GM inequality. We list the problems first and give the solutions later.

Problems

Problem 1. Let a, b, c be positive numbers. Prove that

$$\left(\frac{a}{b}\right)^{1/2} + \left(\frac{b}{c}\right)^{1/3} + \left(\frac{c}{a}\right)^{1/5} > 2. \tag{1}$$

This problem is from *Mathematical Reflection* 2018; it was proposed by Florin Rotaru [1].

Problem 2. Let x, y, z, t be positive real numbers such that $x + y + z + t = 2$. Show that

$$\left(\frac{4}{x^2} - 1\right) \cdot \left(\frac{4}{y^2} - 1\right) \cdot \left(\frac{4}{z^2} - 1\right) \cdot \left(\frac{4}{t^2} - 1\right) \geq 15^4. \tag{2}$$

This problem was published in 2018 in Issue 1 of *AMJ* journal; it was proposed by Mihaela Berindeanu [2].

Problem 3. Let a, b, c, d be positive real numbers. Prove that

$$\begin{aligned} a^4 + b^4 + c^4 + d^4 + 4 \geq 4 \\ \cdot \left((a^2b^2 + 1)(b^2c^2 + 1)(c^2d^2 + 1)(d^2a^2 + 1)\right)^{1/4}. \end{aligned} \tag{3}$$

This problem was proposed by Angel Plaza and published in April 2014 in the *SSMJ* problem corner; it was solved by Albert Stadler [4].

Keywords: AM-GM inequality

Solutions to problems 1, 2, 3

Problem 1. Let a, b, c be positive numbers. Prove that

$$\left(\frac{a}{b}\right)^{1/2} + \left(\frac{b}{c}\right)^{1/3} + \left(\frac{c}{a}\right)^{1/5} > 2.$$

Solution. From the AM-GM inequality we get

$$\begin{aligned}\left(\frac{a}{b}\right)^{1/2} + \left(\frac{b}{c}\right)^{1/3} + \left(\frac{c}{a}\right)^{1/5} &= \left(2\cdot\frac{1}{2}\cdot\frac{a}{b}\right)^{1/2} + 3\cdot\frac{1}{3}\cdot\left(\frac{b}{c}\right)^{1/3} + 5\cdot\frac{1}{5}\cdot\left(\frac{c}{a}\right)^{1/5}\\ &\geq 10\cdot\left(\frac{1}{2^2}\cdot\frac{1}{3^3}\cdot\frac{1}{5^5}\cdot\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}\right)^{1/10}\\ &= 10\cdot\left(\frac{1}{2^2}\cdot\frac{1}{3^3}\cdot\frac{1}{5^5}\right)^{1/10}.\end{aligned}$$

So it suffices to prove that

$$\frac{1}{2^2}\cdot\frac{1}{3^3}\cdot\frac{1}{5^5} > \frac{1}{5^{10}},$$

or $5^5 > 2^2\cdot 3^3$, or $3125 > 108$, which is clearly true.

Problem 2. Let x, y, z, t be positive real numbers such that $x+y+z+t=2$. Show that

$$\left(\frac{4}{x^2}-1\right)\cdot\left(\frac{4}{y^2}-1\right)\cdot\left(\frac{4}{z^2}-1\right)\cdot\left(\frac{4}{t^2}-1\right) \geq 15^4.$$

Solution. From the AM-GM inequality we get:

$$\begin{aligned}(x+y+z+t)+x &\geq 5\left(yztx^2\right)^{1/5},\\ (x+y+z+t)-x &\geq 3\left(yzt\right)^{1/3}.\end{aligned}$$

Hence by multiplication,

$$\begin{aligned}(x+y+z+t)^2 - x^2 &\geq 5\left(yztx^2\right)^{1/5}\cdot 3\left(yzt\right)^{1/3},\\ \therefore\ 4-x^2 &\geq 15\cdot\left(yztx^2\right)^{1/5}\cdot\left(yzt\right)^{1/3}.\end{aligned}$$

In the same way we get:

$$\begin{aligned}4-y^2 &\geq 15\cdot\left(xzty^2\right)^{1/5}\cdot\left(xzt\right)^{1/3},\\ 4-z^2 &\geq 15\cdot\left(yztz^2\right)^{1/5}\cdot\left(xyt\right)^{1/3},\\ 4-t^2 &\geq 15\cdot\left(yzxt^2\right)^{1/5}\cdot\left(xyz\right)^{1/3}.\end{aligned}$$

Multiplication now yields:

$$(4-x^2)\cdot(4-y^2)\cdot(4-z^2)\cdot(4-t^2)\geq 15^4\cdot x^2\cdot y^2\cdot z^2\cdot t^2,$$
$$\therefore\ \left(\frac{4}{x^2}-1\right)\cdot\left(\frac{4}{y^2}-1\right)\cdot\left(\frac{4}{z^2}-1\right)\cdot\left(\frac{4}{t^2}-1\right)\geq 15^4.$$

In [3], I published the following generalization of the above result:

Theorem (Toyesh Sharma)**.** *Let $x_1, x_2, \ldots, x_n$ be n positive real numbers, where $n \geq 2$, and let $x_1 + x_2 + \cdots + x_n = a$; then the following inequality holds;*

$$\left(\frac{a^2}{x_1^2-1}\right)\cdot\left(\frac{a^2}{x_2^2-1}\right)\cdots\left(\frac{a^2}{x_n^2-1}\right)\geq\left(n^2-1\right)^n. \tag{4}$$

Problem 3. Let a, b, c, d be positive real numbers Prove that

$$a^4+b^4+c^4+d^4+4\geq 4\cdot\left((a^2b^2+1)(b^2c^2+1)(c^2d^2+1)(d^2a^2+1)\right)^{1/4}.$$

Solution. By the AM-GM inequality we get:

$$\begin{aligned}a^4+b^4&\geq 2\cdot a^2\cdot b^2,\\ b^4+c^4&\geq 2\cdot b^2\cdot c^2,\\ c^4+d^4&\geq 2\cdot c^2\cdot d^2,\\ d^4+a^4&\geq 2\cdot d^2\cdot a^2.\end{aligned}$$

Adding these inequalities, we obtain

$$a^4+b^4+c^4+d^4+4\geq\left(a^2b^2+1\right)+\left(b^2c^2+1\right)+\left(c^2d^2+1\right)+\left(d^2a^2+1\right).$$

By applying the AM-GM inequality once more, we obtain

$$\begin{aligned}&\left(a^2b^2+1\right)+\left(b^2c^2+1\right)+\left(c^2d^2+1\right)+\left(d^2a^2+1\right)\\ &\geq 4\left(\left(a^2b^2+1\right)\cdot\left(b^2c^2+1\right)\cdot\left(c^2d^2+1\right)\cdot\left(d^2a^2+1\right)\right)^{1/4},\end{aligned}$$

and the claim follows.

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4. Albert Stadler, "Prob. 5303, Angel Plaza", SSMJ problem solution corner, Nov. 2014, pg. 7-9

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An Iterative Approximation of π

M.THAMBAN NAIR

We are taught in school that

The ratio of a circle's circumference to its diameter is a constant.

