Issue 15 | March 2023
Triannual | Bangalore

A publication of Azim Premji University together
with Community Mathematics Centre, Rishi Valley

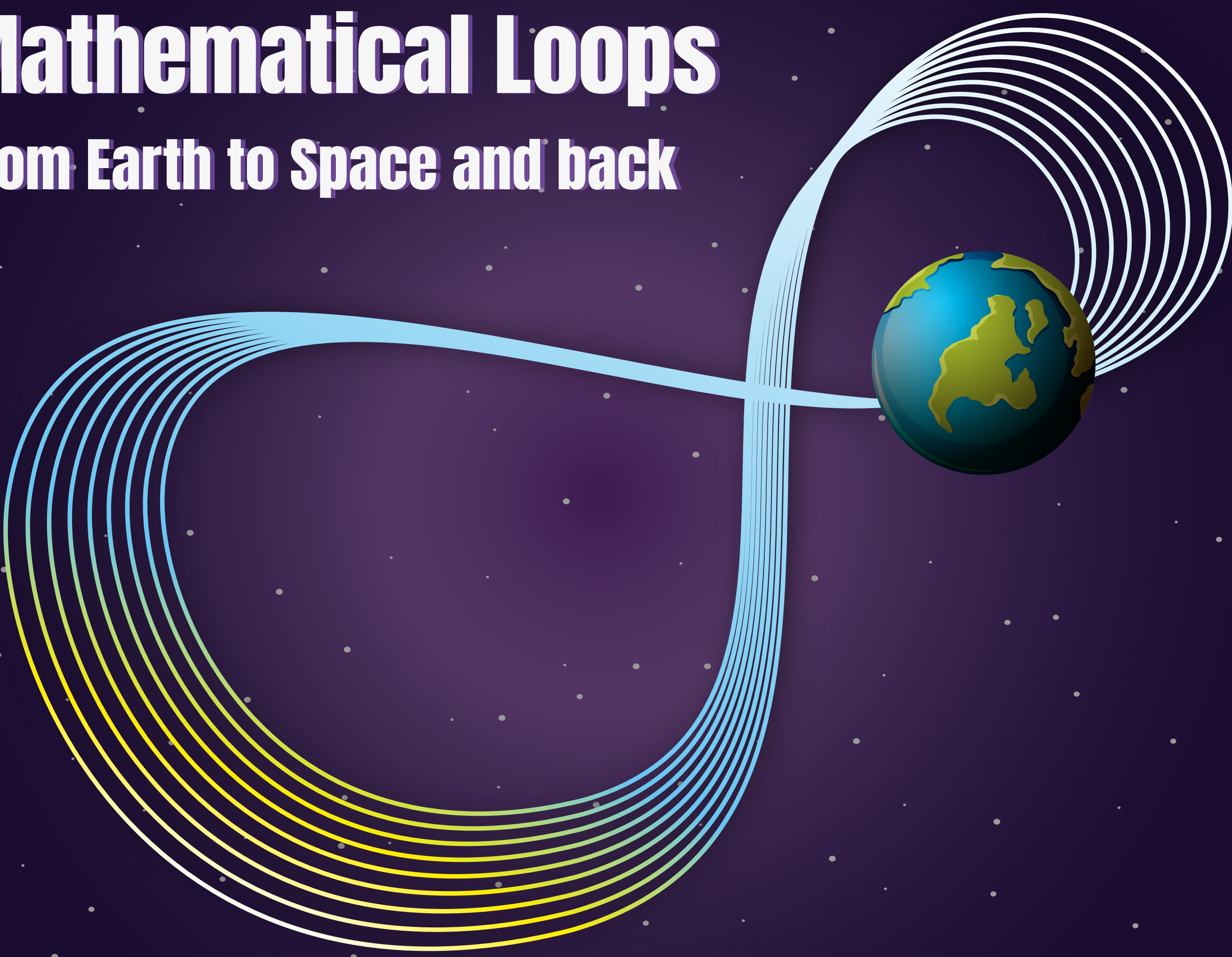
Azim Premji University
At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

RNI No.: KARENG/2018/76619

ISSN 2582-1873

Mathematical Loops
From Earth to Space and back



PULLOUT
SPEED, TIME, DISTANCE & GRAPHS

The Mobius Strip has often been used to represent continuity amidst change, two concepts which are frequently studied in mathematics. To be curious, inventive and creative and at the same time, mathematically rigorous, these are the looping faces of the disposition of a mathematician. How else can we live on earth and attempt to study the Sun and the stars? Wonder about orbital paths and find them in mathematical graphs? Let us be alive to the beauty and truth about us. Else, we will be in this unfortunate situation described so powerfully and vividly in Lockhart's Lament:

All this fussing and primping about which "topics" should be taught in what order, or the use of this notation instead of that notation, or which make and model of calculator to use, for God's sake — it's like rearranging the deck chairs on the Titanic! Mathematics is the music of reason. To do mathematics is to engage in an act of discovery and conjecture, intuition and inspiration; to be in a state of confusion — not because it makes no sense to you, but because you gave it sense and you still don't understand what your creation is up to; to have a breakthrough idea; to be frustrated as an artist; to be awed and overwhelmed by an almost painful beauty; to be alive, damn it. Remove this from mathematics and you can have all the conferences you like; it won't matter. Operate all you want, doctors: your patient is already dead.

From the Editor's Desk . . .

New Year resolutions have just been made and broken, the March 2023 issue comes at a time when we are settling into a new year and at the same time, coming to the end of the academic year. Examinations, marking, report cards, holiday plans….life goes on at a tremendous pace and our PullOut on Speed seems particularly relevant at this time.

Charu Gupta's Feature article on *A (S)trip with a Twist* will have you reaching for the nearest colourful strip of paper; make sure you have paper and pen to record your findings! It sets the stage for the bouquet of articles in this issue, from *A Conversation with Ian* by James Metz to unravelling and extending an algorithm from Vedic mathematics by Progyan Sensowa. New results, new theorems, new understanding of shortcuts prescribed for competitive examinations, there is a lot to read and digest. Shailesh Shirali explains the incredible accuracy of *A Procedure to (Approximately) Trisect an Arbitrary Angle* presented by Mahesh Bubna – approximations to this time-honoured problem seem to be getting better and better. And Prithwijit De suggests that teachers should not let students be satisfied with QED, can we push the limit on this? More power to curiosity which leads to such explorations.

Assessing Mathematical Proficiency at the Secondary Stage, by Math Space, takes a fresh look at the cognitive domain. How are we designing items for the different domains described in the Bloom's Taxonomy? Food for thought for teachers here. Along the way you will encounter *Another Theorem for Congruence of Triangles*, and *Geometric Proofs of Two Trigonometric Identities*. And definitely a lot of inspiration from students for students from our Student Corner, with articles on *Deriving an Equation for the Sun's Path*, number theory problems, and even a mini-review of the *Number Devil*. You are also sure to be fascinated by the review of yet another book on Ramanujan, it promises to be very different from its predecessors.

Do make sure that you are receiving your copies regularly if you have opted for the hard copy, else you can drop a mail to AtRiA.editor@apu.edu.in. This is also the email id for submitting articles and letters to the editor.

Have a great read!

Sneha Titus
Associate Editor

Editorial Office

The Editor, Azim Premji University
Survey No. 66, Burugunte Village,
Bikkanahalli Main Road, Sarjapura,
Bengaluru – 562 125
Phone: 080-66144900
Fax: 080-66144900
Email: publications@apu.edu.in
Website: www.azimpremjiuniversity.edu.in

Publication Team

Shantha K
Programme Manager
shantha.k@azimpremjifoundation.org

Shahanaz Begum
Associate
shahanaz.begum@azimpremjifoundation.org

Print

SCPL
Bengaluru 560 062
www.scpl.net

Design

Zinc & Broccoli
enquiry@zandb.in

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

Problem Corner

Student Corner

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Continue . . .

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Online Articles

A (S)Trip with a Twist

CHARU GUPTA

The Mobius strip has been an intriguing mathematical object since its discovery in the nineteenth century by August Ferdinand Mobius. Quite interestingly, it also appeared in the Draft National Education Policy 2019 to symbolise the "perpetual, developing, and live nature of knowledge"—an ongoing pursuit of knowledge creation and dissemination. The Mobius Strip has been an object full of wonder and surprise that even magicians couldn't resist, and they exploited its properties to play some mysterious tricks on their audience. The exploration described here also began with one such trick, conducted during a session with pre-service elementary teachers during the first year of their course. The tasks related to exploring Mobius strips provided meaningful opportunities for engaging prospective teachers in mathematical processes, including searching for patterns, visualizing, predicting, verifying, formulating generalizations, and posing new problems. It is being assumed that an active, collaborative, and intellectual engagement with such carefully designed tasks would provide pre-service teachers with a wide range of resources to draw upon and enliven their classroom learning environments.

The Magic Trick

Following a magician's trick from Gardner's book *Mathematics, Magic, and Mystery* [1], three long paper bands were presented, and the pre-service teachers were called upon to cut each of these bands along the middle lengthwise. Initially, it seemed like a routine task that would result in two identical pieces. Cutting the first band resulted in two paper bands, as expected. However, they were in for a

Keywords: Exploration, experiential learning, hands-on mathematics

big surprise as cutting the second strip resulted in a single paper band, longer than the original one. And, from the third one, two interlocked paper bands were obtained. An object like a paper strip that seemed so familiar at first is now seen to be behaving in many strange ways. Students were now curious to look for an explanation for the same, setting the stage for an exhilarating ride ahead. Many of them could guess by now that not all the bands were "normal" paper bands they were familiar with. There was a twist (rather, many twists!) to the story.

Let's twist!

Students were guided to prepare the three paper bands whose properties had been exploited to trick the audience. The first band was a simple loop with no twist (resembling a cylinder or a ring). The second was the Mobius Strip, formed by twisting one end 180 degrees with respect to the other before pasting the two ends together, called a 'half-twist'. The third loop was formed by twisting one end 360 degrees relative to the other before pasting the ends. All three bands are shown in Figure 1.

Figure 1. The yellow band is a simple loop, the pink one is the Mobius strip, and the green one is the strip with two half-twists.

Naturally, this led to a question: What difference does adding a half-twist make to the paper strip? This led to an opportunity to explore the properties of the strips. Students were asked to draw a line along each of the strips until they returned to the starting point. On a simple paper band (with no twists), the line ran along the outer surface only. It is a two-sided surface, with an inside and an outside. However, on a Mobius strip, the line ran along both the surfaces. It was with the introduction of this one-sided surface that many exciting mathematical adventures began to unfold.

A mathematician confided
That a Mobius strip is one-sided.
You'll get quite a laugh
If you cut it in half,
For it stays in one piece when divided.
~ Anonymous

Students were amused to discover the difference that a half twist could make to the properties of a paper strip.

A wave of excitement ran through the class as students went on adding more half twists to their paper strips and cutting them along the middle. In the process, they were encouraged to predict the results before cutting these strips and to keep records of their work so as to leave written traces of their ways of thinking and reasoning about the task. Figure 2 shows the resulting strips after cutting the strips in Figure 1 along the mid-line. Pictures 3 and 4 show the strips obtained after cutting the strips with three and four half twists, respectively.

Figure 2

Figure 3. A long knotted strip.

Figure 4. One strip looped twice around the other.

Much to their amazement, the students could now see that the tabulated observations seemed to be unfolding a pattern. Table 1 offers a glimpse of the observations recorded by one of the groups.

Number of half twists in original band	Number of sides	When cut along the middle line
0	2	Two separate but identical bands Length same as original band Width is half the original
1	1	A single band with 4 half twists Length twice the original Width half the original
2	2	Two interlocking bands with 2 half twists in each (i.e., identical with the original) Length same as original band Width half the original
3	1	A single knotted band with 8 half twists Length twice the original Width half the original
4	2	Two interlocking bands with 4 half twists in each Length same as original band Width half the original

Table 1

The generalised expressions based on a strip with n half twists are:

If n is even,

The bands are all two-sided. If this band is cut along the midline between the edges, we obtain two interlocking bands, each of which has n half twists (identical with the original band). One band is looped $n/2$ times around the other. Both bands are, however, narrower.

If n is odd,

The bands are all Mobius-like, i.e., one-sided surfaces. Cutting along the midline in this case, we obtain a single long band with $2n + 2$ half twists. The resulting strips are, therefore, two-sided and not Mobius-like. Also, for $n > 1$, the resulting band is knotted.

The exploration initiated them into imagining lots of cutting experiments and, at the same time, searching for patterns and making a number of generalizations.

According to Lockhart (2009) [2],

> *A good problem is something you don't know how to solve. That's what makes it a good puzzle and a good opportunity. A good problem does not just sit there in isolation but serves as a springboard for other interesting questions.*

The above exploration led to some new questions, like what would happen if the cuts were not made along the middle of the twisted strips, but at some other distance from the edge? Will that change the resulting strip, and how?

Lo and behold!

Students began investigating with the Mobius strip first. In Figure 5, the yellow strip is obtained after cutting a Mobius strip along 1/3rd the distance from the edge, while the pink strip is obtained after cutting along 1/4th the distance.

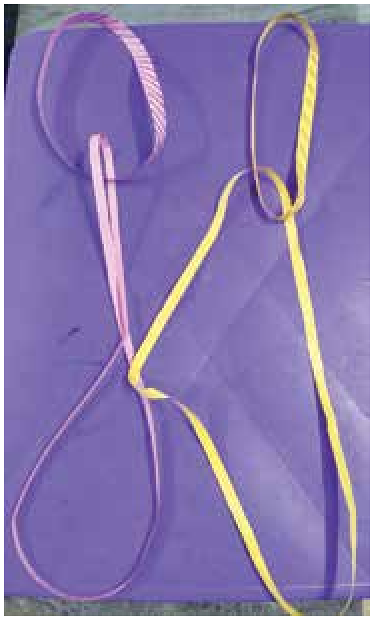


Figure 5

The following observations were made and tabulated as shown in Table 2:

Distance from the Edge	When cut is made
½	A single band with 4 half twists Length twice the original Width half the original
⅓	Two linked bands One Mobius band identical to the original Length same as original; Width one-third of the original One band with 4 half twists Length twice the original; Width one-third of the original
¼	Two linked bands One Mobius band identical to the original Length same as original; Width half of the original One band with 4 half twists Length twice the original; Width one-fourth of the original
⅕	Two linked bands One Mobius band identical to the original Length same as original; Width three-fifths of the original One band with 4 half twists Length twice the original; Width one-fifth of the original

Table 2

Some of the groups could arrive at generalisations based on the observations made. It was expressed as follows:

Cutting a Mobius strip along a $1/m$ distance from the edge resulted in the following linked bands:

The Mobius band obtained has its length same as the original, while the width varied at each stage and is $1 - 2/m$ of the original.

Another band obtained is two-sided and has four half twists in it. Its length is twice the original, while the width is $1/m$ of the original. This band, thus obtained, is identical with (though narrower) the one obtained after cutting the original band along the midline.

A careful look at the cuts being made brought forth an interesting observation. That is, if we start cutting the band at one-third distance from the edge, we traverse the band twice before coming back to the starting point. This explains how the one-third part at the middle of the original band forms a narrower Mobius band identical with the original, along with another band with length twice the original, and width one-third of the original. This is in contrast to cutting the strip along the middle when the strip is traversed only once before coming to the starting point.

A question that arose was: Will it hold for other Mobius-like bands (i.e., bands with an odd number of half twists) too?

This was found to be true for all the bands with an odd number of half twists. We just need to vary the number of half twists in the second band which was again identical (though narrower) with the resulting band when the original band was cut down the middle lengthwise.

The exploration was now being extended to tabulate observations when bands with "n" half twists are cut at $1/m$ distance from the edge.

Number of half twists in original band	Cut along the Distance from the Edge		
	½	⅓	¼
0			
1			
2			
3			

Table 3

The following generalisations were reached:

When the cut is made along $1/m$ distance from the edge:

If the number of half twists is even, 2 interlocking bands, each having n half twists and the same length as the original band, are obtained. The widths of the bands are $1/n$ and $1 - 1/n$.

If the number of half twists is odd, 2 linked bands are obtained. One is Mobius-like, identical with the original, with its width as $1 - 2/n$; another band has a length twice the original, a width of $1/n$, and has $2n + 2$ half twists.

It was quite fascinating how experiments in cutting some twisted strips led to formulating generalizations. Many shifts were witnessed in the students' ways of approaching and engaging with the tasks: from random guessing to being analytical; from simply solving problems to posing new problems; from random note-taking to systematic organising and synthesising information; and towards appreciating beauty in mathematics.

A meaningful engagement in mathematical processes will not only support teachers in emphasising these processes in their own practice but will also prove essential in the sense that these processes cut across different content domains and at all levels. Open-ended tasks allow for multiple entry points and offer multiple pathways for further exploration, charting unfamiliar territories. There is no end to what can be investigated. And so the (s)trip continues...

References

[1] Gardner, M. (1956). *Mathematics, Magic and Mystery*. NY: Dover Publications.

[2] Lockhart, P. (2009). *A Mathematician's Lament.* NY: Bellevue Literary Press.

CHARU GUPTA is currently pursuing a Ph.D. in mathematics education from Jamia Millia Islamia, New Delhi. She has taught courses related to the pedagogy of mathematics in the elementary teacher education programme at the University of Delhi. Her primary area of research interest is foregrounding equity-based issues in mathematics education. Charu may be contacted at charuguptajun30@gmail.com

A Conversation with Ian about Squares and Cubes

IAN KOMENAKA & JAMES METZ

Ian: "What is the difference of the squares of two consecutive positive integers?" (He knew the answer, but wanted to challenge me. He said he had discovered a shortcut.)

James: "I've made a few examples and it looks like you add the two numbers." (I had not observed that before.) "How did you discover this?"

Ian: "When I was building some things, I needed to know some areas, so I just made a multiplication table for the integers from 17 through 28." (See Photograph 1.)

	17	18	19	20	21	22	23	24	25	26	27	28
17	**289**	306	323	340	357	374	391	408	425	442	459	476
18	306	**324**	342	360	378	396	414	432	450	468	486	504
19	323	342	**361**	380	399	418	437	456	475	494	513	532
20	340	360	380	**400**	420	440	460	480	500	520	540	560
21	357	378	399	420	**441**	462	483	504	525	546	567	588
22	374	396	418	440	462	**484**	506	528	550	572	594	616
23	391	414	437	460	483	506	**529**	552	575	598	621	644
24	408	432	456	480	504	528	552	**576**	600	624	648	672
25	425	450	475	500	525	550	575	600	**625**	650	675	700
26	442	468	494	520	546	572	598	624	650	**676**	702	728
27	459	486	513	540	567	594	621	648	675	702	**729**	756
28	476	504	532	560	588	616	644	672	700	728	756	**784**

Table 1. Ian's multiplication table

James: "What did you notice in the table that made you think about this? Do you recall which numbers you first observed?"

Keywords: Pattern, numbers, observation, exploration, connection

Ian: "I looked at 20 and 21 and saw that their sum was the same as 441 – 400 in the table. Then I tried all the others along the diagonal and the same thing happened. The difference of the squares of two consecutive positive integers was always the sum of the two integers. Then I tried some big number like 46 and 47 and some small numbers and it was still true. I tried all the integers from 1 to 50 and they all worked."

James: "You must have had a pretty strong feeling that it is always true."

Ian: "I think so."

James: "Have you tried to prove it?"

Ian: "Not yet."

James: "Why don't you try to prove it now using what you know from algebra?"

Ian wrote, "x is the smaller number; $x + 1$ is the larger number." He then simplified $(x + 1)^2 – x^2$ to $2x + 1$, which is the same as $x + x + 1$, the sum of the two numbers, and he said, "It works!"

James: "Very good. Before we continue, let's do a little geometry to show $7^2 – 6^2$. How can you represent 7^2?"

Ian: "I can draw a square that is 7 by 7."

James: "Good. Please do that."

James: "Now that you have that, outline a 6 by 6 square. If we take away the 6 by 6 square, you see that we have an 'L' shape that is the difference of the squares. How many unit squares are there?" (See Figure 1.)

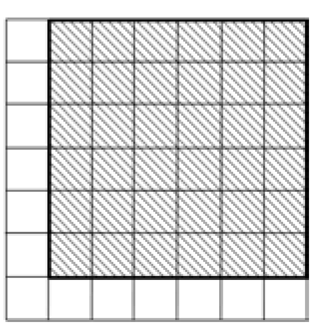


Figure 1. The geometry of $7^2 – 6^2$

Ian: "13."

James: "How can you count them?"

Ian: "7 + 6, or 6 + 6 + 1 which is 2(6) + 1. So, instead of 6, if I use x it will still be true that the difference is $1 + x + x = 2x + 1$." (He could *see* $2x + 1$.)

James: "Great. We will come back to this later. Now I ask you, 'What is the difference of the squares of two consecutive positive even integers?' Try $8^2 – 6^2$ for example."

Ian: "I get 28."

James: "Right. What do 8 and 6 have to do with 28? Remember what you did before."

Ian: "I add them together and then I need to multiply by 2."'

James: "What do you think the rule might be?"

Ian: "2 times the sum. Probably 2 because the difference between them is 2."

James: "Prove it."

Ian simplified $(x + 2)^2 – x^2$ to $4x + 4$ or $2(2x + 2)$ proving that he was correct. He then showed it using the squares on graph paper and generalized from the picture.

James: "You are doing good. What if we have two consecutive positive odd integers?"

Ian: "It will be the same as the evens because they are also separated by 2."

James: "Very good! I have one more question before we move on to something else. What if the difference between the two positive integers is d, where d is a positive integer?"

Ian quickly wrote $(x + d)^2 – x^2$ and announced excitedly, "It is $d(2x + d)$, like I suspected!" He checked a few examples to be satisfied that it worked.

James: "Let's draw a picture for the case $d = 4$, like $7^2 – 3^2$. (See Figure 2.) Can you please describe the "L" shape here?"

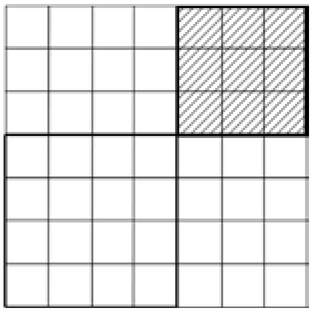


Figure 2. The geometry of $7^2 - 3^2$

Ian looked at the picture and said, "One leg has 3 rows (in general x) with 4 (in general d) squares in each and so does the other one, so there are 2(3)(4) squares (in general $2xd$), plus the corner has 4^2 squares (in general d^2)." Ian generalized easily because he could *see* the rule in the picture.

James: "We have done everything with the difference of the squares of two positive integers, so what shall we do next?"

After thinking about squares for a while, Ian decided to look at the difference of the cubes of two consecutive numbers. He wrote $4^3 - 3^3$ but could not see anything, so he said, "Let me just prove it." He simplified $(x + 1)^3 - x^3$ to $3x^2 + 3x + 1$. (No wonder he could not see any pattern!)

James: "How can we model this with geometry?"

Ian: "We can use cubes."

Using sugar cubes, we looked at $4^3 - 3^3$. After he removed the 27 cubes that made up 3^3, he had 3 yellow "walls", each 3 by 3, 3 white "columns", each 3 by 1 and 1 cube in the corner (not visible in the figure), so he had $3x^2$ (the 3 walls), $3x$ (the 3 columns), and 1 (cube in the corner). (See Figure 3.) The formula made sense.

We decided to call it a day knowing that if we wanted to, we could find the formula for the difference of the cubes of any 2 positive integers.

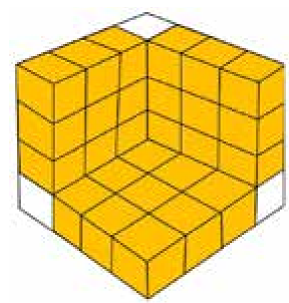
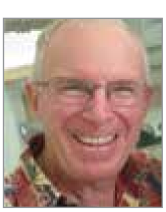


Figure 3. The geometry of $4^3 - 3^3$

Note. At the time of this conversation, Ian was a student in grade 9 in Arizona. Ian's grandmother was the officemate of James for 10 years. The preceding is a transcript of a FaceTime® call. Perhaps the conversation will inspire teachers to delve deeper when students ask questions.

IAN KOMENAKA is currently a student at Arizona College Prep high school in Chandler, Arizona, U.S.A. He has been intrigued by math seemingly for his entire life. He noticed the pattern studied while making a multiplication table for numbers 17-30. He uses math and pattern recognition in his other interests which include chess and football. In the future, he intends to use his interest to pursue a career in statistics. He can be reached at iangk022@gmail.com.



JAMES METZ is a retired mathematics instructor. He volunteers for Teachers Across Borders Southern Africa for one month each year and enjoys doing mathematics. He can be reached at metz@hawaii.edu.

ClassRoom

An Algorithmic Approach to Multiplication using "Ekadhikena Purvena Sutra"

PROGYAN SENSOWA

INTRODUCTION

Vedic mathematics is one of the contributions of India in the field of Mathematics. A saint of the Shankaracharya Order, Swami Bharati Krishna Tirthaji, happened to come across some Ganita Sutras or mathematical texts. He reconstructed them into 16 sutras and 13 upa-sutras. In the sutras he found some patterns of calculations related to topics such as subtraction, multiplication, squaring, square roots, cubes, cube roots, division, simple and quadratic equations, and much more.

The algorithms I am going to describe here are created with the help of the sutra named EKADHIKENA PURVENA. These algorithms stir up a lot of interest and enjoyment. Therefore, they come within the ambit of recreational mathematics.

In this article, we mention some limitations to the Ekadhikena Purvena sutra and describe modified algorithms to overcome the limitations. The limitations are taken as three different cases and the modified algorithms to overcome those limitations are given accordingly.

Keywords: Recreational mathematics, Multiplication of two numbers, Algebraic expressions, Algorithms

The Ekadhikena purvena sutra. This sutra is used to multiply numbers when –

1. The working-base or base is the same in both the numbers.
2. The digits in the units' place add up to ten.

For example: (1) 77×73 (2) 81×89 (3) 63×67, etc.

Note: **Bases** are powers of 10, for example 10, 100, 1000, etc., whereas **working bases** are numbers which are scalar multiples of powers of 10, for example 20, 2300, 4300, etc. If the working base is a scalar multiple of 10, for example $10x$ where x is a natural number, the **base number** will be x .

The process of calculation is as follows.

1. Two columns are made.
2. To get the left side of the answer, multiply the base number of the working base with one more than the same. In Example 1, the working base is 70, so multiply 7 (7 is the base number of 70) with 8 (which is one more than 7) to get 56.
3. To get the right side of the answer, multiply the digits in the units' place of the two numbers. The right-side column must contain two digits. If not, prefix a '0' to the product. In Example 1, the digits in the units' place are 7 and 3 and so the product is 21.
4. After getting the products of the two columns, append the answer in the right column (21) to the answer in the left column (56) to get the required answer (5621).

LIMITATIONS

Here, some questions arise.

1. What to do when the sum of the digits in the units' place is *less than ten*?
2. What to do when the sum of the digits in the units' place *exceeds ten*?
3. What to do when the working bases of the two numbers differ and the digits in the units' place of *both the numbers is 5*?

In response to the above, I place my answers below by using the following algorithms that I have created:

Consider the following examples in the case of **Limitation 1**: 42×43; 63×64; 75×72; 92×96; 223×226.

To explain the algorithm, take the example 42×43. Here, the working base is 40, and the sum of the digits in the units' place is $2 + 3 = 5$, which is less than 10.

Steps – Make two columns

42×43	
$\{40 \times (4 + 1)\} - \{4[10 - (2 + 3)]\}$ In the left column, first multiply the working base, i.e., 40 with 1 more than the base number which is 4 and then from the product, take away 4 times the difference of 10 and the sum of 2 and 3, as shown above.	2×3 In the right column multiply the digits in the units' place, i.e., 2 and 3.
$200 - \{4 \times 5\}$	6
180	6
Now, after getting the products of the two columns, append them to get the answer. And the answer is 1806.	

Here we take another example, i.e., 63×64. Here the working base is 60 and the last two digits add up to 7 which is less than 10. The steps are as follows ——

63×64	
$\{60 \times (6+1)\} - \{6[10-(3+4)]\}$ $420 - \{6 \times 3\}$ $420 - 18$ 402	3×4 12 12 $12 = 10 + 2$
$402 + 1$ Now after getting the products of the two columns shift the tens place digit of the right-side product to the left column and add it as shown above.	2 The right column should only consist of one digit; if not, then carry the left digit of the right product to the left product and add it. The answer is 4032.

Consider the following examples in the case of **Limitation 2**: 53×59, 99×98, 55×56, 999×997, etc.

To explain the algorithm, let us take an example: 99×98. Here both the numbers are of the same working base that is 90 and when the units' digits are added up, it exceeds ten.

As before, make two columns.

99×98	
$\{90 \times (9+1)\} + \{9 \times [(8+9) - 10]\}$ To get the answer of the left column, multiply the working base, i.e., 90 with one more than the base number which is 9 and add it with 9 times the difference of 10 from the sum of 8 and 9.	9×8 To get the answer in the right column multiply 9 with 8. $9 \times 8 = 72$
$900 + \{9 \times 7\}$ $900 + 63$ 963	72 72 72
$963 + 7$ 970 The right column should consist of 1 digit; if not then bring the tens digit of the right product to the left column and add it with the product as shown.	2 2
After getting the results of the two columns, combine them to get the answer 9702.	

Let us take another example: 222×229.

Here we will take the working base as 220 because the two numbers fall under the same working base, i.e., 220 and the units' digits when added up, exceed 10.

222 × 229	
{220(22 + 1)} + {22[(9 + 2) − 10]}	2 × 9
5060 + 22	18
5082	18
5082 + 1	8
5083	8
The final answer is 50838.	

Consider the following examples in the case of **Limitation 3**: The pattern of the numbers would be of the following nature: 25 × 35, 35 × 95, 325 × 465, 105 × 115, 25 × 65, etc.

To view the algorithm let us take the example 25 × 35. Here the two numbers have different working base and the units' digits are 5.

The steps are given below. Make two columns

25 × 35	
$\left\{[2\times(1+3)]+\left[(3-2)\times\frac{1}{2}\right]\right\}\times 100$ On the left side multiply the smaller number, i.e., 2 by one more than the greater number, i.e., 3 and add the result to *half* of the difference of the two numbers, i.e., 2 and 3 as shown above. $(8+0.5)\times 100$	5 × 5 To get the answer of the right column multiply 5 by 5 to get 25. 25
Now multiply 8.5 by 100 to get 850 and bring the last two digits of the answer, i.e., 50 to the right side and add it to 25	
8	25 + 50 = 75
Now combine the results to get the answer 875.	

Let us take another example: 105 × 125.

105 × 125	
$\left\{[10\times(12+1)]+\left[(12-10)\times\frac{1}{2}\right]\right\}\times 100$	5 × 5
(130 + 1) × 100	25
131 × 100	25
131	25 + 00
The answer is 13125.	

These are the new algorithms that overcome the limitations mentioned.

Understanding The Algebra Behind The Third Algorithm

Let $(10x + 5)$ and $(10y + 5)$ be positive integers, where x and y are two single digit natural numbers with $y > x$. Then, the same multiplication can be done using the third algorithm with a twist of algebra in it which is shown below.

$(10x+5)\times(10y+5)=$	$\left\{x(y+1)+\frac{(y-x)}{2}\right\}100$	5×5
$(10x+5)\times(10y+5)=$	$\left\{(xy+x)+\frac{(y-x)}{2}\right\}100$	5×5
$(10x+5)\times(10y+5)=$	$\left\{\frac{(2xy+2x+y-x)}{2}\right\}100$	5×5
$(10x+5)\times(10y+5)=$	$\left\{\frac{(2xy+x+y)}{2}\right\}100$	5×5
$(10x+5)\times(10y+5)=$	$\left\{\frac{(200xy+100x+100y)}{2}\right\}$	5×5
$(10x+5)\times(10y+5)=$	$50(2xy+x+y)$	25

Here, at the end we shift the two last digits of the product of the left column to the right column and add it. Hence, we can also write the same as -

IDENTITY 1. $(10x+5)\times(10y+5)=50(2xy+x+y)+25$

This identity is applicable for all natural values of x and y.

CONCLUSION

The appearance of the illustrated algorithms may give an impression that it is complicated and mind-boggling, but once the algorithm is understood and the calculations are done mentally, it becomes simple and less time consuming. We encourage the readers to explore the algebra behind them. These algorithms make calculations easier for students as they inspire them to explore the manipulative power of numbers. Besides, it takes them away from the boredom of stereotyped calculations. It also increases their interest in mathematics and develops a zeal for it. Further it promotes creativity, curiosity and an explorative frame of mind.

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PROGYAN SENSOWA is an integrated MSc student at NISER, Bhubaneswar, Odisha. He completed his secondary education from Sampoorna Kendra Vidyalaya, Dibrugarh, Assam. He is an aspiring mathematician. He is specially interested in the field of calculus and analysis. He is also a classically trained vocalist and loves to play the tabla. He may be contacted at sensowaprogyan@gmail.com or progyan.sensowa@niser.ac.in.

Derivation of Formulae for Solving Some Common Arithmetic Problems

UTPAL MUKHOPADHYAY

At present, multiple-choice questions (MCQ) and very short answer type questions (VSA) are common features of almost every examination. While they can be solved from first principles, this is often time consuming. In this discussion, some general formulae to solve three common types of arithmetic problems are derived. It is hoped that rather than memorizing these formulae, students will learn to develop and use their own short-cuts as part of their preparation for examinations.

A Problem of Simple and Compound Interest

Problem 1. Find the difference between the simple and compound interest, compounded annually, on ₹10000 for two years when the rate of interest is 5%.

In order to derive a formula for the difference between simple and compound interest, compounded annually, we generalise this problem.

Find the difference between the simple and compound interest, compounded annually, on ₹p, for n years when the rate of interest is r%.

Keywords: Competitive examination questions, shortcuts, rationale

Solution: Compound interest on ₹p for n years at r% rate of interest $= ₹p\left[\left(1+\frac{r}{100}\right)^n - 1\right]$

Simple interest on ₹p for n years at r % rate of interest $= ₹\ \frac{npr}{100}$

Therefore, difference between compound and simple interest

$$= ₹p\left[\left(1+\frac{r}{100}\right)^n - 1 - \frac{nr}{100}\right]$$

$$= ₹p\left[1+\frac{nr}{100}+\frac{n(n-1)}{2!}\left(\frac{r}{100}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{r}{100}\right)^3\right.$$

$$\left.+\ldots\ldots\ldots+\left(\frac{r}{100}\right)^n - 1 - \frac{nr}{100}\right] \text{ [using binomial expansion]}$$

$$= ₹p\left[\frac{n(n-1)}{2!}\left(\frac{r}{100}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{r}{100}\right)^3+\ldots+\left(\frac{r}{100}\right)^n\right]$$

This is the general formula for finding the difference between annually compounded compound interest and simple interest for n years when the principal is ₹p and rate of interest is r%. The above formula clearly shows that, as expected, for one year ($n = 1$) there is no difference between compound and simple interest. The difference starts from the second year ($n = 2$) and it gradually increases with increasing n, as more and more terms of the difference formula come into play.

Thus, for two years ($n = 2$) the difference $= ₹p\left(\frac{r}{100}\right)^2$

For three years ($n = 3$) the difference $= ₹p\left[3\left(\frac{r}{100}\right)^2+\left(\frac{r}{100}\right)^3\right]$ and so on.

In Problem 1, $p = 10000$, $r = 5$ and $n = 2$, so the difference between the interests = ₹25.

A Problem Related to Mixtures

Problem 2. A drum contains 20 litres of milk. 5 litres of milk are taken out from it and replaced by an equal amount of water. With the resulting mixture, this process is repeated 3 more times, i.e., 4 times in all. What will be the ratio of milk to water in the final mixture?

Here also we first generalize the above problem.

A pot contains x litres of milk. From that container, 'a' litres of milk are taken out and replaced by an equal amount of water. With the resulting mixture, this process is repeated $(n - 1)$ times, i.e., n times in all, each time removing 'a' litres of the mixture and replenishing it by the same amount of water.

Solution: After the first operation, the amount of milk in the mixture is $(x - a)$ litres and the amount of water is a litres. Now, $x - a = x\left(1-\frac{a}{x}\right)$. We observe the following pattern:

Op. No.	Milk taken out (litres)	Milk left (litres)	Simplified form for milk left
1	a	$x - a$	$x\left(1-\frac{a}{x}\right)$
2	$\frac{a(x-a)}{x}$	$\left[(x-a)-\frac{a(x-a)}{x}\right]$	$x\left(1-\frac{a}{x}\right)^2$
3	$a\left(\frac{x-a}{x}\right)^2$	$\frac{(x-a)^2}{x}-\frac{a(x-a)^2}{x^2}$	$x\left(1-\frac{a}{x}\right)^3$

Using symmetry, let us suppose that after m operations, the amount of milk left over is

$$x\left(1-\frac{a}{x}\right)^{m} \text{ litres, i.e., } \frac{(x-a)^m}{x^{m-1}} \text{ litres.}$$

Then, during the $(m+1)^{\text{th}}$ operation, the amount of milk taken out $= a\frac{(x-a)^m}{x^m}$ litres.

So, the amount of milk left after $(m+1)$ operations $= \left[\frac{(x-a)^m}{x^{m-1}} - a\frac{(x-a)^m}{x^m}\right]$ litres

$= x\left(1-\frac{a}{x}\right)^{m+1}$ litres.

Thus, we see that if the formula for the remaining amount of milk is valid for $n = m$, then it is true for $n = m+1$ also. But it has already been shown that the result is true for n = 1, 2, 3. Then, by the Principle of Mathematical Induction, the result is true for all positive integral values of n.

Therefore, after n operations the amount of milk left in the mixture $= x\left(1-\frac{a}{x}\right)^n$ litres.

So, the amount of water in the final mixture $= x\left[1-\left(1-\frac{a}{x}\right)^n\right]$ litres. Then, after n operations, the ratio of milk to water in the mixture $= \left(1-\frac{a}{x}\right)^n : \left[1-\left(1-\frac{a}{x}\right)^n\right]$

For Problem 2, $x = 20$, $a = 5$, $n = 4$. So, after 4 operations, the amount of milk in the mixture

$= 20\left(1-\frac{5}{20}\right)^4$ litres $= \frac{405}{64}$ litres.

Amount of water after 4 operations $= \left(20-\frac{405}{64}\right) = \frac{875}{64}$ litres.

So, ratio of milk to water in the final mixture = 81 : 175.

A Problem Related to Percentages

Problem 3. If the price of a watch is increased initially by 10% and then by 6%, then what will be the ultimate percentage increase in the price of the watch?

Here again we first generalize the problem for deriving a formula for direct calculation.

If the price of a commodity is increased in two stages by x% and y% respectively, then what will be the final percentage increase in the price of the commodity?

Solution: Suppose the initial price of the commodity was p.

Then, after x% increase, its price $= p + \frac{px}{100} = p\left(1+\frac{x}{100}\right)$

After y% increase its final price $= p\left(1+\frac{x}{100}\right) + \frac{py}{100}\left(1+\frac{x}{100}\right)$

$= p\left(1+\frac{x}{100}\right)\left(1+\frac{y}{100}\right)$

Therefore, final increase in price $= p\left(1+\frac{x}{100}\right)\left(1+\frac{y}{100}\right) - p$

$= \frac{p}{100}\left(x+y+\frac{xy}{100}\right)$

So, percentage increase in the price of the commodity $= \frac{p}{100}\left(x + y + \frac{xy}{100}\right)\frac{100}{p}$

$= \mathrm{x} + \mathrm{y} + \frac{xy}{100}$

Therefore, if the price of a commodity is increased initially by x% and subsequently by y% then the actual percentage increase of the price $= \left(x + y + \frac{xy}{100}\right)$

Now, for Problem 3, $x = 10$ and $y = 6$.

Therefore, percentage increase in the price of the watch $= \left(10 + 6 + \frac{60}{100}\right) = 16\frac{3}{5}$

Corollary 1. If the price is initially increased by x% and then decreased by y%, then in the above formula y will be replaced by $-y$.

So, percentage change in price $= \left(x - y - \frac{xy}{100}\right)$.

Corollary 2. Similarly, if the price is decreased by x% and then increased by y% , then x will be replaced by $-x$.

So, percentage change in the price $= \left(y - x - \frac{xy}{100}\right)$.

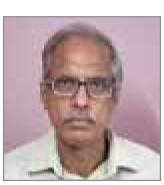
Corollary 3. If the price is decreased in two stages, viz., first by x% and then by y%, then both x and y will have to be replaced by $-x$ and $-y$.

Therefore, percentage change in the price $= \left(\frac{xy}{100} - x - y\right)$.

Note. The above formulae can be used for similar types of problems such as change in the area of a rectangle due to change in its length and breadth, hike and reduction of salary of a person, etc.

Final Remarks

In the present article direct formulae for solving three different types of arithmetic problems have been derived. In Problem 1, the compound interest is considered as compounded annually. What will be the corresponding formulae if the compound interest is compounded half-yearly, quarterly, thrice a year, etc.? Another point is if the formulae derived above are represented graphically, then what will be the nature of those graphs? These questions are left as exercises for interested readers.



UTPAL MUKHOPADHYAY is a retired teacher of Satyabharati Vidyapith, Barasat, West Bengal. He received his Ph. D. from Jadavpur University, Kolkata, by working on accelerating universe and lambda-dark energy. He is actively engaged in science popularisation programme for more than twenty five years through his writings and other outreach activities. He has nearly 250 technical, semi-popular and popular publications in various journals and magazines. He may be contacted at utpalsbv@gmail.com

ClassRoom

Assessing Mathematical Proficiency at the Secondary Stage – I

MATH SPACE

The National Education Policy 2020 emphasizes that *"Mathematical thinking, problem solving, recreational mathematics, connecting classroom mathematics with 'real life mathematics' will be incorporated throughout the school curriculum …. in order to excite children about mathematics and develop the logical skills that are critical throughout school years and indeed throughout life."* The shift towards competency-based teaching and learning in the National Education Policy 2020 will be an important basis for curricular and pedagogical transformation in schools. In keeping with the thrust on competency-based teaching-learning proposed in the National Education Policy 2020, Azim Premji University has supported the Central Board of Secondary Education to develop a 'Learning Framework.' The learning framework is a comprehensive package which provides learning outcomes, assessment frameworks, samples of pedagogical processes, assessment items and marking schemes. Five such frameworks have been developed for English, Hindi, Science, Social Science and Mathematics at the secondary stage which may be accessed at https://cbseacademic.nic.in/cbe/learning-framework.html

This article describes the thinking behind the development of the Mathematics Learning Framework document, recounted with the objective of sharing the learning behind how the mathematics team studied the interface between the Class 9 and 10 NCERT Learning Outcomes, the CBSE mathematics syllabus, the textbooks, the Board examinations,

Keywords: Secondary school, assessment, content domain, cognitive domain, mathematical processes, knowing, applying, reasoning.

and the consequent pedagogical practices of a secondary school mathematics teacher. The Curricular Expectations for the Secondary Stage, along with the Vision for Teaching School Mathematics (from the Position Paper on the Teaching of Mathematics, National Curriculum Framework 2005) were important factors in guiding our thinking, particularly, the mathematical processes described – Problem solving (both informal and formal), use of heuristics, estimation and approximation, optimisation, use of patterns, visualisation and representation, reasoning and proof, making connections, mathematical communication and the skills of Quantification, Abstraction and Modelling culled from the NCTM (National Council of Teachers of Mathematics, USA).

Nature of Mathematics

We began with the elephant in the room, the Nature of Mathematics. Mathematics is an intrinsic part of everyday life, from when we set our alarm for the morning routine to when we cook and eat our dinners. Relationships are made and broken on issues of fair shares or distances. But the relevance of mathematics is more than its utilitarian value. It helps us to think and reason about the world around us and take informed decisions, be it at the individual level to cope with life in various spheres of activity or at the societal level to contribute to technological and socio-economic development.

It is therefore inevitable that mathematics is a compulsory subject at the school level and, in fact, one of the first subjects encountered by the learner entering formal schooling. Sadly, however, the study of mathematics has been reduced to solving problems by remembering procedures and then arriving at 'correct answers.' The focus of school mathematics should be on developing the problem solving and reasoning skills needed to have evolved individuals who contribute to an organised and progressing society. For this, it is important for curriculum and curricular material developers to have a deep understanding of and engagement with, the nature of mathematics.

Mathematical objects and ideas are abstract – created by humans from the needs of science, economics, statistics, and any kind of quantitative analysis needed in daily life. They have no physical properties such as size, colour, sound, smell, etc. For example – A point which does not have breadth or length is represented as a dot by using a sharp pencil.

Mathematical ideas are formed by classifying similarly related and commonly noticed properties. This leads to the pedagogical challenge of making these ideas experiential. For example, Number, which is a root concept, is derived by providing experiences of collections of the same number of objects. The concept of addition is built on the concept of number, and it then becomes the pre-requisite concept for viewing multiplication as repeated addition. This in turn builds on to the understanding of higher concepts. Thus, mathematics builds up from the bottom, i.e., from axioms and definitions in a structured and hierarchical way as a vast network of interlinked concepts.

It is well recognized how rigid mathematics is. This is because mathematical truth, once established and consistent with existing results, lasts forever, and this 'rigid' structure is free from perspectives and subjectivity.

The language of Mathematics is also a part of the abstract nature of this discipline. Mathematics has its special vocabulary and symbols that are very specific to it, and which has been developed over the centuries to represent and communicate mathematical ideas.

We have mathematical propositions that are based on set of conventions, self-evident truths or axioms, definitions, and undefined terms. All the arguments, reasoning and procedures are followed by mathematical justification. This makes mathematics deductive by nature.

Children learn and apply mathematical knowledge by developing abilities through the mathematical processes described above, which, along with mathematical communication are important life skills.

Naturally, it is important to assess the attainment of these abilities. The nature of mathematics makes assessment in mathematics a delicate balance between understanding the difficulties faced by the student in handling abstract concepts (as well as the concise, precise language of mathematics) and the conceptual understanding gained by the student over the course of study. It is important to assess this through the lens of these mathematical abilities rather than simply through their recall of mathematical procedures. Designing assessment which reveals the student's ability to harness the power of mathematics requires that the spectrum of cognitive skills ranging from Knowing to Applying to Reasoning is tested. This requires a clear understanding of the learning outcomes expected from the student at a particular stage of education as well as the opportunities provided by the content of the subject in order to test the attainment of these.

Cognitive domains in Mathematics

As the Math Space team at Azim Premji University studied the design of assessment in mathematics at the secondary level, it became very clear that if assessment was to be mapped to the cognitive levels described in the Revised Bloom's taxonomy of Anderson and Krawthol, then it was important to granularise these and develop a common understanding of the sense in which these granularised levels were used in designing assessment tasks in mathematics. Given below is our first attempt at mapping the questions from a Standard 10 Board examination paper, as we tried to granularise the well-known cognitive domains of Remember, Understand, Apply, Analyse, Evaluate and Create.

Remember		
Remember a specific problem from the textbook	Remember simple facts and formulas For example: Characteristics of a fraction which gives a terminating decimal	Remember a procedure For example: Putting a quadratic in standard form and finding the discriminant
Understand		
Usage of principle, method or a theorem For example: Given a trig ratio, interpret in terms of ratio of sides and find the unknown side and a new ratio using Pythagoras' theorem	Interpret a real-life experiment in mathematical terms. For example: Throwing a dice or picking a card and interpreting the result as favourable or unfavourable and calculate the probability.	To paraphrase and interpret. For example: Paraphrase given data to use relation between roots and coefficients of a quadratic equation
Apply		
Model a given situation and solve a problem. For example: Forms a pair of linear equations using given real-life information	Use given content to interpret a situation and solve a problem. For example: Finding a relation between the coordinates of three collinear points	Applying knowledge to new situation to solve a problem. For example: Taps filling a tank, boats going upstream/downstream
Analyse, Evaluate, Create		
Connect and integrate different pieces of information to solve a problem- analysis. For example: Given data about a real life situation, mathematize it, recognise the value to be calculated and then find it	Connect and integrate different pieces of information to deduce- synthesis. For example: Prove a geometrical result	Judge and/or justify the value or worth of a decision or outcome, or to predict outcomes based on values For example: Justify a conjecture with a valid proof OR Provide a contradiction to invalidate a conjecture OR Compare two proofs for validity

This exercise proved useful in many ways:

- We saw overlaps and fuzziness between the demarcation of the categories
- We realised the gaps in testing the cognitive domains
- Most of all, we realised that all of these categories could easily collapse into the Remember domain, if teachers exhaustively covered all possible problems and did not allow students to develop their muscle for problem solving.

The attempt to granularise also made us prefer three broader bands of Knowing, Applying and Reasoning for the six cognitive domains. Given below are our definitions of these bands and the granularisation of these cognitive domains.

1. Knowing – One of the key curricular expectations for mathematics is the consolidation and generalisation of the concepts learnt so far. This cognitive domain addresses the student's ability to recall, recognize, classify and state concepts, formulas, axioms, postulates and theorems which form the building blocks of the hierarchical structure of mathematics. Armed with these and with a set of heuristics, the student is able to push the levels of understanding and develop confidence in mathematical communication as well as exploration.

Recall	Recall definitions, terminology, properties of different sets of real numbers, rules and properties of arithmetic operations, units of measurement and their inter-conversion, algebraic identities, properties of geometric shapes, statements of theorems, rules and notations (For example: $\sin^2\theta = (\sin\theta)^2$, $a^{p/q} = q^{\text{th}}$ root of a^p), basic geometric constructions
Recognize	Recognize numbers, quantities, expressions, equations Recognize 2D and 3D shapes and their parts. Recognize different orientations of simple geometric figures Recognize special numbers (squares, prime factors, multiples of 2, 3, 5, etc.)
Classify / Order	Classify numbers, quantities, expressions, equations, and shapes by common properties
Compute	Carry out arithmetic operations (addition, subtraction, multiplication, division, exponentiation on real numbers), ratio-proportion and algebraic (substitution, manipulation, solving linear equations in one variable) procedures
Retrieve	Retrieve information from charts, graphs, tables, texts, or other sources Retrieve given and unknown elements of a problem
Measure	Use measuring instruments Choose appropriate units of measurement

2. Applying – This cognitive domain focuses on using knowledge to determine strategies and represent, model or construct objects. Here, students are required to engage in applying knowledge of facts, relationships, processes, concepts to solve problems in real life contexts and expand the dimensions of their acquired knowledge.

Determine	Determine efficient/appropriate operations, strategies, and tools for solving problems Determine variables and their relationships
Represent	Represent situations and relationships with appropriately labeled diagrams, figures, tables, expressions and/or equations
Model	Display data in tables, graphs, geometric figures, or diagrams that model problem situations; identify the variables involved; and create equation(s) based on their relationship(s)
Solve	Solve problems involving familiar mathematical concepts and procedures.
Construct	Construct geometrical figures based on given specifications

3. Reasoning – In this domain, students are engaged in reasoning to analyse data and other information, draw conclusions, and extend their understanding to new situations. In contrast to the more direct applications of mathematical facts and concepts exemplified in the applying domain, learning outcomes in the reasoning domain involve unfamiliar or more complicated contexts. Mathematical reasoning also encompasses pattern recognition, conjecture and proof.

Analyse	Determine, describe, or use relationships among numbers, expressions, quantities, and shapes. Determine steps for a geometric construction
Differentiate	Identifies objects or situations falling under different categories Distinguishes objects using common characteristics Contrasts objects based on their characteristics
Connect	Relates two representations or models or ideas Relates two quantities (measurements), properties, identities or formulas Connects reasoning to properties of an object or to mathematical statements
Integrate/Synthesize	Link different elements of knowledge, related representations, and procedures to solve problems
Evaluate	Evaluate alternative problem-solving strategies and solutions Evaluate validity of an argument or a solution
Draw Conclusions	Make valid inferences on the basis of information and evidence
Generalize	Make statements that represent relationships in more general and more widely applicable terms
Justify	Provide mathematical arguments to support a strategy or a solution or a geometric construction
Create	Pose problems given an equation, graph or other stimulus Find alternative proofs or solutions to problems

The granularisation did not merely give verbs mapped to the three levels – rather, they explicitly indicated what was expected at that level. For example, the verb 'Prove' is used at all three levels, but questions mapped to the three levels can be easily differentiated as:

Remembering: Recalls proofs

Applying: Proves an unfamiliar statement using one or two known facts

Reasoning: Proves using logical reasoning and understanding of properties, laws and theorems

Now that we had granularised the cognitive levels (we, of course, understood that this was a work-in-progress and that our granular levels would evolve as we worked), we had to complete the loop back to connect these to the content of secondary math. It became very apparent that we would have to define learning indicators for each learning outcome; these would specify the content domain and the cognitive domain and would provide a reference point for teachers as they planned their pedagogy and designed their assessment.

In the Learning Framework, the NCERT Learning Outcomes at the Secondary stage as defined by NCF 2005 were the primary point of reference. The nature of mathematics shaped these, and both constrains and gives scope to the Learning Outcomes. Learning Outcomes provide a benchmark on which learning progress can be tracked both quantitatively and qualitatively. The learning outcomes for each subject are expressed in terms of the cognitive skill to be demonstrated and the content to be acquired by the students. In most of the secondary mathematics Learning Outcomes, the content is explicitly delineated. However, the link to the cognitive domain is far more subtle. This has had a deleterious effect on the teaching and assessment of mathematics at the secondary level. Most of the large amount of content to be taught is addressed in the classroom, resulting in assessment being reduced to recalling not just theorems and formulas but actual problems and constructions too.

Our first task was to break down the overarching Learning Outcomes into Content Linked learning Outcomes (CLOs) and then into learning indicators focused on subject specific skills that students need to attain through different concepts addressed in the CBSE curriculum. A clear understanding of the scope of these learning outcomes for each concept dealt within a textbook chapter will be immensely helpful for both teachers and students to plan their teaching and learning better. Having attempted this task for the Learning Framework, we are eagerly anticipating the National Curriculum Framework 2023 which will be released later this year. Once the revised syllabus and learning outcomes are released, we will apply our understanding to the task of granularising these learning outcomes into learning indicators. The next part of this article will describe these.

Conclusion

Understanding this taxonomy helps in enabling teachers to design instruction activities and assessments aligned to learning outcomes, thereby ensuring the attainment of these learning outcomes. Linking the NCERT Learning Outcomes with the curricular expectations and granularising them into learning indicators provides clear direction to teachers to determine pedagogical processes which develop these learning indicators in their students. Further, it guides teachers to develop assessment which gauges how their students are developing. Competency-based learning encourages students to not only acquire knowledge but also apply this knowledge and the skills they have developed to successfully perform tasks in real-life situations. The students of the 21st century must be equipped to handle new and unfamiliar problems and while teachers cannot predict every such problem, they can certainly help their students to face and handle these with confidence and resilience. This goes far beyond 'completing the syllabus' by doing every problem in the textbook in class. It means ensuring that students develop the prescribed competencies and go beyond the scope of the textbook and the classroom.

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[i]**MATH SPACE** is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics -their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space may be reached at mathspace@apu.edu.in

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To Factor or Not to Factor

PRITHWIJIT DE

Factorisation of algebraic expressions is an integral part of school algebra curriculum. Most of the expressions one deals with involve at most two variables and their degrees are either 2 or 3. The process of finding factors relies heavily on the ability to make astute observations and clever reductions to some standard forms such as $u^2 - v^2$ or $u^3 \pm v^3$. Rarely does one encounter expressions of degree 4 or higher with the exception of $x^4 + x^2y^2 + y^4$ which can be reduced to the $u^2 - v^2$ form by writing it as

$$(x^4 + 2x^2y^2 + y^4) - x^2y^2 = (x^2 + y^2)^2 - (xy)^2.$$

An interesting question which arises in the problem of factorising an expression of degree 4 or more is: "When do we terminate the factorization process?" That is, if we succeed in obtaining a linear factor of the given expression, should we be satisfied and stop at that point? Or should we look for more factors? Here is an example where such a question comes up very naturally.

A degree 4 expression arises in geometry as

$$16\Delta^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$$

where Δ is the area of a triangle with sides of lengths a, b and c. This expression can also be reduced to a product of four linear factors involving a, b and c with integer coefficients, by repeated application of $u^2 - v^2$ to suitably adjusted algebraic expressions.

Keywords: Factorisation, difference of two squares

But do we actually check after obtaining a factorisation of an expression that indeed the expressions thus derived cannot be reduced further? Let me make my point with the help of an example. A standard exercise is to factorise

$$a^3 + b^3 + c^3 - 3abc$$

as

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

Students generally stop here because it matches the answer given at the end of the book. But why should one stop here? How does one know that the expression

$$a^2 + b^2 + c^2 - ab - bc - ca$$

cannot be written as a product of two linear factors involving a, b and c with integer coefficients? The only way to know if it can or can't be done is by answering the following question:

> Do there exist integers p, q, r, u, v and w such that
>
> $$a^2 + b^2 + c^2 - ab - bc - ca = (pa + qb + rc)(ua + vb + wc)$$
>
> is an identity in a, b and c?

By equating the coefficients one readily observes that

$$\begin{aligned} & pu = qv = rw = 1 \\ \text{and} \quad & pv + uq = qw + vr = ru + pw = -1. \end{aligned}$$

Since each of p, q, r, u, v and w must be ± 1, each of $pv + uq$, $qw + vr$ and $ru + pw$ is even, and hence cannot be equal to -1. This contradiction establishes the impossibility of finding integers p, q, r, u, v and w such that

$$a^2 + b^2 + c^2 - ab - bc - ca = (pa + qb + rc)(ua + vb + wc)$$

is an identity in a, b and c, and justifies the decision to proceed no further after obtaining

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

Are the students taught this way in schools? This is a point to ponder.

PRITHWIJIT DE is the National Coordinator of the Mathematical Olympiad Programme of the Government of India. He is an Associate Professor at the Homi Bhabha Centre for Science Education (HBCSE), TIFR, Mumbai. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. He may be contacted at de.prithwijit@gmail.com.

Problem Corner

On the Circumcentres of Isosceles Trapezia

A. RAMACHANDRAN

All isosceles trapezia are cyclic, that is, they all have circumcircles. The circumcentre could be inside, on or outside the figure. If it is on the figure, it must coincide with the midpoint of the longer of the parallel sides (generally considered the 'base'). In any case, the circumcentre must lie on the line of symmetry of the figure.

The question we now ask is: Given the lengths of the sides of an isosceles trapezium, is it possible to determine if its circumcentre lies inside, on or outside the figure (without actually constructing it)?

Let us name the longer of the parallel sides as a, the shorter parallel side as b, and each of the equal sides as c. Let us also define $m = a/2$, $n = b/2$ and h as the height of the figure (distance between the parallel sides). The height is related to the sides of the figure by the following equation: $h^2 = c^2 - (a - b)^2/4$ (see Figure 1, where CE is perpendicular to AB).

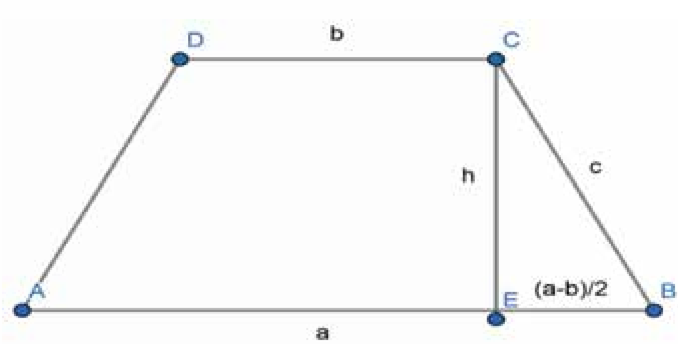


Figure 1

Keywords: Isosceles trapezia, cyclic quadrilaterals, circum-centre, coordinate geometry, pure geometry

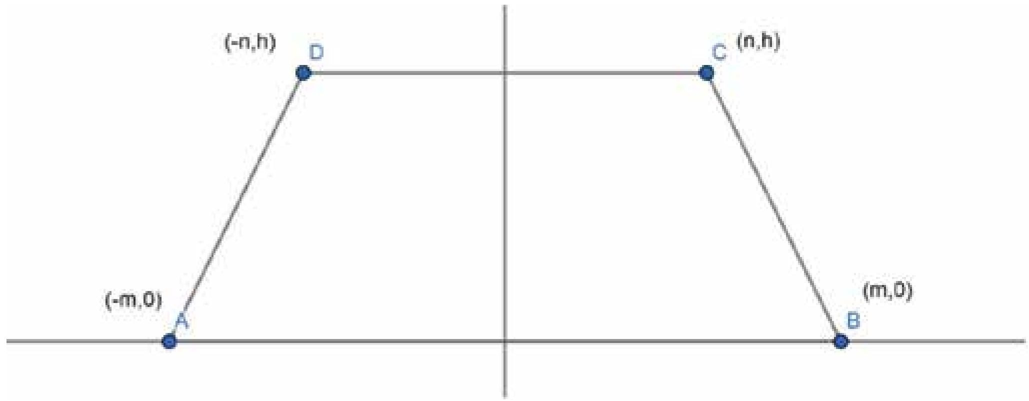


Figure 2

We first take an approach based on coordinate geometry to answer the above question. Let the isosceles trapezium ABCD be placed on the Cartesian plane with the base AB on the X-axis, and with the Y-axis as the line of symmetry (see Figure 2). The coordinates of the vertices are as given in the figure.

The circumcentre of the figure must be at the point of intersection of the perpendicular bisector of BC and the line of symmetry (the Y-axis). The equation of the perpendicular bisector of BC is

$$(x - m)^2 + y^2 = (x - n)^2 + (y - h)^2.$$

As the circumcentre lies on the Y-axis, we can substitute $x = 0$, to find its y-coordinate. Then the above equation simplifies to

$$m^2 + y^2 = n^2 + y^2 + h^2 - 2hy,$$

$$\text{or } y = (n^2 - m^2 + h^2)/2h.$$

If the circumcentre lies on side AB, $y = 0$, i.e.,

$$n^2 - m^2 + h^2 = 0,$$

$$\text{or } h^2 = m^2 - n^2, \text{ or } h^2 = (a^2 - b^2)/4.$$

Substituting for h^2, we have

$c^2 - (a - b)^2/4 = (a^2 - b^2)/4$, which simplifies to $2c^2 = a(a - b)$, the condition for the circumcentre to lie on the base.

If the circumcentre lies inside the isosceles trapezium, $y > 0$, so

$$h^2 > m^2 - n^2, \text{ or } 2c^2 > a(a - b).$$

If the circumcentre lies outside the isosceles trapezium, $y < 0$, so

$$h^2 < m^2 - n^2, \text{ or } 2c^2 < a(a - b).$$

We now obtain the same relations by a pure geometry approach. The length of the diagonal d of an isosceles trapezium can be obtained from the side lengths by the relation $d^2 = ab + c^2$, which follows from Ptolemy's theorem. (This theorem states that in cyclic quadrilateral ABCD, AB.CD + BC.AD = AC.BD.)

Consider ΔABD in isosceles trapezium ABCD (Figure 3). If $\angle$ADB is a right angle, then the circumcentre of ABCD is the midpoint of AB. In that case, $a^2 = c^2 + d^2$, so $a^2 = 2c^2 + ab$, or $2c^2 = a(a - b)$.

If $\angle$ADB is acute, then ABCD occupies a major segment of its circumcircle, with circumcentre inside the figure. In that case,

$$a^2 < c^2 + d^2, \text{ so } a^2 < 2c^2 + ab, \text{ or } 2c^2 > a(a - b).$$

If $\angle$ADB is obtuse, then ABCD is confined to a minor segment of its circumcircle, with circumcentre outside the figure.

In that case, $a^2 > c^2 + d^2$, so $a^2 > 2c^2 + ab$, or $2c^2 < a(a - b)$.

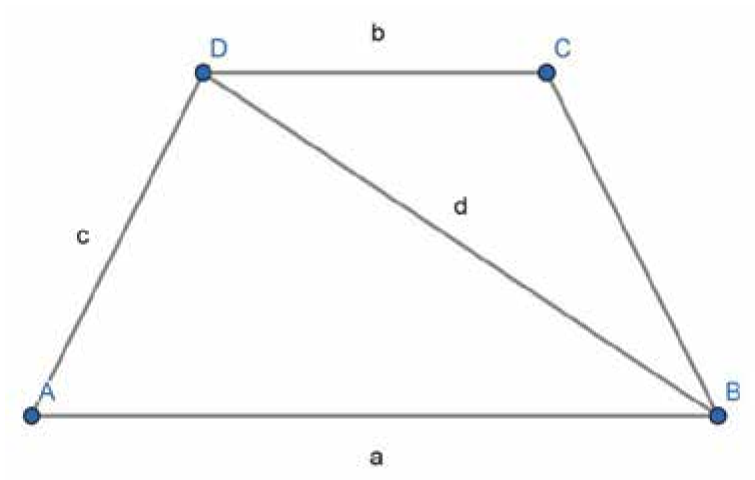
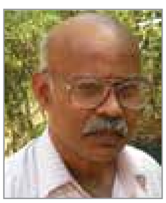


Figure 3



A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

Another Theorem for Congruence of Triangles

KASI RAO JAGATHAPU

We know that any triangle that can be constructed uniquely given three of its six elements gives rise to a congruency relation to other triangles constructed with the same given data.

Based on this core concept, Euclid, in his *Elements-Book 1*, proposed and proved three propositions related to triangle congruency (Proposition-4, Proposition-8 and Proposition-26). These Propositions have appeared as theorems in secondary level Mathematics textbooks as the Side-Angle-Side (SAS), Angle-Side-Angle (ASA) and Side-Side-Side (SSS) congruency theorems respectively. Apart from these, Right Angle-Hypotenuse-Side (RHS) too was introduced in the textbooks.

Are these theorems sufficient to tackle every problem involving the need of establishing congruency between triangles? Let's look at the following problems.

Problem 1

In Figure1, AB and CD are a pair of line segments intersecting each other at K. Segments BC and AD are joined. Prove that if $\angle CKB$ is an obtuse angle, $KB = KD$, and $BC = AD$, then $AB = CD$.

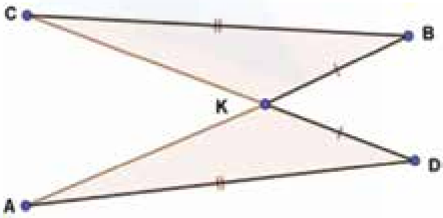


Figure 1

Keywords: Triangles, congruency, sufficient conditions

Here, to prove AB = CD, the need to establish congruency between triangles KAD and KCB is obvious. But none of the theorems we have used so far come to one's aid to prove the required congruency. You can check this for yourself.

Let's examine another problem.

Problem 2

In Figure 2, K is the common centre of two concentric circles. KA is a radius of the inner circle, and AB and AC are a pair of line segments making equal acute angles with AK at A on either side. Prove that AB = AC.

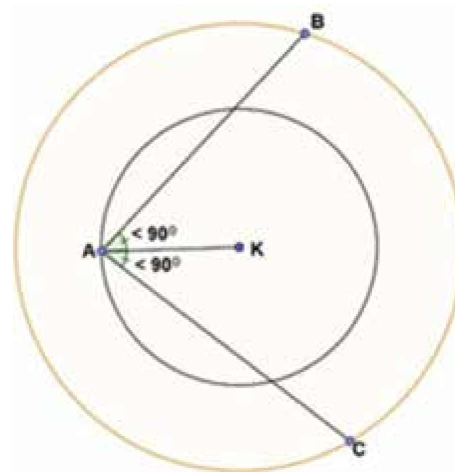


Figure 2

Proof of this result also needs establishment of congruency between the triangles obtained by joining K to B and C. Like Problem 1, this meets the same fate, i.e., no theorem we have in our textbooks will serve the purpose.

Problems such as the above motivate us to introduce and prove additional congruency theorems in secondary level mathematics. To do this, we need to first investigate triangles formed when two sides and a non-included angle are given. The formation of such triangles depends upon the length of the side opposite to the non-included angle relative to the side adjacent to it.

Case-1: When the opposite side is shorter than the adjacent side

Figure 3 shows ΔABC and ΔABC' with BC = BC', with AB and ∠BAC common. It is evident that two different shaped triangles are formed even though there are two pairs of equal sides and a common non-included angle (∠BAC). Here, we note that the side opposite ∠BAC, i.e., BC is *smaller* than the side adjacent to it, i.e., AB. (We consider that adjacent side which is one of the given sides.)