In other words, if r is the radius of a circle and c_r is its circumference, then

$$\frac{c_r}{2r} = \frac{c_1}{2} \quad \text{for all} \quad r > 0,$$

and this constant $c_1/2$ is denoted by the Greek letter π, which is the first letter of the Greek word *perimetros*, for perimeter. Thus,

π is half the length of the circumference of the unit circle.

The number π is ubiquitous in mathematics and in diverse fields of knowledge from ancient times. It is known that π is not a rational number. Also, it is known that it cannot be expressed in a compact form using simple arithmetic operations such as addition, subtraction, multiplication, division, or by taking roots. In other words, π is not an *algebraic number*. (A real number is an algebraic number if it is a zero of a polynomial with integer coefficients.)

Keywords: pi, approximation, iteration, limit, calculus, sequence, algebraic number, polynomial, real number, integer, Archimedes, Madhava, Leibniz

Thus, for practical applications, one uses only its approximations which are correct to a certain number of decimal places. In Wikipedia, one finds the following statements:

> Ancient civilizations, including the Egyptians and Babylonians, required fairly accurate approximations to π for practical computations. Around 250 BC the Greek mathematician Archimedes created an algorithm to approximate π with arbitrary accuracy. In the 5th century AD, Chinese mathematics approximated π to seven digits, while Indian mathematics made a five-digit approximation, both using geometrical techniques. The first exact formula for π, based on infinite series was discovered a millennium later, when in the 14th century, the *Madhava-Leibniz series* was discovered in Indian mathematics. The invention of calculus soon led to the calculation of hundreds of digits of π, enough for all practical scientific computations.

In this note[1] we give an iterative method for getting approximations for π. More precisely, we define an increasing sequence (p_n) of numbers in an iterative manner, involving basic operations of addition, subtraction, multiplication, division and taking square roots, such that

$$p_n \to \pi \quad \text{as} \quad n \to \infty.$$

We shall do this by obtaining a sequence of polygonal lines that approximates the upper part of the unit circle. We shall also give an estimate for the error $|\pi - p_{n+1}|$ as

$$|\pi - p_{n+1}| \leq \frac{2}{3}\left(\frac{1}{4^n}\right) \quad \text{for} \quad n = 1, 2, \dots,$$

with $p_1 = 2\sqrt{2}$.

Analytical definition of π

From calculus, we know that if Γ is a "smooth curve" joins points P and Q in the plane, the length of the curve, denoted by $\ell(\Gamma)$, is defined by

$$\ell(\Gamma) := \lim_{n\to\infty} \sum_{i=1}^{n} |P_i^{(n)} - P_{i-1}^{(n)}|,$$

where $P_1^{(n)}, P_2^{(n)}, \cdots, P_n^{(n)}$ are points on the curve Γ with $P_0^{(n)} = P$, $P_n^{(n)} = Q$ such that

$$\max_{1\leq i\leq n} |P_i^{(n)} - P_{i-1}^{(n)}| \to 0 \quad \text{as} \quad n \to \infty.$$

Here, for points $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ in the plane, we denote by $|X - Y|$ the distance between X and Y, that is, $|X - Y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. Thus,

$$\ell(\Gamma_n) := \sum_{i=1}^{n} |P_i^{(n)} - P_{i-1}^{(n)}|$$

is the length of the polygonal line Γ_n joining the points $P_0^{(n)}, P_1^{(n)}, \cdots, P_n^{(n)}$.

1 This article was communicated when the author was a professor at IIT Madras.

In the above, $P_1^{(n)}, P_2^{(n)}, \cdots, P_n^{(n)}$ are arbitrary points on Γ satisfying the given conditions. Thus, the length of a smooth curve is defined using (a) the idea of lengths of line segments, and (b) the notion of limit.

In the present context, Γ is the upper semi-unit circle, whose length is denoted by the Greek letter π.

Thus,

$$\pi := \lim_{n\to\infty} \sum_{i=1}^{n} |P_i^{(n)} - P_{i-1}^{(n)}|,$$

where $P_1^{(n)}, P_2^{(n)}, \cdots, P_n^{(n)}$ are points on the upper semi-unit circle Γ with $P_0^{(n)}$ and $P_n^{(n)}$ being the right and left end-points of the diameter, such that

$$\max_{1\le i\le n} |P_i^{(n)} - P_{i-1}^{(n)}| \to 0 \quad \text{as} \quad n \to \infty.$$

Approximations for π

We construct polygonal lines $\Gamma_1, \Gamma_2, \ldots$, each of which joins the right end-point P and left end-point Q of the diameter with intermediate points on the upper semicircle, constructed in a particular manner as described below:

- Let P_0 be the right end point of the diameter and P_1 be the point of intersection of the upper semicircle and the y-axis.
- Let L_1 be the line joining the points P_0 and P_1.
- Let L_2 be the line joining the points P_0, P_2, P_1, where P_2 is the point of intersection of the circle with the perpendicular bisector of P_0P_1.
- Let L_3 be the line joining the points $P_0, P_3^1, P_2, P_3^2, P_1$, where P_3^1 and P_3^2 are the points of intersection of the circle with the perpendicular bisectors of P_0P_2 and P_2P_1, respectively.
- Let L_4 be the line joining the points $P_0, P_4^1, P_3^1, P_4^2, P_2, P_4^3, P_3^2, P_4^4, P_1$, where $P_4^1, P_4^2, P_4^3, P_4^4$ are the points of intersection of the circle with the perpendicular bisectors of $P_0P_3^1, P_3^1P_2, P_2P_3^2, P_3^2P_1$, respectively.
- The process continues.

Note that each L_n has 2^{n-1} pieces of line segments of equal lengths, say ℓ_n, with their contact points with the circle constructed using perpendicular bisectors. Let Γ_n be the polygonal line consisting of L_n together with its reflection with respect to the y-axis. Thus,

$$\ell(\Gamma_n) = 2^n \ell_n.$$

We note that

$$\ell_1 = \sqrt{2}.$$

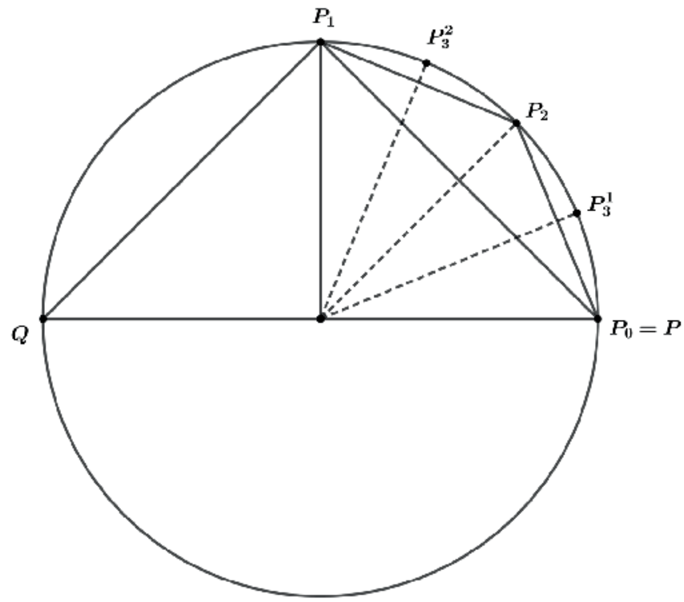

Figure 1

By using Pythagoras theorem, it can be shown that

$$\ell_2^2 = \left(\frac{\ell_1}{2}\right)^2 + \left(1 - \sqrt{1 - \left(\frac{\ell_1}{2}\right)^2}\right)^2.$$

Using the same arguments we obtain

$$\begin{aligned}\ell_{n+1}^2 &= \left(\frac{\ell_n}{2}\right)^2 + \left(1 - \sqrt{1 - \left(\frac{\ell_n}{2}\right)^2}\right)^2 \\ &= 2\left[1 - \sqrt{1 - \left(\frac{\ell_n}{2}\right)^2}\right].\end{aligned}$$

Since $p_n = 2^n \ell_n$, we have

$$p_{n+1}^2 = \frac{2p_n^2}{1 + \sqrt{1 - (\ell_n/2)^2}} = \frac{2p_n^2}{1 + \sqrt{1 - (p_n/2^{n+1})^2}}.$$

Thus, we obtain an iterative formula for p_n^2 for $n \in \mathbb{N}$ as:

$$\boxed{p_{n+1}^2 = \frac{2p_n^2}{1 + \sqrt{1 - (p_n/2^{n+1})^2}}}$$

By definition, p_n is the length of the polygonal line joining P to Q with 2^n intermediate points on the upper part of the unit circle obtained by bisecting the angles each time. Thus, p_n represents a polygonal approximation of half-circumference, that is π, and we know that

$$p_n \to \pi \quad \text{as} \quad n \to \infty.$$

Also, we observe that (p_n) is monotonically increasing. By the construction of the numbers p_n, geometrically, by a repeated application of Pythagoras theorem, we also know that

$$p_n \leq 4 \quad \text{for all} \quad n \in \mathbb{N}.$$

Remark 1. It can be verified, inductively, that

$$p_n^2 \leq 16\left(1 - \frac{1}{2^n}\right) \quad \text{for all} \quad n \in \mathbb{N}.$$

The fact that $p_n \to \pi$ as $n \to \infty$ can be shown using some advanced notions in calculus as follows: We may observe from the manner in which we constructed the sequence (ℓ_n) that

$$\ell_1 = 2\sin(\pi/2^2), \quad \ell_2 = 2\sin(\pi/2^3), \quad \ell_3 = 2\sin(\pi/2^4), \quad \ell_4 = 2\sin(\pi/2^5), \quad \ldots.$$

In general, for any $n \in \mathbb{N}$, we have $\ell_n = 2\sin(\pi/2^{n+1})$. Thus,

$$p_n = 2^n \ell_n = 2^{n+1}\sin(\pi/2^{n+1}) = \pi \frac{\sin(\pi/2^{n+1})}{\pi/2^{n+1}}, \quad n \in \mathbb{N}.$$

Now, using the notion of limit from advanced calculus, we have

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

From this, it can also be proved that for any sequence (θ_n) of non-negative numbers satisfying $\theta_n \to 0$ as $n \to \infty$,

$$\frac{\sin\theta_n}{\theta_n} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, taking $\theta_n := \pi/2^{n+1}$, we obtain

$$p_n = \pi \frac{\sin(\pi/2^{n+1})}{\pi/2^{n+1}} \to \pi \quad \text{as} \quad n \to \infty. \qquad \Diamond$$

Error estimate

In order to know how large n has to be to ensure the accuracy of p_n to π to certain number of decimal places, it is important to determine some estimates for the error $|\pi - p_n|$. In this regard, we first prove the following.

Theorem 2. *The sequence (p_n) is a strictly increasing sequence and*

$$p_{m+1} - p_{n+1} \leq \frac{2}{3}\left(\frac{1}{4^n}\right)$$

for all $m, n \in \mathbb{N}$ with $m \geq n$.

Proof. For $n \in \mathbb{N}$, we have

$$p_{n+1}^2 = \frac{2p_n^2}{1 + \sqrt{1 - (p_n/2^{n+1})^2}}$$

so that

$$\begin{aligned} p_{n+1}^2 - p_n^2 &= p_n^2 \left[\frac{2}{1 + \sqrt{1 - (p_n/2^{n+1})^2}} - 1 \right] \\ &= p_n^2 \left[\frac{1 - \sqrt{1 - (p_n/2^{n+1})^2}}{1 + \sqrt{1 - (p_n/2^{n+1})^2}} \right] \end{aligned}$$

Hence, $p_{n+1} \geq p_n$ and

$$p_{n+1}^2 - p_n^2 \leq p_n^2 \left(1 - \sqrt{1 - (p_n/2^{n+1})^2}\right).$$

Since $1 - \sqrt{1-a} \leq a$ for any $0 < a < 1$, we arrive at

$$p_{n+1}^2 - p_n^2 \leq p_n^2 \left(\frac{p_n}{2^{n+1}}\right)^2,$$

so that,

$$p_{n+1} - p_n \leq \frac{p_n^2}{p_{n+1} + p_n} \left(\frac{p_n}{2^{n+1}}\right)^2 \leq \frac{1}{2} \left(\frac{p_n}{2^{n+1}}\right)^2.$$

Hence, using the fact that $p_n \leq 4$, we have

$$p_{n+1} - p_n \leq 2\left(\frac{1}{2^{2n}}\right) = \frac{2}{4^n}.$$

Now, for $n, m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} p_{m+1} - p_{n+1} &= (p_{m+1} - p_m) + (p_m - p_{m-1}) + \cdots + p_{n+2} - p_{n+1}) \\ &\leq 2\left(\frac{1}{4^m} + \frac{1}{4^{m-1}} + \cdots + \frac{1}{4^{n+1}}\right) \\ &\leq 2\left(\frac{1}{4^{n+1}}\right)\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{m-n-1}}\right) \\ &\leq 2\left(\frac{4}{3}\right)\left(\frac{1}{4^{n+1}}\right) = \left(\frac{2}{3}\right)\left(\frac{1}{4^n}\right). \end{aligned}$$

This completes the proof. ■

By Theorem 2, we know that the sequence (p_n) is a *Cauchy sequence* and hence it converges. In fact, analytically, the limit of (p_n) is the length of the semicircle, which is denoted by the Greek letter π.

We may observe that the right hand side of the inequality in Theorem 2 is independent of m. Hence, by letting m tend to ∞, we obtain the following result.