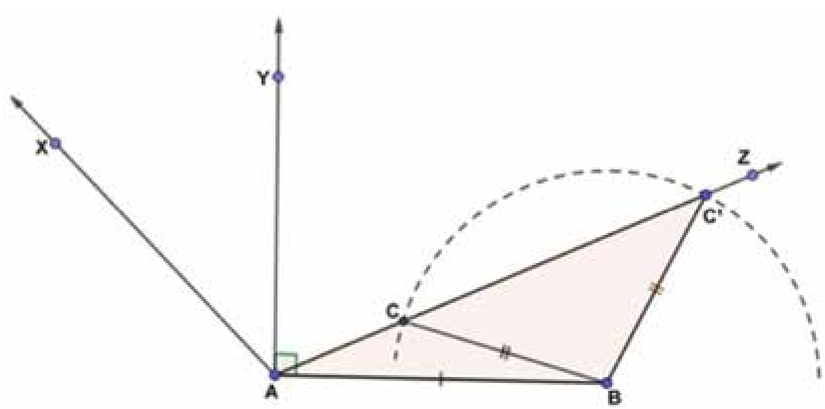


Figure 3

As the triangle so formed is not unique, we conclude that this type of situation for a pair of triangles does not lead to their congruency.

Case-2: When the opposite side is equal to the adjacent side

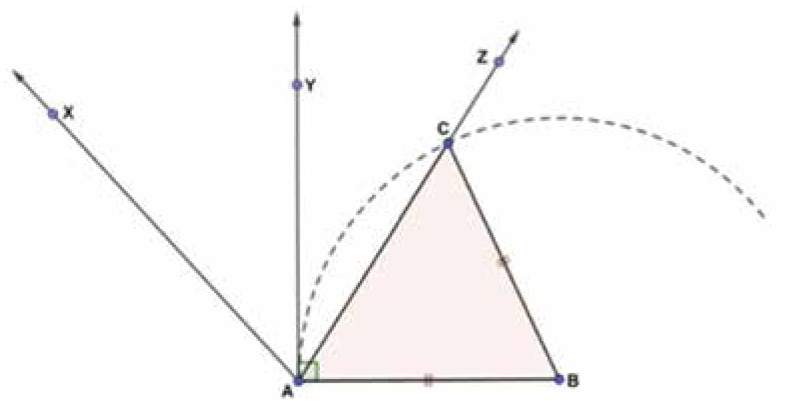


Figure 4

Figure 4 shows that a unique isosceles triangle forms when the side BC opposite to the non-included angle ∠BAC is equal to the side AB adjacent to it. So, this criterion can be considered for establishment of congruency with another triangle of same nature. By SAS congruency criterion, congruency can be established with same measurements. Hence this situation doesn't demand any new proposition.

Case-3: When the opposite side is greater than adjacent side

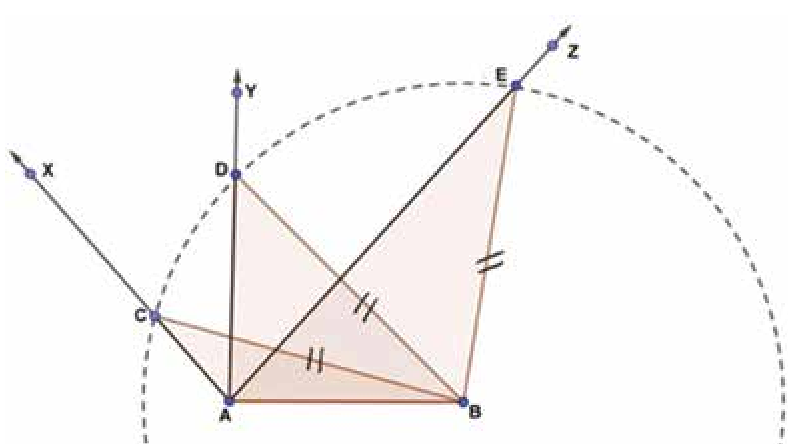


Figure 5

Figure 5 clearly shows that three different triangles – obtuse angled, right angled, and acute-angled can form, and their formation is unique. Hence these kinds of triangles do have the relation of congruency with another triangle of similar nature.

The congruency theorem pertaining to right-angled triangles which is tagged as RHS find place in our textbooks. But the other two results (obtuse-angled and acute-angled cases, with the condition about opposite side to non-included angle) do not appear in our secondary level mathematics curriculum.

The two problems cited at the beginning of this article demand the introduction of theorems of congruency relating to obtuse-angled triangles (can be tagged as 'OSS') and acute-angled triangles (can be tagged as AaSS), just like the RHS congruency theorem.

Obtuse angle - Side - Side Congruency Theorem (OSS)

"If there exists a correspondence between the vertices of two obtuse angled triangles in such a way that two sides and the non-included obtuse angle of one triangle are respectively equal to two corresponding sides and non-included obtuse angle of the other, then the two triangles are congruent."

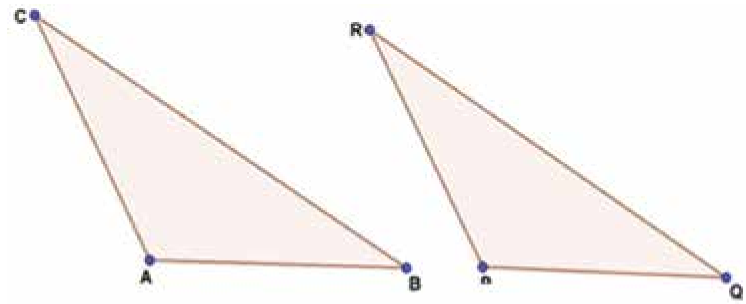


Figure 6

Hypothesis: ΔABC and ΔPQR are two obtuse angled triangles in which

1. Obtuse $\angle BAC$ = Obtuse $\angle QPR$
2. AB = PQ
3. BC = QR

Conclusion: $\Delta ABC \cong \Delta PQR$

PROOF: Assume ΔABC is not congruent to ΔPQR

Because $\angle BAC = \angle QPR$, it is possible to shift the ΔPQR so that P coincides with A, PR in the direction of AC. See Figure 7.

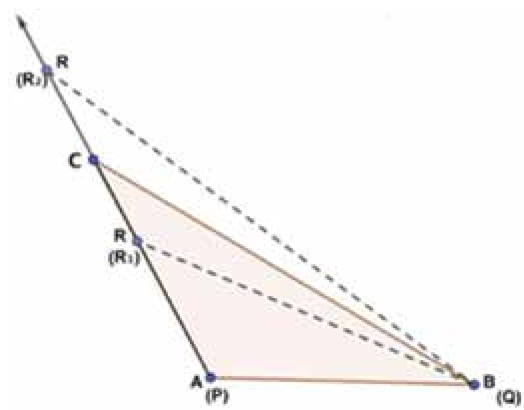


Figure 7

Also, as AB = PQ, Q coincides with B.

As per assumption, triangle ABC is not congruent to triangle PQR, so R does not coincide with C.

Hence vertex R should lie either below C or above C on the line AC. Let the two possibilities be indicated by points R_1 and R_2 respectively.

Consider ΔABR_1, in which $\angle AR_1B$ is an acute angle (since $\angle A$ is obtuse).

In ΔBR_1C, $\angle BR_1C$ is an obtuse angle (linear pair of $\angle AR_1B$), so BC is greater than BR_1.

Now BR_1 is nothing but QR, which implies that BC is greater than QR.

But by hypothesis, BC = QR, which is a contradiction.

So, our assumption that ΔABC is not congruent to ΔPQR is incorrect.

Hence $\Delta ABC \cong \Delta PQR$.

The same logic can be applied to ΔCBR_2 also.

Therefore $\Delta ABC \cong \Delta PQR$.

Note: Right angle - side (Hypotenuse) - Side (RHS) Theorem too can be proved by contradiction using the same logic.

Acute angle - Side - Side (AaSS) Theorem

In Figure 8, ΔABC and ΔPQR are two acute angled triangles in which ∠A = ∠P (both are acute) and AB = PQ, BC = QR. If AC and PR are the longest sides of Δs ABC & PQR respectively, then the two triangles are congruent.

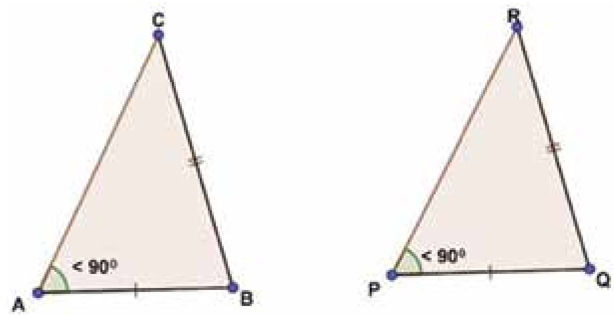


Figure 8

This also can be proved by applying the same logic of proof by contradiction as done in the case of OSS theorem.

The criteria we have covered for congruency in triangles – OSS, RHS and AaSS – may be stated in theorem form as follows:

If two unequal sides of a triangle are equal to two sides of another triangle, and the angles opposite the longer of the two sides are equal, then the two triangles are congruent.

If two equal sides of a triangle are equal to two equal sides of another triangle, and the angles opposite two corresponding equal sides are equal, then the triangles are congruent.

KASI RAO JAGATHAPU is a retired high school headmaster now residing in the town of Kavali in Andhra Pradesh. He taught mathematics for 35 years. He greatly enjoys reading books and finding solutions to problems and puzzles. He received the 'Best Teacher' award in two different districts. He may be contacted at kasiraoj@gmail.com.

Achu's Primes - A Special Class of Weak Primes

SASIKUMAR K

Twin primes. Pairs of prime numbers that differ by 2 are referred to as *twin prime pairs* or simply *twin primes*. Examples of twin prime pairs are $\{3, 5\}$, $\{5, 7\}$, $\{11, 13\}$, $\{17, 19\}$, $\{29, 31\}$, $\{41, 43\}$, $\{59, 61\}$ and $\{71, 73\}$.

It is conjectured that there are infinitely many twin prime pairs. This is known as the *Twin Prime Conjecture*.

The question of whether there exist infinitely many such pairs has been studied for many centuries but it remains open.

It is clear that every prime number beyond 3 must be of one of the forms $6k \pm 1$. This is so because any number exceeding 3 and of any of the forms $6k$, $6k + 2$, $6k + 3$, $6k + 4$ is composite. This implies that every twin prime pair other than $\{3, 5\}$ is of the form $\{6k - 1, 6k + 1\}$ where k is a positive integer.

From this follows a simple claim:

Lemma 1. *If $p \geq 7$, then $p - 2, p, p + 2$ cannot be an arithmetic progression of primes.*

This is true because if $p \geq 7$ and $\{p - 2, p\}$ is a twin prime pair, then $p + 2$ is necessarily a multiple of 3 and hence composite. □

In the definition below, $[z]$ refers to the greatest integer not exceeding z.

Definition 1. A prime number p is said to be an *Achu's prime* if $\left[\dfrac{p^2 + 4}{p + 2}\right]$ is a prime number.

Keywords: Primes, twin primes, weak primes, Achu's primes

For example, 2, 3 and 7 are Achu's primes because

$$\left[\frac{2^2+4}{2+2}\right]=2, \quad \left[\frac{3^2+4}{3+2}\right]=2, \quad \left[\frac{7^2+4}{7+2}\right]=5$$

are all primes. But 5 is not an Achu's prime because

$$\left[\frac{5^2+4}{5+2}\right]=4$$

is not prime. Here are the Achu's primes below 1000:

$$\begin{array}{ccccccccc} 2, & 3, & 7, & 13, & 19, & 31, & 43, & 61, & 73, \\ 103, & 109, & 139, & 151, & 181, & 193, & 199, & 229, & 241, \\ 271, & 283, & 313, & 349, & 421, & 433, & 463, & 523, & 571, \\ 601, & 619, & 643, & 661, & 811, & 823, & 829, & 859, & 883. \end{array}$$

As of now, the largest known Achu's prime is the following number:

$$2996863034895 \times 2^{1290000}+1.$$

Lemma 2. *If $p \geq 7$, then* $\left[\frac{p^2+4}{p+2}\right]=p-2.$

Note that p does not have to be a prime number for this relation to be true.

Proof of Lemma 2. We only need to show the following: if $p \geq 7$ then

$$p-2<\frac{p^2+4}{p+2}<p-1.$$

The inequality on the left side is equivalent to

$$p^2-4<p^2+4,$$

and this is clearly true for all p.

The inequality on the right side is equivalent to

$$p^2+4<p^2+p-2,$$

i.e., to $6<p$, and this is true by supposition, as $p \geq 7$. Hence the statement is true. □

Here is the main result in this paper.

Theorem 1. A prime number $p \geq 7$ is an Achu's prime if and only if $\{p-2, p\}$ is a twin prime pair.

Proof of Theorem 1. Suppose that a prime number $p \geq 7$ is an Achu's prime. Then $\left[\frac{p^2+4}{p+2}\right]$ is a prime number (by definition).

By Lemma 2, $\left[\frac{p^2+4}{p+2}\right]=p-2$. So both $p-2$ and p are prime numbers, i.e., $\{p-2, p\}$ is a twin prime pair.

Next, suppose that $\{p-2, p\}$ is a twin prime pair. Then it means that $\left[\dfrac{p^2+4}{p+2}\right]$ is a prime number. But $\left[\dfrac{p^2+4}{p+2}\right] = p-2$. So p is an Achu's prime. □

Definition 2. A prime number is said to be *weak* if it is less than the arithmetic mean of the two prime numbers immediately below and above it.

That is if

$$p_1 = 2,\ p_2,\ p_3,\ \ldots p_{n-1},\ p_n,\ p_{n+1},\ \ldots$$

is the sequence of prime numbers, then p_k is a weak prime if the following inequality is true:

$$p_k < \frac{p_{k-1}+p_{k+1}}{2}.$$

Here are all the weak primes below 1000:

3,	7,	13,	19,	23,	31,	43,	47,	61,
73,	83,	89,	103,	109,	113,	131,	139,	151,
167,	181,	193,	199,	229,	233,	241,	271,	283,
293,	313,	317,	337,	349,	353,	359,	383,	389,
401,	409,	421,	433,	443,	449,	463,	467,	491,
503,	509,	523,	547,	571,	577,	601,	619,	643,
647,	661,	677,	683,	691,	709,	743,	761,	773,
797,	811,	823,	829,	839,	859,	863,	883,	887,
911,	919,	941,	953,	971,	983,	997.		

Theorem 2. If $p \geq 7$ is an Achu's prime, then p is a weak prime.

Proof of Theorem 2. Let $p \geq 7$ be an Achu's prime, and let q and r be the two prime numbers immediately below it and above p.

Using the results already established, we have $q = p-2$.

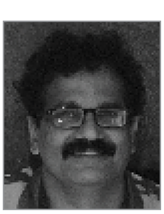
Since $p \geq 7$ and $p-2$ and p are prime numbers, $p+2$ cannot be a prime number. Hence $r > p+2$.

It follows that $q + r > (p-2) + (p+2)$, i.e., $q + r > 2p$, which tells us that p is a weak prime. This completes the proof of the theorem. □

Concluding remarks.

(1) All Achu's primes can be obtained as the maximum of twin prime pairs.

(2) If there are infinitely many twin prime pairs, then there are infinitely many Achu's primes.



SASIKUMAR K is presently working as a PG Teacher in Mathematics at Jawahar Navodaya Vidyalaya, North Goa. He completed his M Phil under the guidance of Dr.K.S.S. Nambooripad. Earlier, he worked as a maths Olympiad trainer for students of Navodaya Vidyalaya Samiti in Hyderabad. He has research interests in Real Analysis and Commutative Algebra. He may be contacted at 112358.ganitham@gmail.com.

Problem Corner

Interesting Infinite, Recurring Decimals

SHAILESH SHIRALI

The case of the infinite decimal 0.1234567…

We wish to find the fraction corresponding to the infinite decimal 0.1234567 . . .. Note that we are *not* simply writing the consecutive numbers one after the other to get the infinite decimal. Rather, there is a 'carry-over' effect to the left as we write the successive numbers: once we get to 10, we must 'carry' the 1 to the left, which means it gets added to the 9; and so on. The natural question to ask is: what kind of number will emerge from this construction? Will it be a rational number? Let us look at the question more closely.

The problem expressed more precisely is to find the fraction corresponding to the following infinite decimal:

$$0.1234567\ldots = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{4}{10^4} + \frac{5}{10^5} + \cdots. \qquad (1)$$

Offhand, it is not at all obvious whether it will result in a recurring decimal. Let's see where the algebra will lead us. Let a denote the above number. Then we have:

$$\begin{aligned} a &= \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{4}{10^4} + \frac{5}{10^5} + \cdots, \\ \therefore\ \frac{a}{10} &= \qquad\ \frac{1}{10^2} + \frac{2}{10^3} + \frac{3}{10^4} + \frac{4}{10^5} + \cdots, \\ \therefore\ \text{(by subtraction)}\ \frac{9a}{10} &= \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \frac{1}{10^5} + \cdots, \\ \therefore\ \frac{9a}{10} &= \frac{1}{9} \qquad \left(\text{since } 0.1111\ldots = \frac{1}{9}\right). \end{aligned} \qquad (2)$$

Keywords: Recurrence, geometric progression, patterns, generalisation

Hence $a = \frac{10}{81}$, i.e.,

$$0.1234567\ldots = \frac{10}{81}. \tag{3}$$

We see that the number in question is indeed a rational number, and it is easy now to find the recurring portion corresponding to it. Here is what we get after a straightforward division:

$$\frac{10}{81} = 0.123456790\ 123456790\ \ldots = 0.\overline{123456790}. \tag{4}$$

Curiously, every digit occurs in the decimal expansion — except for 8!

The case of the infinite decimal 0.01020304050607…

We wish to find the fraction corresponding to the infinite decimal 0.01020304050607 . . .. As earlier, it is assumed that there is a 'carry-over' effect to the left as we write the successive numbers.

The problem expressed more precisely is to find the fraction corresponding to the following infinite decimal:

$$0.01020304050607\ldots = \frac{1}{10^2} + \frac{2}{10^4} + \frac{3}{10^6} + \frac{4}{10^8} + \cdots. \tag{5}$$

As earlier, let's see where the algebra will lead us. Let b denote the above number. Then we have:

$$\begin{aligned} b &= \frac{1}{10^2} + \frac{2}{10^4} + \frac{3}{10^6} + \frac{4}{10^8} + \frac{5}{10^{10}} + \cdots, \\ \therefore\ \frac{b}{10^2} &= \qquad\ \ \frac{1}{10^4} + \frac{2}{10^6} + \frac{3}{10^8} + \frac{4}{10^{10}} + \cdots, \\ \therefore\ \text{(by subtraction)}\ \frac{99b}{10^2} &= \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \frac{1}{10^8} + \frac{1}{10^{10}} + \cdots, \\ \therefore\ \frac{99b}{10^2} &= \frac{1}{99} \qquad \left(\text{since } 0.01010101\ldots = \frac{1}{99}\right). \end{aligned} \tag{6}$$

Hence

$$0.01020304050607\ldots = \frac{100}{99^2} = \frac{100}{9801}. \tag{7}$$

Generalization

In the same way, we obtain:

$$0.001002003004005006007\ldots = \frac{1000}{999^2} = \frac{1000}{998001}, \tag{8}$$

$$0.0001000200030004000500060007\ldots = \frac{10000}{9999^2} = \frac{10000}{99980001}, \tag{9}$$

and so on. The pattern should be clear.

Questions for the reader

What happens when we use the sequence of powers of 2 or 3 (or any other number) to produce an infinite decimal? Do we get rational numbers (and therefore recurring decimals)?

$$0.001\ 002\ 004\ 008\ 016\ 032\ 064\ \ldots = ??$$
$$0.0001\ 0003\ 0009\ 0027\ 0081\ 0243\ 0729\ \ldots = ??$$

Note that the 'carry-over' is assumed to happen in these numbers.

What happens when we use the sequence of Fibonacci numbers to produce an infinite decimal? Do we get rational numbers (and therefore recurring decimals)?

$$0.001\ 001\ 002\ 003\ 005\ 008\ 013\ 021\ \ldots = ??$$
$$0.0001\ 0001\ 0002\ 0003\ 0005\ 0008\ 0013\ 0021\ \ldots = ??$$

We leave these questions for the reader to pursue. As can be seen, many such questions can be posed and explored, and they lead to very pretty results.

SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

FINDING THE MAGICAL DESIGN

Do these dots help to understand the pattern of the carpet shown on page 34

Problem Corner

Geometric Proofs of Two Trigonometric Identities

SHAILESH SHIRALI

There are numerous trigonometric identities involving particular angles that have an extremely appealing form. Here is one such:

$$\cos 36^\circ \cdot \cos 72^\circ = \frac{1}{4}. \qquad (1)$$

Before proceeding, let us convince ourselves using trigonometry that this relation is true. To do this we shall derive expressions for $\cos 36^\circ$ and $\cos 72^\circ$. Let $t = 36^\circ$. Then $3t$ and $2t$ are supplementary angles, so they satisfy the relation

$$\cos 3t = -\cos 2t. \qquad (2)$$

Using the well-known triple angle and double angle identities for the cosine function, we get

$$4\cos^3 t - 3\cos t + 2\cos^2 t - 1 = 0 \qquad (\text{for } t = 36^\circ). \qquad (3)$$

Let $x = \cos t$; then the above equality transforms into a cubic equation in x:

$$4x^3 + 2x^2 - 3x - 1 = 0. \qquad (4)$$

We need to solve this equation. Cubic equations do not form part of the standard school syllabus, so one may wonder how to proceed. Fortunately here, the factor theorem shows the way. The substitution $x = -1$ yields 0 on the left side, which means that $x = -1$ is a root of the equation. This in turn means that $x + 1$ is a factor of the cubic polynomial $4x^3 + 2x^2 - 3x - 1$. By division we easily obtain the other factor which is a quadratic:

$$4x^3 + 2x^2 - 3x - 1 = (x+1) \cdot \left(4x^2 - 2x - 1\right). \qquad (5)$$

Keywords: Trigonometric identity, geometric proof, golden triangle, golden ratio, double angle identity, triple angle identity

The quadratic equation $4x^2 - 2x - 1 = 0$ yields the following two roots:

$$\frac{2 \pm \sqrt{4+16}}{8}, \quad \text{i.e.,} \quad \frac{\sqrt{5}+1}{4} \text{ and } \frac{1-\sqrt{5}}{4}. \tag{6}$$

Hence the roots of the equation $4x^3 + 2x^2 - 3x - 1 = 0$ are -1, $(\sqrt{5}+1)/4$ and $(1-\sqrt{5})/4$. Which of these is the value of $\cos 36^\circ$? Since $0 < \cos 36^\circ < 1$, it must be that

$$\cos 36^\circ = \frac{\sqrt{5}+1}{4}. \tag{7}$$

From this we easily derive an expression for $\cos 72^\circ$, using the double angle identity for cosine, $\cos 2\theta = 2\cos^2\theta - 1$. We obtain:

$$\cos 72^\circ = \frac{\sqrt{5}-1}{4}. \tag{8}$$

Examining (7) and (8), we immediately notice that

$$\cos 36^\circ \cdot \cos 72^\circ = \frac{1}{4}. \tag{9}$$

Observing such a nice-looking result naturally provokes us to ask: Is there a neat geometric argument that will show us why (9) is true, without the use of any trigonometry? Indeed there is, and you can find a very readable version of it at [1]. The proof can be presented as follows.

Roman Andronov's proof of (9) — a proof (almost) without words.

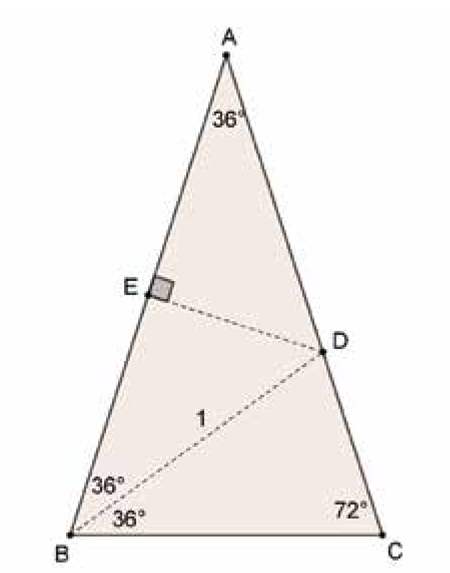


Figure 1

- $\triangle ABC$ has angles $36^\circ, 72^\circ, 72^\circ$.
- BD bisects $\measuredangle ABC$, so $\measuredangle ABD = \measuredangle DAB = 36^\circ$, which means that $\triangle DAB$ is isosceles.
- Draw $DE \perp AB$; so DE is also the perpendicular bisector of AB.
- Let $BD = 1$. Then $BC = 1$ too, since $\triangle BCD$ has angles 36°, 72° and 72°.
- From $\triangle DEB$ we get $BE = 1 \cdot \cos 36^\circ$, and so $AB = 2 \cdot \cos 36^\circ$.
- Hence $BC = 2 \cdot AB \cdot \cos 72^\circ$, i.e., $BC = 4 \cdot \cos 36^\circ \cdot \cos 72^\circ$.
- But we also have $BC = 1$.
- Hence $4 \cdot \cos 36^\circ \cdot \cos 72^\circ = 1$, and (9) follows. □

Figure 1 displays a proof of this equality. Given alongside are the steps of the proof. Note that the proof does not require the expressions for $\cos 36^\circ$ and $\cos 72^\circ$.

Another striking relation. If we look again at (7) and (8), we notice a second relation connecting the two quantities which is just as striking as (9):

$$\cos 36^\circ - \cos 72^\circ = \frac{1}{2}. \tag{10}$$

Let us now set ourselves the challenge of proving (10) geometrically (i.e., without explicitly using the expressions for $\cos 36^\circ$ and $\cos 72^\circ$).

The task turns out to be somewhat more difficult than the earlier one. The best that I have been able to come up with is described below. Perhaps some reader can find a shorter approach.

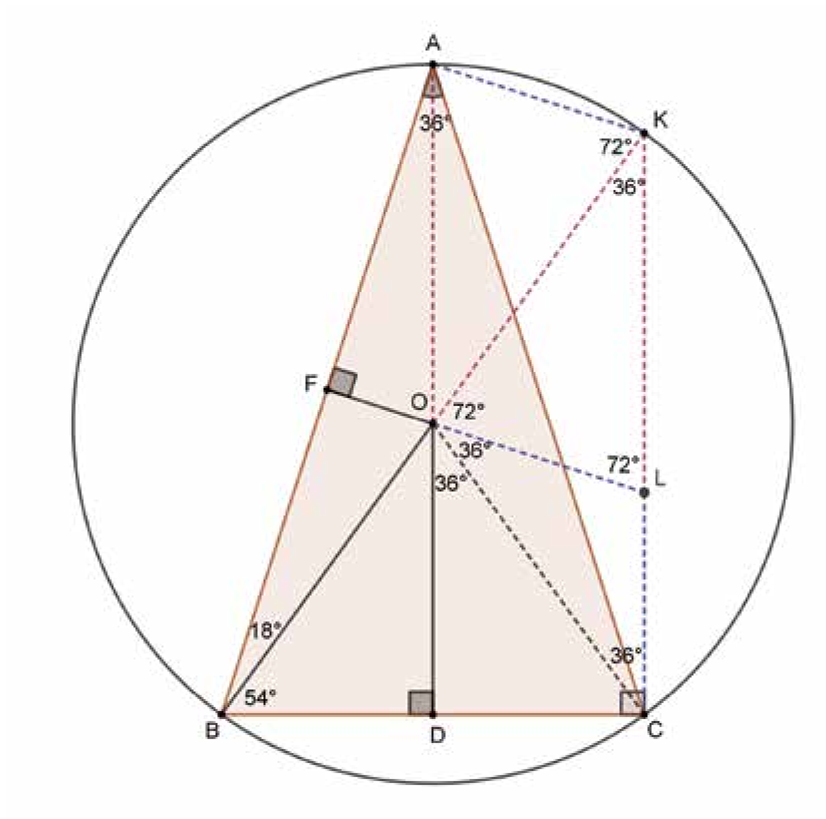


Figure 2

Explanation of the diagram (Figure 2) and steps of the proof.

- $\triangle ABC$ has angles $36^\circ, 72^\circ, 72^\circ$. O is the centre of the circumcircle of $\triangle ABC$. From O drop perpendiculars OD, OF to BC, AB respectively. Then D, F are the midpoints of BC, AB respectively. Draw diameter BK; join AK and KC. Extend FO to meet KC at L. Join OA. Let the radius of the circumcircle be taken as 1 unit.
- From $\measuredangle BAC = 36^\circ$ we get $\measuredangle BOC = 72^\circ$, so $\measuredangle COD = \measuredangle BOD = 36^\circ$. Hence $\measuredangle OBD = 54^\circ$, i.e., $\measuredangle KBC = 54^\circ$, so $\measuredangle BKC = 36^\circ$. Since $BK = 2$, it follows that $KC = 2 \cdot \cos 36^\circ$.
- Also, $\measuredangle OBF = 18^\circ$, so $\measuredangle AKB = 72^\circ$. Hence from $\triangle KAB$, we get $AK = 2 \cdot \cos 72^\circ$.
- Next, observe that $KAOL$ is a parallelogram. (For, $KA \parallel LO$, both being perpendicular to AB; and $AO \parallel KL$, both being perpendicular to BC.) Hence $OL = AK$. So $OL = 2 \cdot \cos 72^\circ$.

- Since $OFBD$ is cyclic, we get $\measuredangle LOD = 72^\circ$, hence $\measuredangle LOC = 36^\circ$. Also, $\measuredangle OCD = 54^\circ$, hence $\measuredangle LCO = 36^\circ$. It follows that $LC = LO$, so $LC = 2 \cdot \cos 72^\circ$.
- Angle-chasing yields $\measuredangle KOL = 72^\circ = \measuredangle KLO$, so $KL = KO$, i.e., $KL = 1$. Hence we have $KC = 1 + 2 \cdot \cos 72^\circ$.
- Earlier we had shown that $KC = 2 \cdot \cos 36^\circ$. It follows that $2 \cdot \cos 36^\circ = 1 + 2 \cdot \cos 72^\circ$.
- Hence $\cos 36^\circ - \cos 72^\circ = 1/2$, as required. □

Remark. A triangle with angles of $36^\circ, 72^\circ, 72^\circ$ is known in the literature as a **golden triangle**. To see why, let $\triangle ABC$ have angles 36°, 72°, 72°. Then we have

$$\frac{AB}{BC} = \frac{1}{2 \cdot \cos 72^\circ} = \frac{2}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{2}. \tag{11}$$

We see that AB/BC is equal to the golden ratio Φ. This explains why it is called the golden triangle.

Postscript. A simpler proof of the second identity

After writing the above proof I came across a vastly simpler proof of (10); it is a true proof-without-words! Here it is.

Proof by Lai Johnny of (10)**.** The proof is from [2]. Curiously, the diagram used is the same as the one used in the proof of (9). We have drawn it afresh here (Figure 3).

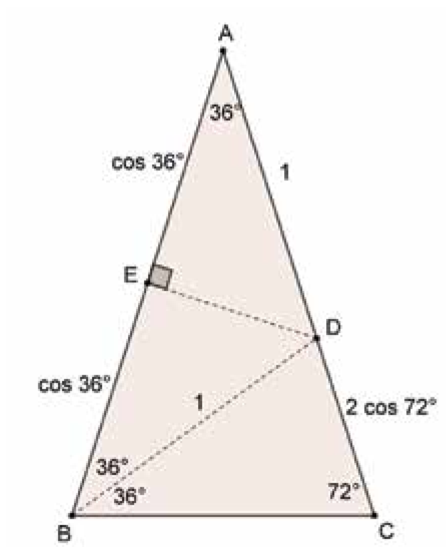


Figure 3. A proof-without-words of the identity $\cos 36^\circ - \cos 72^\circ = 1/2$

From the diagram, it immediately follows that $2 \cdot \cos 36^\circ = 1 + 2 \cdot \cos 72^\circ$, and therefore that

$$\cos 36^\circ - \cos 72^\circ = \frac{1}{2}.$$

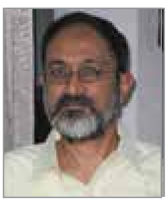
A beautiful proof! □

Appendix : the double-angle and triple-angle identities for the cosine function. These are the following identities valid for all θ; they are needed to prove (3):

$$\cos 2\theta = 2\cos^2\theta - 1,$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

References

1. Roman Andronov, "How do I prove that $\cos 36^\circ \cdot \cos 72^\circ = 1/4$ by using the golden triangle $(36^\circ, 72^\circ, 72^\circ)$?" from https://qr.ae/pv0Dgk
2. Lai Johnny, "What is a simple form of $4\sin 36^\circ \cos 72^\circ \sin 108^\circ$?" from https://qr.ae/prR8a0



SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

A Procedure to (Approximately) Trisect an Arbitrary Angle

MAHESH BUBNA

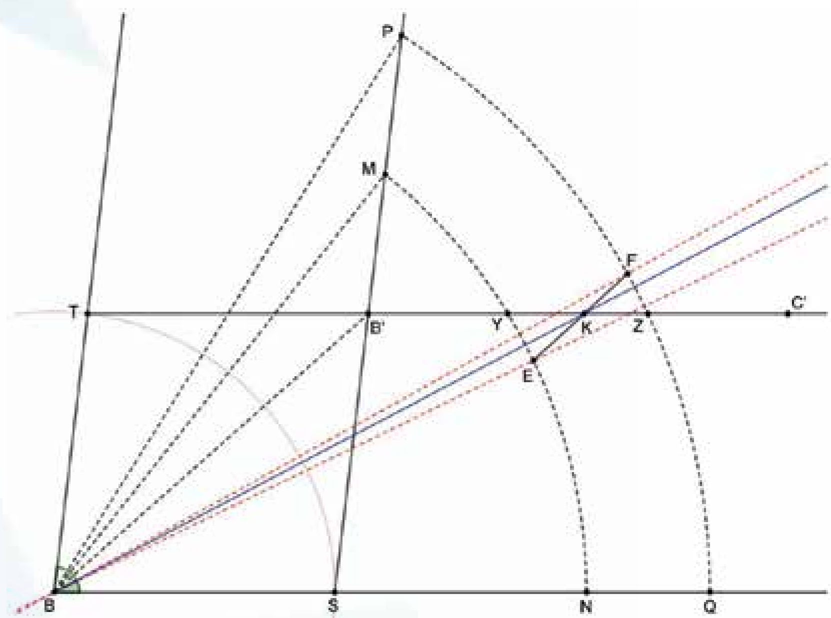


Figure 1

1. Given: an angle to be trisected. Let its vertex be B. Draw an arc with centre B cutting the arms of the angle at T and S. Locate B' such that $TBSB'$ is a parallelogram; see Figure 1. ($TBSB'$ will be a rhombus.)
2. Extend the segments TB', SB' and BS. (That is, draw the rays TB', SB' and BS.)
3. On the extension of TB', locate point C' such that $B'C' = BB'$. Then locate Y and Z, the points of trisection of $B'C'$ (so $B'Y = YZ = ZC' = B'C'/3$).