Theorem 3. *For all $n \in \mathbb{N}$,*

$$0 < \pi - p_{n+1} \leq \left(\frac{2}{3}\right)\left(\frac{1}{4^n}\right).$$

Since $4^5 = 1024$, we have $1/4^5 \leq 10^{-3}$ so that from the above theorem we obtain

$$0 \leq \pi - p_{5n+1} \leq \frac{2}{3}\left(\frac{1}{4^{5n}}\right) < \frac{2}{3} \times 10^{-3n} \quad \text{for all } \ n \in \mathbb{N}.$$

This shows that, at every 5th iterate, we obtain an additional three digits accuracy for the approximation.

Here are a few illustrative examples:

$$\begin{aligned}
\pi - p_2 &\leq \frac{2}{3}\left(\frac{1}{4}\right) \leq 0.1666666667\\
\pi - p_3 &\leq \frac{2}{3}\left(\frac{1}{4^2}\right) \leq 0.0416666667\\
\pi - p_4 &\leq \frac{2}{3}\left(\frac{1}{4^3}\right) \leq 0.0104104167\\
\pi - p_5 &\leq \frac{2}{3}\left(\frac{1}{4^4}\right) \leq 0.0026041042\\
\pi - p_6 &\leq \frac{2}{3}\left(\frac{1}{4^5}\right) \leq 0.0006510417\\
\pi - p_7 &\leq \frac{2}{3}\left(\frac{1}{4^6}\right) \leq 0.0001627605\\
\pi - p_8 &\leq \frac{2}{3}\left(\frac{1}{4^7}\right) \leq 0.0000406902\\
\pi - p_9 &\leq \frac{2}{3}\left(\frac{1}{4^8}\right) \leq 0.0000101726\\
\pi - p_{10} &\leq \frac{2}{3}\left(\frac{1}{4^9}\right) \leq 0.0000025432
\end{aligned}$$

Remark 4. The Madhava-Leibniz series representation for π, due to the Indian Mathematician Madhava, mentioned in the beginning of this article, is

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1} + \cdots\right).$$

A proof of the above series representation of π, using the fact that $1/(2n+1) \to 0$ as $n \to \infty$, is given in [1]. In fact, it is proved in [1] that, for all $n \in \mathbb{N}$,

$$\left|\pi - 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1}\right)\right| \leq \frac{4}{2n+1}.$$

Accordingly,

$$s_n := 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1}\right)$$

is an approximation of π with error at most $4/(2n+1)$. Note that, for $k \in \mathbb{N}$,

$$\frac{4}{2n+1} \leq 10^{-k} \iff n \geq 2(10^k - 1).$$

Thus, in order to ensure an accuracy of two decimal places, we have to take first 198 terms of the series.

Madhava also used another series representation for π, namely,

$$\pi = \sqrt{12}\left(1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 3^3} - \frac{1}{7 \times 3^3} + \cdots + \frac{(-1)^{n-1}}{(2n-1) \times 3^{n-1}} + \cdots\right).$$

Using the first 21 terms of the above series Madhava obtained an approximation for π which is correct up to 11 decimal places [2]. We may observe that, according to the formula suggested in this article, the approximation p_{21} is correct for at least 12 places. ◊

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1. M.Thamban Nair, *Calculus of One Variable*, Second Edition, Ane-Books Pvt. Ltd; (2021) and Springer (2022).
2. Wikipedia, "Approximations of π", https://en.wikipedia.org/wiki/Approximations_of_%CF%80

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7. Provide a compact list of references, with short recommendations.
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9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
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14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
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A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

Foundational topics include, but are not limited to, the following:

- Number systems, patterns and operations
- Fractions, ratios and decimals
- Proportional reasoning
- Integers
- Bridging Arithmetic-Algebra
- Geometry
- Measurement and Mensuration
- Data Handling
- Probability

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Articles in this section may address key questions such as -

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- What approaches that you used were successful?
- What preparations, in terms of knowing mathematics, enacting the tasks and analysing students work were needed for effective instruction?
- What contexts, representations, models did you use that facilitated meaning making by your students?

Send in your articles to
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At Right Angles
A Resource for School Mathematics

MATH PROJECTS
SCALE DRAWINGS

PADMAPRIYA SHIRALI

Azim Premji University

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

MATH PROJECTS

SCALE DRAWINGS

Mathematical projects that involve practical work combined with essential mathematical concepts have the capacity to stimulate interest and create enjoyment of the discipline. In this article, I have shared a few project ideas that integrate the topics of ratio and design in a practical manner. They help students to figure out how enlargement and reduction in terms of size transformation work mathematically, and clarify the concepts and methods involved. Some of them are hands-on activities involving working with cardboard, and some involve the usage of graph paper, dot paper, etc.

Recipes are often used as a context while teaching ratios, and understandably so. Most students can relate to the context of cooking as it is such an essential aspect of our lives. However, scale drawings involving enlargement and reduction are my favourite while teaching ratio and proportion. The topic lends itself to practical activities and can be integrated with social sciences, technical drawing, and art work.

Discuss many examples of enlargement that the students have encountered. Discuss examples of reduction such as toy cars, etc. Increasingly, children play around with various apps and games on mobile phones or computers that expose them to picture enlargements, zooming in, zooming out, and the idea of scale factor.

They are likely to have seen overhead projectors, movie projectors, photocopy machines, TV sets, telescopes, zoom lenses of a camera, etc.

Discuss the kind of enlargements they produce. Are there instruments that reduce the size of an object?

What role does the bulb of a projector play in the projection? Let students ponder over this question as they may not have an understanding of its importance. At a later point, the teacher can show its connection to the centre of enlargement.

What do convex and concave mirrors do to the images?

Let them find out some facts about these instruments and the power of modern electron microscopes.

Many years ago, when I worked with grade 6 on ratios, the class worked on building a small-scale model of the junior school which involved plenty of measuring, estimating, drawing to scale, and maintaining the uniformity of the scale in the 3-D model that we built. It was a project that took four weeks and integrated various mathematical topics including fractions, decimals, and angles. Many drawings involving front view and side views had to be made. It posed challenges for which we needed to think hard! Producing a realistic model developed our ability to select appropriate waste materials for building it and achieving a reasonable likeness. It was great fun!

Figure 1

Keywords: *Real world, design, projects, scale, ratio, measurements, similarity*

PROJECT 1: DESIGNING A HOUSE FOR A PET

Concepts: Measurement, Ratio, Scaling factor
Student groups: 4 students per group.
This activity can be done by any number of groups. Each group can select a pet for which they would like to design a house.
Materials: Old cardboard sheets, metal ruler, scissors or knife, Fevicol and colours.

Children have a natural liking for animals and often have pets at home. They can bring their knowledge and awareness of the animal size, needs and preferences of the animal to the project work. The discussion could extend beyond the intended house design by talking about other attributes like height, weight, diet preferences, animal behaviour, animal care, cruelty to animals, etc.

Figure 2

Here is a specification for a cat house the students could use for their class work. Cats love snuggling up in boxes, especially the right sized ones!

A good sized shelter for a cat should be 2 feet by 3 feet and at least 18 inches high. A large house is not necessarily the best as the heat will disperse quickly and cats need a warm shelter during the monsoon and in cold weather.

1. Students can be instructed to represent a 3D drawing of the house they intend to make, on paper.

How will they represent 2 feet in the drawing? What scale will they use? (Different groups can work with different scales.)

How will they represent 3 feet in the drawing? (Are they using the same scale as before? Discuss what would happen if they use different scales for different dimensions of the model.)

They should record the scale used for the drawing as "The scale of this drawing is …. to …. inches."

How will the roof be? Flat or sloping?

What should the height of the house at its highest point be, if it is to have a sloping roof?

Which side should have the entrance? How big should the entrance be?

2. As a second step, they should work out and list the sizes of the pieces needed to make the house.

Figure 3

Piece 1: 24 inches by 18 inches

Piece 2: … …

If they were to make the cat house from plywood sheet, will one sheet be adequate? A plywood sheet is generally 8 feet by 4 feet in size.

Figure 4

They should make a drawing of the plywood sheet on paper using an appropriate scaling factor.

They should also make scale drawings of the six pieces needed on dot paper.

They can cut out the pieces that they have drawn and try to fit them together on the drawing of the plywood sheet. Do they all fit on it?

PROJECT 2: DESIGNER ROOM

Student groups: 4 students per group.
Material: Dot or graph paper
Concepts: Estimation, scaling factor, representation, layout design

Students can design their bedroom by listing out their needs and prioritising the furniture needed in their bedroom.

Figure 5

Assuming that the child has to share the bedroom with a sibling they can make a list of necessary items. They can make reasonable estimates of the measurements for these pieces of furniture to design their room plan.

Heights can also be considered where required.

Here is a possible list:

- Cots: 3 × 6 feet (2 Nos)
- Cupboard: 2 × 3 feet (1)
- Book shelf:
- Desktop table:
- Computer chair: 1
- Normal chair: 1

Figure 6

Let the groups work with a standard sized bedroom (10 x 10 x 10). The students can use dot or graph paper for their room plan.

What scale will they use?

Have they marked the doorway? Where will the windows be? Will that affect their arrangement?

The students should draw all the furniture pieces on another graph paper using the same scale. They can cut these pieces and try out different arrangements on their room plan to work out a good fit.

Discuss other possibilities.

What if they needed two study tables? What if they needed two cupboards?

Can the use of a bunk bed help to create a play area?

Students can work with a given design to work out the answers for the questions.

If this room in Figure 7 was 12 by 10 feet what is the size of the beds? What is the size of the play area?

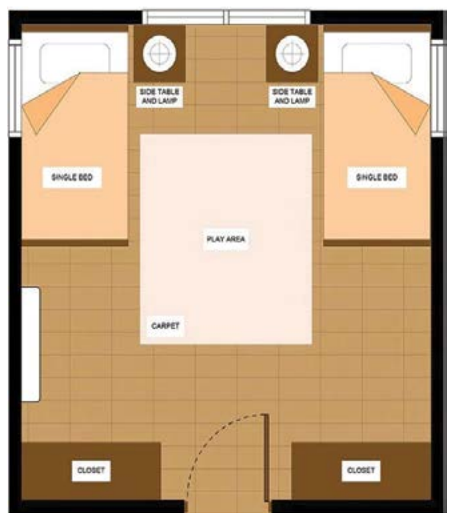

Figure 7

What is the actual length of the door?

PROJECT 3: PHOTOGRAPHS AND COPIES!

Student groups: 4 students per group.
Materials: Pairs of similar rectangles, dissimilar rectangles
Pairs of similar triangles and dissimilar triangles
Similar and dissimilar shapes of varied types.
(They can be printed by using appropriate software or apps.)
Concepts: Observation skills, Similarity, Scaling factor, usage of fractions in measurements

Figure 8

Things which are the same shape but a different size are said to be **similar**.

Talk about similar and dissimilar shapes using geometric materials. What attributes will be used?

Lengths? Angles? Any other attributes?

Which of the shapes in Figures 9-11 are similar to each other? Are there some shapes which are not similar to any other shape? Justify using math properties.

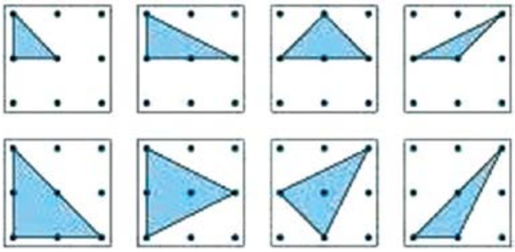

Figure 9

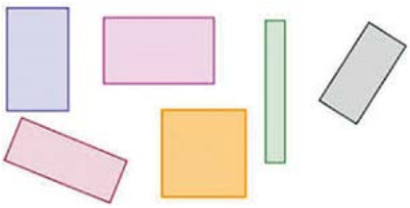

Figure 10

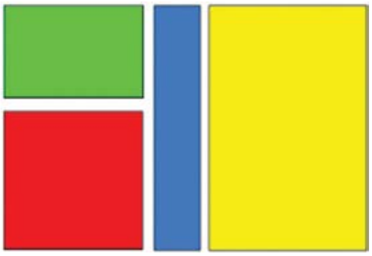

Figure 11

Use the classroom context and objects to talk about similarity. Are the windows in Figure 12 similar?

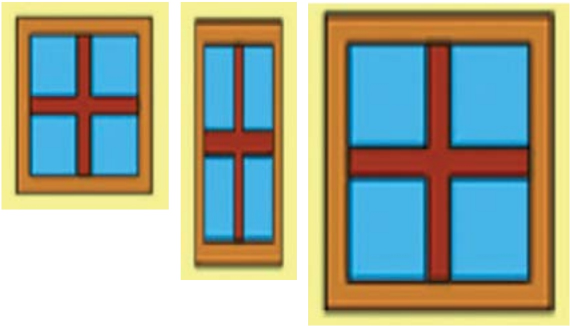

Figure 12

Use outdoor objects of nature like leaves to talk about similarity and dissimilarity in shape. Point out that while all leaves are not similar, functionally they are similar.

Are these dolls in Figure 13 similar? Justify your answer!

Figure 13

Are the pairs of triangles in Figure 14 similar? Use the given measurements to determine similarity.

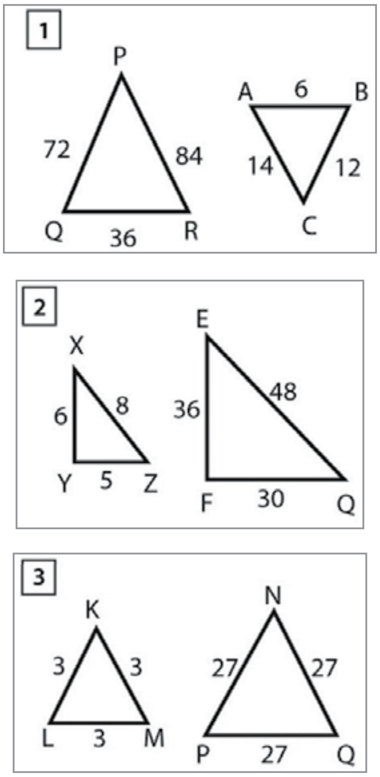

Figure 14

Challenge

Students can be given diagrams that are not drawn to scale and hence they cannot rely on what can be seen or measured. Can they now use mathematical reasoning to identify mathematically similar images?