Keywords: Angle trisection, approximate

(4) With centre B, draw an arc through Y; let it meet ray SB' at M and ray BS at N.

(5) With centre B, draw an arc through Z; let it meet ray SB' at P and ray BS at Q.

(6) Draw the angle bisector of $\measuredangle NBM$. Let it meet arc MN at E.

(7) Draw the angle bisector of $\measuredangle QBP$. Let it meet arc PQ at F.

(8) Locate K, the point of intersection of EF and $B'C'$.

(9) Then BK is the proposed trisector of $\measuredangle TBS$. We claim that $\measuredangle KBS$ is almost exactly equal to ⅓ of $\measuredangle TBS$.

Note by the editor

We give a complete analysis of the procedure in an accompanying article, elsewhere in this issue.

An animation of the procedure in movie form has been uploaded to the following location: https://www.dropbox.com/s/c6ux9r787wdpotf/Trisect%20an%20angle.mp4?dl=0 Do have a look at it!

MAHESH BUBNA is a retired bank officer (retired after a lifetime of service in the Bank of Baroda), now settled in Mumbai. He has been keenly interested in mathematics, particularly geometry, from his early school days. He writes that he has been entangled with the problem of angle trisection for the last four decades. He may be contacted at mbubna24@gmail.com.

EASY MATH

If you return 5 Fancy Fruit bar wrappers, you get one Fancy Fruit bar. If a person has 125 Fancy Fruit bar wrappers, what is the maximum number of Fancy Fruit bars the person can get to eat?

A person buys a new four-wheeler and wishes to start on a 6000 km drive. The salesperson tells him that each tyre would last for only 4000 km. The buyer says, 'I shall then carry two spare tyres, and complete the journey.' How can he manage that?

One morning a person notices that the clock on his wall has stopped working. He is able to start it by changing the battery but cannot set the time correctly as there is no other device to tell the time. He walks to a friend's house, asks for the time, and immediately walks back to his house. On reaching his house he is able to set the time correctly on his wall clock. How does he manage it?

Send in your answers to AtRiA.editor@apu.edu.in

On Trisecting an Angle

SHAILESH SHIRALI

Most students come to know in high school that "it is not possible to trisect an angle" using the methods permitted in geometrical constructions; it has become part of mathematical folklore. On hearing that the problem has not been solved by anyone, one immediately feels tempted to tackle it oneself. Who knows, maybe I will hit upon a method that no one has thought of earlier! Whatever the thinking involved, a phenomenon that continues to this day is that of students coming up with various kinds of procedures that they think (or hope!) will work. These procedures are sometimes so complicated that any kind of analysis becomes daunting. (I receive lots of these!)

The geometrical facts of the situation may be stated as follows.

(a) Using only a compass and an unmarked straight edge, it is not possible to exactly trisect an arbitrary angle. (Note that we use the words "unmarked straight edge" rather than "ruler" as the ruler has markings on it.)

(b) It may be possible to trisect some particular angles, by making use of properties that are special to those angles. (For example, one may trisect an angle measuring 90°, using only a compass and an unmarked straight edge.) But such methods do not work in general.

Keywords: Angle trisection, Archimedean spiral, paper-folding, Maclaurin series, computer algebra system (CAS)

(c) If we are permitted to use a marked straight edge (i.e., a ruler), angle trisection *is* possible.

(d) It is also possible if we are permitted to use a curve known as the *Archimedean spiral*.

(e) Angle trisection is also possible using paper-folding.

(f) In the above three cases ((c), (d) and (e)), the methods do not qualify as Euclidean.

(g) Since exact trisection of an arbitrary angle is not possible (using only a compass and an unmarked straight edge), we naturally look for approximate methods, by means of which we can obtain an angle that is very close to $\frac{1}{3}$ of any given angle. Numerous methods of this kind are available, which work with varying degrees of accuracy. Some of these will be described later in this article.

The obvious approach, and why it does not work

Consider the simplest possible approach to trisecting an angle. Let $\measuredangle AOB$ be the given angle. By drawing an arc of a circle with centre O to intersect the arms of the angle, we may assume that $OA = OB$ (see Figure 1). We now trisect segment AB. Let the points of trisection be C and D (with C closer to A, and D closer to B). Now suppose that someone claims that $\measuredangle AOB$ has been trisected, i.e., that $\measuredangle AOC = \measuredangle COD = \measuredangle DOB = \frac{1}{3}\measuredangle AOB$. How would we check whether this is so or not?

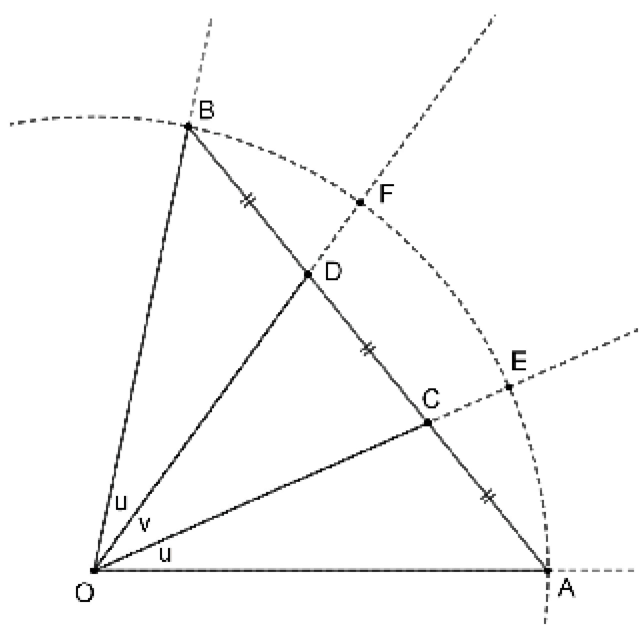


Figure 1

Intuitively, it seems "obvious" that this claim cannot be true. Indeed, it seems obvious that arc EF will be greater in length than arc AE and therefore that $\measuredangle COD > \measuredangle AOC$. But describing something as intuitively obvious does not make it true! We clearly need an argument that is more convincing. (The phrase "this is obvious" has often proved to be quite treacherous; there are numerous instances in the history of mathematics that illustrate this phenomenon.)

To proceed, we first show that $OA > OC$. By design, $OA = OB$, so $\measuredangle OAB = \measuredangle OBA$. Since $\measuredangle ACO$ is an exterior angle to $\triangle OBC$, it follows that $\measuredangle OCA = \measuredangle OBC + \measuredangle BOC$. Therefore $\measuredangle OCA > \measuredangle OBC$. But $\measuredangle OBC = \measuredangle OAC$. Therefore $\measuredangle OCA > \measuredangle OAC$, and so $OA > OC$. (Here we use the known result that in a triangle with two unequal angles, the side opposite the greater angle is longer than the side opposite the smaller angle.)

Using this result, we shall show that $\measuredangle AOC < \measuredangle COD$. For this, we make use of the sine rule (from trigonometry). Let $\measuredangle AOC = u$, and $\measuredangle COD = v$. (Note that $\measuredangle DOB = u$ too.)

Since $\measuredangle ACO$ and $\measuredangle OCD$ are supplementary, $\sin\measuredangle ACO = \sin\measuredangle OCD$. Hence:

$$\frac{\sin u}{\sin v} = \frac{\sin u/\sin\measuredangle ACO}{\sin v/\sin\measuredangle OCD} = \frac{AC/OA}{CD/OD} = \frac{OD}{OA} = \frac{OC}{OA} < 1. \tag{1}$$

Hence $u < v$. Since $2u + v = \measuredangle AOB$, this implies that $u < \frac{1}{3}\measuredangle AOB$ and $v > \frac{1}{3}\measuredangle AOB$.

Unfortunately, from this analysis, we cannot gauge the percentage error in taking u to be $\frac{1}{3}$ of $\measuredangle AOB$. A finer analysis is required for that. We proceed to show how this can be done.

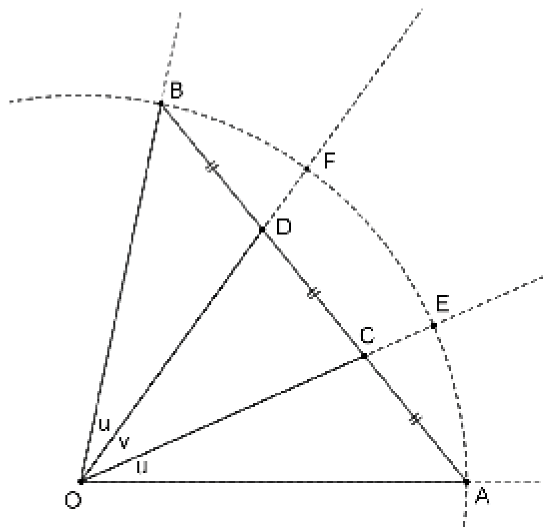


Figure 2

We have redrawn the figure for ease of reading (Figure 2). Let $\measuredangle AOB = t$. Our task now is to express u in terms of t. We now have, from $\triangle AOC$ and $\triangle AOB$,

$$\begin{aligned}\frac{\sin u}{\sin t} &= \left(\frac{(\sin u)/AC}{(\sin t)/AB}\right)\cdot\frac{AC}{AB}\\ &= \left(\frac{(\sin\measuredangle OAC)/OC}{(\sin\measuredangle OAB)/OB}\right)\cdot\frac{1}{3} = \frac{1}{3}\cdot\frac{OB}{OC}.\end{aligned} \tag{2}$$

We now assign coordinates. Take O to be the origin, $O = (0,0)$, and A to be the 'unit point' on the x-axis, $A = (1,0)$. Then $B = (\cos t, \sin t)$. Then we have

$$C = \left(\frac{2+\cos t}{3}, \frac{\sin t}{3}\right). \tag{3}$$

Using this, we determine the length of OC by using the distance formula. We get:

$$OC^2 = \frac{5+4\cos t}{9}, \quad \text{after simplification.} \tag{4}$$

Hence:

$$\sin u = \frac{\sin t}{\sqrt{5+4\cos t}}. \tag{5}$$

We have thus expressed u in terms of t. The formula allows us to compute u for any given t. We may thus generate the values shown in Table 1.

t	9 °	18 °	27 °	36 °	45 °	54 °	63 °	72 °	81 °	90 °
u	2.997 °	5.98 °	8.92 °	11.82 °	14.64 °	17.36 °	19.95 °	22.39 °	24.61 °	26.56 °

Table 1

As can be seen, u is fairly close to $\frac{1}{3}t$ for small values of t. But the error gets steadily larger as t increases, and for $t = 90°$, the error is close to 11.5% (which is unacceptably large).

Using (5) it is easy to compute the Taylor-Maclaurin series for u in terms of t. We get:

$$u = \frac{t}{3} - \frac{t^3}{81} - \frac{t^5}{972} - \frac{7t^7}{87480} + \cdots. \tag{6}$$

We see directly from (6) that u is always less than $\frac{1}{3}$ of t; that the error is small when t is small; but that the error steadily increases as t increases.

Most such procedures can be analysed in a similar way, though the analysis can get complicated and quite challenging if the number of steps is large. Such is surely the case with the procedure devised by Shri Mahesh Bubna, described elsewhere in this issue. On the other hand, the accuracy level of this procedure is truly astonishing.

Trigonometric analysis of Shri Mahesh Bubna's method

We do not repeat the steps here but plunge straight away into the trigonometric analysis. Let $\measuredangle TBS = t$ and let $f(t) = \measuredangle KBS$. We need to express $f(t)$ in terms of t. Here goes …:

(1) $B = (0, 0),\ S = (1, 0),\ \measuredangle TBS = t,\ BS = BT = 1,\ T = (\cos t, \sin t),\ B' = (1 + \cos t, \sin t)$

(2) $BB' = 2\cos\frac{1}{2}t = B'C',\ C' = (1 + 2\cos\frac{1}{2}t + \cos t, \sin t)$

(3) $Y = \left(1 + \frac{2}{3}\cos\frac{1}{2}t + \cos t, \sin t\right),\ Z = \left(1 + \frac{4}{3}\cos\frac{1}{2}t + \cos t, \sin t\right)$

(4) From the above we get:

$$BY^2 = 2 + \frac{4}{9}\cos^2\frac{1}{2}t + \frac{4}{3}\cos\frac{1}{2}t\cos t + \frac{4}{3}\cos\frac{1}{2}t + 2\cos t,$$
$$BZ^2 = 2 + \frac{16}{9}\cos^2\frac{1}{2}t + \frac{8}{3}\cos\frac{1}{2}t\cos t + \frac{8}{3}\cos\frac{1}{2}t + 2\cos t.$$

(5) Let $M = (1 + k_1 \cdot \cos t, k_1 \cdot \sin t)$ where $k_1 = \dfrac{SM}{SB'}$. From $BM^2 = BY^2$ we get

$$1 + k_1^{\,2} + 2k_1 \cdot \cos t = BY^2, \qquad \therefore\ k_1 = -\cos t + \sqrt{\cos^2 t + BY^2 - 1}.$$

(6) $\measuredangle MBN = \arctan\left(\dfrac{k_1 \cdot \sin t}{1 + k_1 \cdot \cos t}\right),\ \measuredangle EBN = \dfrac{1}{2}\arctan\left(\dfrac{k_1 \cdot \sin t}{1 + k_1 \cdot \cos t}\right)$, with k_1 as above.

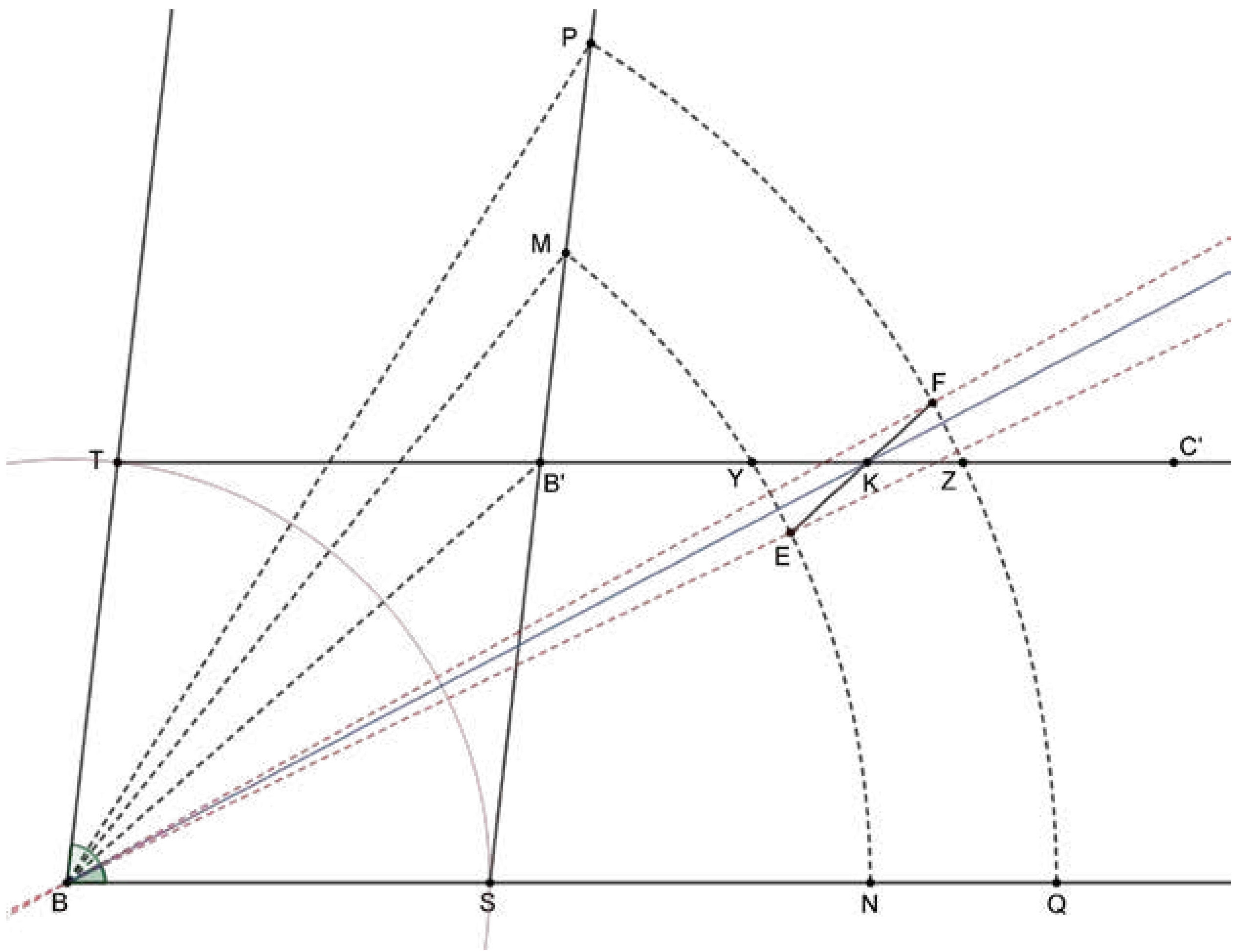


Figure 3

(7) In the same way, let $P = (1 + k_2 \cdot \cos t, k_2 \cdot \sin t)$ where $k_2 = \dfrac{SP}{SB'}$. From $BP^2 = BZ^2$ we get

$$1 + {k_2}^2 + 2k_2 \cdot \cos t = BZ^2, \qquad \therefore\ k_2 = -\cos t + \sqrt{\cos^2 t + BZ^2 - 1}.$$

(8) $\measuredangle PBQ = \arctan\left(\dfrac{k_2 \cdot \sin t}{1 + k_2 \cdot \cos t}\right)$, $\measuredangle FBQ = \dfrac{1}{2}\arctan\left(\dfrac{k_2 \cdot \sin t}{1 + k_2 \cdot \cos t}\right)$, with k_2 as above.

(9) $E = (BY \cdot \cos \measuredangle EBN,\ BY \cdot \sin \measuredangle EBN)$

(10) $F = (BZ \cdot \cos \measuredangle FBQ,\ BZ \cdot \sin \measuredangle FBQ)$

(11) $K = (u, \sin t)$, with u to be determined

(12) $u = BZ \cdot \cos \measuredangle FBQ + \left(\dfrac{BZ \cdot \cos \measuredangle FBQ - BY \cdot \cos \measuredangle EBN}{BZ \cdot \sin \measuredangle FBQ - BY \cdot \sin \measuredangle EBN}\right) \cdot (\sin t - BZ \cdot \sin \measuredangle FBQ)$

(13) $f(t) = \arctan \dfrac{\sin t}{u}$

Using these, we may construct a table of values of $f(t)$, as we did earlier. The result is shown in Table 2.

t	9 °	18 °	27 °	36 °	45 °	54 °	63 °	72 °	81 °	90 °
f (t)	3 °	6 °	8.9997 °	11.999 °	14.998 °	17.997 °	20.995 °	23.993 °	26.989 °	29.985 °

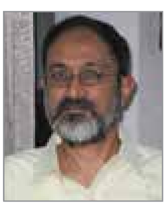
Table 2

The high level of accuracy can easily be seen. When $t = 90°$, the error is just 1 part in 3000. Very impressive!

As earlier we may compute the Taylor-Maclaurin series for $f(t)$. This is difficult to do by hand as the derivation itself is so complicated. We must take recourse to a powerful computer algebra system to do the task. Here is the result:

$$f(t) = \frac{t}{3} - \frac{121t^3}{2073600} - \frac{533179t^5}{159252480000} + \cdots. \qquad (7)$$

The smallness of the coefficients of t^3 and t^5 are convincing demonstrations of the high level of accuracy of this procedure.



SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Student Corner

Four Problems on Surds

TOYESH PRAKASH SHARMA

In this article we look at four problems and provide solutions to them. In high school we encounter square roots but not roots of higher order. Yet these problems can be solved at that level by observing some patterns and using logic.

Problems

1. Evaluate

$$\left[\frac{6}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7}+1\right)\left(\sqrt[8]{7}+1\right)}+1\right]^{16}$$

2. If $x = 1+\sqrt[5]{2}+\sqrt[5]{4}+\sqrt[5]{8}+\sqrt[5]{16}$, find the value of

$$\left(1+\frac{1}{x}\right)^{30}$$

3. Show that

(a) $$\frac{1}{6\sqrt[3]{6\sqrt{3}+10}}+\frac{1}{6\sqrt[3]{6\sqrt{3}-10}} = \frac{1}{\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{-10+6\sqrt{3}}}$$

(b) $$\frac{1}{2\sqrt[3]{6\sqrt{3}-10}}-\frac{1}{2\sqrt[3]{6\sqrt{3}+10}} = \frac{1}{\sqrt[3]{10+6\sqrt{3}}-\sqrt[3]{-10+6\sqrt{3}}}$$

4. Calculate

$$\frac{\sqrt[4]{7-4\sqrt{3}}}{\sqrt[4]{7+4\sqrt{3}}}$$

Solutions

1. We have

$$\left[\frac{6}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7}+1\right)\left(\sqrt[8]{7}+1\right)}+1\right]^{16}$$

$$=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7}+1\right)\left(\sqrt[8]{7}+1\right)\left(\sqrt[8]{7}-1\right)}+1\right]^{16}$$

$$=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7}+1\right)\left(\sqrt[8]{7^2}-1\right)}+1\right]^{16}$$

$$=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7}+1\right)\left(\sqrt[4]{7}-1\right)}+1\right]^{16}$$

$$=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7^2}-1\right)}+1\right]^{16}$$

$$=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{\left(\sqrt{7}+1\right)\left(\sqrt{7}-1\right)}+1\right]^{16}$$

$$=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{6}+1\right]^{16}$$

$$=\left[\left(\sqrt[8]{7}-1\right)+1\right]^{16}$$

$$=\left(\sqrt[8]{7}\right)^{16}$$

$$=49.$$

2. We have

$$x=1+\sqrt[5]{2}+\sqrt[5]{4}+\sqrt[5]{8}+\sqrt[5]{16}$$

$$\Rightarrow \sqrt[5]{2}\cdot x=\sqrt[5]{2}+\sqrt[5]{4}+\sqrt[5]{8}+\sqrt[5]{16}+\sqrt[5]{32}$$

$$\Longrightarrow \sqrt[5]{2}\cdot x=\left(\sqrt[5]{2}+\sqrt[5]{4}+\sqrt[5]{8}+\sqrt[5]{16}+1\right)+1$$

$$\Longrightarrow \sqrt[5]{2}\cdot x=x+1$$

$$\Longrightarrow x\cdot\left(\sqrt[5]{2}-1\right)=1$$

$$\Longrightarrow \sqrt[5]{2}-1=\frac{1}{x}.$$

Hence

$$\left(1+\frac{1}{x}\right)^{30}=\left(1+\sqrt[5]{2}-1\right)^{30}=2^6=64.$$

3. (a) We have

$$\sqrt[3]{10+6\sqrt{3}} = \sqrt[3]{(1)^3+\left(\sqrt{3}\right)^3+3\sqrt{3}\cdot\left(1+\sqrt{3}\right)} = \sqrt[3]{\left(1+\sqrt{3}\right)^3} = \left(1+\sqrt{3}\right),$$

$$\sqrt[3]{-10+6\sqrt{3}} = \sqrt[3]{(-1)^3+\left(\sqrt{3}\right)^3-3\sqrt{3}\cdot\left(-1+\sqrt{3}\right)} = \sqrt[3]{\left(-1+\sqrt{3}\right)^3} = \left(-1+\sqrt{3}\right)$$

Then $\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{-10+6\sqrt{3}} = 2\sqrt{3}$. Now, taking LHS

$$\frac{1}{6\sqrt[3]{6\sqrt{3}+10}}+\frac{1}{6\sqrt[3]{6\sqrt{3}-10}}$$

$$= \frac{1}{6}\left[\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{3}-1}\right] = \frac{1}{6}\left[\frac{\left(\sqrt{3}-1\right)+\left(\sqrt{3}+1\right)}{\left(\sqrt{3}\right)^2-1}\right]$$

$$= \frac{1}{6}\left[\frac{2\sqrt{3}}{2}\right] = \frac{\sqrt{3}}{6},$$

While RHS

$$\frac{1}{\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{-10+6\sqrt{3}}}$$

$$= \frac{1}{\left(\sqrt{3}+1\right)+\left(\sqrt{3}-1\right)} = \frac{1}{2\sqrt{3}} = \frac{1}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{6}.$$

Hence

$$\frac{1}{6\sqrt[3]{6\sqrt{3}+10}}+\frac{1}{6\sqrt[3]{6\sqrt{3}-10}} = \frac{1}{\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{-10+6\sqrt{3}}}.$$

(b) As in the above problem we have seen
$\sqrt[3]{10+6\sqrt{3}} = \left(1+\sqrt{3}\right)$, $\sqrt[3]{-10+6\sqrt{3}} = \left(-1+\sqrt{3}\right)$ and
$\sqrt[3]{10+6\sqrt{3}}-\sqrt[3]{-10+6\sqrt{3}} = 2$. Then, LHS

$$\frac{1}{2\sqrt[3]{6\sqrt{3}-10}}-\frac{1}{2\sqrt[3]{6\sqrt{3}+10}}$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}\right] = \frac{1}{2}\left[\frac{\left(\sqrt{3}+1\right)-\left(\sqrt{3}-1\right)}{\left(\sqrt{3}\right)^2-1}\right]$$

$$= \frac{1}{2}\left[\frac{2}{2}\right] = \frac{1}{2}$$

While RHS

$$\frac{1}{\sqrt[3]{10+6\sqrt{3}}-\sqrt[3]{-10+6\sqrt{3}}}$$

$$= \frac{1}{\left(\sqrt{3}+1\right)-\left(\sqrt{3}-1\right)} = \frac{1}{2}$$

Hence proved.

4. As

$$\left(1+\sqrt{3}\right)^4=\left(\left(1+\sqrt{3}\right)^2\right)^2=\left((1)^2+\left(\sqrt{3}\right)^2+2\,(1)\left(\sqrt{3}\right)\right)^2$$

$$=\left(4+2\sqrt{3}\right)^2=4\cdot\left(2+\sqrt{3}\right)^2=4\cdot\left(4+3+4\sqrt{3}\right)=4\cdot\left(7+4\sqrt{3}\right)$$

then, $\left(7+4\sqrt{3}\right)=\frac{1}{4}\left(1+\sqrt{3}\right)^4\Rightarrow\sqrt[4]{7+4\sqrt{3}}=\frac{1}{\sqrt{2}}\cdot\left(1+\sqrt{3}\right)$ and

$$\left(\sqrt{3}-1\right)^4=\left(\left(\sqrt{3}-1\right)^2\right)^2=\left((1)^2+\left(\sqrt{3}\right)^2-2\,(1)\left(\sqrt{3}\right)\right)^2$$

$$=\left(4-2\sqrt{3}\right)^2=4\cdot\left(2-\sqrt{3}\right)^2=4\cdot\left(4+3-4\sqrt{3}\right)=4\cdot\left(7-4\sqrt{3}\right)$$

Then, $\left(7-4\sqrt{3}\right)=\frac{1}{4}\left(\sqrt{3}-1\right)^4\Rightarrow\sqrt[4]{7-4\sqrt{3}}=\frac{1}{\sqrt{2}}\cdot\left(\sqrt{3}-1\right).$

Now,

$$\frac{\sqrt[4]{7-4\sqrt{3}}}{\sqrt[4]{7+4\sqrt{3}}}$$

$$=\frac{\frac{1}{\sqrt{2}}\cdot(1+\sqrt{3})}{\frac{1}{\sqrt{2}}\cdot(\sqrt{3}-1)}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$=\frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}=\frac{(\sqrt{3}+1)^2}{(\sqrt{3})^2-1}$$

$$=\frac{1+(\sqrt{3})^2+2\sqrt{3}}{2}=\frac{4+2\sqrt{3}}{2}=2+\sqrt{3}.$$

TOYESH PRAKASH SHARMA got interested in Science, Mathematics and Literature when he was in 9th standard. He passed 10th and 12th from St. C.F. Andrews School, Agra. When he was in 11th standard, he got interested in doing Research in mathematics and till now many of his works have found place in different journals such as *Mathematical Gazette*, *Crux Mathematicorum*, *Parabola*, *AMJ*, *Pentagon*, *Octagon*, *At Right Angles*, *Fibonacci Quarterly*, *Mathematical Reflections*, etc. Currently he is doing his undergraduation in Physics and Mathematics from Agra College, Agra, India. He may be contacted at toyeshprakash@gmail.com.

A Problem from the 2019 Zonal Informatics Olympiad

ADHISH KANCHARLA

Introduction

In this article, I describe an application of the powerful technique of *recursion* to an Informatics Olympiad problem. You can find an introduction to this concept in the July 2014 issue of *At Right Angles*[a].

Problem statement

A sequence of positive integers $[a_1, a_2, a_3, \ldots, a_n]$ is called a Special Sequence if a_1 divides a_2, a_2 divides a_3, and so on until a_{n-1} divides a_n, and if all the elements are distinct.

For example, $[2, 4, 8, 32]$ is a Special Sequence.

But $[4, 2, 8]$ is not, because 4 does not divide 2.

Similarly, $[2, 4, 4, 8]$ is also not Special, because the elements are not distinct.

You need to find the number of Special Sequences such that all the elements of the sequence are from the set $\{1, 2, \ldots, K\}$.

Suppose $K = 3$. The Special Sequences possible are $[1], [2], [3], [1, 2], [1, 3]$. So, the answer would be 5.

Find the answer for the following values of K:

a) $K = 15$

b) $K = 19$

c) $K = 22$.

a See "Self-Similarity" by Punya Mishra & Gaurav Bhatnagar, available at 5_Self-Similarity.pdf (azimpremjiuniversity.edu.in)

Keywords: Special Sequence, recursion, investigation, constraints, problem-solving.

Key Observation

Since the elements of the Special Sequence must be distinct and each A_i should divide A_{i+1} for all i in the range $[0, n-1]$, we can see that the elements of the Special Sequence must be strictly increasing.

Thus, the largest element can appear only once and must necessarily appear at the nth position.

Hence the total number of Special Sequences can be obtained by summing up the number of Special Sequences for each possible value of the largest element.

The idea of recursion

A function that is defined in terms of itself is called a *recursive function*. A well-known example is the *Fibonacci sequence* which is defined recursively as $F_n = F_{n-1} + F_{n-2}$ with base cases $F_1 = 1$ and $F_2 = 1$.

The key idea in the solution is to use recursion to compute the number of Special Sequences with a fixed largest element.

Solution

Let f_i be the number of Special Sequences such that the largest element of the sequence is i.

Notice that the final answer would be $\sum_{i=1}^{K} f_i$ because each case is mutually exlusive and there are K possible values of i.

First let us do the easy step of establishing the base case: $f_1 = 1$ because the only possible sequence is [1].

Now we need to find a recurrence relation that helps us find f_i in terms of f_j for $j \neq i$. Here we recall the definition of f. We have fixed the largest and therefore last element of the sequence to be i. Hence, we have a number of options for the other smaller elements of the sequence that come before this last element:

Case 1: There are no other elements in this sequence $\Longrightarrow$ the number of Special Sequences $= 1$.

Case 2: There is at least 1 other element in this sequence.

Let us consider all possible values of j, the next largest element in this sequence. We know that this element must come just before the last element and hence it must be a divisor of the last element.

Further, once we fix this second last element as j we see that we have come across a similar subproblem of the same nature and f_j will give us the number of Special Subsequences with the largest element, j.

Thus, simply adding all f_j for $j < i$ such that j divides i will give us the value of f_i.

A way to look at case 1 and case 2 together

Once we fix the largest element of the sequence (i), we just need to add a Special Subsequence to form its beginning. This subsequence can be empty (case 1) but if it is non empty, it must satisfy the condition that j (the largest element of the subsequence being added to the beginning) is a divisor of i (the largest element of the complete sequence) (case 2).

Final answer

Once we have computed each f_i for each $i < K$, we can add up all of them to get the total number of Special Subsequences with any ending value.

A few values of f_i are listed below:

$f_1 = 1,$

$f_2 = 1 + f_1 = 2,$

$f_3 = 1 + f_1 = 2,$

$f_4 = 1 + f_2 + f_1 = 4,$

$f_5 = 1 + f_1 = 2,$

$f_6 = 1 + f_1 + f_2 + f_3 = 6.$

Thus, we can compute f_i for i upto 22 using a calculator and verify that

$$\sum_{i=1}^{15} f_i = 69$$

$$\sum_{i=1}^{19} f_i = 105$$

$$\sum_{i=1}^{22} f_i = 133$$

ADHISH KANCHARLA is a 11th grade student from Bombay Scottish School in Mumbai. He enjoys solving algorithmic and mathematical problems and has qualified for the American Invitational Math Exam (AIME) and the IOI Training Camp (IOITC) in the 9th and 10th grades. He runs a YouTube channel on competitive programming and has over 3.6K subscribers and 140K views.

Student Corner

A Number Theory Problem from Hungarian Math Olympiad

MURALIDHAR RAO &
PARINITHA M

In this article we discuss a solution to a Number Theory Problem, which was given in the 80th Eötvös-Kürschák Competition, 1980 [1].

Problem. Let $n > 1$ be an odd integer. Prove that a necessary and sufficient condition for the existence of positive integers x and y satisfying

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y}$$

is that n has a prime divisor of the form $4k - 1$.

Solution. We first prove that the given condition is sufficient. (This is the easy part.)

Assume that n has a prime divisor of the form $4k - 1$. Then there exists a positive integer m such that

$$n = (4k - 1)m.$$

We now have:

$$\frac{4}{n} = \frac{4}{(4k-1)m} = \frac{(4k-1)+1}{k(4k-1)m} = \frac{1}{km} + \frac{1}{k(4k-1)m}.$$

Thus we can take $x = km$, $y = k(4k - 1)m$; they are positive integers satisfying the given equation.

Next we prove that the condition is necessary.

Keywords: Number theory, greatest common divisor, prime divisor, Eötvös-Kürschák Competition

Assume that for the odd integer $n > 1$ there exist positive integers x and y such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y}.$$

We shall prove that n has a prime divisor of the form $4k - 1$.

Let $\gcd(x, y) = d$. Then there exist positive integers u and v such that $x = du$ and $y = dv$, and $\gcd(u, v) = 1$ (i.e., u, v are coprime).

Then from the equality

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y}$$

we get

$$n = \frac{4xy}{x+y} = \frac{4d(uv)}{u+v}. \tag{1}$$

We now prove that $\gcd(uv, u+v) = 1$.

Let p be a prime factor of uv and $u + v$. Since p is a divisor of uv, it must divide either u or v. Let us suppose that p is a divisor of u. Since p is a divisor of u as well as $u + v$, it must be that p is a divisor of v too. But this contradicts our supposition that u and v are coprime. Likewise if p is a divisor of v. We conclude that

$$\gcd(uv, u+v) = 1. \tag{2}$$

From (1) we have,

$$n(u+v) = 4d(uv). \tag{3}$$

Therefore, 4 is a divisor of $n(u+v)$. Since n is odd we deduce that

$$4 \text{ is a divisor of } u+v. \tag{4}$$

From (4) it follows that u, v are either both odd or both even. They cannot both be even as u, v are coprime. Hence u, v are both odd.

Further, since 4 is a divisor of $u + v$, it follows that one of u, v is of the form $4a - 1$ while the other is of the form $4b + 1$. From this we deduce that

$$uv \equiv -1 \pmod{4}. \tag{5}$$

This implies that uv has a prime divisor p of the form $4k - 1$. It follows from (3) that p divides $n(u + v)$.

But since $\gcd(uv, u+v) = 1$, it follows that p does not divide $u + v$. From this it follows that p divides n.

We have thus shown that n has a prime divisor p of the form $4k - 1$, as required. □

Another argument to prove that the given condition is necessary (contributed by the second author). For this, we shall assume that all the prime factors of n are of the form $1 \pmod{4}$. From this we proceed to derive a contradiction.

Let $d = \gcd(x, y)$ so that $x = du$ and $y = dv$ where $\gcd(u, v) = 1$.

We have:

$$4xy = n(x+y), \quad \therefore \; 4xy = nd(u+v), \quad \therefore \; 4\left(\frac{x}{d}\right)y = n(u+v), \tag{6}$$

so $4uy = n(u+v)$. As $\gcd(n,4) = 1$, we get

$$u + v \equiv 0 (\text{mod} 4). \tag{7}$$

On the other hand, from $4uy = n(u+v)$, it follows that u is a factor of $n(u+v)$.

Since $\gcd(u, u+v) = 1$, it follows that u is a factor of n.

Similarly, we conclude that v is a factor of n. So both u, v are factors of n.

By assumption, therefore, $u \equiv 1(\text{mod}4)$ and $v \equiv 1(\text{mod}4)$. From these we get

$$u + v \equiv 2(\text{mod}4). \tag{8}$$

This contradicts (7). We have found the desired contradiction.

It follows that n has a prime divisor of the form $4k-1$, as required . □

References

1. *Eötvös-Kürschák Competitions* (Mathematical and Physical Society), compiled by Ercole Suppa.
   http://www.batmath.it/matematica/raccolte_es/ek_competitions/ek_competitions.pdf

MURALIDHAR RAO is a student of class 12 at The Learning Center PU College, Mangalore, Karnataka. He has a deep love for Mathematics, Physics, and Biology. He cleared the NTSE when he was in class 10. He did extremely well in the KVPY Examination 2021, and he qualified in the Indian National Mathematics Olympiad 2021 and the Indian National Astronomy Olympiad 2021. He enjoys playing chess and listening to Indian Classical music. He may be contacted at muralidharrao@tlc.edu.in.

PARINITHA M is a II year BSMS student at IISER, Tirupathi. She is deeply interested in mathematics, especially Analysis, Number theory, and Group theory. She took part in the Simon Marais Math Competition 2022. She is a recipient of the DST Inspire scholarship. She is an active member of The Math Club at IISER and hopes to pursue educational activities related to mathematics. Her other interests include classical instrumental. She may be contacted at parinitha.m.17@gmail.com.

Deriving an Equation for the Sun's Path

AMAN MAKHIJA

Watching the sun rise and set from my balcony each day, I wondered: how easy is it to derive a mathematical formula to describe the path of the sun over the course of the day?

This path varies depending on the observer's latitude and the time of year.

To solve this problem, we must account for the fact that the Earth has two rotational movements: it rotates around its own axis once every 24 hours, and it also revolves around the sun once every year. Both these are relevant to this problem. Luckily for us, they are independent of each other, and we can tackle them one at a time.

Understanding this problem involves three key concepts.

Concept 1: It is helpful to pretend the universe rotates around the Earth's axis, instead of the other way round

The first motion we consider is the Earth's rotation about its axis over a 24-hour period. Since we are interested in the path traced by a celestial body (the sun) from the Earth's perspective, we invoke a clever trick: we pretend the Earth is stationary and consider the universe to be rotating about the Earth's axis in the opposite direction.

From the point of view of an observer, all celestial bodies will appear to trace circles about a fixed point in the sky which lies on the Earth's axis of rotation. We just need a way to identify this point from a given location on Earth.

Keywords: Solar declination, midnight sun, polar night, vector geometry, mathematization, astronomy, vectors, trigonometry, coordinate geometry

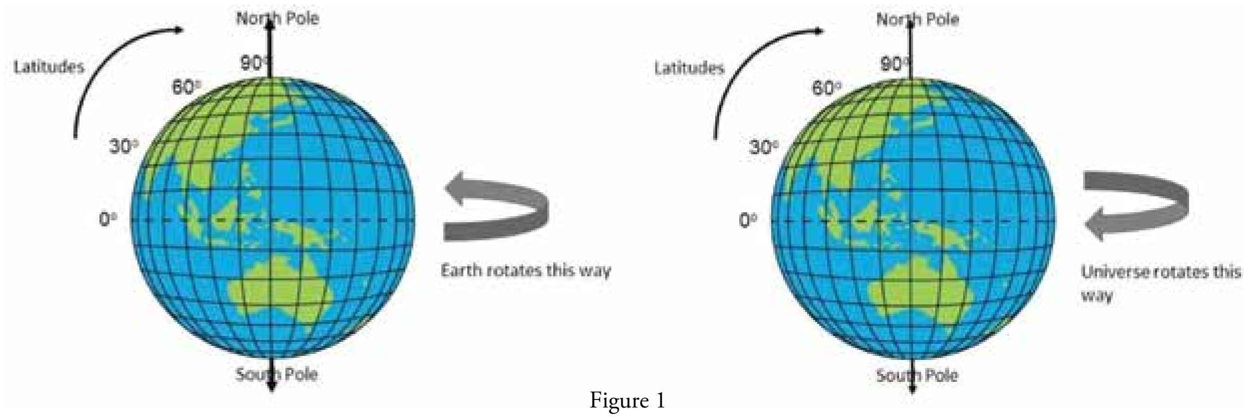


Figure 1

In the proof that follows (Figure 2), it can be seen that the position of this fixed point in the sky depends on the latitude of the observer. In the Northern hemisphere, the point along the axis (known as the North celestial pole or Celestial North) is at an angle of the observer's latitude above his/her northern horizon. This point is marked by the Pole Star (Polaris) in the night sky[1]. In the Southern hemisphere, the fixed point is known as the South Celestial Pole and is elevated at an angle of the observer's latitude above the southern horizon. For our purposes, we will focus on an observer in the Northern hemisphere for convenience.

Proof: Consider a point L on the earth at Latitude ϕ. **OE** is a vector from the centre of the earth to the equator (intersecting at the longitude of point L) and **ON** is the vector pointing from the centre of the Earth to the North Pole (i.e., the Celestial North direction). The local northern horizon is a tangent vector along the earth's surface Let us assume the tangent intersects the vector pointing along the earth's axis at point T.

In Figure 2B, ∠TOL = 90 – ϕ, because **OT** and **OE** are perpendicular to each other. Vector **LT** is tangent to the sphere and therefore perpendicular to the radius.

∴ OLT is a right triangle.

∴ ∠OTL = ϕ = ∠TLN' (interior alternate angles).

In other words, at any given point an observer will see Celestial North at an angle equal to the latitude ϕ above his/her local North, as shown in Figure 2.

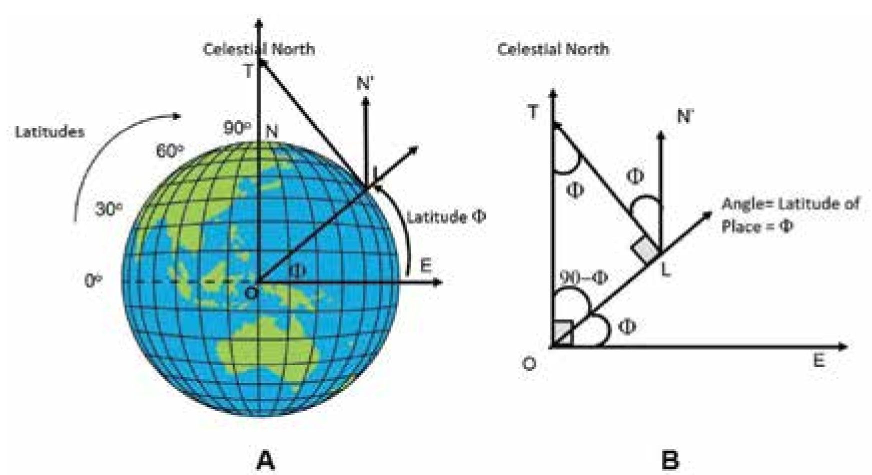


Figure 2. (A) and (B) illustrate why celestial north is at an angle ϕ = the latitude of a place above the local Northern horizon.

[1] Ancient seafarers used the pole star to navigate because its position in the sky is fixed through the day and year, because it lies on the earth's axis.

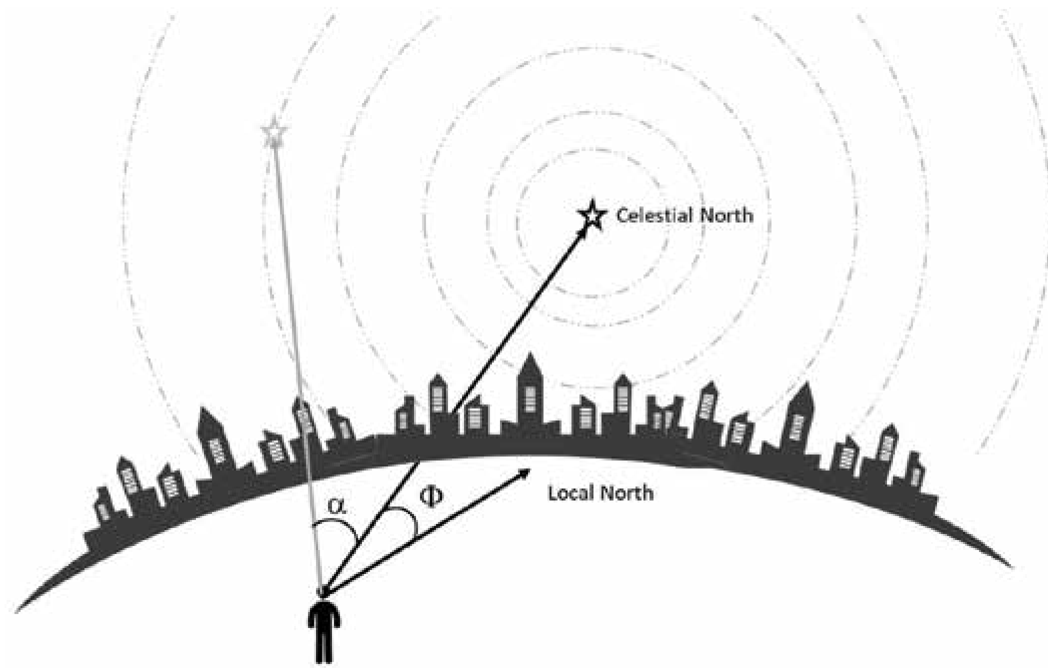


Figure 3. Celestial bodies trace circles around the North Pole.

Concept 2: The size of the circle traced by a celestial body depends on the North Pole Separation Angle of the object in question on that particular day.

Because the Earth's axis is fixed, celestial bodies trace circular paths about the axis of rotation. But what is the exact circular path a given celestial body will trace? This is determined by one specific angle called the "North Pole Separation"[2] of the celestial body (Figure 3).

The North Pole Separation is the angle (α) between the Earth's axis (**ON**) and the vector pointing in the direction of the celestial body. A celestial body with a small α (say 10 degrees), will trace a small circle around celestial north in the northern sky, while a celestial body with a large α (say 60 degrees), will trace a bigger circle around celestial north each day.

Concept 3: For the sun, the North Pole Separation angle (α) changes throughout the year.

For most celestial bodies the North Pole Separation, and hence the paths traced are fixed through the year. The Sun is unique, because its position relative to the Earth changes throughout the year as the Earth revolves.

This revolution and the Earth's axial tilt causes the North Pole separation angle of the sun to change over the course of the year, causing the daily path of the Sun to change.

So, our goal is to solve for α in terms of the time of year.

Let us start by defining our coordinate system, as follows:

- The centre of the Earth is at the point (0,0,0), which we call O.
- X-axis: Axis lying on the plane of revolution and the plane containing the Earth's rotational axis and the Z-axis. It's the direction of the sun on the June solstice.
- Y-axis is thus defined to be perpendicular to both the X and Z axes.
- Z- axis: Axis perpendicular to the plane of revolution and passing through the centre of the Earth.

[2] Astronomy texts will generally refer to the complementary angle to the North Pole separation, known as declination. But for the purposes of this problem, using the North Pole separation is more convenient.

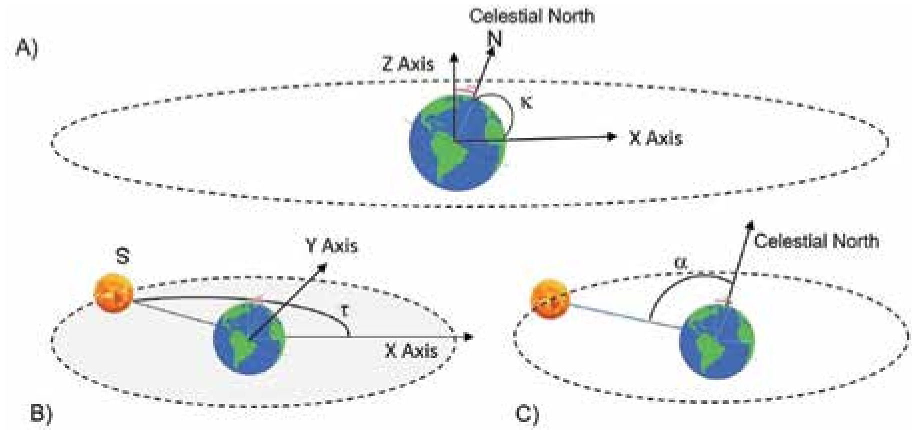


Figure 4. The coordinate system

We assume that the sun moves anti-clockwise along the unit circle centred at O, completing one complete cycle every year. The sun is along the X-axis on the June solstice. The XY plane is the plane of the sun's revolution.

- The unit vector in the direction of the Earth's rotational axis (in the Northern direction) is N. The angle between the Earth's rotational axis and the plane of revolution is a constant κ = 66. 5 degrees (Figure 4A).
- τ is the angle traced out by the Sun in its orbit since June 21 in radians (Figure 4B). For example, 37 days after summer solstice, τ = (37/365) × 360 degrees.
- The unit vector from the centre of the Earth in the direction of the sun is called S.
- α is the angle between S and N or the north pole separation of the sun (Figure 4C).

As the circular path traced by the sun depends on the North Pole Separation (α), the mathematical question is, what is the exact formula for α at a given time of the year (represented by τ)?

Figure 5 shows that α is the smallest on the June solstice (it is 66.5 degrees on this day) when the northern hemisphere is tilted towards the sun, and largest on the December solstice when the northern hemisphere is tilted away from the sun (it is 113.5 degrees). On the equinoxes, α is 90 degrees.

We can find α in terms of τ by using vector analysis.

Using the coordinate system defined, the position vector of the sun on any given day is:

$$S = [cos(\tau), sin(\tau), 0]$$

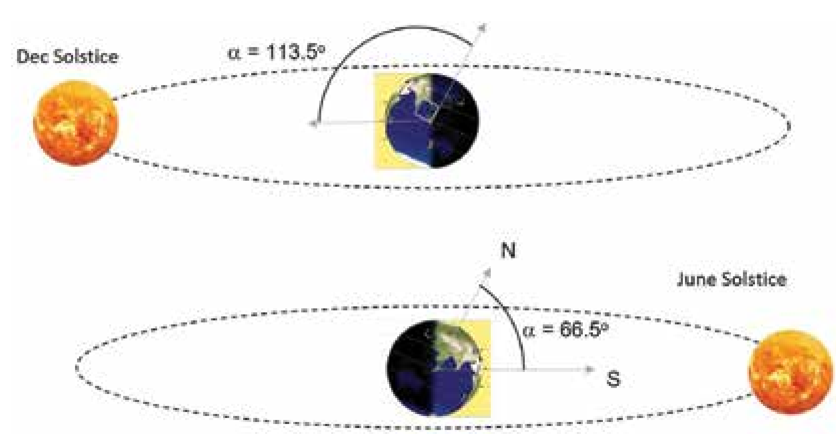


Figure 5. The North Pole separation angle α, varies through the year.

Since the earth is fixed in our convention system, N is constant with time, with the y coordinate as 0 by definition. Its x coordinate is cos (κ) and the z coordinate is $\sin(\kappa)$, so the position vector N is

$$N = [cos(\kappa), 0, sin(\kappa)]$$

In order to find the cosine of the angle between S and N, we take the dot product of the two vectors. Since the y and z component products are clearly 0, we end up with only x-component product:

$$cos(\alpha) = cos(\tau) * cos(\kappa)$$

We have now found a formula for α, the angle of separation, in terms of time of year (τ).

Discussion: What does it mean? The overhead sun.

From Figure 3, we can draw some conclusions about specific cases of the position of the celestial object.

For example, if $\alpha + \phi = 90°$ then the object will pass overhead.

For the sun, we can use the formula $cos(\alpha) = cos(\tau) * cos(\kappa)$ and set $\alpha = 90° - \phi$ to get

$$cos(\tau) = sin(\phi)/cos(\kappa)$$

We can then solve for τ to find the day of the year when the sun passes overhead between the tropics.

For example, let us find the dates on which the sun passes overhead in Bengaluru, Karnataka, India (at a latitude of $\phi = 12.8°$ N). We get $\cos(\tau) = \sin 12.8°/\cos 66.5°$ which gives $\tau = 56°$ or $\tau = 304°$.

Since τ is the proportion of the year completed since June Solstice (June 21), we get that the sun passes overhead in Bengaluru $\tau/360 * 365$ days after June 21.

This corresponds to 17 August and 25 April, for the two values of τ [3] (slight variation in the time of summer solstice may cause it to vary – but by at most a day!).

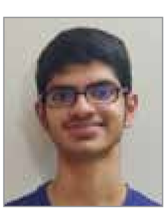
The same formula can be used to explain the phenomena of the "midnight sun" and "polar night" in far-North latitudes $(\phi > 66.5°)$.

From Figure 3, we see that the case with $\alpha < \phi$ in the Northern hemisphere for a particular celestial body would mean it would never set below the local horizon over the 24-hour period.

In the case of the sun, this occurs near the June Solstice (Summer) in far-North locations; this is referred to as the "midnight sun" because the sun is above the horizon, throughout the day.

Similarly, the case where $\alpha + \phi > 180$ degrees occurs near the December Solstice (Winter); this is referred to as the "polar night", because the sun never rises over the course of the 24-hour period.

[3] https://bangaloremirror.indiatimes.com/bangalore/others/its-a-zero-shadow-day-in-bengaluru-at-noon-today-the-sun-will-be-exactly-overhead/articleshow/63887232.cms



AMAN S. MAKHIJA is a grade 12 student living in Bangalore. An aspiring mathematician, he loves geometry and linear algebra. He is also an avid stargazer and music lover. He can be contacted at amanmakhija1010@gmail.com

Real World Implications of a KVPY Problem

MAHIT WARHADPANDE & ABHRONEEL GHOSH

Abstract: Where do the intriguing problems seen in puzzle books and competitive exams come from? Are they mere mathematical curiosities or do they represent something in the real world? If the latter, then does our awareness of the real-world connection provide any deeper insights?

Introduction

The following question (see Figure 1) appeared in the Kishore Vaigyanik Protsahan Yojana (KVPY) 2017 SX/SB question paper [1].

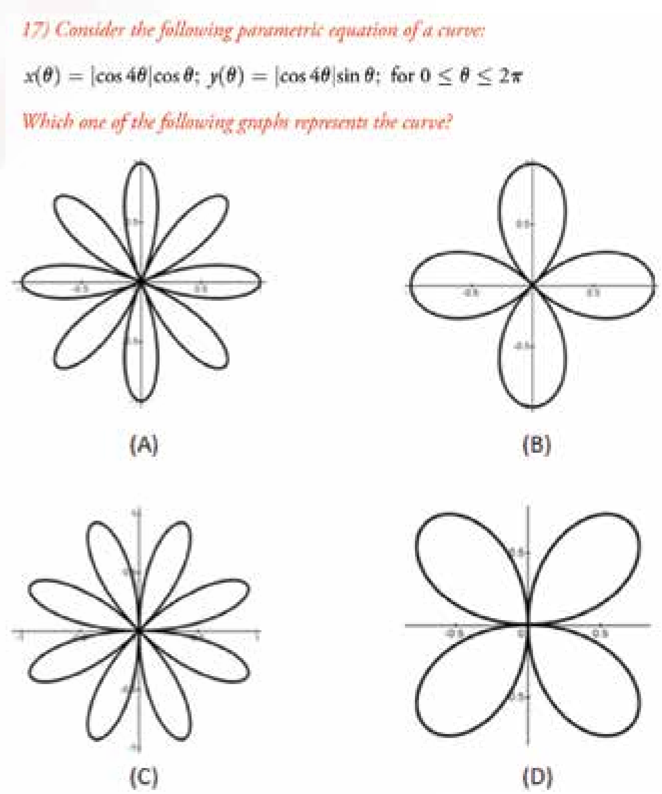


Figure 1. Question 17 in KVPY 2017 for SX/SB stream.

Keywords: KVPY, retrograde motion, Desmos, Geometric Chuck.

One common approach to solving this problem would be to set $\theta = \pi/2$ so that $x(\theta) = 0$ and $y(\theta) = 1$. This eliminates options (C) and (D). To choose between (A) and (B) we could make another "intelligent" guess and set $\theta = \pi/4$ so that $x(\theta) = 1/\sqrt{2} = y(\theta)$. This eliminates option (B) since it does not have any point on the line $y = x$ other than $(0, 0)$. End of problem.

Or is it? Why would anyone come up with equations or graphs like this? Do they describe any real phenomenon or are they purely mathematical constructs whose sole application is as examination questions? Here is an account of how we were able to relate this particular question to a real-world phenomenon. This not only helped us answer the KVPY question (well, almost!), but also allowed us to understand several properties of these curves and equations through the behaviour of the corresponding physical system and vice-versa. We used the Desmos graphing calculator [3] to study the graphs produced. See the Appendix for directions on using this software.

Apparent Planetary Motion

Figure 2 shows a simplified heliocentric model for planetary motion: all planetary orbits are circular and all planets start at zero phase in their respective orbits.

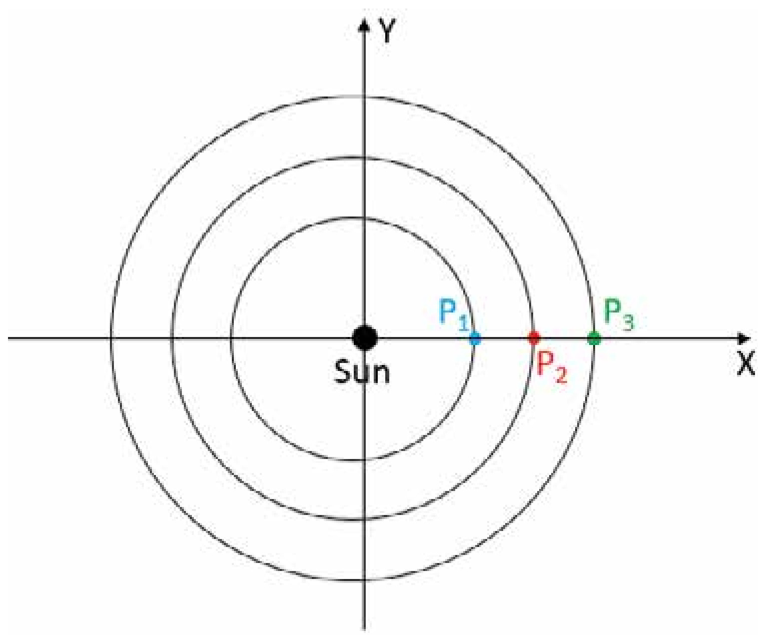


Figure 2. Simplified heliocentric planetary model. All planets (P_1, P_2, P_3) have circular orbits centred at the sun. At time = 0, all planets are located on the x-axis (i.e., they have "zero phase").

As shown in Figure 3, if the orbital radii of observed planet (P) and the Earth (E) be R_P and R_E respectively, then the coordinates of the observed planet *in Earth's frame of reference* are:

$$x_{PE} = x_P - x_E = R_P \cos\theta_P - R_E \cos\theta_E$$
$$y_{PE} = y_P - y_E = R_P \sin\theta_P - R_E \sin\theta_E$$

If the planet P completes one full revolution (2π radians) around the sun in time T_P, then in time t, it would have covered an angle $\theta_P = 2\pi t/T_P$. We say that T_P is the orbital period of the planet P. Similarly, if the orbital period of Earth E is T_E, then in the same time t, it will have covered an angle $\theta_E = 2\pi t/T_E$.

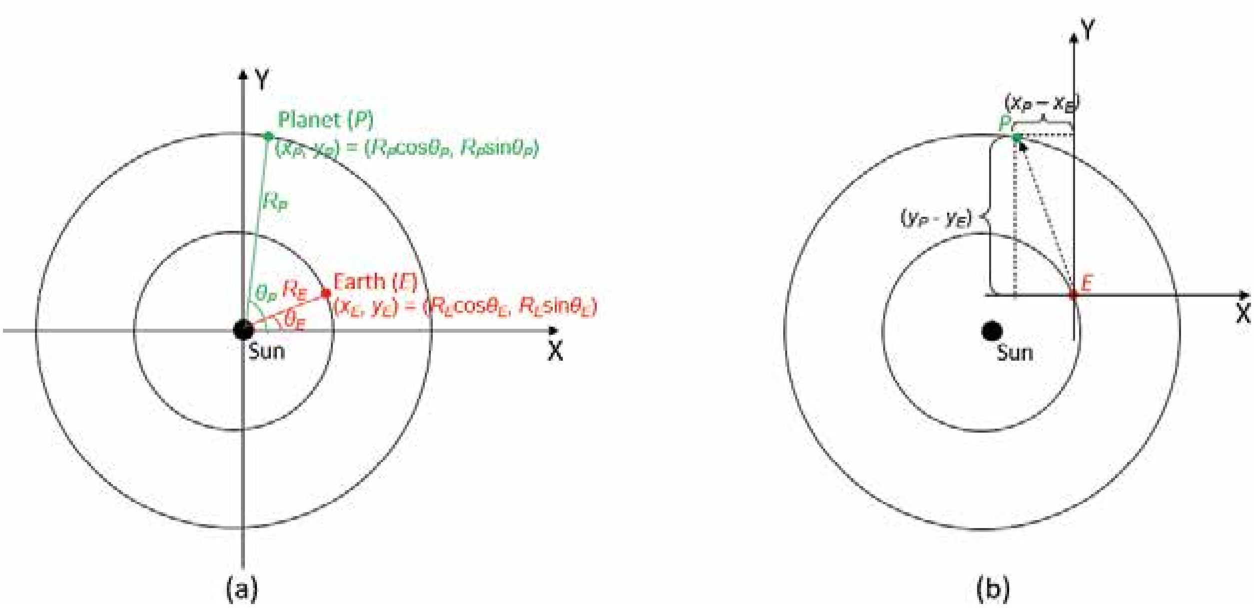


Figure 3. (a) The coordinates of planets E and P with the Sun as the origin. (b) The coordinates of planet P with planet E as the origin.

So, we can rewrite:

$$x_{PE} = x_P - x_E = R_P \cos(2\pi t/T_P) - R_E \cos(2\pi t/T_E)$$
$$y_{PE} = y_P - y_E = R_P \sin(2\pi t/T_P) - R_E \sin(2\pi t/T_E)$$

Note that after a period $T = \text{LCM}(T_P, T_E)$, both planets will have completed an integer number of revolutions and will be back in their initial positions on their respective orbits. A plot of (x_{PE}, y_{PE}) will therefore, also return to its starting point after this time and then start repeating itself beyond this time.

Finally, we also note that in the above equations, θ increases with t for all planets. The implication is that all planets orbit the sun in the same direction. This is indeed true for our solar system.

Mars and Venus

If the orbital radii of Earth, Mars and Venus be respectively R_E, R_M and R_V and their orbital periods respectively be T_E, T_M and T_V, then $R_M \approx 1.5R_E$, $R_V \approx 0.7R_E$, $T_M \approx 1.9T_E$, and $T_V \approx 0.6T_E$ [2, pp. 388–389]. Using these values in the above equations, we plot the orbit of Mars for $\text{LCM}(1.9, 1) = 19$ years and the orbit of Venus for $\text{LCM}(0.6, 1) = 3$ years as seen from Earth. Figure 4 shows the results obtained using the Desmos online graphing calculator [3]. The small loops highlighted in the insets represent periods of ‘apparent retrograde motion’ where the observed planet seems to reverse its ‘usual’ direction of motion against the backdrop of the distant stars.

Correlating the Math with the Physical System

We define the planets to be at ‘perigee’ when they are closest to each other and at ‘apogee’ when they are furthest from each other. As shown in Figure 5, the ‘perigee’ distance between the two planets P and E will be $R_P - R_E$ and the apogee distance between them will be $R_P + R_E$.

Indeed, we can confirm from Figure 4 that, the minimum (perigee) distance between the Earth (at origin) and the observed planet (on the curve) is $|R_P - R_E|$ (which is 0.5 for Mars and 0.3 for Venus) and the

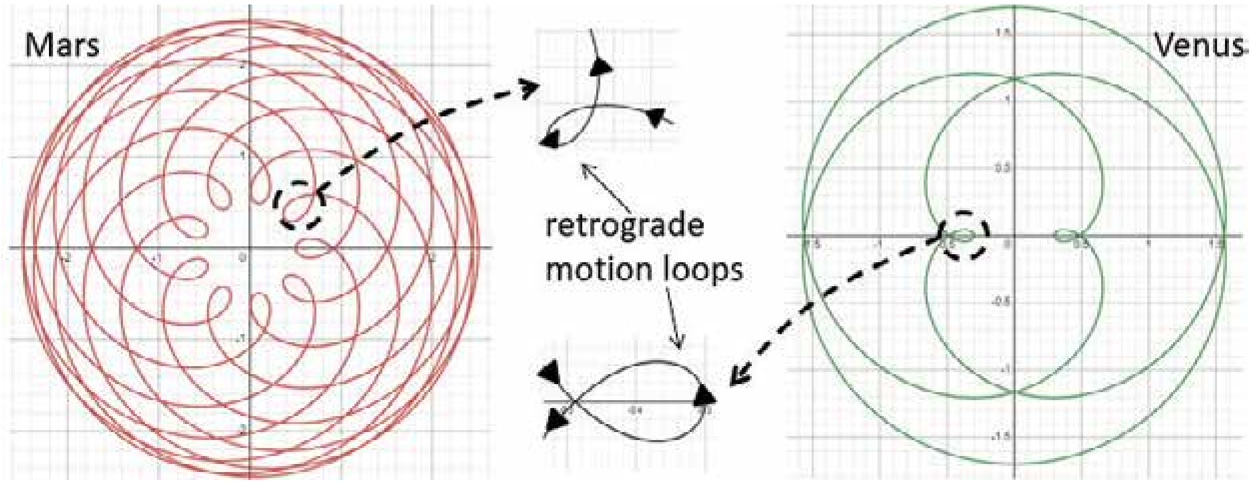


Figure 4. Apparent motion of Mars and Venus observed from the Earth in our model.