Original Image:

Figure 15

Which of the three images (Figures 16) is similar to the original image? Justify your answer. What methods can the students use?

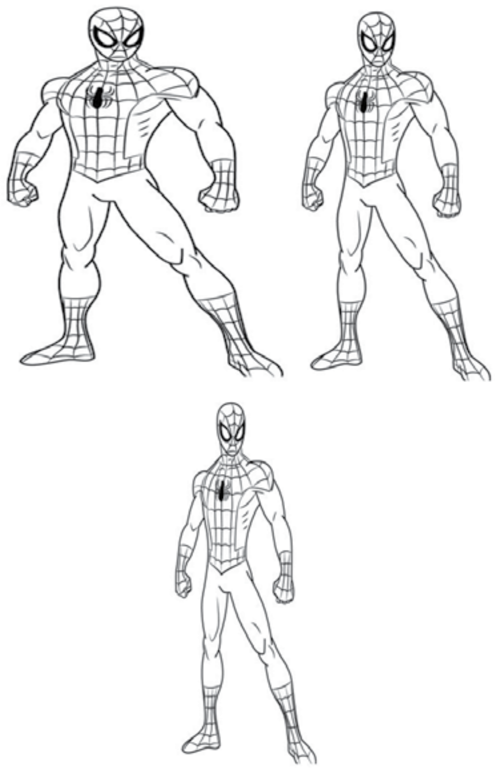

Figure 16

PROJECT 4: ENLARGEMENTS

Student groups: 4 students per group.
Materials: Square dot paper, similar mathematical shapes
Concepts: Enlargement, Scale factor, Similarity, Properties of similar figures

Students can be shown examples of enlarged figures to understand the principles of enlargement. Discuss the pictures (Figure 17) with them. Study the figures in terms of length to contrast and gain an understanding of the scale factor.

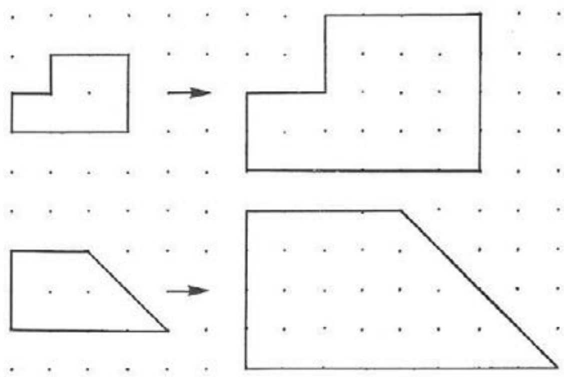

Figure 17

On a square dot paper, the students can make some letters of the alphabet (Figure 18) and make an enlargement by a scale factor of two or three.

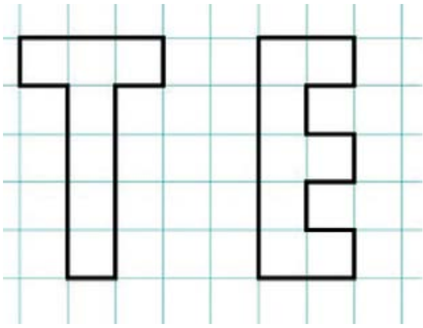

Figure 18

Students compare two given similar triangles (Figure 19) and note down their observations.

Triangle B is an enlargement of triangle A by a scale factor of …

The angles of triangle A and the angles of triangle B are …

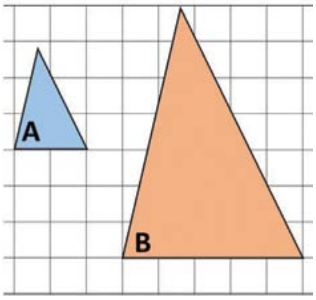

Figure 19

Students can compare two similar rectangles and verify if the scale factor holds for the diagonal.

Are these two rectangles (Figure 20) similar to each other? Justify.

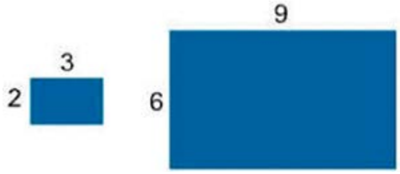

Figure 20

Are any of the shapes B, C, D an enlargement of shape A (Figure 21)?

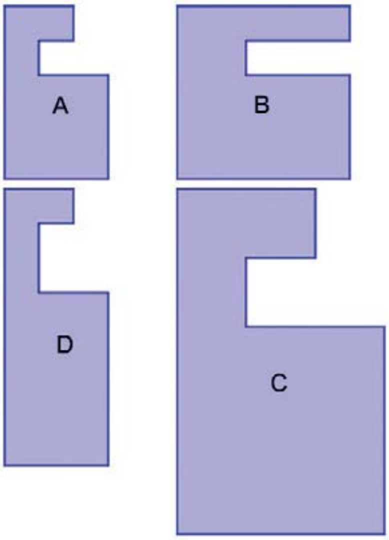

Figure 21

PROJECT 5: BLOW UP SHAPES

Student groups: 4 students per group.
Materials: Set of similar pictures

In Figure 22, measure the wingspan of the butterfly (wing tip to wing tip, measured horizontally) from one end to the other in picture *XY*.

Picture *X′Y′* is an enlargement of picture *XY*. Measure the wingspan in picture *X′Y′* from one end to the other.

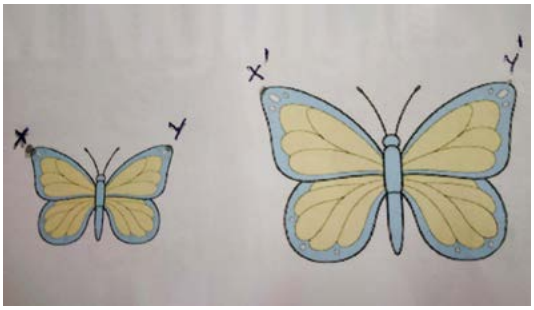

Figure 22

What is the scale factor of the enlargement?

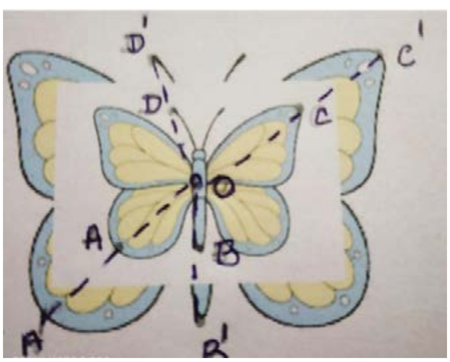

Figure 23: The dotted lines connecting corresponding points all meet at point O

The large butterfly is an enlargement of the small butterfly.

Measure OA and OA′. What is the scale factor?

Is it the same for OB and OB′?

Check the other extensions.