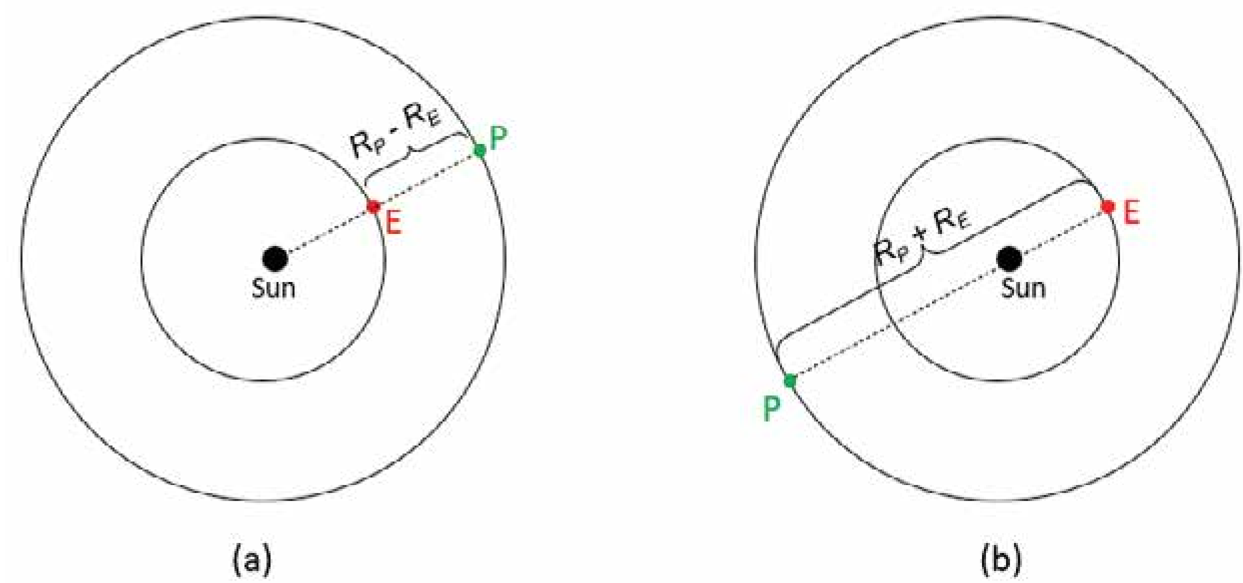


Figure 5. (a) Perigee: when the planets are closest to one another.
(b) Apogee: when the planets are furthest from one another.

maximum (apogee) distance is $R_P + R_E$ (which is 2.5 for Mars and 1.7 for Venus). We now analyze some more features of the graphs obtained in Figure 4.

Retrograde loops

Let us define P to be an *inner planet* if $R_P < R_E$ and an *outer planet* if $R_P > R_E$. (The usual nomenclature is 'inferior' and 'superior'. Note that though Mars is to be treated as an "outer" planet for this article, it is otherwise considered an inner planet as it is inside the asteroid belt.) At equally spaced times t_1, t_2 and t_3, let the Earth E be at locations E_1, E_2 and E_3 and the observed outer planet P be at locations P_1, P_2 and P_3 respectively, with both E and P orbiting the sun in the same (anticlockwise) direction. Figure 6 shows three possible configurations for the planets' positions at t_1, t_2 and t_3 as case A, case B and case C.

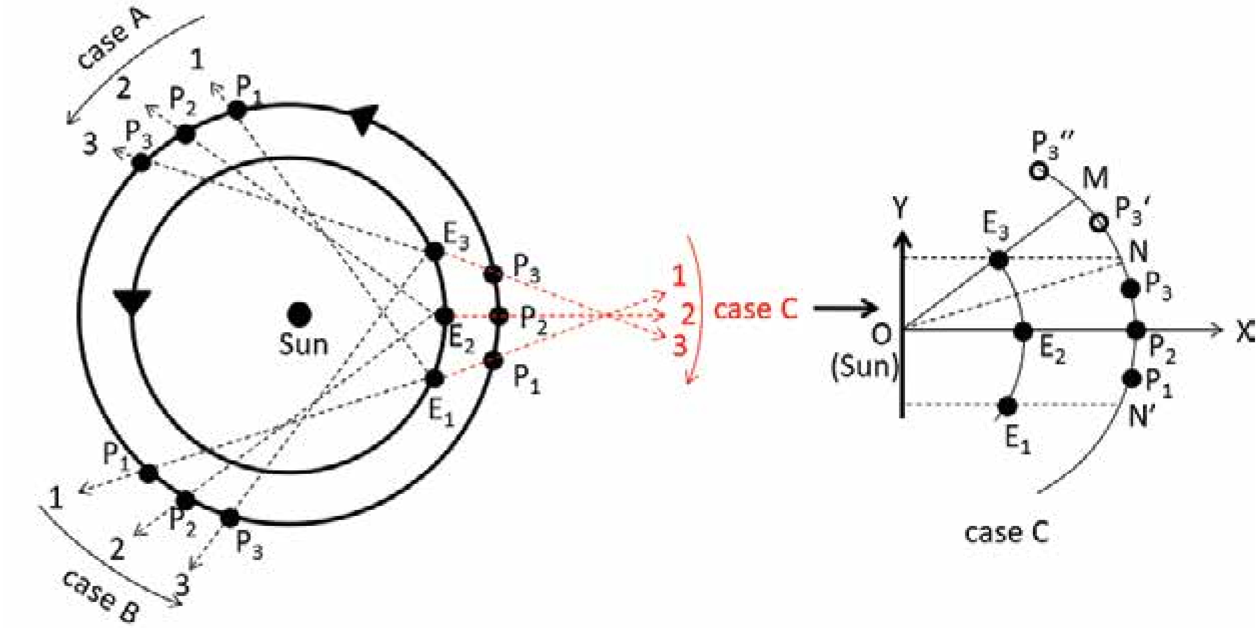


Figure 6. Apparent retrograde motion occurs when planets are positioned as in case C.

In case A and case B, the apparent motion of P as seen from E seems to be anticlockwise. In case C however, P appears to have reversed its motion and moves clockwise when seen from E. Such retrograde motion is only possible at the perigee if some additional constraints are met as discussed below.

With reference to the inset for Case C in Figure 6, if the lines E_3N and E_1N' be parallel to the line OE_2P_2, then apparent retrograde motion will only occur if P_3 lies between P_2 and N and by symmetry P_1 lies between P_2 and N'. If P were to move faster so that it reaches position P_3' between N and M at t_3, then no retrograde motion would be seen. In particular, if P were to reach position P_3'' beyond M at t_3, it would mean that the orbital period of P is less than that of the Earth. Thus, for retrograde motion we want the y-coordinate of P_3 to be less than that of N or equivalently, E_3, i.e., $R_P \sin(2\pi t/T_P) < R_E \sin(2\pi t/T_E)$, where $t = t_3 - t_2$. Since $\sin\theta \approx \theta$ for small θ, we can write the condition for retrograde motion for at least an infinitesimal time for outer planets as:

$$1 < R_P/R_E < T_P/T_E$$

A similar analysis yields the condition for retrograde motion for inner planets as $1 > R_P/R_E > T_P/T_E$. By symmetry, this analysis applies to all perigee locations.

All the planets in the solar system show retrograde motion because they satisfy these conditions as a consequence of Kepler's Law: $T_E^2/R_E^3 = T_P^2/R_P^3$ which in turn is a consequence of Newton's theory of Gravitation [2, pp. 388-389, p. 404].

If $T = LCM(T_P, T_E)$, then the number of retrograde loops formed will depend on how many times the outer planet gets lapped by the inner planet in time T. Thus, number of retrograde loops in the plot will be $|T/T_E - T/T_P|$. These loops will be evenly distributed in the 360° angle around the Earth. This explains why in Figure 4, the Mars plot has $|19 - 10| = 9$ retrograde loops at every $360°/9 = 40°$, while the Venus plot has $|3 - 5| = 2$ loops at every $360°/2 = 180°$.

Initial Phase

We assumed that both the Earth and the observed planet start in their orbits with zero initial phase. In this scenario, as shown in Figure 7, we would expect at least one perigee to occur in the $+x$ direction for outer planets and along the $-x$ direction for inner planets. These have been highlighted for Mars and Venus in Figure 7. Since Venus has its two perigees separated by 180°, its other perigee also ends up on the x-axis, along the positive x-axis.

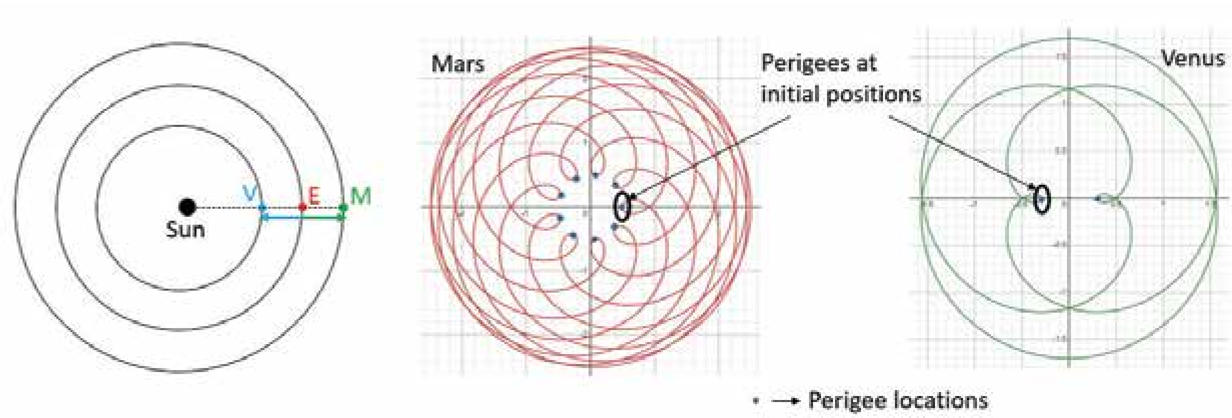


Figure 7. Initial positions are positions of perigee along the x-axis when all planets start with zero phase.

Now, let the initial phases of the Earth and the observed planet be $\varnothing_E$ and $\varnothing_P$ respectively, then the first perigee location will be reached when $\frac{2\pi t}{T_P} + \varnothing_P = \frac{2\pi t}{T_E} + \varnothing_E$. For example, if Venus starts at phase 0 and Earth starts at phase 90°, then a perigee will occur when $\frac{2\pi t}{0.6} = 2\pi t + \pi/2$, i.e., $t = 3/8$ and the phase for both planets at this time is 225°. The whole orbit of Venus as seen from Earth in this situation would then appear to be rotated w.r.t. Figure 4 by this angle. This is borne out by Figure 8.

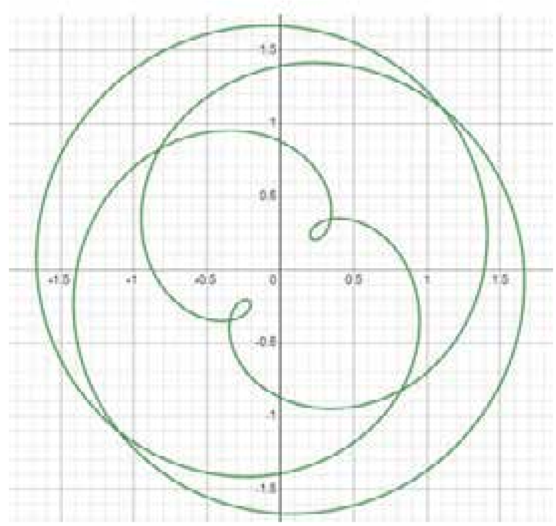


Figure 8. The graph is rotated in accordance with the relative initial phase of the planets.

Hypothetical planets

Figure 9 shows what would have happened in the hypothetical scenario where Mars and Venus orbit the sun in a direction *opposite* to that of the Earth.

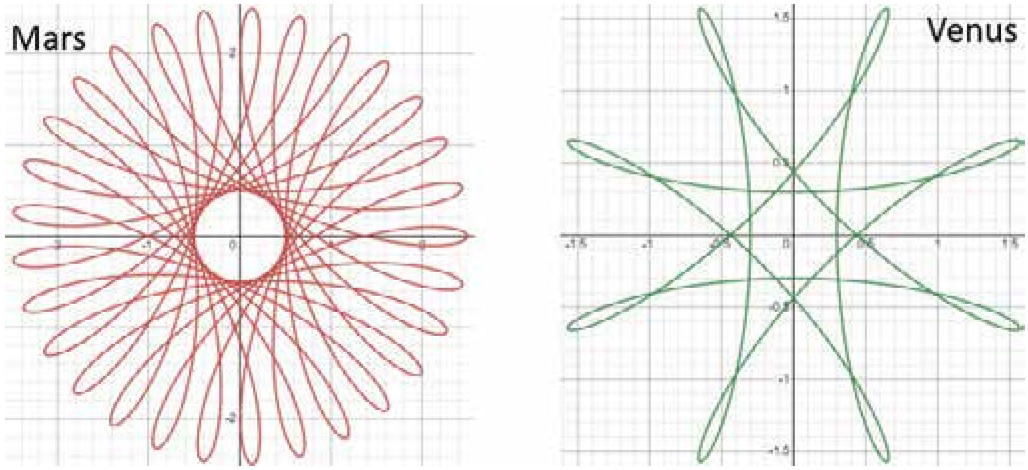


Figure 9. Mars and Venus plots from Earth if orbiting in opposite direction to Earth.

The conditions for the formation of the retrograde motion loops remain the same, except that they are now formed at the apogee. This causes the retrograde loops to face outward. The number of times the planets are at the apogee in time $T = LCM(T_E, T_P)$ in this case, is $(T/T_E + T/T_P)$. Hence, Mars now shows $19 + 10 = 29$ loops of retrograde motion while Venus shows $3 + 5 = 8$ such loops.

The KVPY Planets

Let us now turn back to the function given in the KVPY question. Ignoring the modulus operation (we will come back to this later), we can rewrite [4]:

$$x(\theta) = \cos 4\theta \cos \theta = 0.5(\cos 5\theta + \cos 3\theta) = 0.5(\cos 5\theta - \cos(\pi - 3\theta))$$
$$y(\theta) = \cos 4\theta \sin \theta = 0.5(\sin 5\theta - \sin 3\theta) = 0.5(\sin 5\theta - \sin(\pi - 3\theta))$$

Relating this to our model of apparent planetary motion, we can now deduce:

1. The orbital radii of the two "planets" are same, therefore, the nearest distance between them will be zero. All curves satisfy this since they pass through $(0, 0)$.
2. While one planet's motion is determined by $+\theta$, the other's changes as $-\theta$, i.e., these planets are orbiting in opposite directions. Therefore, the retrograde motion loops must face outward. This is also satisfied by all the curves.
3. The number of retrograde motion loops should be $(5 + 3) = 8$. This eliminates options (B) and (D).
4. Since the reference planet has initial phase π while the observed planet has initial phase 0, the starting position must be an apogee position *along the x-axis*. This does not occur in (C). Thus, the answer must be (A).

Can you now work out what the equations could be for the remaining curves?

What about the Modulus?

With the help of the DESMOS calculator, we plotted the graph for the functions with and without the modulus operation for increasing ranges of θ. The results are shown in Figure 10.

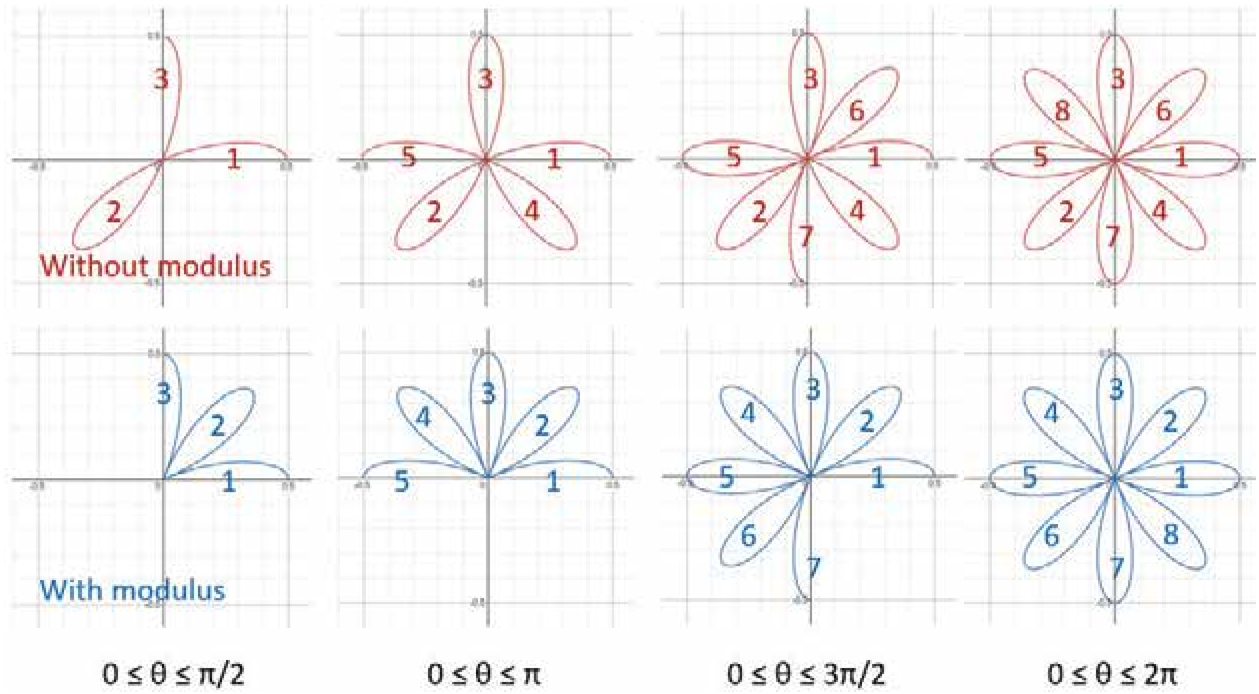


Figure 10. Difference caused by modulus operation.

The sequence in which the retrograde loops are produced is different in the two cases. Once θ has covered its full period of 2π though, the graphs are indistinguishable. We wonder what physical system could correspond to the equations with the modulus?

Conclusions

Our analysis of planetary motion helped us appreciate the KVPY question discussed in this article from a completely different perspective. We wonder if other equations and graphs that we come across in puzzle books and examinations are also related to some everyday physical phenomena. We believe an awareness of such connections will hugely enrich our learning of both, the physical phenomena and its underlying mathematics. In that context, here are a few more points to ponder as extensions of the discussion in this article (ignore the modulus operation to begin with):

1. We said that the graph loops back to its starting point and then repeats after time $T = \text{LCM}\,(T_P, T_E)$. What is T if T_P/T_E is not rational? What happens to the graph in this case?
2. What if both planets start with the same but non-zero initial phase?
3. Given a function, can we predict the sequence in which the retrograde loops will be generated as θ increases (see Figure 10)?
4. Can we predict how many "layers" of intersections the curve will have and the angles along which these intersections will lie (see Figure 11)?
5. What if $1 < T_P/T_E < R_P/R_E$ for outer planets? Does the curve have concavities or is it fully convex or does it depend on exactly how much R_P/R_E is greater than T_P/T_E?

Trivia: The *Geometric Chuck* is a mechanical instrument that generates the types of curves discussed in this article [5]. Such curves are equivalent to the ancient (geocentric) epicycle model of the solar system which, understandably, had good success in explaining the retrograde motion of planets [6].

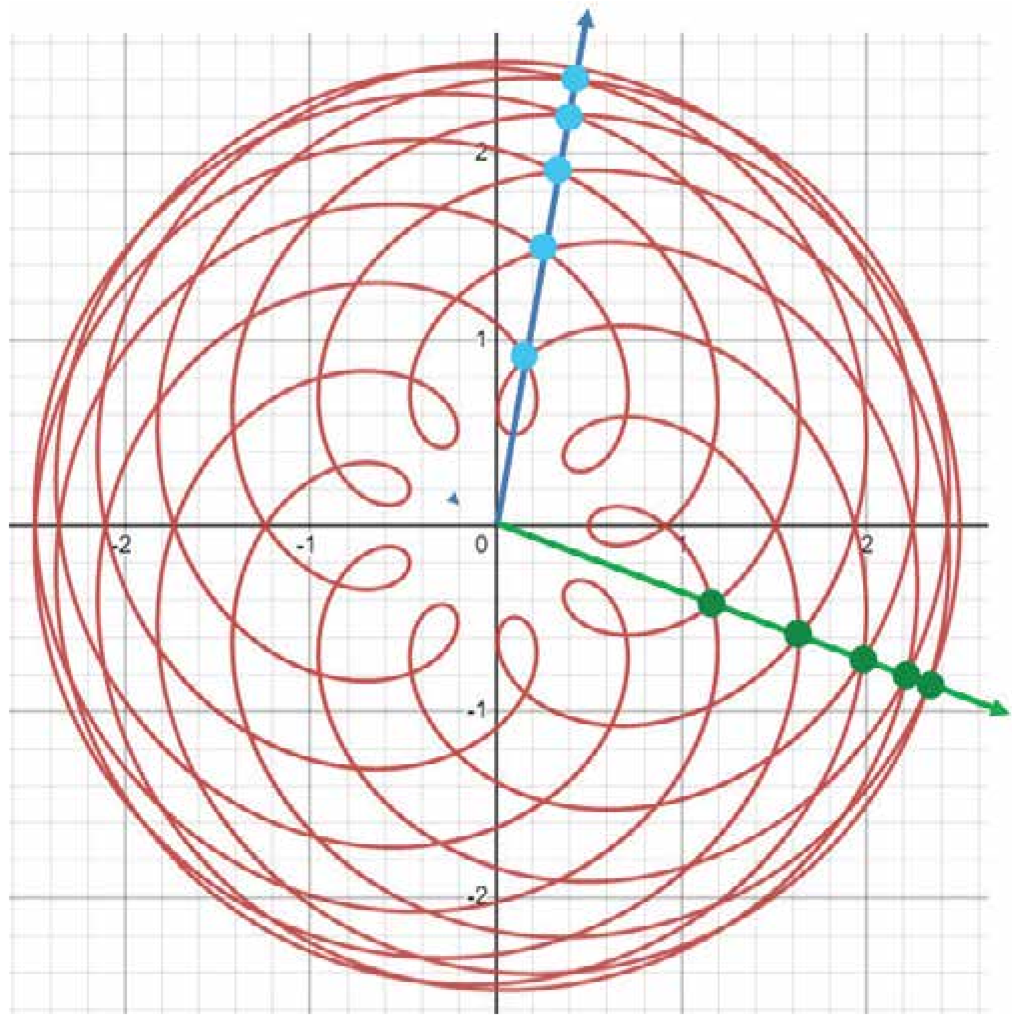


Figure 11. The graph for Mars intersects itself in 5 "layers" (marked by the 5 dots in blue and green colours). The intersections exactly align along radial lines of two types: (a) going through the centre of the retrograde loop and (b) going exactly between two neighbouring retrograde loops.

Appendix: Desmos graphing calculator

The Desmos online graphing calculator can be accessed at: https://www.desmos.com/calculator. The user interface is quite intuitive. One can either directly type the equation to be plotted in the box provided or use the inbuilt keyboard option. Settings are available to alter the appearance of the grid and of the plotted curve. Extensive documentation and a broad compilation of example graphs can also be accessed from the tool itself (see Figure A1).

In this article we have used the feature for plotting *parametric* curves. One can refer to the 'Parametric: Introduction' example in the Desmos example list to get started. The parametric form allows us to express the x and y coordinates as a function of the parameter t, i.e., $x = f(t)$ and $y = g(t)$. The coordinates of the curve may then be entered as $(f(t), g(t))$ in the box provided for the input equation. For example, the straight line equation $y = 3x$ can be plotted in parametric form as $(t, 3t)$ as shown in Figure A2. The default Desmos range of the parameter t is $0 \leq t \leq 1$, which can be changed as per our needs.

For this article, we need the coordinates of, for example, Mars (M) with the Earth (E) as the origin. Since both planets are assumed to start with the same (zero) phase, be in circular orbits with the sun at the center

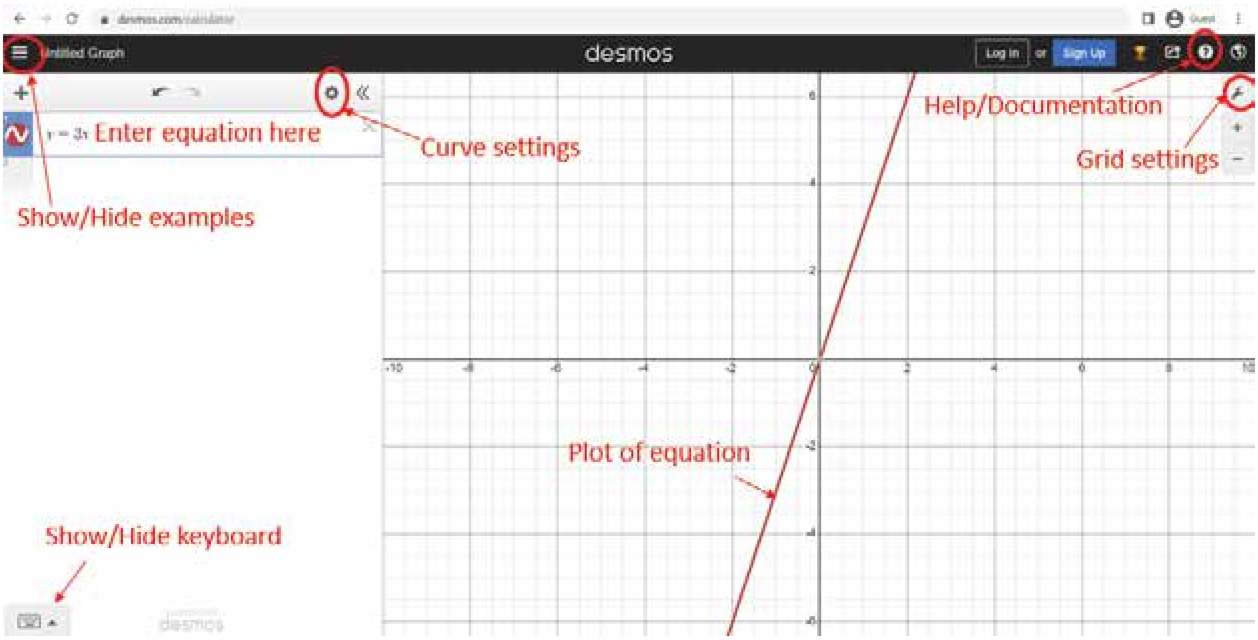


Figure A1. Desmos user interface.

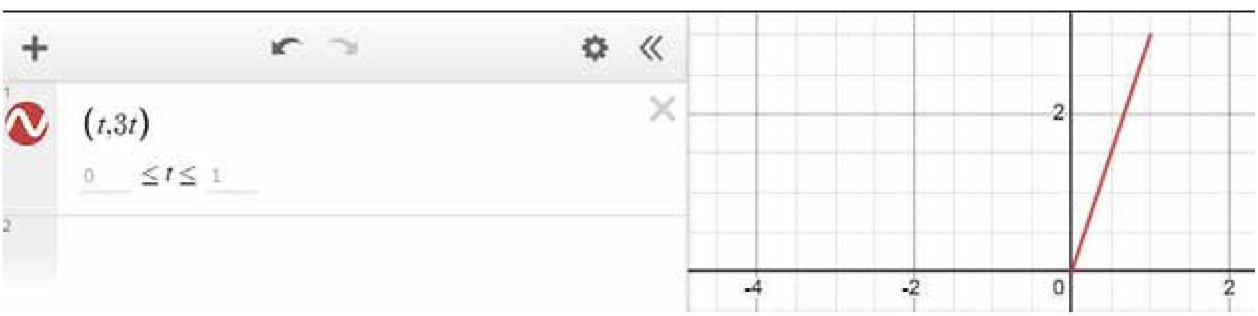


Figure A2. Parametric form of equation $y = 3x$ expressed as $(t, 3t)$ with $0 \leq t \leq 1$.

and revolve in the same direction, their coordinates, with the sun as the origin, at any time t, can be represented as shown in Figure A3.

With T_M and T_E being respectively the orbital periods of Mars and Earth, both θ_M and θ_E can be expressed in terms of time t. The coordinates of Mars w.r.t. Earth at any time t can then be written as:

$$x_{ME} = x_M - x_E = R_M \cos(2\pi t/T_M) - R_E \cos(2\pi t/T_E)$$

$$y_{ME} = y_M - y_E = R_M \sin(2\pi t/T_M) - R_E \sin(2\pi t/T_E)$$

Further, knowing that $R_M \approx 1.5R_E$, we can put $R_E = 1$ and $R_M = 1.5$. Similarly, knowing that $T_M \approx 1.9T_E$, we can put $T_E = 1$ and $T_M = 1.9$. Thus, we can write the coordinates for Mars in Earth's frame of reference in Desmos as:

$$(1.5\cos(2\pi t/1.9) - \cos(2\pi t), 1.5\sin(2\pi t/1.9) - \sin(2\pi t))$$

This generates the graph shown in Figure A4 since the range used for t is still set to the default. If we extend this range from 0 to $\text{LCM}(T_M, T_E) = 19$, we will get the curve for Mars as shown in Figure 4.

Similarly, the curve for Venus (V) shown in Figure 4 is generated using the coordinates:

$$(0.7\cos(2\pi t/0.6) - \cos(2\pi t), 0.7\sin(2\pi t/0.6) - \sin(2\pi t))$$

where t ranges from 0 to $\text{LCM}(T_V, T_E) = 3$.

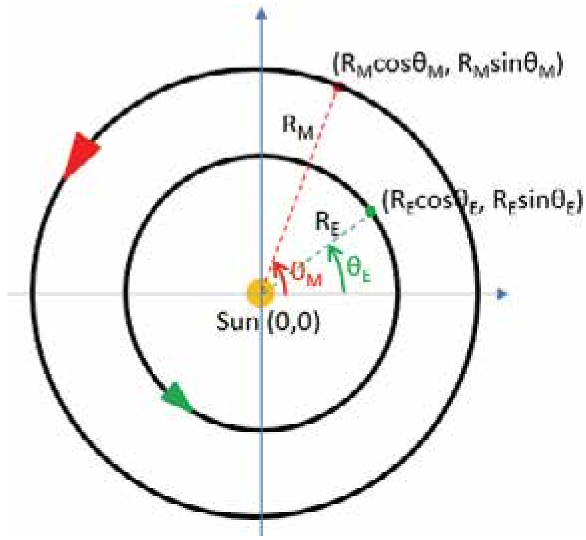


Figure A3. Coordinates of planets E and M with the Sun assumed to be at origin.

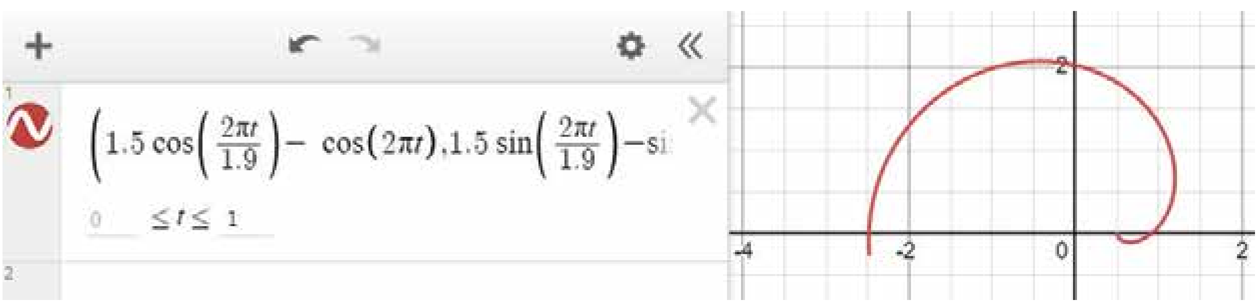


Figure A4. Plot of Mars w.r.t. Earth with default range for t.

To generate the curve for Figure 8, it was assumed that Earth had an initial phase of $\pi/2$. The coordinates of Venus w.r.t. Earth are then given by:

$$(0.7\cos(2\pi t/0.6) - \cos(2\pi t + \pi/2), 0.7\sin(2\pi t/0.6) - \sin(2\pi t + \pi/2))$$

For Figure 9, we have to assume that Mars and Venus revolve opposite to Earth's direction, hence their θ changes as $-t$ while Earth's changes with t. The coordinates for Mars and Venus w.r.t. Earth then respectively become:

$$(1.5\cos(2\pi(-t)/1.9) - \cos(2\pi t), 1.5\sin(2\pi(-t)/1.9) - \sin(2\pi t))$$
$$(0.7\cos(2\pi(-t)/0.6) - \cos(2\pi t), 0.7\sin(2\pi(-t)/0.6) - \sin(2\pi t))$$

The coordinates and the parameter ranges used to generate Figure 10 have already been provided in the main article (to enter them in Desmos, use t instead of θ).

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MAHIT WARHADPANDE a.k.a. the Jigyasu Juggler, retired after a 16-year career at Texas Instruments, Bengaluru, to pursue his interests at leisure. These include Mathematics and Juggling, often in combination (see http://jigyasujuggler.com/blog/). He may be contacted at jigyasujuggler@gmail.com

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ABHRONEEL GHOSH Currently pursuing a B.Stat at the Indian Statistical Institute, Kolkata, he was a student of Grade XII in Delhi Public School Bengaluru East at the time this work was done. He enjoys exploring various interesting and brain-addling topics across fields, with a particular interest in astronomy and science. He may be contacted at abhroneel.ghosh@gmail.com

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Written by Shaheen Suhail
Pursuing Masters from JK Institute of Mathematical Sciences, Srinagar.

Mathematicians like to talk about the beauty of mathematics. This beauty is seen in the harmony, patterns, and structures of numbers and forms – classical ideals of balance and symmetry. While experienced mathematicians can envision tangible representations of notations made on a page, mathematical beauty is not so well accessed by the non-mathematician.

Have you ever read a book, watched a movie, or played a video game that you really got into? Just everything about it is extremely engaging and you lose yourself in that world. Then you reach that one part, where there's some incredible scene, some unexpected twist, or some ingenious plot development that blows your mind, and you get excited. You run into the other room where you find a friend, and you excitedly tell them all about what just happened. And they sort of shrug and say, "Oh, that's cool I guess." And you're a bit disappointed that your friend isn't as excited as you are, but you know deep down that you've gotten so deeply involved with this story that you're seeing it from a much different perspective than everyone else.

Mathematics has a reading protocol all its own, and just as we learn to read literature, we should learn to read mathematics. Students need to learn how to read mathematics, in the same way they learn how to read a novel or a poem, listen to music, or view a painting.

Students of Mathematics must know that **"A three-line proof of a subtle theorem is the distillation of years of activity."** A Math article usually tells only a small piece of a much larger and longer story. The author usually spends months discovering things. At the end, he organizes it all into a story that covers up all the mistakes and presents the completed idea with a clean, neat flow.

Mathematics says a lot with a little. The student must participate. At every stage, he/she must decide whether or not the idea being presented is clear. Ask yourself these questions:

- Why is this idea true? Do I really believe it?
- Could I convince someone else that it is true?
- Why didn't the author use a different argument?
- Do I have a better argument or method of explaining the idea?
- Why didn't the author explain it the way that I understand it? Is my way wrong?
- Do I really get the idea? Am I missing something?
- Did this author miss a particular argument?
- If I can't understand the point, perhaps I can understand a similar but simpler idea? Which simpler idea?
- Is it really necessary to understand this idea?
- Can I accept this point without understanding the details of why it is true?
- Will my understanding of the whole story suffer from not understanding why the point is true?

Mathematicians often say that to understand something you must first read it, then write it down in your own words, then teach it to someone else. In a nutshell, besides the book, you need paper and a pen. You must do the exercises of the book, and convince yourself, convince a friend, and convince a sceptic.

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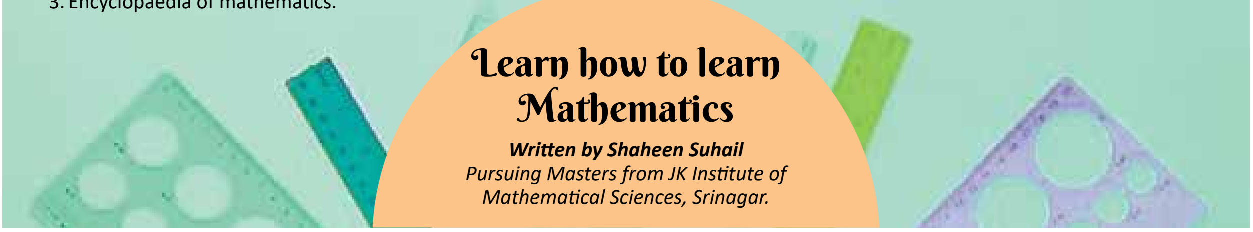

Review

Beware of the Devil in your Dreams!

Reviewed by Vishnu Lakshman

"Pure mathematics is, in its way, the poetry of logical ideas"

– Albert Einstein

As an eleven-year-old, I rarely read. I hardly ever found entertainment in reading books. I'd rather have been playing outside, climbing rocks or doing anything that didn't require me to sit and stare at words.

However, during this time, I was introduced to a book called the Number Devil by my school librarian. The book was the odd one out among the books on its shelf, tall, broad and had a peculiar devilish-looking man on the cover. I browsed through it and saw that it contained intriguing illustrations and text in a huge font. So I decided to read it.

I fell in love with the book the very first time I read it. And for someone who doesn't read books very often, I have read this book five times!

The book is about Robert, a ten-year-old boy who hates math and his treacherous math teacher. He doesn't understand what happens in class nor is his mind able to handle all the jumbled numbers in his head. The only absolute that he believes to be true is that he hates anything mathematical!

I fell in love with the book the very first time I read it because it was able to simplify mathematical concepts which were surely way beyond my level, such as exponents, irrational numbers and prime numbers. The advanced mathematical concepts became more straightforward but also more importantly, applicable. An example is the way the number devil introduces permutations and combinations through the different ways in which Robert's friends can be seated in class.

Keywords: Mathematics, children's books, number patterns, curiosity, exploration

I'm pretty sure I didn't understand all the concepts completely the first time I read it. For example, the sum to infinity concept was particularly hard to grasp. More often, though, I was just fascinated by results like when you start with half (½), divide by two repeatedly and then add them all together, the sum is equal to one! And in every one of the five times I read it, I understood a concept I hadn't been able to earlier, or was able to grasp more of the mathematics behind a result.

Along with being tired of math, Robert was also fed up of dreaming. He was always getting swallowed by a large fish, or it would be the bike or computer game that he never had that would haunt him. However, one night it was different. No getting swallowed by a fish, no bike, no computer - instead he meets a man proclaiming to be the 'number devil'. At first, he is slightly apprehensive, and moreover, can't stand the thought of getting homework in his dream! Once Robert trusts the number devil with his intentions, he quickly gets hooked on his catchy ways of teaching, such as using coconuts to introduce triangle numbers, or using rabbits to teach him the Fibonacci numbers. Together they make their way from the basics of maths like exponents and prime numbers, to more adventurous topics which Robert could have never dreamed of understanding just a short while back before meeting his new friend.

It is unreasonable and probably impossible to force someone, especially a young kid, to enjoy a subject they are having difficulty with. For mathematics, the reasons could be as simple as that it is too hard, or too abstract, or boring and repetitive, and therefore a child gradually loses interest. Teachers should read this book and possibly use it in their class as a way to introduce children to slightly more complex topics that could give them a break from what they are currently learning, because it could lead to a re-sparking of interest in the subject. They may also want to read chapters of the book aloud in class. The story-telling writing style is easy, and the plot is straightforward and easy to follow. Students will like the devil's devil-may-care attitude! The large page-filling illustrations are fascinating and feed into the imagination of a young child. The fun comes from simple ideas, like the number devil jumping onto trees and counting fruits to make a complex maths concept come true.

The first time I read this book was with my dad, which made a huge difference to the extent to which I understood the content. So if you are reading it when you're eleven or twelve, then I recommend you read it with an adult, whereas at an older age you can read it alone.

One criticism of the book is that certain chapters can get long, when many concepts surrounding the same topic are being covered (for example, the chapter on permutations and combinations). Smaller chapters where only one concept is being explained could make for an easier read. In addition, the explanation in certain chapters can get verbose. However, some could argue that this is better because, after all, Robert is eleven years old, so he is going to need some extra explanation!

The novelty of the concept of meeting a number devil in your dreams really intrigued me, especially the first time. I often wished I could have someone appear in *my* dreams, someone to help me in all the subjects I struggled with! And in fact, the story of Robert shows that we all could have struggles in any avenue of life, but there is always a number devil out there to help us with our problems and they may appear in the most unusual ways!

VISHNU LAKSHMAN is a 17-year-old studying at Centre for Learning in Bengaluru. His interests are varied - the environment, math and the outdoors (he can be outside and play all day). He enjoys reading non-fiction books, exploring new and exciting topics and ends up rereading treasures like *The Number Devil* too many times. He loves his time on long walks with his dog, Kivi, where they explore the wilderness around home together and lose track of all time.

Review

Ramanujan: From Zero to Infinity

Reviewed by Anand Mathew Kurien

The book titled 'Ramanujan: From Zero to Infinity' is written by Arundhati Venkatesh and illustrated by Priya Kuriyan. The main character of this children's storybook is the Indian mathematician Srinivasa Ramanujan (1887 – 1920), as a precocious ten-year-old. While the story is a work of historical fiction, many of the details used in it were inspired by true events of Ramanujan's life, as mentioned in the 'Afterword' of the book.

Introduction

The first thing that I do when I get a book is scan through the contents page. To my surprise, this book did not have one! The most probable reason for the missing page was clearer when I browsed through the pages curiously. I noticed that while there were twenty-three short chapters with titles, there were no direct chapter numbers. Instead, each one was written as a mathematical expression that would give the correct chapter number if solved. Even more interestingly, the expression itself was written in a surprising and mathematically aesthetic manner which I will let the reader explore. These simple mathematical puzzles at the beginning of each chapter should get children curious enough to solve them themselves or could be used by teachers in classrooms as a fun learning exercise.

Many numerical relations are introduced in the first chapter as relations that Ramanujan had thought of while having a bath! These are simple relations that students above the ages of 11 or 12 will be able to understand. For example,

$$100^2 + 75^2 = 25^3 = 120^2 + 35^2 = 117^2 + 44^2.$$

Keywords: Ramanujan, childhood stories, achievements, social issues

This is one of many such numerical relationships given in the book. Surely, none of *us* think like this! It may even prompt the question "Who thinks like that?!" or to the less mathematically inclined, "So what?" The author intends to give us a glimpse into the deep relationship that Ramanujan had with numbers and his fondness for playing with them. It reminds me of the quote by J. E. Littlewood, one of Ramanujan's colleagues from Trinity College in Cambridge, *"Every positive integer is one of Ramanujan's personal friends"* [1]. The genre of historical fiction must attempt to make extrapolations of historical facts into fiction. This book captures, rather well, the essence of how Ramanujan's mind would have worked while he was ten years old.

The Story

The book begins by introducing us to Ramanujan, his mother, and their home, briefly touching upon his extended family in the first chapter. In the following chapters, we meet his friends – a familiar, fun-loving, noisy bunch – and his school. Ramanujan and his friends are all in the same class, the first form. The story follows the young friends and describes the events surrounding their lives at home and in school, the primary theme being a mathematics contest that is to be conducted that year between two teams in their class. As a children's storybook, it captures themes that would interest children. The light-hearted fun that ten, and eleven-year-old boys have is well depicted, as they pull each other's legs, say silly things, laugh uproariously, tease each other, and on occasion are mean to one another. Their interests, fears, and frustrations come through, bringing the characters to life. Ramanujan is portrayed as a child who lives almost perpetually in his head with mathematical and numerical ideas, speaking mostly only to share his hilarious – often bilingual – puns, or puzzles with his friends. This is almost always followed by them making fun of him or mentioning how they are tired of his obsessive mathematical nagging.

The relationship between Ramanujan and his mother is portrayed in vivid detail in the book and appears in several chapters. His mother knows that her 'Chinnaswami', as she calls him, is uniquely gifted, and puts up with his idiosyncrasies and does her best to provide for him. She takes immense pride in him and attributes his gifts to the Gods.

The early chapters introduce the characters and the context. Subsequently, the author alternates the narrative between the main theme and sub-themes that cover social issues and short incidents that take place in the neighbourhood. The final few chapters bring a logical end to the contest with a very heart-warming finish. As mentioned earlier, the 'Afterword' at the end of the book gives context to these subplots, sifting fact from fiction and mentioning the facts that were used in the story, but under a different context. The book also dedicates a couple of pages, in the end, to providing solutions to the mathematical puzzles mentioned in the book.

Use of Language and Style

From a stylistic point of view, the author's descriptions of events and places are so vivid that it takes little effort to imagine the setting, the surroundings, the people, and their mood in the scene. The characters' spoken English, coloured with the influence of Tamizh, is true to life. The ten illustrations in the book add to the flavour and contain accurate details as in the textual description. The book also introduces the reader to many customs, traditions, and a host of traditional names of Tamizh Naadu – of people, places, streets, temples and food, with one chapter containing a rich and delightful description of sweets and savouries.

One of the moments that I enjoyed is in Chapter 14, where one of Ramanujan's friends speaks with his nose pinched closed. The author describes how he speaks so lucidly that I found myself reading it out loud, with my nose closed, to check if it really sounded like that. It did!

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[1] *https://en.wikiquote.org/wiki/John_Edensor_Littlewood*

The use of vocabulary is such that even adults will be able to enjoy the book while teenagers will have good fun reading and learning some new words that may not yet be part of their repertoire. Overall, there is drama – sometimes to a slightly unrealistic degree, but well within the confines of artistic freedom – a pinch of suspense, and lots of values such as friendship and forgiveness, which combine to make it an enjoyable work of historical fiction.

The 'Uncommon' Sense

My favourite moment in the book is when during one of the competitions in the math contest, in Chapter 14, Ramanujan comes up with a brilliant solution while his friend comes up with an offbeat 'uncommon sense' solution which leaves Ramanujan and his friends dumbstruck. Ramanujan says, "I hadn't thought of that!" Amid all the mathematical brilliance of Ramanujan displayed in the book, the author makes it a point to emphasize that one does not always need to be brilliant to come up with a solution to a problem!

This offbeat solution reminded me of *"Angels on the Head of a Pin: A Modern Parable"* by Alexander Calandra[2] where a physics question is posed: "Show how it is possible to determine the height of a tall building with the aid of a barometer." While there are many creative solutions to the problem, the simplest and most 'uncommon' sense one is quoted as – *Probably the best is to take the barometer to the basement and knock on the superintendent's door. When the superintendent answers, you speak to him as follows: "Mr. Superintendent, here I have a fine barometer. If you tell me the height of this building, I will give you this barometer."*

Social Issues Addressed in the Book

The author has dedicated three chapters to bring to light prominent social concerns of the time, that remain relevant to Indian society even today, including patriarchy and injustice against women and girl children. Considering that the story is set over a century ago, it is a rather clear indicator of how we, as a society, are over a century old in our mindsets when it comes to certain aspects of our social outlook. These three chapters intertwine social issues with the story very naturally. One of these (Chapter 21) was particularly moving, depicting with sincerity the agony and anguish of a ten-year-old boy. The author also brings out with clarity the hypocritical inconsistencies in the behaviour of adults portraying how they don't live up to the standards that they expect their children or others to live up to. The most important takeaway here is that these social issues have not been resolved in the intervening century and remain predominant in our society even today.

Ramanujan's Genius

Srinivasa Ramanujan's story is unconventional and one of a kind. He was self-taught and had a unique approach to mathematics. So profound were his abilities that he became one of the youngest Fellows of the Royal Society and the first Indian to be elected a Fellow of Trinity College, Cambridge. He could see connections between numbers that no one else could see, which is well depicted in the book's first chapter. In early 1913 when Ramanujan reached out to mathematician G.H. Hardy in Cambridge through letters with a long list of his mathematical discoveries, Hardy immediately recognized his genius and invited him to come to England to work with him. J.E. Littlewood said that the discoveries must be true as no one would have the imagination to invent them, and Hardy stated that with just a single look he knew that it could only be written by a mathematician of the highest class and compared him with other mathematical geniuses such as Euler and Jacobi.

Ramanujan had deep intuition and is considered the 'greatest intuitionist' that the mathematical world has seen. His ability to come up with solutions to conjectures without proof was

[2] http://www.rbs0.com/baromete.htm

prolific given that he only lived for 32 years. It was not a few dozen or a few hundred solutions to conjectures that he came up with, but as the Afterword of this book states, *"it took some of the world's best mathematicians years to make sense of the 3000 theorems, without proofs, Ramanujan left behind in his notebooks".* How he did this, nobody knows. Ramanujan himself credited his deep insights to his family Goddess, Naamagiri.

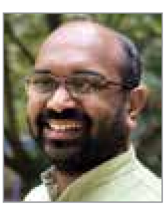
For those who would like to learn more about the life and work of Ramanujan, Robert Kanigel's book, *"The Man Who Knew Infinity"* is highly recommended. This book was reviewed in At Right Angles (Volume 2, No. 3, November 2013).

Errata

One typographic error was spotted on page 91 (*'amin'* instead of *'admin'*). There was also a mathematical error on page 123 ('sin45° = 1' instead of 'tan45° = 1') which, given the number of mathematical ideas and equations used in the book, is surely a slip of the pen.

Conclusion

A substantial amount of research, thought, and creativity has gone into bringing out the breadth of themes covered in this book. Yet the author makes it seem so simple in this small, 160-page children's book with vivid illustrations. The author, in the process of telling a story of friendship, intertwines language, mathematical riddles, puns, puzzles, fun stories, humour, and important social issues in this book. It also introduces a reader, unfamiliar with Tamizh Naadu, to its traditions, culture, and food, along with a host of traditional names. All of this is done while the author gives us a glimpse into the mind of the strange and rare genius phenomenon of nature called Srinivasa Ramanujan! It was surely no easy task to maintain the graceful balance of all the themes covered in the book and ensure the book is also fun to read, but the author has managed it masterfully.



ANAND MATHEW KURIEN is a faculty member in the Teacher Education group at Azim Premji University in Bengaluru. Before this role, he taught physics and music in schools in Chennai for over a decade. Though his core passions are conveying physics and music to an uninitiated audience and making video content for online platforms, he continues to be an ardent mathematics enthusiast. He uses his channel www.youtube.com/@anandmkurien to share ideas. He may be contacted at anand.kurien@apu.edu.in

Problem Corner

Products of Sums of All Co-Primes from 1 till Any Given Number

ANAND PRAKASH

We are familiar with the following problem: *given a positive integer n, find the sum g(n) of all the positive integers from 1 to n that are coprime to n.* The formula for the required sum is simple:

$$g(n) = \begin{cases} 1, & \text{for } n = 1, \\ \frac{1}{2} \times n \times \Phi(n), & \text{for } n > 1, \end{cases} \tag{1}$$

where $\Phi(n)$ is the Euler totient function (it counts the positive integers from 1 to n that are coprime to n). The reason for formula (1) should be clear: for $n > 1$, if $1 \leq a < n$ and we have $\gcd(a, n) = 1$, then we also have $1 \leq n - a < n$ and $\gcd(n - a, n) = 1$. This means that the positive integers between 1 and n that are coprime to n can be paired with one another such that the sum of the numbers in each pair is n. Hence the stated result.

In this article, I explore the following problem:

Problem. Find the product $f(n)$ of all such sums up to a given positive integer n.

In other words, we want the formula for the following in terms of the prime factorization of n.

$$f(n) := \prod_{k=1}^{n} g(k). \tag{2}$$

Keywords: Prime number, coprime, greatest common divisor (gcd), Euler's totient function

Example. Let $n = 10$. The sums of all the co-primes for $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ are (respectively) 1, 1, $1+2$, $1+3$, $1+2+3+4$, $1+5$, $1+2+3+4+5+6$, $1+3+5+7$, $1+2+4+5+7+8$, $1+3+7+9$, i.e., $1, 1, 3, 4, 10, 6, 21, 16, 27, 20$, respectively. Hence:

$$\begin{aligned} f(10) &= \text{product of the sums of the co-primes from 1 to 10} \\ &= 1 \times 1 \times 3 \times 4 \times 10 \times 6 \times 21 \times 16 \times 27 \times 20 \\ &= 2^{10} \times 3^6 \times 5^2 \times 7. \end{aligned}$$

It is easy to obtain $f(n)$ for small values of n. However, it takes too much time for large numbers. We seek a usable formula for $f(n)$.

Theorem. For $n \geq 9$ we have:

$$f(n) = \frac{1}{2^{n-1}} \times (n!)^2 \times \left(\frac{\prod_{k=2}^{k=n} \prod_{p|k} (p-1)}{\prod_{k=2}^{k=n} \prod_{p|k} p} \right). \tag{3}$$

Proof of theorem. We have $\Phi(1) = 1$, and for $n > 1$,

$$\Phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{4}$$

Hence, from (1), for $n > 1$:

$$g(n) = \frac{1}{2} \times n \times \Phi(n) = \frac{1}{2} \times n^2 \times \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{5}$$

Hence:

$$\begin{aligned} f(n) &= \prod_{k=1}^{k=n} g(k) \\ &= \prod_{k=2}^{k=n} \frac{1}{2} \times k^2 \times \prod_{p|k} \left(1 - \frac{1}{p}\right) \\ &= \frac{1}{2^{n-1}} \times (n!)^2 \times \prod_{k=2}^{k=n} \prod_{p|k} \left(1 - \frac{1}{p}\right) \\ &= \frac{1}{2^{n-1}} \times (n!)^2 \times \frac{\prod_{k=2}^{k=n} \prod_{p|k} (p-1)}{\prod_{k=2}^{k=n} \prod_{p|k} p}. \end{aligned} \tag{6}$$

Using (6), we may compute the values of f. See Table 1 for a list of some values of the function.

n	f(n)
1	1
2	1
3	3
4	$2^2 \times 3$
5	$2^3 \times 3 \times 5$
6	$2^4 \times 3 \times 5$
7	$2^4 \times 3^3 \times 5 \times 7$
8	$2^8 \times 3^3 \times 5 \times 7$
9	$2^8 \times 3^6 \times 5 \times 7$
10	$2^{10} \times 3^6 \times 5^2 \times 7$
11	$2^{10} \times 3^6 \times 5^3 \times 7 \times 11$
12	$2^{13} \times 3^7 \times 5^3 \times 7 \times 11$
13	$2^{14} \times 3^8 \times 5^3 \times 7 \times 11 \times 13$
14	$2^{15} \times 3^9 \times 5^3 \times 7^2 \times 11 \times 13$
15	$2^{17} \times 3^{10} \times 5^4 \times 7^2 \times 11 \times 13$

Table 1. Values of $f(n)$ for $1 \leq n \leq 15$, expressed in terms of the primes $\leq n$

To find an explicit formula for f, we need to work out the power to which each prime $p \leq n$ occurs in the above quantity. We look at each term in (6) separately.

Let $p_1 = 2, p_2 = 3, p_3, \ldots, p_s$ be the primes not exceeding n. (So there are s prime not exceeding n.)

- For $q \in \{p_1, p_2, \ldots, p_s\}$, the power to which q divides $n!$ is

$$\left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{q^2} \right\rfloor + \left\lfloor \frac{n}{q^3} \right\rfloor + \cdots .$$

  This follows from Legendre's formula [1] which gives an expression for the exponent of the largest power of a prime q that divides $n!$.

  Hence the power to which q divides $(n!)^2$ is

$$2\left\lfloor \frac{n}{q} \right\rfloor + 2\left\lfloor \frac{n}{q^2} \right\rfloor + 2\left\lfloor \frac{n}{q^3} \right\rfloor + \cdots . \qquad (7)$$

- Next, the power to which q divides $\prod_{k=1}^{k=n} \prod_{p|k} p$ is $\left\lfloor \frac{n}{q} \right\rfloor$.

  To see why, we simply count the number of occurrences of q in the expression $\prod_{k=1}^{k=n} \prod_{p|k} p$.

- Hence, the power to which q divides $\frac{(n!)^2}{\prod_{k=1}^{k=n} \prod_{p|k} p}$ is

$$\left\lfloor \frac{n}{q} \right\rfloor + 2\left\lfloor \frac{n}{q^2} \right\rfloor + 2\left\lfloor \frac{n}{q^3} \right\rfloor + 2\left\lfloor \frac{n}{q^4} \right\rfloor + \cdots. \tag{8}$$

- Now we must compute the power to which an arbitrary prime number q divides the quantity

$$\prod_{k=1}^{k=n} \prod_{p|k} (p-1). \tag{9}$$

  Further investigation will have to be undertaken to give an explicit formula for (9). The formula may be somewhat complex.

References

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ANAND PRAKASH runs a small garment shop at Kesariya village in the state of Bihar. He has a keen interest in number theory and recreational mathematics and has published many papers in international journals in these fields. He also has a deep interest in classical Indian music as well as cooking. In addition, he has written a large number of poems in Hindi. He may be contacted at prakashanand805@gmail.com.

Problem Corner

A Divisibility Chain Problem

K M SASTRY

Notation. Let u and v be positive integers. By $u \mid v$ we mean "u is a divisor of v" and by $u \nmid v$ we mean: "u is not a divisor of v". For example, $4 \mid 12$, but $5 \nmid 12$.

Problem

Let a and b be positive integers such that

$$a \mid b^2, \quad b^2 \mid a^3, \quad a^3 \mid b^4, \quad b^4 \mid a^5, \quad a^5 \mid b^6, \quad \ldots . \qquad (1)$$

Prove that $a = b$.

Solution

We make use of the following auxiliary result (such a preliminary step is also called a 'lemma'):

Lemma. *Let m and n be positive integers such that*

$$m \le 2n \le 3m \le 4n \le 5m \le 6n \le \cdots . \qquad (2)$$

Then $m = n$.

Proof of lemma. The inequalities $m \le 2n \le 3m \le 4n \le \cdots$ imply that $(2k-1)m \le 2kn$ for every positive integer k. Hence we have

$$\frac{m}{n} \le 1 + \frac{1}{2k-1} \quad \text{for every positive integer } k.$$

Since

$$\frac{1}{2k-1} \to 0 \quad \text{as } k \to \infty,$$

it follows that

$$\frac{m}{n} \le 1,$$

and so $m \le n$.

Keywords: Prime number, divisible, lemma

The same inequalities also imply that $2kn \leq (2k+1)m$ for every positive integer k. Hence we have

$$\frac{n}{m} \leq 1 + \frac{1}{2k} \quad \text{for every positive integer } k.$$

Reasoning the same way as we did earlier, we conclude that

$$\frac{n}{m} \leq 1,$$

and so $n \leq m$.

Since $m \leq n$ and $n \leq m$, it follows that $m = n$. □

Solution of problem. The divisibility conditions imply that for any prime number p, if $p \mid a$ then $p \mid b$ as well; and in the same way, if $p \mid b$ then $p \mid a$ as well. Hence a and b are divisible by exactly the same set of primes.

Let p be any prime number dividing a, b. Let p^u be the highest power of p that divides a, and let p^v be the highest power of p that divides b. That is, we have $p^u \mid a$ but $p^{u+1} \nmid a$; and $p^v \mid b$ but $p^{v+1} \nmid b$. Here $u > 0$ and $v > 0$. Then from the given conditions we argue as follows:

- $a \mid b^2$, so $p^u \mid p^{2v}$, so $u \leq 2v$;
- $b^2 \mid a^3$, so $p^{2v} \mid p^{3u}$, so $2v \leq 3u$;
- $a^3 \mid b^4$, so $p^{3u} \mid p^{4v}$, so $3u \leq 4v$;
- $b^4 \mid a^5$, so $p^{4v} \mid p^{5u}$, so $4v \leq 5u$;

and so on. Hence:

$$u \leq 2v \leq 3u \leq 4v \leq 5u \leq \cdots.$$

Invoking the lemma proved above, we deduce that $u = v$. So the highest power of p that divides a is identical to the highest power of p that divides b.

Since the same is true for every prime number that divides a and b, it follows that a and b have identical prime factorization. This implies that $a = b$. □

Student Corner

Divergence and Convergence of Two Closely Related Infinite Series

TOYESH PRAKASH SHARMA

There are two kinds of infinite series: those that are *divergent*, and those that are *convergent*. The sum of the terms of a divergent series increases without limit as more terms are taken; the sum of the terms of a convergent series approaches a definite number as more terms are taken. A well-known divergent series is the *harmonic series*, which is the sum of the reciprocals of the positive integers, $\sum \frac{1}{n}$. It was first proved to be divergent by Nicole Oresme (1323-1382). Others gave proofs of the same statement, e.g., Cauchy, Augustus De Morgan, Johann Bernoulli, Euler, etc. Here I put forward my way for proving the divergence of $\sum_{n=1}^{\infty} 1/n$.

Theorem: $\sum_{n=1}^{\infty} 1/n$ is divergent.

Proof: Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, it follows that if $x > 0$, then $e^x \geq 1 + x$ or $x \geq \ln(1 + x)$. Substituting $x = 1/n$ where n is a positive integer, we get

$$\frac{1}{n} \geq \ln\left(1 + \frac{1}{n}\right) = \ln(n + 1) - \ln n.$$

Keywords: Series, convergence, divergence

Taking summation on both sides for $n = 1, 2, \ldots, k$, we get

$$\sum_{n=1}^{k} \frac{1}{n} > (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \cdots + (\ln (k+1) - \ln k)$$

$$\Rightarrow \sum_{n=1}^{k} \frac{1}{n} > \ln (1 + k) .$$

Taking limits on both sides, $k \longrightarrow \infty$, we get

$$\sum_{n=1}^{\infty} \frac{1}{n} > \lim_{k \to \infty} \ln (1 + k) = \infty.$$

Hence $\sum_{n=1}^{\infty} 1/n$ is divergent.

In the series considered above, all the terms are positive. Another kind of series is one where the terms alternate in sign; they are called *alternating series*. We now examine a well-known alternating series closely related to the harmonic series: $\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$.

Theorem: $\sum_{n=1}^{\infty} (-1)^{n-1}/n$ is convergent.

Proof: Consider the following sequence of inequalities which are all clearly true:

$$\frac{1}{3} - \frac{1}{4} < \frac{1}{2} - \frac{1}{3}$$
$$\frac{1}{5} - \frac{1}{6} < \frac{1}{3} - \frac{1}{4}$$
$$\frac{1}{7} - \frac{1}{8} < \frac{1}{4} - \frac{1}{5}$$
$$\ldots\ldots\ldots$$

We also have, trivially:

$$1 - \frac{1}{2} = 1 - \frac{1}{2}.$$

Adding the corresponding sides of all these statements, we get

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \ldots$$
$$< \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots,$$

which yields, after simplifying the expression on the right side,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots < 1.$$

It follows that the sum of this alternating series is bounded above by 1. By converting the series into a sum of positive terms as follows,

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\ldots=\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\frac{1}{56}+\ldots,$$

we conclude that the series is convergent. (Mathematically, it is more accurate to say that the series is "conditionally convergent.")

TOYESH PRAKASH SHARMA got interested in Science, Mathematics and Literature when he was in 9^{th} standard. He passed 10^{th} and 12^{th} from St. C.F. Andrews School, Agra. When he was in 11^{th} standard, he got interested in doing Research in mathematics and till now many of his works have found place in different journals such as *Mathematical Gazette*, *Crux Mathematicorum*, *Parabola*, *AMJ*, *Pentagon*, *Octagon*, *At Right Angles*, *Fibonacci Quarterly*, *Mathematical Reflections*, etc. Currently he is doing his undergraduation in Physics and Mathematics from Agra College, Agra, India. He may be contacted at toyeshprakash@gmail.com.

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1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
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7. Provide a compact list of references, with short recommendations.
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9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
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11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
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14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
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Printed and Published by Manoj P on behalf of Azim Premji Foundation for Development;
Printed at Suprabha ColorGrafix (P) Ltd., No. 10, 11, 11-A, J.C. Industrial Area, Yelachenahalli, Kanakapura Road, Bengaluru 560062.
Published at Azim Premji University, Survey No. 66, Burugunte Village, Bikkanahalli Main Road, Sarjapura, Bengaluru – 562 125;
Editor: Shailesh Shirali

A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

Foundational topics include, but are not limited to, the following:

- Number systems, patterns and operations
- Fractions, ratios and decimals
- Proportional reasoning
- Integers
- Bridging Arithmetic-Algebra
- Geometry
- Measurement and Mensuration
- Data Handling
- Probability

We invite articles from teachers, teacher educators and others that are helpful in designing and implementing effective instruction. We strongly encourage submissions that draw directly on experiences of teaching. This is an opportunity to share your successful teaching episodes with AtRiA readers, and to reflect on what might have made them successful. We are also looking for articles that strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

Articles in this section may address key questions such as -

- What challenges did your students face while learning these fundamental mathematical topics?
- What approaches that you used were successful?
- What preparations, in terms of knowing mathematics, enacting the tasks and analysing students work were needed for effective instruction?
- What contexts, representations, models did you use that facilitated meaning making by your students?

Send in your articles to
AtRiA.editor@apu.edu.in

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The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

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FORM IV

1. **Place of publication**: Azim Premji University Survey No. 66, Burugunte Village, Bikkanahalli, Main Road, Sarjapura, Bengaluru – 562125
2. **Periodicity of its publication:** Triannual
3. **Printer's Name:** Manoj P
   Nationality: Indian
   Address: Azim Premji University Survey No. 66, Burugunte Village, Bikkanahalli, Main Road, Sarjapura, Bengaluru – 562125
4. **Publisher's Name:** Manoj P
   Nationality: Indian
   Address: Azim Premji University Survey No. 66, Burugunte Village, Bikkanahalli, Main Road, Sarjapura, Bengaluru – 562125
5. **Editor's Name:** Shailesh Shirali
   Nationality: Indian
   Address: Community Mathematics Centre, Rishi Valley School, Rishi Valley – 517352, Madanapalle, A.P
6. **Name of the owner**: Azim Premji Foundation for Development
   Address: #134 Doddakannelli, Next to Wipro Corporate Office, Sarjapur Road, Bangalore – 560035

I, Manoj P, hereby declare that the particulars given above are true to the best of my knowledge and belief.

AZIM PREMJI UNIVERSITY BENGALURU

Date: 1st March 2023

Signature of Publisher

The Closing Bracket . . .

The Gubbachi Learning Community was registered as a Section 8, not-for-profit company, on 7 September, 2015. Their interventions focus on both quality education programmes for marginalised children and community empowerment programmes for poor migrant communities in Bangalore City. The education programmes are in partnership with the Department of Education (DoE), Govt of Karnataka (GoK) and are focused on:

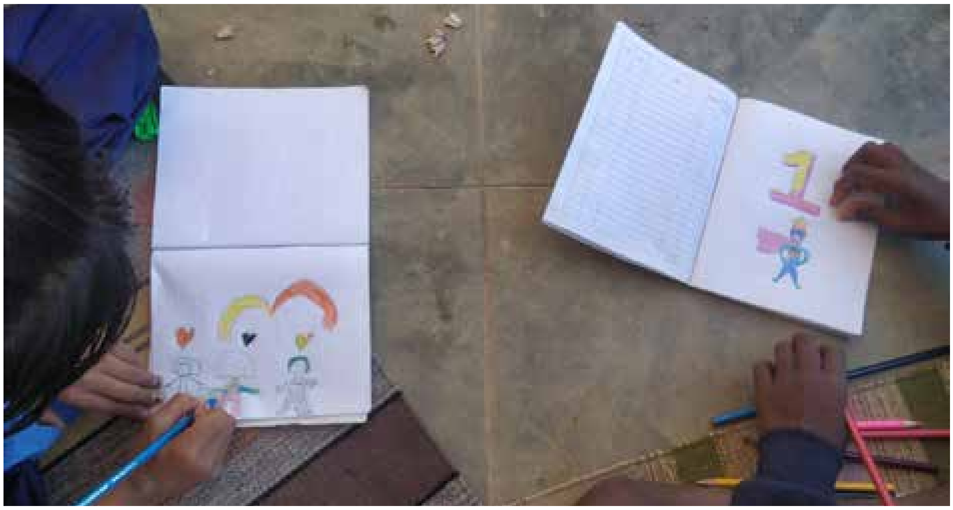
- Creating access and bridging the learning gaps for out-of-school children so they can join mainstream government schools or work through open schooling for their 10th grade attainment, and
- Delivering high quality foundational learning in grades 1,2 and 3 in government schools following the flagship Nali Kali (Multi-Grade-Multi-Level) pedagogy of the Department of Education, Government of Karnataka.

The community empowerment programmes are focused on creating awareness of access to welfare schemes and affordable public health in marginalised communities; good hygiene and sanitation; on social issues like alcoholism, domestic violence, child labour and child marriage.

Unstructured counting gets integrated with identifying the types of trees in and around the school, looking at shapes of the leaves and classifying them.

What does a mathematics classroom look like in this community? At Gubbachi, learning is squarely centred on the learner i.e., the context and age group of the child. For example, in the Nali Kali classrooms (ages 6-8 years), the classroom processes are supported by a stated curriculum and methodology. The children are in similar zones of development with the language of transaction being Kannada. In a bridge classroom, where the children are also in multi-age multi-level classrooms, differences in age and language backgrounds could be huge. There could be a 12 year old Hindi speaking child who has entered the classroom for the first time along with an eight year old Kannada speaking child. Here the teacher uses trans-languaging frequently, i.e., along with Kannada there are key words in Hindi as well or sometimes she only uses Hindi/Telugu. This is frequently evident in children who were out of school till the age of 11-12 years and from non-Kannada backgrounds and are now working towards 10th grade through open schooling. The trans-languaging of the teacher straddles between Hindi, English, Telugu, often within the same sentence.