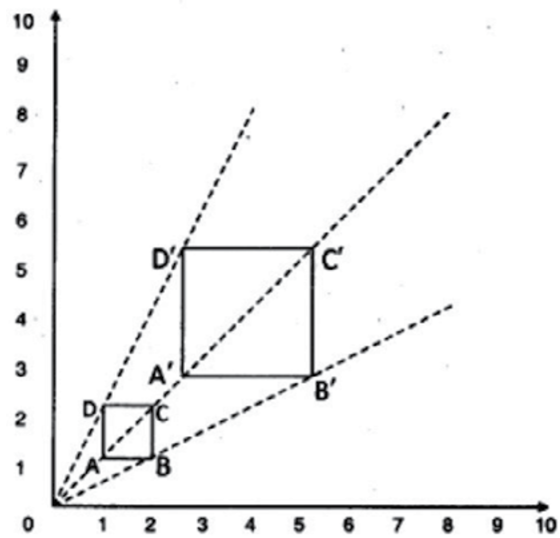

Figure 24

Can we say that a picture is an enlargement of another based on a single measurement?

Point O is called the centre of enlargement. In Figure 24, the larger square is an enlargement of the smaller square, the centre of enlargement being the origin.

The centre of enlargement need not be inside the picture. It can lie anywhere.

Draw any quadrilateral ABCD. Mark a point O inside it (Figure 25).

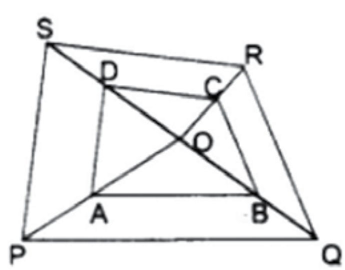

Figure 25

Draw lines from O to the four vertices A, B, C, D.

To enlarge the quadrilateral by a scale of 1.5, measure the length from O to A and extend the line to P by scaling the length by a factor of 1.5. Do the same for all the other lengths, OB, OC, OD.

The quadrilateral ABCD is now enlarged by a scale factor of 1.5.

The centre of enlargement can lie anywhere, as shown in Figure 26. The process of extending the lines is the same.

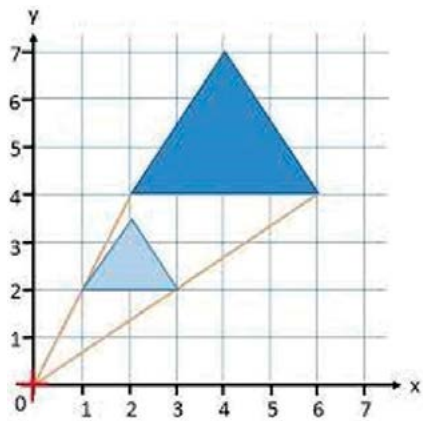

Figure 26

Enlargement can also be done by using coordinates. In Figure 26 the centre of enlargement is at the origin.

In Figure 27 the coordinates of the vertices of quadrilateral ABCD are A (1, 4), B (4,4), C (2,2), D (1,2).

The coordinates of quadrilateral PQRS are P (4,5), Q (10, 5), R (6,1), S (4,1).

Where is the centre of enlargement in Figure 27? What do you think is the scale factor?

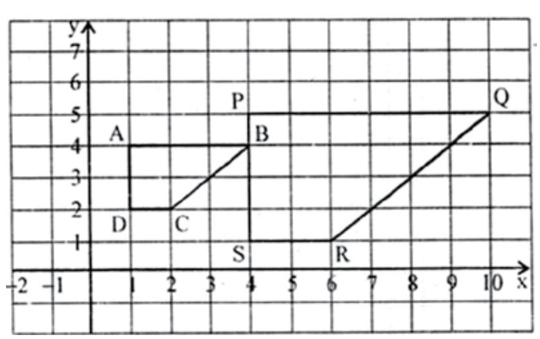

Figure 27

Figure 28 shows a small yellow triangle which has been enlarged.

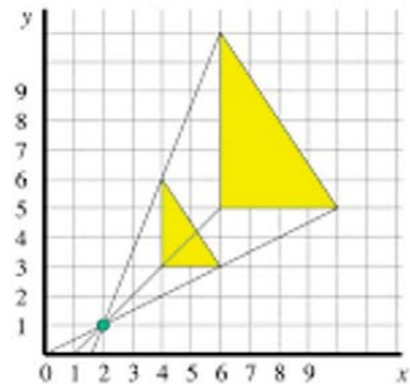

Figure 28

What is its scale factor?

Can you enlarge it by a scale factor of 3?

Is the red outlined figure in Figure 29 an enlargement of the black outlined figure?

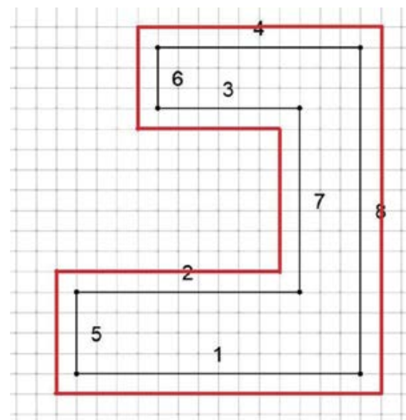

Figure 29

Can you enlarge the black figure by a scale factor of 2?

Students can be helped to articulate what enlargement does to the points in an object by asking leading questions.

When an object is enlarged what happens to the vertices of the object? How do they move? Do they move in a straight line?

What happens to the other points on it and in it? Do they also move away in straight lines?

What happens to the points that are furthest away? How does a bigger scale factor influence the movement?

Do the students see that to enlarge a shape, we only need to know how the crucial points behave?

It will be appropriate to discuss how the centre of enlargement behaves, using different examples.

PROJECT 6: SHRINKING SHAPES

Student groups: 4 students per group.
Materials: Set of similar pictures

Reduction or shrinking is the opposite of enlargement.

Students can experiment with drawing reduced copies of a figure on graph paper, using a scale factor of 1/2 or 1/3.

Figure 30

Can they reduce the given figure by a factor of 1/2?

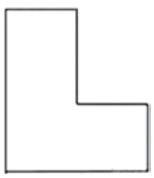

Figure 31

By what scale factor has the original stamp design been reduced to the actual size?

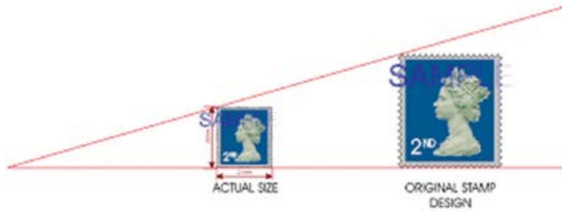

Figure 32

By what scale factor is each cuboid shrinking in height?

Figure 33

PROJECT 7: GROWTH PATTERNS

Student groups: 4 students per group.
Materials: Set of similar pictures of a shape in different sizes

Here is a trapezium that has been enlarged several times.