Mathematical thinking is also mediated through the specialised language of mathematics. The joy of facilitating learning in these multi-language, multi-level, multi-age groups are the complexities straddling language and content of mathematics.



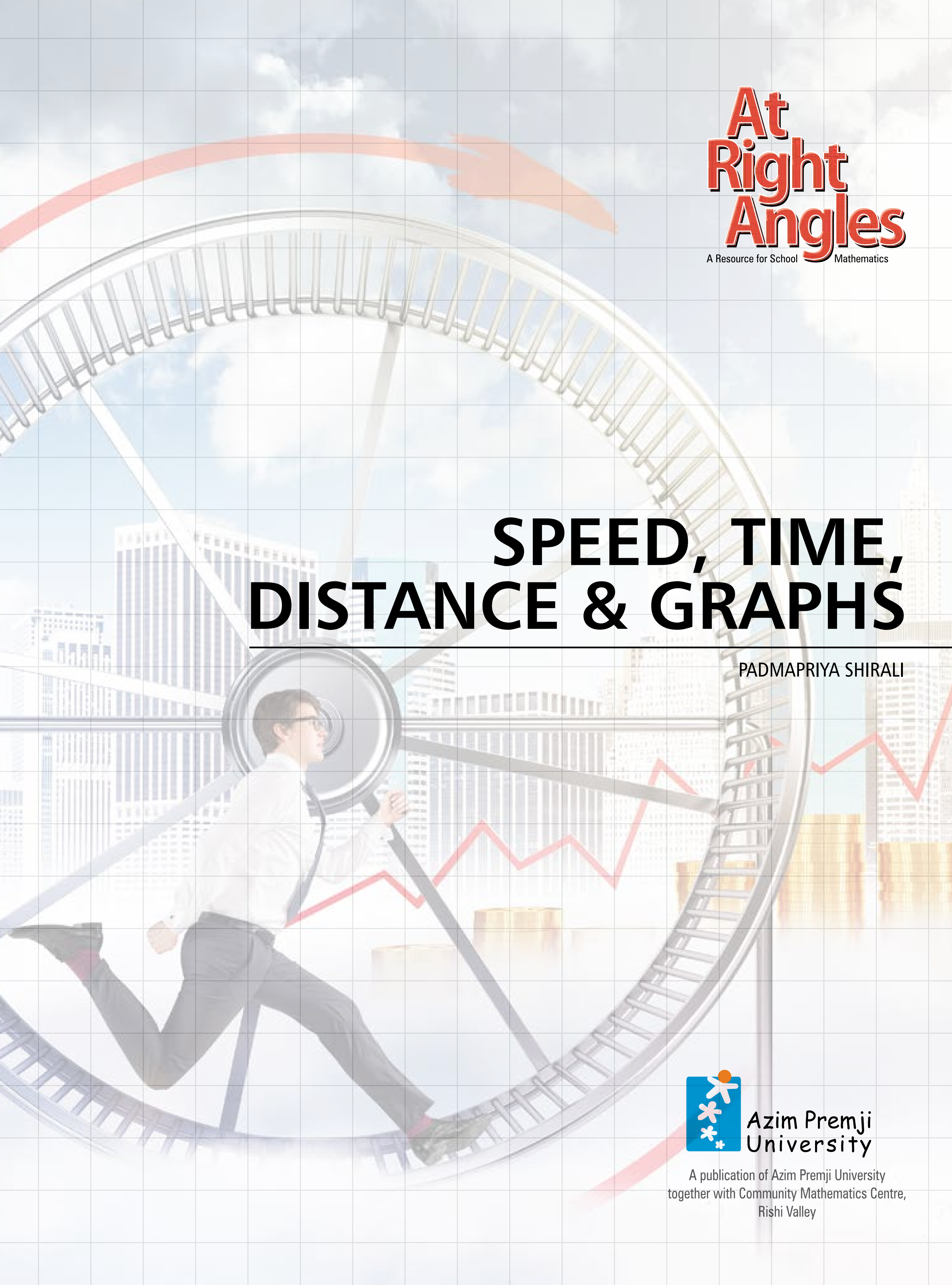
Making my own number book because that's the only way to own learning.

Somya Nand, class of 2014, M.A. Ed, Azim Premji University, who is one of the co-founders of this community writes:

> *Just the other day 10–12-year-olds, first time learners, who had entered a classroom for the first time in March 2022, were discussing the large number of stones on the school field, because of which children get hurt. We were collectively wondering if we could do something to reduce the pain of the children. While the mathematical objective of the class was to work on the idea of quantity, we were hoping to sow a seed of empathy. The children suggested collecting the stones from the field and putting them away in a separate space. We also wondered how many stones we will be able to collect in one class. Someone suggested 100. Now the class was hugely focused on solving a problem – the problem of getting hurt-which they were also a part of. And by the way, I was working on unstructured counting with children. Listening to articulation of numbers, observing team-work when one child had a few less than 100 and another helping with the balance quantity, learners building a sense of quantity. Often children counting in Hindi up to a number then taking forward in English or starting in English and then continuing in Hindi. Or they might have gotten an opportunity to count in Hindi up to 25-30 in the home context and then when they got an opportunity to be in a formal learning space at Gubbachi, they learnt to count on in English. Or younger children who got exposed to counting in English first and then picked up counting in the home language. Or genuinely thinking the name "seventy" means 17 so counting out 17 stones and saying "seventy". And then wondering aloud how "seventy" comes again after 69.*
>
> *Recently, we were wondering how prices of uniforms were negotiated at Gubbachi. This was with a group of learners who are doing grade 7 level math but at the same time writing grade 10 level exams for computers. We were discussing uniform prices because for the first time children had received comfortable, colourful t-shirts and track pants. They were on an emotional high. It just seemed right to engage with the idea of percentages. Doing comparisons of absolute values and then percentages came truly naturally. For me it was an "ah-ha" teacher moment. Pre-requisite to learning – the learner has to have a valid reason to learn.*

Bertrand Russel once said, "Even if the open windows of science at first make us shiver after the cosy indoor warmth of traditional humanizing myths, in the end, the fresh air brings vigour and the great spaces have a splendour of their own." How poignantly beautiful that the Gubbachi community is doing exactly this, opening the world for children, who want to and can fly, if they are given the opportunity to learn.

At Right Angles
A Resource for School Mathematics



SPEED, TIME, DISTANCE & GRAPHS

PADMAPRIYA SHIRALI

Azim Premji University
A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

SPEED, TIME, DISTANCE AND GRAPHS

In my experience of teaching problems related to speed, time and distance in grades 8 and 9, I have often found students using formulae with insufficient understanding of the relationship between these three variables. As the problems get more complex, involving movement in opposite directions, against a current, winds, etc., many resort to the usage of formulae and methods which rely on memory rather than a clear understanding born of experience. It does seem puzzling that something that is experienced in everyday life poses a challenge when expressed in an algebraic form.

Students have an intuitive understanding of the speed concept through travel experiences. More and more families even in distant, rural areas use vehicles such as motorcycles, autos, buses and cars, and children become familiar with ideas of speed and develop an understanding of the relationship between speed and distance, and between speed and time. They notice the movement of the needle on the speedometer. Problems of time, speed and distance arise naturally in a contextual form in all journeys.

It is helpful to introduce the topic through activities which bring out the relationships in a clear manner and to verbalise the understanding of concepts associated with speed in an explicit form. It is also worthwhile to present the topic of speed through graphs; this supports the visual learner and serves as a visual aid. By focusing on experience — recording data, converting it into a graphical form, and then studying the visual representation — the student's understanding is enhanced. It is important to gradually increase the complexity of such graphs in the context of speed.

What is speed in a general sense? It can be thought of as the *rate* of an action. For example, the number of words that one can type in one minute. It is a very useful concept, as one's speed in performing an action helps in predicting the time needed to complete that action. Speed is a rate: distance travelled per unit of time.

The first few activities described below do not require any calculations involving speed, but do cover the general ideas associated with speed, time and distance. Subsequent activities use graphs liberally to illustrate various scenarios arising out of speed problems. Children experience sport- track and field events which are a big part of a student's life. They are fascinated by speed and fast vehicles. It makes for a lot of fun to use data from the world of bikes, cars, trains, aircraft and rockets!

Learning to interpret graphs, after some experience in drawing graphs, revisits and reinforces the concept and the relationship between the concept and its visual presentation.

ACTIVITY 1

Concepts: Understanding the relationship between speed and distance, when the time is fixed
Materials: Measuring tape

Figure 1

Let four students run on a track for a given time, say 10 seconds. Other students can measure the distance covered by the four students and record the results (Figure 2).

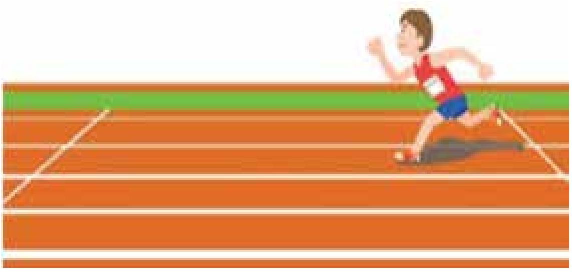


Figure 2

Use the data collected to raise and discuss questions.

Who travelled the longest distance? Why?

Who travelled the shortest distance? Why?

Help students to see that higher the speed greater the distance covered if the time taken is fixed. It is a *direct* relationship.

ACTIVITY 2

Objective: Understanding the relationship between speed and time, when the distance is fixed.
Materials: Stop watch, digital watch

Let four students run on a 100-metre track; others measure the time taken by each student. Help them to record the timings in a sheet.

Use the data collected to raise and discuss questions.

Who arrived first? What is the time taken?

Who arrived second? What is the time taken?

Help students to see that higher the speed, lesser the time taken, when the distance is fixed. It is an *inverse* relationship.

ACTIVITY 3

Objective: Inferring fastest, slowest from given information through use of logic
Materials: Information from Train/ Bus timetables

Information can be presented either in a pictorial or a tabular form. Students can study the picture and make inferences about the fastest or the slowest vehicle (Figure 3).

For example,

- Azadpur to Sarai Rohilla: 6 km: 10 min
- Kirtinagar to Sarai Rohilla: 4 km: 30 min
- Tilaknagar to Sarai Rohilla: 8 km: 75 min

Which train is the fastest? Which train is the slowest?

Figure 3

ACTIVITY 4

Objective: Inferring fastest, slowest through the use of logic
Materials: Chalk drawings of straight tracks, 2 toy cars in different colours.

Children love to play with cars and we can use this as an opportunity to teach concepts of speed. Students place two toy cars on chalk lines representing racing tracks, with a starting point and a finish point marked. The placement of the cars can be the starting point of a discussion to raise questions.

Why is the red car ahead of the green car? (Assume that both started at the same time from the starting point.)

If the green car has moved ahead, which car is going faster?

Rearrange the two cars to face each other (coming in opposite directions).

If they meet at the halfway point (assuming they started at the same time), which one is faster?

If they meet at one-fourth the distance from the left end, which car is faster: the one starting from the left or from the right? And so on.

ACTIVITY 5

Objective: Understanding the need for a uniform rate as a measure of speed
Materials: Stop watch, digital watch

Let four students run different distances: 100 m, 250 m, 500 m, 1000 m. Let the other students time them and record the timings. Ask: "How can we compare the speeds of the four students?"

Is it possible to compare the results directly? Do the students use proportions in order to arrive at a uniform rate to compare the speeds of the four students? What is the unit measure that they arrive at? Is it distance travelled per minute? Could there be other units of measure for this example?

ACTIVITY 6

Objective: Reading speed from straight tracks (metres per second)
Materials: Linear track with metre markings (to be made)

Let four students run on a track with metre markings for a given time, say 5 seconds. Other students can read the distance covered by the four students against the metre markings and record the results (Figure 4).

Use the data collected to discuss the way speed is expressed. Suppose the fastest runner covers 20 metres in 5 seconds, then we say that the speed is 4 metres per second and it is written as 4 m/s.

Students can run again for 5 more seconds from their earlier positions where they had stopped. Data can be collected to make comparisons of their speed with the earlier measurement. How do the new figures compare with the earlier figures?

Why is it different in some or all the cases?

The teacher can initiate a discussion on why speed is not a constant figure.

If a person can run 100 m in 12 seconds, does it mean that the person will be able to run 1000 m in 2 minutes? Discuss.

Students can list various factors that affect the speed of a person, a vehicle, etc.

Figure 4

Do trains and airplanes go at constant speed for some stretches?

On a clear straight road can a bicycle go at a constant speed?

Can a car go at a constant speed in a city?

Have they noticed what happens when we cycle uphill? How does going uphill affect the speed? How does going downhill affect the speed?

ACTIVITY 7

Objective: To discover the distance formula

Student 1 runs at a constant speed of 60 metres per 10 seconds. Student 2 runs at a constant speed of 110 metres per 20 seconds. Student 3 runs at a constant speed of 200 metres per 25 seconds.

How much distance will each student cover in 1 minute?

Students may solve it using unitary method or proportional method. The unitary method can be used to derive the formula, Distance = Speed × time.

Student 1: 10 seconds for 60 metres. In 1 second it will be 6 metres. In 60 seconds, it will be 6 × 60 = 360 metres.

Student 2: 20 seconds for 110 metres. In 1 second it will be 5.5 metres. In 60 seconds, it will be 5.5 × 60 = 330 metres.

Student 3: 25 seconds for 200 metres. In 1 second it will be 8 metres. In 60 seconds, it will be 8 × 60 = 480 metres.

Help the students to arrive at the distance formula, Distance = Speed × time.

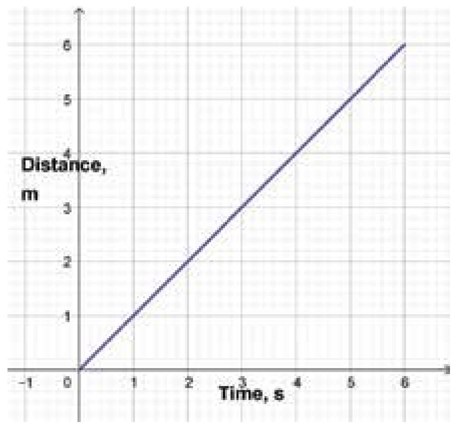


Figure 5

ACTIVITY 8

Objective: Graphing data of walking/running at constant speed

Students can work in pairs to collect data, graph the data and compare them.

Let students measure their walking rate per second and build a table and draw a distance-time graph.

Student A

Time in Seconds	0	1	2	3	4	5
Distance in Metres	0	5	10	15	20	25

Student B

Time in Seconds	0	1	2	3	4	5
Distance in Metres	0	2	4	6	8	10

Time distance graph of two people, A and B (Figure 6).

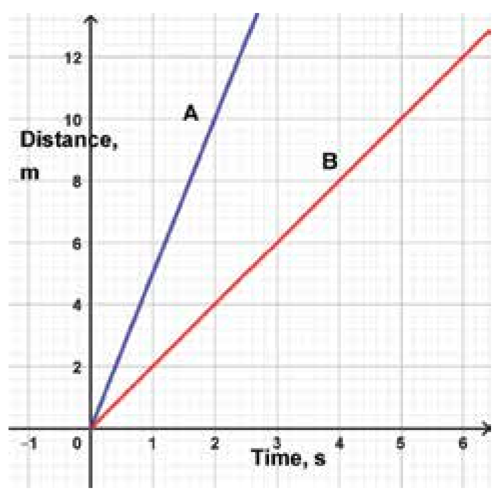


Figure 6

Student pairs can exchange the graphs for reading and interpretation.

Who is faster? Who is slower?

How far has A gone after one second? How far has B gone after one second? How much is the gap between A and B after one second? After 2 seconds? After 3 seconds?

How is the gap increasing? What would be the gap between the two after 10 seconds?

Describe A's speed in your own words.

If there is a third person C faster than both, where could C's line lie on the graph?

If C is slower than both, where could C's line lie?

If C is faster than one and slower than the other, where could C's line lie on the graph?

Discuss:

If A's speed is 10 metres per second, what might be the speeds of B and C (Figure 7)?

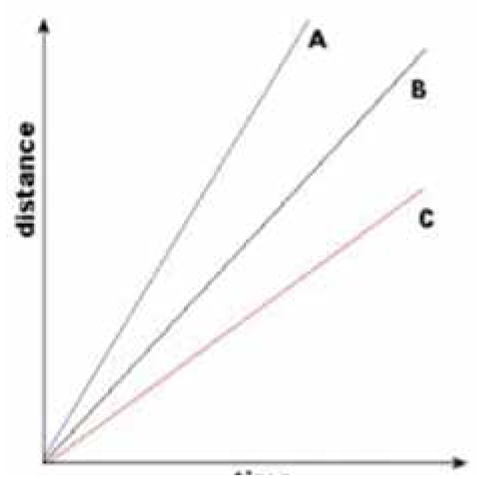


Figure 7

ACTIVITY 9

Objective: Graphing data of a vehicle at varying speed. Ideas of acceleration and deceleration.

What is acceleration? Acceleration is the rate of change of speed of an object.

A car was going initially at 10 metres per second. The car speeds up and is now moving at 15 metres per second. We say that *the car has accelerated*.

If the car slows down to 8 metres per second, the car has *decelerated*. Deceleration is the opposite of acceleration.

Here is a speed-time graph of a body moving with constant speed (Figure 8).

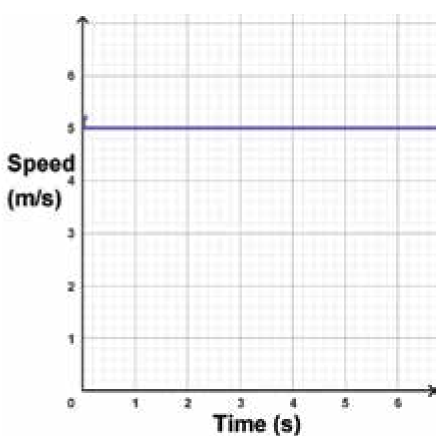


Figure 8

What do we notice? It is a straight line parallel to the time axis. We can find the distance travelled by the vehicle in a given time.

Here is the speed-time graph (Figure 9) of an accelerating vehicle which is accelerating at a constant rate.

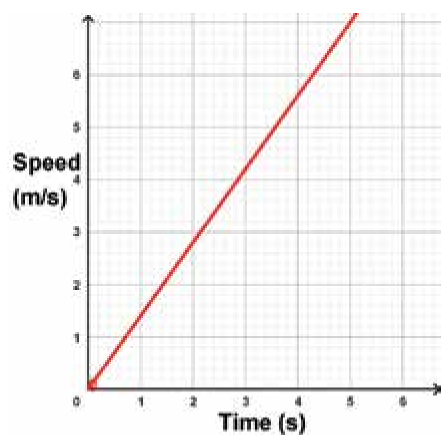


Figure 9

Here is the speed time graph (Figure 10) of a decelerating vehicle which is decelerating at a constant rate.

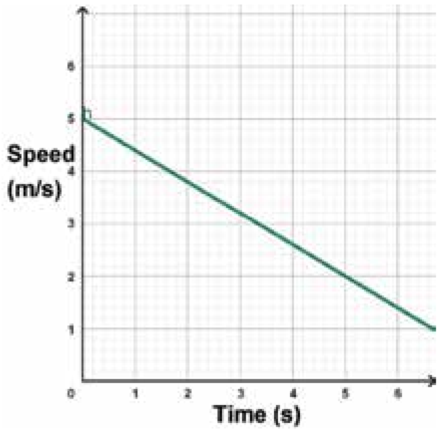


Figure 10

A rocket goes at a speed of 400 m/s for 20 seconds and at a speed of 800 m/s after that. What would the distance-time graph look like?

The students can build a graph for this data.

Time in Seconds	0	10	20	30	40	50
Distance in Km	0	4	8	16	24	32

Does the data show constant speed or acceleration?

What kind of movement is shown by the speed-time graph in Figure 11?

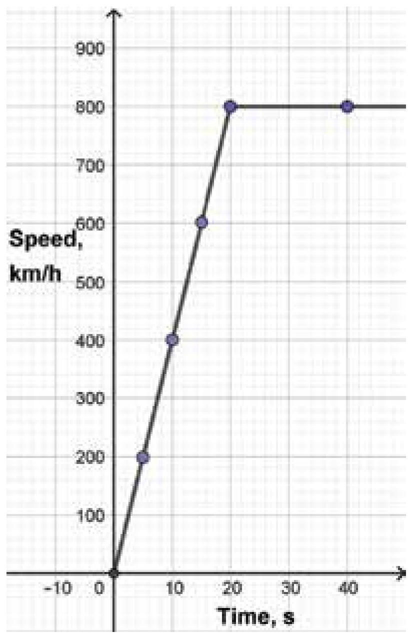


Figure 11

ACTIVITY 10

Objective: Real world data of an aeroplane or any vehicle with multiple levels of speed.

Big commercial planes generally fly at roughly 900 km per hour, but their speeds while landing and taking off are roughly 250 km per hour.

Here is a speed-time graph (Figure 12) of the flight of an airplane. The teacher can get students to interpret the flight graph.

How many times did the plane accelerate? Was the acceleration time longer or shorter than the deceleration time?

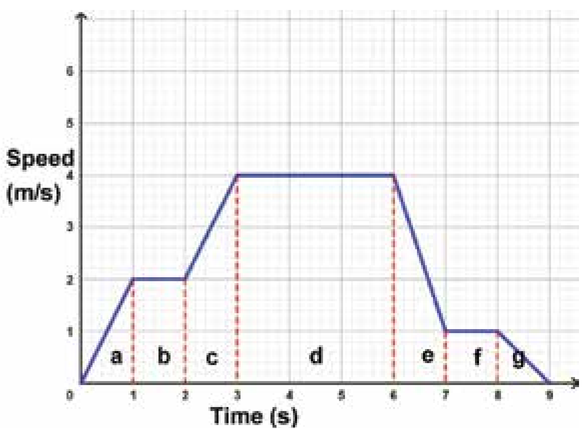


Figure 12

ACTIVITY 11

Objective: Comparing speeds of various stages.

See Figure 13. What was the speed of the car from 9:00 till 11:00?

How is it different from the speed of the car from 12:00 till 13:00?

What was happening between 11:00 and 12:00? Did the car come back to the starting point?

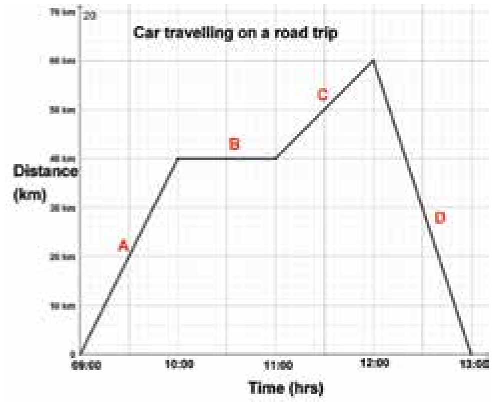


Figure 13

ACTIVITY 12

Objective: Comparing graphs of three levels of speed.

Here is a graph showing the movements of a biker, a runner and a walker (Figure 14).

What can the students say about the speed of each person?

How much more time did the runner take than the biker to reach the same distance?

How much time is the walker likely to take to reach the same distance?

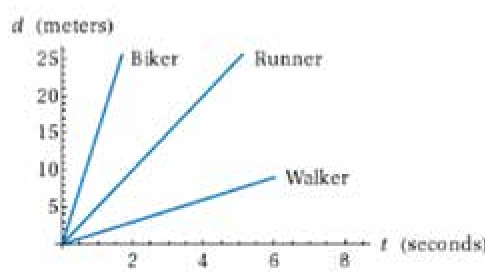


Figure 14

ACTIVITY 13

Objective: Identifying speed phases from a given graph.

Here is a graph which shows the speed of 4 vehicles (Figure 15).

Which one corresponds to the following: speeding up; going fast; slowing down; going slow?

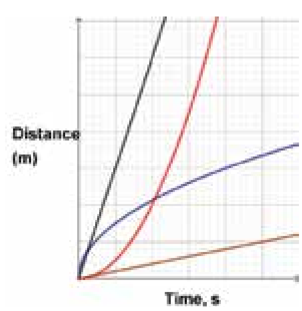


Figure 15

ACTIVITY 14

Objective: Concept of average speed.

Discuss the experience of travelling by a car, a bus or a train. Students perceive that the speed does not stay fixed during their journey.

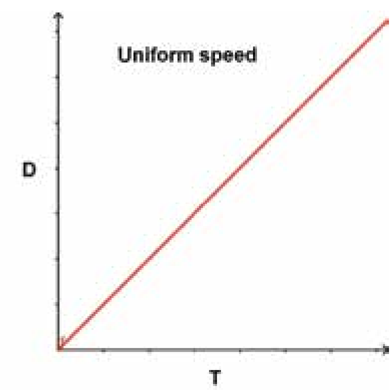
Discuss how while riding a bicycle one goes faster at certain times and slower at other times. Also, one may stop at a traffic light. Yet if someone asks how fast you go on your bike, they do not expect an answer which lists out the speeds at different times. A more understandable answer would be the *average* of all these times. It is the appropriate response when the individual figures are not important. The average helps in calculating the total time required for a journey.

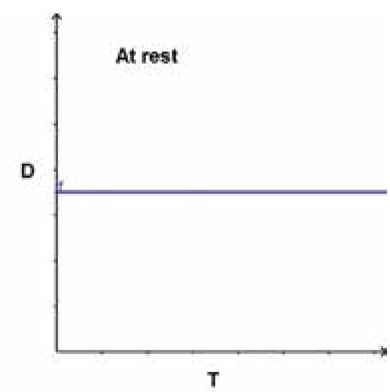
Riva rides to school which is 16 km from her home. She takes 20 minutes in the morning to reach the school and 40 minutes in the evening to come back home. How do we find her average speed? Riva travels a total of 32 kms in 60 minutes. Therefore, her average speed is 32 km per hour. Her speed may vary during the journey. But in 1 hour she covers as much distance as she would if her speed were 32 km per hour all through the journey.

ACTIVITY 15

Objective: Uniform and non-uniform speed graphs.

Discuss how a graph would look when the speed is uniform, when the vehicle is at rest, and when the speed is not uniform. Plot graphs based on data for each of the situations (Figure 16).





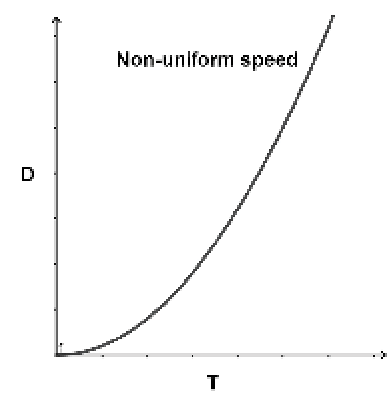


Figure 16

ACTIVITY 16

Objective: Contrast study of two graphs.

Two flights with different take off times can be plotted on a graph and studied to contrast their ascent and descent.

Similarly, two trains travelling from one terminus to another can be studied in this manner (see Figure 17).

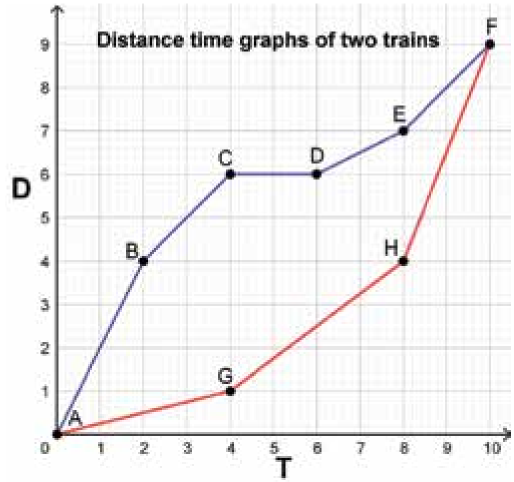


Figure 17

PICTURE PROBLEMS

Students should solve a large number of picture problems before attempting word problems without any pictures.

Here is the graph of Aditya cycling to school (Figure 18).

What is the distance from home to school?

At one point he stopped to buy a pen at the store. At what time did he stop? How long did he stop?

How do you compare his cycling speed before and after the stop?

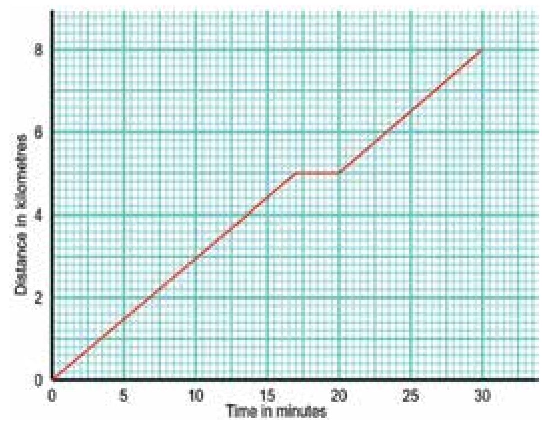


Figure 18

ACTIVITY 17

Objective: Understand the formula for vehicles moving in opposite directions

Pairs of students run on a 100-metre track from both ends towards each other. Speed of each student can be recorded. Timing of their meeting can be recorded.

The results could be studied to notice the relationship between the sum of the speeds and the time taken.

FUN FACTS FOR SHARING!

The speed of light is very close to 300,000 km/s. It takes light more than 8 minutes to travel from the Sun to the Earth! And it takes light more than 40 minutes to travel from the Sun to Pluto.

The speed of sound in air is approximately 340 m/s.

The speed of sound in water is approximately 1480 m/s, so it travels more than four times as fast in water as in air!

The speed of sound in iron is approximately 4910 m/s: close to 5 kilometres per second!

References

"The School Mathematics Project." Books 1-3, 3T, 4-5 and Teacher's guides. Internet Archive.

PADMAPRIYA SHIRALI

PADMAPRIYA SHIRALI is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. In the 1990s, she worked closely with the late Shri P K Srinivasan. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box.' Padmapriya may be contacted at padmapriya.shirali@gmail.com.

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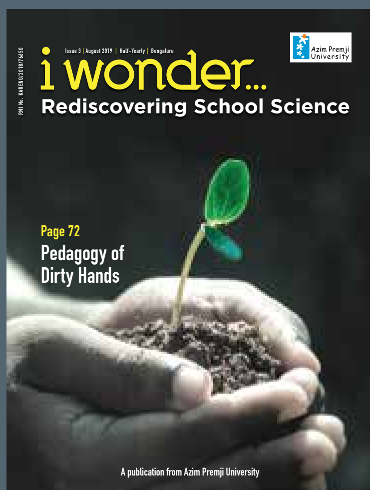

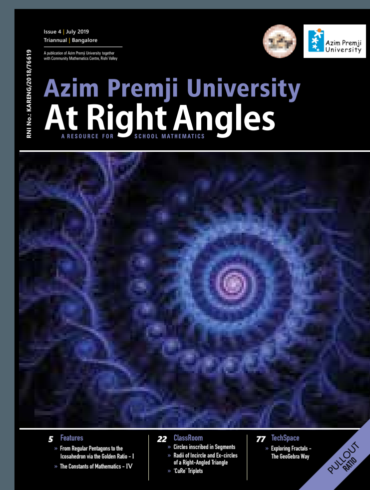

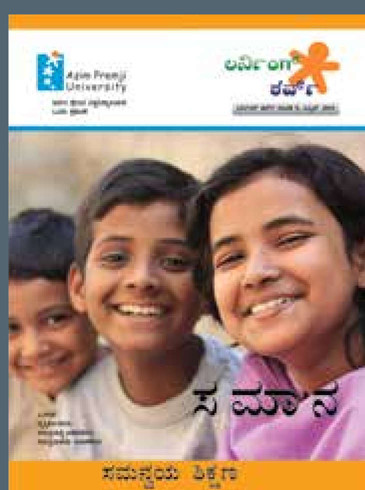

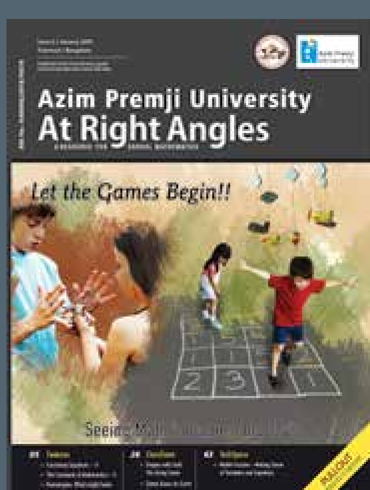

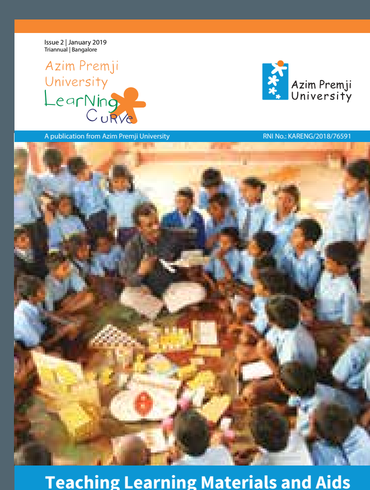

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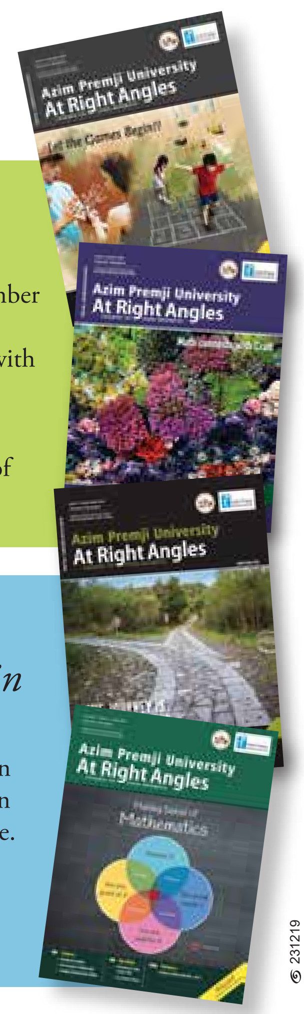

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Azim Premji University
Survey No. 66, Burugunte Village, Bikkanahalli
Main Road, Sarjapura, Bengaluru – 562 125

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