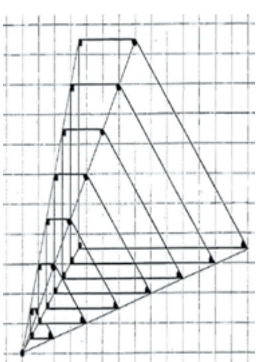

Figure 34

Create a table for the growing trapezia in Figure 34.

Scale factor	Original length	Corresponding new length	Original area	New area	Ratio of original area to new area

What do you notice about the ratios in columns 1 and 6?

Investigate how changing the side of a square by a scale factor of 3 affects its area.

Materials: Set of 3D objects in increasing size

Here is a set of cubes in increasing size.

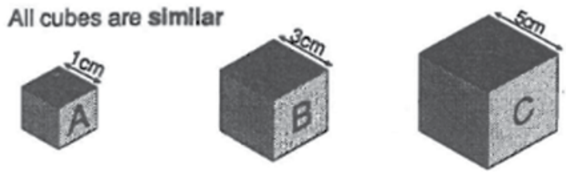

Figure 35

What do you notice about the ratios?

Investigate how changing the side of a cube by a scale factor of 2 affects its volume.

Investigate a rectangle which has been enlarged by a scale factor of 3. How are the perimeter and the area of the new rectangle affected?

Create a table for the growing cubes.

Scale factor	Original length	Corre-sponding new length	Ratio of the original surface area to the new surface area	Ratio of the original volume to the new volume

How does doubling the height and base of a triangle affect the area of the triangle?

What happens to the proportions of the new shape?

PROJECT 8: NEGATIVE ENLARGEMENT

Student groups: 4 students per group.
Materials: Set of pictures with negative enlargement

Discuss with the students about what is happening in the negative enlargements.

Do they notice that the figure is on the other side of the centre of enlargement?

When we draw the lines in the opposite direction, it is called a negative enlargement (Figure 36).

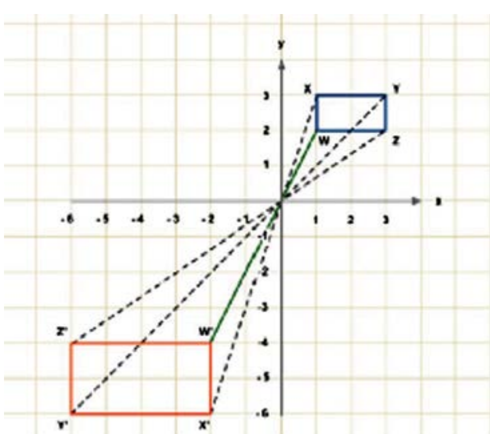

Figure 36

Discuss with students the meaning of a negative scale factor.

A **negative scale** factor -2 indicates double the distance from the centre of enlargement, but in the opposite direction. Here is one such example.

Let the students study the following diagrams (Figures 37-39) carefully.

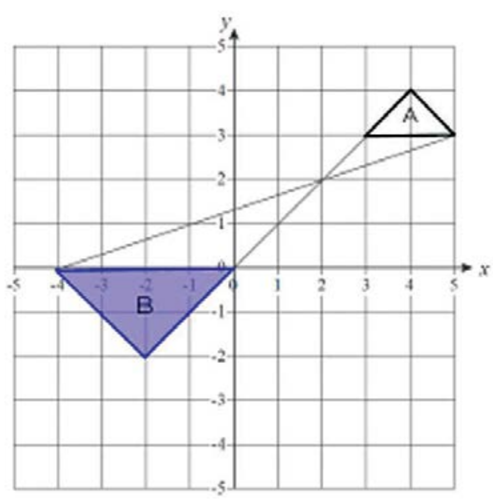

Figure 37

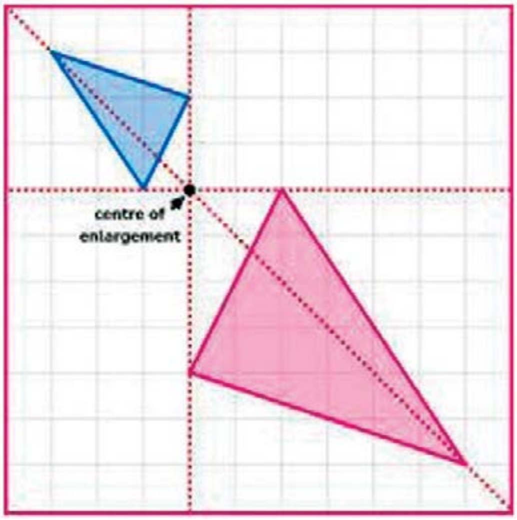

Figure 38

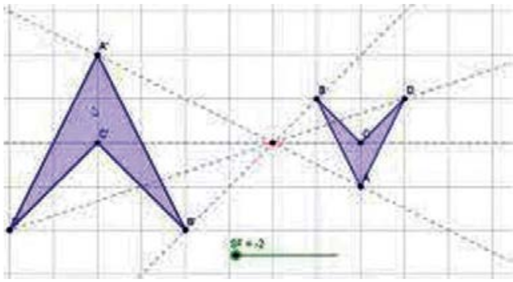

Figure 39

Do the students notice that a negative enlargement causes the image to be inverted or turned 'upside down'?

The students can make a figure like a kite and make a negative enlargement, say by a factor of -3.

What happens when the scale factor is greater than 1?

What happens when the scale factor is equal to 1?

What happens when the scale factor is between -1 and 1?

Notice that the shape reduces in size.

What happens when the scale factor is negative?

Investigation

Students can select any shape and experiment with different centres (one lying outside, one lying inside, one lying on the shape) to draw enlargements of uniform positive scale to observe what difference it makes.

What will remain the same? Size or shape?

What changes? Their location.

They can also experiment with negative scale factors.

Acknowledgements

School Mathematics project (SMP) https://generic.wordpress.soton.ac.uk/smp2/access-to-smp/

Smile: https://mypages.iit.edu/~smile/mathinde.html

https://ng.siyavula.com/read/maths/jss3/similar-shapes/09-similar-shapes

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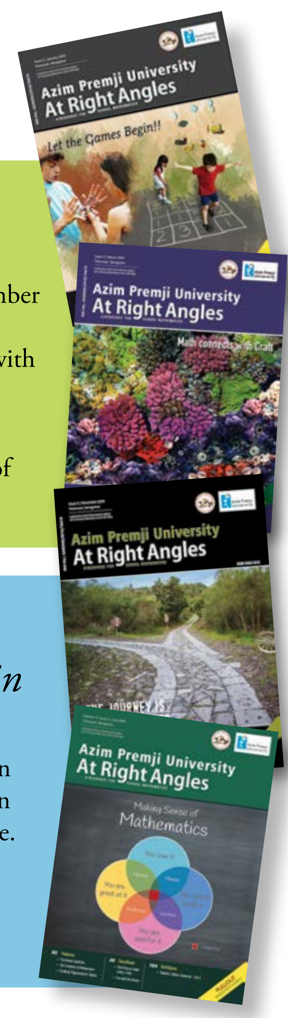

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